Local splitting families of hyperelliptic pencils, I.

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Introduction. The aim of this paper is to construct splitting families of degenerations of hyperelliptic curves so that the original complicated singular fiber decompose into several simple singular fibers by these local deformations. Moreover we define certain invariants of germs of singular fibers and study their properties. The historical background of our study is the following:

In 1977, Moishezon [Mo] showed that any degenerate elliptic curve splits by a local deformation into several singular fibers of only one of two types. One is a rational curve with a node (type I_1 in Kodaira's table [Kod1]) and the other is a multiple fiber whose reduced scheme is a nonsingular elliptic curve (type ${}_mI_0$). Y. Matsumoto [Ma1] and Ue [Ue1] studied the similar problem for good torus fibrations which are topological analogue of elliptic fibrations in some sense. These results are available for studying diffeomorphism classes of global elliptic surfaces or torus fibrations ([FM,Ma2,Ue2]etc.).

In 1988, Horikawa [Ho4] showed that any degenerate genus two curve splits into several fibers of type I_1 in his table [Ho2] (a stable curve with two elliptic components with a node) and several fibers of type 0 (the fibers "arising from rational double points" in some sense, which include many topologically different types of fibers). At the same time, he remarked that the sum of a certain invariant (Horikawa index for genus 2 in our terminology) is conserved in his splitting families.

In 1990, inspired by the discussion with Xiao Gang, Miles Reid [Re] proposed this type of problem – Morsification for fiber germs in his terminology – from the viewpoint of the study of relative canonical algebra for pencils of curves such as Mendes-Lopes'es work [Men]. Especially he pointed out the importance to seek for "atomic fibers", i.e. the units of degenerations.

On the other hand, Nakayama [Na] studied certain two-parameter splitting degenerations of elliptic curves from the viewpoint of threefold minimal model theory.

Now in this paper, we show that any degenerate hyperelliptic curve splits into several singular fibers belong to very small classes called type 0_0, class I and class II defined in §1 step by step by preserving the relative hyperelliptic involution of pencils. (Theorem 3.3.) The fiber of type 0_0 is an irreducible stable curve with one node. The fiber in class I is a stable curve with two components with two nodes. The fiber in class II is defined by a certain strong condition for the germs of singularities of the branch curve arising from it's relative hyperelliptic involution. We only classify them for small genus case, i.e. it contains one, three and three types for genus 2, 3 and 4 respectively (Proposition 1.7).

The method of the construction of our splitting families is as follows: We first construct a one-parameter family of normal surfaces which is a double cover over a family of local projective line bundles, and second we resolve it simultaneously. A bad singular fiber corresponds to bad singularities of the branch curve of the double cover which pass through the projective line over the origin. Therefore we first construct a splitting family of branch curve singularities so that bad singularities decompose into several more simple singularities, and then we lift it to a family of local hyperelliptic pencils.

The idea of our construction of local analytic splitting family of this curve singularity essentially comes from the well-known method for the Morsification of singularity so called A'Campo–Gusein Zade theory [Ac,G], that is, a perturbation method along the way of it's resolution process.

The method of the lifting to the family of pencils is to write down the equation itself case by case according to the situation of the family of branch curves. All our methods are completely explicit so that we can describe which types of singular fibers appear in our splitting families.

In the last section, we define two invariants for the singular fiber of a hyperelliptic pencil, i.e. the Horikawa index and the local signature. Moreover we show that the sum of these invariants are conserved in our splitting families. (Theorem 4.10.)

The Horikawa index is defined to be the fractional number which measure the contribution of the fiber to the distance from the geographical lower bound of existence of hyperellitic pencils of fixed genus. From the viewpoint of singularity theory, it measures the total badness of singularities of the branch curve apart from simple (ADE) singularities.

The local signature is defined directly from the Horikawa index and the topological Euler contribution, and the usual global signature of a compact surface with hyperelliptic pencil is the sum of these local signatures. (Proposition 4.7.) For the genus 2 case, Matsumoto [Ma4] observed this fact by using the speciality of Meyer's signature cocycle [Mey]. (Note that the general pencil does not satisfy this localized property of the signature, we refer to [At2, Kod3, Hi2] etc.) Matsumoto also calculated the local signature of two types of singular fibers of Lefschetz fibrations, which coincides with the "atomic fibers" of genus 2 by [Ho4] and a special case of Corollary 4.11. Therefore the global signature of genus 2 fibration is written as a simple linear combination of two terms as in Corollary 4.13, which give a precise "third" solution of the the negative signature problem posed by Persson [P2]. (Note that Xiao[X1] and Ueno[U] solved this problem.) Corollary 4.13 gives us similar information for genus 3 and 4 cases.

We lastly remark that the indecomposability of the singular fibers, the conservation of invariants of the fiber for general splitting families and the decomposition rule of homotopical monodromy for splitting degenerations will be forthcoming themes.

1 Special classes of germs of singular fiber

In this section, we define very special classes of degenerations of hyperelliptic curves. We will show in §3 that any degeneration of hyperelliptic curves splits by local deformations into several fibers in these classes step by step.

1.1. Let S be a 2-dimensional complex manifold and $\Delta_\epsilon = \{t \in \mathbf{C} \,; |t| < \epsilon\}$ be a complex 1-dimensional open disk with radius ϵ. Let $f : S \to \Delta_\epsilon$ be a relatively minimal proper surjective holomorphic map. If f satisfies the conditions
(i) For any $t \in \Delta_\epsilon \setminus \{0\}$, the fiber $f^{-1}(t)$ is a nonsingular hyperelliptic curve of genus $g \geq 2$,
(ii) $F = f^{-1}(0)$ is a singular curve,
we call f a degeneration of hyperelliptic curves of genus g, and F a singular fiber of f.

Now let $f' : S' \to \Delta_{\epsilon'}$ be another degeneration of hyperelliptic curves of genus g. If there is a positive real number ϵ'' with $\epsilon'' \leq \epsilon$ and $\epsilon'' \leq \epsilon'$ such that the restrictions $f|_{\Delta_{\epsilon''}} : S|_{\Delta_{\epsilon''}} \to \Delta_{\epsilon''}$ and $f'|_{\Delta_{\epsilon''}} : S|_{\Delta_{\epsilon''}} \to \Delta_{\epsilon''}$ are isomorphic to each other as analytic fiber spaces, then we call f and f' are equivalent. The equivalence class of f, which we write $[f, F]$, is called the germ of singular fiber associated with f.

The study of the germ $[f, F]$ is translated into the study of the singularities of the branch curve of the double cover arising from the relative hyperelliptic involution in the following way: Let $\pi : W = \mathbf{P}^1 \times \Delta_\epsilon \to \Delta_\epsilon$ be a trivial $\mathbf{P}^1$-bundle and set $\Gamma_t = \pi^{-1}(t)$ for $t \in \Delta_\epsilon$. Let B be a reduced divisor on W which satisfies
i) $B\Gamma_t = 2g + 2$ $(t \in \Delta_\epsilon)$,
ii) The *bad points* of B are at most on Γ_0, where the bad point P of B is the point which satisfies $I_P(B, \Gamma_{\pi(P)}) \geq 2$ (i.e. a singular point of B or a tangential point to $\Gamma_{\pi(P)}$).

Let B' be another divisor on $W' = \mathbf{P}^1 \times \Delta\epsilon'$ which satisfies the same conditions as in the above i) and ii). We say (W, B) and (W', B') are equivalent if there is a positive real number ϵ'' with $\epsilon'' \leq \epsilon$ and $\epsilon'' \leq \epsilon'$ such that the restrictions $(W, B)|_{\Delta_{\epsilon''}}$ and $(W', B')|_{\Delta_{\epsilon''}}$ coincide with each other modulo a finite succession of elementary transformations in the sense of [Ho2, §2]. We write $[W, B]$ the equivalence class of (W, B) and call it the *germ of branch curve* of a degeneration of hyperelliptic curves. It is well-known that there exists a one-to-one correspondence between the set of germs of singular fibers $\{[f, F]\}$ and those of branch curves $\{[W, B]\}$ (cf. [Ho2]).

We remark that there exist examples such that $[f, F]$ and $[f', F']$ are mutually different as germs of singular fiber but F and F' have the same weighted dual graph, for instance compare $2I_{0-m}$ with II^*_{n-0} in Namikawa-Ueno's table [NU,p.159,172] for $g = 2$ case. We also refer to Matsumoto-Montesinos [MM, §8] for its topological meaning. Therefore in this paper, we always express the germ of fiber by a suitable representative of the corresponding branch curve.

We introduce some notations for a germ of branch curve (W, B). We set

$$\widetilde{B} = B - \Gamma_0 \;\text{ if }\; \Gamma_0 \subset B, \quad \widetilde{B} = B \;\text{ otherwise},$$

and call $\widetilde{B}$ the *horizontal part* of B. Let

$$W = W_0 \overset{\tau_1}{\longleftarrow} W_1 \overset{\tau_2}{\longleftarrow} \cdots \overset{\tau_r}{\longleftarrow} W_r \tag{1.1.1}$$

be a succession of blow-ups such that the center $P_{i-1} \in W_{i-1}$ of τ_i $(1 \le i \le r)$ is infinitely near to a bad point of B. Let $E_{i,i}$ be the exceptional curve for τ_i. For $0 \le i < j \le r$, let $E_{i,j}$ be the proper transform of $E_{i,i}$ by the map $\tau_{i,j} = \tau_{i+1} \circ \cdots \circ \tau_j : W_j \longrightarrow W_i$. We also write $E_{0,j}$ the proper transform of Γ_0 by $\tau_{0,j}$. Then the reduced scheme of the total transform of Γ_0 by $\tau_{0,j}$ is written by $\mathbf{E}_j = \sum_{i=0}^{j} E_{i,j}$. We put $B_0 = B$ and let B_i $(1 \le i \le r)$ be the *even* proper transform of B_{i-1} in the sense of [Ho1, P1], that is, $B_i = \tau_i^* B_{i-1} - 2[m_i/2]E_{i,i}$ where m_i is the multiplicity of B_{i-1} at P_{i-1}.

For choosing one representative among the equivalence class of it in a normalized form in some sense, the following lemma is useful:

Lemma 1.2. ([X3,Lemma 5.1.2],[Ho3], [P2]) *Let $[f, F]$ be a germ of singular fiber of degeneration of hyperelliptic curves of genus g. Then there exists a representative (W, B) of the branch curve of $[f, F]$ which satisfies the following condition $(*)$;*
$(*)$ *For any bad point P of B, the multiplicity of the horizontal part $\mathrm{mult}_P\widetilde{B}$ is not greater than $g + 1$.*
Moreover such (W, B) is uniquely determined by $[f, F]$ except for the following case $(**)$;
$(**)$ *There exists just two bad point P_1, P_2 of B such that $\mathrm{mult}_{P_1}\widetilde{B} = \mathrm{mult}_{P_2}\widetilde{B} = g + 1$.*

We call (W, B) which satisfies $(*)$ the germ of the *normalized branch curve.* Note that the branch curve is not necessarily normalized in this paper, otherwise mentioned.

1.3. Now we define very special classes of germs of singular fiber.

1.3.1. Assume B is smooth on W, meet nonsingular transversally to Γ_0 except for one point P and is contact of order two to Γ_0 at P. Then the associated singular fiber F is an irreducible curve of genus $g - 1$ with one ordinary double point. (The genus of a component means that of its normalization.) If the normalized branch curve (W, B) has the above property, we call $[f, F]$ a fiber germ of *type* 0_0.

1.3.2. Assume B does not contain Γ_0 and meet nonsingular transversally to Γ_0 except for one point P and has an ordinary singularity at P of multiplicity $2g' + 2$, where g' is an integer with $1 \le g' \le [(g - 1)/2]$. Then F has an irreducible decomposition $F' + F''$ where F' and F'' are nonsingular curves of genus g' and $g - g' - 1$ respectively meeting each other transversally at two points, whose self-intersection numbers in S are $(F')^2 = (F'')^2 = -2$. If the normalized branch curve (W, B) has the above property, we say that $[f, F]$ belongs to *class I*. In fact, this class contains $[(g - 1)/2]$ types according the number g'.

1.3.3. Assume B contains Γ_0 and the following conditions (a), (b), (c) hold:

(a) We produce the succession of blow-ups (1.1.1) such that any singularity of the even proper transform B_r is ordinary. Note that such a process is uniquely determined by B up to the order of the choice of the centers of blow-ups. Then we have:
(a1) B_r contains $\mathbf{E}_r$,
(a2) Any singularity of B_r has even multiplicity,
(a3) For any $0 \leq i \leq r$, any singularity on B_i with even multiplicity is ordinary.
(b) If a bad point P on B is not an ordinary singularity, then at least one local branch $B^{(1)}$ of B at P is tangential to Γ_0, that is, $I_P(B^{(1)}, \Gamma_0) > \mathrm{mult}_P B^{(1)}$.
(c) If B has a ordinary double point at Q, then the horizontal part $\widetilde{B}$ meets Γ_0 at another point R such that $\widetilde{B}$ is contact to Γ_0 at R of order at least three (that is, after the three times blow-ups at the infinitely near point of R, at least one component of the proper transform of $\widetilde{B}$ and Γ_0 still meet each other.)

We say a germ of fiber $[f, F]$ belong to *class II* if *any* representative of the branch curve (W, B) with $B \supset \Gamma_0$ of $[f, F]$ has the above properties.

1.4. Let (C, P) be an analytic germ of a plane curve singularity. If (C, P) satisfies the following condition for some $m \geq 2$ and $n \geq 2$, we call it an *m^n point*; C has m irreducible local branches $C^{(1)}, \cdots, C^{(m)}$ such that $C^{(i)}$ is nonsingular and $I_P(C^{(i)}, C^{(j)}) = n$ for any $1 \leq i < j \leq m$. (The typical equation is $x^m + y^{mn} = 0$.)

LEMMA 1.5 *Let P be a bad point of the branch curve B which defines a singular fiber in class II.*
(i) *If $\mathrm{mult}_P B$ is even, then (B, P) is an ordinary singularity.*
(ii) *If $\mathrm{mult}_P B = 3$, then (B, P) is a 3^2 point.*
(iii) *If $\mathrm{mult}_P B = 5$, then (B, P) is a 5^2 point.*

PROOF The assertion (i) is clear by the definition. We consider (ii) and (iii). Let $B^{(1)}, \cdots, B^{(k)}$ be the set of local analytic components of B at P. Let $\tau_1 : W_1 \longrightarrow W$ be the blow-up at P and we put $P_1 = E_{0,1} \cap E_{1,1}$. Let $\widetilde{B}^{(i)}$ $(1 \leq i \leq k)$ be the proper transform of $B^{(i)}$ by τ_i.

Assume $\mathrm{mult}_P B = 3$. We suppose $k = 1$. Since $B^{(1)}$ is tangential to Γ_0 by the condition (b), (B_1, P_1) is a triple or a quadruple point. But (B_1, P_1) is not an ordinary quadruple point, and so it is a triple point by the condition (a3). Then the even proper transform B_r in (1.3.3.a) has an ordinary triple point, which contradict the condition (a2). Suppose $k = 2$. If one of $B^{(i)}$ $(i = 1, 2)$ is tangential to Γ_0 and the another is not, then B_r also has an ordinary triple point. Hence both of $B^{(i)}$ $(i = 1, 2)$ are tangential, and so P_1 is a quadruple point, which must be ordinary. Therefore we have the assertion (ii).

Assume $\mathrm{mult}_P B = 5$. We first suppose that all of $B^{(i)}$ $(1 \leq i \leq k)$ are tangential to Γ_0. If $k = 4$, then the multiplicity at P_1 of B_1 is 6, and so (B_1, P_1) is ordinary. Therefore (B, P) is a 5^2 point. We will show that $1 \leq k \leq 3$ is impossible.

Assume $k = 3$. We may suppose $\mathrm{mult}_P B^{(1)} = 2$ and the other two are nonsingular. Then $\widetilde{B}^{(1)}$ is nonsingular, otherwise (B_1, P_1) is a non-ordinary 6-pl point. Therefore

$(B^{(1)}, P)$ is a simple cusp, i.e. $t^2 + x^3 = 0$. Then after the blow-up at P_1, B_3 has an ordinary triple point at $P_2 = E_{1,2} \cap E_{2,2}$. This is impossible.

Assume $k = 2$. We first consider the case where $\mathrm{mult}_P B^{(1)} = 3$ and $B^{(2)}$ is nonsingular. Since P_1 cannot be a 6-pl point, we have $\mathrm{mult}_{P_1} \widetilde{B}^{(1)} \leq 2$. Then $(B^{(1)}, P)$ is either $t^3+x^4 = 0$ or $t^3 + x^5 = 0$. In each case, after 2-times blow ups, (B_2, P_2) has a triple point which is not a 3^2 point. This is impossible. In the case where $\mathrm{mult}_P B^{(1)} = \mathrm{mult}_P B^{(2)} = 2$, the argument is similar. $k = 1$ is also impossible by a similar argument.

It remains to consider that at least one of the components, say $B^{(1)}$, is not tangential. Then $\widetilde{B}^{(1)}$ pass through a point $Q \in E_{1,1} \backslash E_{0,1}$. Since $\mathrm{mult}_Q B_1 \leq 4$ by the condition (b), it follows from (i) and (ii) that (B_1, Q) is an ordinary quadruple point or a 3^2 point or ordinary double point. In the first case, B_1 has a triple point at P_1 which is not a 3^2 point. This is impossible. The second case contradicts the condition (b). In the third case, we produce the elementary transformation so that the image of $E_{1,1}$ become a new fiber. Then the condition (c) is not satisfied. Therefore we have the assertion (iii). Q.E.D.

1.6. For low genus cases $2 \leq g \leq 4$, we have the classification of fibers in class II. We prepare some notations for writing them. We express the information of the set of bad points of a representative of the branch curve B inside the bracket $\{\ \}$ by writing that of $\widetilde{B}$. $\{m, \cdots\}$ means that B has an ordinary $(m+1)$-pl point. $\{m^n, \cdots\}$ means that B has an $(m+1)^n$ point. We express the weighted dual graph of the minimal singular fiber as follows: The component of the singular fiber is always nonsingular in our list, The Gothic numbers $\mathbf{O}$, $\mathbf{I}$, $\mathbf{II}$ $\cdots$ mean that the genus of the components are 0, 1, 2, $\cdots$ respectively. The coefficient means the multiplicity of the component. The lower suffix means the self-intersection number of the component, but we omit it if it is -2. The symbol "—" means "intersects transversally".

For instance, $\{3, 3, 2^2\}$ means that the bad points of B are two ordinary quadruple points and one 3^2 point.(See the Figure 1.) This really appears as (g=4, iii) in Proposition 1.7. In this case, the even resolution consists of 4 times blow-ups , and the minimal singular fiber is $F = 2F_0 + F_1 + F_2 + 2F_3$ where $g(F_0) = 0$, $g(F_1) = g(F_2) = g(F_3) = 1$, $F_0F_1 = F_0F_2 = F_0F_3 = 1$, $F_0^2 = F_1^2 = F_2^2 = -2$ and $F_3^2 = -1$. This is easily seen by the usual double covering method (cf.[P1]).

Proposition 1.7. *The classification of singular fiber germs of class II for genus* $2 \leq g \leq 4$ *are the followings: We have one, three and three types for* $g = 2, 3$ *and* 4 *respectively;*

$(g = 2)$ (i) $\{3, 3\}$ $\mathbf{I}_{-1} - \mathbf{I}_{-1}$

$(g = 3)$ (i) $\{3, 5\}$ $\mathbf{I}_{-1} - \mathbf{II}_{-1}$ (ii) $\{2^2, 2^2\}$ $2\mathbf{I}_{-1} - 2\mathbf{O} - 2\mathbf{I}_{-1}$ (iii) $\{4^2\}$ $2\mathbf{II}_0$

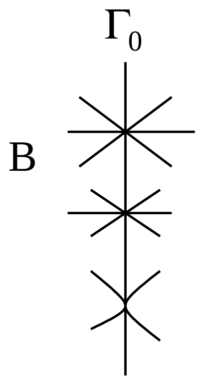

Figure 1: branch curve of type $\{3,3,2^2\}$

$(g=4)$ (i) $\{3,7\}$ $\mathbf{I}_{-1} - \mathbf{III}_{-1}$ (ii) $\{5,5\}$ $\mathbf{II}_{-1} - \mathbf{II}_{-1}$ (iii) $\{3,3,2^2\}$

$$\begin{array}{l} \mathbf{I} \\ | \\ 2\mathbf{O} - 2\mathbf{1}_{-1} \\ | \\ \mathbf{I} \end{array}$$

Proof Let (W, B) be the normalized branch curve of the germ of fiber. If B contains Γ_0, lemma 1.5 easily implies the list in the assertion.

Assume B does not contain Γ_0. We set $B \cap \Gamma_0 = \{P_1, P_2, \cdots, P_k\}$. We may assume that (B, P_1) is a triple or a 5-pl point and B intersect Γ_0 nonsingular transversally at $P_2, \cdots, P_k$. Otherwise, we choose another branch curve (W', B') such that B' contains the fiber by elementary transformations. Then the condition (a3) in 1.3.3 is not satisfied for (W', B').

Now let (W'', B'') be the branch curve obtained by the elementary transformation at P_1. Then B'' contains the new fiber Γ_0''. Let Q be the point on Γ_0 which is the image of $E_{0,1}$ by the contraction map. Since B does not contain Γ_0, the multiplicity of (B'', Q) is even. Therefore (B'', Q) is an ordinary singularity. Other bad points of B'' have their multiplicity at most 6. Then we apply lemma 1.5, and obtain the assertion. Q.E.D.

2 Local analytic splitting of plane curve singularities

Let C be a normal analytic curve defined on an open set $V_0 = \{(X, Y) \in \mathbf{C}^2 \ ; |X| < \epsilon, |Y| < \epsilon, \}$ for a sufficiently small ϵ such that $P = (0, 0)$ is the unique singular point of C. In this section, we explicitly construct certain local analytic deformations $\{C_u\}_{u \in \Delta_\delta}$ of

C such that C_u ($u \neq 0$) has several isolated singular points. The idea is essentially same as the method in A'Campo, Gusein-Zade theory ([Ac, G]).

2.1. Let $C = \sum_{j=1}^{s} C^{(j)}$ be the local analytic irreducible decomposition of C at P. Let $V_0 \xleftarrow{\tau_1} V_1 \xleftarrow{\tau_2} \cdots \xleftarrow{\tau_r} V_r$ be the succession of blow-ups whose centers are infinitely near to P and let $\widetilde{C}_r$ (resp. $\widetilde{C}_r^{(j)}$, $1 \leq j \leq s$) be the proper transform of C (resp. $C^{(j)}$) by $\tau_{0,r} := \tau_1\tau_2\cdots\tau_r$. Let Q be an isolated singular point of $\widetilde{C}_r$.

We first assume that Q is a smooth point of the (reduced) exceptional set $\mathcal{E}_r = \sum_{j=1}^{r} E_j$. Assume Q is on E_{j_0} ($1 \leq j_0 \leq r$). Let $U(x,y)$ be the canonical open coordinate neighborhood on V_r containing Q, that is, $U \cap \mathcal{E}_r$ coincides with the union of the complements of one point of E_{j_0} and that of E_{j_1} ($1 \leq j_0, j_1 \leq r$, $j_0 \neq j_1$) which are defined by $y = 0$ and $x = 0$, respectively. Let $(x,y) = (\alpha, 0)$ ($\alpha \neq 0$) be the coordinate of Q in U. Since the map $(\tau_{0,r})|_U : U \to U_0$ is written by $(x,y) \mapsto (X = x^{r_1}y^{s_1}, Y = x^{r_2}y^{s_2})$ for some non-negative integers r_1, s_1, r_2 and s_2, U is written by

$$U = \{(x,y) \in \mathbf{C}^2 \; ; |x^{r_1}y^{s_1}| < \epsilon, |x^{r_2}y^{s_2}| < \epsilon\}.$$

Let $\widetilde{C}_r^{(1)}, \ldots \widetilde{C}_r^{(k_0)}$ ($k_0 \leq s$) be the components of $\widetilde{C}_r$ which pass through Q (by changing the order of them if necessary). Since ϵ is sufficiently small, we may assume that the support of the divisor $\sum_{j=0}^{k_0} \widetilde{C}_r^{(j)}$ on V_r is contained in U. Let

$$g(x-\alpha, y) = \prod_{j=1}^{k_0} g^{(j)}(x-\alpha, y) = 0$$

be the equation of $\sum_{j=0}^{k_0} \widetilde{C}_r^{(j)}$ on U. Now let $\mathcal{D}$ be the divisor on $V_r \times \Delta_\delta$ which satisfies the following conditions:
(i) The support of $\mathcal{D}$ is contained in $U \times \Delta_\delta$ and the equation of $\mathcal{D}$ is given by

$$g(x - \alpha - \varphi(u), y - \psi(u)) = 0 \tag{2.1.1}$$

where $\varphi(u)$, $\psi(u)$ are generic holomorphic functions of u which satisfy $\varphi(0) = \psi(0) = 0$,
(ii) $\mathcal{D}$ does not meet the divisor $\mathcal{D}' = \sum_{j=k_0+1}^{s} \widetilde{C}_r^{(j)} \times \Delta_\delta$ on $V_r \times \Delta_\delta$,
(iii) For any $u \in \Delta_\delta$, let $\mathcal{D}_u$ be the restriction of $\mathcal{D}$ to $V_r \times \{u\}$ and we consider it as a curve on V_r. Then $\mathcal{D}_u$ intersects $\mathcal{E}_r$ transversally at $I_Q(\widetilde{C}_r, E_{j_0})$ mutually distinct points. Moreover the point $Q_u = \{(x,y) = (\alpha + \varphi(u), \psi(u))\}$ on U is the unique singular point of $\mathcal{D}_u$.

The existence of $\mathcal{D}$ is clear by taking ϵ and δ sufficiently small. Now we define a divisor M on $V_0 \times \Delta_\delta$ by

$$M = (\tau_{0,r})_*\mathcal{D} + \sum_{j=k_0+1}^{s} \widetilde{C}^{(j)} \times \Delta_\delta.$$

The natural fibration $\boldsymbol{\pi} : M \to \Delta_\delta$ has the following properties:
(a) $M_0 = \boldsymbol{\pi}^{-1}(0)$ coincides with C.

(b) For any $u \in \Delta_\delta \setminus \{0\}$, the fibers M_u has just two isolated singular points at P and $R_u = \tau_{0,r}(Q_u)$. Moreover the germ of singularity (M_u, R_u) coincides with $(\widetilde{C}_r, Q)$.

We call this situation that (C, P) *splits into* (C_u, P) *and* (C_u, R_u) *by the perturbation at the infinitely near point* Q, and we call the fibration $\boldsymbol{\pi}$ *the splitting family of* (C, P) *of type A*.

EXAMPLE 2.2. The singularity $x^3 + y^n = 0$ $(0 < 3k < n)$ splits into $x^3 + y^{3k} = 0$ and $x^3 + y^{n-3k} = 0$ by the perturbation at the infinitely near point which appear after k times blow- ups.

2.3. Assume Q is a double point of $\mathcal{E}_r$. We construct local deformations of (C, p) by the following two different ways:

(1) We assume $Q = E_{j_0} \cap E_{j_1}$ and we put $\alpha = 0$ in the same situation as in 2.1. Then we can construct the fibration $\boldsymbol{\pi} : M \to \Delta_\delta$ such that the only one condition (iii) is replaced by the following (iii)':
(iii)' For any $u \in \Delta_\delta$, $\mathcal{D}_u$ intersects $\mathcal{E}_r$ only at nonsingular points of $E_{j_0} + E_{j_1}$. $\mathcal{D}_u$ intersects E_{j_k} $(k = 0, 1)$ transversally at $I_Q(\widetilde{C}_r, E_{j_k})$ mutually distinct points. Moreover the point $Q_u = \{(x, y) = (\varphi(u), \psi(u))\}$ on U is the unique singular point of $\mathcal{D}_u$.
We also call $\boldsymbol{\pi}$ a splitting family of *type A*.

(2) On the equation (2.1.1), we put $\alpha = 0$ and $\psi(u) \equiv 0$. Then $\mathcal{D}_u$ $(u \neq 0)$ has a singularity at $Q_u = \{(x, y) = (\psi(u), 0)\}$ on E_{j_0} and intersects E_{j_1} at $I_Q(\widetilde{C}_r, E_j)$ distinct points transversally. We construct $\boldsymbol{\pi} : M \to \Delta_\delta$ in the same way. Then M_u $(u \neq 0)$ has one isolated singular point at P such that the germ (M_u, P) is different from (C, P), because their embedded resolution processes are distinct each other. We call π a splitting family of *type B*.

EXAMPLE 2.4. $x^5 + y^3 = 0$ splits into $(x^4 + y^2)(x + y) = 0$ as follows: After 2 times blow-ups, the proper transform become nonsingular which meet E_1 transversally and contacts of order 2 to E_2. We perturb it to the direction of generic point of E_1, and then blow down 2 times.

REMARK 2.5. The above deformations are not necessarily well-defined in the algebraic category, because the local analytic reducibility does not induce the algebraic reducibility. (This was pointed out to the authors by Prof. Miyanishi.)

3 Splitting families of degenerations

In this section, we construct splitting families of degenerations of hyperelliptic curves. Our aim is not only to prove the existence of splitting families but also to describe explicitly the germs of singular fibers in our families.

Definition 3.1. *Let $[f, F]$ be a germ of singular fiber of a degeneration of hyperelliptic curves of genus g. Assume there exist positive real numbers ϵ and δ, a 3-dimensional complex manifold Z and a flat surjective holomorphic map $h : Z \to \Delta_\epsilon \times \Delta_\delta$ such that*
(i) *The fibration $h_0 : Z_0 \to \Delta_\epsilon \times \{0\}$ arising from the restriction of h over $0 \in \Delta_\delta$ has an unique singular fiber over $0 \in \Delta_\epsilon$ such that the germ of fiber $[h_0, h_0^{-1}(0)]$ coincide with $[f, F]$,*
(ii) *For any $u \in \Delta_\delta \setminus \{0\}$, the fibration $h_u : Z_u \to \Delta_\epsilon \times \{u\}$ has l $(l \geq 2)$ singular fibers. The number l is independent on u.*
Then we call h a splitting family of $[f, F]$.

Definition 3.2. *Let $\mathbf{G}$ be a certain subset of the set of all germs of singular fiber of degenerations of hyperelliptic curves of genus g. We say the germ $[f, F]$ is reduced to $\mathbf{G}$ via several splitting families if the following conditions are satisfied:*

First $[f, F]$ has splitting family $h : Z \to \Delta_\epsilon \times \Delta_\delta$. If any germ of singular fiber $[h_u, h_u^{-1}(t)]$ $(t \in \Delta_\epsilon \times \{u\})$ of any general fibration $h_u : Z_u \to \Delta_\epsilon \times \{u\}$ $(u \in \Delta_\delta \setminus \{0\})$ belongs to $\mathbf{G}$, then we stop our reducing process. Assume some of them do not belong to $\mathbf{G}$. Let $[h_u, h_u^{-1}(t_0)]$ $(t_0 \in \Delta_\epsilon \times \{u\})$ be any germ of singular fiber of h_u which does not belong to $\mathbf{G}$. Then $[h_u, h_u^{-1}(t_0)]$ has a splitting family $h' : Z' \to \Delta_{\epsilon'} \times \Delta_{\delta'}$. If any germ of singular fiber of any general fibration of h' belongs to $\mathbf{G}$, then we stop our process. If not, the germ which does not belong to $\mathbf{G}$ has also a splitting family.

Then this process terminate for finitely many times, that is, any germ of singular fiber of any general fibration of the terminating splitting families belong to $\mathbf{G}$

The main result of this paper is the following:

Theorem 3.3. *Let $\mathbf{G}$ be the set of the union of the germs of singular fiber of type 0_0, classes I and II defined in §1. Then any germ of singular fiber of a degeneration of hyperelliptic curves is reduced to $\mathbf{G}$ via several splitting families.*

Corollary 3.4. *Any degeneration of hyperelliptic curves of genus two (resp. three, resp. four) is reduced via several splitting families to germs of singular fibers of two types (resp. five types, resp. five types) whose tables are listed in 1.3.1, 1.3.2 and proposition 1.7.*

For the proof of theorem 3.3, we start to prepare some lemmas. Let (W, B) be a representative of the branch curve of the germ of fiber $[f, F]$, where $W = \mathbf{P}^1 \times \Delta_\epsilon$ for sufficiently small ϵ. Let P be a bad point of B. Let $W = W_0 \xleftarrow{\tau_1} W_1 \xleftarrow{\tau_2} \cdots \xleftarrow{\tau_r} W_r$ be a succession of blow-ups at infinitely near points of P. We use the same notations as in 1.1.

Lemma 3.5. *Assume there is a component $E_{j_1,r}$ $(0 \leq j_1 \leq r)$ and a point Q on $E_{j_1,r}$ such that (i) Q is not a double point of $\mathbf{E}_r$, (ii) $E_{j_1,r}$ is not contained in B_r and (iii)*

B_r has a singularity at Q or B_r is tangential to $E_{j,r}$ at Q. Then $[f,F]$ has a splitting family.

PROOF Let $B^\sharp$ be the sum of components of B which pass through P. Since ϵ is sufficiently small, $B^\sharp$ does not pass through any other point of Γ_0 and we have $B^\sharp \cap (B - B^\sharp) = \phi$. Now let V_0 be a sufficiently small classical open neighborhood of P on W, and let

$$V_0 \times \Delta_\delta \supset \mathbf{B}^\sharp \xrightarrow{\boldsymbol{\pi}} \Delta_\delta$$

be the splitting family of type A of the germ $(B^\sharp, P)$ arising from the perturbation at Q constructed in 2.1. Since ϵ and δ are sufficiently small and the functions $\varphi(u)$, $\psi(u)$ in 2.1 are general, we may assume that the analytic closure of $B^\sharp$ in $W_0 \times \Delta_\delta$ coincides with $B^\sharp$ itself, which we denote the same symbol $B^\sharp$. Now we define a divisor $\mathbf{B}$ on $\mathbf{W} = W \times \Delta_\delta$ by

$$\mathbf{B} := \mathbf{B}^\sharp + (B - B^\sharp) \times \Delta_\delta.$$

Let $\rho : \mathbf{B} \to \Delta_\delta$ be the natural morphism. We consider the fiber $\mathbf{B}_u = \rho^{-1}(u)$ as a divisor on $W \cong W \times \{u\}$. Then $\mathbf{B}_0$ coincides with B and $\mathbf{B}_u$ $(u \neq 0)$ has the following properties:
i) $\mathbf{B}_u \Gamma_t = 2g + 2$ $(t \in \Delta_\epsilon)$,
ii) $\mathbf{B}_u$ has two bad points P on Γ_0 and R_u on $\Gamma_{\pi(R_u)}$, where the point R_u is described in 2.1.

Since the problem is local for the parameter space, we may assume that there is a line bundle $\mathbf{L}$ on $\mathbf{W}$ which satisfies $[\mathbf{B}] \cong 2\mathbf{L}$. On $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{O}_W \oplus \mathcal{O}_W(\mathbf{L}))$, we construct a double cover $\mu : \mathcal{S} \longrightarrow \mathbf{W}$ branched along $\mathbf{B}$. In order to resolve the singularities on $\mathcal{S}$ and to obtain a desired smooth family, we construct a *relative even resolution* for the family ρ of branch curves in the following way:

By the construction of ρ and Tomari's lemma [As, Lemma 3.8], the first r-times multiplicity sequence of the even resolution of the germ (B_u, P) coincides with $\{m_1, \ldots, m_r\}$ for any $u \in \Delta_\delta$. Now let

$$\mathbf{W} = \mathbf{W}_0 \xleftarrow{\boldsymbol{\tau}_1} \mathbf{W}_1 \xleftarrow{\boldsymbol{\tau}_2} \cdots \xleftarrow{\boldsymbol{\tau}_r} \mathbf{W}_r$$

be the succession of blow-ups such that the center of $\boldsymbol{\tau}_i$ $(1 \leq i \leq r)$ is $\{P_{i-1}\} \times \Delta_\delta$. Let $\mathbf{B}_i = \boldsymbol{\tau}_i^* \mathbf{B}_{i-1} - 2[m_i/2]\mathcal{C}_i$ $(1 \leq i \leq r)$ be the even proper transform of $\mathbf{B}_{i-1}$, where $\mathbf{B}_0 = \mathbf{B}$ and $\mathcal{C}_i$ is the exceptional set for $\boldsymbol{\tau}_i$. Note that $\mathbf{B}_r$ does not contain $E_{j_1,r} \times \Delta_\delta$ by our assumption. Therefore on the coordinate open neighborhood $U \times \Delta_\delta = \{(x,y,z)\}$ of $W \times \Delta_\delta$ described in 2.1, the equation of $\mathbf{B}_r$ is written by $g(x - \alpha - \varphi(u), y - \psi(u)) = 0$. Now let $\mathbf{Q}$ be the curve on $W_r \times \Delta_\delta$ defined by the image of the holomorphic map

$$\begin{array}{ccc} \Delta_\delta & \longrightarrow & U \times \Delta_\delta \subset W_r \times \Delta_\delta \\ u & \longmapsto & (x,y,u) = (\varphi(u), \psi(u), u). \end{array}$$

Then $\mathbf{B}_r$ is equisingular along the locus $\mathbf{Q}$ in the sence that the restricted germs of singularity $\left((\mathbf{B}_r)_{|\pi_r^{-1}(u)}, \mathbf{Q}_{|\pi_r^{-1}(u)}\right)$ have the same even resolution process for any $u \in \Delta_\delta$, where $\pi_r : \mathbf{B}_r \longrightarrow \Delta_\delta$ is the natural map. Therefore we have the similar process for $\mathbf{Q}$. Lastly the singularities of $(B - B^\sharp) \times \Delta_\delta$ are of course simultaneously resolved and we complete the relative even resolution for the family ρ. Namely, by the succession of blow-ups $\boldsymbol{\tau}^\flat : \mathbf{W}^\flat \to \mathbf{W}$, the even proper transform $\mathbf{B}^\flat$ of $\mathbf{B}$ become nonsingular and the restriction of $\boldsymbol{\tau}^\flat$ over any $u \in \Delta_\delta$ coincides with the even resolution of $\mathbf{B}_u$. Since $\mathbf{B}^\flat$ is an even divisor on $\mathbf{W}^\flat$, we can construct the nonsingular double covering $\boldsymbol{\pi} : \mathbf{M}^\flat \to \mathbf{W}^\flat$ which branches along $\mathbf{B}^\flat$ and obtain natural morphisms $\mathbf{M}^\flat \xrightarrow{\rho_1} \Delta_\epsilon \times \Delta_\delta \xrightarrow{\rho_2} \Delta_\delta$.

Now assume a certain fiber of the fibration $\mathbf{M}^\flat_{|(\rho_1\rho_2)^{-1}(u_0)} \longrightarrow \Delta_\epsilon \times \{u_0\}$ contains a (-1)-curve for some $u_0 \in \Delta_\delta$. Then by Kodaira's stability theorem [Kod2], every fiber of $\mathbf{M}^\flat_{|(\rho_1\rho_2)^{-1}(u)} \longrightarrow \Delta_\epsilon \times \{u\}$ contains a (-1)-curve for each $u \in \Delta_\delta$, and those (-1)-curves are simultaneously contracted by Fujiki-Nakano [FN]. Repeating these processes, we reach a family of relatively minimal fibrations $\mathbf{M} \to \Delta_\epsilon \times \Delta_\delta \to \Delta_\delta$. By our construction, this family has the following properties:
(i) $\mathbf{f}_0 : \mathbf{M}_0 \to \Delta_\epsilon \times \{0\}$ has a unique singular fiber over $0 \in \Delta_\epsilon$ whose fiber germ coincides with $[f, F]$.
(ii) For any $u \in \Delta_\delta \setminus \{0\}$, $\mathbf{f}_u : \mathbf{M}_u \to \Delta_\epsilon \times \{u\}$ has just two singular fibers $\mathbf{f}_u^{-1}(0)$ and $\mathbf{f}_u^{-1}(\pi(R_u))$.
Hence we have the assertion. Q.E.D.

Example 3.6. Assume $B \not\supset \Gamma_0$, $B\Gamma_0 = 6$ and B has a triple point $x^3 + t^{6n} = 0$ at $P \in \Gamma_0$ with $I_P(B, \Gamma_0) = 3$ and meet transversally to Γ_0 at other three points. By $2k$-times blowing-ups $(1 \le k \le n-1)$, the infinitely near point Q of P become $x^3 + t'^{6(n-k)} = 0$ and B_{2k} does not contain $E_{2k,2k}$. Then we use the method in Lemma 3.5. $\mathbf{B}_u$ has two bad points $x^3 + t^{6k} = 0$ and $x^3 + t^{6(n-k)} = 0$ at mutually distinct two points as in Example 2.2. (In the Figure 2, the solid line is a component of the branch locus and the dotted line is not so.) The associated family of genus two fibrations $\mathbf{f} : M \to \Delta_\delta$ has the following properties:
$\mathbf{f}_0$ has an unique singular fiber

$$\mathbf{I}_{-1} - \overbrace{\mathbf{O} - \cdots - \mathbf{O}}^{n} - \mathbf{I}_{-1}$$

and $\mathbf{f}_u$ $(u \neq 0)$ has two singular fibers

$$\mathbf{I}_{-1} - \overbrace{\mathbf{O} - \cdots - \mathbf{O}}^{k} - \mathbf{I}_{-1}\,, \quad \mathbf{I}_{-1} - \overbrace{\mathbf{O} - \cdots - \mathbf{O}}^{n-k} - \mathbf{I}_{-1}.$$

Lemma 3.7. *Under the same conditions i), ii) as in Lemma 3.5, we further assume $Q = E_{j_0,r} \cap E_{j_1,r}$ for some $0 \le j_0 < j_1$. Then $[f, F]$ has a splitting family.*

Proof If $E_{j_0,r} \not\subset B$, then we can construct a splitting family as same as in Lemma 3.5 by using the splitting family of type A for (B, P) defined in 2.3 (1).

Assume $E_{j_0,r} \subset B$. Then we use the splitting family of type B for (B, P) defined in 2.3 (2) arising from the perturbation of the branch curve at Q into the generic point of $E_{j_0,r}$,

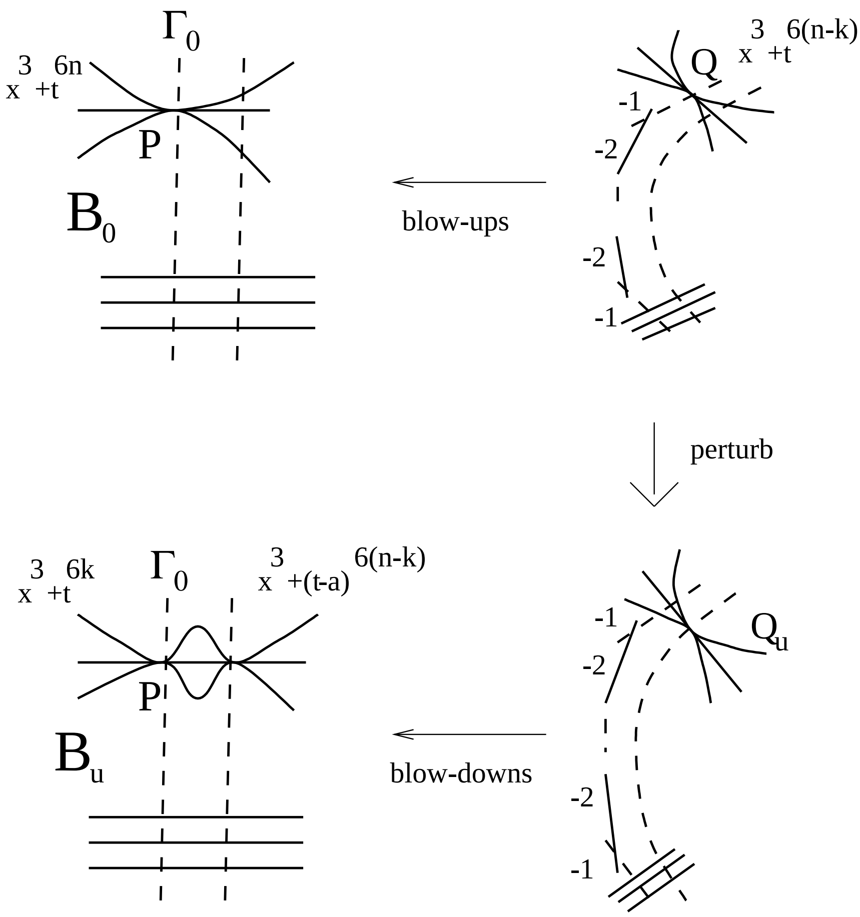

Figure 2: Construction of the branch curve in Example 3.6

and construct a family $\rho : \mathbf{B} \to \Delta_\delta$ of branch curves as in Lemma 3.5. ρ has a relative even resolution and the associated family of relatively minimal hyperelliptic fibrations $\mathbf{f} : \mathbf{M} \to \Delta_\epsilon \times \Delta_\delta \to \Delta_\delta$ has the following properties:
(i) $\mathbf{f}_0 : \mathbf{M}_0 \to \Delta_\epsilon \times \{0\}$ has a unique singular fiber over $0 \in \Delta_\epsilon$ whose fiber germ coincides with $[f, F]$,
(ii) For any $u \in \Delta_\delta \setminus \{0\}$, the fibration $\mathbf{f}_u : \mathbf{M}_u \to \Delta_\epsilon \times \{u\}$ has $k+1$ singular fibers, where $k = I_Q(B_r - E_{j_0,r}, E_{j_1,r})$. One of them is $\mathbf{f}_u^{-1}(0)$ and the others are fibers of type 0_0.

Indeed, we denote by $(\mathbf{B}_u)_i$ $(1 \leq i \leq r, u \neq 0)$ the even proper transform by $\tau_{1,i}$ of $\mathbf{B}_u = \rho^{-1}(u)$ on $W \cong W \times \{u\}$. Since $E_{j_1,r}$ is contracted to $Q' = \tau_{j_1,r}(Q)$ by the map $\tau_{j_1,r} : W_r \to W_{j_1-1}$, the image by $\tau_{j_1,r}$ of the components of $(\mathbf{B}_u)_r$ which intersect to $E_{j_1,r}$ pass through Q'. Since the family ρ arises from the perturbation of type B at $Q = E_{j_0,r} \cap E_{j_1,r}$, $(\mathbf{B}_u)_{j_1-1}$ has at least k nonsingular local components $C^{(1)}, \ldots, C^{(k)}$ at Q' which intersect transversally each other at this point. Moreover $C^{(j)}$ $(1 \leq j \leq k)$ also pass through the point $Q'_u = \tau_{j_1,r}(Q_u)$ where Q_u is defined in 2.3 (2). Now let U' be a small classical open set of W_{j_1-1} containing Q' and Q'_u, and we consider the restricted fibration $\pi_{j-1}|_U : W_{j-1}|_U \to \Delta_\epsilon$. Then it is easy to see that each $C^{(j)}$ $(1 \leq j \leq k)$ contacts of order 2 at a point of certain mutually distinct fibers $\pi|_U^{-1}(t_j)$ $(t_j \neq 0)$. Therefore $\mathbf{B}_u$ also contacts of order 2 at just k points of fibers Γ_{t_j} $(j = 1, \ldots, k)$ of $\pi_0 : W \to \Delta_\delta$, and intersects transversally to the fiber of π_0 at other points. Hence the fibration $\mathbf{f}$ has the above properties. Q.E.D.

Example 3.8. Assume $B \supset \Gamma_0$, $\widetilde{B}\Gamma_0 = 8$, $\widetilde{B}$ has a triple point $x^6 + t^3 = 0$ at P with $I_P(\widetilde{B}, \Gamma_0) = 6$ and intersects transversally to Γ_0 at other two points. Let $\tau_1 : W_1 \to W$ be the blow-up at P. Then $B_1 \not\supset E_{1,1}$, $B_1 \supset E_{0,1}$ and $\widetilde{B}_1 = B_1 - E_{0,1}$ has an ordinary triple point at $Q = E_{0,1} \cap E_{1,1}$. By the method in Lemma 3.7, $\mathbf{B}_u$ $(u \neq 0)$ has two ordinary triple points on Γ_0 and three tangential points on mutually distinct three fibers. (see Figure 3.) The associated family of genus three fibrations $\mathbf{f} : \mathbf{M} \to \Delta_\delta$ has the following properties: $\mathbf{f}_0$ has an unique singular fiber

$$\begin{array}{ccccccc} \mathbf{O} & - & 2\mathbf{O} & - & 2\mathbf{I} & - & \mathbf{O} \\ & & | & & | & & \\ & & \mathbf{O} & & \mathbf{O} & & \end{array}$$

and $\mathbf{f}_u$ $(u \neq 0)$ has four singular fibers. One of them is

$$\begin{array}{ccccc} & & \mathbf{O} & & \\ & & | & & \\ \mathbf{O} & - & 2\mathbf{O} & - & \mathbf{I} \\ & & | & & \\ & & \mathbf{I} & & \end{array}$$

and the other three are of type 0_0.

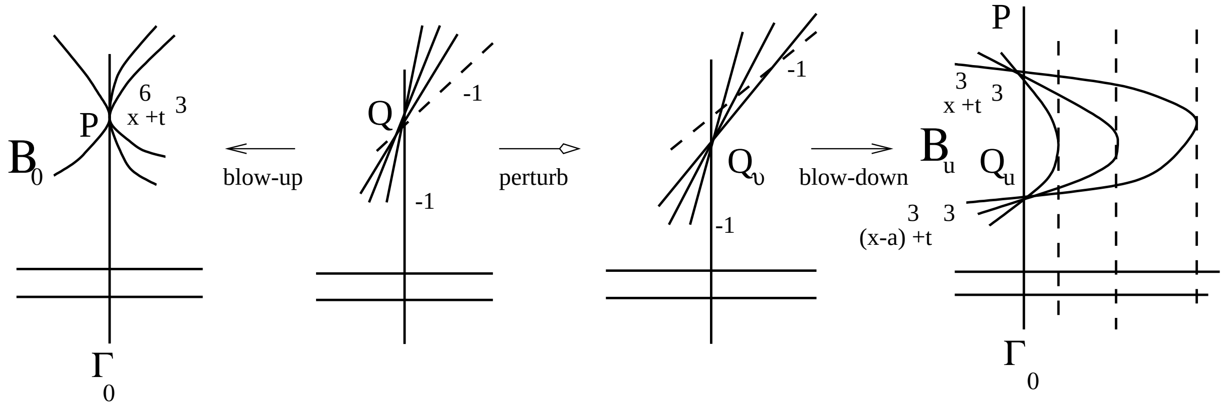

Figure 3: Construction of the branch curve in Example 3.8

LEMMA 3.9. *Assume $B \not\supset \Gamma_0$ and B has at least two bad points on Γ_0. Then $[f, F]$ has a splitting family.*

PROOF Let $P_1, \ldots, P_s$ be bad points of B. Let $h_i : \Delta_\epsilon \to W$ $(i = 1, 2)$ be sections of $\pi : W \to \Delta_\epsilon$ such that the image $h_i(\Delta_\epsilon)$ does not pass through any of $P_1, \ldots, P_s$. Then the pair $(h_1 : h_2)$ defines a homogeneous fiber coordinate of the $\mathbf{P}^1$-bundle π. Let (x, t) be the coordinate of the open set $U := W - h_2(\Delta_\epsilon) \cong \Delta_\epsilon \times \mathbf{C}$, where $x = h_1/h_2$. We may assume that the support of the divisor B on W is contained in U. We set $P_1 = \{(x, t) = (\alpha, 0)\}$ $(\alpha \in \mathbf{C})$. Then the equation of B in U is written by $f_1(x - \alpha, t)f_2(x - \alpha, t) = 0$ where f_1, f_2 are holomorphic functions of (x, t) which satisfy $f_1(\alpha, 0) = 0$ and $f_2(\alpha, 0) \neq 0$. For a sufficiently small δ, we define a divisor $\mathbf{B}'$ on $\mathbf{W} = W \times \Delta_\delta$ by

$$f_1(x - \alpha, t - cu)f_2(x, t) = 0,$$

where c is a complex number whose absolute value is sufficiently small. Then the analytic closure $\mathbf{B}$ of $\mathbf{B}'$ in W coincides with $\mathbf{B}'$ itself. Let $\boldsymbol{\varphi} : \Delta_\delta \longrightarrow \mathbf{W}$ be the map defined by $u \mapsto (x, t, u) = (\alpha, cu, u)$. Then the family of branch curves $\mathbf{B} \to \Delta_\delta$ on W has a relative even resolution by the blow-ups whose centers are infinitely near to the locus $\boldsymbol{\varphi}(\Delta_\delta)$ and $\{P_i\} \times \Delta_\delta$ $(2 \leq i \leq s)$. Then we obtain a family of hyperelliptic fibrations $\mathbf{M} \to \Delta_\epsilon \times \Delta_\delta$ such that $\mathbf{M}_u \to \Delta_\epsilon \times \{u\}$ $(u \neq 0)$ has two singular fibers. One of them has one bad point on its branch curve, and the other has $(s - 1)$ bad points. Q.E.D.

LEMMA 3.10. *We assume (i) $B \supset \Gamma_0$, (ii) the even proper transform B_1 of B by $\tau_1 : W_1 \to W$ contains $\mathbf{E}_1 = E_{0,1} + E_{1,1}$, (iii) $\widetilde{B}_1 = B_1 - \mathbf{E}_1$ does not pass through $P_1 = E_{0,1} \cap E_{1,1}$ and (iv) $\widetilde{B}_1$ intersects both $E_{0,1}$ and $E_{1,1}$. Then $[f, F]$ has a splitting family.*

PROOF We take the coordinate (t_1, x) on the open set $U \cong \Delta_\epsilon \times \mathbf{C}$ of W_1 such that the support of the divisor $\widetilde{B}_1$ on W_1 is contained in U, the equation of $E_{0,1}$ and

$E_{1,1}$ on U are $t_1 = 0$ and $x = 0$ respectively and the map $\pi\tau_1 : W_1|_U \to \Delta_\epsilon$ is given by $(t_1, x) \mapsto t = t_1 x$. Let $(t_1, x) = (\alpha_1, 0), \cdots (\alpha_s, 0), (0, \beta_1), \cdots, (0, \beta_{s'})$ be the coordinates on U of the intersection points of $\widetilde{B}_1$ to $\mathbf{E}_1$ $(\alpha_i \neq 0, \beta_j \neq 0)$. Then the equation of B_1 on U is written by

$$t_1 x \prod_{i=1}^{s} f_i(t_1 - \alpha_1, x) \prod_{j=1}^{s'} f_{s+j}(t_1, x - \beta_j) = 0.$$

Now we define a divisor $\mathbf{B}'$ on the open coordinate neighborhood $U \times \Delta_\delta = \{(t_1, x, u)\}$ by

$$(t_1 x - cu^2) \prod_{i=1}^{s} f_i(t_1 - \alpha_1, x - \frac{cu^2}{\alpha_1}) \prod_{j=1}^{s'} f_{s+j}(t_1 - \frac{cu^2}{\beta_1}, x - \beta_j) = 0$$

where c is a complex number whose absolute value is sufficiently small. Let $\mathbf{B}_1$ be the closure of $\mathbf{B}^1$ in $W_1 \times \Delta_\delta$. Let $\boldsymbol{\varphi}_i$ $(1 \leq i \leq s)$, $\boldsymbol{\psi}_j$ $(1 \leq j \leq s')$, be the map from Δ_δ to $U \times \Delta_\delta \subset W_1 \times \Delta_\delta$ defined by

$$\boldsymbol{\varphi}_i : u \longmapsto (\alpha_i, \frac{cu}{\alpha_i}, u), \quad \boldsymbol{\psi}_j : u \longmapsto (\frac{cu}{\beta_j}, \beta_j, u).$$

Then by the succession of blow-ups at the infinitely near locus of $\boldsymbol{\varphi}_i(\Delta_\delta)$ and $\boldsymbol{\psi}_i(\Delta_\delta)$, we have a relative even resolution of the family of branch curves $W_1 \times \Delta_\delta \supset \mathbf{B} \longrightarrow \Delta_\delta$ except the point P_1. Let $\widetilde{\mathbf{M}} \to W \times \Delta_\delta$ be the double cover branched along $\widetilde{\mathbf{B}}$. Then $\widetilde{\mathbf{M}}$ is smooth except for one point $\widetilde{P}_1$, which is a 3-dimensional A_1-singularity. $\widetilde{P}_1$ is resolved without affecting the general fiber of $\widetilde{\mathbf{M}} \longrightarrow \Delta_\delta$ by the method of Atiyah [At1] (see also [B]). Let $\mathbf{M} \to \Delta_\epsilon \times \Delta_\delta \to \Delta_\delta$ be the family of relatively minimal hyperelliptic fibrations obtained from the relative (–1) contractions of the resolved threefold of $\widetilde{\mathbf{M}}$. Then $\mathbf{M}_u \to \Delta_\epsilon \times \{u\}$ has two singular fibers $\mathbf{f}_u^{-1}(0)$ and $\mathbf{f}_u^{-1}(u)$ for any $u \neq 0$. Q.E.D.

Lemma 3.11. *Assume (i) $B \supset \Gamma_0$, (ii) $\widetilde{B} = B - \Gamma_0$ intersects nonsingular transversally to Γ_0 at a point P and (iii) $\widetilde{B}$ has no intersection point where $\widetilde{B}$ contact of order more than two to Γ_0. Then $[f, F]$ has a splitting family.*

Proof We set $\widetilde{B} \cap \Gamma_0 = \{P, P_1, \ldots, P_s\}$. Let $U \cong \mathbf{C} \times \Delta_\epsilon = \{(x, t)\}$ be an open coordinate on W such that $\widetilde{B} \cap \Gamma_0$ is contained in U, the map is given by $(x, t) \mapsto t$ and the local branch of $\widetilde{B}$ passing through P is given by $x = 0$. Then the equation of B on U is written by $txf_1(x - \alpha_1, t) \cdots f_s(x - \alpha_s, t) = 0$ where $f_i(x - \alpha_i, t)$ is the equation of the equation of $\widetilde{B}$ passing through $P_i = \{(x, t) = (\alpha_i, 0)\}$ $(1 \leq i \leq s)$. Now we define a divisor $\mathbf{B}'$ on $U \times \Delta_\delta$ by the equation

$$(tx - u^2) \prod_{i=1}^{s} f_i \left(x - \alpha_i, t - \frac{u^2}{\alpha_i} + \frac{u^2}{\alpha_i^2}(x_i - \alpha_i) \right) = 0.$$

Then $\mathbf{B}'$ pass through the point $\widetilde{P}_i = (\alpha_i, \frac{u^2}{\alpha_i}, u)$ $(1 \leq i \leq s)$. Since the tangent line of the curve $tx - u^2 = 0$ at $\widetilde{P}_i$ coincides with $t - \frac{u^2}{\alpha_i} + \frac{u^2}{\alpha_i^2}(x_i - \alpha_i) = 0$ (where u is fixed) and

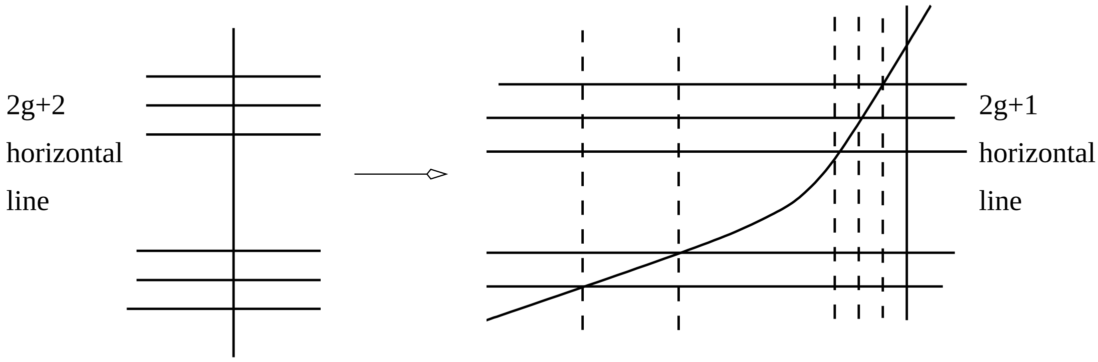

Figure 4: Construction of the branch curve in Example 3.12

this curve contacts of order 2 to this tangent line, the germs of singularities (B, P_i) and $(\mathbf{B}_u, \widetilde{P}_i)$ have the same even resolution process by the assumption iii).

Let $\mathbf{B}$ be the closure of $\mathbf{B}'$ on $W \times \Delta_\delta$. Then $\mathbf{B}$ coincides with $\mathbf{B}' \cup \{\infty\}$ where ∞ is a certain point on Γ_0. The family $W \times \Delta_\delta \supset \mathbf{B} \to \Delta_\delta$ has a relative even resolution except for P, and the A_1-singularity over P on its double cover is resolved by [At]. The induced splitting family of hyperelliptic fibrations has s singular fibers on the general fibration. Q.E.D.

Example 3.12. In the situation of Lemma 3.11, we assume that $\widetilde{B}$ is given by $x(x^{2g+1} - 1) = 0$. Then the divisor $\mathbf{B}'$ is written by $(tx - u^2)(x^{2g+1} - 1) = 0$. Note that if $u \neq 0$, then $\mathbf{B}'_u$ has $2g + 1$ ordinary double points at $(x, t) = (\zeta^\nu, u^2\zeta^{-\nu})$ $(0 \leq \nu \leq 2g)$ where ζ is a primitive $(2g + 1)$-th root of unity. (See Figure 4.) The associated family of genus g fibrations $\mathbf{f} : \mathbf{M} \to \Delta_\delta$ has the following properties:

$\mathbf{f}_0$ has a unique singular fiber of the form $F = 2E_0 + E_1 + \cdots + E_{2g+2}$ where E_j's are nonsingular rational curves with $E_0^2 = -g - 1$, $E_j^2 = -2$, $E_0E_j = 1$ $(1 \leq j \leq 2g + 2)$ and $E_iE_j = 0$ $(1 \leq i < j \leq 2g + 2)$.

$\mathbf{f}_u$ $(u \neq 0)$ has $2g + 1$ singular fibers of the following forms: $F' = E_1 + E_2$ where E_1 is a nonsingular curve of genus $g - 1$ and E_2 is a nonsingular rational curve, $E_1^2 = E_2^2 = -2$ and $E_1E_2 = 2$.

Lemma 3.13. *Assume (i) $B \not\supset \Gamma_0$ and (ii) B has an ordinary double point at a point $P \in \Gamma_0$. Then $[f, F]$ has a splitting family.*

Proof Let $C^{(i)}$ $(i = 1, 2)$ be the components of B passing through P. Then the equation of B on an open set U of W is written by $f_1(x, t)f_2(x, t)g(x, t) = 0$ where $f_i(x, t)$ $(i = 1, 2)$ is the equation of $C^{(i)}$. Let $\mathbf{B}'$ be the divisor on $U \times \Delta_\delta$ defined by

$$\left(f_1(x, t)f_2(x, t) - u^2\right) g(x, t) = 0.$$

Then the closure of $\mathbf{B}'$ in $W \times \Delta_\delta$ similarly induce a splitting family $\mathbf{f} : \mathbf{M} \to \Delta_\epsilon \times \Delta_\delta$ as in Lemma 3.11. The singular fibers of $\mathbf{f}_u$ $(u \neq 0)$ are as follows:
i) if B has another bad points except P, then $\mathbf{f}_u$ has three singular fibers. Moreover two of them are of type 0_0.
ii) If P is the unique bad points of B, then $\mathbf{f}$ has two singular fibers of type 0_0. Q.E.D.

EXAMPLE 3.14 Let $([f, F])$ be the germ given in example 3.12. By combining example 3.12 and lemma 3.13, $([f, F])$ splits step by step into $4g + 2$ fibers of type 0_0. (Compare it with [Ma4, p.140],[Ma5].)

LEMMA 3.15. *We assume B_r has an ordinary singularity Q of odd multiplicity $k \geq 3$. Then $[f, F]$ has a splitting family.*

PROOF We only consider the case where Q is contained in a unique $E_{j_1,r}$ $(1 \leq j_1 \leq r)$ and $E_{j_1} \subset B_r$ (The other cases are similar and we omit them). By the canonical coordinate $U(x, y)$ as in 2.1, B_r is written by $yf_1(x - \alpha, y) \cdots f_{k-1}(x - \alpha, y)g(x, y) = 0$, where $f_i(x - \alpha, y) = 0$ $(1 \leq i \leq k - 1)$ is the equation of each branch curves of $B_r - E_{j_1,r}$ passing through $P = (\alpha, 0)$ and $E_{j_1,r}$ is defined by $y = 0$. Then we define a divisor $\mathbf{B}'$ on $U \times \Delta_\delta$ by

$$yf_1\left(x - \alpha - \varphi(u), y - \psi(u)\right) \prod_{i=2}^{k-1} f_i(x - \alpha, y)g(x, y) = 0$$

for generic holomorphic functions $\varphi(u)$, $\psi(u)$ with $\varphi(0) = \psi(0) = 0$. Then $\mathbf{B}' \subset U \times \{u\}$ has $k - 1$ ordinary double points. The analytic closure $\mathbf{B}$ on $W_r \times \Delta_\delta$ of $\mathbf{B}'$ has the natural properties similar as in 2.1.

Note that the locus $(x, y, u) = (\alpha, 0, u)$ on $U \times \Delta_\delta$ has the same multiplicity sequence for the even resolution for any $u \in \Delta_\delta$. Therefore the family of the branch curves $W_r \times \Delta_\delta \subset \mathbf{B} \to \Delta_\delta$ has the relative even resolution except for the $m - 1$ ordinary double points, which can be also resolved by [At]. Therefore we have a splitting family $\mathbf{f} : M \to \Delta_\delta$ such that $\mathbf{f}_u$ $(u \neq 0)$ has k singular fibers and $k - 1$ of them are of the following type: the branch curve does not contain the fiber and the bad point is an ordinary double point. The fiber F' is written by $F' = A + B$ where A is a nonsingular curve of genus $g - 1$ and B is a nonsingular rational curve, and $A^2 = B^2 = -2$, $AB = 2$. Q.E.D.

PROOF OF THEOREM 3.3

Let (W, B) be a representative of the branch curve of $[f, F]$ which has no splitting family.

Step 1 Assume $B \supset \Gamma_0$. Then we will show that the germ $[f, F]$ belongs to class II. Lemma 3.15 implies the condition (a2) in 1.3.3 and lemmas 3.5 and 3.7 imply (a3) and (a1). Lemma 3.11 implies (c).

We consider (b). Assume B has a bad point P of odd multiplicity such that the local branch of B at P does not contain any tangential component to Γ_0. By the blow-up

τ_1 at P, we get B_1, which contains $E_{0,1} + E_{1,1}$ and does not pass through $E_{0,1} \cap E_{1,1}$. Hence we conclude that B has no other bad point, otherwise $[f, F]$ has a splitting family by Lemma 3.10. Let $\tau^{(1)} : W_1 \to W_0^{(1)}$ be the contraction of $E_{0,0}$ and we set $B^{(0)} = \tau_*^{(1)} B_1$. The pair $(W_0^{(1)}, B^{(1)})$ is also a representative of the branch curve of $[f, F]$ and $B^{(1)}$ contains the new fiber Γ_0 of $W_0^{(1)}$. If $B^{(1)}$ has a bad point $P^{(1)}$ of odd multiplicity without tangential component, $P^{(1)}$ is the unique bad point of $B^{(1)}$. Then we produce the elementary transformation $(W_0^{(2)}, B^{(2)})$ of $(W_0^{(1)}, B^{(1)})$ at $P^{(1)} \cdots$. After finitely many times, this situation terminates. Otherwise, $\{P, P^{(1)}, P^{(2)}, \ldots\}$ coincides with the set of infinitely near points of P and their multiplicities drop for finitely many steps. On the other hand, $P^{(i)}$ is the unique bad point of $B^{(i)}$ without tangential component such that $B^{(i)}$ contains Γ_0. This implies $(B^{(i)} - \Gamma_0)\Gamma_0 < 2g + 2$ for some i, which is absurd. Therefore after finite times of those elementary transformations, any bad point of the obtained branch curve $B^{(r)}$ either has even multiplicity or has no tangential component. In the former case, $B^{(r)}$ has ordinary singularity at that point by (a2). Hence (b) holds.

Step 2 Assume $B \not\supset \Gamma_0$. Then the bad point P of B is unique by Lemma 3.9. If $m = \mathrm{multi}_P B \geq 1$ is odd, then the problem is reduced to Step 1 by the elementary transformation at P.

Assume m is even. Then P is an ordinary singularity by lemmas 3.5 and 3.7.

If $m = 2$, then $[f, F]$ splits into two singular fibers of type 0_0 by Lemma 3.13. If $m \geq 4$, then $[f, F]$ belongs to class I. Hence we proved that any germ $[f, F]$ which does not belong to classes 0_0, I or II has a splitting family.

Step3 It remains to prove the finiteness of our reducing process to class **G**. Let $\mathbf{f} : \mathbf{M} \to \Delta_\epsilon \times \Delta_\delta \to \Delta_\delta$ be any splitting family constructed in this section for a given (f, F), and let $(\mathbf{f}_u, F')$ be any singular fiber germ of the fibration $\mathbf{f}_u$ $(\forall u \neq 0)$. Then there are some representatives of branch curves (B, W) and (B', W') of (f, F) and $(\mathbf{f}_u, F')$ respectively so that, for any bad point P' of B', there exists a bad point P of B such that (B', P') is a (perturbed) deformation of (B, P) in our sense. Then one of the following conditions is satisfied:

(i) The germ of the singularity (B', P') coincides with (B, P),

(ii) $\mathrm{mult}_{P'} B' < \mathrm{mult}_P B$,

(iii) $\mathrm{mult}_{P'} B' = \mathrm{mult}_P B$ and the minimal number of times of blow-ups for (B', P') such that the reduced scheme of the total transform of the horizontal component $\widetilde{B}'$ of B' is strictly less than that of (B, P).

Therefore the assertion follows. Q.E.D.

4 Horikawa index and signature

In this section, we define the Horikawa index and the local signature of a germ of singular fiber of a degeneration of hyperelliptic curves. By using these notions, we again present the result of theorem 3.3 and apply it to the study of compact complex surfaces with hyperelliptic pencils, especially to their signature problem.

4.1. Let $f : S \longrightarrow \Delta_\epsilon$ be a degeneration of hyperelliptic curves of genus g with an unique singular fiber F. Let (W, B) be a representative of the branch curve of the germ $[f, F]$. Let $\tau_{r,0} : W_r \longrightarrow W_0 = W$ be the succession of blow-ups such that the even proper transform B_r is nonsingular. Assume the number r of times of blow-ups is taken to be minimal which enjoy the above property. Let $m_1, \cdots, m_r$ be the multiplicity sequence of $\tau_{r,0}$ as in 1.1. Let $S' \longrightarrow W$ be the normal double cover branched along B on the total space of the line bundle of the square root of B. Let $\{Q_i\}_{1\le i\le \ell_0}$ be the set of isolated singular points on S'. Let $\rho : \widetilde{S} \longrightarrow S'$ be the canonical resolution of S', i.e. $\widetilde{S}$ is the double cover of W_r branched along B_r on the total space of the line bundle of the square root of B_r. The natural fibration $\widetilde{S} \longrightarrow \Delta_\epsilon$ is not relatively minimal in general. Let $\alpha_{(W,B)}$ be the number of times of contraction maps of (-1) curves from $\widetilde{S}$ to the original relative minimal model S. Let $p_g(S', Q_i)$ and $K^2_{(S',Q_i)}$ be the geometric genus and the square of the canonical cycle for the resolution ρ of the germ of singularity (S', Q_i) $(1 \le i \le \ell_0)$ respectively. Note that the formula in [Ho1, Lemma 6] implies that

$$\sum_{i=1}^{\ell} p_g(S', Q_i) = \frac{1}{2}\sum_{i=1}^{r}[\frac{m_i}{2}]([\frac{m_i}{2}]-1), \quad \sum_{i=1}^{\ell} K^2_{(S',Q_i)} = -2\sum_{i=1}^{r}([\frac{m_i}{2}]-1)^2.$$

Definition-Lemma 4.2. *We put*

$$\mathcal{H}([f,F]) := \sum_{i=1}^{\ell}\{\frac{4(g-1)}{g}p_g(S',Q_i)+K^2_{(S',Q_i)}\}+\alpha_{(W,B)} = \frac{2}{g}\sum_{i=1}^{r}([\frac{m_i}{2}]-1)(g-[\frac{m_i}{2}])+\alpha_{(W,B)}.$$

This is well-defined as an invariant for $[f, F]$, i.e. is determined independently of the choice of a representative (W, B) of the branch curve of $[f, F]$. We call it Horikawa index *(or* H-index *for short) of $[f, F]$.*

Since this notion is essential in our discussion, we give two proofs for the well-definedness. One comes from a global compactification argument, and the other comes from a local computational argument.

The first proof of lemma 4.2. Let $P_1^{(0)}, \cdots, P_{\ell_0}^{(0)}$ be all the bad points of B. By Artin's theorem [Ar], we may assume that the germ of singularity $(B, P_i^{(0)})$ $(1 \le i \le \ell_0)$ is algebraic. We fix an inclusion $W = \mathbf{P}^1 \times \Delta_\epsilon \hookrightarrow \mathbf{P}^1 \times \mathbf{P}^1 := \Sigma_0$. We denote by $\bar{\pi} : \Sigma_0 \longrightarrow \mathbf{P}^1$ the extension of the projection $\pi : W \longrightarrow \Delta_\epsilon$ and identify the original fiber Γ_0 with $\bar{\pi}^{-1}(0)$. Then it is easy to see that there exists a reduced divisor $\bar{B}^{(0)}$ on Σ_0 such that

(i) $\bar{B}^{(0)}$ is linearly equivalent to $(2g + 2)C_0 + 2k_0\Gamma_0$ for a sufficiently large integer k_0 where C_0 is a fiber for another fibration of Σ_0,

(ii) The germs of bad points of $\bar{B}^{(0)}$ on Γ_0 coincide with $(B, P_1^{(0)}), \cdots, (B, P_{\ell_0}^{(0)})$.
Let $(W^{(1)}, B^{(1)})$ be a representative of the branch curve of $[f, F]$ obtained from (W, B) by one-time elementary transformation. This transformation is globalized from $(\Sigma_0, \bar{B}^{(0)})$ to $(\Sigma_1, \bar{B}^{(1)})$ where $\bar{B}^{(1)}$ is a divisor on Hirzebruch surface Σ_1 of degree 1. Let $S^{(i)}$ be the normal double cover of Σ_i branched along $\bar{B}^{(i)}$ on the total space of the line bundle of the square root of $\bar{B}^{(i)}$ for $i = 0, 1$. By usual double covering method (cf.[P1]), we easily have

$$\omega_{S^{(i)}}^2 = \frac{4(g-1)}{g}(\chi(\mathcal{O}_{S^{(i)}}) - g - 1)$$

where $\omega_{S^{(i)}}$ is the dualizing sheaf.

Now let $Q_1^{(i)}, \cdots, Q_{\ell_i}^{(i)}$ be the singular points on the fiber over the origin of the fibration $S^{(i)} \longrightarrow \mathbf{P}^1$. Let $\widetilde{S^{(i)}} \longrightarrow S^{(i)}$ be the canonical resolution with respect to these singularities. Let $\widetilde{S^{(i)}} \longrightarrow S^{*(i)}$ be the succession of contractions of (-1) curves such that the fiber over the origin of the fibration $S^{*(i)} \longrightarrow \mathbf{P}^1$ does not contain any (-1) curve. Let $\alpha^{(i)}$ be the number of times of this (-1) contractions. Then we have

$$\chi(\mathcal{O}_{S^{*(i)}}) - \chi(\mathcal{O}_{S^{(i)}}) = -\sum_{j=1}^{\ell_i} p_g(S^{(i)}, Q_j^{(i)}), \;\; \omega_{S^{*(i)}}^2 - \omega_{S^{(i)}}^2 = \sum_{j=1}^{\ell_i} K^2_{(S^{(i)}, Q_j^{(i)})} + \alpha^{(i)}$$

for $i = 0, 1$. On the other hand, the fibrations $S^{*(i)} \longrightarrow \mathbf{P}^1$ for $i = 0, 1$ are isomorphic to each other over $\mathbf{P}^1 \backslash \{0\}$, and after resolving the singularities on these locuses and contracting (-1) curves, we obtain the same surface by the uniqueness of the relative minimal model. Therefore we have

$$\chi(\mathcal{O}_{S^{*(0)}}) = \chi(\mathcal{O}_{S^{*(1)}}), \;\; \omega_{S^{*(0)}}^2 = \omega_{S^{*(1)}}^2.$$

From these, we have

$$\begin{aligned} & \textstyle\sum_{j=1}^{\ell_0} \{\frac{4(g-1)}{g} p_g(S^{(0)}, Q_j^{(0)}) + K^2_{(S^{(0)}, Q_i^{(0)})}\} + \alpha^{(0)} \\ = & \textstyle\sum_{j=1}^{\ell_1} \{\frac{4(g-1)}{g} p_g(S^{(1)}, Q_j^{(1)}) + K^2_{(S^{(1)}, Q_i^{(1)})}\} + \alpha^{(1)}. \end{aligned}$$

Hence the assertion follows. Q.E.D.

The second proof of lemma 4.2. Let (W', B') be a resulting pair after one-time elementary transformation of (W, B). Let $W'_{r'} \longrightarrow W'$ be the minimal even resolution of B', and let $m'_1, \cdots, m'_{r'}$ be their multiplicity sequence. We put

$$\mathcal{H} = \frac{2}{g}\sum_{i=1}^{r}([\frac{m_i}{2}] - 1)(g - [\frac{m_i}{2}]) + \alpha, \;\; \mathcal{H}' = \frac{2}{g}\sum_{i=1}^{r'}([\frac{m'_i}{2}] - 1)(g - [\frac{m'_i}{2}]) + \alpha'$$

where $\alpha = \alpha_{(W,B)}$ and $\alpha' = \alpha_{(W',B')}$. It suffices to prove $\mathcal{H} = \mathcal{H}'$. Note that $0 \leq m_1 \leq 2g + 3$.

(i) Suppose $m_1 = 0$, that is, B does not pass through P_1. We may assume that $m_1' = 2g+2$, $m_2' = m_1$, $m_3' = m_2 \ldots$. Since there are two more (-1) curves over the proper transform of the fiber on the double cover of $W_{r'}$, we have $\alpha' = \alpha + 2$. Hence $\mathcal{H}' = \mathcal{H} + \frac{2}{g}([\frac{m_1'}{2}] - 1)(g - [\frac{m_1'}{2}]) + 2 = \mathcal{H}$.
(ii) Suppose $m_1 = 1$ and B does not contain Γ_0. We may assume $m_1' = 2g+2$, $m_2' = 2$, $m_3' = m_1$, $m_4' = m_2 \ldots$ and we have $\alpha' = \alpha + 2$. Hence we similarly have $\mathcal{H}' = \mathcal{H}$.
(iii) Suppose $m_1 = 1$ and B contains Γ_0. We may assume $m_1' = 2g+3$, $m_2' = 2$, $m_3' = m_1$, $m_4' = m_2 \ldots$ and we have $\alpha' = \alpha + 2$. Hence $\mathcal{H}' = \mathcal{H}$.
(iv) Suppose $2 \leq m_1 \leq 2g+1$. In this case, we may assume the i-th even proper transform of B and B' coincides with each other for $i \geq 1$. Especially we have $m_2' = m_2$, $m_3' = m_3 \ldots$ and $\alpha' = \alpha$. Moreover we have the following equality:

$$m_1' = \begin{cases} 2g+2-m_1 & \text{if } \Gamma_0 \not\subset B \text{ and } m_1 \equiv 0 \ (2) \\ 2g+3-m_1 & \text{if } \Gamma_0 \not\subset B \text{ and } m_1 \equiv 1 \ (2) \\ 2g+3-m_1 & \text{if } \Gamma_0 \subset B \text{ and } m_1 \equiv 0 \ (2) \\ 2g+4-m_1 & \text{if } \Gamma_0 \subset B \text{ and } m_1 \equiv 1 \ (2) \end{cases}$$

Indeed, we first assume $\Gamma_0 \not\subset B$ and m_1 is even. Let $W \xleftarrow{\tau_1} W_1 \xrightarrow{\tau_1'} W'$ be the elementary transformation. The 1-st even proper transform B_1 coincides with the proper transform of B by τ_1,and also coincides with the proper transform of B' by τ_1'. The proper transform $E_{0,1}$ of Γ_0 by τ_1 coincides with the exceptional curve for τ_1' and vice versa for $E_{1,1}$. Therefore we have $m_1' = B_1 E_{0,1} = (\tau_1^* B - m_1 E_{1,1}) E_{0,1} = B\Gamma_0 - m_1 = 2g+2-m_1$. The other cases are similar. In any case, we have $([\frac{m_1'}{2}] - 1)(g - [\frac{m_1'}{2}]) = ([\frac{m_1}{2}] - 1)(g - [\frac{m_1}{2}])$, which induce $\mathcal{H}' = \mathcal{H}$.
(v) Suppose $m_1 = 2g+2$ or $2g+3$. Then $(W', B') \to (W, B)$ is also an elementary transformation whose center has multiplicity less than two in B'. Hence the assertion follows from (i), (ii) and (iii). Q.E.D.

Remark 4.3. The notion of H-index was introduced for the first time by Horikawa [Ho2] for genus 2 case. He classified germs of singular fibers of genus 2 into six types $0, I_k, II_k, III_k, IV_k$ and V, and their H-indices are $0, 2k-1, 2k, 2k-1, 2k$ and 1 respectively. (see also Reid [Re]).

H-index is related to the invariants of compact surfaces by the following theorem, whose proof is essentially seen in Horikawa [Ho3, Theo.2.1] and Persson [P2, Prop.2.12] (see also Xiao[X2]):

Theorem 4.4. (Horikawa-Persson)
(i) For any germ $[f, F]$ of a degeneration of hyperelliptic curves, we have $\mathcal{H}([f, F]) \geq 0$.
(ii) Let $\varphi : V \longrightarrow C$ be a global hyperelliptic pencil of genus g, that is, h is a surjective proper holomorphic map from a nonsingular compact surface to a nonsingular compact

curve such that the general fiber is a hyperelliptic curve of genus g. Let $F_1, \cdots, F_k$ be all its singular fibers. Then we have

$$K_{V/C}^2 = \frac{4(g-1)}{g}\chi_\varphi + \sum_{i=1}^{k} \mathcal{H}([\varphi, F_i]),$$

where $K_{V/C}$ is the relative canonical sheaf and $\chi_\varphi = \deg \varphi_* K_{V/C}$.

EXAMPLE 4.5. The calculation of $\mathcal{H} := \mathcal{H}([f, F])$ for the germs defined in §1 is the following:
(1) If $[f, F]$ is of type 0_0, then $\mathcal{H} = 0$.
(2) If $[f, F]$ belong to class I with $1 \leq g' \leq [(g-1)/2]$, then $\mathcal{H} = 2g'(g - g' - 1)/g$.
(3) If $[f, F]$ belong to class II with $2 \leq g \leq 4$, then $\mathcal{H}$ of the germs corresponding to proposition 1.7 are as follows:
$(g = 2)$ (i) $\mathcal{H} = 1$,
$(g = 3)$ (i) $\mathcal{H} = 5/3$, (ii) $\mathcal{H} = 10/3$, (iii) $\mathcal{H} = 8/3$,
$(g = 4)$ (i) $\mathcal{H} = 2$, (ii) $\mathcal{H} = 3$, (iii) $\mathcal{H} = 4$.

Next we study the signature of compact surfaces with hyperelliptic pencils from the viewpoint of H-index.

DEFINITION 4.6. *We put*

$$\sigma([f, F]) := \frac{1}{2g+1}\{g\mathcal{H}([f, F]) - (g+1)\mathcal{E}([f, F])\}$$

and call it the local signature *of the germ $[f.F]$, where $\mathcal{E}([f, F]) := \chi_{\mathrm{top}}(F) - 2g + 2$ is the Euler contribution of F* (cf.[P3])*.*

PROPOSITION 4.7. *Let $\varphi : V \longrightarrow C$ be a global hyperelliptic pencil of genus g, and let $F_1, \cdots, F_k$ be all its singular fibers. Then we have*

$$\sigma(V) = \sum_{i=1}^{k} \sigma([\varphi, F_i]),$$

where $\sigma(V)$ is the global signature of V, i.e. the signature of the intersection form of the $2-$homology of V.

PROOF We use the relative formulation of Xiao [X2,X3]. Set $e_\varphi = \sum_{i=1}^{k} \mathcal{E}(F_i)$. Noether's formula and Hirzebruch's signature theorem ([Hi1]) says that

$$12\chi_\varphi = K_{V/C}^2 + e_\varphi, \quad 3\sigma(V) = K_{V/C}^2 - 2e_\varphi.$$

We set $\mathcal{H}(V) = \sum_{i=1}^{k} \mathcal{H}([\varphi, F_i])$. It follows from theorem 4.4 (ii) and the above formulas that

$$\mathcal{H}(V) = \frac{2g+1}{3g} K_{V/C}^2 - \frac{g-1}{3g} e_\varphi = \frac{2g+1}{g}\sigma(V) + \frac{g+1}{g} e_\varphi.$$

Therefore the assertion follows. Q.E.D.

EXAMPLE 4.8. The calculation of $\sigma := \sigma([f,F])$ for the germs defined in §1 is the following:
(1) If $[f,F]$ is of type 0_0, then $\sigma = -(g+1)/(2g+1)$.
(2) If $[f,F]$ belong to class I with $1 \le g' \le [(g-1)/2]$, then $\sigma = 2(gg' - g'^2 - g - g' - 1)/(2g+1)$.
(3) If $[f,F]$ belong to class II with $2 \le g \le 4$, then σ of the germs corresponding to proposition 1.7 are as follows:
($g=2$) (i) $\sigma = -1/5$,
($g=3$) (i) $\sigma = 1/7$, (ii) $\sigma = -6/7$, (iii) $\sigma = 0$,
($g=4$) (i) $\sigma = 1/3$, (ii) $\sigma = 7/9$, (iii) $\sigma = -1$.

DEFINITION 4.9. *Let $X([f,F])$ be a certain invariant of the germ of singular fiber $[f,F]$. Let $h : Z \longrightarrow \Delta_\epsilon \times \Delta_\delta$ be a splitting family of $[f,F]$ in the sense of definition 3.1. We say the family h* conserves *the invariant X if the following condition is satisfied: Let $F_{1,u}, \cdots, F_{\ell,u}$ be all singular fibers of the fibration $h_u : Z_u \longrightarrow \Delta_\epsilon \times \{u\}$ for any $u \in \Delta_\delta$. Then we have*

$$X([f,F]) = X([h_u, F_{1,u}]) + \cdots + X([h_u, F_{\ell,u}]).$$

The following is a more precise version of theorem 3.3.

THEOREM 4.10. *Let $\mathbf{G}$ be the union of the sets of germs of singular fibers of type 0_0, in class I and in class II defined in 1.3. Then for any germ $[f,F]$ of a degeneration of hyperelliptic curves, there exists a reduction of $[f,F]$ to $\mathbf{G}$ via splitting families which conserves the H-index and the local signature, that is, these invariants are conserved in any splitting family for this reduction.*

PROOF *Step 1* Let $h : Z \longrightarrow \Delta_\epsilon \times \Delta_\delta$ be any splitting family constructed in §3. By our construction, h is induced by the double cover over a family of local $\mathbf{P}^1$-bundle $\mathbf{W} = W \times \Delta_\delta$ whose branch locus is a family of divisors $\mathbf{W} \supset \mathbf{B} \xrightarrow{\rho} \Delta_\delta$. Let $= \{m_{1,u}, \cdots, m_{r_u,u}\}$ be the multiplicity sequence of the minimal even resolution of $(\mathbf{W}_u, \mathbf{B}_u)$ for $u \in \Delta_\delta$. We note that, bad points on several fibers of ρ_u contribute to this set for $u \neq 0$, while bad points are on an unique fiber for $u = 0$. Now we define the subsequence $\mathbf{T}_u$ of the sequence $\{[m_{1,u}/2], \cdots, [m_{r_u,u}/2]\}$ which consists of the elements of numbers greater than 1, and we consider it as a set, i.e. we neglect its order. (Note that the multiplicity with $[m_{u,i}/2] \le 1$ does not contribute the H-index of the fiber.) Then it follows from our construction that the splitting family h conserves this set, i.e. we have

$$\mathbf{T}_0 = \mathbf{T}_u$$

for any $u \in \Delta_\delta$. Moreover the number of times of blow-downs of (-1) curves from the canonical resolution to its relative minimal model is independent of u by [Kod2]. Therefore h conserves the H-index.

Step 2 Next we show the conservation of the Euler contribution, which induce the conservation of the local signature by the definition and Step 1.

We first consider the splitting family of lemma 3.5. We use the notations in this lemma. By the stability of (-1) curves, we may identify $\mathbf{M}^\flat$ with $\mathbf{M}$. Let $\widetilde{E}$ be the set of components of F which dominates $E_{j,r}$ by the generically two-to-one map $h_0 : \mathbf{M}_0 \longrightarrow W_r$. Note that $\widetilde{E}$ is irreducible or consists of two irreducible components. Let $A_1, \cdots, A_s, A'_1, \cdots, A'_{s'}, D$ be all the connected components of $F - \widetilde{E}$, where A_i $(1 \leq i \leq s)$ intersect $\widetilde{E}$ at one point, A'_j $(1 \leq j \leq s)$ intersect $\widetilde{E}$ at two points and $h_0(D) = Q$. Then we have

$$\chi_{\mathrm{top}}(F) = \chi_{\mathrm{top}}(\widetilde{E}) + \sum_{i=1}^{s} \chi_{\mathrm{top}}(A_i) + \sum_{j=1}^{s'} \chi_{\mathrm{top}}(A'_j) - s - 2s' + \chi_{\mathrm{top}}(D) - \epsilon(D),$$

where $\epsilon(D)$ (1 or 2) is the number of the points of the intersection of D and $\widetilde{E}$.

Now we consider singular fibers $F' = \mathbf{f}_u^{-1}(0)$, $F'' = \mathbf{f}_u^{-1}(\pi(R_u))$ of $h_u : \mathbf{M}_u \longrightarrow W_r \times \{u\}$ for $u \neq 0$. Let $\widetilde{E}'$ be the set of components of F' which dominates $E_{j,r} \times \{u\}$ by h_u. Then $F' - \widetilde{E}'$ has connected components which are isomorphic to $A_1, \cdots, A_s, A'_1, \cdots, A'_{s'}$, and we have $\chi_{\mathrm{top}}(F') = \chi_{\mathrm{top}}(\widetilde{E}') + \sum_{i=1}^{s} \chi_{\mathrm{top}}(A_i) + \sum_{j=1}^{s'} \chi_{\mathrm{top}}(A'_j) - s - 2s'$. Moreover our method of the construction implies

$$\chi_{\mathrm{top}}(\widetilde{E}') = \chi_{\mathrm{top}}(\widetilde{E}) - 2[\frac{k}{2}],$$

where $k = I_Q(B_r, E_{j,r})$. On the other hand, F'' consists of the irreducible component $\widetilde{\Gamma}$ which dominates the fiber $\mathbf{P}^1$ by h_u and the connected component which is isomorphic to D. We have $\chi_{\mathrm{top}}(F'') = \chi_{\mathrm{top}}(\widetilde{\Gamma}) + \chi_{\mathrm{top}}(D) - \epsilon(D)$. Moreover our construction implies

$$\chi_{\mathrm{top}}(\widetilde{\Gamma}) = 2 - 2g + 2[\frac{k}{2}].$$

Therefore we have $\mathcal{E}(F) = \mathcal{E}(F') + \mathcal{E}(F'')$. For other splitting families constructed in §3, we have the assertion by the similar argument. We omit it.

Hence the claim of the theorem follows from theorem 3.3. Q.E.D.

COROLLARY 4.11. *Assume the germ $[f, F]$ has vanishing H-index. Then $[f, F]$ is reduced to germs of type 0_0 via splitting families.*

PROOF Note that the germ has vanishing H-index if and only if it has a representative of branch curve which has at most simple singularities, in other words whose normal double cover has at most rational double points. Especially a germ in class II has positive H-index by the definition. Therefore example 4.5 says that a germ which belongs to $\mathbf{G}$ has vanishing H-index if and only if it is of type 0_0. Thus the assertion follows from theorem 4.10. Q.E.D.

Remark 4.12 The surface of general type with a pencil whose slope has minimal value automatically has a hyperelliptic pencil such that all the singular fibers have vanishing H-indices ([X2, Kon]). Therefore, considering the above corollary, one can ask whether such a surface has a reduction to surfaces with at most 0_0 fibers by global splitting families or not. This type of problem is called "global Morsification problem".

Corollary 4.13. *Let $\varphi : V \longrightarrow C$ be a global hyperelliptic pencil of genus $2 \leq g \leq 4$. Then the global signature $\sigma(V)$ is written as follows:*

($g = 2$) $\sigma(V) = -(3/5)k_{0_0} - (1/5)k_{(II,i)}$,

($g = 3$) $\sigma(V) = -(4/7)k_{0_0} - (6/7)k_{(I,1)} + (1/7)k_{(II,i)} - (6/7)k_{(II,ii)}$,

($g = 4$) $\sigma(V) = -(5/9)k_{0_0} - (2/3)k_{(I,1)} + (1/3)k_{(II,i)} + (7/9)k_{(II,ii)} - k_{(II,iii)}$,

where $k_{0_0}, \cdots, k_{(II,iii)}$ are the sum of numbers of germs of the corresponding type which are obtained by applying every singular fibers of φ to our reducing process via local splitting families. For instance, $k_{(I,1)}$ is the number of germs in class I with $g' = 1$, and $k_{(II,i)}$ is the number of germs of type (i) in class II written in proposition 1.7.

Proof The assertion is clear from proposition 4.7, theorem 4.10, proposition 1.7 and example 4.8. Q.E.D.

Remark 4.14. For $g = 2$, The above formula due to Matsumoto, Horikawa and corollary 4.11 as we explained in Introduction. For $g \geq 3$, corollary 4.13 and example 4.8 tell us that there are many singular fibers with positive local signature. In some sense, this fact support the existence of surfaces with hyperelliptic pencils whose global signatures are positive, which was discovered by Xiao and Chen [C].

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