RIEMANN-HILBERT FACTORIZATION AND INTEGRABILITY

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ABSTRACT

I will try to review some ideas that have first appeared in the work of Soviet mathematicians in the 1970s-1980s and have further been developed in the 1980s-1990s "in the West", concerning the connection of so-called completely integrable systems (KdV, NLS, Toda, etc.) and Riemann-Hilbert factorization problems. At first, such ideas have been most fruitful in enabling us to compute rigorous asymptotics (long time, semiclassical, etc.) for integrable equations. But Riemann-Hilbert problems have also enriched the theory of orthogonal polynomials, random matrices, and combinatorics (and not only concerning asymptotics).

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I. INTEGRABILITY

Archetypal examples of integrable equations are:

1. The KdV equation

$$u_t - 6uu_x + \epsilon^2 u_{xxx} = 0, \tag{1}$$

where $\epsilon > 0$.

2. The nonlinear Schrödinger equation (focusing (+) and defocusing (-) cases)

$$i\epsilon u_t + \frac{\epsilon^2}{2} u_{xx} \pm |u|^2 u = 0. \tag{2}$$

Integrable equations always admit a "Lax pair" representaion: they can be written in the form

$$L_t = [L, B] = LB - BL, \tag{3}$$

where L, B are operators on a Hilbert space. The existence of a Lax pair does not guarantee integrability. It is only the first step.

Integrability machinery is based on the scattering and inverse scattering theory for equations like (1) and (2). The Lax operator L associated to (1) is the Schrödinger operator

$$L = -\epsilon^2 \frac{d^2}{dx^2} + u(x) \tag{4}$$

while

$$B = 4\epsilon^3 \frac{d^3}{dx^3} - 3\epsilon[u\frac{d}{dx} + \frac{d}{dx}u].$$

The NLS equations are associated to the Dirac operators

$$\begin{pmatrix} i\epsilon\partial_x & \mp iu(x) \\ -iu^*(x) & -i\epsilon\partial_x \end{pmatrix}, \tag{5}$$

where sign "+" corresponds to the defocusing case, while sign "-" corresponds to the focusing case. * denotes complex conjugation. It is important to note that, although (4) gives a self-adjoint operator (here u is always real), (5) is self-adjoint in the defocusing case and non-self-adjoint in the focusing case.

II. RIEMANN-HILBERT FACTORIZATION PROBLEMS

The solution of an integrable system via the inverse scattering transform can always be reduced to the solution of a Riemann-Hilbert factorization problem. In particular, the asymptotic analysis of the given equation is thus reduced to the asymptotic analysis of a Riemann-Hilbert problem.

Consider, for example, the nonlinear Schrödinger equation (defocusing case), that is equation (2) with the "-" sign. Assume $\epsilon = 1$ for the moment. Suppose we want to solve (2) under some initial data $u_0(x)$ that, for simplicity, are considered to belong to the Schwartz class.

Let $r(\lambda)$ be the reflection coefficient corrsponding to u_0. It is a fact of direct scattering theory that r is also Schwartz, and that $|r(\lambda)| < 1,\ for\ all\ \lambda \in \mathbb{R}$.

Let Q solve

$$Q_+(\lambda) = Q_-(\lambda) \begin{pmatrix} 1 - |r(\lambda)|^2 & -r^*(\lambda)e^{-2i\lambda x - 4i\lambda^2 t} \\ r(\lambda)e^{2i\lambda x + 4i\lambda^2 t} & 1 \end{pmatrix},\ Im\lambda = 0, \tag{8}$$

Note that the jump matrix has determinant 1. Also

$$Q(\lambda = \infty) = I. \tag{9}$$

This is a Riemann-Hilbert factorization problem: Q is a matrix function, analytic in the complement of the real line, satisfying the jump condition and the asymptotics above.

The solution of the Riemann-Hilbert problem enables us to recover $u(x,t)$. Indeed

$$u(x,t) = -2lim_{\lambda\to\infty}\lambda Q_{12}. \tag{10}$$

Thus, the initial value problem for (2) is reduced to the above Riemann-Hilbert factorization problem.

Conversely, suppose we are given a Riemann-Hilbert problem of the form

$$Q_+(\lambda) = Q_-(\lambda)e^{i\theta(x,t,\lambda)ad\sigma_3}v(\lambda),\ \lambda \in \Sigma, \tag{11}$$

where $v(\lambda)$ is independent of x,t. In (8) above, for example, $\theta(x,t,\lambda)=2i\lambda x+4i\lambda^2 t$ and

$$v(\lambda)=\begin{pmatrix} 1-|r(\lambda)|^2 & -r^*(\lambda)\\ r(\lambda) & 1\end{pmatrix}.$$

Define $m(x,t,\lambda)=Q(\lambda)e^{i\theta(x,t,\lambda)\sigma_3}$. Then $m_+=m_-v$ for $\lambda\in\Sigma$. Differentiating with respect to x or t, we also get $\partial_x m_+=\partial_x m_- v$ and $\partial_t m_+=\partial_t m_- v$, for $\lambda\in\Sigma$. Hence both $\partial_x m\cdot m^{-1}$ and $\partial_t m\cdot m^{-1}$ have no jumps, so they are entire functions. By calculating $\partial_x m$ and $\partial_t m$ at infinity, we find

$$m_x=U(x,t,\lambda)m,$$

$$m_t=V(x,t,\lambda)m,$$

where U,V are polynomials in λ [quadratic in the NLS case].

This is essentially the Lax pair arising from the Riemann-Hilbert problem (11). The compatibility condition $m_{xt}=m_{tx}$ leads to an integrable equation.

III. FREDHOLM DETERMINANTS AND RANDOM MATRICES

PROPOSITION (Its, Izergin, Korepin, Slavnov). Let

$$\tau=det(I+K_x),$$

where K_x is the integral operator on $L^2(\Sigma)$ with kernel

$$K_x(z,z')=\frac{A(z')B(z)+A'(z)B'(z')}{z-z'},$$
$$such\ that\ \frac{d}{dx}K_x(Z,z')=A(z)A'(z')+A'(z)A(z')$$
$$and\ A(z)B(z)+A'(z)B'(z)=0.$$

The determinant above is a Fredholm determinant.

Consider the following Riemann-Hilbert problem on the contour Σ. Let m be a 2x2 matrix function analytic in the complement of Σ, and let m_+,m_-, the limits of m on Σ from left and right, respectively, satisfy

$$m_+(z)=m_-(z)\begin{pmatrix} 1+2\pi iA'(z)B'(z) & -2\pi iA(z)B'(z)\\ -2\pi iA'(z)B(z) & 1+2\pi iA(z)B(z)\end{pmatrix},\ for\ z\in\Sigma,$$
$$and\ lim_{z\to\infty}m(z)=I.$$

Then

$$(log\tau)_x = i(m^1_{11} - m^1_{22}),$$

where $m(z) = I + \frac{m^1}{z} + O(z^{-2})$ is the expansion of $m(z)$ near infinity and the suffixes 11 and 22 denote the diagonal entries of m^1.

PROPOSITION (Mehta, Gaudin, Dyson). Let $J = \cup_{k=0}^n J_k$, where $J_k = (a_k, b_k)$ are $n+1$ disjoint intervals in $\mathbb{R}$. Let $K(z,z') = \frac{sinx(z-z')}{z-z'}$ be the kernel of the operator K acting on $L^2(J, dz)$. Then the determinant $\tau = det(I+K)$ is the probability of finding no eigenvalues in the union of intervals $\frac{xJ}{\pi} = \cup_{k=0}^n(\frac{xa_k}{\pi}, \frac{xb_k}{\pi})$ for a random Hermitian matrix chosen from the so-called Gaussian unitary ensemble (GUE), in the bulk scaling limit with mean spacing 1.

PROPOSITION (des Cloizeaux, Mehta). If $J = (-1,1)$, then as $x \to \infty$, then $logP = -x^2/2 - logx/4 + c + o(1)$.

REMARK. The Gaussian Unitary Ensemble consists of Hermitian matrices M with the following probability distribution

$$P_n(M)dM = c \cdot exp(-tr(M^2))\Pi_i M_{ii} \Pi_{i<j} Re(M_{ij}) Im(M_{ij}),$$

such that c is a normalization constant.

In general one can consider distributions

$$P_n(M)dM = c \cdot exp(-F(M))\Pi_i M_{ii} \Pi_{i<j} Re(M_{ij}) Im(M_{ij}),$$

where $F(M)$ is not too rapidly growing at infinity. If we impose the symmetry $P_n(UMU^*) = P_n(M)$ for any unitary U, then $F(M)$ must be a symmetric function of the eigenvalues of M. For GUE we actually choose $F(M)$ to be the trace of a polynomial in M. Then, it must actually be a quadratic polynomial, and through renormalization, $F(M) = tr(M^2)$.

By the way the Universality results in Random Matrix Theory imply that the distribution of the eigenvalues of large matrices with distribution as above is independent of F (for a fairly general class of F).

One of the most famous applications of the theory of GUE matrices and in particular the asymptotic distribution of their eigenvalues (for large matrix dimension) is the so-called Montgomery-Odlyzko law for the distribution of zeros of the Riemann zeta function.

Montgomery found that on assuming the Riemann Hypothesis one could partially determine the pair correlation function for the zeros, and the result matched the pair correlation function for eigenvalues of GUE random hermitian matrices.

More precisely, let $\cdots \leq \gamma_{-1} < 0 < \gamma_1 \leq \cdots \leq \gamma_n \leq \cdots$ be the sequence of the imaginary parts of the nontrivial zeros of ζ $(\gamma_{-i} = -\gamma_i)$. For $i \geq 1$, put $\hat{\gamma}_i = \gamma_i \log \gamma_i / 2\pi$, so that the $\hat{\gamma}_i$ have mean spacing 1 (renormalization process). Assuming the Riemann hypothesis (RH), Montgomery, in order to study the statistics of the normalized spacings, computed the limiting pair correlation distribution and got the formula

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{1 \leq i \neq j \leq N} f(\hat{\gamma}_j - \hat{\gamma}_i) = \int_{-\infty}^{+\infty} f(x)(1 - ((\sin \pi x)/\pi x)^2) dx,$$

valid for any f whose Fourier transform is compactly supported in $(-1, 1)$; he also conjectured that the above equality should hold without restriction on the size of the support of $\hat{f}$ (this was known as the pair correlation conjecture: it was later largely confirmed numerically by Odlyzko). The density $1 - ((\sin \pi x)/\pi x)^2$ matches the one found for the pair correlation distribution of the eigenvalues of a typical large GUE matrix!

IV. RIEMANN-HILBERT DEFORMATION. A SINGULAR INTEGRAL EQUATION AND THE BEALS-COIFMAN FORMULA

In the scalar case, a Riemann-Hilbert problem can be explicitly solved. For example, let d solve

$$d_+(z) = d_-(z)q(z),$$

$$lim_{z\to\infty} d(z) = 1,$$

where q is a scalar $1 + L^1 \cap L^\infty$ function on the real line. Taking logarithms

$$log d_+(z) = log d_-(z) + log q(z),$$

and by Plemelj's formula

$$log d = \int_{\mathbb{R}} \frac{log q(s)}{2\pi i(s-z)} ds,$$

hence

$$d = exp[\int_{\mathbb{R}} \frac{log q(s)}{2\pi i(s-z)} ds].$$

Note that indeed

$$lim_{z\to\infty} d(z) = 1.$$

However, it is not a priori clear that the integral $\int_{\mathbb{R}} \frac{log q(s)}{2\pi i(s-z)} ds$ is defined. For this we need to have ensured that the index of q is zero, i.e $\int_{\mathbb{R}} log q(s) ds = 0$.

In any case, scalar problems, to the extent that they are solvable, are explicitly solvable. This is not so for matrix problems.

The solution of a Riemann-Hilbert problem can be reduced to the solution of a singular integral equation (see [B-C]). In our case $\Sigma = \mathbb{R}$. Define the matrix Cauchy operators as follows:

$$C_\pm : (L^2(\Sigma))_{2x2} \to (L^2(\Sigma))_{2x2}$$
$$(C_\pm f)(z) = \frac{1}{2\pi i} \int_\Sigma \frac{f(\zeta) d\zeta}{\zeta - z_\pm}$$

where $z \in \Sigma$ and the signs "+" and "-" conote normal limits as $z \to \Sigma$, from the right and the left of Σ respectively (with respect to its orientation).

It is easy to see that C_+ and C_- are bounded. It is a simple calculation to show that

$$C_+ - C_- = I.$$

Note that the jump in (8) can be factorized as

$$(b_-)^{-1} b_+$$

where

$$b_+ = \begin{pmatrix} 1 & -r^*(\lambda) e^{2i\lambda x + 4i\lambda^2 t} \\ 0 & 1 \end{pmatrix},$$
$$b_- = \begin{pmatrix} 1 & 0 \\ -r(\lambda) e^{-2i\lambda x - 4i\lambda^2 t} & 1 \end{pmatrix}.$$

Define the "Cauchy" operators C:

$$\text{for } z \notin \Sigma, \quad \text{let } (Cf)(z) = (2\pi i)^{-1} \int_\Sigma \frac{f(w) dw}{w - z} \quad \text{and}$$

Let the operator C_w be defined by

$$C_w f = C_+(f w_-) + C_-(f w_+)$$

for a 2x2 matrix valued f, where

$$w_+ = b_+ - I \quad \text{and } w_- = I - b_-.$$

Let μ solve the singular integral equation

$$\mu = I + C_w \mu. \tag{12}$$

THEOREM Let Q be defined by the integral formulae

$$Q_+ = I + C_+(\mu w)$$

$$Q_- = I + C_-(\mu w)$$

$$Q = I + C(\mu w) \quad \text{on } \mathbb{C} \setminus \Sigma,$$

where $w = w_+ + w_-$. Suppose $detb_- = detb_+$. Then Q provides the unique solution of the following Riemann-Hilbert problem.

(*)
$$Q_+(z) = Q_-(z) b_-^{-1}(z) b_+(z), \;\; z \in \Sigma,$$
$$lim Q(z) = I,$$

as $z \to \infty$ in compact subsets of $\mathbb{C} \setminus \Sigma$.

PROOF: Uniqueness: The jump matrix in has determinant 1. Hence $det\ Q$ is holomorphic. It is also equal to 1 at infinity. By Liouville's theorem it is constant, and hence equal to 1 everywhere. In particular Q^{-1} exists. Now, suppose there was another solution, say $\tilde{Q}$. Then $\tilde{Q}Q^{-1}$ exists, is holomorphic, and is I at infinity. By Liouville, it has to be identically I.

To show that Q defined above solves (*) note that

$$Q_+(I + w_+)^{-1} = (I + C_+(\mu w))(I + w_+)^{-1} =$$
$$(I + C_+(\mu w_+) + C_+(\mu w_-))(I + w_+)^{-1} =$$
$$(I + \mu w_+ + C_-(\mu w_+) + C_+(\mu w_-))(I + w_+)^{-1},$$
$$since\ C_+ - C_- = I,$$
$$= (I + \mu w_+ + C_w \mu)(I + w_+)^{-1},\ by\ the\ definition\ of\ C_w,$$
$$= (\mu w_+ + \mu)(I + w_+)^{-1} = \mu,\ by\ (12).$$
$$Similarly\ Q_-(I - w_-)^{-1} = (I + C_-(\mu w))(I - w_-)^{-1} =$$
$$(I + C_-(\mu w_+) + C_-(\mu w_-))(I - w_-)^{-1} =$$
$$(I - \mu w_- + C_-(\mu w_+) + C_+(\mu w_-))(I - w_-)^{-1} =$$
$$(I - \mu w_- + C_w \mu)(I - w_-)^{-1} = (\mu - \mu w_-)(I - w_-)^{-1} = \mu.$$

Hence $Q_+(I + w_+)^{-1} = Q_-(I - w_-)^{-1}$ and thus $Q_-^{-1} Q_+ = (I - w_-)^{-1}(I + w_+) = (b_-)^{-1} b_+$ and thus $Q_+ = Q_-(b_-)^{-1} b_+$.

The above theory is useful if one seeks asymptotics of the solution of a Riemann-Hilbert problem with respect to a parameter [e.g. time t in (8)-(9)]. The idea is to "deform" the factorization problem (*) into one that can be solved explicitly, in a series of steps each of which involves either an extension or a deletion of a part of the factorization contour, or a substitution of $w_\pm$ by approximate (for large times) matrices, or an appropriate conjugation. The choice of the contour extensions or deletions depends on the phase of the exponentials appearing in $b_\pm$. For each step, the final factorization problem is an approximation to the initial one in the sense that the solution as given by the corresponding integral formula is close to the exact solution of (*).

There are basically two kind of deformations used. Deformations of a "geometric type" involve a contour deformation. This is allowed if the jump matrix on that contour is analytic. These deformations are exact, not approximate. There are also "analytic" type deformations that involve perturbing the formula

$$Q = I + C(\mu w) = I + \int_\Sigma \frac{(I - C_w)^{-1}(I)(s)w(s)}{2\pi i(s-z)}ds$$

with respect to w while keeping the contour Σ fixed. Such deformations are only approximate, with respect to some underlying parameter (like t for example).

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