Curvature Estimate for the Volume Growth of Globally Minimal Surfaces

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Introduction.

It is well known that in each homology class of a Riemannian manifold there exists a cycle of the least volume (or simply speaking, a globally minimal surface). These globally minimal cycles yield many information of geometry and topology of their ambient manifold, however, to detect them the existence (and almost regularity) theorems can not help us so much. Intuitively, one knows that globally minimal surfaces would occupy a position of "maximal curvature" in their ambient manifold. In A.T.Fomenko's and author's announcement [LF] we gave a mathematical formulation of this conjecture. The aim of this note is to complete the proof of our announcement [LF]. In particular, we obtain an estimate for the volume growth of globally minimal surfaces in Riemannian manifolds, new isoperimetric inequalities for these surfaces, an explicit formula of the least volumes of closed surfaces in symmetric spaces. As a result, we prove that every Helgason's sphere in a compact irreducible simply connected symmetric space is a globally minimal surface. In connection with the application of integral geometry to minimal surfaces [Le 2] we note that the technique of Fomenko's method of geodesic nullity employed in this note is very close to the technique in [Le 2]. In some sense, the method of integral geometry in the theory of minimal surfaces is a bridge between the calibration method [HL] and the method of geodesic nullity [Fo 1].

§1. Geodesic nullity of Riemannian manifolds and the volume of globally minimal submanifolds.

a) Let $B_r(x)$ be the ball of radius r in a tangent space T_xM. Recall that the injective radius $R(x)$ of a Riemannian manifold M at a point x is defined as follows: $R(x) = sup\{r|\ Exp : B_r(x) \longrightarrow M$ is a diffeomorphism $\}$. The injective radius $R(M)$ of M is defined as: $R(M) = \inf_{x\in M} R(x)$. Now we fix a point $x_0 \in M$. We define *k-dimensional deformation coefficient* $\chi_k(x > x_0)$ as follows (cf.[Fo 2]). Suppose that Π_x^{k-1} is a $(k-1)$-plane through x in the tangent space T_xM. Denote D_ε^{k-1} the disk of radius ε in Π_x^{k-1}, and by S_ε the disk $Exp(D_\varepsilon^{k-1})$. We consider the cone CS_ε formed by geodesics joining the vertex x_0 and the base S_ε. We put

$$\chi(x > x_0, \Pi_x^{k-1}) = \lim_{\varepsilon\to 0} \frac{vol_k CS_\varepsilon}{vol_{k-1} S_\varepsilon},$$

$$\chi(x > x_0) = \max_{\Pi^{k-1} \subset T_x M} \chi_k(x, \Pi^{k-1}).$$

b)Let $f(x)$ be the function which measures the distance between point $x \in M$ and the fixed point x_0. We set

$$q(x_0, r) = \exp\left(\int_0^r \left(\max_{x \in \{f=t\}} \chi_k(x > x_0)\right)^{-1} dt\right). \tag{1.1}$$

We put

$$\Omega_k(x_0) = \lambda_k q(x_0, R(x_0)),$$
$$\Omega_k = \inf_{x_0 \in M} \Omega_k(x_0),$$

where λ_k is the volume of the ball of radius 1 in R^k.

The defined value is called *the k^{th} geodesic nullity* of Riemannian manifold M. The following theorem was obtained by Fomenko in 1972 [Fo 2].

Theorem 1.1. *Let $X^k \subset M^n$ be a globally minimal surface. Then the following inequality holds*

$$vol_k(X^k) \geq \Omega_k \geq 0.$$

Remark. Theorem 1.1 has a clear geometric interpretation. It is a consequence of the fact that the derivative of logarithm of the volume function exhausting a globally minimal surface X in M is greater than the function under integral in (1.1). This derivative $d/dt(ln\, vol X_t)$ equals the "isoperimetric" relation $vol\, \partial X_t / vol\, X_t$ (see also Proof of Theorem 2.3). The injective radius of M is involved, because X is a globally minimal surface in M.

§2. Lower bound for geodesic nullities of Riemannian manifolds. New isoperimetric inequalities.

Suppose that the section curvature of manifold M in any 2-plane is not greater then a^2 ($a \in R$ or $a \in \sqrt{-1} \otimes R$).

Theorem 2.1 [LF]. *Lower bound of geodesic nullity.*

a) If $a^2 \geq 0$ and $Ra \leq \pi$ then we have:

$$\Omega_k(M) \geq k\, \lambda_k\, a^{1-k} \int_0^R (\sin at)^{k-1}\, dt.$$

b) If $a^2 > 0$ and $Ra > \pi$ then we have:

$$\Omega_k(M) > vol(S^k(r = 1/a)).$$

c) If $a = 0$ then we have $\Omega_k(M) \geq \lambda_k\, R^k$.

d) If $a^2 \leq 0$ then we have:

$$\Omega_k(M) \geq k\,\lambda_k\,|a|^{1-k} \int_0^R (\sinh|a|t)^{k-1}\,dt.$$

Theorem 2.2 [LF]. *Upper bound of the deformation coefficient. Let r be the distance between x and x_0.*

a) If $a^2 \geq 0$ and $r \leq \pi/a$ then we have:

$$\chi_k(x > x_0) \leq \frac{\int_0^r (\sin at)^{k-1} dt}{(\sin ar)^{k-1}}$$

b) If $a = 0$ then we have:

$$\chi_k(x > x_0) \leq \frac{r}{k}$$

c) If $a^2 \leq 0$ then we have

$$\chi_k(x > x_0) \leq \frac{\int_0^r (\sinh\,|a|)^{k-1}\,dt}{(\sinh|a|r)^{k-1}}.$$

Theorem 2.3 [LF]. *Isoperimetric inequality. Assume that X^k is a globally minimal surfaces through a point $x \in M$. Let $B_x(r)$ be the geodesic ball of radius r and with its center at x. Denote A_r^{k-1} the boundary of the intersection $X^k \bigcap B_x(r) = X_r^k$.*

a) If $a^2 > 0$ and $r \leq \min(R, \pi/a)$ then we have:

$$\frac{vol(A_r^{k-1})}{vol(X_r^k)} \geq \frac{\sin(ar)^{k-1}}{\int_0^r (\sin at)^{k-1} dt}.$$

Consequently, the following inequality holds

$$vol(A_r^{k-1}) \geq k\,\lambda_k\,a^{1-k}\,\sin^{k-1}(ar).$$

b) If $a = 0$ and $r \leq R$ then we have:

$vol(A_r^{k-1}) \geq k\,\lambda_k\,r^{k-1}$ = the volume of the standard k-dimensional sphere S^k of radius r.

Hence we imply the following inequalities:

$$vol(A_r^{k-1}) \geq (kr)^{-1}\,vol(X_r^K),$$

$$vol(X_r^k) \leq (k)^{\frac{k}{1-k}}\,(\lambda_k)^{\frac{1}{1-k}}(vol_{k-1}A_r)^{\frac{k}{k-1}}.$$

c) If $a^2 < 0$ then we have:

$$\frac{vol(A_r^{k-1})}{vol(X_r^k)} \geq \frac{(\sinh|a|r)^{k-1}}{\int_0^r (\sinh|a|t)^{k-1}\,dt}.$$

Hence we get

$$vol(A_r^{k-1}) \geq k\,\lambda_k\, sh^{k-1}(|a|t)/|a|^{k-1}.$$

The estimates in Theorems 2.1 and 2.2 are sharp, that is, in many cases they become equalities. Roughly speaking, these theorems tell us that globally minimal surfaces tend to a position of "maximal curvature" in their ambient manifold. Now we show some consequences of Theorem 2.1.

Corollary 2.4. *If M is a compact simply-connected symmetric space of sectional curvature not greater than a, then the volume of any non-trivial cycle is not less than the volume of k-dimensional sphere of curvature a.*

Corollary 2.5. *The length of a homologically non-trivial loop in a manifold M is not less then the double injective radius of M.*

Corollary 2.6 *Lower bound for the volume of a manifold.*

a) If $a^2 > 0$ then we get:

$$vol(M^n) \geq n\,\lambda_n\, a^{1-n} \int_0^R (\sin at)^{n-1}\,dt.$$

b) If $a = 0$, then we get: $vol(M^n) \geq n\,\lambda_n R^n$.

c) If $a^2 < 0$ then we get

$$vol(M^n) \geq n\,\lambda_n |a|^{1-n} \int_0^R (\sinh|a|t)^{n-1}\,dt.$$

Remark. The estimate in Corollary 2.6 coincides with that of Bishop's theorem [BC].

Now we infer from Theorems 2.2 and 2.3 the following consequence on the volume growth of globally minimal surfaces.

Corollary 2.7. *Let X^k be a globally minimal surface in a complete non-compact Riemannian manifold M of non-positive curvature. Then the function $V(r) = vol_k B_X(r)$ grows at least as a polynomial of r of degree k, where $B_X(r)$ is a geodesic ball of radius r in X^k. If the curvature of M has an upper bound strictly less than zero then the function $V(r)$ grows at least as the exponent of r.*

Remark. It is well known that there is a close relationship between the curvature of a Riemannian manifold M and the growth of its volume [BC]. As a consequence, we obtain the estimate for the growth of its fundamental group (see [M]), and other topological and geometrical invariants of M such as the

Betti numbers, the eigenvalues of the Laplace operator and the Gromov invariants [Br 1, Br 2, Gr 1, Gr 2, Gr 3].

Proof of Theorems and Corollaries. Let us write down an explicit formula for the coefficient $\chi_k(x > x_0, \Pi_x^{k-1})$. Suppose $\lambda(t)$ is the shortest geodesic curve joining the points $x_0 = \lambda(0)$ and $x = \lambda(r)$. So, for $0 < t < r$, point $\lambda(t)$ is not conjugated with x_0. We now consider the case if $x = \lambda(r)$ is not conjugated with x_0 (otherwise, we should take the limit). Choose an orthonormal basis of vectors $Y_1(r), ..., Y_{k-1}(r)$ in the plane $\Pi^{k-1} \subset T_xM$. (Let us recall that by definition Π^{k-1} must to be orthogonal to $\dot{\lambda}(r)$). We denote K_ρ the (k-1)-dimensional cube in Π_x^{k-1} with the edges $\rho Y(r)$. Then the formula for deformation coefficient $\chi_k(x > x_0)$ can be rewritten as follows:

$$\chi_k(x > x_0, \Pi_x^{k-1}) = \lim_{\rho \to 0} \frac{vol_k(C\tilde{K}_\rho)}{vol_{k-1}\tilde{K}_\rho},$$

here we set $\tilde{K}_\rho = Exp_x K_\rho$.

We denote λ_{st}^j the s-geodesic, joining points x_0 and $Exp_x(sY_j(r))$. Put

$$Y_j(t) = \frac{d}{ds}_{|s=0} \lambda_{st}^j.$$

Then $Y_j(t)$ is an Jacobian vector field with the data $Y_j(0) = 0$, $Y_j(r)$ - the chosen vector in Π^{k-1}, and besides, for every t we have $Y_j(t) \perp \dot{\lambda}(t)$. We note that the tangent plane to the orthogonal section $\tilde{K}_{t\rho}$ of the cone $C\tilde{K}_\rho$ at the point $\lambda(t)$ possesses the basis of vectors $Y_1(t), .., Y_{k-1}(t)$. Hence,

$$vol_{k-1}(\tilde{K}_{t\rho}) = \rho^{k-1}(|Y_1(t) \wedge ... \wedge Y_{k-1}(t)|) + o(\rho^{k-1}).$$

This yields

$$\chi_k(x > x_0, \Pi^{k-1}) = \lim_{\rho \to 0} \frac{vol(C\tilde{K}_\rho)}{vol_{k-1}(\tilde{K}_\rho)} =$$

$$= \lim_{\rho \to 0} \frac{\int_0^r vol_{k-1}\tilde{K}_{t\rho}\, dt}{vol_{k-1}\tilde{K}_{r\rho}} = \frac{\int_0^r |Y_1(t) \wedge ... \wedge Y_{k-1}(t)|\, dt}{|Y_1(r) \wedge ... \wedge Y_{k-1}(r)|}. \tag{2.1}$$

Proof of Theorem 2.2. Put $F(t) = |Y_1(t)| \cdot ... \cdot |Y_{k-1}(t)|$. Since $|Y_1(t) \wedge ... \wedge Y_{k-1}(t)| \leq F(t)$, and this inequality becomes an equality at $t = r$, the formula(2.1) yields

$$\chi_k(x > x_0, \Pi^{k-1}) \leq \frac{\int_0^r F(t)\, dt}{F(r)}. \tag{2.2}$$

We need the following lemmas.

Lemma 2.8. *Suppose $F(t)$ be in(2.2). If for all t and Y_j the section curvature $S(\dot\lambda(t), Y_j(t)) \le a^2$, where $a > 0$, then the function $F(t)/G(t)$ increases on the interval $[0, r]$. Here $G(t) = (\sin at)^{k-1}/(\sin ar)^{k-1}$.*

Lemma 2.9. *Suppose the function $F(t)$ and $G(t)$ be in the Lemma 2.8. Then the following inequality holds*

$$\frac{\int_0^r F(t)\,dt}{F(r)} \le \frac{\int_0^r G(t)\,dt}{G(r)}.$$

Proof of Lemma 2.8. The Rauch's comparison Theorem [BC] states that the function $f_j(t) = |Y_j(t)|/\sin at$ increases on the interval $[0, r]$. Hence, the function $F(t)/G(t) = \prod f_j$ is such a function.

Proof of Lemma 2.9. Since the function $F(t)/G(t)$ increases on the interval $[0, r]$, we get $F(x_i)G(r) \le G(x_i)F(r)$ for every $0 \le x_i \le r$. Hence we obtain

$$\sum_{k=0}^{n} F(kr/n)G(r) \le \sum_{k=0}^{n} G(kr/n)F(r).$$

Letting $n \to \infty$ we easily infer Lemma 2.10 from the above inequality.

Let us continue the proof of Theorem 2.2.

Taking into account (2.2) and lemmas 2.8, 2.9 we get

$$\chi_k(x, \Pi^{k-1}) \le \frac{\int_0^r F(t)\,dt}{F(r)} \le \frac{\int_0^r (\sin at)^{k-1}\,dt}{(\sin ar)^{k-1}}.$$

The proof of the first part in Theorem 2.2 is completed. In the same way we can prove the rest parts (b) and (c).

Proof of Theorem 2.1. Let us recall the definition

$$\Omega_k(x_0, r) = \lambda_k \exp \int_0^r (\max_{x\in\{f=t\}} \chi_k(x > x_0))^{-1}\,dt.$$

Theorem 2.1 (a) yields

$$\Omega_k(x_0, r) \ge \lambda_k \exp \int_0^r \frac{\sin at)^{k-1}\,dt}{\int_0^t (\sin a\tau)^{k-1}\,d\tau}.$$

Put

$$\Phi_k(r) = \lambda_k \exp \int_0^r \frac{(\sin at)^{k-1}\,dt}{\int_0^t (\sin a\tau)^{k-1}\,d\tau}.$$

Clearly , we can infer Theorem 2.1(a) from the following identity

$$\Phi_k(r) = k\,\lambda_k\,a^{1-k} \int_0^r (\sin at)^{k-1}\,dt. \tag{2.3}$$

Proof of Formula (2.3). Put $\Phi_k^*(r)$ equal the right hand side in(2.3). We observe that the functions $\Phi_k(r)$ and $\Phi_k^*(r)$ satisfy the same differential equation:

$$\frac{\Phi_k(r)}{(\partial/\partial r)\Phi_k(r)} = \frac{\Phi_k^*(r)}{(\partial/\partial r)\Phi_k^*(r)} = \frac{\int_0^r (\sin at)^{k-1}\, dt}{(\sin ar)^{k-1}}. \tag{2.4}$$

Let us consider the limit

$$\lim_{r\to 0} \frac{\Phi_k(r)}{\Phi_k^*(r)} = \lim_{r\to 0} \frac{\lambda_k \, \exp \int_0^r (\int_0^t (\sin a\tau)^{k-1}\, d\tau)^{-1} (\sin at)^{k-1}\, dt}{k\, \lambda_k a^{1-k} \int_0^r (\sin at)^{k-1}\, dt}. \tag{2.5}$$

Taking into account the increaseness of the function $(a\tau/\sin a\tau)^{k-1}$ on the interval $[0,t]$, where $0 \le t \le \pi/a$, and using Lemma 2.10 we obtain

$$\frac{(\sin at)^{k-1}}{\int_0^t (\sin a\tau)^{k-1}\, d\tau} < \frac{t^{k-1}}{\int_0^t \tau^{k-1}\, d\tau} = \frac{k}{t}. \tag{2.6}$$

Combining (2.5) and (2.6) yields the following inequality

$$\lim_{r\to 0} \frac{\Phi_k(r)}{\Phi_k^*(r)} \le \lim_{r\to 0} \frac{\lambda_k \, \exp \int_0^r k\, t^{-1}\, dt}{k\, \lambda_k \, a^{1-k} \int_0^r (\sin at)^{k-1}\, dt}.$$

Fix $\varepsilon > 0$. Since $\lim_{y\to 0}(\sin at/\, at) = 1 > 1-\varepsilon$ we get the following inequality.

$$\lim \frac{\Phi_k(r)}{\Phi_k^*(r)} \le \lim_{r\to 0} \frac{\exp \int_0^r (k/t)\, dt}{k \, \int_0^r (1-\varepsilon)^{k-1} (at)^{k-1}\, a^{1-k}\, dt} =$$

$$= \lim_{r\to 0} \frac{r^k}{r^k (1-\varepsilon)^{k-1}} = (1-\varepsilon)^{1-k}. \tag{2.7}$$

Since the inequality (2.7) holds for all $\varepsilon > 0$ we have

$$\lim_{r\to 0} \frac{\Phi_k(r)}{\Phi_k^*(r)} \le \lim_{\varepsilon\to 0} (1-\varepsilon)^{1-k} = 1. \tag{2.8}$$

On the other hand, applying the inequality $\sin at < at$ to (2.5) we get

$$\lim_{r\to 0} \frac{\Phi_k(r)}{\Phi_k^*(r)} \ge \lim_{r\to 0} \frac{\lambda_k \, \exp(\int_0^r (\sin ay)^{k-1}\, a^{1-k}\, k\, y^{-k}\, dy}{k\, \lambda_k \, \int_0^r t^{k-1}\, dt}.$$

Fixed ε as above we have

$$\lim_{r\to 0} \frac{\Phi_k(r)}{\Phi_k^*(r)} \ge \lim_{r\to 0} \exp\Big(\int_0^r \frac{(1-\varepsilon)(ay)^{k-1}\, dy}{a^{k-1}\cdot y^k \cdot k^{-1}}\Big)\, r^{-k} =$$

$$= \lim_{r\to 0} r^{-k} \exp\left(\int_0^r \frac{(1-\varepsilon)^{k-1}\, k\, dy}{y}\right) = r^{k((1-\varepsilon)^{k-1}-1)}. \tag{2.9}$$

Letting $\varepsilon \to 0$ we infer from (2.9)

$$\lim_{r\to 0} \frac{\Phi_k(r)}{\Phi_k^*(r)} \geq \lim_{\varepsilon\to 0} r^{k((1-\varepsilon)^{k-1}-1)} = 1. \tag{2.10}$$

Now we obtain from (2.8) and (2.10)

$$\lim_{r\to 0} \frac{\Phi_k(r)}{\Phi_k^*(r)} = 1. \tag{2.11}$$

The differential equation (2.4) for $\Phi_k(r)$ and $\Phi_k^*(r)$ has the same initial data (2.11). So we get the identity $\Phi_k^* = \Phi_k$, that completes the proof of Theorem 2.2 (a).

The rest parts (c), (d) can be proved in the same way. The part (b) follows from that fact if $R > \pi/a$ then we have $\Omega_k(M) > \Omega_k(x_0, \pi/a) \geq vol(S^k, 1/a)$. This completes the proof of Theorem 2.2.

Proof of Theorem 2.3. Let r be as in Theorem 2.3. We denote CA_r^{k-1} the geodesic cone of base A_r^{k-1} and with its vertex at the point x. Since X_r^k is a globally minimal surface, and the cone CA_r^{k-1} is homological to X_r^k, we have $vol(X_r^k) \leq vol(CA_r^{k-1})$. Hence we conclude

$$\frac{vol(A_r^{k-1})}{vol(X_r^k)} \geq \frac{vol(A_r^{k-1})}{vol(CA_r^{k-1})} \geq$$

$$\geq (\max_{y\in A_r} \chi_k(y > x))^{-1} \geq \frac{(\sin ar)^{k-1}}{\int_0^r (\sin at)^{k-1}\, dt}. \tag{2.12}$$

(The second inequality in (2.12) is inferred from the following formula

$$vol(CA_r^{k-1}) = \int_{A_r^{k-1}} \chi_k(y > x, \Pi_y^{k-1})\, dy,$$

where Π_y^{k-1} denotes the tangent space to A_y^{k-1} at y. The third inequality in (2.12) is a consequence of Theorem 2.2(a).)

We infer from (2.12) the following inequality

$$vol(CA_r^{k-1}) \geq vol(X_r^k) \frac{(\sin ar)^{k-1}}{\int_0^r (\sin at)^{k-1}\, dt} \geq \Omega_k(r) \frac{(\sin ar)^{k-1}}{\int_0^r (\sin at)^{k-1}\, dt}. \tag{2.13}$$

Combining (2.13) and Theorem 1.1(a) yields

$$vol\, A_r^{k-1} \geq k\, \lambda_k\, a^{1-k}\, (\sin ar)^{k-1}.$$

This completes the proof of Theorem 2.3(a). The rest part of Theorem 2.3 can be proved in the same way.

Proof of Corollary 2.4. It is well known that a compact simply-connected symmetric space satisfies the relation: $Ra = \pi$. So we get Corollary 2.4 from Theorem 2.1.

Proof of Corollary 2.5.. Clearly, $\lambda_1 = 2$. So we obtain $\Omega_1(M) \geq 2\int_0^R 1\,dt = 2R$.

Proof of Corollary 2.6.a. Let $\dim(M) = m$. Then $vol(M) \geq \Omega_m(M)$. With the help of (2.3) we obtain $\Omega_m(M) \geq \Phi_m(R) = k\,\lambda_k a^{1-k} \int_0^r (\sin at)^{k-1}\,dt$. This completes the proof Corollary 2.6.a. The rest assertions can be proved in the same way.

§3. Explicit formula for geodesic nullities of symmetric spaces. Global minimality of Helgason's spheres.

Suppose M is a compact symmetric space. Let us compute the deformation coefficient associated with fixed point $e \in M$. Without loss of generality we compute this coefficient at point $Exp\,tx \in M$, where x is a vector in a Cartan space H_{lM} of the tangent space lM to M at e. We shall redenote $\chi_k(Exp\,rx) = \chi_k(Exp\,rx > e)$.

Theorem 3.1. *Let $\{\alpha_i\}$ be the roots systems of symmetric space M with respect to H_{lM}. Suppose x is a vector of unit length in H_{lM}. Without loss of generality we assume that $\alpha_1(x) \geq ... \geq \alpha_p(x) = 0 = \alpha_{p+1}(x) = ...$.*

a) If $k < p$ then the following equality holds

$$\chi_k(Exp\,rx) = \frac{\int_0^r \sin(\alpha_1(x)t) \cdot ... \cdot \sin(\alpha_k(x)t)\,dt}{\sin \alpha_1(x)r \cdot ... \cdot \sin \alpha_k(x)r}.$$

b) If $k \geq p$ then the following inequality holds

$$\chi_k(Exp\,rx) = \frac{\int_0^r \sin(\alpha_1(x)t) \cdot ... \cdot \sin(\alpha_{p-1}(x)t)t^{k-p}\,dt}{\sin(\alpha_1(x)r) \cdot ... \cdot \sin(\alpha_{p-1}(x)r)r^{k-p}}.$$

Lemma 3.2. *Let $\{v_1, ..., v_k\} \in M$ be an orthonormal frame which consisting of the eigenvectors of eigenvalues $\alpha_1^2(x), ..., \alpha_p^2(x), , , , 0, ...0$ of the operator ad_x^2. Denote $V_i(t)$ the parallel vector field along the geodesics $Exp\,tx$ such that $V_i(0) = v_i$, and denote $W_i(t)$ the Jacobian vector field along $Exp\,tx$ such that $W_i(0) = v_i$. Then we have the following relation*

- *if $i < p$ then $W_i(t) = \alpha_i(x)^{-1} \sin(\alpha_i(x)t)V(t)$,*

- *if $i \geq p$ then $W_i(t) = tV_i(t)$.*

Proof of Lemma 3.2. In the tangent space lM the vector field tv_i is a Jacobian field along the ray tx. It is well known that the vector field $dExp_{|tx}(tv_i)$ is also a Jacobian vector field along the geodesic $Exp\,tx \subset M$ [He]. Let us write an explicit formula for the differential of the exponential mapping at the point tx ([He]). We will identify M with the quotient G/U, moreover, the tangent space lM with the orthogonal complement to the algebra lU in the algebra lG. We denote exp the exponential mapping from the algebra to the group. Then $\exp tx$ is an element in G acting on M and we denote $d\tau(\exp tx)$ the differential of this action. We have

$$dExp_{|tx}(tv) = d\tau(\exp tx)\sum_{n=0}^{\infty}\frac{t^2\, ad_x^2(tv_i)}{(2n+1)!} =$$

$$= d\tau\,(\exp tx)\sum_{n=0}^{\infty}\frac{(t^2\alpha_i^2(x))^n(-1)^n}{(2n+1)!}(tv_i) =$$

$$= d\tau(\exp tx)\frac{\sin\alpha_i(x)t}{t\,\alpha_i(x)}tv_i = \frac{sin(\alpha_i(x)t)}{\alpha_i(x)}(d\tau(\exp tx)v_i) \qquad (3.1)$$

Now we observe that the parallel vector field V_i is obtained from the vector v_i by the shift $d\tau(exp)$ along the geodesic $Exp\,tx$, that is, $V_i(t) = d\tau(\exp tx)\,v_i$. Hence we get Lemma 3.2 from (3.1).

Proof of Theorem 3.1. Now we compute the coefficient $\chi_k(Exp\,rx, \Pi^{k-1})$. We observe that the tangent space $\Pi^{k-1}(t)$ to the normal section of the cone CD_ε^{k-1} at the point $Exp\,tx$ can be represented as the sum $\sum a_i\Pi_i^{k-1}(t)$, where a_i are constant, and $\Pi_i^{k-1}(t)$ is the basis in the space $\Lambda_{k-1}(T_{Exp\,tx}M)$ such that $\Pi_i^{k-1}(t)$ is generated by the orthonormal frame of vectors $W_i(t) \in T_{Exp\,tx}M$. Using formula (2.1) we get

$$\chi_k(Exp\,rx, \Pi^{k-1}) = \frac{\int_0^r |\Pi^{k-1}(t)|\,dt}{|\Pi^{k-1}(r)|} = \frac{\sum_i \int_0^r a_i\,|\Pi_{k-1}^i(t)|\,dt}{\sum_i a_i\,|\Pi_{k-1}^i(r)|}.$$

Hence we obtain

$$\chi_k(Exp\,rx) = \max_i \frac{\int_0^r |\Pi_{k-1}^i(t)|\,dt}{\Pi_{k-1}^i(r)}.$$

Combining Lemma 3.2 , Lemma 2.8 and Lemma 2.9 we get

$$\max_i \frac{\int_0^r |\Pi_i^{k-1}(t)|dt}{|\Pi_{k-1}^i(r)|} = \frac{\int_0^r sin(\alpha_1(x)t)\cdot\ldots\cdot sin(\alpha_k(x)t)\,dt}{sin(\alpha_1(x)r)\cdot\ldots\cdot sin(\alpha_k(x)r)}$$

if $k < p$. In the same way we can prove the theorem in the case $k \geq p$. The proof of Theorem 3.1 is complete.

Corollary 3.3. *If M is a symmetric space of rank = 1 ,that is, $dim\, H_{lM} = 1$, then the deformation coefficient $\chi_k(Exp\, rx)$ depends only on r.*

a) For $M = S^n$ (or RP^n) we have $\chi_k(r) = \int_0^r (\sin t)^{k-1}\, dt/(\sin r)^{k-1}$.

b) For $M = CP^n$ we have

$$\chi_k(r) = \frac{\int_0^r (\sin\sqrt{2}t)(\sin t)^{2k-2}\, dt}{\sin\sqrt{2}r(\sin r)^{2k-2}}.$$

c) For $M = HP^n$ we have

$$\chi_k(r) = \frac{\int_0^r (\sin\sqrt{2}t)^3(\sin t)^{4k-4}\, dt}{(\sin\sqrt{2}r)^3(\sin r)^{4k-4}}.$$

We immediately obtain the following consequence.

Corollary 3.4. [Fo 1]. *For any $k \le n$ the standardly embedded space RP^n (and CP^n, HP^n resp.) has the volume $= \Omega_k(RP^n)$ (and $\Omega_{2k}(CP^n), \Omega_{4k}(HP^n)$ resp.), therefore it is a globally minimal submanifold.*

Let us now compute geodesic nullities $\Omega_k(RP^n), \Omega_{2k}CP^n, \Omega_k(HP^n)$. Clearly, $\Omega_k(RP^n) = \frac{1}{2}vol(S^k(1))$ can be computed from the following formulas. First, we take integration over parallel sections of the unit ball

$$\lambda_k = 2\lambda_{k-1}\int_0^{\pi/2} cos^k\alpha\, d\alpha.$$

Taking into account (2.3) we get

$$vol\, S^k(1) = (k+1)\lambda_{k+1} = 2\lambda_k k\int_0^{\pi/2} sin^{k-1}\alpha\, d\alpha.$$

Hence we obtain the following identity

$$k+1 = \frac{2k\lambda_k\int_0^{\pi/2} sin^{k-1}\alpha\, d\alpha}{2\lambda_k\int_0^{\pi/2} cos^{k+1}\alpha\, d\alpha}. \tag{3.2}$$

We infer from (3.2) the following equation

$$\int_0^{\pi/2} sin^{k+1}\alpha\, d\alpha = \frac{k}{k+1}\int_0^{\pi/2} sin^{k-1}\alpha\, d\alpha. \tag{3.3}$$

Using (3.3) we easely get

$$\lambda_{2k} = \frac{\pi^k}{k!}, \qquad \lambda_{2k+1} = \frac{\pi^k 2^{k+1}}{(2k+1)!!}.$$

Let us compute $\Omega_{2k}(CP^n) = vol(CP^k)$. Using Corollary 3.3 and taking into account $R(CP^n) = \pi/\ sqrt2$ we get

$$\Omega_k(CP^n) = \lambda_{2k} \exp \int_0^{\pi/sqrt2} \frac{dt}{(\sin\sqrt{2}t)(\sin t)^{2k-1}} =$$

$$. \quad = 2k\lambda_{2k} \int_0^{\pi/sqrt2} (\sin\sqrt{2}t/\sqrt{2})(\sin t)^{2k-2}\, dt =$$

$$= 2^k 2k\, \lambda_{2k} \int_0^1 x^{2k-1}\, dx = \pi^{2k}/k!.$$

In the same way we compute $\Omega_{4k}(HP^n) = vol(HP^k)$. We have

$$\Omega_{4k}(HP^n) = \lambda_{4k} \exp \int_0^{\pi/\sqrt{2}} \frac{dt}{(\sin\sqrt{2}t)^3(\sin t)^{4k-4}} =$$

$$= 4k\lambda_{4k} \int_0^{\pi/\sqrt{2}} (\sin\sqrt{2}t/\sqrt{2})^3(\sin t)^{4k-4}\, dt =$$

$$= 2^{2k} 4k\lambda_{4k} \int_0^1 y^{4k-4}(1-y^2)\, dy =$$

$$= \pi^{2k}\, 2^{2k}/(2k+1)!.$$

Remark. Operator ad_x^2 coincides with the Ricci transformation $R_x : y \to R_{xy}x$ in the tangent space lM. Therefore, the deformation coefficient $\chi_k(Exp\, rx)$ get the maximal value, if and only if the plane Π^{k-1} is an eigenspace with the maximal eigenvalue of the induced Ricci transformation in the space $\Lambda_{k-1}T_{Exp\, rx}M$. Roughly speaking, the curvature at point $Exp\, rx$ in direction (rx, Π^{k-1}) get the maximal value.

It is well known that in a simply connected irreducible compact symmetric space M there are totally geodesic spheres of curvature a^2, where a^2 is the upper bound of section curvature on M. Further, any such sphere lies in some totally geodesic Helgason's sphere of maximal dimension $i(M)$. All Helgason's spheres are equivalent under the action of the isometry group $Iso(M)$. Moreover, they are of the same curvature a^2. Now we immediately get from Corollary 2.8 the following Proposition.

Proposition 3.5. *If a Helgason sphere $S(M)$ realizes a non-trivial cycle in a homology group of space M, then it is a globally minimal submanifold in M.*

First, we write the list of Helgason's spheres realizing a non-trivial cycles in real homologies of compact irreducible simply-connected symmetric spaces.

1) If M is a simple compact group, then $i(M) = 3$, and $S(M)$ is a subgroup associated to a highest root of the group M.
2) $M = SU_{l+m}/S(U_l \times U_m)$, $\quad i(M) = 2$, $\quad S(M) = SU_2/S(U_1 \times U_1)$.

3) $M = SO_{l+2}/SO_l \times SO_2$, $i(M) = 2$, $S(M) = SO_3/SO_2$.
4) $M = SU_{2n}/Sp_n$, $i(M) = 5$, $S(M) = SU_4/Sp_2$.
5) $M = Sp_{m+n}/Sp_m \times Sp_n$, $i(M) = 4$; $, S(M) = HP^1$.
6) $M = SO_{2n}/U_n$, $i(M) = 2$, $S(M) = SO_4/U_2$.
7) $M = Sp_n/U_n$, $i(M) = 2$, $S(M) = Sp_1/U_1$.
8) $M = F_4/Spin_9$, $i(M) = 8$, $S(M) = Spin_9/Spin_8$.
9) $M = Ad\,E_6/T^1\,Spin_{10}$, $i(M) = 2$, $S(M) = SU_2/T^1$.
10) $M = Ad\,E_7/T^1\,E_6$, $i(M) = 2$, $S(M) = SU_2/T^1$.
11) $M = E_6/F_4$, $i(M) = 9$, $S(M) = Spin_{10}/Spin_9$.

Remark. In all listed cases, if the dimension of Helgason's spheres i(M) =2, the corresponding symmetric space are Kälerian manifolds, so their Helgason's spheres are diffeomorphic to CP^1.The global minimality of the Helgason sphere in 1) was first proved by A.T.Fomenko [Fo 1], and then by Dao Chong Thi [Da 1], H.Tasaki [Ts] the author [Le 1] by the calibration method. The global minimality of the Helgason sphere in 8) was proved by A.T.Fomenko [Fo 1] by the method of geodesic nullity and by M.Berger [Be] by the calibration method. It would be interesting to find calibrations which calibrate the Helgason spheres in 4) and 11). It is well known that all characteristic classes on spaces M in 4) and 11) are trivial[Ta 2]. We think a suitable calibration may be chosen among induced invariant differential forms from the isometry group $I(M)$ to M (see also the proof below). We also conjecture that all Helgason's spheres are M^*-minimal submanifolds (see [Le 2]).

Proof of our classification. By looking at the table of real homologies of irreducible globally symmetric spaces [Ta 1, Ta 2], and the table of Helgason's spheres in these spaces [O], comparing dimensions, we conclude that all other Helgason's spheres not in the above list are trivial cycles in real homologies of their ambient spaces. By the above remark, to complete the classification, it suffices to show that the Helgason's spheres in 4) and 11) are non-trivial cycles. First, we consider the case 4) $S(M) = SU_4/Sp_2 \longrightarrow SU_{2n}/Sp_n$. We have the following commutative diagram

$$\begin{array}{ccc} SU_4/Sp_2 & \longrightarrow & SU_{2n}/Sp_n \\ \downarrow & & \downarrow \\ SU_4 & \longrightarrow & SU_{2n}. \end{array}$$

Here the embedding $SU_{2k}/Sp_k \longrightarrow SU_{2k}$, k=2 or n, is the Cartan embedding of symmetric spaces. We note that $S^5 = SU_4/Sp_2$ realizes a non-trivial cycle in SU_4, since so does the corresponding subgroup Sp_2. Therefore, the sphere S^5 also realizes a non-trivial cycle in SU_{2n}, because the subgroup SU_4 is totally non-homologous to zero in SU_{2n}. Hence we conclude that the Helgason sphere S^5 realizes a non-trivial cycle of real homologies of SU_{2n}/Sp_n.

The fact, that the Helgason's sphere S^9 realizes a non-trivial cycle of real homologies of E_6/F_4 was proved in Dao Trong Thi's paper [D 2]. To see it we consider the following sequence of mappings

$$S^9 \longrightarrow E_6/F_4 \longrightarrow E_6 \longrightarrow SU_{27}.$$

It is easy to see that the resulting map $\rho : S^9 \longrightarrow SU_{27}$ is a composition of two maps ρ_1 and ρ_2, where $\rho_1(S^9) \subset Spin_{10}$ is a primitive cycle, and ρ_2 is a spinor representation of $Spin_{10}$ which sends the primitive cycle S^9 to a non-trivial cycle in SU_{27} [Dy], [Da 2]. Therefore, we conclude that the Helgason sphere S^9 realizes a non-trivial cycle of real homologies of E_6/F_4.

Theorem 3.6. *Every Helgason's sphere in a compact irreducible simply connected symmetric space is a globally minimal surface in its Z_2 homology class.*

Remark. As a simple corollary of our theorem we obtain that all Helgason spheres in irreducible simply connected symmetric spaces are stable minimal. This corollary was obtained by Ohnita [O] with the help of analyzing the spectrum of the Jacobi operator on these spheres.

Proof. In view of our classification it suffices to show that the Helgason's spheres not in the above list realize non-trivial cycles of Z_2 homologies in their ambient symmetric spaces. All of them are of dimension 2 ([He], [O]). Since their ambient spaces M are simply connected and besides, in the considered cases we have $\pi_2(M) = \mathbf{Z_2}$ [Ta 1], it suffices to show that these spheres realize non-trivial elements of the second homotopy group $\pi_2(M)$. Let $M = G/U$, where G is a simply connected group. Our proof is based on the exact sequence [Ta 1]

$$O = \pi_2(G) \longrightarrow \pi_2(G/U) \longrightarrow \pi_1(U) \longrightarrow \pi_1(G) = 0.$$

Thus, the map $j : \pi_2(g/U) \longrightarrow \pi_1(U)$ is an isomorphism. Therefore, the Helgason sphere realizes a non-trivial element in $\pi_2(G/U)$, if and only if its image via j is a non-trivial circle $S^1 \subset U$ in the fundamental group $\pi_1(U)$. Let us recall a geometrical realization of the map j. Assume S^2 is a sphere in G/U. Fix a point $x \in S^2$. Let us realize the sphere S^2 as a suspension over S^1 such that one of its vertices is the fixed point x, and the other one is some point $y \in S^2$. This means that we are given a homotopy $F : [0,1] \times S^1 \longrightarrow S^2$ such that $F(0 \times S^1) = x$, and $F(1 \times S^1) = y \in S^2$. Let $\tilde{y}$ be a point in G whose projection $p(y) = y$. According to the covering homotopy theorem there exists a homotopy $\tilde{F} : [0,1] \times S^1 \longrightarrow G$ such that $F(1 \times S^1) = \tilde{y}$, and $p \cdot \tilde{F} = F$. Clearly, $\tilde{F}$ realizes a relative sphere whose boundary S^1 lies in the fiber $p^{-1}(x)$. Hence, this circle is the image of sphere S^2 via the map j. With the above geometric realization j_F of the map j we will show that the image $j_F(S^2)$ of the Helgason sphere $S^2 \in G/U$ may be chosen as a geodesic circle $S^1 \subset U$. To do this we consider the following orthogonal decomposition of the Lie algebra $lG = lU \oplus V$, where V is identified with the tangent space of the symmetric space G/U. We note that the totally geodesic subspace $\exp V$ coincides with the Cartan embedding $C(G/U)$ of symmetric space G/U into G. Consider a highest root α of the algebra lG. It is known that its restricted root $\bar{\alpha}$ is a highest root of the symmetric space G/U. Fix a Cartan algebra $H_V \subset V$. Let $h_{\bar{\alpha}} \in H_V$ be the dual vector to $\bar{\alpha}$, and $v_{\bar{\alpha}} \in V$ the corresponding eigenvector. This implies that

$$h_{\bar{\alpha}} = \sqrt{-1}(1/2)(H_\alpha - H_{\alpha^\theta}),\ \mathbf{R}v_{\bar{\alpha}} = V \cap \mathbf{C}(X_\alpha - \theta X_\alpha), \qquad (3.4),$$

where H_α denotes the vector in the Cartan algebra $H_{\mathbf{C}lG}$ corresponding to the root α, $X_\alpha \in \mathbf{C}lG$ is the corresponding eigenvector, and θ is the involutive authomorphism defining the symmetric space G/U [He]. Recall that in our case Hegalson's sphere is of dimension 2. Therefore, the multiplicity of $\bar{\alpha}$ equals 1 and $v_{\bar{\alpha}}$ is defined uniquely, moreover, the plane $span(h_{\bar{\alpha}}, v_{\bar{\alpha}})$ is a Lie triple. Indeed, this plane is the tangent plane to the Helgason sphere $S^2 \subset G/U$, it is also the tangent space to the Cartan embedding $C(S^2)$ of this sphere into G. Now we put $w_\alpha = [h_{\bar{\alpha}}, v_{\bar{\alpha}}]$. Since the multiplicity of $\bar{\alpha}$ equals 1 we have $w_\alpha \in lU \cap \mathbf{C}(X_\alpha + \theta X_\alpha)$ (see [He, p.336]). Taking into account (3.4) we see that the vectors $h_{\bar{\alpha}}, v_{\bar{\alpha}}, w_\alpha$ form a basis of the Lie subalgebra in lG corresponding to the root α. Denote $SU_2(\alpha)$ the corresponding subgroup in G. We note that the subgroup $SU_2(\alpha)$ contains the sphere $C(S^2)$. Further, we observe that the intersection between groups $SU_2(\alpha)$ and U is an one-dimensional compact subgroup $S^1(\alpha)$ generated by the vector w_α.

Lemma 3.7. *There exists a geometrical realization F_j such that F_j sends the Helgason sphere S^2 to the geodesic circle $S^1(\alpha)$.*

Proof. Let $\bar{e}$ denotes the antipodal point of e in the sphere $C(S^2)$. Let $S^1(\bar{e})$ be the equator on $C(S^2)$, consisting of those points $g \in S^2_C \subset G$ such that $g^2 = \bar{e}$. We claim that the natural projection $q : G \longrightarrow G/U$ sends this equator to a point. In fact, this claim is a consequence of the following assertion

Proposition 3.8 [Fo 2, p.124]. *Let gU be an arbitrary coset relative to U in G, and besides, $g \in C(G/U)$. Then $gU \cap C(G/U) = \{\sqrt{g^2}\} \cap C(G/U)$.*

This assertion can be obtained from the following explicit expression for the Cartan embedding $C : G/U \longrightarrow G$; $gU \rightarrow g\sigma(g^{-1})$, where σ denotes the corresponding involutive automorphism of the group G.

From Proposition 3.8 and the above claim we immediately get that the semi-sphere $S^{2+} \subset C(S^2)$ with boundary $S^1(\bar{e})$ and containing point e is a relative sphere of the fibration $U \longrightarrow G \longrightarrow G/U$, moreover, its projection into G/U coincides with the Helgason sphere $S^2 \subset G/U$. Now, it is easy to see that there exists a geometric realization F_j which sends the Helgason sphere S^2 to the equator $S^1(\bar{e})$. Suppose z is a point of $S^1(\bar{e})$. Then the shift L_z^{-1} sends the equator $S^1(\bar{e})$ to a geodesic circle $T^1(\alpha)$. By definition $T^1(\alpha)$ is also a geometric realization of the image $j(S^2)$. To complete the proof of Lemma

3.7 it suffices to show that $T^1(\alpha) = S^1(\alpha)$. In fact, the shift L_z^{-1} sends the fiber containing $S^1(\bar{e})$ to the subgroup U and on the other hand, the subgroup $SU_2(\alpha)$ is invariant under the action L_z^{-1}. Hence, $T^1(\alpha)$ belongs to the intersection between $SU_2(\alpha)$ and U. This implies that $T^1(\alpha) = S^1(\alpha)$.

Corollary 3.9. *$S^1(\alpha)$ is a shortest closed geodesic on group G, and therefore, on group U.*

Proof. By construction $SU_2(\alpha)$ is the subgroup corresponding to the highest root α of G. Since G is simply connected the circle $S^1(\alpha)$ is of minimal length [He].

Let $U = SO_n$. It is known that a shortest closed geodesic on SO_n is conjugate under the action of the group $Iso(SO_n)$ with the standardly embedded subgroup SO_2 which generates a non-trivial element in the fundamental group $\pi_1(SO_n)$. Hence, from Corollary 3.9 we immediately get the following consequence.

Corollary 3.10. *Helgason's spheres in symmetric spaces SU_n/SO_n; E_8/SO_{16}, G_2/SO_4 realize non-trivial elements in Z_2-homologies of their ambient spaces.*

In other cases we have to look more carefully. Our aim is to show that the geodesic circle $S^1(\alpha)$ realizes a non-trivial element in the fundamental group $\pi_1(U)$. Let w_α belong to a Cartan algebra H_{lU} which is contained in a Cartan algebra H_{lG}. Let $h_\alpha \in \mathbf{R}w_\alpha$ be the vector corresponding to the root α. It is known that the vector $h(\alpha) = 4\pi\, h_\alpha/|\alpha|^2$ belongs to the unit lattice $\Gamma(G, H_{lG})$ of the group G. Let $\tilde{U}$ denote the universal covering of the group U. The fact that the geodesic circle $S^1(\alpha)$ realizes a non-trivial element in $\pi_1(U)$ is equivalent to that $h(\alpha)$ does not belong to the unit lattice $\Gamma(\tilde{U}, H_{lU})$ of the group $\tilde{U}$. It is known that the unit lattice Γ of the simply connected group $\tilde{U}$ is $span_{\mathbf{Z}}\{h(\beta_j)\}$, where $\{\beta_j\}$ is a fundamental systems of roots of lU, and $h(\beta_j) = 4\pi h_{\beta_j}/|\beta_j|^2$ (see [He], [Ta 1]).

Let us now consider a symmetric space $M = G/U$, where lU is a direct sum of 2 simple Lie algebras lU_1 and lU_2. In our case M is one of the following spaces: $SO_{m+n}/(SO_n \times SO_m)$, $E_6/(SU_2 \cdot SU_6)$, $E_7/(SU_2 \cdot Spin_{12})$, $E_8/(SU_2 \cdot E_7)$, $F_4/SU_2 \cdot Sp_3$. (Except the case of real grassmannians, other products listed above, $U = U_1 \cdot U_2$, are not direct. Namely, the intersection of U_1 and U_2 consists of 2 points [Ta 1]). We note that the vector $h(\alpha)$ does not lie in any algebra lU_i, $i = 1, 2$, otherwise, the subgroup $SU_2(\alpha)$ lies in the group $U_i \subset U$ entirely. This contradicts to our observation that $SU_2(\alpha)$ meets U at only a circle $S^1(\alpha)$. Hence, in case $lU = so_n \oplus so_m$, the root α can be written as $x_i \pm x_j$, where $x_i \in H^*_{so_n}$ and $x_j \in H^*_{so_m}$. Thus, $h(\alpha)$ does not belong to the unit lattice of $Spin_n \times Spin_m$. In the same way we verify that for all listed above M the Helgason sphere S^2 realizes a non-trivial element in $\pi_2(M) = H_2(M, \mathbf{Z}) = H_2(M, \mathbf{Z_2}) = \mathbf{Z_2}$. To complete the proof of Theorem 3.6 we need to consider the cases $M = E_6/PSp_4$ and $M = E_7/SU_8^*$. Straightforward calculation shows that if a closed geodesic of minimal length in group $\tilde{U}/\{\pm 1\}$, $\tilde{U} = Sp_4, SU_8$, then it is conjugate under $I_0(U)$ with either the circle $S^1(\beta)$ generated by highest root β or the closed geodesic S^1_* whose pull back into the covering group U is the shortest geodesic joining two element $(+1) = (e)$ and (-1). Since the group $SU_2(\alpha)$ does not lie in U, we get that α is not a highest root of $lU_1 \oplus lU_2$. Hence, we easily obtain that the circle $S^1(\alpha)$ is conjugate with S^1_*. Thus, $S^1(\alpha)$ realizes a non-trivial element in $\pi_1(U)$. This completes the proof.

In conclusion we show a consequence of Theorem 2.1 for non-compact symmetric spaces. It is well known that the upper bound of section curvature of these spaces is zero [He].

Proposition 3.11. *Let X be a flat totally geodesic submanifold in a non-compact symmetric space M. Then X is a globally minimal submanifold.*

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CURVATURE ESTIMATE FOR THE VOLUME GROWTH OF GLOBALLY MINIMAL SURFACES

LE Hong Van

Introduction.

It is well known that in each homology class of a Riemannian manifold there exists a cycle of the least volume (or simply speaking, a globally minimal surface). These globally minimal cycles yield many information of geometry and topology of their ambient manifold, however, to detect them the existence (and almost regularity) theorems can not help us so much. Intuitively, one knows that globally minimal surfaces would occupy a position of "maximal curvature" in their ambient manifold. In A.T.Fomenko's and author's announcement [LF] we gave a mathematical formulation of this conjecture. The aim of this note is to complete the proof of our announcement [LF]. In particular, we obtain an estimate for the volume growth of globally minimal surfaces in Riemannian manifolds, new isoperimetric inequalities for these surfaces, an explicit formula of the least volumes of closed surfaces in symmetric spaces. As a result, we prove that every Helgason's sphere in a compact irreducible simply connected symmetric space is a globally minimal surface. In connection with the application of integral geometry to minimal surfaces [Le 2] we note that the technique of Fomenko's method of geodesic nullity employed in this note is very close to the technique in [Le 2]. In some sense, the method of integral geometry in the theory of minimal surfaces is a bridge between the calibration method [HL] and the method of geodesic nullity [Fo 1].

§1. Geodesic nullity of Riemannian manifolds and the volume of globally minimal submanifolds.

a) Let $B_r(x)$ be the ball of radius r in a tangent space T_xM. Recall that the injective radius $R(x)$ of a Riemannian manifold M at a point x is defined as follows: $R(x) = sup\{r|\ Exp : B_r(x) \longrightarrow M$ is a diffeomorphism $\}$. The injective radius $R(M)$ of M is defined as: $R(M) = \inf_{x\in M} R(x)$. Now we fix a point $x_0 \in M$. We define *k-dimensional deformation coefficient* $\chi_k(x > x_0)$ as follows (cf.[Fo 2]). Suppose that Π_x^{k-1} is a $(k-1)$-plane through x in the tangent space T_xM. Denote D_ε^{k-1} the disk of radius ε in Π_x^{k-1}, and by S_ε the disk $Exp(D_\varepsilon^{k-1})$. We consider the cone CS_ε formed by geodesics joining the vertex x_0 and the base S_ε. We put

$$\chi(x > x_0, \Pi_x^{k-1}) = \lim_{\varepsilon\to 0} \frac{vol_k CS_\varepsilon}{vol_{k-1} S_\varepsilon},$$

$$\chi(x > x_0) = \max_{\Pi^{k-1} \subset T_x M} \chi_k(x, \Pi^{k-1}).$$

b)Let $f(x)$ be the function which measures the distance between point $x \in M$ and the fixed point x_0. We set

$$q(x_0, r) = \exp(\int_0^r (\max_{x \in \{f=t\}} \chi_k(x > x_0))^{-1}\, dt). \tag{1.1}$$

We put

$$\Omega_k(x_0) = \lambda_k q(x_0, R(x_0)),$$
$$\Omega_k = \inf_{x_0 \in M} \Omega_k(x_0),$$

where λ_k is the volume of the ball of radius 1 in R^k.

The defined value is called *the* k^{th} *geodesic nullity* of Riemannian manifold M. The following theorem was obtained by Fomenko in 1972 [Fo 2].

Theorem 1.1. *Let $X^k \subset M^n$ be a globally minimal surface. Then the following inequality holds*

$$vol_k(X^k) \geq \Omega_k \geq 0.$$

Remark. Theorem 1.1 has a clear geometric interpretation. It is a consequence of the fact that the derivative of logarithm of the volume function exhausting a globally minimal surface X in M is greater than the function under integral in (1.1). This derivative $d/dt(ln\, vol X_t)$ equals the "isoperimetric" relation $vol\, \partial X_t / vol\, X_t$ (see also Proof of Theorem 2.3). The injective radius of M is involved, because X is a globally minimal surface in M.

§2. Lower bound for geodesic nullities of Riemannian manifolds. New isoperimetric inequalities.

Suppose that the section curvature of manifold M in any 2-plane is not greater then a^2 ($a \in R$ or $a \in \sqrt{-1} \otimes R$).

Theorem 2.1 [LF]. *Lower bound of geodesic nullity.*

a) If $a^2 \geq 0$ and $Ra \leq \pi$ then we have:

$$\Omega_k(M) \geq k\, \lambda_k\, a^{1-k} \int_0^R (\sin at)^{k-1}\, dt.$$

b) If $a^2 > 0$ and $Ra > \pi$ then we have:

$$\Omega_k(M) > vol(S^k(r = 1/a)).$$

c) If $a = 0$ then we have $\Omega_k(M) \geq \lambda_k\, R^k$.

d) If $a^2 \le 0$ then we have:

$$\Omega_k(M) \ge k\,\lambda_k\,|a|^{1-k} \int_0^R (\sinh|a|t)^{k-1}\,dt.$$

Theorem 2.2 [LF]. *Upper bound of the deformation coefficient. Let r be the distance between x and x_0.*

a) If $a^2 \ge 0$ and $r \le \pi/a$ then we have:

$$\chi_k(x > x_0) \le \frac{\int_0^r (\sin at)^{k-1} dt}{(\sin ar)^{k-1}}$$

b) If $a = 0$ then we have:

$$\chi_k(x > x_0) \le \frac{r}{k}$$

c) If $a^2 \le 0$ then we have

$$\chi_k(x > x_0) \le \frac{\int_0^r (\sinh\,|a|)^{k-1}\,dt}{(\sinh|a|r)^{k-1}}.$$

Theorem 2.3 [LF]. *Isoperimetric inequality. Assume that X^k is a globally minimal surfaces through a point $x \in M$. Let $B_x(r)$ be the geodesic ball of radius r and with its center at x. Denote A_r^{k-1} the boundary of the intersection $X^k \bigcap B_x(r) = X_r^k$.*

a) If $a^2 > 0$ and $r \le \min(R, \pi/a)$ then we have:

$$\frac{vol(A_r^{k-1})}{vol(X_r^k)} \ge \frac{\sin(ar)^{k-1}}{\int_0^r (\sin at)^{k-1} dt}.$$

Consequently, the following inequality holds

$$vol(A_r^{k-1}) \ge k\,\lambda_k\,a^{1-k}\,\sin^{k-1}(ar).$$

b) If $a = 0$ and $r \le R$ then we have:

$vol(A_r^{k-1}) \ge k\,\lambda_k\,r^{k-1}$ = the volume of the standard k-dimensional sphere S^k of radius r.

Hence we imply the following inequalities:

$$vol(A_r^{k-1}) \ge (kr)^{-1}\,vol(X_r^K),$$

$$vol(X_r^k) \le (k)^{\frac{k}{1-k}}\,(\lambda_k)^{\frac{1}{1-k}}(vol_{k-1}A_r)^{\frac{k}{k-1}}.$$

c) If $a^2 < 0$ then we have:

$$\frac{vol(A_r^{k-1})}{vol(X_r^k)} \geq \frac{(\sinh|a|r)^{k-1}}{\int_0^r (\sinh|a|t)^{k-1}\,dt}.$$

Hence we get

$$vol(A_r^{k-1}) \geq k\,\lambda_k\, sh^{k-1}(|a|t)/|a|^{k-1}.$$

The estimates in Theorems 2.1 and 2.2 are sharp, that is, in many cases they become equalities. Roughly speaking, these theorems tell us that globally minimal surfaces tend to a position of "maximal curvature" in their ambient manifold. Now we show some consequences of Theorem 2.1.

Corollary 2.4. *If M is a compact simply-connected symmetric space of sectional curvature not greater than a, then the volume of any non-trivial cycle is not less than the volume of k-dimensional sphere of curvature a.*

Corollary 2.5. *The length of a homologically non-trivial loop in a manifold M is not less then the double injective radius of M.*

Corollary 2.6 *Lower bound for the volume of a manifold.*

a) If $a^2 > 0$ then we get:

$$vol(M^n) \geq n\,\lambda_n\, a^{1-n} \int_0^R (\sin at)^{n-1}\,dt.$$

b) If $a = 0$, then we get: $vol(M^n) \geq n\,\lambda_n R^n$.

c) If $a^2 < 0$ then we get

$$vol(M^n) \geq n\,\lambda_n |a|^{1-n} \int_0^R (\sinh|a|t)^{n-1}\,dt.$$

Remark. The estimate in Corollary 2.6 coincides with that of Bishop's theorem [BC].

Now we infer from Theorems 2.2 and 2.3 the following consequence on the volume growth of globally minimal surfaces.

Corollary 2.7. *Let X^k be a globally minimal surface in a complete non-compact Riemannian manifold M of non-positive curvature. Then the function $V(r) = vol_k B_X(r)$ grows at least as a polynomial of r of degree k, where $B_X(r)$ is a geodesic ball of radius r in X^k. If the curvature of M has an upper bound strictly less than zero then the function $V(r)$ grows at least as the exponent of r.*

Remark. It is well known that there is a close relationship between the curvature of a Riemannian manifold M and the growth of its volume [BC]. As a consequence, we obtain the estimate for the growth of its fundamental group (see [M]), and other topological and geometrical invariants of M such as the

Betti numbers, the eigenvalues of the Laplace operator and the Gromov invariants [Br 1, Br 2, Gr 1, Gr 2, Gr 3].

Proof of Theorems and Corollaries. Let us write down an explicit formula for the coefficient $\chi_k(x > x_0, \Pi_x^{k-1})$. Suppose $\lambda(t)$ is the shortest geodesic curve joining the points $x_0 = \lambda(0)$ and $x = \lambda(r)$. So, for $0 < t < r$, point $\lambda(t)$ is not conjugated with x_0. We now consider the case if $x = \lambda(r)$ is not conjugated with x_0 (otherwise, we should take the limit). Choose an orthonormal basis of vectors $Y_1(r), ..., Y_{k-1}(r)$ in the plane $\Pi^{k-1} \subset T_x M$. (Let us recall that by definition Π^{k-1} must to be orthogonal to $\dot{\lambda}(r)$). We denote K_ρ the (k-1)-dimensional cube in Π_x^{k-1} with the edges $\rho Y(r)$. Then the formula for deformation coefficient $\chi_k(x > x_0)$ can be rewritten as follows:

$$\chi_k(x > x_0, \Pi_x^{k-1}) = \lim_{\rho \to 0} \frac{vol_k(C\tilde{K}_\rho)}{vol_{k-1}\tilde{K}_\rho},$$

here we set $\tilde{K}_\rho = Exp_x K_\rho$.

We denote λ_{st}^j the s-geodesic, joining points x_0 and $Exp_x(sY_j(r))$. Put

$$Y_j(t) = \frac{d}{ds}_{|s=0} \lambda_{st}^j.$$

Then $Y_j(t)$ is an Jacobian vector field with the data $Y_j(0) = 0$, $Y_j(r)$ - the chosen vector in Π^{k-1}, and besides, for every t we have $Y_j(t) \perp \dot{\lambda}(t)$. We note that the tangent plane to the orthogonal section $\tilde{K}_{t\rho}$ of the cone $C\tilde{K}_\rho$ at the point $\lambda(t)$ possesses the basis of vectors $Y_1(t), .., Y_{k-1}(t)$. Hence,

$$vol_{k-1}(\tilde{K}_{t\rho}) = \rho^{k-1}(|Y_1(t) \wedge ... \wedge Y_{k-1}(t)|) + o(\rho^{k-1}).$$

This yields

$$\chi_k(x > x_0, \Pi^{k-1}) = \lim_{\rho \to 0} \frac{vol(C\tilde{K}_\rho)}{vol_{k-1}(\tilde{K}_\rho)} =$$

$$= \lim_{\rho \to 0} \frac{\int_0^r vol_{k-1}\tilde{K}_{t\rho}\, dt}{vol_{k-1}\tilde{K}_{r\rho}} = \frac{\int_0^r |Y_1(t) \wedge ... \wedge Y_{k-1}(t)|\, dt}{|Y_1(r) \wedge ... \wedge Y_{k-1}(r)|}. \tag{2.1}$$

Proof of Theorem 2.2. Put $F(t) = |Y_1(t)| \cdot ... \cdot |Y_{k-1}(t)|$. Since $|Y_1(t) \wedge ... \wedge Y_{k-1}(t)| \leq F(t)$, and this inequality becomes an equality at $t = r$, the formula(2.1) yields

$$\chi_k(x > x_0, \Pi^{k-1}) \leq \frac{\int_0^r F(t)\, dt}{F(r)}. \tag{2.2}$$

We need the following lemmas.

Lemma 2.8. *Suppose $F(t)$ be in(2.2). If for all t and Y_j the section curvature $S(\dot\lambda(t), Y_j(t)) \leq a^2$, where $a > 0$, then the function $F(t)/G(t)$ increases on the interval $[0, r]$. Here $G(t) = (\sin at)^{k-1}/(\sin ar)^{k-1}$.*

Lemma 2.9. *Suppose the function $F(t)$ and $G(t)$ be in the Lemma 2.8. Then the following inequality holds*

$$\frac{\int_0^r F(t)\,dt}{F(r)} \leq \frac{\int_0^r G(t)\,dt}{G(r)}.$$

Proof of Lemma 2.8. The Rauch's comparison Theorem [BC] states that the function $f_j(t) = |Y_j(t)|/\sin at$ increases on the interval $[0, r]$. Hence, the function $F(t)/G(t) = \prod f_j$ is such a function.

Proof of Lemma 2.9. Since the function $F(t)/G(t)$ increases on the interval $[0, r]$, we get $F(x_i)G(r) \leq G(x_i)F(r)$ for every $0 \leq x_i \leq r$. Hence we obtain

$$\sum_{k=0}^{n} F(kr/n)G(r) \leq \sum_{k=0}^{n} G(kr/n)F(r).$$

Letting $n \to \infty$ we easily infer Lemma 2.10 from the above inequality.

Let us continue the proof of Theorem 2.2.

Taking into account (2.2) and lemmas 2.8, 2.9 we get

$$\chi_k(x, \Pi^{k-1}) \leq \frac{\int_0^r F(t)\,dt}{F(r)} \leq \frac{\int_0^r (\sin at)^{k-1}\,dt}{(\sin ar)^{k-1}}.$$

The proof of the first part in Theorem 2.2 is completed. In the same way we can prove the rest parts (b) and (c).

Proof of Theorem 2.1. Let us recall the definition

$$\Omega_k(x_0, r) = \lambda_k \exp \int_0^r (\max_{x \in \{f=t\}} \chi_k(x > x_0))^{-1}\,dt.$$

Theorem 2.1 (a) yields

$$\Omega_k(x_0, r) \geq \lambda_k \exp \int_0^r \frac{\sin at)^{k-1}\,dt}{\int_0^t (\sin a\tau)^{k-1}\,d\tau}.$$

Put

$$\Phi_k(r) = \lambda_k \exp \int_0^r \frac{(\sin at)^{k-1}\,dt}{\int_0^t (\sin a\tau)^{k-1}\,d\tau}.$$

Clearly , we can infer Theorem 2.1(a) from the following identity

$$\Phi_k(r) = k\,\lambda_k\,a^{1-k} \int_0^r (\sin at)^{k-1}\,dt. \tag{2.3}$$

Proof of Formula (2.3). Put $\Phi_k^*(r)$ equal the right hand side in(2.3). We observe that the functions $\Phi_k(r)$ and $\Phi_k^*(r)$ satisfy the same differential equation:

$$\frac{\Phi_k(r)}{(\partial/\partial r)\Phi_k(r)} = \frac{\Phi_k^*(r)}{(\partial/\partial r)\Phi_k^*(r)} = \frac{\int_0^r (\sin at)^{k-1}\, dt}{(\sin ar)^{k-1}}. \tag{2.4}$$

Let us consider the limit

$$\lim_{r\to 0}\frac{\Phi_k(r)}{\Phi_k^*(r)} = \lim_{r\to 0}\frac{\lambda_k \exp\int_0^r (\int_0^t (\sin a\tau)^{k-1}\, d\tau)^{-1}(\sin at)^{k-1}\, dt}{k\,\lambda_k a^{1-k}\int_0^r (\sin at)^{k-1}\, dt}. \tag{2.5}$$

Taking into account the increaseness of the function $(a\tau/\sin a\tau)^{k-1}$ on the interval $[0,t]$, where $0 \le t \le \pi/a$, and using Lemma 2.10 we obtain

$$\frac{(\sin at)^{k-1}}{\int_0^t (\sin a\tau)^{k-1}\, d\tau} < \frac{t^{k-1}}{\int_0^t \tau^{k-1}\, d\tau} = \frac{k}{t}. \tag{2.6}$$

Combining (2.5) and (2.6) yields the following inequality

$$\lim_{r\to 0}\frac{\Phi_k(r)}{\Phi_k^*(r)} \le \lim_{r\to 0}\frac{\lambda_k \exp\int_0^r k\,t^{-1}\, dt}{k\,\lambda_k\, a^{1-k}\int_0^r (\sin at)^{k-1}\, dt}.$$

Fix $\varepsilon > 0$. Since $\lim_{y\to 0}(\sin at/\, at) = 1 > 1-\varepsilon$ we get the following inequality.

$$\lim \frac{\Phi_k(r)}{\Phi_k^*(r)} \le \lim_{r\to 0}\frac{\exp\int_0^r (k/t)\, dt}{k\,\int_0^r (1-\varepsilon)^{k-1}(at)^{k-1}\, a^{1-k}\, dt} =$$

$$= \lim_{r\to 0}\frac{r^k}{r^k(1-\varepsilon)^{k-1}} = (1-\varepsilon)^{1-k}. \tag{2.7}$$

Since the inequality (2.7) holds for all $\varepsilon > 0$ we have

$$\lim_{r\to 0}\frac{\Phi_k(r)}{\Phi_k^*(r)} \le \lim_{\varepsilon\to 0}(1-\varepsilon)^{1-k} = 1. \tag{2.8}$$

On the other hand, applying the inequality $\sin at < at$ to (2.5) we get

$$\lim_{r\to 0}\frac{\Phi_k(r)}{\Phi_k^*(r)} \ge \lim_{r\to 0}\frac{\lambda_k \exp(\int_0^r (\sin ay)^{k-1}\, a^{1-k}\, k\, y^{-k}\, dy}{k\,\lambda_k\,\int_0^r t^{k-1}\, dt}.$$

Fixed ε as above we have

$$\lim_{r\to 0}\frac{\Phi_k(r)}{\Phi_k^*(r)} \ge \lim_{r\to 0}\exp(\int_0^r \frac{(1-\varepsilon)(ay)^{k-1}\, dy}{a^{k-1}\cdot y^k \cdot k^{-1}})\, r^{-k} =$$

$$= \lim_{r\to 0} r^{-k} \exp\Big(\int_0^r \frac{(1-\varepsilon)^{k-1}\, k\, dy}{y}\Big) = r^{k((1-\varepsilon)^{k-1}-1)}. \tag{2.9}$$

Letting $\varepsilon \to 0$ we infer from (2.9)

$$\lim_{r\to 0} \frac{\Phi_k(r)}{\Phi_k^*(r)} \geq \lim_{\varepsilon\to 0} r^{k((1-\varepsilon)^{k-1}-1)} = 1. \tag{2.10}$$

Now we obtain from (2.8) and (2.10)

$$\lim_{r\to 0} \frac{\Phi_k(r)}{\Phi_k^*(r)} = 1. \tag{2.11}$$

The differential equation (2.4) for $\Phi_k(r)$ and $\Phi_k^*(r)$ has the same initial data (2.11). So we get the identity $\Phi_k^* = \Phi_k$, that completes the proof of Theorem 2.2 (a).

The rest parts (c), (d) can be proved in the same way. The part (b) follows from that fact if $R > \pi/a$ then we have $\Omega_k(M) > \Omega_k(x_0, \pi/a) \geq vol(S^k, 1/a)$. This completes the proof of Theorem 2.2.

Proof of Theorem 2.3. Let r be as in Theorem 2.3. We denote CA_r^{k-1} the geodesic cone of base A_r^{k-1} and with its vertex at the point x. Since X_r^k is a globally minimal surface, and the cone CA_r^{k-1} is homological to X_r^k, we have $vol(X_r^k) \leq vol(CA_r^{k-1})$. Hence we conclude

$$\frac{vol(A_r^{k-1})}{vol(X_r^k)} \geq \frac{vol(A_r^{k-1})}{vol(CA_r^{k-1})} \geq$$
$$\geq (\max_{y\in A_r} \chi_k(y > x))^{-1} \geq \frac{(\sin ar)^{k-1}}{\int_0^r (\sin at)^{k-1}\, dt}. \tag{2.12}$$

(The second inequality in (2.12) is inferred from the following formula

$$vol(CA_r^{k-1}) = \int_{A_r^{k-1}} \chi_k(y > x, \Pi_y^{k-1})\, dy,$$

where Π_y^{k-1} denotes the tangent space to A_y^{k-1} at y. The third inequality in (2.12) is a consequence of Theorem 2.2(a).)

We infer from (2.12) the following inequality

$$vol(CA_r^{k-1}) \geq vol(X_r^k) \frac{(\sin ar)^{k-1}}{\int_0^r (\sin at)^{k-1}\, dt} \geq \Omega_k(r) \frac{(\sin ar)^{k-1}}{\int_0^r (\sin at)^{k-1}\, dt}. \tag{2.13}$$

Combining (2.13) and Theorem 1.1(a) yields

$$vol\, A_r^{k-1} \geq k\, \lambda_k\, a^{1-k} (\sin ar)^{k-1}.$$

This completes the proof of Theorem 2.3(a). The rest part of Theorem 2.3 can be proved in the same way.

Proof of Corollary 2.4. It is well known that a compact simply-connected symmetric space satisfies the relation: $Ra = \pi$. So we get Corollary 2.4 from Theorem 2.1.

Proof of Corollary 2.5.. Clearly, $\lambda_1 = 2$. So we obtain $\Omega_1(M) \geq 2\int_0^R 1\, dt = 2R$.

Proof of Corollary 2.6.a. Let $\dim(M) = m$. Then $vol(M) \geq \Omega_m(M)$. With the help of (2.3) we obtain $\Omega_m(M) \geq \Phi_m(R) = k\,\lambda_k a^{1-k}\int_0^r (\sin at)^{k-1}\, dt$. This completes the proof Corollary 2.6.a. The rest assertions can be proved in the same way.

§3. Explicit formula for geodesic nullities of symmetric spaces. Global minimality of Helgason's spheres.

Suppose M is a compact symmetric space. Let us compute the deformation coefficient associated with fixed point $e \in M$. Without loss of generality we compute this coefficient at point $Exp\,tx \in M$, where x is a vector in a Cartan space H_{lM} of the tangent space lM to M at e. We shall redenote $\chi_k(Exp\,rx) = \chi_k(Exp\,rx > e)$.

Theorem 3.1. *Let $\{\alpha_i\}$ be the roots systems of symmetric space M with respect to H_{lM}. Suppose x is a vector of unit length in H_{lM} . Without loss of generality we assume that $\alpha_1(x) \geq ... \geq \alpha_p(x) = 0 = \alpha_{p+1}(x) = ...$.*

a) If $k < p$ then the following equality holds

$$\chi_k(Exp\,rx) = \frac{\int_0^r \sin(\alpha_1(x)t)\cdot ... \cdot \sin(\alpha_k(x)t)\, dt}{\sin\alpha_1(x)r \cdot ... \cdot \sin\alpha_k(x)r}.$$

b) If $k \geq p$ then the following inequality holds

$$\chi_k(Exp\,rx) = \frac{\int_0^r \sin(\alpha_1(x)t)\cdot ... \cdot \sin(\alpha_{p-1}(x)t)t^{k-p}\, dt}{\sin(\alpha_1(x)r)\cdot ... \cdot \sin(\alpha_{p-1}(x)r)r^{k-p}}.$$

Lemma 3.2. *Let $\{v_1, ..., v_k\} \in M$ be an orthonormal frame which consisting of the eigenvectors of eigenvalues $\alpha_1^2(x), ..., \alpha_p^2(x), , , , 0, ...0$ of the operator ad_x^2. Denote $V_i(t)$ the parallel vector field along the geodesics $Exp\,tx$ such that $V_i(0) = v_i$, and denote $W_i(t)$ the Jacobian vector field along $Exp\,tx$ such that $W_i(0) = v_i$. Then we have the following relation*

- if $i < p$ then $W_i(t) = \alpha_i(x)^{-1}\sin(\alpha_i(x)t)V(t)$,

- if $i \geq p$ then $W_i(t) = tV_i(t)$.

Proof of Lemma 3.2. In the tangent space lM the vector field tv_i is a Jacobian field along the ray tx. It is well known that the vector field $dExp_{|tx}(tv_i)$ is also a Jacobian vector field along the geodesic $Exp\,tx \subset M$ [He]. Let us write an explicit formula for the differential of the exponential mapping at the point tx ([He]). We will identify M with the quotient G/U, moreover, the tangent space lM with the orthogonal complement to the algebra lU in the algebra lG. We denote exp the exponential mapping from the algebra to the group. Then $\exp tx$ is an element in G acting on M and we denote $d\tau(\exp tx)$ the differential of this action. We have

$$dExp_{|tx}(tv) = d\tau(\exp tx) \sum_{n=0}^{\infty} \frac{t^2\, ad_x^2(tv_i)}{(2n+1)!} =$$

$$= d\tau\,(\exp tx) \sum_{n=0}^{\infty} \frac{(t^2\alpha_i^2(x))^n(-1)^n}{(2n+1)!}(tv_i) =$$

$$= d\tau(\exp tx)\frac{\sin\alpha_i(x)t}{t\,\alpha_i(x)}tv_i = \frac{sin(\alpha_i(x)t)}{\alpha_i(x)}(d\tau(\exp tx)v_i) \tag{3.1}$$

Now we observe that the parallel vector field V_i is obtained from the vector v_i by the shift $d\tau(exp)$ along the geodesic $Exp\,tx$, that is, $V_i(t) = d\tau(\exp tx)\,v_i$. Hence we get Lemma 3.2 from (3.1).

Proof of Theorem 3.1. Now we compute the coefficient $\chi_k(Exp\,rx, \Pi^{k-1})$. We observe that the tangent space $\Pi^{k-1}(t)$ to the normal section of the cone CD_ϵ^{k-1} at the point $Exp\,tx$ can be represented as the sum $\sum a_i\Pi_i^{k-1}(t)$, where a_i are constant, and $\Pi_i^{k-1}(t)$ is the basis in the space $\Lambda_{k-1}(T_{Exp\,tx}M)$ such that $\Pi_i^{k-1}(t)$ is generated by the orthonormal frame of vectors $W_i(t) \in T_{Exp\,tx}M$. Using formula (2.1) we get

$$\chi_k(Exp\,rx, \Pi^{k-1}) = \frac{\int_0^r |\Pi^{k-1}(t)|\,dt}{|\Pi^{k-1}(r)|} = \frac{\sum_i \int_0^r a_i\,|\Pi_{k-1}^i(t)|\,dt}{\sum_i a_i\,|\Pi_{k-1}^i(r)|}.$$

Hence we obtain

$$\chi_k(Exp\,rx) = \max_i \frac{\int_0^r |\Pi_{k-1}^i(t)|\,dt}{\Pi_{k-1}^i(r)}.$$

Combining Lemma 3.2 , Lemma 2.8 and Lemma 2.9 we get

$$\max_i \frac{\int_0^r |\Pi_i^{k-1}(t)|dt}{|\Pi_{k-1}^i(r)|} = \frac{\int_0^r sin(\alpha_1(x)t)\cdot\ldots\cdot sin(\alpha_k(x)t)\,dt}{sin(\alpha_1(x)r)\cdot\ldots\cdot sin(\alpha_k(x)r)}$$

if $k < p$. In the same way we can prove the theorem in the case $k \geq p$. The proof of Theorem 3.1 is complete.

Corollary 3.3. *If M is a symmetric space of rank = 1 ,that is, dim $H_{lM} = 1$, then the deformation coefficient $\chi_k(Exprx)$ depends only on r.*

a) For $M = S^n$ (or RP^n) we have $\chi_k(r) = \int_0^r (\sin t)^{k-1}\,dt/(\sin r)^{k-1}$.

b) For $M = CP^n$ we have

$$\chi_k(r) = \frac{\int_0^r (\sin\sqrt{2}t)(\sin t)^{2k-2}\,dt}{\sin\sqrt{2}r(\sin r)^{2k-2}}.$$

c) For $M = HP^n$ we have

$$\chi_k(r) = \frac{\int_0^r (\sin\sqrt{2}t)^3(\sin t)^{4k-4}\,dt}{(\sin\sqrt{2}r)^3(\sin r)^{4k-4}}.$$

We immediately obtain the following consequence.

Corollary 3.4. [Fo 1]. *For any $k \le n$ the standardly embedded space RP^n (and CP^n, HP^n resp.) has the volume $= \Omega_k(RP^n)$ (and $\Omega_{2k}(CP^n), \Omega_{4k}(HP^n)$ resp.), therefore it is a globally minimal submanifold.*

Let us now compute geodesic nullities $\Omega_k(RP^n), \Omega_{2k}CP^n, \Omega_k(HP^n)$. Clearly, $\Omega_k(RP^n) = \frac{1}{2}vol(S^k(1))$ can be computed from the following formulas. First, we take integration over parallel sections of the unit ball

$$\lambda_k = 2\lambda_{k-1}\int_0^{\pi/2} cos^k\alpha\,d\alpha.$$

Taking into account (2.3) we get

$$vol\,S^k(1) = (k+1)\lambda_{k+1} = 2\lambda_k k\int_0^{\pi/2} sin^{k-1}\alpha\,d\alpha.$$

Hence we obtain the following identity

$$k+1 = \frac{2k\lambda_k\int_0^{\pi/2} sin^{k-1}\alpha\,d\alpha}{2\lambda_k\int_0^{\pi/2} cos^{k+1}\alpha\,d\alpha}. \tag{3.2}$$

We infer from (3.2) the following equation

$$\int_0^{\pi/2} sin^{k+1}\alpha\,d\alpha = \frac{k}{k+1}\int_0^{\pi/2} sin^{k-1}\alpha\,d\alpha. \tag{3.3}$$

Using (3.3) we easely get

$$\lambda_{2k} = \frac{\pi^k}{k!}, \qquad \lambda_{2k+1} = \frac{\pi^k 2^{k+1}}{(2k+1)!!}.$$

Let us compute $\Omega_{2k}(CP^n) = vol(CP^k)$. Using Corollary 3.3 and taking into account $R(CP^n) = \pi/\ sqrt2$ we get

$$\Omega_k(CP^n) = \lambda_{2k} \exp \int_0^{\pi/sqrt2} \frac{dt}{(\sin\sqrt{2}t)(\sin t)^{2k-1}} =$$

$$= 2k\lambda_{2k} \int_0^{\pi/sqrt2} (\sin\sqrt{2}t/\sqrt{2})(\sin t)^{2k-2}\, dt =$$

$$= 2^k 2k\, \lambda_{2k} \int_0^1 x^{2k-1}\, dx = \pi^{2k}/k!.$$

In the same way we compute $\Omega_{4k}(HP^n) = vol(HP^k)$. We have

$$\Omega_{4k}(HP^n) = \lambda_{4k} \exp \int_0^{\pi/\sqrt{2}} \frac{dt}{(\sin\sqrt{2}t)^3(\sin t)^{4k-4}} =$$

$$= 4k\lambda_{4k} \int_0^{\pi/\sqrt{2}} (\sin\sqrt{2}t/\sqrt{2})^3(\sin t)^{4k-4}\, dt =$$

$$= 2^{2k} 4k\lambda_{4k} \int_0^1 y^{4k-4}(1-y^2)\, dy =$$

$$= \pi^{2k}\, 2^{2k}/(2k+1)!.$$

Remark. Operator ad_x^2 coincides with the Ricci transformation $R_x : y \to R_{xy}x$ in the tangent space lM. Therefore, the deformation coefficient $\chi_k(Exp\, rx)$ get the maximal value, if and only if the plane Π^{k-1} is an eigenspace with the maximal eigenvalue of the induced Ricci transformation in the space $\Lambda_{k-1}T_{Exp\,rx}M$. Roughly speaking, the curvature at point $Exp\,rx$ in direction (rx, Π^{k-1}) get the maximal value.

It is well known that in a simply connected irreducible compact symmetric space M there are totally geodesic spheres of curvature a^2, where a^2 is the upper bound of section curvature on M. Further, any such sphere lies in some totally geodesic Helgason's sphere of maximal dimension $i(M)$. All Helgason's spheres are equivalent under the action of the isometry group $Iso(M)$. Moreover, they are of the same curvature a^2. Now we immediately get from Corollary 2.8 the following Proposition.

Proposition 3.5. *If a Helgason sphere $S(M)$ realizes a non-trivial cycle in a homology group of space M, then it is a globally minimal submanifold in M.*

First, we write the list of Helgason's spheres realizing a non-trivial cycles in real homologies of compact irreducible simply-connected symmetric spaces.

1) If M is a simple compact group, then $i(M) = 3$, and $S(M)$ is a subgroup associated to a highest root of the group M.
2) $M = SU_{l+m}/S(U_l \times U_m)$, $\quad i(M) = 2$, $\quad S(M) = SU_2/S(U_1 \times U_1)$.

3) $M = SO_{l+2}/SO_l \times SO_2$, $i(M) = 2$, $S(M) = SO_3/SO_2$.
4) $M = SU_{2n}/Sp_n$, $i(M) = 5$, $S(M) = SU_4/Sp_2$.
5) $M = Sp_{m+n}/Sp_m \times Sp_n$, $i(M) = 4$; $, S(M) = HP^1$.
6) $M = SO_{2n}/U_n$, $i(M) = 2$, $S(M) = SO_4/U_2$.
7) $M = Sp_n/U_n$, $i(M) = 2$, $S(M) = Sp_1/U_1$.
8) $M = F_4/Spin_9$, $i(M) = 8$, $S(M) = Spin_9/Spin_8$.
9) $M = Ad\,E_6/T^1\,Spin_{10}$, $i(M) = 2$, $S(M) = SU_2/T^1$.
10) $M = Ad\,E_7/T^1\,E_6$, $i(M) = 2$, $S(M) = SU_2/T^1$.
11) $M = E_6/F_4$, $i(M) = 9$, $S(M) = Spin_{10}/Spin_9$.

Remark. In all listed cases, if the dimension of Helgason's spheres i(M) =2, the corresponding symmetric space are Kālerian manifolds, so their Helgason's spheres are diffeomorphic to CP^1.The global minimality of the Helgason sphere in 1) was first proved by A.T.Fomenko [Fo 1], and then by Dao Chong Thi [Da 1], H.Tasaki [Ts] the author [Le 1] by the calibration method. The global minimality of the Helgason sphere in 8) was proved by A.T.Fomenko [Fo 1] by the method of geodesic nullity and by M.Berger [Be] by the calibration method. It would be interesting to find calibrations which calibrate the Helgason spheres in 4) and 11). It is well known that all characteristic classes on spaces M in 4) and 11) are trivial[Ta 2]. We think a suitable calibration may be chosen among induced invariant differential forms from the isometry group $I(M)$ to M (see also the proof below). We also conjecture that all Helgason's spheres are M^*-minimal submanifolds (see [Le 2]).

Proof of our classification. By looking at the table of real homologies of irreducible globally symmetric spaces [Ta 1, Ta 2], and the table of Helgason's spheres in these spaces [O], comparing dimensions, we conclude that all other Helgason's spheres not in the above list are trivial cycles in real homologies of their ambient spaces. By the above remark, to complete the classification, it suffices to show that the Helgason's spheres in 4) and 11) are non-trivial cycles. First, we consider the case 4) $S(M) = SU_4/Sp_2 \longrightarrow SU_{2n}/Sp_n$. We have the following commutative diagram

$$\begin{array}{ccc} SU_4/Sp_2 & \longrightarrow & SU_{2n}/Sp_n \\ \downarrow & & \downarrow \\ SU_4 & \longrightarrow & SU_{2n}. \end{array}$$

Here the embedding $SU_{2k}/Sp_k \longrightarrow SU_{2k}$, k=2 or n, is the Cartan embedding of symmetric spaces. We note that $S^5 = SU_4/Sp_2$ realizes a non-trivial cycle in SU_4, since so does the corresponding subgroup Sp_2. Therefore, the sphere S^5 also realizes a non-trivial cycle in SU_{2n}, because the subgroup SU_4 is totally non-homologous to zero in SU_{2n}. Hence we conclude that the Helgason sphere S^5 realizes a non-trivial cycle of real homologies of SU_{2n}/Sp_n.

The fact, that the Helgason's sphere S^9 realizes a non-trivial cycle of real homologies of E_6/F_4 was proved in Dao Trong Thi's paper [D 2]. To see it we consider the following sequence of mappings

$$S^9 \longrightarrow E_6/F_4 \longrightarrow E_6 \longrightarrow SU_{27}.$$

It is easy to see that the resulting map $\rho : S^9 \longrightarrow SU_{27}$ is a composition of two maps ρ_1 and ρ_2, where $\rho_1(S^9) \subset Spin_{10}$ is a primitive cycle, and ρ_2 is a spinor representation of $Spin_{10}$ which sends the primitive cycle S^9 to a non-trivial cycle in SU_{27} [Dy], [Da 2]. Therefore, we conclude that the Helgason sphere S^9 realizes a non-trivial cycle of real homologies of E_6/F_4.

Theorem 3.6. *Every Helgason's sphere in a compact irreducible simply connected symmetric space is a globally minimal surface in its Z_2 homology class.*

Remark. As a simple corollary of our theorem we obtain that all Helgason spheres in irreducible simply connected symmetric spaces are stable minimal. This corollary was obtained by Ohnita [O] with the help of analyzing the spectrum of the Jacobi operator on these spheres.

Proof. In view of our classification it suffices to show that the Helgason's spheres not in the above list realize non-trivial cycles of Z_2 homologies in their ambient symmetric spaces. All of them are of dimension 2 ([He], [O]). Since their ambient spaces M are simply connected and besides, in the considered cases we have $\pi_2(M) = \mathbf{Z_2}$ [Ta 1], it suffices to show that these spheres realize non-trivial elements of the second homotopy group $\pi_2(M)$. Let $M = G/U$, where G is a simply connected group. Our proof is based on the exact sequence [Ta 1]

$$O = \pi_2(G) \longrightarrow \pi_2(G/U) \longrightarrow \pi_1(U) \longrightarrow \pi_1(G) = 0.$$

Thus, the map $j : \pi_2(g/U) \longrightarrow \pi_1(U)$ is an isomorphism. Therefore, the Helgason sphere realizes a non-trivial element in $\pi_2(G/U)$, if and only if its image via j is a non-trivial circle $S^1 \subset U$ in the fundamental group $\pi_1(U)$. Let us recall a geometrical realization of the map j. Assume S^2 is a sphere in G/U. Fix a point $x \in S^2$. Let us realize the sphere S^2 as a suspension over S^1 such that one of its vertices is the fixed point x, and the other one is some point $y \in S^2$. This means that we are given a homotopy $F : [0,1] \times S^1 \longrightarrow S^2$ such that $F(0 \times S^1) = x$, and $F(1 \times S^1) = y \in S^2$. Let $\tilde{y}$ be a point in G whose projection $p(y) = y$. According to the covering homotopy theorem there exists a homotopy $\tilde{F} : [0,1] \times S^1 \longrightarrow G$ such that $F(1 \times S^1) = \tilde{y}$, and $p \cdot \tilde{F} = F$. Clearly, $\tilde{F}$ realizes a relative sphere whose boundary S^1 lies in the fiber $p^{-1}(x)$. Hence, this circle is the image of sphere S^2 via the map j. With the above geometric realization j_F of the map j we will show that the image $j_F(S^2)$ of the Helgason sphere $S^2 \in G/U$ may be chosen as a geodesic circle $S^1 \subset U$. To do this we consider the following orthogonal decomposition of the Lie algebra $lG = lU \oplus V$, where V is identified with the tangent space of the symmetric space G/U. We note that the totally geodesic subspace $\exp V$ coincides with the Cartan embedding $C(G/U)$ of symmetric space G/U into G. Consider a highest root α of the algebra lG. It is known that its restricted root $\bar{\alpha}$ is a highest root of the symmetric space G/U. Fix a Cartan algebra $H_V \subset V$. Let $h_{\bar{\alpha}} \in H_V$ be the dual vector to $\bar{\alpha}$, and $v_{\bar{\alpha}} \in V$ the corresponding eigenvector. This implies that

$$h_{\bar{\alpha}} = \sqrt{-1}(1/2)(H_\alpha - H_{\alpha^\theta}),\ \mathbf{R}v_{\bar{\alpha}} = V \cap \mathbf{C}(X_\alpha - \theta X_\alpha), \qquad (3.4),$$

where H_α denotes the vector in the Cartan algebra $H_{\mathbf{C}lG}$ corresponding to the root α, $X_\alpha \in \mathbf{C}lG$ is the corresponding eigenvector, and θ is the involutive authomorphism defining the symmetric space G/U [He]. Recall that in our case Hegalson's sphere is of dimension 2. Therefore, the multiplicity of $\bar{\alpha}$ equals 1 and $v_{\bar{\alpha}}$ is defined uniquely, moreover, the plane $span(h_{\bar{\alpha}}, v_{\bar{\alpha}})$ is a Lie triple. Indeed, this plane is the tangent plane to the Helgason sphere $S^2 \subset G/U$, it is also the tangent space to the Cartan embedding $C(S^2)$ of this sphere into G. Now we put $w_\alpha = [h_{\bar{\alpha}}, v_{\bar{\alpha}}]$. Since the multiplicity of $\bar{\alpha}$ equals 1 we have $w_\alpha \in lU \cap \mathbf{C}(X_\alpha + \theta X_\alpha)$ (see [He, p.336]). Taking into account (3.4) we see that the vectors $h_{\bar{\alpha}}, v_{\bar{\alpha}}, w_\alpha$ form a basis of the Lie subalgebra in lG corresponding to the root α. Denote $SU_2(\alpha)$ the corresponding subgroup in G. We note that the subgroup $SU_2(\alpha)$ contains the sphere $C(S^2)$. Further, we observe that the intersection between groups $SU_2(\alpha)$ and U is an one-dimensional compact subgroup $S^1(\alpha)$ generated by the vector w_α.

Lemma 3.7. *There exists a geometrical realization F_j such that F_j sends the Helgason sphere S^2 to the geodesic circle $S^1(\alpha)$.*

Proof. Let $\bar{e}$ denotes the antipodal point of e in the sphere $C(S^2)$. Let $S^1(\bar{e})$ be the equator on $C(S^2)$, consisting of those points $g \in S^2_C \subset G$ such that $g^2 = \bar{e}$. We claim that the natural projection $q : G \longrightarrow G/U$ sends this equator to a point. In fact, this claim is a consequence of the following assertion

Proposition 3.8 [Fo 2, p.124]. *Let gU be an arbitrary coset relative to U in G, and besides, $g \in C(G/U)$. Then $gU \cap C(G/U) = \{\sqrt{g^2}\} \cap C(G/U)$.*

This assertion can be obtained from the following explicit expression for the Cartan embedding $C : G/U \longrightarrow G;\ gU \rightarrow g\sigma(g^{-1})$, where σ denotes the corresponding involutive automorphism of the group G.

From Proposition 3.8 and the above claim we immediately get that the semisphere $S^{2+} \subset C(S^2)$ with boundary $S^1(\bar{e})$ and containing point e is a relative sphere of the fibration $U \longrightarrow G \longrightarrow G/U$, moreover, its projection into G/U coincides with the Helgason sphere $S^2 \subset G/U$. Now, it is easy to see that there exists a geometric realization F_j which sends the Helgason sphere S^2 to the equator $S^1(\bar{e})$. Suppose z is a point of $S^1(\bar{e})$. Then the shift L_z^{-1} sends the equator $S^1(\bar{e})$ to a geodesic circle $T^1(\alpha)$. By definition $T^1(\alpha)$ is also a geometric realization of the image $j(S^2)$. To complete the proof of Lemma

3.7 it suffices to show that $T^1(\alpha) = S^1(\alpha)$. In fact, the shift L_z^{-1} sends the fiber containing $S^1(\bar{e})$ to the subgroup U and on the other hand, the subgroup $SU_2(\alpha)$ is invariant under the action L_z^{-1}. Hence, $T^1(\alpha)$ belongs to the intersection between $SU_2(\alpha)$ and U. This implies that $T^1(\alpha) = S^1(\alpha)$.

Corollary 3.9. *$S^1(\alpha)$ is a shortest closed geodesic on group G, and therefore, on group U.*

Proof. By construction $SU_2(\alpha)$ is the subgroup corresponding to the highest root α of G. Since G is simply connected the circle $S^1(\alpha)$ is of minimal length [He].

Let $U = SO_n$. It is known that a shortest closed geodesic on SO_n is conjugate under the action of the group $Iso(SO_n)$ with the standardly embedded subgroup SO_2 which generates a non-trivial element in the fundamental group $\pi_1(SO_n)$. Hence, from Corollary 3.9 we immediately get the following consequence.

Corollary 3.10. *Helgason's spheres in symmetric spaces SU_n/SO_n; E_8/SO_{16}, G_2/SO_4 realize non-trivial elements in Z_2-homologies of their ambient spaces.*

In other cases we have to look more carefully. Our aim is to show that the geodesic circle $S^1(\alpha)$ realizes a non-trivial element in the fundamental group $\pi_1(U)$. Let w_α belong to a Cartan algebra H_{lU} which is contained in a Cartan algebra H_{lG}. Let $h_\alpha \in \mathbf{R}w_\alpha$ be the vector corresponding to the root α. It is known that the vector $h(\alpha) = 4\pi\, h_\alpha/|\alpha|^2$ belongs to the unit lattice $\Gamma(G, H_{lG})$ of the group G. Let $\tilde{U}$ denote the universal covering of the group U. The fact that the geodesic circle $S^1(\alpha)$ realizes a non-trivial element in $\pi_1(U)$ is equivalent to that $h_(\alpha)$ does not belong to the unit lattice $\Gamma(\tilde{U}, H_{lU})$ of the group $\tilde{U}$. It is known that the unit lattice Γ of the simply connected group $\tilde{U}$ is $span_{\mathbf{Z}}\{h(\beta_j)\}$, where $\{\beta_j\}$ is a fundamental systems of roots of lU, and $h(\beta_j) = 4\pi h_{\beta_j}/|\beta_j|^2$ (see [He], [Ta 1]).

Let us now consider a symmetric space $M = G/U$, where lU is a direct sum of 2 simple Lie algebras lU_1 and lU_2. In our case M is one of the following spaces: $SO_{m+n}/(SO_n \times SO_m)$, $E_6/(SU_2{\cdot}SU_6)$, $E_7/(SU_2{\cdot}Spin_{12})$, $E_8/(SU_2{\cdot}E_7)$, $F_4/SU_2 \cdot Sp_3$. (Except the case of real grassmannians, other products listed above, $U = U_1 \cdot U_2$, are not direct. Namely, the intersection of U_1 and U_2 consists of 2 points [Ta 1]). We note that the vector $h(\alpha)$ does not lie in any algebra lU_i, $i = 1, 2$, otherwise, the subgroup $SU_2(\alpha)$ lies in the group $U_i \subset U$ entirely. This contradicts to our observation that $SU_2(\alpha)$ meets U at only a circle $S^1(\alpha)$. Hence, in case $lU = so_n \oplus so_m$, the root α can be written as $x_i \pm x_j$, where $x_i \in H^*_{so_n}$ and $x_j \in H^*_{so_m}$. Thus, $h(\alpha)$ does not belong to the unit lattice of $Spin_n \times Spin_m$. In the same way we verify that for all listed above M the Helgason sphere S^2 realizes a non-trivial element in $\pi_2(M) = H_2(M, \mathbf{Z}) = H_2(M, \mathbf{Z_2}) = \mathbf{Z_2}$. To complete the proof of Theorem 3.6 we need to consider the cases $M = E_6/PSp_4$ and $M = E_7/SU_8^*$. Straightforward calculation shows that if a closed geodesic of minimal length in group $\tilde{U}/\{\pm 1\}$, $\tilde{U} = Sp_4, SU_8$, then it is conjugate under $I_0(U)$ with either the circle $S^1(\beta)$ generated by highest root β or the closed geodesic S^1_* whose pull back into the covering group U is the shortest geodesic joining two element $(+1) = (e)$ and (-1). Since the group $SU_2(\alpha)$ does not lie in U, we get that α is not a highest root of $lU_1 \oplus lU_2$. Hence, we easily obtain that the circle $S^1(\alpha)$ is conjugate with S^1_*. Thus, $S^1(\alpha)$ realizes a non-trivial element in $\pi_1(U)$. This completes the proof.

In conclusion we show a consequence of Theorem 2.1 for non-compact symmetric spaces. It is well known that the upper bound of section curvature of these spaces is zero [He].

Proposition 3.11. *Let X be a flat totally geodesic submanifold in a non-compact symmetric space M. Then X is a globally minimal submanifold.*

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