Moduli of Hyper–Kählerian Manifolds II

(Torelli Problem)

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One has the following 2-cocycle on $\mathfrak{a}_\infty^F$ [KP, DJKM]:

$$\begin{cases}\psi(E_{ij}, E_{ji}) = -\psi(E_{ji}, E_{ij}) = 1 & \text{if} \quad i \leq 0, \quad j > 0 \\ \psi(E_{ij}, E_{rs}) = 0 & \text{otherwise},\end{cases}$$

whose cohomology class generates $H^2_{\mathrm{cont}}(\mathfrak{a}_\infty^F, \mathbb{C}) \cong \mathbb{C}$.

Another way of expressing this cocycle is the following. Given $(a_{ij}) \in \mathfrak{a}_\infty^F$, write $f(z,w) = \sum a_{ij} z^{i-1} w^{-j}$ and let

$$f_{-+} = \sum_{\substack{i \leq 0 \\ j > 0}} a_{ij} z^{i-1} w^{-j}, \qquad f_{+-} = \sum_{\substack{i > 0 \\ j \leq 0}} a_{ij} z^{i-1} w^{-j}.$$

Both f_{+-} and f_{-+} are polynomials, and given

$$g = \sum b_{ij} z^{i-1} w^{-j}, \qquad (b_{ij}) \in \mathfrak{a}_\infty^F,$$

we have

$$\psi((a_{ij}),(b_{ij})) = \operatorname*{Res}_{\substack{z=0 \\ w=0}} (f_{-+} g_{+-} - g_{-+} f_{+-}).$$

Pulling back the cocycle ψ via ϕ_n we get a cocycle $\phi_n^*(\psi)$ on $\mathscr{D}^F$ which works out to be

$$\phi_n^*(\psi)(d_j, d_k) = -\delta_{j,-k} \frac{j^3 - j}{6}(6n^2 - 6n + 1),$$

$$\phi_n^*(\psi)(z^j, z^k) = \delta_{j,-k} \; j,$$

$$\phi_n^*(\psi)(z^j, d_k) = -\delta_{j,-k}(n - \tfrac{1}{2}) j(j-1).$$

Restricting to $\mathbf{d}^F$ we get cocycles $\varrho_n^*(\psi)$ which satisfy the relation

$$\varrho_n^*(\psi) = (6n^2 - 6n + 1)\varrho_0^*(\psi). \tag{0.1}$$

Recall that the cohomology class of $\varrho_0^*(\psi)$ generates $H^2(\mathbf{d}^F, \mathbb{C}) \cong \mathbb{C}$; a less well-known fact is that $H^2(\mathscr{D}^F, \mathbb{C}) \cong \mathbb{C}^3$.

On the other hand, let $\pi: \mathscr{C} \to S$ be a family of genus g compact Riemann surfaces and let $\omega_{\mathscr{C}/S}$ be the relative dualizing sheaf of π. Denote by λ_n the determinant line bundle of $\omega^n_{\mathscr{C}/S}$ on S. Then, as observed by Mumford [Mu], the Grothendieck-Riemann-Roch theorem for the family π gives the following relation between Chern classes:

$$c_1(\lambda_n) = (6n^2 - 6n + 1)c_1(\lambda_1). \tag{0.2}$$

One of the main objectives of the present paper is to explain the coincidence of (0.1) and (0.2). In order to achieve this it is therefore of central importance to us to find a relationship between extensions of our Lie algebras and line bundles on moduli spaces.

Let us briefly introduce the moduli spaces involved in our construction. First of all the moduli space $\mathscr{M}_g$ of smooth curves of genus g, then the moduli space $\mathscr{M}_g''$ of triples (C, p, v) when C is a genus g Riemann surface, p a point on C, and v a non-zero tangent vector to C at p. We also consider the moduli space $\mathscr{F}_h''$ of quadruples (C, p, v, L), where L is a degree h line bundle on C and $(C, p, v) \in \mathscr{M}_g''$.

Furthermore, we construct an infinite dimensional complex manifold $\hat{\mathscr{M}}_g$ which is a moduli space of triples (C, p, z), where z is a local parameter at p. Finally, we construct another infinite dimensional complex manifold $\hat{\mathscr{F}}_h$ parametrizing quintuples $(C, p, z, L, [\phi])$, where C, p, z, L are as above, ϕ is a local trivialization of L at p and $[\phi]$ is the class of ϕ modulo non-zero multiplicative constants. Of course, we have natural projections

$$\mathscr{M}_g'' \to \mathscr{M}_g, \qquad \hat{\mathscr{M}}_g \to \mathscr{M}_g'', \qquad \hat{\mathscr{F}}_h \to \mathscr{F}_h''.$$

The first projection induces an isomorphism in second cohomology [actually, Harer, Ann. Math. **121**, 215–249 (1985), has proven that $\mathscr{M}_g''$ has the same cohomology as $\mathscr{M}_g$ for g large], the remaining two are homotopy equivalences.

By using the Kodaira-Spencer deformation theory on the infinite dimensional manifolds $\hat{\mathscr{M}}_g$ and $\hat{\mathscr{F}}_{g-1}$ we get natural Lie algebra homomorphisms

$$\mathbf{d} \to \mathrm{Vect}(\hat{\mathscr{M}}_g),$$

$$\mathscr{D} \to \mathrm{Vect}(\hat{\mathscr{F}}_{g-1}),$$

where $\mathbf{d}$ and $\mathscr{D}$ are suitable analytic analogues of $\mathbf{d}^F$ and $\mathscr{D}^F$ in which $\mathbf{d}^F$ and $\mathscr{D}^F$ are dense. The above homomorphisms have the property that for every $x \in \hat{\mathscr{M}}_g$ (respectively $\hat{\mathscr{F}}_{g-1}$) the evaluation map

$$p_{\mathbf{d},x}: \mathbf{d} \to T_x(\hat{\mathscr{M}}_g) \qquad (p_{\mathscr{D},x}: \mathscr{D} \to T_x(\hat{\mathscr{F}}_{g-1}))$$

is surjective.

From this one gets that the tangent bundle $T(\hat{\mathscr{M}}_g)$ (respectively $T(\hat{\mathscr{F}}_{g-1})$) is canonically a quotient of the trivial bundle $\hat{\mathscr{M}}_g \times \mathbf{d}$ (respectively $\hat{\mathscr{F}}_{g-1} \times \mathscr{D}$). Similar results have been obtained in [BMS].

This allows us to define a canonical homomorphism

$$\mu: H^2(\mathscr{D}) \to H^1(\Omega^1_{\hat{\mathscr{F}}_{g-1}}) = \mathrm{Ext}^1(\mathscr{T}_{\hat{\mathscr{F}}_{g-1}}, \mathscr{O}_{\hat{\mathscr{F}}_{g-1}})$$

(the case of $\hat{\mathscr{M}}_g$ is analogous). The definition of μ is as follows. Given a central extension

$$0 \longrightarrow \mathbb{C} \longrightarrow \tilde{\mathscr{D}} \overset{\varrho}{\longrightarrow} \mathscr{D} \longrightarrow 0$$

we can lift canonically the inclusion

$$\mathscr{D}_x =: \operatorname{Ker} p_{\mathscr{D},x} \hookrightarrow \mathscr{D}$$

to an inclusion

$$\mathscr{D}_x \hookrightarrow \tilde{\mathscr{D}}.$$

For this we use the following two facts:

i) $\qquad \mathscr{D}_x = [\mathscr{D}_x, \mathscr{D}_x]$,

ii) $\qquad \varrho|_{\mathscr{D}_x}$ is the trivial extension.

Using the inclusion $\mathscr{D}_x \hookrightarrow \tilde{\mathscr{D}}$ we can construct an extension of the tangent bundle $T(\hat{\mathscr{F}}_{g-1})$ whose fiber at x is $\tilde{\mathscr{D}}/\mathscr{D}_x$. Thus dualizing and passing to cohomology classes we get μ.

MODULI OF HYPER--KÄHLERIAN MANIFOLDS II.
(TORELLI PROBLEM)
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INTRODUCTION.

This is the second part of the article

"MODULI OF HYPER--KÄHLERIAN MANIFOLDS I".

In this part we will prove Torelli type theorem for ALGEBRAIC HYPER-KÄHLERIAN MANIFOLDS. We will use the same notations as in the first part. Namely we partially comactify the moduli space of marked polarized Hyper-Kähleriau manifolds. Then we prove that the period map is a proper surjective and étale. Since the period domain is simply connected we deduce that the period map has degree one on each component of the moduli space.

The content of this articles is exactly that of #4 announced in

"MODULI OF HYPER--KÄHLERIAN MANIFOLDS I".

The partial compactification that we propose here has a close resemblence with the article of D. Morrison. See [8].

#4. TORELLI PROBLEM FOR ALGEBRAIC HYPER-KÄHLERIAN MANIFOLDS.

THEOREM 3. Let

$$\pi_L : \mathfrak{X}_L \to \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

be one of the components of the universal family of marked Hyper-Kählerian manifolds with fixed class of polarization

$$L \in H^2(X,\mathbf{Z}) \cap H^{1,1}(X_t,\mathbf{R})$$

for every $t \in \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$

then there exists a universal partial compactification $\overline{\pi}_L : \overline{\mathfrak{X}}_L \to \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ of

$$\pi_L : \mathfrak{X}_L \to \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \text{ (See \#3.)}$$

such that

A)
$$\begin{array}{ccc} \mathfrak{X}_L & \subset & \overline{\mathfrak{X}}_L \\ \downarrow & & \downarrow \\ \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})} & \subset & \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \end{array}$$

and for all

$$t \in \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \setminus \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

$\overline{\pi}_L^{-1}(t) = X_t$ is birationally isomorphic to a nonsingular Hyper-Kählerian manifold, even more for each $t \in \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$, X_t is embedded in $\mathbf{P}^N$, where N is fixed.

B) The period map $p : \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \to \Omega(L)$ can be prolonged to a biholomorphic isomorphism:

$$\overline{p} : \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \to \Omega(L).$$

PROOF: We will prove A) & B) simultaniously. The proof of THEOREM 3 is based on THEOREM 1.

LEMMA 4.1. There exists a family $\tilde{\mathfrak{X}}_L \to \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ such that

a)
$$\begin{array}{ccc} \mathfrak{X}_L & \subset & \tilde{\mathfrak{X}}_L \\ \downarrow & & \downarrow \\ \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})} & \subset & \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \end{array}$$

b) The period map $p:\mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \to \Omega(L)$ can be prolonged to a surjective holomorphic map

$$\overline{p}:\overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \to \Omega(L)$$

PROOF:

Construction of the family $\tilde{\mathfrak{X}} \to \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$.

Let $H_L \overset{def}{=} \{u \in \mathbf{P}(H^2(X,\mathbf{Z})\otimes\mathbf{C}) \mid \langle u,L\rangle = 0\}$ and let $\mathfrak{S} \to \mathfrak{K}$ be the Kuranishi family of some Hyper-Kählerian manifold X. From local Torelli Theorem it follows that we may suppose that

$$\mathfrak{K} \subset \Omega \subset \mathbf{P}(H^2(X,\mathbf{Z})\otimes\mathbf{C})$$

Let

$$\overline{\mathfrak{K}}_L = H_L \cap \mathfrak{K}$$

The restriction of $\mathfrak{S} \to \mathfrak{K}$ to $\overline{\mathfrak{K}}_L$ we will denote by

$$(*) \qquad \tilde{\mathfrak{S}}_L \to \overline{\mathfrak{K}}_L$$

So $(*)$ will be the local universal deformations of all Hyper-Kählerian manifolds for which L is a class of type (1,1) on each fibre of $(*)$. This fact follows since

$$\langle \omega_u(2,0), L\rangle = 0 \Leftrightarrow L \text{ is of type } (1,1).$$

Notice that in #3 we demanded that L to be the class of an imaginary part of a Hodge metric, which is much stronger condition than L to be just of type (1,1).

From Lemma 3.1. it follows that we can glue all families

$$\{\tilde{\mathfrak{S}}_L \to \overline{\mathfrak{K}}_L\}$$

by identifying isomorphic marked Hyper-Kählerian manifolds for which L is a class of type (1,1), so we get familiy

$$\begin{array}{ccc} \mathfrak{X}_L & \subset & \tilde{\mathfrak{X}}_L \\ \downarrow & & \downarrow \\ \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})} & \subset & \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \end{array}$$

and all its fibres are non-singular manifolds. Local Torelli Theorem tells us that

$$\overline{p}:\overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \to \Omega(L)$$

is an ètale map.

Proof of the surjectivity of $\overline{p}$:

In the proof of Teorem 1 we used the fact that

$$\Omega(L)\backslash p(\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})})=V'$$

where V' is a countable union of analytic subsets.(See 2.5.5.4.).

Let D be a disk in $\Omega(L)$ such that $D\bigcap V'=\{o\}=\{\text{one point}\}$. We suppose that

$$D^*=D\backslash\{o\}\subset\mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

From the arguments in #2.6. it follows that over D^* we have a family $\mathfrak{X}^*\to D^*$ that fulfills the conditions of THEOREM 1. and so is contained in a family $\mathfrak{X}\to D$, where all fibres are non-singular manifolds. Clearly for $\pi^{-1}(o)=X_o$ L will be a class of type (1,1). This and the way we constructed

$$\tilde{\mathfrak{X}}\to\overline{\mathfrak{M}}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

proves that $\overline{p}$ is surjective. Q.E.D.

REMARK. Mukai's elementary transformations

$$\tilde{\mathfrak{X}}_L\to\overline{\mathfrak{M}}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

defines on $\overline{\mathfrak{M}}_{(L;\gamma_1,\dots,\gamma_{b_2})}$ a non-Houssdorf topology, since over the disk D we can have two families $\mathfrak{X}\to D$ & $\mathfrak{Y}\to D$ such that they are not isomorphic but over $D^*=D\backslash\{o\}$ are isomorphic. (See [7].)

We will construct a new family

$$\overline{\mathfrak{X}}_L\to\overline{\mathfrak{M}}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

such that it induces a Haussdorf topology on $\overline{\mathfrak{M}}_{(L;\gamma_1,\dots,\gamma_{b_2})}$.

LEMMA 4.2. Let $\tilde{\mathfrak{X}}_L\to\overline{U}_L$ be the family constructed in LEMMA 4.1., where $\overline{U}_L\subset\mathfrak{X}\bigcap H_L$ is a polycylinder, then there exists a holomorphic map

$$\begin{array}{ccc}\tilde{\phi}_L:\tilde{\mathfrak{X}}_L & \to & \mathbf{P}^N\times\overline{U}_L\\ \downarrow & & \downarrow\\ \overline{U}_L & \equiv & \overline{U}_L\end{array}$$

such that:

A) Let $U_L=\{t\in\overline{U}_L|\tilde{\pi}^{-1}(t)=X_t$ be such that L corresponds to a Chern class of a very ample divisor on $X_t\}$ and let $\mathfrak{X}_L\rightarrow U_L$ be the restriction of the family $\tilde{\mathfrak{X}}_L\rightarrow\overline{U}_L$ to U_L, then if ϕ_L is the restrition of $\tilde{\phi}_L$ to the family $\mathfrak{X}_L\rightarrow U_L$. ϕ gives an embedding:

$$\begin{array}{ccc} \phi_L: \mathfrak{X}_L & \subset & \mathbf{P}^N x U_L \\ \downarrow & & \downarrow \\ U_L & \equiv & U_L \end{array}$$

B) for $\forall t\in\overline{U}_L\setminus U_L\tilde{\phi}_L|_{X_t}=\phi_t$ is a holomorphic and birational map.

Proof: Let me remind You that in #2.5. we have proved that

$$\Omega(L)\setminus p(\mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})})=V'$$

where V' is a countable union of analytic subsets. So we may suppose that $\overline{U}_L\setminus U_L=\mathcal{A}$ is analytic subset in $\overline{U}_L$.

The following Proposition holds:

Proposition 4.2.1. There exists an embedding:

$$\begin{array}{ccc} \phi_L:\mathfrak{X}_L & \subset & \mathbf{P}^N x U_L \\ \downarrow & & \downarrow \\ U_L & \equiv & U_L \end{array}$$

Proof of 4.2.1.:

Let $\{U_i\}$ be a covering of U_L by polycilinders. Since $U_i\subset\overline{\mathfrak{X}}_L=\mathfrak{X}\bigcap H_L$, where $\mathfrak{X}$ is the Kuranishi space, then it is a well known fact that

$$U_i\subset \mathrm{Hilb}_{X/\mathbf{P}^N} \text{ (See 2.5.)}$$

i.e. U_i is a polycilinder of maximal dimension transversal to the orbits of the action of

$$\mathbf{G}/G_O \text{ on } \mathrm{Hilb}_{X/\mathbf{P}^N}. \text{ (See [6])}$$

Since the group of the automorphisms of an algebraic Hyper-Kählerian manifold X that preserve the polarization class L is finite, it follows that all the orbits have one and the same dimension. All these are standart facts. (See [6].) So this local slice exists. From this fact and the fact that we know that we have a family over $\mathrm{Hilb}_{X/\mathbf{P}^N}$, i.e.

$$\begin{array}{ccc} \mathfrak{Y} & \subset & \mathrm{Hilb}_{X/\mathbf{P}^N} x \mathbf{P}^N \\ \downarrow & & \downarrow \\ \mathrm{Hilb}_{X/\mathbf{P}^N} & \equiv & \mathrm{Hilb}_{X/\mathbf{P}^N}. \text{ (See [SGA].)} \end{array}$$

We may suppose that $U_i \subset \mathrm{Hilb}_{\mathbf{X}/\mathbf{P}^N}$, then it follows that we have:

$$\begin{array}{ccc} \mathfrak{X}_i & \subset & U_i \times \mathbf{P}^N \\ \downarrow & & \downarrow \\ U_i & \equiv & U_i \end{array}$$

From here and the fact that

$\{U_i\}$ is a covering of U_L

we obtain that we have the following embedding

$$\begin{array}{ccc} \phi_L : \mathfrak{X}_L & \subset & U_L \times \mathbf{P}^N \\ \downarrow & & \downarrow \\ U_L & \equiv & U_L \end{array}$$

So 4.2.1. is proved.

Q.E.D.

Cor. 4.2.1.a. Over $\mathfrak{X}_L$ there exists a relatively very ample line bundle $\mathcal{L}$ such that

$$c_1(\mathcal{L}|_{X_t}) = L$$

and

$$\mathcal{L} = \phi^*(\mathcal{O}_{U_L} \otimes \mathcal{O}(1))$$

Cor. 4.2.1.b.

We may suppose that $U_L \subset \mathrm{Hilb}_{\mathbf{X}/\mathbf{P}^N}$, may be after shrinking the polycylinder $\overline{U}_L$.

Proof of Cor. 4.2.1.a.: This is a standart fact from Algebraic Geometry. (See [6].)

Q.E.D.

Proof of Cor. 4.2.1.b.:

From the universal properties of the family

$$\mathfrak{Y} \to \mathrm{Hilb}_{\mathbf{X}/\mathbf{P}^N}$$

it follows from 4.2.1.a. that there exists a map $i: U_L \to \mathrm{Hilb}_{\mathbf{X}/\mathbf{P}^N}$.

From the Proof of 4.2.1. it follows that i is locally an embedding. From the construction of $\overline{U}_L$, i.e. from the fact that $\overline{U}_L \subset \mathfrak{X}$, where $\mathfrak{X}$ is the Kuranishi space, we get that all the points of $\mathfrak{X}$ corresponds to non-isomorphic Hyper-Kählerian manifolds, so from here and the definition of $\mathrm{Hilb}_{\mathbf{X}/\mathbf{P}^N}$ 4.2.1.b. follows directly.

Q.E.D.

Proposition 4.2.2.

The relative ample line bundle $\mathcal{L}$ over $\mathfrak{X}_L \to U_L$ can be prolonged to a line bundle $\tilde{\mathcal{L}}$ over the family $\tilde{\mathfrak{X}}_L \to \overline{U}_L$ such that $c_1(\tilde{\mathcal{L}}|_{X_t})=L$ for every $t\in\overline{U}_L$.

Proof:: From 4.2.1. we know that we have the following embedding:

$$\begin{array}{ccc} \mathfrak{X}_L \subset & U_L \times \mathbf{P}^N \\ \downarrow & \downarrow \\ U_L \subset & U_L \end{array}$$

and that this embedding is given by the relative ample sheaf $\mathcal{L}$ on $\mathfrak{X}_L$.

Let $\sigma(t)\in\Gamma(\mathfrak{X}_L,\mathcal{L})$ & $t\in U_L$ and H_t be the zeroes of $\sigma(t)$ on each $X_t\subset\mathbf{P}^N$ i.e. H_t is a divisor on X_t.

We will need the following Sublemma 4.2.2.a.

Sublemma 4.2.2.a. Let $\{t_n\}$ be a sequence of points such that

a) $t_n\in U_L$

b) $\lim_{n\to o} t_n=o$, where $o\in\overline{U}_L\setminus U_L$. Then $\lim_{t_n\to o} H_t=H_o$ exists, where H_o is a divisor on X_o and

$$[H_o]=L^* \text{ in } H_{4n-2}(X,\mathbf{Z})$$

where L^* is the Poincare dual class of L.

Proof of 4.2.2.a.:

We will use the following THEOREM of Bishop:

THEOREM (Bishop). Let X be a complex manifold and let $\{Y_n\}$ be a family of complex analytic subsets in X, let on X there exists a hermitian metric $\{h_{\alpha,\overline{\beta}}\}$ such that with respect to this metric for each n

$$\mathrm{Vol}(Y_n)<C(\text{constant}),$$

then we can find a subseqence $\{t_{n_k}\}$ such that $\lim_{n_k\to\infty} Y_{n_k}=Y_o$ exists as a complex analytic subset in X.

Proof: See [2]. Q.E.D.

First we will construct the so called Harvey-Lawson metric.

Definition. Let X be a compact complex manifolds and let $(h_{\alpha,\overline{\beta}})$ be a Hermitian metric on X.

Then $(h_{\alpha,\overline{\beta}})$ we will call a Harvey-Lawson metric if there exists a form of type (1,0) $\alpha^{1,O}$ such that

$$d(\partial\alpha^{1,O}+\omega^{1,1}+\overline{\partial\alpha^{1,1}})=O, \text{ where } \mathrm{Im}(h_{\alpha,\overline{\beta}})=\omega^{1,1}$$

In the Apendix we will prove the following THEOREM:

THEOREM.. Let X be a compact complex manifold such that:

A) $\dim_{\mathbf{C}} H^2(X,\mathcal{O}_X)=1$

B) On X there exists a closed non-degenerate holomorphic two form

Then on X exists a Harvey-Lawson metric.

Also in the Apendix we prove that the form

$$\partial\alpha^{1,O}+\omega^{1,1}+\overline{\partial\alpha^{1,1}}=\omega$$

defines a non zero class of cohomology $[\omega]$ in $H^2(X,\mathbf{R})$.

Using these two results we can conclude that on the family

$$\tilde{\mathfrak{X}}_L \to \tilde{U}_L$$

there exists a relative Harvey-Lawson metric, i.e.on each fibre there exists a closed real form ω_t such that $\omega_t^{1,1}$ is strictly positive at each point of every fibre X_t, where $\omega_t^{1,1}$ is the (1,1) component of ω. ω_t depends on C^∞ manner on t. Now using this relative Harvey-Lawson metric and the standart metric on the polycylinder we will get a metric which will fulfill the conditions of Bishop's Theorem, indeed it is easy to see that

$$\mathrm{Vol}(H_t)=<\wedge^{2n-1}[\omega_t^{1,1}],c_1(\mathcal{L}|_{X_t})>=\int_X \wedge^{2n-1}[\omega_t^{1,1}]\wedge c_1(\mathcal{L}|_{X_t})\leq<\wedge^{2n-1}[\omega_t],L>$$

so $\mathrm{Vol}(H_t)$ is a bounded function on $\overline{U}_L$. So Sublemma 4.2.2.a.is proved. Q.E.D.

From Sublemma 4.2.2.a. Proposition 4.2.2. follows immediately. Since each divisor on a complex compact manifold defines a line bundle. Q.E.D.

Proposition 4.2.3. The line bundle $\tilde{\mathcal{L}}$ gives a holomorphic map g:

$$\begin{array}{ccc} \tilde{\mathfrak{X}}_L & \to & \mathbf{P}^N \times \overline{U}_L \\ \downarrow & & \downarrow \\ \overline{U}_L & \equiv & \overline{U}_L \end{array}$$

Proof: If we prove that for any point $o\in\tilde{\mathfrak{X}}_L$ there exisits a section

$$\sigma\in\Gamma(\tilde{\mathfrak{X}}_L,\tilde{\mathcal{L}})$$

such that

$$\sigma(o)\neq O$$

then Proposition 4.2.3. will be proved.

If $o\in\mathfrak{X}_L$, then Proposition 4.2.3. was proved in 4.2.1., so we may suppose that

$$o\in X_t, \text{ where } t\in\overline{U}_L\backslash U_L.$$

Sublemma 4.2.3.1. Let o be any point on $\tilde{\mathfrak{S}}_L$ such that $o \in X_t$, where $t \in \bar{U}_L \setminus U_L$ then there exists $\sigma \in \Gamma(\tilde{\mathfrak{S}}_L, \tilde{\mathcal{L}})$ such that $\sigma(o) \neq O$.

Proof: From Cor. 4.2.1.b. we know that

$$U_L \subset \mathrm{Hilb}_{X/\mathbf{P}^N}$$

so we conclude that

$$\bar{U}_L \subset \overline{\mathrm{Hilb}_{X/\mathbf{P}^N}}$$

where $\overline{\mathrm{Hilb}_{X/\mathbf{P}^N}}$ is a projective variety, i.e. the component of the Hilbert scheme

that contains U_L and $\bar{U}_L$ is a polycylynder such that $\bar{U}_L \setminus U_L$ is a complex analytic set See (#2.5.).

We know from [SGA] that

1) $\overline{\mathrm{Hilb}_{X/\mathbf{P}^N}}$ is a projective variety

2) There exists a universal family

$$\begin{array}{ccc} \bar{\mathfrak{Y}} & \subset & \mathbf{P}^N \times \overline{\mathrm{Hilb}_{X/\mathbf{P}^N}} \\ \downarrow & & \downarrow \\ \overline{\mathrm{Hilb}_{X/\mathbf{P}^N}} & \equiv & \overline{\mathrm{Hilb}_{X/\mathbf{P}^N}} \end{array}$$

(See [SGA].)

So from these general results it follows that we have a family over $\bar{U}_L$, $\bar{\mathfrak{Y}} \to \bar{U}_L$ such that

$$\begin{array}{ccc} \bar{\mathfrak{Y}} \subset & \bar{U}_L \times \mathbf{P}^N \\ \downarrow & \downarrow \\ \bar{U}_L & \equiv & \bar{U}_L \end{array}$$

So over $\bar{\mathfrak{Y}}$ we have a relative ample line bundle $\bar{\mathcal{L}}$ such that $\bar{\mathcal{L}}$ restricted to the family

$$\mathfrak{S}_L \to U_L$$

is just $\mathcal{L}$. From the definition of the point o, i.e.

$$o \in X_t \text{ where } t \in \bar{U}_L \setminus U_L$$

and the fact that $\bar{U}_L \setminus U_L$ is a complex analytic subset in the polycylinder $\bar{U}_L$ it follows that there exists a sequence of points $\{x_n\} \in \mathfrak{S}_L$ such that

$$\lim_{n\to\infty} x_n = o \in X_t \ \& \ t \in \overline{U}_L \setminus U_L$$

Even more it is easy to see that we can chose $\{x_n\}$ in such way that in $\overline{\mathfrak{Y}}$ we have

$$\lim_{n\to\infty} x_n = y_o \in \overline{\mathfrak{Y}}$$

Since the line bundle $\overline{\mathcal{L}}$ is a relatively very ample line bundle we can find

$$\overline{\sigma} \in \Gamma(\overline{\mathfrak{Y}}, \overline{\mathcal{L}}) \ \& \ \overline{\sigma}(y_o) \neq O$$

Let σ be the restriction of $\overline{\sigma}$ on $\mathfrak{X}_L$. From Bishop's Theorem we know that we can continue $\sigma \in \Gamma(\mathfrak{X}_L, \mathcal{L})$ to a section $\tilde{\sigma} \in \Gamma(\tilde{\mathfrak{X}}_L, \tilde{\mathcal{L}})$.

From the definitions of

$$\sigma, \ \overline{\sigma} \text{ and } \tilde{\sigma}$$

and the definition of a section of a vector bundle we get that

$$\lim_{n\to\infty} \sigma(x_n) = \lim_{n\to\infty} \overline{\sigma}(x_n) = \lim_{n\to\infty} \tilde{\sigma}(x_n) = \overline{\sigma}(y_o) \neq O$$

So Proposition 4.2.3. is proved. Q.E.D.

Proposition 4.2.4.

Let X be a Hyper-Kählerian manifold such that on X there exists a line bundle $\mathcal{L}$ such that $\mathcal{L}$ defines a holomorphic map $\phi_{\mathcal{L}}: X \to \mathbb{P}^N$ such that

$$\dim_{\mathbf{C}} \phi_{\mathcal{L}}(X) = \dim_{\mathbf{C}} X$$

then $\mathcal{L}^{\otimes 3(2n-1)}$ defines a holomorphic birational map. $2n = \dim_{\mathbf{C}} X$.

Proof: Since

$$\dim_{\mathbf{C}} \phi_{\mathcal{L}}(X) = \dim_{\mathbf{C}} X = 2n$$

we can apply Bertinni's Theorem, which says that for generic

$$\sigma \in \Gamma(X, \mathcal{L})$$

the divisor of σ $(\sigma) = H_o$ on X is a non-singular subvariety. (See [6].)

Repeating these arguments $2n-1$ times we will get a non-singular curve

$$C = H_o \cap \ldots \cap H_{2n-1} \subset X$$

From the adjunction formula we get that

$$\mathcal{L}^{\otimes 3(2n-1)}|_C = 3K_C$$

where K_C is the canonical divisor on C. Since $3K_C$ gives an embedding

$$C \subset \mathbb{P}^{5g(C)-4}$$

we will get that $\phi_{\mathcal{L}}$ has degree 1. So 4.2.4. is proved. Q.E.D.

From 4.2.4. it follows that Lemma 4.2. is proved. Q.E.D.

4.3. THE CONSTRUCTION OF THE FAMILY: $\overline{\pi}_L: \overline{\mathfrak{X}}_L \to \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$.

Proposition 4.3.1. On the family

$$\tilde{\mathfrak{X}}_L \to \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

there exists a line bundle $\tilde{\mathcal{L}}$ such that

a) it gives a holomorphic map ϕ:

$$\begin{array}{ccc} \tilde{\mathfrak{X}}_L & \to & \mathbf{P}^N \times \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \\ \downarrow & & \downarrow \\ \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} & \equiv & \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \end{array}$$

b) The restriction of ϕ on $\mathfrak{X}_L \to \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ gives an embedding, i.e.

$$\begin{array}{ccc} \mathfrak{X}_L & \subset & \mathbf{P}^N \times \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \\ \downarrow & & \downarrow \\ \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})} & \subset & \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \end{array}$$

c) for all $t \in \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \setminus \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$ ϕ restricted to X_t is a holomorphic and

birational map on its image in $\mathbf{P}^N$.

Proof: Proposition 4.3.1. follows directly from Lemma 4.2. by glueing the line bundles $\tilde{\mathcal{L}}$ on all families $\{\tilde{\mathfrak{S}}_L \to \overline{U}_L\}$

Q.E.D.

Definition 4.3.2. Let

$$\overline{\mathfrak{X}}_L \overset{\text{def}}{=} \phi(\tilde{\mathfrak{X}}_L) \subset \mathbf{P}^N \times \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

then

$$\overline{\pi}_L: \overline{\mathfrak{X}}_L \to \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

we will call The partial compactification of the universal family of marked Hyper-Kählerian manifolds

$$\pi_L: \mathfrak{X}_L \to \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

Proposition 4.3.3. The family

$$\overline{\pi}_L : \overline{\mathfrak{X}}_L \to \overline{\mathfrak{M}}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

induces on

$$\overline{\mathfrak{M}}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

a Hausdorff topology.

Proof: We need to prove the following Sublemma 4.3.3.1. from which 4.3.3. follows directly.

Sublemma 4.3.3.1. Let

$$\mathfrak{X}_L \to U \text{ and } \mathfrak{Y}_L \to U$$

be two families of marked Hyper-Kählerian manifolds over the polycylinder U as a parameter space so that both are diffeomorphically identified with a trivial family UxX and they have the following properties:

a) All the fibres of both families are non-singular manifolds.

b) For each fibre in both families L is a class of cohomology of type (1,1), where $L \in H^2(X,\mathbf{Z})$ is a fixed class of cohomology

c) There exists an open and everywhere dense subset $U^o \subset U$ such that the restrictions of the families

$$\mathfrak{X}_L \to U \text{ and } \mathfrak{Y}_L \to U,$$

$$\mathfrak{X}^o_L \to U^o \quad \mathfrak{Y}^o_L \to U^o$$

are isomorphic as families of marked polarized Hyper-Kählerian manifolds and for each fibre in both families over U^o, L is a Chern class of a very ample line bundle on this fibre.

Then for each $t \in U \setminus U^o$ there exists a birational map

$$\phi_t : X_t \to Y_t$$

such that

$$(\phi_t)^*(\mathcal{L}_t) = \mathcal{L}_t$$

where $\mathcal{L}_t$ is the line bundle that corresponds to the class L in both X_t and Y_t.

Remarks.

a) The existence of the line bundle $\mathcal{L}_t$ on both X_t and Y_t is proved in (4.2.2.)

b) For each $t \in U^o$ we have a biholomorphic map

$$\phi_t : X_t \to Y_t$$

which preserve both the marking and the polarization class L. From here and using the C^∞ trivializations of the both families we conclude that

$$(\phi_t)^*: H^2(X,\mathbf{Z}) \rightarrow H^2(X,\mathbf{Z})$$

is just the identity map, i.e. $(\phi_t)^* = id$

Proof of 4.3.3.1.:

Minilemma 4.3.3.1.a.

Let $\{t_n\}$ be a sequence of points such that

a) $t_n \in U^o$

b) $\lim_{n\to\infty} t_n = t_o \in U \setminus U^o$

For $t \in U^o$ let $\Gamma_t \subset X_t x Y_t$ be the graph of the biholomorphic map between

$$\phi_t: X_t \rightarrow Y_t$$

stated in condition c) of 4.3.3.1., i.e. ϕ_t preserve the marking and the polarization class.

Then

$$\lim_{n\to\infty} \Gamma_{t_n} = \Gamma_{t_o}$$

exists as complex analytic subset in $X_{t_o} x Y_{t_o}$.

Proof: From the THEOREM proved in the Appendix we know that on both X_{t_o} and Y_{t_o} there exist two d closed real two forms

$$\omega_{Y_o} \text{ and } \omega_{X_o}$$

such that their (1,1) components

$$\omega_{Y_o}(1,1) \text{ and } \omega_{X_o}(1,1)$$

with respect to the complex structures

$$X_{t_o} \text{ and } Y_{t_o}$$

are positive definite at every point.

By continuity arguments we may suppose that both (1,1) components of $\omega_{X_o}(t)$ and

$$\omega_{Y_o}(t),\ \omega_{Y_o}(1,1)(t) \text{ and } \omega_{X_o}(1,1)(t)$$

are positive definite at every point of X_t and Y_t for all $t \in U$.

Let η be a positive definite (1,1)-form on U. The collection of (1,1) forms

$$\eta,\ \omega_{Y_o}(1,1)(t) \text{ and } \omega_{X_o}(1,1)(t)$$

defines a Hermitian metric H on

$$\mathfrak{X}_L \times_U \mathfrak{Y}_L$$

Let

$$H \text{ be the (1,1) form on } \mathfrak{X}_L x_U \mathcal{Y}_L$$

associated to the Hermitian metric H. Then the pullback of H to the submanifold

$$X_t x Y_t \subset \mathfrak{X}_L x_U \mathcal{Y}_L$$

is equal to

$$\omega_{Y_o}(1,1)(t)+\omega_{X_o}(1,1)(t)$$

where for notational simplicity we use

$$\omega_{Y_o}(1,1)(t) \text{ and } \omega_{X_o}(1,1)(t)$$

to denote their pullbacks under the projections from $X_t x Y_t$ to X_t and Y_t respectively.

We want to compute the volume of (Γ_{t_n}) with respect to

$$H \text{ on } \mathfrak{X}_L x_U \mathcal{Y}_L$$

and to show that it is bounded as $t_n \to t_o$ so that we can apply Bishop's Theorem to conclude the convergence of the subvariety

$$\Gamma_{t_n} \text{ in } \mathfrak{X}_L x_U \mathcal{Y}_L \text{ as } t_n \to t_o.$$

(*) $\mathrm{Vol}(\Gamma_t)<C$, where Γ_t is the graph of the biholomorphic map ϕ_t for $\forall t \in U$.

<u>Proof of (*):</u> It is easy to see that:

$$\mathrm{Vol}(\Gamma_t)=\int_{X_t} \wedge^{2n}((\phi_t)^*(\omega_{Y_o}(1,1))+\omega_{X_t}(1,1))$$

Let

$$f(t) \overset{\text{def}}{=} \int_{X_t} \wedge^{2n}((\phi_t)^*(\omega_{Y_o}(t)+\omega_{X_o}(t))) \tag{1}$$

We will prove that the following inequalities hold:

$$\mathrm{Vol}(\Gamma_t)\leq f(t)<C$$

First we will compute

$$\wedge^{2n}(h_t(2,0)+h_t(1,1)+\overline{h_t(2,0)})$$

where

$$(\phi_t)^*(\omega_{Y_o}(t)+\omega_{X_o}(t)) \overset{\text{def}}{=} h_t(2,0)+h_t(1,1)+\overline{h_t(2,0)}$$

where

$$h_t(1,1)=\mathrm{Im}H \text{ and } H \text{ is the Hermitian metric on } X_t x Y_t$$

defined as above.

Clearly

$$\wedge^{2n}(h_t(2,0)+h_t(1,1)+\overline{h_t(2,0)})=\sum_{k=1}^{2n}(\wedge^k h_t(2,0))\wedge(\overline{\wedge^k h_t(2,0)})\wedge(\wedge^{2n-2k}h_t(1,1)) \tag{2}$$

15

From the following Lemma:

<u>LEMMA.</u> If η is a primitive form of type (p,q), then

$$*\eta=\frac{(\sqrt{-1}\,)^{p-q}}{(2n-p-q)}(-1)^{\frac{(p+q)(p+q+1)}{2}}L^{2n-p-q}$$

where $*$ is the Hodge star operator. See [3].

we get that

$$*(\wedge^k h_t(2,0))=(\wedge^k h_t(2,0))\wedge(\wedge^{2n-2k}h_t(1,1)) \tag{3}$$

where $*$ is the Hodge star operator with respect to

$$\mathrm{Im}H=h_t(1,1)$$

From (1), (2) and (3) we get

$$f(t)\overset{\text{def}}{=}\int_{X_t}\wedge^{2n}((\phi_t)^*(\omega_{Y_o}(t)+\omega_{X_o}(t)))= \tag{4}$$

$$=\int_{X_t}(\sum_{k=1}^{2n}(\wedge^k h_t(2,0))\wedge(\overline{\wedge^k h_t(2,0)})\wedge(\wedge^{2n-2k}h_t(1,1)))=$$

$$=\sum_{k=1}^{2n}\|\wedge^k h_t(2,0)\|^2+\mathrm{Vol}(\Gamma_t)>O$$

From (4) we get that

$$\mathrm{Vol}(\Gamma_t)<f(t) \tag{5}$$

From the definition of f(t) it follows that since

$$(\phi)^*=\mathrm{id}$$

f(t) is a continuous function on U and so it is bounded. From (5) we obtain that

$$\mathrm{Vol}(\Gamma_t)<C \ \forall t\in U$$

So (*) is proved.

Q.E.D.

For a subvariety Z of pure codimension in a complex manifold X, we denote by [Z] the current on X defined by Z. Now we invoke Bishop's Theorem (See [2]) and conclude that the current $[\Gamma_{t_n}]$ converges weakly to a current on $X_{t_o}xY_{t_o}$ of the form

$$\sum_{i=1}^{k} m_i[\Gamma^i]$$

where m_i is a positive integer and Γ^i is an irreducible subvariety of complex dimension 2n on $X_{t_o}xY_{t_o}$.

So Minilemma 4.3.3.1.a. is proved.

Q.E.D.

<u>The end of the proof of 4.3.3.1.</u>

For any closed 4n-current Θ on X_txY_t, define a linear map

$$\Theta_*:H^*(X_t,\mathbf{C})\rightarrow H^*(Y_t,\mathbf{C})$$

of cohomology rings as follows. A cohomology class defined by a closed p-form α on X_t is mapped by Θ_* to the cohomology class defined by the closed p-current

$$(pr_2)_*(\Theta\wedge(pr_1)^*\alpha) \text{ on } Y_t$$

where pr_1 and pr_2 are respectively the projections of X_txY_t onto the first and the second factors and $(pr_1)_*$ and $(pr_2)^*$ mean respectively the corresponding pushforward and pullback maps. By reversing the rules of X_t and Y_t, we define analogously a linear map

$$\Theta_*:H^*(Y_t,\mathbf{C})\rightarrow H^*(X_t,\mathbf{C})$$

The map $[\Gamma_{t_n}]_*$ defined by the 4n-current

$$[\Gamma_{t_n}] \text{ in } X_{t_n}xY_{t_n}$$

clearly agrees with the map from $H^*(X,\mathbf{C})$ to $H^*(X,\mathbf{C})$ defined by

$$\phi_{t_n}:X_{t_n}\rightarrow Y_{t_n}.$$

Since ϕ_t defines the identity map on $H^2(X,\mathbf{Z})$, by passing to limit along the sequence $\{t_n\}$ we conclude that)

$$(\sum_{i=1}^{k} m_i[\Gamma^i])_*$$

is the identity map on $\wedge H^2(X,\mathbf{Z})$(exterior algebra of $H^2(X,\mathbf{Z})$)

Let

$$\omega_o(2n,0)=\wedge^n \omega_{X_{t_o}}(2,0)$$

be the non-zero holomorphic 2n form, which has no zeroes on X_{t_o}. Since

$$(\sum_{i=1}^{k} m_i[\Gamma^i])_*$$

is the identity map on $\wedge^n H^2(X,\mathbf{C})$ it follows that the 2n-current

$$(pr_2)_*(\sum_{i=1}^{k} m_i[\Gamma^i]\wedge(pr_1)^*\omega_o(2n,0))$$

on Y_{t_o}(which is automatically a holomorphic 2n-form on Y_{t_o}) can not be zero. Hence there must be some Γ^j which is projected both onto X_{t_o} and Y_{t_o}. There can only be one such Γ^j and moreover, $m_j=1$ and its projections onto X_{t_o} and onto Y_{t_o} are both of degree one, because both

$$(\sum_{i=1}^{k} m_i[\Gamma^i])_* \text{ and } (\sum_{i=1}^{k} m_i[\Gamma^i])^*$$

must leave fixed the class in $H^o(X,\mathbf{C})$ which is defined by the function on X with constant values. So Γ^j defines a birational morphism

$$\phi_{t_o}:X_{t_o}\rightarrow Y_{t_o}$$

since

$$\Gamma^j\subset X_{t_o}xY_{t_o}$$

and the projection on both factors has degree 1. On the other hand since

$$\lim_{n\rightarrow\infty}\phi_{t_n}=\phi_{t_o}$$

and the fact that

$$(\phi_{t_n})^*=id, \text{ where } (\phi_{t_n})^*:H^2(X,\mathbf{Z})\rightarrow H^2(X,\mathbf{Z})$$

we obtain that

$$(\phi_{t_o})^*=id, \text{ where } (\phi_{t_o})^*:H^2(X,\mathbf{Z})\rightarrow H^2(X,\mathbf{Z})$$

So since

$$(\phi_{t_o})^*(L)=L\Rightarrow(\phi_{t_o})^*(\mathcal{L}_{X_{t_o}})=\mathcal{L}_{Y_{t_o}}$$

where

$$\mathcal{L}_{X_{t_o}} \text{ and } \mathcal{L}_{X_{t_o}}$$

are the line bundles that corresponds to the class L in both X_{t_o} and Y_{t_o}.

So Sublemma 4.3.3.1. is proved.

Q.E.D.

In order to finish the proof of Proposition 4.3.3. we need to show that

(4.3.3.2.) LEMMA. $\Delta \subset \overline{\mathfrak{M}}_{(L;\gamma_1,\dots,\gamma_{b_2})} \times \overline{\mathfrak{M}}_{(L;\gamma_1,\dots,\gamma_{b_2})}$ (Δ is the diagonal)

is closed in the topology induced by the family

$$\overline{\pi}:\overline{\mathfrak{X}} \rightarrow \overline{\mathfrak{M}}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

Proof of (4.3.3.2.):

We will consider two cases.

First case.

$$U \subset \mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

First notice that if

$$U \subset \mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

and

$$\mathfrak{X}_U \rightarrow U \ \& \ \mathfrak{Y}_U \rightarrow U$$

are two families of marked polarized Hyper-Kählerian manifolds then they are isomorphic, i.e. there exists an isomorphism between those two families which preserve the marking and the polarization. This follows from the universal properties of the family constructed in #3

$$\mathfrak{X}_L \rightarrow \mathfrak{M}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

Indeed let $o \in U$ and X_o and Y_o are the fibers in

$$\mathfrak{X}_U \rightarrow U \ \& \ \mathfrak{Y}_U \rightarrow U$$

and they are isomorphic as marked polarized Hyper-Kählerian manifolds. Let

$$\phi_o : X_o \rightarrow Y_o$$

be an isomorphism that preserve the marking and the polarization class. Since the polarization class

$$L \in H^2(X,\mathbf{Z})$$

defines a line bundles

$$\{\mathcal{L}_U\}_{1,2}$$

on both families

$$\mathfrak{X}_U \to U \ \& \ \mathfrak{Y}_U \to U$$

and those two line bundles

$$\{\mathcal{L}_U\}_{1,2}$$

gives the embeddings:

$$\begin{array}{ccccc} \mathfrak{X}_U \subset & \mathbf{P}^N x U & \& & \mathfrak{Y}_U \subset & \mathbf{P}^N x U \\ \downarrow & \downarrow & & \downarrow & \downarrow \\ U \equiv & U & & U \equiv & U \end{array}$$

Since ϕ_o preserve the line bundles defined by the class L, then it follows that we may suppose that ϕ_o is induced by

$$g \in \mathbf{PGL}(N).$$

This is so since we may suppose that

$$\mathcal{L}_{o,1} = \mathcal{L}_{U,1}|_{X_o} \ \& \ \mathcal{L}_{o,2} = \mathcal{L}_{U,2}|_{Y_o}$$

gives embeddings of X_o and Y_o in one and the same $\mathbf{P}^N$.

Remember that we may suppose that both families are mapped to the universal family

$$\mathfrak{Y} \to \mathrm{Hilb}_{\mathbf{X}/\mathbf{P}^N}$$

i.e. they are obtained from two different imbeddings

$$i_1 : U \subset \mathrm{Hilb}_{\mathbf{X}/\mathbf{P}^N} \ \text{ and } \ i_2 : U \subset \mathrm{Hilb}_{\mathbf{X}/\mathbf{P}^N}$$

where $i_1(U)$ and $i_2(U)$ are transversal to the orbits of the action

$$\mathbf{PGL}(N) \text{ on } \mathrm{Hilb}_{\mathbf{X}/\mathbf{P}^N}$$

From here we get that there exists a holomorphic map

$$g : U \to \mathbf{PGL}(N)$$

such that g induces an isomorphism between the marked polarized Hyper-Kählerian manifolds

$$\mathfrak{X}_U \to U \ \& \ \mathfrak{Y}_U \to U$$

So the First case of 4.3.3. is proved.

Q.E.D.

<u>Second case.</u>

Let

$$\overline{\mathfrak{X}}_U \to U \text{ and } \overline{\mathfrak{Y}}_U \to U$$

be two families such that

$$U \subset \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \text{ and } U \cap (\overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \setminus \mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})}) \neq \emptyset$$

and both are subfamilies of

$$\overline{\mathfrak{X}} \to \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

then we must show that they are isomorphic.

Let

$$U^o \overset{\text{def}}{=} U \cap (\mathfrak{M}_{(L;\gamma_1,\ldots,\gamma_{b_2})})$$

and let

$$\mathfrak{X} \to U^o \text{ and } \mathfrak{Y} \to U^o$$

be the restrictions of

$$\overline{\mathfrak{X}}_U \to U \text{ and } \overline{\mathfrak{Y}}_U \to U$$

on U^o.

Notice that

$$\mathfrak{X} \to U^o \text{ and } \mathfrak{Y} \to U^o$$

are two isomorphic marked polarized families of Hyper-Kählerian manifolds, i.e. there exists

$$g^o : U^o \to \mathbf{PGL}(N)$$

such that g induces the isomorphism between those two families. This is so since we may suppose that

$$\begin{array}{ccccccc} \mathfrak{X} & \subset & \mathbf{P}^N \mathrm{x} U^o & \text{and} & \mathfrak{Y} & \subset & \mathbf{P}^N \mathrm{x} U^o \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ U^o & \equiv & U^o & & U^o & \equiv & U^o \end{array}$$

From the construction of the partial compactification

$$\overline{\mathfrak{X}} \to \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

it follows that

$$\begin{array}{ccc} \bar{\mathfrak{X}} \subset \mathbf{P}^N \times U & \text{and} & \bar{\mathfrak{Y}} \subset \mathbf{P}^N \times U \\ \downarrow \quad \downarrow & & \downarrow \quad \downarrow \\ U \equiv U & & U \equiv U \end{array}$$

Let me just remind You that both families $\bar{\mathfrak{X}} \to U$ and $\bar{\mathfrak{Y}} \to U$ are obtained in the following way: Let

$$\tilde{\mathfrak{X}} \to U \text{ and } \tilde{\mathfrak{Y}} \to U$$

be two families of marked Hyper-Kählerian manifolds and all its fibres are nonsingular and in each fibre L is a class of type (1,1). We proved that on $\tilde{\mathfrak{X}}$ and $\tilde{\mathfrak{Y}}$ there exist line bundles $(\tilde{\mathcal{L}}_1)$ and $(\tilde{\mathcal{L}}_2)$ such that

a) $c_1(\tilde{\mathcal{L}}_1|_{X_t})=L$ and $c_1(\tilde{\mathcal{L}}_1|_{Y_t})=L$ for all $t \in U$

b) The line bundles $(\tilde{\mathcal{L}}_1)$ and $(\tilde{\mathcal{L}}_2)$ give a holomorphic maps

$$\begin{array}{ccc} \tilde{\mathfrak{X}} \to \mathbf{P}^N \times U & \text{and} & \tilde{\mathfrak{Y}} \to \mathbf{P}^N \times U \\ \downarrow \quad \downarrow & & \downarrow \quad \downarrow \\ U \equiv U & & U \equiv U \end{array}$$

such that for each $t \in U^o$ X_t and Y_t are embedded in $\mathbf{P}^N$ and for $t \in U \setminus U^o$ the restriction of those two bundles on X_t and Y_t gives a holomorphic birational maps. Applying 4.3.3.1. to the this situation we obtain that

$$g^o: U^o \to \mathbf{PGL}(N)$$

can be prolonged to

$$g: U \to \mathbf{PGL}(N)$$

Indeed let $\{t_n\}$ be a sequence of points in U^o such that

$$\lim_{n \to \infty} t_n = t_o \in U \setminus U^o$$

then

$$\lim_{n \to \infty} g_n = g_o$$

where g_o is a birational map between

$$X_{t_o} \to Y_{t_o}$$

such that

$$(g_o)^*(\tilde{\mathcal{L}}_2|_{Y_{t_o}}) = \tilde{\mathcal{L}}_1|_{X_{t_o}}$$

From here it is easy to obtain directly that g_o induces an isomorphism between

$$\overline{X}_{t_o} \text{ and } \overline{Y}_{t_o}$$

i.e.

$$g_o \in \mathbf{PGL}(N)$$

This is so since both $\overline{X}_{t_o}$ and $\overline{Y}_{t_o}$ are submanifolds in $\mathbf{P}^N$ and g_o preserve the standart line bundle $\mathcal{O}_{\mathbf{P}^N}(1)$. So from here we obtain that the holomorphic map

$$g^o : U^o \to \mathbf{PGL}(N)$$

can be prolonged to a map

$$g : U \to \mathbf{PGL}(N)$$

and so g induces an isomorphism between

$$\begin{array}{ccccccc} \overline{\mathfrak{X}} \subset \mathbf{P}^N x U & \text{and} & \overline{\mathfrak{Y}} \subset \mathbf{P}^N x U \\ \downarrow \quad\quad \downarrow & & \downarrow \quad\quad \downarrow \\ U \equiv U & & U \equiv U \end{array}$$

This proves 4.3.3.

Q.E.D.

LEMMA 4.4. The period map $\overline{p}:\overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \to \Omega(L)$ is a proper map.

PROOF: The proof is based on the following criteria due to Grothendieck. (See [SGA]):

(*) Suppose that

$$f : X \to Y$$

is a holomorphic map between two complex manifolds. Let

$$D = \{t \in \mathbf{C} \mid |t| < 1\} \text{ and } D^* = D \setminus \{O\}$$

Then the map f will be a proper one if for each holomorphic map $g : D \to Y$ there exists a holomorphic map

$$h^* : D^* \to X$$

such that the following diagram is commutative one:

$$\begin{array}{ccc} D^* & \subset & D \\ h^* \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

and h^* can be prolonged to a map $h : D \to X$ such that the following diagramm is a commutative one

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ h\downarrow & & \downarrow g \\ D & \equiv & D \end{array}$$

We need to prove that if

$$\phi: D \to \Omega(L)$$

is any holomorphic map such that the following diagram is commutative:

$$\begin{array}{ccc} D^* & \subset & D \\ \psi^*\downarrow & & \downarrow\phi \\ \overline{\mathfrak{M}}_{(L;\gamma_1,\dots,\gamma_{b_2})} & \xrightarrow{\overline{p}} & \Omega(L) \end{array}$$

Then ψ^* can be prolonged to a map

$$\psi: D \to \overline{\mathfrak{M}}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

such that the following diagram is commutative one:

$$\begin{array}{ccc} D & \equiv & D \\ \psi\downarrow & & \downarrow\phi \\ \overline{\mathfrak{M}}_{(L;\gamma_1,\dots,\gamma_{b_2})} & \xrightarrow{\overline{p}} & \Omega(L) \end{array}$$

(Without loss of generality we may suppose that $D \subset \Omega(L)$)

Since we have a family

$$\overline{\mathfrak{X}} \to \overline{\mathfrak{M}}_{(L;\gamma_1,\dots,\gamma_{b_2})}$$

the map ψ^* will induce a family

$$\psi^*: \mathfrak{S}^* \to D^*$$

On the other hand the period map

$$p^*: D^* \to \Omega(L)$$

can be prolonged to a map

$$p: D \to \Omega(L)$$

iff the monodromy of the family

$$\psi^*: \mathfrak{S}^* \to D^*$$

is the trivial one. (See [4].) So $\psi^*: \mathfrak{S}^* \to D^*$ has a trivial monodromy. From THEOREM 1 we deduce that the family $\psi^*: \mathfrak{S}^* \to D^*$ can be embedded into a family $\psi: \mathfrak{S} \to D$, where all fibres are nonsingular Hyper-Kählerian manifolds. From 4.3.1. it follows that there exists a map

$$\begin{array}{ccc} \phi_L: \mathfrak{S} & \rightarrow & \mathbf{P}^N \times D \\ \downarrow & & \downarrow \\ D & \equiv & D \end{array}$$

such that

a) ϕ_L^* is an embedding, where ϕ_L^* is the restriction of ϕ_L to $\mathfrak{S}^* \rightarrow D^*$, i.e we have

$$\begin{array}{ccc} \phi_L^*: \mathfrak{S}^* & \subset & \mathbf{P}^N \times D^* \\ \downarrow & & \downarrow \\ D^* & \equiv & D^* \end{array}$$

b) $\phi_L|_{X_o}: X_o \rightarrow \mathbf{P}^N$ is a holomorphic and birational map. From the universal properties of

$$\overline{\mathfrak{X}} \rightarrow \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

and the way we constructed

$$\overline{\mathfrak{X}} \rightarrow \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

we obtain the map

$$\psi: D \rightarrow \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})}$$

with the desired properties, i.e. ψ is a map such that the following diagram is commutative:

$$\begin{array}{ccc} D & \equiv & D \\ \psi \downarrow & & \downarrow \phi \\ \overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} & \xrightarrow{\overline{p}} & \Omega(L) \end{array}$$

This proves that the map $\overline{p}$ is a proper map.

Q.E.D.

<u>The end of the proof of THEOREM 3.</u>

Since

$$\Omega(L) \doteq SO(2,b_2-3)/SO(2) \times SO(b_2-3)$$

is a bounded domain and so it is simply connected. On the other hand

$$\overline{\mathfrak{M}}_{(L;\gamma_1,\ldots,\gamma_{b_2})} \xrightarrow{\overline{p}} \Omega(L)$$

is a proper surjective and étale map between complex analytic manifolds, so p is an isomorphism on any of the components of the moduli space of marked polarized Hyper-Kählerian manifolds.

Q.E.D.

APENDIX.

DEFINITION. Suppose that X is a compact complex manifold such that:

1) On X there exists a closed holomorphic two form $\omega_X(2,0)$ such that at each point $x\in X$ $\omega_X(2,0)$ is a non-degenerate skew symmetric matrix, i.e. everywhere $\omega_X(2,0)$ has a maximal rank equal to $2n=\dim_{\mathbf{C}}X$.

2) $\dim_{\mathbf{C}}H^2(X,\mathcal{O}_X)=1$

3) $\dim_{\mathbf{C}}X\geq 4$

then X will be called a holomorphic symplectic manifold.

TEOREM. Let X be a holomorphic symplectic manifold, then X admits a real closed two form

$$\omega=\omega^{2,0}+\omega^{1,1}+\omega^{0,2}$$

such that

a) $\omega^{2,0}=\partial\alpha^{1,0}$, $\omega^{0,2}=\overline{\partial\alpha^{1,0}}$,

b) $\omega^{1,1}$ is positive definite at each point $x\in X$.

PROOF:

The proof is based on the following Theorem of Harvey and Lawson:

TEOREM. (See [5].) Suppose that X is a compact complex manifold, then X admits a real closed two form

$$\omega=\omega^{2,0}+\omega^{1,1}+\omega^{0,2}$$

such that

a) $\omega^{2,0}=\partial\alpha^{1,0}$, $\omega^{0,2}=\overline{\partial\alpha^{1,0}}$

b) $\omega^{1,1}$ is positive definite at each point $x\in X$

if and only if X does not support a non-trivial, d-closed current which is the bidimension (1,1) component of a boundary.

We need to check that if X is a holomorphic symplectic manifold then X satisfies the conditions of the Theorem of R. Harvey and B. Lawson Jr.

Let

$$\mu=\sqrt{-1}\sum\mu^{i,\bar{j}}\frac{\partial}{\partial z^{i}}\wedge\overline{\frac{\partial}{\partial z^{j}}}$$

be an exact real (1,1) positive current on X. Since on X we have a closed holomorphic form $\omega_X(2,0)$ which is non-degenerate at each point $x\in X$ we get immediately from μ an exact (2n-1,2n-1) current η in the following way:

$$\eta=\mu\perp((\wedge^{n-1}\omega^*_X(2,0))\wedge((\wedge^{n-1}\omega^*_X(2,0))$$

<u>Definition of $\omega^*_X(2,0)$.</u>

Since $\omega^*_X(2,0)$ is a non-degenerate closed holomorphic form, we can repeat the arguments of Darboux Lemma (See [1].) and we will get a local coordinate system

$$(z^1,\ldots,z^n,\ldots,z^{2n})$$

such that locally

$$\omega_X(2,0)=\sum_{i=1}^{n} dz^i\wedge dz^{i+n}$$

then

$$\omega^*_X(2,0):=\sum_{i=1}^{n}\frac{\partial}{\partial z^i}\wedge\frac{\partial}{\partial z^{i+n}}$$

Let

$$\eta=dj^*$$

Clearly

$$\alpha=\eta\perp(\wedge^n(\omega_X(2,0))\wedge(\wedge^n(\omega_X(0,2))=$$

$$d(j^*\perp(\wedge^n(\omega_X(2,0))\wedge(\wedge^n(\omega_X(0,2))))=dj$$

where α is a real two form of type (1,1) with distribution coefficients and j is also a real one form. We can write

$$j=\beta+\overline{\beta}$$

where β is a (1,0)-form on X. Since α is of type (1,1) it follows that

$$\alpha=\overline{\partial}\beta+\partial\overline{\beta} \text{ and } \overline{\partial}\overline{\beta}=0$$

So from

$$\overline{\partial}\overline{\beta}=0$$

it follows that

$$\overline{\beta}\in H^1(X,\mathcal{O}_X)$$

<u>Proposition 1.</u> If X is a holomorphic symplectic manifold, then

$$H^1(X,\mathcal{O}_X)=0$$

if $\dim_{\mathbf{C}}X\geq 4$.

<u>Proof:</u>

<u>Case 1.</u>

<u>Suppose that $\dim_{\mathbf{C}}H^1(X,\mathcal{O}_X)=1$.</u>

Let $H^1(X,\mathcal{O}_X)=\mathbf{C}\alpha$, where α is a form of type (0,1) and

$$\overline{\partial}\alpha=0$$

Consider the map:

$$\psi\colon \alpha\rightarrow\alpha\wedge(\wedge^{n-1}\overline{\omega_X(2,0)}\wedge(\wedge^n\omega_X(2,0))$$

Since $\omega_X(2,0)$ is a non degenerate holomorphic two form it follows that ψ gives an isomorphism between

$$H^1(X,\mathcal{O}_X) \text{ and } H^{2n-1}(X,\Omega^{2n})$$

From Serre's duality we know that the pairing

$$H^1(X,\mathcal{O}_X)\times H^{2n-1}(X,\Omega^{2n})\rightarrow\mathbf{C}$$

given by

$$(\alpha,\beta)\rightarrow\int_X\alpha\wedge\beta$$

is a non-degenerate. On the other hand α generates $H^1(X,\mathcal{O}_X)$ and

$$H^{2n-1}(X,\Omega^{2n})\cong\mathbf{C}\alpha\wedge(\wedge^{n-1}\overline{\omega_X(2,0)}\wedge(\wedge^n\omega_X(2,0)))$$

Since

$$\alpha\wedge\alpha\wedge(\wedge^{n-1}\overline{\omega_X(2,0)}\wedge(\wedge^n\omega_X(2,0)))=0$$

we get a contradiction with Serre's duality. So if X is a symplectic holomorphic manifold we have two possibilities in case of $\dim_{\mathbf{C}}X\geq 4$; either

$$H^1(X,\mathcal{O}_X)=0 \text{ or } \dim_{\mathbf{C}}H^1(X,\mathcal{O}_X)\geq 2.$$

Case 2.

$$\underline{\dim_{\mathbf{C}} H^1(X,\mathcal{O}_X)\geq 2.}$$

Sublemma. Suppose X is a holomorphic symplectic manifold and $\dim_{\mathbf{C}} X\geq 4$. Let

$$\alpha,\ \beta\in H^1(X,\mathcal{O}_X)$$

then

$$\alpha\wedge\beta=\overline{\partial}\mu$$

where μ is a (0,1) form.

Proof: Clearly

$$\alpha\wedge\beta\in H^2(X,\mathcal{O}_X)$$

Since

$$\dim_{\mathbf{C}} H^2(X,\mathcal{O}_X)=1 \text{ and } H^2(X,\mathcal{O}_X)=\mathbf{C}\overline{\omega_X(2,0)}$$

it follows that

$$\alpha\wedge\beta=c\overline{\omega_X(2,0)}+\overline{\partial}\mu$$

So we need to prove that c=0. Clearly we have

$$\alpha\wedge\beta\wedge\alpha\wedge\beta=0=c^2\wedge^2\overline{\omega_X(2,0)}+c\overline{\partial}\mu\wedge\overline{\omega_X(2,0)}+\overline{\partial}\mu\wedge\overline{\partial}\mu$$

From

$$\dim_{\mathbf{C}} X\geq 4$$

it follows that

$$\int_X \alpha\wedge\beta\wedge\alpha\wedge\beta\wedge(\wedge^{n-2}\overline{\omega_X(2,0)}\wedge(\wedge^n\omega_X(2,0))=$$
$$c^2\int_X(\wedge^n\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))+c\int_X\overline{\partial}\mu\wedge(\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))+$$
$$\int_X\overline{\partial}\mu\wedge\overline{\partial}\mu\wedge(\wedge^{n-2}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=0$$

From

$$\overline{\partial}\mu\wedge(\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0)=d(\mu\wedge(\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))$$

$$\overline{\partial}(\mu\wedge\overline{\partial}\mu\wedge(\wedge^{n-2}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=$$
$$d(\mu\wedge\overline{\partial}\mu\wedge(\wedge^{n-2}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))$$

and Stoke's Theorem we get that

$$\int_X \overline{\partial}\mu\wedge(\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=0$$

.

So

$$\int_X \overline{\partial}\mu\wedge\overline{\partial}\mu\wedge(\wedge^{n-2}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=0$$

$$c^2\int_X(\wedge^n\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=0$$

From the fact that $\omega_X(2,0)$ is a non-degenerate form it follows that

$$\int_X(\wedge^n\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))>0$$

So from

$$c^2\int_X(\wedge^n\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=0\Rightarrow c=0$$

Q.E.D.

We know already that every element of $H^{2n-1}(X,\Omega^{2n})$ can be expressed as

$$\beta\wedge((\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))$$

where $\beta\in H^1(X,\mathcal{O}_X)$.

By Serre's duality the pairing

$$(\alpha,\beta\wedge((\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=\int_X\alpha\wedge\beta\wedge((\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))$$

is a non-degenerate belinear map. On the other hand since

$$\alpha\wedge\beta=\overline{\partial}\mu$$

we get that

$$\alpha\wedge\beta\wedge((\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=$$
$$d(\mu\wedge((\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))$$

From Stoke's Theorem we get

$$\int_X \alpha\wedge\beta\wedge((\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=$$

$$\int_X d(\mu\wedge((\wedge^{n-1}\overline{\omega_X(2,0)})\wedge(\wedge^n\omega_X(2,0))=O$$

So this is a contradiction to Serre's duality. This proves that $H^1(X,\mathcal{O}_X)=0$.

Q.E.D.

<u>Proposition 2.</u> Suppose that η is a positive (1,1) current and $\eta=dj^*$, then $\eta=0$.

<u>Proof:</u> Let

$$\eta=dj^*$$

Clearly

$$\alpha=\eta\perp(\wedge^n(\omega_X(2,0))\wedge(\wedge^n(\omega_X(0,2))=$$

$$d(j^*\perp(\wedge^n(\omega_X(2,0))\wedge(\wedge^n(\omega_X(0,2))))=dj$$

where α is a real two form of type (1,1) with distribution coefficients and j is also a real one form. We can write

$$j=\beta+\overline{\beta}$$

where β is a (1,0)-form on X. Since α is of type (1,1) it follows that

$$\alpha=\overline{\partial}\beta+\partial\overline{\beta} \text{ and } \overline{\partial}\,\overline{\beta}=0$$

So from

$$\overline{\partial}\,\overline{\beta}=0$$

it follows that

$$\overline{\beta}\in H^1(X,\mathcal{O}_X)=0\Rightarrow\overline{\beta}=\overline{\partial}\sigma$$

where σ is a (0,0) current on X. Hence

$$\alpha=\sqrt{-1}\partial\overline{\partial}\tau,\text{ where } \tau=\sqrt{-1}(\overline{\sigma}-\sigma)$$

The positivity of the (1,1) current on X implies that τ is a plurisubharmonic function on X. By the compactness of X and the maximum principle we get

$$\tau\equiv\text{const}$$

So

$$\alpha=\partial\overline{\partial}\,\text{const}\equiv O$$

So $\eta=O$

Q.E.D.

<u>Proposition 3.</u>

Suppose that η is a positive (1,1) current and $\eta=(d\alpha)(1,1)$(i.e. η is a (1,1) component of a boundary), then $\eta\equiv0$.

Proof: The existence of the closed holomorphic two form $\omega_X(2,0)$ which is a non-degenerate form on X shows that we may consider η as a form of type (1,1) on X. Since

$$d\eta=0 \text{ and } \eta=\overline{\partial}\alpha^{1,0}+\partial\alpha \Rightarrow \partial\overline{\partial}\alpha^{1,0}=-\overline{\partial}\partial\alpha^{1,0}=0$$

and the regularity of the $\overline{\partial}$ operator we get that $\partial\alpha^{1,0}$ is a holomorphic form on X. It is easy to see that if $\partial\alpha^{1,0}\neq 0$, indeed

$$\int_X \partial\alpha^{1,0}\wedge\overline{\partial\alpha^{1,0}}\wedge(\wedge^{n-1}(\omega_X(2,0))\wedge(\wedge^{n-1}(\omega_X(0,2))>0$$

On the other hand we have

$$d(\alpha^{1,0}\wedge\overline{\partial\alpha^{1,0}}\wedge(\wedge^{n-1}(\omega_X(2,0))\wedge(\wedge^{n-1}(\omega_X(0,2))=$$

$$\partial\alpha^{1,0}\wedge\overline{\partial\alpha^{1,0}}\wedge(\wedge^{n-1}(\omega_X(2,0))\wedge(\wedge^{n-1}(\omega_X(0,2))$$

From Stoke's Theorem it follows that

$$0<\int_X \partial\alpha^{1,0}\wedge\overline{\partial\alpha^{1,0}}\wedge(\wedge^{n-1}(\omega_X(2,0))\wedge(\wedge^{n-1}(\omega_X(0,2))=$$

$$\int_X d(\alpha^{1,0}\wedge\overline{\partial\alpha^{1,0}}\wedge(\wedge^{n-1}(\omega_X(2,0))\wedge(\wedge^{n-1}(\omega_X(0,2))=0$$

So we get a contradiction with $\partial\alpha\neq 0$. So $\partial\alpha=0$.

Q.E.D.

We know that $\partial\alpha^{1,0}=0$ and so

$$\eta=d\alpha$$

From Proposition 3 we obtain that

$$\eta\equiv 0$$

So the conditions of The Theorem of Harvey and Lawson are fulfilled for holomorphic symplectic manifolds.

Q.E.D.

Cor. The form ω defined in the THEOREM defines a non-zero class of cohomology $[\omega]$ in $H^2(X,\mathbf{R})$.

Proof: From the computations on page 23 formula (4) it follows that

$$\int_X \wedge^{2n}(\omega)=\sum_{k=1}^{n} \|\wedge^k\omega\|^2+\mathrm{Vol}(X)$$

where the Vol(X) is defined with respect to the metric $\omega^{1,1}$. On the other hand if

$$\omega=d\mu$$

then from Stoke's Theorem it follows that

$$\int_X (\omega)\wedge(\wedge^{2n-1}(\omega))=\int_X (d\mu)\wedge(\wedge^{2n-1}(\omega)))=0$$

So $[\omega]\neq 0$ in $H^2(X,\mathbf{R})$.

Q.E.D.

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