HALF-DERIVATIVES OF MODULAR FORMS AND INVARIANTS OF 3-MANIFOLDS

Don Zagier (MPI Bonn)

In the classical theory of modular forms, one associates to a modular (cusp) form

$$f(z) \;=\; \sum_{n>0} a(n)\, q^n \qquad (z \in \mathfrak{H} = \text{ upper half-plane, } q = e^{2\pi i z}\,)$$

of integral weight $k \geq 2$ on a subgroup $\Gamma \subset SL(2,\mathbb{Z})$ its *Eichler integral*

$$\widetilde{f}(z) \;=\; \sum_{n>0} \frac{a(n)}{n^{k-1}}\, q^n \qquad (z \in \mathfrak{H})\,,$$

which satisfies $(d/dz)^{k-1}\widetilde{f}(z) \;=\; (\text{const.}) \cdot f(z)$ and, as a consequence, also $(cz+d)^{k-2}\widetilde{f}(\gamma(z)) = \widetilde{f}(z) + P_\gamma(z)$ for each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, with $P_\gamma(z)$ a polynomial of degree $\leq k-2$ in z. The function $\widetilde{f}(z)$ has finite limiting values as z tends to a limit $\alpha \in \mathbb{Q}$, and in fact $\widetilde{f}(\alpha + i\varepsilon)$ is C^∞ from the right at $\varepsilon = 0$ and has a terminating Taylor expansion $\sum_{j=0}^{k-2} \widetilde{f}_j(\alpha)\varepsilon^j$ as $\varepsilon \searrow 0$ for all $\alpha \in \mathbb{Q}$, where the $f_j : \mathbb{Q} \to \mathbb{C}$ are functions with interesting arithmetic properties (generalized modular symbols, Eichler-Shimura-Manin theory of periods).

The lecture discusses an analogue when f has weight $1/2$, or more generally half-integral weight. For $k = 1/2$, the $(k-1)$-fold integration defining the Eichler integral $\widetilde{f}$ becomes the half-differentiation of the title. As an example, let $\eta(z) \;=\; q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ be Dedekind's eta-function, a modular form of weight $1/2$ on the full modular group. Euler's famous identity $\prod_{n=1}^{\infty}(1-q^n) = \sum_{n\in\mathbb{Z}}(-1)^n q^{(3n^2+n)/2}$ can be rewritten

$$\eta(z) = \sum_{n>0} \chi_{12}(n)\, q^{n^2/24}\,, \qquad \chi_{12}(n) = \Big(\frac{12}{n}\Big) \;=\; \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{12}, \\ -1 & \text{if } n \equiv \pm 5 \pmod{12}, \\ 0 & \text{otherwise,} \end{cases}$$

so its (suitably normalized) "half-derivative" can be written

$$\widetilde{\eta}(z) \;=\; \sum_{n>0} \chi_{12}(n)\, n\, q^{n^2/24} \;=\; q^{1/24}\,(1 - 5q - 7q^2 + 11q^5 + 13q^7 - \cdots)\,.$$

This function satisfies amusing identities like

$$\frac{\widetilde{\eta}(z)}{\eta(z)} \;=\; 1 - 2\sum_{n=1}^{\infty} \frac{q^n}{1-q^n}\left(1 + \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}\right),$$

but the more interesting properties concern the behavior at or near rational points.

A couple of years ago, Kontsevich wrote down the sum

$$F(q) \;=\; \sum_{n=0}^{\infty} (1-q)(1-q^2)\cdots(1-q^n)$$

and observed that, although the series diverges both for $|q| < 1$ and $|q| > 1$, and also for most points on the unit circle, it terminates (and hence converges!) at all roots of unity, giving interesting values

$$F(1)=1, \quad F(-1)=3, \quad F(i)=8-3i, \quad \ldots\,, \quad F(q) \in \mathbb{Z}[q]$$

whenever q is a root of unity. He also found experimentally an asymptotic formula

$$F\big(e^{2\pi i/k}\big) \;\sim\; e^{-\pi i(k-3+1/k)/12}\, k^{3/2} + 1 + (-\frac{2\pi i}{k}) + \frac{3}{2!}(-\frac{2\pi i}{k})^2 + \frac{19}{3!}(-\frac{2\pi i}{k})^3 + \cdots$$

as $k \to \infty$. One can also note that all (formal) derivatives of F make sense at all roots of unity, so that one obtains, for instance, expansions

$$\begin{aligned} F(q) &= 1 + x + 2x^2 + 5x^3 + 15x^4 + 53x^5 + \cdots && (q = 1-x,\ x \to 0), \\ F(q) &= 1 + \frac{t}{1!} + 3\,\frac{t^2}{2!} + 19\,\frac{t^3}{3!} + 207\,\frac{t^4}{4!} + \cdots && (q = e^{-t},\ t \to 0), \\ q^{1/24}F(q) &= 1 \,+\, 23\,\frac{u}{1!} + 1681\,\frac{u^2}{2!} \,+\, 257543\,\frac{u^3}{3!} \,+\, \cdots && (q = e^{-24u},\ u \to 0) \end{aligned}$$

as $q \to 1$. Each of these expressions turns out to have an interesting interpretation (all three of which were first observed experimentally, but are now proved): the coefficients 1, 1, 2, 5, 15, ... of the first expansion are equal to the numbers of "regularized linearized chord diagrams" as defined and computed by Stoimenow [3], and hence provide an upper bound, considerably better than the previously known ones, for the number of Vassiliev invariants of degree n; the coefficients 1, 1, 3, 19, ... of the second expansion are the same as those occurring in Kontsevich's empirical asymptotic expansion of $F(e^{2\pi i/k})$; and the coefficients 1, 23, 1681, ... of the third expansion are equal to the "T-numbers" defined by Glaisher in 1898 by the generating function

$$\frac{\sin 2X}{2\,\cos 3X} \;=\; \frac{X}{1!} \,+\, 23\,\frac{X^3}{3!} \,+\, 1681\,\frac{X^5}{5!} \,+\, \cdots\,,$$

and are also equal (up to a factor -2) to the values at negative even integers of the Dirichlet L-series associated to the character χ_{12}. The third relation is the one that connects with our "half-derivatives of modular forms": these coefficient are the same as the coefficients in the power series expansion with respect to ε of the "Eichler integral" $\widetilde{\eta}(z)$ at $z = i\varepsilon$ for $\varepsilon \searrow 0$. More generally, one has the "identity"

$$F(q) \;=\; -\frac{1}{2}\,\widetilde{\eta}(z) \qquad (q = e^{2\pi i z})$$

in which the two sides are never defined simultaneously but where the meaning is that the Taylor expansions (as power series in ε) of both sides for $z = \alpha + i\varepsilon$ agree to all orders in ε for every rational number α. Another consequence is a modular property of the function $\widetilde{\eta}_0 : \mathbb{Q} \to \mathbb{C}$, which turns out to be a kind of modular form of weight $3/2$ on $\mathbb{Q}$ modulo restrictions to $\mathbb{Q}$ of smooth functions on $\mathbb{R}$.

The same phenomenon of "functions of q with interesting behavior (only) at roots of unity" occurs in various quantum-theoretically inspired constructions in mathematics, and in particular in connection with quantum invariants of 3-dimensional manifolds. The second half of the lecture was devoted to this topic. Very briefly, the WRT (Witten-Reshitikhin-Turaev) invariant of a 3-manifold M is a "function" $W_M(q)$ which was "defined" by Witten [4] in 1989 by a path integral over an infinite-dimensional space of connections on M to which no rigorous mathematical meaning can be assigned, and then defined combinatorially and rigorously by Reshitikhin and Turaev [2] in 1991 for q a root of unity. In recent work with Ruth Lawrence, it was shown that in a certain number of cases the function $\mathbb{Q} \to \mathbb{C}$ defined by $\alpha \mapsto W_M(e^{2\pi i\alpha})$ agrees with or is related to the Eichler integral in the sense described above of a modular form of half-integral weight. In particular, for the Poincaré homology sphere the values of $W_M(q)$ are given by the Eichler integral of a theta series of weight 3/2 associated to an odd Dirichlet character of conductor 60. The details are given in [1].

References

[1] R. Lawrence and D. Zagier, *Modular forms and quantum invariants of 3-manifolds.* Asian J. Math. **3** (1999).

[2] N.Y. Reshetikhin and V. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups.* Invent. Math. **103** (1991), 547–597.

[3] A. Stoimenow,*Enumeration of chord diagrams and an upper bound for Vassiliev invariants.* J. of Knot Theory and its Ramifications **7** (1998), 93–114.

[4] E. Witten, *Quantum Field Theory and the Jones Polynomial.* Commun. Math. Phys. **121** (1989), 351–399.

[5] D. Zagier, *Vassiliev invariants and a strange identity related to the Dedekind eta-function.* Preprint, MPI 1999.