IDEAL WEIGHTS: DOUBLING AND ABSOLUTE CONTINUITY WITH ASYMPTOTICALLY OPTIMAL BOUNDS

Michael Brian Korey

Max-Planck-Arbeitsgruppe
„Partielle Differentialgleichungen und komplexe Analysis"
Universität Potsdam
14415 Potsdam
GERMANY

MPI 96-2

2.5. **Doubling and halving.** Note that all of the above proofs are purely measure-theoretic. They use nothing about the geometry of cubes, the symmetry of Lebesgue measure on $\mathbf{R}^n$, or the uniformity of the A_∞ or A_p bounds of a weight over all cubes. Of importance are only two facts: that Jensen's inequality applies and that the collection of measurable sets is rich enough for a given set to be partitionable into two (equimeasurable) halves, in each of which the values of the weight in question are only on one side of a median value. In order to extend the key estimates of the previous sections to doubling measures, we isolate this partitioning property in the form of a definition.

A measure space is *halving* if each set of finite measure contains a subset of exactly half its measure. Of course, by continuity this means that it is not only possible to divide such a set into halves, but into two pieces the ratios of whose measures is arbitrary.[17] With this in mind, we can re-state the basic implication (2.7) underlying Theorem 1, for example, in a more general form.

Proposition 8. *Let (X, μ) be a halving probability measure space. If f is a measurable function, $0 \le \varepsilon \le 1$, and*

$$\left(\int_X \exp f \, d\mu\right) \Big/ \left(\exp \int_X f \, d\mu\right) = 1 + \varepsilon, \tag{2.18}$$

then

$$\int_X |f - m_{X,\mu}(f)| \, d\mu \le c\sqrt{\varepsilon}. \tag{2.19}$$

Here $m_{X,\mu}(f)$ is a median value of f over X with respect to μ, and c is a universal constant.

Proof. As indicated, the crux of the matter is the question of how to partition X into halves E and F on which the values of f are (respectively) no smaller and no larger than the median. For once this is done, we can follow the proof of Theorem 1, simply replacing Q everywhere by X and $dx/|Q|$ by $d\mu$.

Let $Y = \{x \in X : f(x) = m_{X,\mu}(f)\}$ and $E' = \{x \in X : f(x) > m_{X,\mu}(f)\}$. If Y has zero measure, then the partition is evident: Take $E = E'$ and $F = X \setminus E$. Otherwise, $\mu(Y) > 0$ and $\mu(E') < 1/2$, while $\mu(Y \cup E') > 1/2$; the halving property then allows us to add to E' a portion of Y of measure exactly $1/2 - \mu(E')$. □

Our main interest in this partitioning property stems from the next observation.

Lemma 9. *Every doubling measure is halving.*

Proof. Fix a doubling measure ν and a set Y of finite measure. We can without loss of generality assume that Y is contained within some large cube Q. Consider $g(r) = \nu(Y \cap rQ)$ as a function of r, when $0 \le r \le 1$. This is non-decreasing, $g(0) = 0$, and $g(1) = \mu(Y)$, so that it suffices to see that g is continuous. Were this not the case, then g would have a jump discontinuity at some r_0. But then $\nu(Y \cap \partial(r_0 Q))$ would be positive, violating the fact that boundaries of cubes are sets of zero measure (see Corollary 2 in §1.4). □

[17] Let (X, μ) be a halving measure space and suppose that Y has finite, positive measure. Let $D = \{j2^{-k} : k \in \mathbf{N}, j = 0, 1, \dots, 2^k\}$ be the set of all dyadic rationals in $[0, 1]$. By iteration there exists an increasing family $\{Y_s\}_{s \in D}$ of measurable subsets of Y such that $Y_0 = \emptyset$, $Y_1 = Y$, and $\mu(Y_s) = s\mu(Y)$ for each s in D. When s is not in D, set $Y_s = \cup_{r<s, r \in D} Y_r$; then continuity insures that $\mu(Y_s) = s\mu(Y)$. for all s in $[0, 1]$.

Remark. One can obtain the correct asymptotic result for A_2 by at least two other methods. The first relies upon a simple inequality from calculus:

$$2 + x^2 \leq e^x + e^{-x}, \quad \text{for all } x \in \mathbf{R}.$$

Suppose that $w \in A_2$. With $x = \log(w/w_Q)$, this inequality yields

$$⨍_Q \left(2 + \left(\log \frac{w}{w_Q}\right)^2\right) \leq ⨍_Q \frac{w}{w_Q} + ⨍_Q \frac{w_Q}{w}.$$

Hence

$$⨍_Q |\log w - \log(w_Q)|^2 \leq 1 + w_Q (w^{-1})_Q - 2,$$

and so

$$⨍_Q |\log w - \log(w_Q)| \leq \sqrt{A_2(w) - 1}.$$

As $A_2(w) \to 1$, the radical is $O\big(\sqrt{\log A_2(w)}\big)$.

Alternatively, one can derive the asymptotic A_2 results in the manner of Theorem 1 and Corollary 3 by first analyzing the ratio of the arithmetic and harmonic means of two numbers, in place of their arithmetic and geometric means. The function in question (the ratio when the two numbers are 1 and t) is then $F(t) = (1+t)(1+t^{-1})/4$; like the function in the second proof of Lemma 2, this F satisfies the implication that $F(t) = 1 + \varepsilon$ entails $t = 1 + O(\sqrt{\varepsilon})$.[16]

The formal limit of the A_p condition as $p \to 1$ is the requirement that

$$⨍_Q w \leq K \operatorname*{ess\,inf}_Q w \tag{2.17}$$

uniformly over all cubes Q. This is equivalent to a weak-type bound for the Hardy-Littlewood maximal operator M on $L^1(w\,dx)$, i.e., the estimate

$$w(\{x \in \mathbf{R}^n : Mf(x) > \lambda\}) \leq C\lambda^{-1} \int_{\mathbf{R}^n} f w\,dx.$$

A weight that satisfies (2.17) is said to be in the class A_1, and $A_1(w)$ likewise denotes the smallest constant K. For such weights the corresponding sharp estimates are slightly different.

Corollary 7. *As* $A_1(w) \to 1$, *then*

$$\|\log w\|_* = O\big(\log A_1(w)\big) \quad \textit{and} \quad \log Db(w) = O\big(\log A_1(w)\big).$$

Proof. That the inequality $(1+t)/\big(2\min(1,t)\big) \leq (1+\varepsilon)$ implies $|t-1| \leq c\varepsilon$ is the underlying numerical fact here. Applied in place of Lemma 2, this fact gives the sharp result. □

[16] This was Sarason's technique for proving the doubling estimate; he used a different method to obtain the (non-sharp) BMO estimate.

Consequently, all the prior results of Section 2 are automatically valid, with the same constants, when all averages are formed (and medians taken) with respect to a doubling measure ν in place of Lebesgue measure. In particular, when

$$A_\infty^\nu(w) = \sup_Q \left(\frac{1}{\nu(Q)} \int_Q w\,d\nu \right) \Big/ \left(\frac{1}{\nu(Q)} \exp \int_Q \log w\,d\nu \right)$$

and the BMO_ν norm is given as in §1.3, then

$$\|\log w\|_{*,\nu} = O\left(\sqrt{\log A_\infty^\nu(w)} \right), \quad \text{as } A_\infty^\nu(w) \to 1. \tag{2.20}$$

The results above likewise automatically hold when all averages are formed over balls or over arbitrary intervals in $\mathbf{R}^n$ (the so-called "product setting"), rather than over cubes.

2.6. **B_q and BMO.** The situation for B_q weights ("reverse Hölder weights") is slightly more complicated. While the sharp, asymptotic control of the doubling constant follows by an argument analogous to that for A_∞ above, the corresponding dominance of the mean oscillation of $\log w$ is somewhat trickier.

Let us begin with the doubling result.

Theorem 10. *Fix any number q larger than 1. Then*

$$\log Db(w) = O\left(\sqrt{\log B_q(w)} \right), \quad \textit{as } B_q(w) \to 1. \tag{2.21}$$

To prove this, we once again convert a statement about functional averages to an arithmetic one. In this case, we compare the ℓ^1 and ℓ^q means of two numbers.

Lemma 11. *Suppose that a and b are positive, $0 \le \varepsilon \le 1$, and $1 < q < \infty$. If*

$$\left(\frac{a^q + b^q}{2} \right)^{1/q} \le (1+\varepsilon) \frac{a+b}{2},$$

then

$$1 - c_q\sqrt{\varepsilon} \le \frac{a}{b} \le 1 + c_q\sqrt{\varepsilon},$$

for some constant c_q dependent only on q.

Proof. Simply examine the asymptotic behavior of $F(t) = 2^{1-1/q}(1+t^q)^{1/q}/(1+t)$ near $t = 1$, as in the second proof of Lemma 2. □

Proof of the theorem. To prove (2.21), split a cube Q (by measure) into any two halves, E and F, and set

$$a = \left(\fint_E w^q \right)^{1/q}, \qquad b = \left(\fint_F w^q \right)^{1/q},$$
$$a' = \fint_E w, \qquad b' = \fint_F w.$$

Then

$$\left(\frac{a^q + b^q}{2} \right)^{1/q} = \left(\fint_Q w^q \right)^{1/q} \le B_q(w) \fint_Q w = B_q(w) \frac{a' + b'}{2}.$$

Jensen's inequality once again gives the relations $a' \le a$ and $b' \le b$. If $B_q(w) = 1+\varepsilon$, then Lemma 11 yields $a'/b' = w(E)/w(F) = 1 + O(\sqrt{\varepsilon})$. To complete the proof, take E itself to be a cube within Q and iterate, as in the A_∞ case (Corollary 3). □

So much for doubling. On the other hand, we cannot expect the dominance of the BMO norm of $\log w$ by $\log B_q(w)$ to be a purely measure-theoretic, "single-cube" estimate, as was the case for the corresponding A_∞ and A_p results. For, on any one cube, the B_q condition restricts a priori only the distribution of the large values of w, not simultaneously that of its small values.[18] Consider, for example, the function w defined on the interval $I = [0, 1]$ by

$$w(x) = \begin{cases} 1, & \varepsilon \le x \le 1; \\ \exp(-1/\varepsilon^2) & 0 \le x < \varepsilon. \end{cases}$$

Then $\int_I w^2/\left(\int_I w\right)^2 = 1 + O(\varepsilon)$, since $\int_I w \ge 1 - \varepsilon$ and $\int_I w^2 \le 1$. On the other hand, $\int_I |\log w - (\log w)_I| = O(1/\varepsilon)$. So no uniform bound of the desired type can follow from non-iterative calculations over a single cube.

With the help of the John-Nirenberg inequality, it is possible, however, to recover the analogous asymptotic estimate.[19]

Theorem 12. *Fix a number q larger than 1. Then*

$$\|\log w\|_* = O\left(\sqrt{\log B_q(w)}\right), \quad \text{as } B_q(w) \to 1. \tag{2.22}$$

Proof. The argument has three components. First, note that a weight w is in B_q if and only if its power w^{q-1} is in A_q^ν, with $d\nu = w\,dx$. Indeed, a calculation shows that

$$A_q^\nu(w^{q-1}) = \left(B_q(w)\right)^q, \quad \text{for } d\nu = w\,dx.$$

Second, suppose that $B_q(w) = 1 + \varepsilon$. By Theorem 1, which holds for ν in place of Lebesgue measure, as we have noted in (2.20), then $\log w^{q-1} \in \mathrm{BMO}_\nu$ and

$$\|\log w^{q-1}\|_{*,\nu} = O\left(\sqrt{\log(1+\varepsilon)^q}\right).$$

This means that there is a constant c, depending only on q, such that

$$\|\log w\|_{*,\nu} \le c\sqrt{\varepsilon}, \tag{2.23}$$

provided that ε is sufficiently small.

Third, recall that every B_q weight w is also in A_p, for some p depending only on q and $B_q(w)$; in particular (cf. [30, Chap. 5, §5.1]), it is possible to choose constants p and K larger than 1 such that $A_p(w) \le K$ whenever $B_q(w) \le 2$.[20] Apply this in the customary way (see [25] or [16]) to convert the weighted BMO_ν estimate (2.23) to an unweighted BMO estimate (with respect to Lebesgue measure). That is, fix a cube Q and let c_Q denote the mean value of $\log w$ over Q with respect to the measure ν, i.e., $c_Q = (\log w)_{Q,\nu}$. Express 1 as the product $w^{1/p}w^{-1/p}$, for

[18] As Carleson [5, p. 13] observed, the A_p condition separately restricts the first power and some negative power of w, and thus "the large and small values of $w(x)$ do not interact." The point above is that the B_q condition is, by contrast, a restriction on two *positive* powers of a weight; the fact that this restriction holds uniformly over all cubes leads—via the deep demonstration by Coifman and Fefferman [7] that each B_q weight also is in A_∞, and hence in some A_p class—to a restriction on the small values of w.

[19] An earlier statement of this result appears in Politis [27].

[20] This is the place in the proof that is not simply a measure-theoretic, single-cube estimate, as it invokes the Calderón-Zygmund decomposition.

the index p specified above, apply Hölder's inequality, and use the John-Nirenberg inequality in the form (1.5). With $1/p+1/p'=1$, then

$$\begin{aligned}\frac{1}{|Q|}\int_Q |\log w - c_Q|\,dx &= \frac{1}{|Q|}\int_Q |\log w - c_Q| w^{1/p} w^{-1/p}\,dx \\ &\le \left(\frac{1}{|Q|}\int_Q |\log w - c_Q|^p\,d\nu\right)^{1/p}\left(\frac{1}{|Q|}\int_Q w^{-p'/p}\,dx\right)^{1/p'} \\ &\le C_p Db(w)^2 \|\log w\|_{*,\nu} A_p(w)^{1/p}.\end{aligned}$$

By assumption, $B_q(w)=1+\varepsilon<2$; by Theorem 10, the doubling constant $Db(w)$ is then bounded by some constant $K'=K'(q)$. Combine this with (2.23) and the last estimate to obtain

$$\frac{1}{|Q|}\int_Q |\log w - c_Q|\,dx \le C_p(K')^2 c\sqrt{\varepsilon}K^{1/p}.$$

Thus, $\|\log w\|_* = O(\sqrt{\varepsilon})$ when $B_q(w)=1+\varepsilon$, which completes the proof. □

Remark. Note that the estimates in this section also hold for B_q weights defined with respect to an arbitrary doubling measure. Consideration of step functions once again shows that the square root is the sharp power.

2.7. **Sharp embedding results for weights.** Coifman and Fefferman [7] showed that each A_∞ weight satisfies a reverse Hölder inequality. A well-known consequence of this—related to the work of Gehring [12] on quasiconformal mappings—is the "self-improving" property of A_∞ weights: If w is in the class A_p, then w actually belongs to $A_{\bar p}$, for some $\bar p$ less than p. The results in the previous section allow us to refine the statement of this property. For if $A_p(w)$ is near its optimal value 1, then both the smaller index $\bar p$ and the bound $A_{\bar p}(w)$ can be taken suitably close to 1. Likewise, an A_p weight with small bound must also have B_q bound near the optimal value 1, for large q. For general A_∞ weights, the first explicit statement of this result with the correct asymptotic behavior of the bounds and indices appears in Politis [27]. The result is as follows:

Theorem 13. *There exist positive constants ε_0 and K, depending only on n, such that if $w\in A_\infty$ and $A_\infty(w)<1+\varepsilon_0$, then both*

$$w\in A_p, \quad \textit{for } p = 1+K\sqrt{\log A_\infty(w)},$$

and

$$w\in B_q, \quad \textit{for } q=\left(K\sqrt{\log A_\infty(w)}\right)^{-1}.$$

Furthermore, both $A_p(w)$ and $B_q(w)$ are smaller than $1+K\sqrt{\log A_\infty(w)}$.

Proof. Suppose that the bound $A_\infty(w)$ is close to 1. Apply Theorem 1; then $\log w\in \mathrm{BMO}$ and $\|\log w\|_*\le k\sqrt{\log A_\infty(w)}$, for some constant k. Now use the John-Nirenberg inequality to conclude that w belongs to a finite A_p class.[21] For with c and C the constants appearing in the estimate (1.4) and $\log w$ substituted for f there, then

$$\text{(2.24)}\qquad \int_Q \exp(r|\log w - (\log w)_Q|)\le 1+C, \quad \text{when } r=c/(2\|\log w\|_*).$$

[21] This was first observed, implicitly, by Moser [22]. Sarason [29] invoked the same argument to show that $\log A_2(w)=O(\|\log w\|_*)$, as $\|\log w\|_*\to 0$.

If $r \geq 1$, then Hölder's inequality shows that

$$\begin{aligned}\left(\fint_Q w\right)\left(\fint_Q w^{-r}\right)^{1/r} &\leq \left(\fint_Q w^r\right)^{1/r}\left(\fint_Q w^{-r}\right)^{1/r} \\ &\leq \left(\fint_Q e^{r\left(\log w-(\log w)_Q\right)}\right)^{1/r}\left(\fint_Q e^{-r\left(\log w-(\log w)_Q\right)}\right)^{1/r} \\ &\leq (1+C)^{1/r}(1+C)^{1/r}.\end{aligned}$$

For $\bar{p}-1=(1/r)$, we thus have an estimate for $A_{\bar{p}}(w)$ in terms of $\|\log w\|_*$, namely that

$$A_{\bar{p}}(w) \leq (1+C)^{(4/c)\|\log w\|_*}, \quad \text{when } \bar{p}=1+(2/c)\|\log w\|_* \leq 2.$$

We must now merely keep track of the various constants in order to convert this into the statement of the theorem; for this the choice $K=(2k/c)\max(1,4\log(1+C))$ suffices.

The proof of the second embedding likewise follows from the John-Nirenberg inequality. Simply remove the absolute values from (2.24) to conclude that

$$\fint_Q \exp(r\log w) \leq (1+C)\exp\left(r(\log w)_Q\right).$$

Take the rth root of both sides and apply Jensen's inequality. Then

$$B_r(w) \leq (1+C)^{(2/c)\|\log w\|_*}, \quad \text{when } 1 \leq r = c/(2\|\log w\|_*).$$

Similar manipulations with the constants give the desired estimate. □

Since $A_p(w) \leq A_\infty(w)$, the theorem immediately extends to A_p weights with nearly optimal bounds. We can also use Theorem 12 in place of Theorem 1 to show the converse embedding: Each B_q weight with bound close to the optimal value 1 is also in A_p, with both p and $A_p(w)$ suitably close to 1.

Corollary 14. *Let q be any number larger than 1. There exist positive constants ε_0 and K, depending only on n and q, such that if $w \in B_q$ and $B_q(w) < 1+\varepsilon_0$, then both*

$$w \in A_p, \quad \text{for } p = 1+K\sqrt{\log B_q(w)},$$

and

$$w \in B_{\bar{q}}, \quad \text{for } \bar{q} = \left(K\sqrt{\log B_q(w)}\right)^{-1}.$$

Furthermore, both $A_p(w)$ and $B_{\bar{q}}(w)$ are smaller than $1+K\sqrt{\log B_q(w)}$.

The last result, on the higher integrability of reverse Hölder weights with small bounds, is due to Wik [34]. By considering power weights, Politis [27] has shown that the asymptotic behavior of the embeddings described in the theorem and its corollary is sharp, in the sense that the square root cannot be replaced by a higher power.

Several other variants are possible. For instance, when we consider A_1 weights with nearly optimal bounds, then Corollary 7 shows that the corresponding higher integrability result has the first power of $A_1(w)$ in place of the square root. Kinnunen [20] found the exact version of this last result in the product setting: a formula for the exact higher integrability index q for the finest class B_q to which a given A_1 weight can belong, as well as a precise formula for the bound $B_q(w)$, both in terms of $A_1(w)$.

3. Asymptotic weight conditions

3.1. Convergence of the asymptotic conditions. Section 2 began with a list of equivalent formulations of the A_∞ condition. We shall now show that these various characterizations remain equivalent when the bounds in question approach their optimal values over small scales. In particular, although the weight classes A_p and B_q are distinct for different p and q, these distinctions collapse when we demand that the weight bounds behave optimally in the asymptotic limit. The next theorem proves these assertions.[22]

Theorem 1. *Let w be a weight and p, q numbers larger than 1. Then the following conditions are equivalent:*

(a) $\limsup_{|Q|\to 0}(f_Q w)/(\exp f_Q \log w) = 1$.
(b) $\limsup_{|Q|\to 0}(f_Q w)(f_Q w^{-1/(p-1)})^{p-1} = 1$.
(c) $\limsup_{|Q|\to 0}(f_Q w^q)^{1/q}/(f_Q w) = 1$.
(d) *For each $\varepsilon > 0$ there is a $\delta > 0$ such that whenever E is a measurable subset of Q, with $|E| = |Q|/2$ and $|Q| < \delta$, then $w(E)/w(Q) \le (1+\varepsilon)/2$.*
(e) *For each $\varepsilon > 0$ there is a $\delta > 0$ such that whenever E is a measurable subset of Q and $|Q| < \delta$, then $w(E)/w(Q) \le (1+\varepsilon)(|E|/|Q|)^{1-\varepsilon}$.*
(f) $\limsup_{|Q|\to 0} f_Q |\log w - (\log w)_Q| = 0$.

An A_∞ weight w that satisfies any of these conditions is termed *asymptotically absolutely continuous*, and we write $w \in A_{\infty,as}$.

Proof. The equivalence of the first three conditions with the last is a consequence of the local form[23] of the key results in the Section 2: the dominance of $\|\log w\|_*$ by $\log A_\infty(w)$ and $\log B_q(w)$ in Theorems 1 and 12, and the converse dominance relations (using the John-Nirenberg inequality) in Theorem 13.

Condition (e) follows from the validity of (c) for all finite indices q. Indeed, if $E \subseteq Q$ and w is a B_q weight, then Hölder's inequality shows that

$$w(E)/w(Q) \le B_q(w)\big(|E|/|Q|\big)^{(q-1)/q}. \tag{3.1}$$

When ε is given and $A_\infty(w)$ is sufficiently close to 1, then Theorem 13 guarantees that w is in B_q, with the index q so large and the bound $B_q(w)$ so close to 1 that both $(q-1)/q > 1-\varepsilon$ and $B_q(w) < 1+\varepsilon$. The local form of this shows that (c) implies (e).

Now (e) implies (d), and the challenge is to show that (d) is actually equivalent to one of the other conditions in the list. The proof of this implication in the non-asymptotic case, essentially the proof that A_∞ is the union of the A_p classes for finite p, uses the Calderón-Zygmund decomposition; this decomposition, however, does not keep the bounds which enter close to their optimal values. The key to the proof is rather the interplay between mean and median values and especially the good behavior of the latter under composition with monotone functions (compare Corollary 5 in §2.3).

[22] The asymptotic conditions in the theorem are labeled to correspond to their non-asymptotic counterparts in §2.1. Note that the equivalence of (b) and (f) above was first shown by Sarason [29] for $p = 2$; Jerison and Kenig [16] adapted Sarason's argument to the reverse Hölder condition (c), for $q = 2$. The case of arbitrary, finite indices p and q readily follows. The equivalence of these three conditions with the other three is new.

[23] By this is meant that all references to $\sup_Q$ and all seminorms such as $\|f\|_*$ or quantities such as $A_\infty(w)$ are understood to apply to all cubes Q within a fixed cube Q_0.

In fact, we shall now show that if w fulfills (d), then $v = \sqrt{w}$ satisfies the reverse Hölder condition (c), for the index $q = 2$. To see this, suppose that $|Q| < \delta$ and partition Q into two halves E and F that straddle a median value $m_Q(w)$ (with the larger values of w in E, as above). By assumption, $w(F)/w(Q) \geq (1-\varepsilon)/2$, so that $w_Q \leq (1-\varepsilon)^{-1} w_F$. Substitute v^2 for w and use the fact that $v \leq m_Q(v)$ pointwise on F to pull out one power of v. Then

$$(v^2)_Q \leq (1-\varepsilon)^{-1}(v^2)_F \leq (1-\varepsilon)^{-1} v_F m_Q(v).$$

But $m_Q(v) = \sqrt{m_Q(w)}$, by (2.15), and $m_Q(w)/w_Q \leq w_E/w_F \leq (1+\varepsilon)/(1-\varepsilon)$, by (d). Hence

$$(v^2)_Q = \big(1+O(\varepsilon)\big) v_F \sqrt{w_Q} = \big(1+O(\varepsilon)\big) v_Q \sqrt{(v^2)_Q}.$$

That is, $(v^2)_Q = \big(1+O(\varepsilon)\big)(v_Q)^2$. So v satisfies (c) for $q = 2$, as claimed. Thus v also satisfies (f), and so does w. This completes the proof of the theorem. $\square$

Remark. Reducing the multiplicative conditions (a)–(e) to the additive condition (f) has several advantages. For one, it is now clear that we could have used balls rather than cubes (without altering the class of weights in question). For another, condition (f) ties $A_{\infty,as}$ to the space VMO of functions with *vanishing mean oscillation*, which was introduced by Sarason in [29]. Recall that VMO comprises all the functions f in BMO for which $\limsup_{|Q|\to 0} ⨍_Q |f - f_Q| = 0$. Any bounded, uniformly continuous function on $\mathbf{R}^n$ is in VMO, and VMO is actually the closure of the set of all such functions under the BMO seminorm $\|\cdot\|_*$. The equivalence of (a) and (f) in the prior theorem means that *the logarithm of a weight is in* VMO *just in case some small positive power*[24] *of it is in* $A_{\infty,as}$. One consequence of this is that while asymptotically absolutely continuous weights cannot have jump discontinuities, they can be unbounded; the weight w given by

$$w(x) = \exp\sqrt{\log^+(1/|x|)}$$

is an example, for $\log^+|x| = \max(\log|x|, 0)$.[25]

3.2. **An equivalence relation.** Coifman and Fefferman [7] showed that the A_∞ condition can be viewed as an equivalence relation on doubling measures. In fact, the same is true for $A_{\infty,as}$, as we shall now show.

Toward this end, note first that condition (e) in the last theorem implies the corresponding condition with the opposite bound, namely:

(e') For each $\varepsilon > 0$ there is a $\delta > 0$ such that whenever E is a measurable subset of Q and $|Q| < \delta$, then $|E|/|Q| \leq (1+\varepsilon)\big(w(E)/w(Q)\big)^{1-\varepsilon}$.

[24]The restriction to small powers arises only from global considerations. For the asymptotic condition (a) alone assures, for some finite p, that $A_p(w)$ is uniformly bounded when measured over all sufficiently small cubes; the same is thus true for the $A_{p'}$ bound of the conjugate weight $w^{-p'/p}$, when $1/p + 1/p' = 1$. But then the doubling bounds of w and $w^{-p'/p}$ are uniformly controlled over small cubes, by Corollary 3 of §2.3, and these bounds can be combined to give a uniform A_p bound for w over *all* cubes within an arbitrary, given cube Q_0. In the local case, therefore, we can correctly write the equality VMO $= \log A_{\infty,as}$.

[25]See the calculations in [26] for the corresponding facts about VMO.

Indeed, as (e) implies (b) for all p larger than 1, then (e') follows from the local form of the inequality

$$|E|/|Q| \le \left(A_p(w)\right)^{1/p}\left(w(E)/w(Q)\right)^{1/p}. \tag{3.2}$$

This is the A_p estimate analogous to the B_q estimate (3.1).

We can use condition (d) in the prior theorem as the basis for formulating the desired equivalence relation. Suppose that μ and ν are doubling measures. For any cube Q, let $\mathcal{H}_\nu(Q)$ denote the collection of all "halves" of Q with respect to ν, i.e., all subsets E of Q for which $\nu(E) = \nu(Q)/2$. We say that $\mu \preceq \nu$ if

$$\limsup_{|Q|\to 0} \sup_{E\in\mathcal{H}_\nu(Q)} \frac{\mu(E)}{\mu(Q)} = 1/2. \tag{3.3}$$

Now (3.3) insures that $d\mu = w\,d\nu$, for some weight w for which the local A_∞^ν bounds depend only on the size of the cube under consideration. Since the embedding results of §2.7 that underlie the proof of the prior theorem apply to weight classes defined with respect to doubling measures in place of Lebesgue measure, then we can repeat the steps in the proof of the theorem, with $d\mu$ and $d\nu$ replacing $w\,dx$ and dx, respectively, to obtain a strengthened form of (3.3) on the model of (e) and (e'). In particular, $\mu \preceq \nu$ implies that for each $\varepsilon > 0$,

$$(1-\varepsilon)\left(\nu(E)/\nu(Q)\right)^{1+\varepsilon} \le \mu(E)/\mu(Q) \le (1+\varepsilon)\left(\nu(E)/\nu(Q)\right)^{1-\varepsilon}, \tag{3.4}$$

provided that E is a measurable subset of Q and that the diameter of Q is sufficiently small. From (3.4) it is clear that $\preceq$ is an equivalence relation.

3.3. **Summary.** The material here sheds new light on the nature of the difference between the A_∞ and doubling conditions (cf. [9]). To illustrate this difference, let us consider all of the possible ways to divide a cube Q into two (equimeasurable) halves, E and F. As noted in §2.3, a weight w is in A_∞ just in case the mass of w over the two halves is always comparable; that is, if and only if the ratio $w(E)/w(F)$ is uniformly bounded over all cubes and all such partitions. The doubling condition, on the other hand, requires that this be true only for those special partitions in which at least one of the two halves is itself a cube.

Proposition 4 in §1.5 and Theorem 1 in §3.1 extend this observation to the asymptotic case. In fact, the asymptotic forms of A_∞ and doubling arise precisely when we further demand that the ratio $w(E)/w(F)$—considered in each case over the appropriate class of "halving" partitions—uniformly approach 1 as $|Q| \to 0$.[26]

3.4. **Means and medians are not alone enough.** Section 2.3 contained yet another criterion for A_∞, the uniform comparability of mean and median values over all cubes. We conclude by noting that, unlike the other formulations that appear in Theorem 1 above, this one does not lead to the class $A_{\infty,as}$ when the constant of comparability approaches 1 over ever-smaller scales. While (2.14) insures that means and medians do converge for $A_{\infty,as}$ weights, the example below shows how the reverse can fail.

[26] In fact, it is not necessary to analyze w over all such halving partitions. For asymptotic doubling, it actually suffices by Proposition 4 in §1.5 to consider on each cube Q only the 2^n partitions in which E is a cube that has half the measure and a vertex in common with Q. And for $A_{\infty,as}$, one need only consider within each cube Q a single partition straddling a median value, since any such partition makes the ratio $w(E)/w(F)$ extremal.

Example. Call a closed interval within $[0,1]$ *triadic* if it can be written in the form $[j3^{-k},(j+1)3^{-k}]$, for $k = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots, 3^k - 1$; denote the collection of all triadic intervals by $\mathcal{T}$. On each I in $\mathcal{T}$ define the "extended" Haar function H_I by the rule

$$H_I = \begin{cases} +1 & \text{on the left third of } I; \\ 0 & \text{on the middle third of } I; \\ -1 & \text{on the right third of } I. \end{cases}$$

Note that—in contrast to the case for the standard, dyadic Rademacher functions—the orthogonal set $\{1\} \cup \{H_I : I \in \mathcal{T}\}$ does not form a complete basis for $L^2([0,1])$.

Next, let $\widetilde{\mathcal{T}}$ be the subcollection of those triadic intervals that are not the middle thirds of other intervals in $\mathcal{T}$. Then $\widetilde{\mathcal{T}}$ consists of the intervals

$$[0,1], [0,\tfrac{1}{3}], [\tfrac{2}{3},1], [0,\tfrac{1}{9}], [\tfrac{2}{9},\tfrac{1}{3}], [\tfrac{2}{3},\tfrac{7}{9}], [\tfrac{8}{9},1], \ldots .$$

Construct a sequence $\{w_k\}$ of weights, as follows: With $\widetilde{\mathcal{T}}_k$ the collection of all intervals in $\widetilde{\mathcal{T}}$ of length at least 3^{1-k}, set $w_0 = 1$ and

$$w_k = \prod_{I \in \widetilde{\mathcal{T}}_k} (1 + H_I/2), \qquad k \in \mathbf{N}. \tag{3.5}$$

Each weight w_k has integral 1 over $[0,1]$. The sequence $\{w_k\}$ converges pointwise a.e. to the function $w = \prod_{I \in \widetilde{\mathcal{T}}}(1 + H_I/2)$; note that the infinite product has only finitely many factors (different than 1) at a.e. point in the unit interval. Furthermore, the convergence to w is actually in L^1, as $\{w_k\}$ is a Cauchy sequence in L^1. Indeed,

$$\int |w_{k+1} - w_k| = \left(\frac{1}{2} + \frac{3}{2}\right)^k \left(\frac{1}{3}\right)^{k+1} = \frac{1}{3}\left(\frac{2}{3}\right)^k,$$

so that if $l > k$, then $\int |w_l - w_k| \leq (2/3)^k$.

The limit function w is piecewise constant on the complement of the Cantor set in $[0,1]$. Moreover, w has the special property that on each triadic interval I its mean and median values agree. By the self-similarity[27] of w, it suffices to verify this on the full unit interval alone. Since the mean value of w over $[0,1]$ is 1, we must show that 1 is also a median value there. To see this, we need only consider the values of w over intervals removed in the formation of the Cantor set. Then w assumes the value 1 on $1/3$ of $[0,1]$, the value $3/2$ on $1/9$ of $[0,1]$, and, in general, the value $(3/2)^k$ on $(1/3)^{k+1}$ of $[0,1]$. Summing up, we find that w is no smaller than 1 on a set of measure at least $1/3 + 1/9 + \cdots = 1/2$. When we consider the values $(1/2)^k$, we likewise find that w is no larger than 1 on at least half of the unit interval. Thus, 1 is both the mean value of w on $[0,1]$ and a median value there.[28]

[27] In particular, the values of w, and hence its mean and median values, change by a factor of $3/2$ when we pass from one generation's left-most interval to the next. Explicitly, $w(\tau 3^{-k}) = (3/2)w(\tau 3^{1-k})$ when $k \in \mathbf{N}$ and $0 < \tau < 1$. A similar statement holds when we pass to the right; the factor $1/2$ then replaces $3/2$.

[28] In fact, 1 is the unique median value. Uniqueness requires the additional argument that the values of w, namely products of powers of $1/2$ and $3/2$, include rationals arbitrarily close to 1. This is due to a basic number-theoretic fact on the approximability of irrational numbers (such as $\xi = \log_2 3$) by rationals: For each positive integer N there are integers m and n such that $0 < n \leq N$ and $|\xi - (m/n)| < 1/(Nn)$; see, e.g., [6, Chap. 3].

On the other hand, the ratio of the arithmetic and geometric means of w does not approach 1 over ever-smaller (triadic) intervals, as would be the case for a weight in $A_{\infty,as}$. For I in $\mathcal{T}$, the logarithm of this ratio (cf. [10] in the dyadic case) is

$$\begin{aligned}\log(w_I) - (\log w)_I &= -\frac{1}{|I|}\sum_{\substack{J\in\widetilde{\mathcal{T}}\\ J\subseteq I}}\int_I \log\left(1+\frac{H_J}{2}\right) \\ &= -\frac{1}{|I|}\sum_{\substack{J\in\widetilde{\mathcal{T}}\\ J\subseteq I}}\left(\frac{1}{3}\log\frac{3}{2}+\frac{1}{3}\log 1+\frac{1}{3}\log\frac{1}{2}\right)|J| \\ &= \frac{1}{3}\left(\log\frac{4}{3}\right)\left(\frac{1}{|I|}\sum_{\substack{J\in\widetilde{\mathcal{T}}\\ J\subseteq I}}|J|\right).\end{aligned}$$

The last factor (the sum divided by $|I|$) is 0 for each middle-third interval I in $\widetilde{\mathcal{T}}\setminus\mathcal{T}$, but is $1+(2/3)+(2/3)^2+\cdots = 3$ for each I in $\widetilde{\mathcal{T}}$. Thus, $w_I/m_I(w) = 1$ for all triadic I, although the ratio defining the A_∞ bound of w does not approach 1 as $|I| \to 0$.

Remark. The weight w has another property worthy of note. Recall from the John-Nirenberg inequality that a function f is in BMO if and only if

$$\sup_Q \fint_Q |f-f_Q|^p < \infty$$

when $p \geq 1$. By refinements of this due to John [17] and Strömberg [31], the same is true for any positive p.

What happens as p approaches 0? The weight w constructed above has the property that it satisfies the limiting condition

$$\sup_Q \exp\left(\frac{1}{|Q|}\int_Q \log|w-w_Q|\right) < \infty \tag{3.6}$$

(over all triadic intervals $Q = I$ in $[0,1]$) while not itself being in BMO. Indeed, w assumes its mean value over any triadic interval identically on the entire middle third of that interval—this is because the products in (3.5) were formed over the special intervals in $\widetilde{\mathcal{T}}$ rather than over all of $\mathcal{T}$—so that the left-hand side in (3.6) vanishes when the supremum runs over all triadic I. On the other hand, examining the scaling of w over the intervals $\{[0,3^{-k}]\}$ shows that w is not in (triadic) BMO.

It should be noted, however, that a slight modification of (3.6) does lead to a correct characterization of BMO. In fact, one can use the methods of [31] to show that

$$f \in \text{BMO} \quad \text{if and only if} \quad \sup_Q \exp\left(\frac{1}{|Q|}\int_Q \log^+|f-f_Q|\right) < \infty.$$

For details, see [21, Chap. 3].

References

[1] A. Beurling and L. Ahlfors, The boundary correspondence under quasiconformal mappings, *Acta Math.* 96 (1956), 125–142.

[2] S. M. Buckley, Estimates for operator norms on weighted spaces and reverse Jensen inequalities, *Trans. Amer. Math. Soc.* 340 (1993), 253–272.

[3] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa, Boundary behavior of non-negative solutions of elliptic operators in divergence form, *Indiana U. Math. J.* 30 (1981), 621–640.

[4] L. Carleson, On mappings, conformal at the boundary, *J. Analyse Math.* **10** (1967), 1–13.
[5] L. Carleson, BMO—10 years' development, *18th Scandanavian Congress of Mathematicians (Aarhus, 1980)*, Progress in Mathematics **11**, Birkhäuser (1981), 3–21.
[6] K. Chandrasekharan, *Introduction to Analytic Number Theory*, Grund. der math. Wiss. **148**, Springer-Verlag (1968).
[7] R. R. Coifman and C. Fefferman, Weighted norm inequalities for maximal functions and singular integrals, *Studia Math.* **51** (1974), 241–250.
[8] B. Dahlberg, On estimates for harmonic measure, *Arch. Rat. Mech. Anal.* **65** (1977), 272–288.
[9] C. Fefferman and B. Muckenhoupt, Two nonequivalent conditions for weight functions, *Proc. Amer. Math. Soc.* **45** (1974), 99–104.
[10] R. Fefferman, C. E. Kenig, and J. Pipher, The theory of weights and the Dirichlet problem for elliptic equations, *Ann. of Math.* (2) **134** (1991), 65–124.
[11] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland (1985).
[12] F. W. Gehring, The L^p-integrability of the partial derivatives of a quasiconformal mapping, *Acta Math.* **130** (1973), 265–277.
[13] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Univ. Press (1934).
[14] S. V. Hruščev, A description of weights satisfying the A_∞ condition of Muckenhoupt, *Proc. Amer. Math. Soc.* **90** (1984), 253–257.
[15] R. Hunt, B. Muckenhoupt, and R. Wheeden, Weighted norm inequalities for the conjugate function and Hilbert transform, *Trans. Amer. Math. Soc.* **176** (1973), 227–251.
[16] D. Jerison and C. E. Kenig, The logarithm of the Poisson kernel for a C^1 domain has vanishing mean oscillation, *Trans. Amer. Math. Soc.* **273** (1982), 781–794.
[17] F. John, Quasi-isometric mappings, *Seminari 1962/63*, Instituto Nazionali di Alta Matematica (1965), 462–473.
[18] F. John and L. Nirenberg, On functions of bounded mean oscillation, *Comm. Pure Appl. Math.* **14** (1961), 415–426.
[19] C. E. Kenig, *Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems*, CBMS Regional Conf. Ser. in Math. **83**, American Math. Society (1994).
[20] J. Kinnunen, Sharp results on reverse Hölder inequalities, *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes* **95** (1994).
[21] M. B. Korey, Ideal weights: doubling and absolute continuity with asymptotically optimal bounds, Ph.D. Thesis, University of Chicago (1995).
[22] J. Moser, On Harnack's theorem for elliptic partial differential equations, *Comm. Pure Appl. Math.* **14** (1961), 577–591.
[23] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.* **165** (1972), 207–226.
[24] B. Muckenhoupt, The equivalence of two conditions for weight functions, *Studia Math.* **49** (1974), 101–106.
[25] B. Muckenhoupt and R. Wheeden, Weighted bounded mean oscillation and the Hilbert transform, *Studia Math.* **54** (1976), 221–237.
[26] U. Neri, Some properties of functions with bounded mean oscillation, *Studia Math.* **61** (1977), 63–75.
[27] A. Politis, Sharp results on the relation between weight spaces and BMO, Ph.D. Thesis, University of Chicago (1995).
[28] H. M. Reimann and T. Rychener, *Funktionen beschränkter mittlerer Oszillation*, Lecture Notes in Math. **487**, Springer-Verlag (1975).
[29] D. Sarason, Functions of vanishing mean oscillation, *Trans. Amer. Math. Soc.* **207** (1975), 391–405.
[30] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press (1993).
[31] J.-O. Strömberg, Bounded mean oscillation with Orlicz norms and duality of Hardy spaces, *Indiana U. Math. J.* **28** (1979), 511–544.
[32] J.-O. Strömberg and A.Torchinsky, Weights, sharp maximal functions and Hardy spaces, *Bull. Amer. Math. Soc.* **3** (1980), 1053–1056.
[33] J.-O. Strömberg and A.Torchinsky, *Weighted Hardy Spaces*, Lecture Notes in Math. **1381**, Springer-Verlag (1989).

[34] I. Wik, Reverse Hölder inequalities with constant close to 1, *Ricerche Mat.* **39** (1990), 151–157.

MAX-PLANCK-ARBEITSGRUPPE "PARTIELLE DIFFERENTIALGLEICHUNGEN UND KOMPLEXE ANALYSIS", UNIVERSITÄT POTSDAM, 14415 POTSDAM, GERMANY
E-mail address: mike@mpg-ana.uni-potsdam.de

IDEAL WEIGHTS: DOUBLING AND ABSOLUTE CONTINUITY WITH ASYMPTOTICALLY OPTIMAL BOUNDS

MICHAEL BRIAN KOREY

ABSTRACT. Sharp relations between weight bounds (from the doubling, A_p, and reverse Hölder conditions) and the BMO norm are obtained, when the former are near their optimal values. In particular, the BMO norm of the logarithm of a weight is seen to be controlled by the square root of the logarithm of its A_∞ bound. Coifman and Fefferman's formulation [7] of the A_∞ condition as an equivalence relation on doubling measures is extended to the setting in which all bounds become optimal over small scales.

Oh! the little more, and how much it is!
And the little less, and what worlds away![1]

INTRODUCTION

This work focuses on two conditions for non-negative measures: doubling and the scale-invariant form of absolute continuity known as the A_∞ condition. Both conditions restrict the rate of growth of a measure over a nested sequence of sets, but the latter does so far more stringently. In the simplest context of intervals on the real line, a measure is doubling when the measures of the left and right half of each interval agree up to some fixed factor. The A_∞ condition requires more: Such uniform comparability must still hold whenever an interval is divided into two sets of equal length, not just into its left and right half.[2] This additional requirement actually guarantees that the measure in question and Lebesgue measure are mutually absolutely continuous, as Coifman and Fefferman [7] observed.

Both conditions arise often in varied contexts of mathematical analysis. In the setting of complex analysis, for example, Beurling and Ahlfors [1] gave a criterion for when an increasing homeomorphism $x \mapsto F(x)$ of the real line can be extended to a quasiconformal mapping on the upper half-plane; the criterion is exactly that dF be a doubling measure.

Harmonic analysis in non-smooth domains presents another setting. Consider an elliptic operator in divergence-form,

$$Lu(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right).$$

1991 *Mathematics Subject Classification.* Primary 42B25 26D15; Secondary 26B35.

Key words and phrases. Doubling measure, bounded mean oscillation, A_∞ condition, reverse Hölder inequality, Muckenhoupt A_p condition, arithmetic-geometric inequality.

Supported by the Max-Planck-Gesellschaft. This work is a revised form of part of the author's dissertation, which was written under Professor Carlos E. Kenig at the University of Chicago.

[1] Robert Browning, cited in [13, p. ii].

[2] Length means here Lebesgue measure; for this simple formulation of A_∞ see §2.3.

Suppose that the coefficient functions $\{a_{ij}\}$ form a real, symmetric, and strictly positive-definite matrix, but are otherwise only bounded, measurable functions; the pullback of the Laplacian from a starlike, Lipschitz domain about the origin to the unit ball B yields an operator of this type. Solutions u of the classical Dirichlet problem with continuous boundary data,

$$\begin{cases} Lu = 0 & \text{in } B \\ u|_{\partial B} = f \in C(\partial B), \end{cases}$$

give rise to the so-called "harmonic measure" ω_L for L, defined by the representing formula

$$u(0) = \int_{\partial B} f \, d\omega_L.$$

Harmonic measure is always doubling (over surface balls on ∂B) for all such elliptic operators, but it satisfies the stronger A_∞ condition with respect to surface measure σ only when the Dirichlet problem can be meaningfully solved for all boundary data in some $L^p(\partial B, d\sigma)$ space, not merely for continuous functions f (see [3] and [8]). Much recent work has been devoted to finding reasonable conditions on the coefficients of the operator that guarantee that this is the case (see [19] for an overview).

The situation of interest in the present work occurs when the doubling or A_∞ behavior of a measure becomes optimal in the asymptotic limit ("over small scales"). On the line, a measure is said to have asymptotic doubling, for example, when the ratio of the measures of the left and right half of each interval is not only bounded, but approaches 1 as we examine ever-smaller intervals. Optimal asymptotic behavior of this type was considered by Carleson [4], in the context of quasiconformal mappings that approach conformality at the boundary; by Sarason [29], for the space of functions of vanishing mean oscillation; and by Jerison and Kenig [16], via the sharp regularity for the Poisson kernel in a C^1 domain. Each of these settings motivates a portion of the present work.

The structure of this paper is as follows. The first section introduces several formulations of the doubling condition; in particular, doubling is seen to be equivalent to a multiplicative version of the continuity property for measures. The class of asymptotically doubling measures is also studied.

Section 2 focuses on the theory of A_∞ weights, as developed by Muckenhoupt, Coifman, and Fefferman, and extends this theory to the asymptotic case of weights with bounds that approach the optimal value 1 over ever-smaller scales. A key result is the sharp relation between the A_∞ bound of a weight and the norm of its logarithm in the space of functions of bounded mean oscillation (BMO), when both quantities are near their smallest possible values; the proof is a purely measure-theoretic argument. This is used, in conjunction with the fundamental inequality of John and Nirenberg [18], to obtain embedding results between the A_p and reverse Hölder weight classes that keep all constants of inequality close to their optimal values.[3]

Much attention is paid throughout to the precise relations between different types of functional averages, especially between arithmetic means, geometric means, and median values. The variety of sharp estimates developed bears fruit in allowing us

[3]The thesis of Politis [27] focuses on these embedding results and contains the first proof of the sharp BMO-A_∞ result alluded to above.

to show in the final section that the various classical formulations of A_∞ remain equivalent when we demand optimal bounds in their respective asymptotic limits. We use this to extend the formulation of A_∞ in [7] as an equivalence relation on doubling measures to the asymptotic setting.

1. An overture on doubling

1.1. **Notation.** For reference, we record here the notation that is used throughout this work. The symbol $|E|$ denotes the Lebesgue measure of the set E in $\mathbf{R}^n$. The Lebesgue integral of the function f over E is written $\int_E f$ or $\int_E f\,dx$; if the region of integration is not shown, it is understood to be all of $\mathbf{R}^n$. When $0 < |E| < \infty$, the symbol f_E and the "barred" integral $⨍_E f$ both represent the mean value of f over E; that is, $f_E = ⨍_E f = (\int_E f)/|E|$. If ν is a Borel measure on $\mathbf{R}^n$, then ν_E likewise denotes the corresponding mean value, i.e., $\nu_E = \nu(E)/|E|$.

An interval in $\mathbf{R}$ is always assumed to be closed and of finite, positive length. An interval in $\mathbf{R}^n$ is a Cartesian product of n such intervals in $\mathbf{R}$; cubes are intervals all of whose sides have the same length. The dyadic intervals in the line are all those of the form $[k2^l, (k+1)2^l]$, for arbitrary integers k and l. Products of n dyadic intervals of the same length constitute the collection $\mathcal{D}$ of all dyadic cubes in $\mathbf{R}^n$. Similarly, the dyadic subcubes of an arbitrary cube in $\mathbf{R}^n$ are all those obtained by dividing it into 2^n congruent cubes of half its length, dividing each of these into 2^n congruent cubes, and so on.

The side-length of the cube Q is written $l(Q)$. The symbol mQ denotes the m-fold dilation of Q, that is, the cube with the same center as and m times the side-length of Q; the notation mB likewise indicates the ball concentric with the ball B and having m times its radius. The diameter, interior, and closure of a set E are abbreviated $\operatorname{diam}(E)$, $\operatorname{int}(E)$, and $\bar{E}$, respectively. Finally, two sets are said to be non-overlapping if the intersection of their interiors is empty.

1.2. **The basics of doubling.** A non-negative, locally-finite Borel measure ν on $\mathbf{R}^n$ is *doubling* if the mean values of ν over each cube and over the concentric double of the cube are uniformly comparable; that is, if there is a constant C such that for all cubes Q in $\mathbf{R}^n$,

$$C^{-1}\nu_Q \le \nu_{2Q} \le C\nu_Q. \tag{1.1}$$

Since $|2Q|/|Q| = 2^n$ and $\nu \ge 0$, the first equality is automatically true. What the doubling condition asserts is rather the second inequality, which is usually expressed by the requirement that the ratio $\nu(2Q)/\nu(Q)$ be uniformly bounded over all cubes. We choose the formulation (1.1), in terms of averages, because we shall be particularly interested in the case when the behavior of the measure ν closely resembles that of Lebesgue measure, for which the constant C can be taken to be exactly 1. The smallest C in (1.1) is termed the *doubling constant* $Db(\nu)$ of ν.

Note that there is nothing sacred about the choice of cubes in this context. Since the inscribed and circumscribed balls of a cube have comparable volume, the definition could as well have been stated in terms of balls, for example. More generally, let us say that the two sets E and F in $\mathbf{R}^n$ form an r-*regular pair*, for some fixed number r larger than 1, if each contains a cube whose r-fold dilation engulfs the union $E \cup F$. Repeated application of the doubling property shows that the averages of a doubling measure over all pairs E, F of r-regular sets are uniformly comparable. So, for instance, the averages of a doubling measure over all pairs of congruent, adjacent cubes (those with a face in common) are uniformly comparable. The same is also true for the averages over consecutive pairs within a sequence of similar annuli,

$$(2B\backslash B), (4B\backslash 2B), \ldots, (2^k B\backslash 2^{k-1}B), \ldots$$

where B is an arbitrary ball.

1.3. **BMO in the context of doubling.** As has often been observed, many classical results in harmonic analysis that were first noted for function spaces defined in terms of Lebesgue measure continue to hold when these spaces are defined in terms of doubling measures. Such is the case for the space BMO, and, for reference, we state here several such results that will later be of use.

Let ν be a non-negative Borel measure on $\mathbf{R}^n$. A locally ν-integrable, real-valued function f on $\mathbf{R}^n$ has *bounded mean oscillation* with respect to ν if the quantity

$$\|f\|_{*,\nu} = \sup_Q \frac{1}{\nu(Q)} \int_Q |f - f_{Q,\nu}|\, d\nu$$

is finite; the supremum here runs over all cubes Q, and $f_{Q,\nu}$ denotes the average $(\int_Q f\, d\nu)/\nu(Q)$. The set of all such functions is written BMO_ν. (If ν is Lebesgue measure, then we write simply BMO and $\|\cdot\|_*$.) Note that the class of functions is not changed if we minimize the mean oscillation about all real constants, not just about the mean value, for the triangle inequality shows that

$$\frac{1}{\nu(Q)} \int_Q |f - f_{Q,\nu}|\, d\nu \leq 2 \inf_{c\in\mathbf{R}} \frac{1}{\nu(Q)} \int_Q |f - c|\, d\nu \tag{1.2}$$

on each cube Q.

All bounded functions are in BMO. The even logarithm $x \mapsto \log|x|$ is an example of an unbounded BMO function; in fact, functions with singularities no worse than logarithmic are paradigmatic for BMO, as the following noted result implies.

John-Nirenberg Inequality ([18]). *Let ν be a doubling measure and f a function in BMO_ν. Then for every $\lambda > 0$ and every cube Q,*

$$\nu(\{x \in Q : |f(x) - f_{Q,\nu}| > \lambda\}) < C \exp\left(\frac{-c\lambda}{Db(\nu)^2 \|f\|_{*,\nu}}\right) \nu(Q). \tag{1.3}$$

Here C and c are constants depending only on the dimension n, not on f, Q, ν, or λ.[4]

An iterative use of the Calderón-Zygmund decomposition gives a direct proof of (1.3) with respect to Lebesgue measure; for the modifications of this necessary for the general case of doubling measures, see [25] or [28, Chap. 2].

The estimate (1.3) has a number of important consequences. First, each BMO_ν function, when raised to a sufficiently small power, is (locally) exponentially integrable. In fact, if $f \in \mathrm{BMO}_\nu$, then

$$\frac{1}{\nu(Q)} \int_Q \exp(r|f - f_{Q,\nu}|)\, d\nu \leq 1 + C, \quad \text{when } r = c/(2Db(\nu)^2 \|f\|_{*,\nu}). \tag{1.4}$$

[4] The exponent 2 of $Db(\nu)$ is inessential; it arises because $Db(\nu)$ was defined in terms of a cube and its concentric double, not in terms of the dyadic double that enters in the proof.

Second, although the mean oscillation of a function was defined above in terms of the L^1 norm, the L^1 and L^p mean oscillations of BMO functions are actually equivalent, for all finite p larger than 1. In other words, when $1 < p < \infty$, then

$$(1.5) \qquad \|f\|_{*,\nu} \leq \sup_Q \left(\frac{1}{\nu(Q)} \int_Q |f - f_{Q,\nu}|^p \, d\nu \right)^{1/p} \leq C_{n,p} Db(\nu)^2 \|f\|_{*,\nu},$$

for all f in BMO_ν.

1.4. **A continuity criterion.** The definition of doubling in §1.2 seems in one sense rather coarse. Suppose, for instance, that we wish to compare the mean values of a doubling measure over two cubes of identical size that largely overlap; it appears that we can only predict that the means lie within a factor of $2^n Db(\nu)$ of one another, since the concentric double of each cube contains the other. But as we shall presently show, the constant of comparability actually approaches 1 as the degree of overlap becomes total, and this property characterizes doubling.

Recalling a continuity property for general measures helps to clarify this formulation of the doubling condition. Indeed, for any (closed) cube Q and any (locally-finite) measure ν, continuity insures that $\nu(rQ)$ converges to $\nu(Q)$ as r decreases to 1. If ν is doubling, however, then this convergence of differences can be strengthened to a convergence of ratios.[5]

Proposition 1. *A measure ν is doubling if and only if $\nu(rQ)/\nu(Q) \to 1$ as $r \to 1$, uniformly over all cubes Q. An analogous statement holds for balls.*

Proof. By translation and dilation invariance, it suffices to take Q to be $[-1,1]^n$. For $k = 1, 2, 3, \ldots$, let E_k be the outermost annular band of width 2^{-k} within this cube, i.e., $E_k = \overline{Q \setminus (r_k Q)}$, when $r_k = 1 - 2^{-k}$. Our aim is to show that $\nu(E_k)/\nu(Q) \to 0$ as $k \to \infty$, with a rate that depends only on the doubling constant of ν.

An elementary geometric argument suffices. Let Ω_k be the "inner half" of the annular region E_k, i.e., $\Omega_k = \overline{(r_{k+1}Q) \setminus (r_k Q)}$. This set is actually a finite union of non-overlapping cubes of side-length 2^{-k-1}, the 3-fold dilations of which together cover E_k. Taking the central half of each of these cubes leads to a pairwise disjoint collection, fully within the interior of Ω_k, whose 6-fold dilations cover E_k. Hence, $\nu(\operatorname{int} \Omega_k) > \theta \nu(E_k)$, for some constant θ less than 1 that depends only on $Db(\nu)$. Then $\nu(E_{k+1}) < (1-\theta)\nu(E_k)$, and so $\nu(E_k) < (1-\theta)^{k-1}\nu(Q)$, by iteration. Since $(1-\theta)^{k-1}$ vanishes as $k \to \infty$, we obtain the desired result.

The sufficiency of the criterion is straightforward, and the argument for balls is similar. □

From this we recover the familiar fact that doubling measures vanish over the faces of a cube (see [11, §4.2]).

Corollary 2. *Boundaries of balls and cubes are null sets for all doubling measures.*

Formulated in the language of convolutions, the doubling condition (1.1) becomes the requirement that

$$(1.6) \qquad C^{-1} \nu * \chi_t \leq \nu * \chi_{2t} \leq C \nu * \chi_t,$$

uniformly for all positive t. Here χ is the characteristic function of the unit cube $[-1,1]^n$ or of the unit ball, χ_t is the mass-preserving dilation of χ to scale t

[5] This is implicit in Buckley [2].

(i.e., $\chi_t(x) = t^{-n}\chi(x/t)$), and the functional inequality (1.6) is understood to hold uniformly over all of the underlying domain $\mathbf{R}^n$.[6] The previous proposition then implies that small dilations and translations of the averaging kernel χ have a negligible effect on the averages of a doubling measure. With T^λ denoting the operator of translation by λ, we can express this fact as follows:

Corollary 3. *Suppose ν is doubling. Then for each $\varepsilon > 0$, there are constants $\rho_0 = \rho_0(Db(\nu), \varepsilon) > 1$ and $\lambda_0 = \lambda_0(Db(\nu), \varepsilon) > 0$ such that the estimates*

$$(1+\varepsilon)^{-1}\nu * \chi_t \le \nu * \chi_{\rho t} \le (1+\varepsilon)\nu * \chi_t \tag{1.7}$$

and

$$(1+\varepsilon)^{-1}\nu * \chi_t \le \nu * (T^\lambda\chi)_t \le (1+\varepsilon)\nu * \chi_t \tag{1.8}$$

hold uniformly for positive t, whenever ${\rho_0}^{-1} \le \rho \le \rho_0$ and $\lambda \in \mathbf{R}^n$ satisfies $|\lambda| \le \lambda_0$. Conversely, if there exists a single $\rho \ne 1$ and a single positive ε such that (1.7) *holds uniformly for positive t, then the measure ν is doubling.*[7]

1.5. **Asymptotic doubling.** The various characterizations of doubling in the prior section are scale-invariant. What happens, however, if we demand of a measure that its doubling behavior improves over finer scales? The optimal improvement in this regard would be for the doubling constant to approach 1 over smaller and smaller scales, and this is exactly the condition we now examine.

A doubling measure ν is *asymptotically doubling* if the averages of ν over every pair Q, Q' of sufficiently small, 3-regular cubes agree up to a factor arbitrarily close to 1. That is, for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$(1+\varepsilon)^{-1}\nu_Q < \nu_{Q'} < (1+\varepsilon)\nu_Q, \tag{1.9}$$

whenever the three conditions $Q \subseteq 3Q'$, $Q' \subseteq 3Q$, and $\max(l(Q), l(Q')) \le \delta$ jointly hold.

Though this definition requires that ν closely resemble Lebesgue measure over small scales, such a ν can still be purely singular with respect to Lebesgue measure, as Carleson showed in [4] by means of modified Riesz products. The A_∞ condition, a criterion that guarantees absolute continuity and is thus stronger than doubling, will be the focus of attention beginning in the next section.

The choice of the number 3 in the above definition was, of course, arbitrary; by iteration, we could just as well have considered pairs of r-regular cubes, for any other fixed r larger than 1. Thus, in terms of convolution with the characteristic function of the unit cube, a doubling measure ν is asymptotically doubling if, for each given, *fixed* range of dilation and translation factors, say $1/2 \le \rho \le 2$ and $|\lambda| \le 1$, the estimates (1.7) and (1.8) together hold uniformly for all sufficiently small, positive t.[8] For a generic doubling measure, by contrast, Corollary 3 asserts that these estimates hold only over *some* range ${\rho_0}^{-1} \le \rho \le \rho_0$ and $|\lambda| \le \lambda_0$, where ρ_0 and λ_0 depend on the doubling constant of ν.

---

[6] This will be a standing assumption for such convolution inequalities in the sequel.

[7] Likewise, if for some positive ε, the condition (1.8) holds uniformly over all positive t and all λ in a neighborhood of the origin, then ν is doubling. How to formulate a similarly strong converse in terms of only finitely many discrete translations is not immediately apparent, although Proposition 4 below gives a partial answer.

[8] For definiteness, the convolution kernel χ is in this case taken to be the characteristic function of the unit cube.

II. Doubling over dyadic neighbors. By assumption, the averages of ν over two dyadic cubes of the same size that share a common face agree up to the factor $1+\varepsilon$. It is easy to see that a similar result holds for same-sized dyadic cubes that have (at least) a vertex in common. Indeed, any two such dyadic cubes Q and Q' are connectable by a chain $Q, Q_1, \ldots, Q_m, Q'$ of at most $n+1$ congruent, dyadic cubes, each of which is adjacent to its immediate neighbors in the chain. Hence, $\nu_Q/\nu_{Q'} \leq (1+\varepsilon)^n = 1+O(\varepsilon)$, for all such pairs Q, Q'.

III. Special enveloping cubes. Returning to the problem at hand, let Q_0 be a fixed (arbitrary) cube and let $\hat{Q}_0$ denote any cube with following properties:

(i) $Q_0 \subset \hat{Q}_0$
(ii) $4l(Q_0) \leq l(\hat{Q}_0) < 8l(Q_0)$
(iii) $\hat{Q}_0$ is the union of 4^n dyadic cubes of side length $l(\hat{Q}_0)/4$.

Such a cube $\hat{Q}_0$, while not necessarily dyadic, contains Q_0 and can be decomposed into a finite union of dyadic pieces, each of which is approximately the size of Q_0. Any such cube $\hat{Q}_0$ is termed a *special enveloping cube* of Q_0. Since each 3-regular pair of cubes has such a special enveloping cube in common, to prove the proposition it suffices to show that the mean values of ν over Q_0 and any such enveloping cube $\hat{Q}_0$ agree to within a factor of $1+O(\varepsilon)$.

By (1.12) and (ii), each dyadic cube Q in $\mathcal{D}_k(Q_0)$ satisfies the size condition

$$2^{k+2}l(Q) \leq l(\hat{Q}_0) < 2^{k+4}l(Q)$$

As both $l(Q)$ and $l(\hat{Q}_0)$ are powers of 2, the latter length is either 2^{k+2} or 2^{k+3} times the former. Hence Q is one of the cubes obtained from dividing $\hat{Q}_0$ into $2^{(k+2)n}$ or $2^{(k+3)n}$ dyadic pieces. Paragraph II thus shows that

$$\nu_Q = \big(1+O(\varepsilon)\big)^{k+3}\nu_{\hat{Q}_0}, \quad \text{for each } Q \in \mathcal{D}_k(Q_0).$$

We now use the convergence of the approximations shown in Paragraph I to sum up this last estimate over all the maximal dyadic cubes Q within Q_0. In fact,

$$\begin{aligned}
\nu(Q_0) &= \sum_{k=0}^{\infty} \sum_{Q\in\mathcal{D}_k(Q_0)} \nu(Q) \\
&= \sum_{k=0}^{\infty} \sum_{Q\in\mathcal{D}_k(Q_0)} \nu_Q|Q| \\
&= \sum_{k=0}^{\infty} \sum_{Q\in\mathcal{D}_k(Q_0)} \big(1+O(\varepsilon)\big)^{k+3}\nu_{\hat{Q}_0}|Q| \\
&= \sum_{k=0}^{\infty} \big(1+O(\varepsilon)\big)^{k+3}|\Omega_k(Q_0)|\nu_{\hat{Q}_0}.
\end{aligned}$$

Then

$$\frac{\nu_{Q_0}}{\nu_{\hat{Q}_0}} = \frac{\nu(Q_0)}{|Q_0|\nu_{\hat{Q}_0}} = |Q_0|^{-1}\sum_{k=0}^{\infty}\big(1+O(\varepsilon)\big)^{k+3}|\Omega_k(Q_0)|.$$

The choice of the number $r = 3$ is nevertheless convenient, because it is just large enough for the composite condition

$$Q \subseteq rQ' \quad \text{and} \quad Q' \subseteq rQ$$

to include the case when Q and Q' are adjacent cubes of the same size. It turns out that we need only compare the averages of a measure over such special pairs of cubes in order to analyze its asymptotic doubling behavior. This fact follows directly from the next result.

Proposition 4. *Assume that $0 < \varepsilon < 4^{-n}$. Suppose that*

$$(1+\varepsilon)^{-1}\nu_{Q'} \leq \nu_Q \leq (1+\varepsilon)\nu_{Q'} \tag{1.10}$$

for all pairs Q, Q' of adjacent, dyadic cubes of the same size. Then

$$(1+C\varepsilon)^{-1}\nu_{Q'} \leq \nu_Q \leq (1+C\varepsilon)\nu_{Q'} \tag{1.11}$$

for all (not necessarily dyadic) pairs Q, Q' of 3-regular cubes. Here C is a purely dimensional constant.

Proof. A one-dimensional proof is given in [4]. The proof that follows is a variant of this, using a substitute construction to get around the difficulty that functions in higher dimensions do not have primitives. As modified, the proof has three parts.

I. Dyadic approximation. Each cube in $\mathbf{R}^n$ is a union of non-overlapping dyadic cubes. Ordering the dyadic cubes by size gives us a means of approximating any cube in $\mathbf{R}^n$ by finite unions of dyadic cubes. More precisely, let $\mathcal{D}$ denote the collection of dyadic cubes in $\mathbf{R}^n$; let $\tilde{Q}$ denote the "dyadic double" of a cube Q in $\mathcal{D}$, that is, the unique cube in $\mathcal{D}$ containing Q and with twice its side-length. Fix an arbitrary (not necessarily dyadic) cube Q_0 and let $\mathcal{D}(Q_0)$ consist of all the maximal dyadic cubes within Q_0, i.e.,

$$\mathcal{D}(Q_0) = \{Q \in \mathcal{D} : Q \subseteq Q_0, \tilde{Q} \not\subseteq Q_0\}.$$

Group the cubes in $\mathcal{D}(Q_0)$ according to size: For each $k = 0, 1, 2, \ldots$, set

$$\mathcal{D}_k(Q_0) = \{Q \in \mathcal{D}(Q_0) : 2^{-k-1}l(Q_0) < l(Q) \leq 2^{-k}l(Q_0)\}. \tag{1.12}$$

Finally, let

$$\Omega_k(Q_0) = \bigcup_{Q \in \mathcal{D}_k(Q_0)} Q \quad \text{and} \quad R_k(Q_0) = \bigcup_{j \leq k} \Omega_j(Q_0).$$

Note that each $R_k(Q_0)$ is an n-dimensional interval within Q_0.

Now, not only do the intervals $\{R_k(Q_0)\}$ approximate Q_0, but the convergence occurs at a fixed, exponential rate; that is, the difference in measure between Q_0 and $R_k(Q_0)$ decays exponentially in k. To see this, note first that each of the side-lengths of the interval $R_k(Q_0)$ does not exceed $l(Q_0)$. On the other hand, when $k \geq 2$, each of these lengths must be larger than $(1-2^{-k+1})l(Q_0)$. For otherwise,,,, at least one of the sides of Q_0 would be longer than the corresponding side of $R_k(Q_0)$ by at least $2^{-k+1}l(Q_0)$. Since $R_k(Q_0)$ is an interval, this would mean that some dyadic cube of side-length at least $2^{-k}l(Q_0)$ lay within Q_0 but not within $R_k(Q_0)$, an impossibility. Thus,

$$|R_k(Q_0)| \geq \left((1-2^{1-k})l(Q_0)\right)^n = (1-2^{1-k})^n|Q_0|,$$

so that

$$|Q_0 \setminus R_k(Q_0)| = O(2^{-k})|Q_0|, \quad \text{as } k \to \infty. \tag{1.13}$$

Proof. The first claim follows immediately from the local form of the preceding corollary. It does not seem possible to give a direct, geometric proof of the converse on the model of Proposition 4, and we thus give a different argument.

Let χ and $\bar{\chi}$ be constant multiples of the characteristic function of the unit ball and unit cube $[-1,1]^n$, normalized so that $\int \chi = \int \bar{\chi} = 1$. Suppose that the measure ν has asymptotic doubling with respect to balls. Given a small ε, use Corollary 3 to choose a number ρ just smaller than 1 so that

$$\nu * \bar{\chi}_t < (1+\varepsilon)\nu * \bar{\chi}_{\rho t} \tag{1.15}$$

for all (positive) scales t. Suppose that ρ is also so close to 1 that there is a small, positive r such that

$$\bar{\chi}_\rho < (1+\varepsilon)\bar{\chi} * \chi_r; \tag{1.16}$$

this is possible by a direct calculation. Convert (1.16) to scale t and combine it with the previous estimate (1.15). Then

$$\nu * \bar{\chi}_t < (1+\varepsilon)\nu * \bar{\chi}_{\rho t} < (1+\varepsilon)^2 \nu * (\bar{\chi} * \chi_r)_t .$$

This last term may be written at x as

$$(1+\varepsilon)^2 \nu * \bar{\chi}_t * \chi_{rt}(x) = (1+\varepsilon)^2 \int_{B_t(x)} \Big(\nu * \chi_{rt}(y)\Big) \bar{\chi}_t(x-y)\, dy,$$

for $B_t(x) = \{y \in \mathbf{R}^n : |y-x| \le t\}$. Note that this has the form of an average of an average.[10]

To this point, the argument has been scale-invariant. At small scales t, however, the first factor in the last integrand is nearly constant over the (bounded) region of integration. Indeed, since ν is assumed to have asymptotic doubling over balls, then

$$\nu * \chi_{rt}(y) < (1+\varepsilon)\nu * \chi_t(x) \tag{1.17}$$

uniformly for all y in the ball $B_t(x)$, when t is sufficiently small.[11] Since $\int \bar{\chi}_t = 1$, inserting (1.17) into the integrand leads to the conclusion that

$$\nu * \bar{\chi}_t(x) < (1+\varepsilon)^3 \int_{B_t(x)} \nu * \chi_t(x) \bar{\chi}_t(x-y)\, dy = (1+\varepsilon)^3 \nu * \chi_t(x),$$

when t is small.

The reverse estimate controlling $\nu * \chi_t$ by $\nu * \bar{\chi}_t$ follows similarly. Thus, the averages of ν over small balls approximate its averages over small cubes,

$$\frac{\nu * \chi_t}{\nu * \bar{\chi}_t} \to 1, \quad \text{as } t \to 0,$$

which is the desired result. □

Remark. The argument we have just given can be adapted to characterize asymptotic doubling in terms of convolution with certain non-compactly supported kernels, such as the Gaussian (see [21, Chap. 4]).

---

[10] The idea for this method stems from Jerison and Kenig [16]. There, truncations of the Poisson kernel (and other kernels with polynomial decay) are compared to their averages formed over small scales.

[11] In the terminology of §1.2, this follows from the geometric observation that that the unit ball $B_1(0)$ and the ball $B_r(z)$, for an arbitrary z in $B_1(0)$, are an m-regular pair, for some fixed $m = m(r)$.

Since $|Q_0| = \sum_{k=0}^{\infty} |\Omega_k(Q_0)|$, then the mean-value theorem[9] and (1.13) yield the estimate

$$\begin{aligned}\left|\frac{\nu_{Q_0}}{\nu_{\hat{Q}_0}} - 1\right| &= \left||Q_0|^{-1}\sum_{k=0}^{\infty}(1+O(\varepsilon))^{k+3}|\Omega_k(Q_0)| - 1\right| \\ &= |Q_0|^{-1}\sum_{k=0}^{\infty}|\Omega_k(Q_0)|\,\left|(1+O(\varepsilon))^{k+3} - 1\right| \\ &= O(\varepsilon)\sum_{k=0}^{\infty} O(2^{-k})(k+3)\left(\frac{3}{2}\right)^{k+2}.\end{aligned}$$

The series converges and hence the ratio $\nu_{Q_0}/\nu_{\hat{Q}_0}$ is $1+O(\varepsilon)$. This completes the proof. □

The argument of the proposition is local: It still holds if both the assumption and the conclusion refer only to cubes within a fixed cube, not to all cubes in $\mathbf{R}^n$. Consequently, to determine whether a measure is asymptotically doubling, we must only compare its average over each dyadic cube Q with that over the other $2n$ "nearby" dyadic cubes of the same size.

The definition of asymptotic doubling used above was with reference to cubes, and—as in the case of simple doubling—we might suppose that we could equivalently have used balls. This supposition is correct, but its justification is not as straightforward as in the earlier case, because all constants of comparability must now be kept arbitrarily close to 1. To see that asymptotic doubling (over cubes) in fact implies a corresponding doubling condition over balls, we can modify the last proposition in the following manner:

Corollary 5. *Under the assumptions of Proposition 4, the estimate*

$$(1.14) \qquad (1+C\varepsilon)^{-1}\nu_{B'} \le \nu_B \le (1+C\varepsilon)\nu_{B'}$$

holds for all pairs B, B' of 3-regular balls.

Proof. The argument is similar to that just given. Let $R_k(B)$ be the kth dyadic approximation of a ball B, that is, the union of all maximal dyadic cubes within B with sides no larger than $2^{-k}\operatorname{diam}(B)$. While it is no longer true that each $R_k(B)$ is an n-dimensional interval, it is still true that the approximation occurs exponentially fast:

$$|B \setminus R_k(B)| = O(2^{-k})|B|, \quad \text{as } k \to \infty.$$

Indeed, there is a dimensional constant C such that

$$(1 - C2^{-k})B \subseteq R_k(B), \quad \text{for all large } k.$$

As each ball B is contained within a cube $\hat{Q}$ comprising 4^n dyadic subcubes, each of which is comparable in size to B, then the rest of the proof follows analogously. □

Corollary 6. *Every measure that is asymptotically doubling with respect to cubes also has this property over balls, and vice versa.*

[9]The appropriate estimate is $(1+t)^{k+3} - 1 \le (k+3)(3/2)^{k+2}$, when $0 \le t \le 1/2$.

When this holds, we write $w \in B_q$ and use the notation $B_q(w)$ for the smallest constant K. Note that $B_p \subseteq B_q$ when $p \geq q$, again by Hölder's inequality.

Now, not only is A_∞, so defined, the formal limit of A_p, it is also the actual union of the various A_p classes. This is indeed but one of several standard formulations of the A_∞ condition.

Characterization of A_∞. *The following statements are equivalent:*

(a) *The weight w is in A_∞.*
(b) *The weight w is in A_p for some p larger than 1.*
(c) *The weight w is in B_q for some q larger than 1.*
(d) *There exist constants α and β, both less than 1, such that*

$$\text{whenever } E \subset Q \text{ and } |E|/|Q| \leq \alpha, \quad \text{then } w(E)/w(Q) \leq \beta.$$

Here Q is an arbitrary cube, E a measurable subset, and $w(E) = \int_E w$.

(e) *There exist constants C and θ such that, with the same notation as in the last item,*

$$w(E)/w(Q) \leq C\left(|E|/|Q|\right)^\theta.$$

See [11, Chap. 4] or [30, Chap. 5] for the proof of these assertions.[12] Yet another characterization of A_∞ appears in §2.3 below.

Note that the bounds $A_p(w)$, $B_q(w)$, and $A_\infty(w)$ are never smaller than 1, by Hölder's (or Jensen's) inequality. When any one of these is equal to 1, then all are, and the weight w must be a.e. constant. Our focus in the remainder of Section 2 is on the properties of weights w with "nearly optimal" bounds, those for which $A_p(w)$, $B_q(w)$, or $A_\infty(w)$ is close to the value 1.

2.2. **The key estimate.** The first result is fundamental in this regard. It shows that each A_∞ weight w with small bound can oscillate only mildly, for not only must $\log w$ have bounded mean oscillation, but the BMO norm of $\log w$ must also be close to the optimal value 0.

Theorem 1. *If w is an A_∞ weight, then*

$$\|\log w\|_* \leq \log\left(2A_\infty(w)\right).$$

Moreover,[13]

$$\|\log w\|_* = O\left(\sqrt{\log A_\infty(w)}\right), \quad \text{as } A_\infty(w) \to 1. \tag{2.3}$$

[12]The equivalence of the last four conditions is due to Coifman and Fefferman [7]; the equivalence of (b) and (d) was found simultaneously by Muckenhoupt [24]. The characterization (a), which we have taken as the definition of A_∞ and used to define the quantitative bound $A_\infty(w)$, is due to Reimann and Rychener [28, p. 52]; this criterion was later found independently by Hruščev [14] and García-Cuerva and Rubio de Francia [11] and is usually attributed to these latter authors. The characterization in (e) is a quantitative statement of absolute continuity, uniform at all scales, and is particularly useful in PDE problems in which the class of operators under consideration is scale-invariant (see [19]).

[13]The asymptotic estimate (2.3) was conjectured by the author in an early draft of his thesis. A proof was subsequently found by Politis [27] using the dyadic martingale characterization of A_∞ in [10]; shortly thereafter, the direct, measure-theoretic proof given here was found independently by the author. As noted in §2.5 below, the latter proof is valid in the general setting of probability measures.

2. Weights with nearly optimal bounds

We begin this section by recalling the fundamental aspects of the theory of weights, as developed by Muckenhoupt, Coifman, and Fefferman. Thereafter we focus on the situation in which the weight bounds in question approach their optimal values.

2.1. The basic theory of weights.

It is well-known that the Hardy-Littlewood maximal operator M is bounded on the Lebesgue spaces $L^p(dx)$, when $p > 1$. Here M is defined by the rule

$$Mf(x) = \sup_{Q:\, x\in Q} \frac{1}{|Q|}\int_Q |f|, \quad \text{for } f \in L^1_{\text{loc}}(dx).$$

The fundamental, "mixed-measure" question of determining for which measures $d\nu$ the operator M is bounded on $L^p(d\nu)$ was resolved by Muckenhoupt [23], who gave the following characterization (see also [15] and [7]):

Weighted Maximal Theorem. *Let M be the Hardy-Littlewood maximal operator, ν a non-negative Borel measure, and p a number larger than 1. Then M is bounded on $L^p(d\nu)$ if and only if (jointly) ν is absolutely continuous with respect to Lebesgue measure, $d\nu = w\,dx$, and the function w satisfies the inequality*

$$\left(⨍_Q w\right)\left(⨍_Q w^{-1/(p-1)}\right)^{p-1} \le K, \quad \textit{for all cubes } Q \textit{ in } \mathbf{R}^n. \tag{2.1}$$

Since the theorem picks out absolutely continuous measures, we choose in the sequel largely to focus on functions rather than measures. In general, we use the term *weight* for any non-negative, locally-integrable function that is non-zero on at least some set of positive measure. A weight w is said to be doubling or asymptotic doubling if the associated measure $d\nu = w\,dx$ is; in this case we shall write $Db(w)$ for the doubling constant $Db(\nu)$ and $w(E)$ for $\nu(E) = \int_E w\,dx$.

The criterion (2.1) is referred to as the A_p *condition*, and A_p denotes the collection of weights that satisfy it. For example, a power weight w, with $w(x) = |x|^\alpha$, is in A_p if and only if $-n < \alpha < n(p-1)$; this happens exactly when the two functions in the integrand of (2.1) are locally integrable on $\mathbf{R}^n$. We let $A_p(w)$ denote the smallest constant K for which (2.1) holds and refer to this as the A_p *bound* of w.

Hölder's inequality shows that $A_p \subseteq A_q$ when $p \le q$, and so it seems reasonable to consider the formal limit of the A_p condition as $p \to \infty$. The inner exponent $-1/(p-1)$ in (2.1) then tends to 0 from below, and thus the second factor there converges (see [13, §6.8]) to $\exp ⨍_Q \log(1/w)$. The formal limit of (2.1) is therefore the condition

$$⨍_Q w \le K \exp\left(⨍_Q \log w\right), \quad \text{for all cubes } Q \text{ in } \mathbf{R}^n, \tag{2.2}$$

which is precisely the requirement that the arithmetic and geometric means of a weight be uniformly comparable at all scales. This is known as the A_∞ *condition*, with A_∞ the corresponding class and $A_\infty(w)$ denoting the smallest constant K.

Relatedly, we say that a weight w satisfies a *reverse Hölder inequality of index q*, for some number q larger than 1, if

$$\left(⨍_Q w^q\right)^{1/q} \le K ⨍_Q w, \quad \text{for all cubes } Q \text{ in } \mathbf{R}^n.$$

Jensen's inequality implies that $a' \leq a$ and $b' \leq b$, so that condition (2.4) of the lemma holds for the pairs (a, b) and (a', b') of (2.8) and (2.9). Applied to the latter pair, the conclusion of the lemma is then

$$\frac{a'}{b'} = \exp\left(\fint_E f - \fint_F f \right) \leq 1 + c\sqrt{\varepsilon};$$

hence

$$\fint_E f - \fint_F f \leq c\sqrt{\varepsilon}.$$

But since $f \geq 0$ on E and $f \leq 0$ on F, with $|E| = |F| = |Q|/2$, then

$$2\fint_Q |f| = \fint_E f - \fint_F f \leq c\sqrt{\varepsilon},$$

as claimed in (2.7). To obtain the asymptotic part of the theorem, simply set $f = \log w$, for w an A_∞ weight with small bound.

The same technique also proves the general estimate valid for all A_∞ weights. Indeed, replacing ε by $A_\infty(w) - 1$ in (2.6) leads to the inequality

$$\|\log w\|_* \leq \frac{1}{2} \log\left(2A_\infty(w)^2 - 1 + 2A_\infty(w)\sqrt{A_\infty(w)^2 - 1} \right).$$

The right-hand side is smaller than $\log(2A_\infty(w))$, as claimed. This completes the proof of the theorem. □

Note that the square root in the theorem is the sharp power. For if w is a function that assumes each of the two values $1 + \varepsilon$ and $1 - \varepsilon$ on exactly half of the cube Q, then

$$\log \fint_Q w - \fint_Q \log w = \log \frac{1}{\sqrt{1 - \varepsilon^2}},$$

while

$$\fint_Q |\log w| = \frac{1}{2} \log \frac{1 + \varepsilon}{1 - \varepsilon}.$$

As $\varepsilon \to 0$, the first expression is $O(\varepsilon^2)$, while the second is $O(\varepsilon)$. A moment's reflection shows that this calculation simply recapitulates the numerical estimate of the lemma.

2.3. **Extensions.** The method of the theorem yields a number of other sharp estimates. The next result, for instance, is a quantitative version of the well-known fact that every A_∞ weight is doubling.

Corollary 3. *If w is an A_∞ weight, then*

$$Db(w) \leq 2^n \left(A_\infty(w) \right)^{2n}.$$

Moreover,

$$\log Db(w) = O\left(\sqrt{\log A_\infty(w)} \right), \quad \text{as } A_\infty(w) \to 1. \tag{2.11}$$

Proof. The argument is as in the theorem. Here, however, we can take E and F to be any two complementary halves of Q (by measure), irrespective of a median

The key to the proof of the theoremis the simple observation that if the ratio of the arithmetic and geometric means of two numbers is close to 1, then so must be the ratio of the two numbers.

Lemma 2. *Suppose a and b are positive numbers and $0 \le \varepsilon \le 1$. If*

$$\frac{a+b}{2} \le (1+\varepsilon)\sqrt{ab}, \tag{2.4}$$

then

$$1 - c\sqrt{\varepsilon} \le \frac{a}{b} \le 1 + c\sqrt{\varepsilon}, \tag{2.5}$$

for some absolute constant c.

Proof. The proof of the lemma is straightforward: If (2.4) holds for some positive a, b, and ε, then the quadratic formula yields

$$\left|\frac{a}{b} - (1 + 4\varepsilon + 2\varepsilon^2)\right| \le \sqrt{(1 + 4\varepsilon + 2\varepsilon^2)^2 - 1}. \tag{2.6}$$

The radical is $O(\sqrt{\varepsilon})$ for small ε. □

An alternative argument will be useful in later situations in which the quadratic formula does not apply. Consider the function F implicit in (2.4), namely

$$F(t) = (1+t)/(2\sqrt{t});$$

this is the ratio of the arithmetic and geometric means of the numbers 1 and t. Note that F is increasing over $1 \le t < \infty$ and is symmetric about 1, in the sense that $F(t) = F(1/t)$. A calculation shows that $F(1+\sqrt{\varepsilon}) = 1 + O(\varepsilon)$ for small ε; hence, $F(t) \le 1+\varepsilon$ implies that $|t-1| \le c\sqrt{\varepsilon}$. With $t = a/b$, this is exactly the content of the lemma.

Proof of the theorem. To prove the asymptotic estimate (2.3) we shall show the following implication:

$$\left(⨍_Q \exp f\right) \Big/ \left(\exp ⨍_Q f\right) = 1 + \varepsilon \Longrightarrow ⨍_Q |f - m_Q(f)| \le c\sqrt{\varepsilon}. \tag{2.7}$$

Here $m_Q(f)$ is a *median value* of f over the cube Q, that is, any real number λ such that the Lebesgue measure of each of the two sets $\{x \in Q : f(x) > \lambda\}$ and $\{x \in Q : f(x) < \lambda\}$ does not exceed half of the measure of the cube Q. Our technique is to use median values to reduce the functional averages of the theorem to the numerical averages of the lemma.

To prove (2.7), assume that $m_Q(f) = 0$, adding a scalar to f, if necessary. Divide Q into two halves, in each of which the values of f are on only one side of the median; that is, choose two subsets E and F of Q, each with measure $|Q|/2$, such that $E \subseteq \{x \in Q : f(x) \ge 0\}$ and $F \subseteq \{x \in Q : f(x) \le 0\}$. Let

$$a = ⨍_E \exp f, \qquad b = ⨍_F \exp f, \tag{2.8}$$

$$a' = \exp ⨍_E f, \qquad b' = \exp ⨍_F f. \tag{2.9}$$

Then

$$\left(⨍_Q \exp f\right) \Big/ \left(\exp ⨍_Q f\right) = \frac{a+b}{2\sqrt{a'b'}} \le 1 + \varepsilon. \tag{2.10}$$

value $m_Q(w)$. If $f = \log w$ and $A_\infty(w) = 1+\varepsilon$, then for the pair (a,b) we obtain from (2.10) and Lemma 2 that

$$\frac{a}{b} = \left(\fint_E w\right)\Big/\left(\fint_F w\right) = 1 + O(\sqrt{\varepsilon}); \tag{2.12}$$

for a generic A_∞ weight, we likewise obtain

$$(4A_\infty(w)^2-1)^{-1} \le \left(\fint_E w\right)\Big/\left(\fint_F w\right) \le 4A_\infty(w)^2 - 1. \tag{2.13}$$

If E is itself a cube within Q of half the latter's measure, then the last two estimates imply that $w_Q/w_E = 1 + O(\sqrt{\varepsilon})$ or $w_Q/w_E \le 2A_\infty(w)^2$, respectively. Iterating n times (to compare the mean of w over the cube Q with that over any cube within Q of half its side-length) completes the proof. □

Remark. Note that the last argument also gives a new proof of how (d) follows from (a) among the characterizations of A_∞ in §2.1. For the argument in the preceding paragraph shows exactly that if $(\fint_Q w)/(\exp\fint_Q \log w) = A$, with $E \subset Q$ and $|E|/|Q| \le 1/2$, then $w(E)/w(Q) \le (4A^2-1)/(4A^2)$. The characterization (d) of A_∞ can thus be simplified to include only subsets of a cube of half the cube's measure (see also §2.5 below).

Consideration of median values actually leads to another criterion for A_∞.[14]

Theorem 4. *A weight w is in A_∞ if and only if its mean and median values are uniformly comparable over all cubes. In addition,*

$$\sup_Q \left|\log \frac{m_Q(w)}{w_Q}\right| = O\left(\sqrt{\log A_\infty(w)}\right), \quad \text{as } A_\infty(w) \to 1. \tag{2.14}$$

Proof. We have effectively already proven (2.14) in demonstrating the last result. For suppose that the ratio of the arithmetic and geometric means of w over Q is $1+\varepsilon$. Split Q again into two halves E and F that straddle a median value $m_Q(w)$. Then

$$w_Q = \frac{w_E + w_F}{2} \le \frac{1 + c\sqrt{\varepsilon} + 1}{2} w_F \le (1 + c\sqrt{\varepsilon}) m_Q(w);$$

the first inequality is from (2.12) and the second holds because $w \le m_Q(w)$ pointwise on F. Similarly,

$$w_Q \ge \frac{1 - c\sqrt{\varepsilon} + 1}{2} w_E \ge (1 - c\sqrt{\varepsilon}) m_Q(w).$$

Together these two estimates give the asymptotic statement (2.14).

For a generic A_∞ weight w the same argument, with (2.13) in place of (2.12), yields the estimate $w_Q/m_Q(w) \le 2A_\infty(w)^2$. Since alone the non-negativity of w shows that the mean always dominates the median,

$$w \ge 0 \Longrightarrow m_Q(w) \le 2w_Q,$$

then these two kinds of averages are always comparable for an A_∞ weight.

Conversely, suppose that there is a constant C for which $w_Q \le C m_Q(w)$ for all cubes Q. Then

$$|\{x \in Q : w(x) < C^{-1} w_Q\}| \le |\{x \in Q : w(x) < m_Q(w)\}| \le |Q|/2.$$

[14] The characterization is not new; it appears in [32], which cites the earlier announcement [33]. This previous work was unknown to the author when he found the (new) asymptotic estimate (2.14).

If E is a measurable subset of a cube Q, then

$$\begin{aligned} |E| &= |\{x \in E : w(x) < C^{-1}w_Q\}| + |\{x \in E : w(x) \geq C^{-1}w_Q\}| \\ &\leq |Q|/2 + Cw(E)/w_Q \\ &\leq \left(\frac{1}{2} + C\frac{w(E)}{w(Q)}\right)|Q|. \end{aligned}$$

So whenever $w(E)/w(Q) < 1/(4C)$, then $|E|/|Q| < 3/4$; thus w is in A_∞.[15] □

Since many of the standard characterizations of A_∞ depend upon comparisons of two types of integral averages, it is perhaps not surprising that we can define A_∞ by a comparison of median and mean values, as in the last theorem. One virtue of the criterion for A_∞ just given, however, is that medians, unlike means, are well-behaved under composition. In particular, if Φ is a monotone function, then

$$m_Q(\Phi \circ w) = \Phi(m_Q(w)). \tag{2.15}$$

This observation permits a simple proof of the next result.

Corollary 5. *Suppose that $\Phi\colon [0,\infty) \to [0,\infty)$ is a convex, continuous, strictly increasing function that vanishes at 0. If $\Phi \circ w \in A_\infty$, then $w \in A_\infty$.*

Proof. By the preceding theorem, we know that there is a constant C such that

$$(\Phi \circ w)_Q \leq C m_Q(\Phi \circ w) \tag{2.16}$$

uniformly over all cubes Q. So

$$\Phi(w_Q) \leq (\Phi \circ w)_Q \leq C m_Q(\Phi \circ w) = C\Phi(m_Q(w)),$$

by Jensen's inequality, the assumption (2.16), and the observation (2.15), respectively. Now, since Φ^{-1} is concave, $\Phi^{-1}(0) = 0$, and (without loss of generality) $C \geq 1$, then $\Phi^{-1}(Ct) \leq C\Phi^{-1}(t)$ for all $t \geq 0$. Hence

$$w_Q = \Phi^{-1} \circ \Phi(w_Q) \leq \Phi^{-1}(C\Phi(m_Q(w)) \leq C\Phi^{-1}(\Phi(m_Q(w)) = Cm_Q(w).$$

As the converse inequality $m_Q(w) \leq 2w_Q$ holds automatically, another application of Theorem 4 shows that $w \in A_\infty$. □

2.4. **A_p and BMO.** By Jensen's inequality, $A_\infty(w) \leq A_p(w)$. The bounds on the mean oscillation $\|\log w\|_*$ and the doubling constant $Db(w)$ in Theorem 1 and Corollary 3 are thus immediately valid for A_p weights.

Corollary 6. *If w is an A_p weight and $1 < p < \infty$, then*

$$\|\log w\|_* \leq \log(2A_p(w)) \quad \textit{and} \quad Db(w) \leq 2^n \left(A_p(w)\right)^{2n}.$$

Moreover, as $A_p(w) \to 1$, then

$$\|\log w\|_* = O\left(\sqrt{\log A_p(w)}\right) \quad \textit{and} \quad \log Db(w) = O\left(\sqrt{\log A_p(w)}\right).$$

That the latter asymptotic estimates are sharp follows, once again, from considering a step function with the values $1+\varepsilon$ and $1-\varepsilon$. For $p = 2$, Sarason [29] obtained the weaker BMO estimate $\|\log w\|_* = O(\sqrt[3]{\log A_p(w)})$ along with the sharp estimate for the doubling constant.

[15] This is because w satisfies condition (d) in §2.1, with $\alpha = 1/4$ and $\beta = (4C-1)/(4C)$. The proof of the converse above is analogous to the (A'_∞) condition in [7].