ON SOME SYSTEMS OF DIFFERENCE EQUATIONS

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§0. Foreword.

This is a translation in English of the first part of my paper [59]. The goal of this translation is to present the recurrent relations, on the basis of which the results of my papers [43], [54], [55], [56], [60], [65], [66], [72], [69], [74] have been obtained. As in these papers, below we use the Mejers functions (see [19], ch. 5 or [54] – [55])

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§1. My auxiliary functions.

Let

$$|z| \geq 1, -3\pi/2 < \arg(z) \leq \pi/2, \log(z) = \ln(|z|) + i\arg(z).$$

Then $\log(-z) = \log(z) - i\pi$, if $\Re(z) > 0$ and $\log(z) = \log(-z) - i\pi$, if $\Re(z) < 0$.

Let

(1) $$f^{\vee}_{l,1}(z,\nu) = f_{l,1}(z,\nu) = -(-1)^{\nu l} \times$$

$$G^{(1,2+l)}_{4+2l,4+2l}\left(-(-1)^l z\left|\begin{array}{l}\overbrace{-\nu,\quad \dots \quad ,-\nu,}^{l+2 \text{ times}}\overbrace{\nu+1,\quad \dots \quad ,\nu+1}^{l+2 \text{ times}}\\ 0,\quad \dots \quad ,0,\quad 0,\quad \dots \quad ,0\end{array}\right.\right)=$$

$$\frac{-(-1)^{l\nu}}{2i\pi}\int\limits_{L_1} g^{(1,2+l)}_{4+2l,4+2l}\times$$

$$\left(-(-1)^l z\left|\begin{array}{l}\overbrace{-\nu,\quad \dots \quad ,-\nu,}^{l+2 \text{ times}}\overbrace{\nu+1,\quad \dots \quad ,\nu+1}^{l+2\text{times}}\\ 0,\quad \dots \quad ,0,\quad 0,\quad \dots \quad ,0\end{array}\right|s\right)ds,$$

where $l=0,1,2$, $\nu\in[0,+\infty)\cap\mathbb{Z}$,

$$g^{(1,2+l)}_{4+2l,4+2l}=g^{(1,2+l)}_{4+2l,4+2l}(s)=$$

$$g^{(1,2+l)}_{4+2l,4+2l}\left(-(-1)^l z\left|\begin{array}{l}\overbrace{-\nu,\quad \dots \quad ,-\nu,}^{l+2 \text{ times}}\overbrace{\nu+1,\quad \dots \quad ,\nu+1}^{l+2 \text{ times}}\\ 0,\quad \dots \quad ,0,\quad 0,\quad \dots \quad ,0\end{array}\right|s\right)=$$

$$(-(-1)^l z)^s\Gamma(-s)(\Gamma(1+s))^{-3-2l}(\Gamma(1+\nu+s))^{2+2l}(\Gamma(1+\nu-s))^{-2-2l},$$

and the curve L_1 passes from $+\infty$ to $+\infty$ in the negative direction such that the set $[0,+\infty)\cap\mathbb{Z}$ lies to the right from it, but the set $(-\infty,-1)\cap\mathbb{Z}$ lies to the left from it. The set of all unremovable singular points of $g^{(1,2+l)}_{4+2l,4+2l}(s)$ encircled by the curve L_1, consists of the points $s=0,\dots,\nu$, each of these points is a pole of the first order. Therefore

$$Res(g^{(1,2+l)}_{4+2l,4+2l};k)=$$

$$\lim_{s\to k}((s+k)g^{(1,2+l)}_{4+2l,4+2l}(s)),$$

where $l=0,1,2$ and $k=0,\dots,\nu$. Let

$$s=k+u,\ H_{l,1}(u,k,\nu)=$$

$$g^{(1,2+l)}_{4+2l,4+2l}(k+u)=$$

$$(-(-1)^l z)^{k+u}\Gamma(-k-u)(\Gamma(1+k+u))^{-3-2l}\times$$

$$(\Gamma(1+\nu+k+u))^{2+2l}(\Gamma(1+\nu-k-u))^{-2-2l}=$$

$$\prod_{\kappa=1}^{k}(-k+\kappa-u)^{-1}(-(-1)^l z)^{k+u}\times$$

$$\Gamma(1-u)(\Gamma(1+k+u))^{-3-2l}(\Gamma(1+\nu+k+u))^{2+2l}(\Gamma(1+\nu-k-u))^{-2-2l},$$

where $l=0,1,2$ and $k\in[0,+\infty)\cap\mathbb{Z}$. Therefore

$$Res(g^{(1,2+l)}_{4+2l,4+2l}; k) = \lim_{u\to 0}(ug^{(1,2+l)}_{4+2l,4+2l}(k+u)) =$$

$$-(-1)^{lk}(k!)^{-4-2l}((\nu+k)!)^{2+l}((\nu-k)!)^{-2-l} =$$

$$-(-1)^{l}kz^{k}\binom{\nu}{k}^{2+l}\binom{\nu+k}{k}^{2+2l}((\nu-k)!)^{-2-2l},$$

where $l = 0,\ 1, 2$ and $k = 0,\ \dots,\ \nu$. Consequently, the function $f_{l,1}(z,\nu)$ is equal to a finite sum

$$f_{l,1}(z,\nu) = \sum_{k=0}^{\nu}(-1)^{(\nu+k)l}(z)^{k}\binom{\nu}{k}^{2+l}\binom{\nu+k}{\nu}^{2+l}, \tag{2}$$

where $l = 0,\ 1,\ 2,\ \nu \in [0,+\infty)\cap\mathbb{Z}$.

Let

$$f_{l,2}(z,\nu)^{\vee} = f_{l,2}(z,\nu) = -(-1)^{l\nu}\times \tag{3}$$

$$G^{(3+l,2+l)}_{4+2l,4+2l}\left(-z\left|\begin{matrix}\overbrace{-\nu,\quad\dots\quad,-\nu,}^{l+2\text{ times}}\overbrace{\nu+1,\quad\dots\quad,\nu+1}^{l+2\text{ times}}\\ 0,\quad\dots\quad,0,\quad 0,\quad\dots\quad,0\end{matrix}\right.\right) =$$

$$\frac{-(-1)^{l\nu}}{2i\pi}\int\limits_{L_2} g^{(3+l,2+l)}_{4+2l,4+2l}\left(-z\left|\begin{matrix}\overbrace{-\nu,\quad\dots\quad,-\nu,}^{l+2\text{ times}}\overbrace{\nu+1,\quad\dots\quad,\nu+1}^{l+2\text{ times}}\\ 0,\quad\dots\quad,0,\quad 0,\quad\dots\quad,0\end{matrix}\right|s\right)ds,$$

where $l = 0,\ 1,\ 2,\ \nu \in [0,+\infty)\cap\mathbb{Z}$,

$$g^{(3+l,2+l)}_{4+2l,4+2l} = g^{(3+l,2+l)}_{4+2l,4+2l}(s) =$$

$$g^{(3+l,2+l)}_{4+2l,4+2l}\left(-z\left|\begin{matrix}\overbrace{-\nu,\quad\dots,\quad-\nu,}^{l+2\text{ times}}\overbrace{\nu+1,\quad\dots\quad,\nu+1}^{l+2\text{ times}}\\ 0,\quad\dots\quad,0,\quad 0,\quad\dots\quad,0\end{matrix}\right|s\right) =$$

$$(-z)^{s}(\Gamma(-s))^{3+l}(\Gamma(1+s))^{-1-l}(\Gamma(1+\nu+s))^{2+l}(\Gamma(1+\nu-s))^{-2-l},$$

and the curve L_2 passes from $-\infty$ to $-\infty$ in the positive direction such that the set $[0,+\infty)\cap\mathbb{Z}$ lies to the right from it but the set $\mathbb{Z}\cap(-\infty,-1]$ lies to the left from it. The set of all unremovable singular points of $g^{(3+l,2+l)}_{4+2l,4+2l}(s)$ encircled by L_2, consists of all the $s = -1-\nu-k$ with $k\in[0,+\infty)\cap\mathbb{Z}$; each of these points is a pole of the first order. Therefore

$$Res(g^{(3+l,2+l)}_{4+2l,4+2l}; -1-d_1\nu-k) = \lim_{s\to-\nu-1-k}((s+\nu+1+k)g^{(3+l,2+l)}_{4+2l,4+2l}(s)),$$

where $l = 0,\ 1,\ 2$ and $k\in[0,+\infty)\cap\mathbb{Z}$. Let $\nu\in[0,+\infty)\cap\mathbb{Z}$, Let

$$s = -\nu-1-k+u, H^{*}_{l}(u,k,\nu) = (\Gamma(\nu+1+k-u))^{4+2l}\times$$

$$(\Gamma(1+k-u))^{-2-l}(\Gamma(2+2\nu+k-u))^{-2-l},$$

and

$$R(t,\nu)=\frac{\prod\limits_{j=1}^{\nu}(t-j)}{\prod\limits_{j=0}^{\nu}(t+j)}. \tag{4}$$

where $\nu\in[0,+\infty)\cap\mathbb{Z}$. Let further $T=-s=\nu+1+k-u$. Then

$$H_l^*(u,k,\nu)=\left(\frac{\prod\limits_{j=1}^{\nu}(1+\nu+k-j-u)}{\prod\limits_{j=0}^{\nu}(1+\nu+k+j-u)}\right)^{2+l}=(R(T,\nu))^{2+l}, \tag{5}$$

where $\nu\in[0,+\infty)\cap\mathbb{Z}$. Let

$$H_{l,2}(u,k,\nu)=g^{(3+l,2+l)}_{4+2l,4+2l}(-\nu-1-k+u)=(-z)^{-\nu-1-k+u}(\Gamma(\nu+1+k-u))^{3+l}\times$$
$$(\Gamma(-\nu-k+u))^{-1-l}(\Gamma(-k+u))^{2+l}(\Gamma(2+2\nu+k-u))^{-2-l}=$$
$$(-1)^{k+(l+1)nu}(-z)^{-\nu-1-k+u}\frac{\pi}{\sin(u\pi)}(\Gamma(\nu+1+k-u))^{4+2l}\times$$
$$(\Gamma(1+k-u))^{-2-l}(\Gamma(2+2\nu+k-u))^{-2-l}=$$
$$(-1)^{k+(l+1)nu}(-z)^{-\nu-1-k+u}\frac{\pi}{\sin(u\pi)}H_l^*(u,k,\nu),$$

where $l=0,\ 1,\ 2$ and $k\in[0,+\infty)\cap\mathbb{Z}$. Therefore

$$Res(g^{(3+l,2+l)}_{4+2l,4+2l};-1-\nu-k)=$$
$$\lim_{u\to0}(uH_{l,2}(u,k,\nu))=$$
$$-(-1)^{l\nu}z^{-(1+\nu+k)}(R(1+\nu+k,\nu))^{2+l},$$

where $l=0,\ 1,\ 2$ and $k\in[0,+\infty)\cap\mathbb{Z}$. Consequently,

$$f_{l,2}(z,\nu)=\sum_{k=0}^{+\infty}z^{-(1+\nu+k)}(R(1+\nu+k,\nu))^{2+l}, \tag{6}$$

where $l=0,1,2$ and $k\in[0,+\infty)\cap\mathbb{Z}$. Let $t=1+\nu+k$ with $k\in[0,+\infty)\cap\mathbb{Z}$; in view of (4) and (5), it follows that

$$f_{l,2}(z,\nu)=\sum_{t=1+\nu}^{+\infty}z^{-t}(R(t,\nu))^{2+l}, \tag{7}$$

where $l=0,\ 1,\ 2$ and $\nu\in[0,+\infty)\cap\mathbb{Z}$. Since $(R(t,\nu))^{2+l}$ for $\nu\in\mathbb{N}$ has in the points $t=1,\ \dots\ ,\ \nu$, the zeros of the order $2+l$, it follows that

$$f_{l,2}(z,\nu)=\sum_{t=1}^{+\infty}z^{-t}(R(t,\nu))^{2+l}, \tag{8}$$

where $l = 0,\ 1,\ 2$ and $\nu \in [0, +\infty) \cap \mathbb{Z}$. Let

(9) $$f^{\vee}_{l,3}(z, \nu) = f_{l,3}(z, \nu) = (-1)^{l\nu} \times$$

$$G^{(4+l,2+l)}_{4+2l,4+2l}\left(z\left|\begin{matrix}\overbrace{-\nu, & \ldots & , -\nu,}^{l+2 \text{ times}} & \overbrace{\nu+1, & \ldots & , \nu+1}^{l+2 \text{ times}} \\ 0, & \ldots & , 0, & 0, & \ldots & , 0\end{matrix}\right.\right) =$$

$$\frac{(-1)^{l\nu}}{2i\pi}\int\limits_{L_2} g^{(4+l,2+l)}_{4+2l,4+2l}\left(-z\left|\begin{matrix}\overbrace{-\nu, & \ldots & , -\nu,}^{l+2 \text{ times}} & \overbrace{\nu+1, & \ldots & , \nu+1}^{l+2 \text{ times}} \\ 0, & \ldots & , 0, & 0, & \ldots & , 0\end{matrix}\right|s\right) ds,$$

where $l = 0,\ 1,\ 2,\ \nu[0, +\infty) \cap \mathbb{Z}$,

$$g^{(4+l,2+l)}_{4+2l,4+2l} = g^{(4+l,2+l)}_{4+2l,4+2l}(s) =$$

$$g^{(4+l,2+l)}_{4+2l,4+2l}\left(-z\left|\begin{matrix}\overbrace{-\nu, & \ldots & , -\nu,}^{l+2 \text{ times}} & \overbrace{\nu+1, & \ldots & , \nu+1}^{l+2 \text{ times}} \\ 0, & \ldots & , 0, & 0, & \ldots & , 0\end{matrix}\right|s\right) =$$

$$(z)^s(\Gamma(-s))^{4+l}(\Gamma(1+s))^{-l}(\Gamma(1+\nu+s))^{2+l}(\Gamma(1+\nu-s))^{-2-l}.$$

The set of all unremovable singular points of the function $g^{(4+l,2+l)}_{4+2l,4+2l}(s)$ encircled by the curve L_2, consists of the points $s = -1-\nu-k$ with $k \in [0, +\infty)\cap\mathbb{Z}$; each of these points is a pole of the second order. Therefore

$$Res(g^{(4+l,2+l)}_{4+2l,4+2l}; -1-\nu-k) =$$

$$\lim_{s\to-\nu-1-k}\left(\frac{\partial}{\partial s}((s+\nu+1+k)^2 g^{(4+l,2+l)}_{4+2l,4+2l}(s))\right),$$

where $l = 0,\ 1,\ 2$ and $k \in [0, +\infty) \cap \mathbb{Z}$. Let

$$s = -\nu-1-k+u,\ H_{l,3}(u, k, nu) = g^{(4+l,2+l)}_{4+2l,4+2l}(-\nu-1-k+u) = (z)^{-\nu-1-k+u}\times$$

$$(\Gamma(\nu+1+k-u))^{4+l}(\Gamma(-\nu-k+u))^{-l}(\Gamma(-k+u))^{2+l}(\Gamma(2+2\nu+k-u))^{-2-l} =$$

$$(-1)^{l\nu}\left(\frac{\pi}{\sin(u\pi)}\right)^2(z)^{-\nu-1-k+u}(\Gamma(\nu+1+k-u))^{4+2l}\times$$

$$(\Gamma(1+k-u))^{-2-l}(\Gamma(2+2\nu+k-u))^{-2-l} =$$

$$(-1)^{l\nu}\left(\frac{\pi}{\sin(u\pi)}\right)^2(z)^{-\nu-1-k+u}H^*_l(u, k, nu) =$$

$$(-1)^{l\nu}\left(\frac{\pi}{\sin(u\pi)}\right)^2(z)^{-\nu-1-k+u}(R(T, \nu))^{2+l},$$

where $T = \nu+1+k-u, l = 0,\ 1,\ 2$ and $k \in [0, +\infty) \cap \mathbb{Z}$. Therefore

$$Res(g^{(4+l,2+l)}_{4+2l,4+2l}; -1-d_1\nu-k) =$$

$$\lim_{u\to 0}\left(\frac{\partial}{\partial u}(u^2 H_{l,2}(u))\right) =$$
$$(-1)^{l\nu}(\log(z))(z)^{-\nu-1-k}H^*_l(0,k,\nu)+$$
$$(-1)^{l\nu}(z)^{-\nu-1-k}\left(\frac{\partial}{\partial u}H^*_l\right)(0,k,\nu) =$$
$$(-1)^{l\nu}(\log(z))(z)^{-\nu-1-k}(R(1+\nu+k,\nu))^{2+l}-$$
$$(-1)^{l\nu}(z)^{-\nu-1-k}\left(\frac{\partial}{\partial t}(R)^{2+l}\right)(1+\nu+k,\nu)$$

because $(\pi u/(\sin(\pi u))^2$ is a even function. Thus

$$f_{l,3}(z,\nu) = (\log(z))\sum_{t=1+\nu}^{+\infty} z^{-t}(R(t,\nu))^{2+l} - \tag{10}$$
$$\sum_{t=1+\nu}^{+\infty} z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu),$$

where $l = 0, 1, 2$ and $\nu \in [0,+\infty)\cap\mathbb{Z}$. Since $(R(t,\nu))^{2+l}$ for $\nu\in\mathbb{N}$ have in the points $t = 1, \dots, \nu$, the zeros of the order $2+l$, it follows that

$$f_{l,3}(z,\nu) = (\log(z))\sum_{t=1}^{+\infty} z^{-t}(R(t,\nu))^{2+l} - \tag{11}$$
$$\sum_{t=1}^{+\infty} z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu),$$

where $l = 0, 1, 2$ and $\nu \in [0,+\infty)\cap\mathbb{Z}$. Puting

$$f_{l,4}(z,\nu) = -\sum_{t=1}^{+\infty} z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu), \tag{12}$$

where $l = 0, 1, 2$ and $\nu \in [0,+\infty)\cap\mathbb{Z}$, we come to equality

$$f_{l,3}(z,\nu) = (\log(z))f_{l,2}(z,\nu) + f_{l,4}(z,\nu) \tag{13}$$

Let

$$f^{\vee}_{l,5}(-z,\nu) = -(-1)^{l\nu}\times \tag{14}$$
$$G^{(5+l,2+l)}_{4+2l,4+2l}\left(z\left|\begin{matrix}\overbrace{-\nu, & \dots & , -\nu,}^{l+2\text{ times}} & \overbrace{\nu+1, & \dots & , \nu+1}^{l+2\text{ times}}\\ 0, & \dots & , 0, & 0, & \dots & , 0\end{matrix}\right.\right) =$$
$$\frac{-(-1)^{l\nu}}{2i\pi}\int_{L_2} g^{(5+l,2+l)}_{4+2l,4+2l}\left(-z\left|\begin{matrix}\overbrace{-\nu, & \dots & , -\nu,}^{l+2\text{ times}} & \overbrace{\nu+1, & \dots & , \nu+1}^{l+2\text{ times}}\\ 0, & \dots & , 0, & 0, & \dots & , 0\end{matrix}\right|s\right)ds,$$

where $l = 1,\ 2,\ \nu \in [0, +\infty) \cap \mathbb{Z}$,

$$g^{(5+l,2+l)}_{4+2l,4+2l} = g^{(5+l,2+l)}_{4+2l,4+2l}(s) =$$

$$g^{(5+l,2+l)}_{4+2l,4+2l}\left(-z\left|\begin{matrix}\overbrace{-\nu, & \dots & , -\nu,}^{l+2 \text{ times}} & \overbrace{\nu+1, & \dots & , \nu+1}^{l+2 \text{ times}} \\ 0, & \dots & , 0, & 0, & \dots & , 0\end{matrix}\right| s\right) =$$

$$(-z)^s(\Gamma(-s))^{5+l}(\Gamma(1+s))^{-l+1}(\Gamma(1+\nu+s))^{2+l}(\Gamma(1+\nu-s))^{-2-l}.$$

The set of all unremovable singular points of the function $g^{(5+l,2+l)}_{4+2l,4+2l}(s)$ encircled by the curve L_2, consists of the points $s = -1-\nu-k$ with $k \in [0, +\infty) \cap \mathbb{Z}$; each of these points is a pole of the third order. Therefore

$$Res(g^{(5+l,2+l)}_{4+2l,4+2l}; -1-\nu-k) =$$

$$\frac{1}{2}\lim_{s\to -\nu-1-k}\left(\left(\frac{\partial}{\partial s}\right)^2((s+\nu+1+k)^3 g^{(5+l,2+l)}_{4+2l,4+2l}(s))\right),$$

where $l = 1, 2$ and $k \in [0, +\infty) \cap \mathbb{Z}$. Let

$$s = -\nu-1-k+u,\ H_{l,5}(u, k, nu) =$$

$$g^{(5+l,2+l)}_{4+2l,4+2l}(-\nu-1-k+u) = (-z)^{-\nu-1-k+u}(\Gamma(\nu+1+k-u))^{5+l}\times$$

$$(\Gamma(-\nu-k+u))^{-l+1}(\Gamma(-k+u))^{2+l}(\Gamma(2+2\nu+k-u))^{-2-l} =$$

$$(-1)^{(k+\nu)(l-1)}\left(\frac{\pi}{\sin(u\pi)}\right)^3(-z)^{-\nu-1-k+u}(\Gamma(\nu+1+k-u))^{4+2l}\times$$

$$(-1)^{k(2+l)}(\Gamma(1+k-u))^{-2-l}(\Gamma(2+2\nu+k-u))^{-2-l} =$$

$$(-1)^{k+(l-1)\nu}\left(\frac{\pi}{\sin(u\pi)}\right)^3(-z)^{-\nu-1-k+u}H^*_l(u, k, \nu) =$$

$$(-1)^{k+(l-1)\nu}\left(\frac{\pi}{\sin(u\pi)}\right)^3(-z)^{-\nu-1-k+u}(R(T, \nu))^{2+l},$$

where $T = \nu+1+k-u, l = 1, 2$ and $k \in [0, +\infty) \cap \mathbb{Z}$. Therefore

$$Res(g^{(5+l,2+l)}_{4+2l,4+2l}; -1-\nu-k) = \tag{15}$$

$$\frac{1}{2}\lim_{u\to 0}\left(\left(\frac{\partial}{\partial u}\right)^2(u^3 H_{l,5}(u, k, \nu))\right) =$$

$$2^{-1}\pi^2(-1)^{l\nu+1}(z)^{-\nu-1-k}H^*_l(0, k, \nu)+$$

$$2^{-1}(-1)^{l\nu+1}(\log(-z))^2 z^{-\nu-1-k}H^*_l(0, k, \nu)+$$

$$(-1)^{l\nu+1}z^{-\nu-1-k}(\log(-z))\left(\frac{\partial}{\partial u}H^*_l\right)(0, k, \nu)+$$

$$2^{-1}(-1)^{l\nu+1}z^{-\nu-1-k}\left(\left(\frac{\partial}{\partial u}\right)^2 H^*_l\right)(0, k, \nu).$$

We put now

(16) $$f_{l,5}(z,\nu) = 2^{-1}(\log(z))^2 \sum_{t=\nu+1}^{\infty} z^{-t}R(t,\nu)^{2+l} -$$

$$(\log(z)) \sum_{t=\nu+1}^{\infty} z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu)+$$

$$2^{-1} \sum_{t=\nu+1}^{\infty} z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^2 (R^{2+l})\right)(t,\nu),$$

(17) $$f_{l,6}(z,\nu) = 2^{-1} \sum_{t=\nu+1}^{\infty} z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^2 (R^{2+l})\right)(t,\nu).$$

Since $(R(t,\nu))^{2+l}$ for $\nu \in \mathbb{N}$ have in the points $t = 1, \ldots, \nu$, the zeros of the order $2+l$, and $l = 1, 2$ now, it follows that

(18) $$f_{l,6}(z,\nu) = 2^{-1} \sum_{t=1}^{\infty} z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^2 (R^{2+l})\right)(t,\nu).$$

Then in view of (15), (8), (11) – (14), (16), (17)

(19) $$f^{\vee}_{l,5}(z,\nu) = -i\pi(\log(z)) \sum_{t=\nu+1}^{\infty} z^{-t}R(t,\nu)^{2+l} +$$

$$2^{-1}(\log(z))^2 \sum_{t=\nu+1}^{\infty} z^{-t}R(t,\nu)^{2+l}+$$

$$i\pi \sum_{t=\nu+1}^{\infty} z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu)-$$

$$(\log(z)) \sum_{t=\nu+1}^{\infty} z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu)+$$

$$2^{-1} \sum_{t=\nu+1}^{\infty} z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^2 (R^{2+l})\right)(t,\nu) =$$

$$-i\pi f_{l,3}(z,\nu) + f_{l,5}(z,\nu),$$

(20) $$f_{l,5}(z,\nu) =$$

$$2^{-1}(\log(z))^2 f_{l,2}(z,\nu) + (\log(z))f_{l,4}(z,\nu) + f_{l,6}(z,\nu) =$$
$$= -2^{-1}(\log(z))^2 f_{l,2}(z,\nu) + +(\log(z))f_{l,3}(z,\nu) + f_{l,6}(z,\nu)$$

Let

(21) $$f^{\vee}_{l,7}(z,\nu) = (-1)^{l\nu} \times$$

$$G^{(6+l,2+l)}_{4+2l,4+2l}\left(z\left|\begin{matrix}\overbrace{-\nu, & \dots & , -\nu,}^{l+2\text{ times}} & \overbrace{\nu+1, & \dots & , \nu+1}^{l+2\text{ times}}\\ 0, & \dots & , 0, & 0, & \dots & , 0\end{matrix}\right.\right) =$$

$$\frac{-(-1)^{l\nu}}{2i\pi}\int\limits_{L_2} g^{(6+l,2+l)}_{4+2l,4+2l}\left(z\left|\begin{matrix}\overbrace{-\nu, & \dots & , -\nu,}^{l+2\text{ times}} & \overbrace{\nu+1, & \dots & , \nu+1}^{l+2\text{ times}}\\ 0, & \dots & , 0, & 0, & \dots & , 0\end{matrix}\right|s\right) ds,$$

where $l = 2$, $\nu \in [0, +\infty) \cap \mathbb{Z}$,

$$g^{(5+l,2+l)}_{4+2l,4+2l} = g^{(6+l,2+l)}_{4+2l,4+2l}(s) =$$

$$g^{(5+l,2+l)}_{4+2l,4+2l}\left(-z\left|\begin{matrix}\overbrace{-\nu, & \dots & , -\nu,}^{l+2\text{ times}} & \overbrace{\nu+1, & \dots & , \nu+1}^{l+2\text{ times}}\\ 0, & \dots & , 0, & 0, & \dots & , 0\end{matrix}\right|s\right) =$$

$$(z)^s(\Gamma(-s))^{6+l}(\Gamma(1+s))^{-l+2}(\Gamma(1+\nu+s))^{2+l}(\Gamma(1+\nu-s))^{-2-l}.$$

The set of all unremovable singular points of the function $g^{(6+l,2+l)}_{4+2l,4+2l}(s)$ encircled by the curve L_2, consists of the points $s = -1-\nu-k$ with $k \in [0, +\infty)\cap\mathbb{Z}$; each of these points is a pole of the fourth order. Therefore

$$Res(g^{(6+l,2+l)}_{4+2l,4+2l}; -1-\nu-k) =$$

$$\frac{1}{6}\lim_{s\to-\nu-1-k}\left(\left(\frac{\partial}{\partial s}\right)^3((s+\nu+1+k)^4 g^{(5+l,2+l)}_{4+2l,4+2l}(s))\right),$$

where $l = 2$ and $k \in [0, +\infty) \cap \mathbb{Z}$. Let

$$s = -\nu - 1 - k + u,\ H_{l,7}(u, k, nu) =$$

$$g^{(6+l,2+l)}_{4+2l,4+2l}(-\nu-1-k+u) = (z)^{-\nu-1-k+u}(\Gamma(\nu+1+k-u))^{6+l}\times$$

$$(\Gamma(-\nu-k+u))^{-l+2}(\Gamma(-k+u))^{2+l}(\Gamma(2+2\nu+k-u))^{-2-l} =$$

$$(-1)^{l\nu}\left(\frac{\pi}{\sin(u\pi)}\right)^4(z)^{-\nu-1-k+u}(\Gamma(\nu+1+k-u))^{4+2l}\times$$

$$(\Gamma(1+k-u))^{-2-l}(\Gamma(2+2\nu+k-u))^{-2-l} =$$

$$(-1)^{l\nu}\left(\frac{\pi}{\sin(u\pi)}\right)^4(z)^{-\nu-1-k+u}H^*_l(u, k, \nu) =$$

$$(-1)^{l\nu}\left(\frac{\pi}{\sin(u\pi)}\right)^4(z)^{-\nu-1-k+u}(R(T, \nu))^{2+l},$$

where $T = \nu + 1 + k - u, l = 2$ and $k \in [0, +\infty) \cap \mathbb{Z}$. Therefore

(22) $$Res(g^{(6+l,2+l)}_{4+2l,4+2l}; -1-\nu-k) =$$

$$\frac{1}{6}\lim_{u\to 0}\left(\left(\frac{\partial}{\partial u}\right)^4(u^4 H_{l,5}\right)(u, k, \nu)) =$$

$$\frac{1}{6}(-1)^{l\nu}\left(\frac{\pi}{\sin(u\pi)}\right)^4(z)^{-\nu-1-k}(\log(z))^3H_l^*(0,k,\nu)+$$
$$\frac{1}{2}(-1)^{l\nu}(z)^{-\nu-1-k}(\log(z))^2\left(\left(\frac{\partial}{\partial u}\right)H_l^*\right)(0,k,\nu)+$$
$$\frac{1}{2}(-1)^{l\nu}(z)^{-\nu-1-k}(\log(z))\left(\left(\frac{\partial}{\partial u}\right)^2H_l^*\right)(0,k,\nu)+$$
$$\frac{1}{6}(-1)^{l\nu}(z)^{-\nu-1-k}\left(\left(\frac{\partial}{\partial u}\right)^3H_l^*\right)(0,k,\nu)+$$
$$\pi^2\frac{2}{3}(-1)^{l\nu}(z)^{-\nu-1-k}(\log(z))H_l^*(0,k,\nu)+$$
$$\pi^2\frac{2}{3}(-1)^{l\nu}(z)^{-\nu-1-k}\left(\frac{\partial}{\partial u}H_l^*\right)(0,k,\nu).$$

We put now

(23)
$$f_{l,7}(z,\nu)=6^{-1}(\log(z))^3\sum_{t=\nu+1}^{\infty}z^{-t}R(t,\nu)^{2+l}-$$
$$2^{-1}(\log(z))^2\sum_{t=\nu+1}^{\infty}z^{-t}\left(\frac{\partial}{\partial t}(R^{2+l})\right)(t,\nu)+$$
$$2^{-1}(\log(z))\sum_{t=\nu+1}^{\infty}z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^2(R^{2+l})\right)(t,\nu)-$$
$$6^{-1}\sum_{t=\nu+1}^{\infty}z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^3(R^{2+l})\right)(t,\nu)=$$
$$6^{-1}(\log(z))^3f_{l,2}(z,\nu)+2^{-1}(\log(z))^2f_{l,4}(z,\nu)+(\log(z))f_{l,6}(z,\nu)+f_{l,8}(z,\nu)=$$
$$-3^{-1}(\log(z))^3f_{l,2}(z,\nu)+2^{-1}(\log(z))^2f_{l,3}(z,\nu)+f_{l,8}(z,\nu)+$$
$$(\log(z))(f_{l,5}(z,\nu)+2^{-1}(\log(z))^2f_{l,2}(z,\nu)-(\log(z))f_{l,3}(z,\nu))=$$
$$6^{-1}(\log(z))^3f_{l,2}(z,\nu)-2^{-1}(\log(z))^2f_{l,3}(z,\nu)+(\log(z))f_{l,5}(z,\nu)+f_{l,8}(z,\nu),$$

where

(24)
$$f_{l,8}(z,\nu)=-6^{-1}\sum_{t=\nu+1}^{\infty}z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^3(R^{2+l})\right)(t,\nu),$$

and, since $(R(t,\nu))^{2+l}$ for $\nu\in\mathbb{N}$ have in the points $t=1,\dots,\nu$, the zeros of the order $2+l$, and $l=2$ now, it follows that

(25)
$$f_{l,8}(z,\nu)=-6^{-1}\sum_{t=1}^{\infty}z^{-t}\left(\left(\frac{\partial}{\partial t}\right)^3(R^{2+l})\right)(t,\nu).$$

Then, clearly,

(26)
$$f_{l,7}^{\vee}(z,\nu)=f_{l,7}(z,\nu)+(2\pi^2/3)f_{l,3}(z,\nu).$$

Let

(27) $$S_i(t,\nu)=\left(\frac{\partial}{\partial t}\right)^i\left(\left(\sum_{k=1}^{\nu}\frac{1}{t-k}\right)-\sum_{k=0}^{\nu}\frac{1}{t+k})\right)=$$

$$(-1)^{i-1}(i-1)!\left(\left(\sum_{k=1}^{\nu}\frac{1}{(t-k)^i}\right)-\sum_{k=0}^{\nu}\frac{1}{(t+k)^i})\right),$$

(28) $$H_i(\nu)=S_i(\nu,\nu-1)-S_i(\nu+1,\nu)=$$

$$(-1)^{i-1}(i-1)!\left(-\frac{2}{(\nu)^i}+\frac{1}{(2\nu)^i}+\frac{1}{(2\nu+1)^i}\right).$$

Then

(29) $$\left(\frac{\partial}{\partial t}\right)(R(t,\nu))^{2+l})=(R(t,\nu))^{2+l}(2+l)S_1(t,\nu),$$

(30) $$\left(\frac{\partial}{\partial t}\right)^2(R(t,\nu))^{2+l})=$$

$$(R(t,\nu))^{2+l}((2+l)^2(S_1^2(t,\nu))+(2+l)S_2(t,\nu)),$$

(31) $$\left(\frac{\partial}{\partial t}\right)^3(R(t,\nu))^{2+l}=$$

$$(R(t,\nu))^{2+l}((2+l)^3S_1^3(t,\nu))^{2+l}+3(2+l)^2S_1^2(t,\nu))S_2(t,\nu)+(2+l)S_3(t,\nu)).$$

§2. Some relations. The case $|z|>1$.

Let

$$P_0(t)=1-4t+8t^2-12t^3,\ Q_0(t)=16+12t,$$

$$P_1(t)=-1+6t-18t^2+38t^3-66t^4+102t^5,\ Q_1(t)=-146-240t-102t^2,$$

$$P_2(t)=1-8t+32t^2-88t^3+192t^4-360t^5+608t^6-952t^7,$$

$$Q_2(t)=1408+3640t+3200t^2+952t^3.$$

It is not difficult to verify that

(32) $$0=T_l(t):=(t+1)^{2+l}P_l(t)+t^{4+2l}Q\ (t)-(t-1)^{2+l}$$

for $l=0,\ 1,\ 2$. Let (see (3.1.52) in [56] with $c_1(\nu)=c_2(\nu)=\nu,\ m=n=2+l$)

(33) $$D_l(z,\nu,w)=z(w-\nu)^{2+l}(w+\nu+1)^{2+l}-w^{4+2l},$$

where $l=0,\ 1,\ 2,\ \nu\in\mathbb{Z}$, and w is independent variable. Clearly,

(34) $$D_l(z,\nu,w)=D_l(z,-\nu-1,w)$$

for $l = 0, 1, 2$, $\nu \in \mathbb{Z}$, Let further $\delta := z\frac{\partial}{\partial z}$. Then (see (3.1.64) in [56])

$$D_l(z, \nu, \delta) f^{\vee}_{l,k}(z, \nu) = 0, \tag{35}$$

where $\nu \in [0, +\infty) \cap \mathbb{Z}$, $l = 0, 1, 2$ with $k = 1, 2, 3$ for $l = 0$, with $k = 1, 2, 3, 5$ for $l = 1$ and with $k = 1, 2, 3, 5, 7$ for $l = 2$. Further we have (see (3.1.53) – (3.1.54) with $d^{\vee} = d^{\wedge} = 1$ and (3.1.56) in [56])

$$(\delta + \nu + 1)^{2+l} f^{\vee}_{l,k}(z, \nu) = (\delta - \nu - 1)^{2+l} f^{\vee}_{l,k}(z, \nu + 1), \tag{36}$$

where $\nu \in [0, +\infty) \cap \mathbb{Z}$, $l = 0, 1, 2$ with $k = 1, 2, 3$ for $l = 0$, with $k = 1, 2, 3, 5$ for $l = 1$ and with $k = 1, 2, 3, 5, 7$ for $l = 2$,

$$(\delta + \nu)^{2+l} f^{\vee}_{l,k}(z, \nu - 1) = (\delta - \nu)^{2+l} f^{\vee}_{l,k}(z, \nu), \tag{37}$$

where $\nu \in \cap\mathbb{N}$, $l = 0, 1, 2$ with $k = 1, 2, 3$ for $l = 0$, with $k = 1, 2, 3, 5$ for $l = 1$ and with $k = 1, 2, 3, 5, 7$ for $l = 2$. Let

$$f^{\vee}_{l,k}(z, -\nu - 1) = f^{\vee}_{l,k}(z, \nu), \tag{38}$$

where $\nu \in [0, +\infty) \cap \mathbb{Z}$, $l = 0, 1, 2$ with $k = 1, 2, 3$ for $l = 0$, with $k = 1, 2, 3, 5$ for $l = 1$ and with $k = 1, 2, 3, 5, 7$ for $l = 2$. Then (37) holds also for $\nu = 0$. Therefore, if $\nu = -\kappa - 1$ with $\kappa \in [0, +\infty) \cap \mathbb{Z}$, then

$$f_{l,k}(z, \nu) = f_{l,k}(z, \kappa) = f_{l,k}(z, -\nu - 1),$$

$$(\delta + \nu + 1)^{2+l} f^{\vee}_{l,k}(z, \nu) = (\delta - \kappa)^{2+l} f^{\vee}_{l,k}(z, \kappa) =$$

$$(\delta + \kappa)^{2+l} f^{\vee}_{l,k}(z, \kappa - 1) = (\delta - \nu - 1)^{2+l} f^{\vee}_{l,k}(z, \nu + 1),$$

$$D_l(z, \nu, \delta) f^{\vee}_{l,k}(z, \nu) = D_l(z, -\kappa - 1, \delta) f^{\vee}_{l,k}(z, \kappa) D_l(z, \kappa, \delta) f^{\vee}_{l,k}(z, \kappa) = 0,$$

where $l = 0, 1, 2$ with $k = 1, 2, 3$ for $l = 0$, with $k = 1, 2, 3, 5$ for $l = 1$ and with $k = 1, 2, 3, 5, 7$ for $l = 2$. So, (35) – (38) hold for all the $\nu \in \mathbb{Z}$. We consider w and ν as independent variables now and we put $t = w/\nu$,

$$P^*_l(\nu; w) = \nu^{3+2l} P_l(w/nu),\ Q^*_l(\nu; w) = \nu^{1+l} Q_l(w/\nu),$$

$$T^*_l(\nu; w) = \nu^{5+3l} T_l(w/\nu).$$

Then

$$P^*_0(\nu; w) = \nu^3 - 4\nu^2 w + 8\nu w^2 - 12w^3,\ Q^*_0(\nu, w) = \tag{39}$$

$$16\nu + 12w,$$

$$P^*_1(\nu; w) = -\nu^5 + 6\nu^4 w - 18\nu^3 w^2 + 38\nu^2 w^3 - 66\nu w^4 + \tag{40}$$

$$102\nu^3 w^5,\ Q^*_1(\nu; w) = -146\nu^2 - 240\nu w - 102w^2,$$

$$P^*_2(\nu; w) = \nu^7 - 8\nu^6 w + 32\nu^5 w^2 - 88\nu^4 w^3 + 192\nu^3 w^4 - \tag{41}$$

$$360\nu^2 w^5 + 608\nu w^6 - 952w^7,\ Q^*_2(\nu; w) = 1408\nu^3 + 3640\nu^2 w + 3200\nu w^2 + 952w^3.$$

In view of (39) – (41),

$$z^{\nu+1}Q_0^*(\nu;\delta+1)z^{-\nu-1} = 16\nu - 12(\nu+1) + 12 = 4\nu,$$

$$z^{\nu+1}Q_1^*(\nu;\delta+1)z^{-\nu-1} = -8\nu^2,$$

$$z^{\nu+1}Q_2^*(\nu;\delta+1)z^{-\nu-1} = 16\nu^3,$$

(42) $$z^{\nu+1}Q_l^*(\nu;\delta+1)z^{-\nu-1} = (-1)^l 2^{l+2}\nu^{l+1},$$

(43) $$z^{\nu+1}Q_l^*(\nu;\delta+1)(\delta-\nu)^{2+l}z^{-\nu-1} = 2^{l+2}\nu^{l+1}(2\nu+1)^{2+l},$$

for $l = 0, 1, 2,$

$$\frac{\partial(zQ_l^*(\nu;w+1)(w-\nu)^{2+l})}{\partial w}) =$$

$$z\frac{\partial Q_l^*(\nu;w+1)}{\partial w}(w-\nu) + (2+l)Q_l^*(\nu;w+1))(w-\nu)^{1+l},$$

$$\frac{\partial(zQ_0^*(\nu;w+1)(w-\nu)^2)}{\partial w}) =$$

$$z(12(w-\nu) + 2(16\nu + 12(w+1))(w-\nu),$$

$$z(12(\delta-\nu) + 2(16\nu + 12(\delta+1))(\delta-\nu)z^{-\nu-1} =$$

$$-(-12(2\nu+1) + 8\nu)(2\nu+1)z^{-\nu} = 4(4\nu+3)(2\nu+1)z^{-\nu},$$

$$\frac{\partial(zQ_1^*(\nu;w+1)(w-\nu)^3)}{\partial w}) =$$

$$z((-240\nu-204(w+1))(w-\nu)+3(-146\nu^2-240\nu(w+1)-102(w+1)^2)(w-\nu)^2,$$

$$z((-240\nu - 204(\delta+1))(\delta-\nu)+$$

$$3(-146\nu^2 - 240\nu(\delta+1) - 102(\delta+1)^2)(\delta-\nu)^2 z^{-\nu-1} =$$

$$(36\nu(2\nu+1) - 24\nu^2)(2\nu+1)^2 = 12\nu(4\nu+3)(2\nu+1)^2 z^{-\nu},$$

$$z\frac{\partial Q_2^*(\nu;w+1)}{\partial w}(w-\nu) + 4Q_2^*(\nu;w+1))(w-\nu)^{1+l} =$$

$$z(3640\nu^2 + 6400\nu(w+1) + 2856(w+1)^2)(w-\nu)+$$

$$4(1408\nu^3 + 3640\nu^2(w+1) + 3200\nu(w+1)^2 + 952w^3))(w-\nu)^3,$$

$$z(3640\nu^2 + 6400\nu(\delta+1) + 2856(\delta+1)^2)(\delta-\nu)+$$

$$4(1408\nu^3 + 3640\nu^2(\delta+1) + 3200\nu(\delta+1)^2 + 952(\delta+1)^3))(\delta-\nu)^3 z^{-\nu-1} =$$

$$-(-96\nu^2(2\nu+1) + 64\nu^3)(2\nu+1)^3 z^{-\nu} = 32\nu^2(4\nu+3)(2\nu+1)^3 z^{-\nu},$$

(44)
$$\left.\frac{\partial zQ_l^*(\nu;w+1)(w-\nu)^{2+l}}{\partial w}\right|_{w=\delta} z^{-\nu-1} = 4(2^l + l2^{l-1})\nu^l(4\nu+3)(2\nu+1)^{l+1} =$$

$$2^{l+1}(2+l)\nu^l(4\nu+3)(2\nu+1)^{l+1}$$

for $l = 0, 1, 2$. Clearly,

$$\left(\frac{\partial}{\partial w}\right)^2 \left(zQ_l^*(\nu; w+1)(w-\nu)^{2+l}\right) =$$

$$\left(z\left(\frac{\partial}{\partial w}\right)^2 Q_l^*(\nu; w+1)\right)(w-\nu)^{2+l}+$$

$$2(2+l)\left(z\left(\frac{\partial}{\partial w}\right) Q_l^*(\nu; w+1)\right)(w-\nu)^{1+l}+$$

$$(1+l)(2+l)zQ_l^*(\nu; w+1)(w-\nu)^l = (w-\nu)^l\times$$

$$\left(\left(z\left(\frac{\partial}{\partial w}\right)^2 Q_l^*(\nu; w+1)\right)(w-\nu)^2+\right.$$

$$2(2+l)\left(z\left(\frac{\partial}{\partial w}\right) Q_l^*(\nu; w+1)\right)(w-\nu)+$$

$$\left.(1+l)(2+l)zQ_l^*(\nu; w+1)\right),$$

$$\left(\frac{\partial}{\partial w}\right)^2 \left(zQ_1^*(\nu; w+1)(w-\nu)^3\right) = (w-\nu)\times$$

$$(-204(w-\nu)^2 + 6(-240\nu - 204(w+1))(w-\nu)+$$

$$6(-146\nu^2 - 240\nu(w+1) - 102(w+1)^2)),$$

$$\left(\frac{\partial}{\partial w}\right)^2 \left(zQ_1^*(\nu; w+1)(w-\nu)^3\right)\Big|_{w=\delta} z^{-\nu-1} =$$

$$-(2\nu+1)(-204(2\nu+1)^2 + 216\nu(2\nu+1) - 48\nu^2) =$$

$$12(2\nu+1)(17(2\nu+1)^2 - 18\nu(2\nu+1) + 4\nu^2)z^{-\nu} =$$

$$12(2\nu+1)^3\nu^2(68/(2\nu)^2 - 36/(2\nu(2\nu+1)) + 4/(2\nu+1)^2)z^{-\nu} =$$

$$48(2\nu+1)^3\nu^2(17/(2\nu)^2 - 9/(2\nu(2\nu+1)) + 1/(2\nu+1)^2)z^{-\nu}$$

$$\left(\frac{\partial}{\partial w}\right)^2 \left(zQ_2^*(\nu; w+1)(w-\nu)^4\right) = (w-\nu)^2\times$$

$$((6400\nu + 5712(w+1))(w-\nu)^2 + 8(3640\nu^2 + 6400\nu(w+1)+$$

$$2856(w+1)^2)(w-\nu) + 12(1408\nu^3 + 3640\nu^2(w+1) + 3200\nu(w+1)^2 + 952(w+1)^3),$$

$$(1/2)\left(\left(\frac{\partial}{\partial w}\right)^2 \left(zQ_2^*(\nu; w+1)(w-\nu)^4\right)\right)\Big|_{w=\delta} z^{-\nu-1} =$$

$$(2\nu+1)^2(688\nu(2\nu+1)^2 - 768\nu^2(2\nu+1) + 192\nu^3)z^{-\nu}/2 =$$

$$8(2\nu+1)^2(43\nu(2\nu+1)^2 - 48\nu^2(2\nu+1) + 12\nu^3)z^{-\nu} =$$

$$32(2\nu+1)^4\nu^3(43/(2\nu)^2 - 24(2\nu) + 24/(2\nu+1) + 3/(2\nu+1)^2)z^{-\nu} =$$

$$\left(\left(\frac{\partial}{\partial w}\right)^2 (z^{\nu+1}Q_l^*(\nu; w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} z^{-\nu-1} = \tag{45}$$

$$\nu^{1+l}(2\nu+1)^{2+l}2^{2+l}(2+l)\times$$

$$((25+9l)/(2\nu)^2 + (1+l)/(2\nu+1)^2 - 6(2+l)/(2\nu) + 6(2+l)/(2\nu+1))$$

for $l = 1$, 2. Further we have

$$\left(\frac{\partial}{\partial w}\right)^3 \left(zQ_l^*(\nu; w+1)(w-\nu)^{2+l}\right) =$$

$$\left(z\left(\frac{\partial}{\partial w}\right)^3 Q_l^*(\nu; w+1)\right)(w-\nu)^{2+l}+$$

$$3(2+l)\left(z\left(\frac{\partial}{\partial w}\right)^2 Q_l^*(\nu; w+1)\right)(w-\nu)^{1+l}+$$

$$3(1+l)(2+l)\left(z\left(\frac{\partial}{\partial w}\right) Q_l^*(\nu; w+1)\right)(w-\nu)^{l}+$$

$$l(1+l)(2+l)zQ_l^*(\nu; w+1)(w-\nu)^{l-1},$$

and, if $l = 2$, then

$$\left(\frac{\partial}{\partial w}\right)^3 \left(zQ_2^*(\nu; w+1)(w-\nu)^4\right) =$$

$$z\Bigg(\left(\left(\frac{\partial}{\partial w}\right)^3 Q_2^*(\nu; w+1)\right)(w-\nu)^4+$$

$$12\left(\left(\frac{\partial}{\partial w}\right)^2 Q_2^*(\nu; w+1)\right)(w-\nu)^3+$$

$$36\left(\left(\frac{\partial}{\partial w}\right) Q_2^*(\nu; w+1)\right)(w-\nu)^2+$$

$$24Q_2^*(\nu; w+1)(w-\nu)\Bigg) =$$

$$z\Bigg(5712(w-\nu)^4+$$

$$12(6400\nu + 5712(w+1))(w-\nu)^3+$$

$$36(3640\nu^2 + 6400\nu(w+1) + 2856(w+1)^2)(w-\nu)^2+$$

$$24(1408\nu^3 + 3640\nu^2(w+1) + 3200\nu(w+1)^2 + 952(w+1)^3)(w-\nu)\Bigg) =$$

$$48z\Bigg(119(w-\nu)^4+$$

$$(1600\nu + 1428(w+1))(w-\nu)^3 +$$
$$3(910\nu^2 + 1600\nu(w+1) + 714(w+1)^2)(w-\nu)^2 +$$
$$\left.(704\nu^3 + 1820\nu^2(w+1) + 1600\nu(w+1)^2 + 476(w+1)^3)(w-\nu)\right),$$

(46) $$\left(\left(\frac{\partial}{\partial w}\right)^3 (z^{\nu+1}Q_2^*(\nu; w+1)(w-\nu)^4\right)\Bigg|_{w=\delta} z^{-\nu-1} =$$

$$48(119(2nu+1)^4 - 172\nu(2nu+1)^3 + 72\nu^2(2nu+1)^2 - 8(2nu+1))\nu^3) =$$
$$48(2nu+1)^4 nu^3\Bigg(952/(2\nu)^3 - 688/((2\nu)^2(2nu+1)) +$$
$$144/(2\nu(2nu+1)^2) - 8/(2nu+1)^3\Bigg) =$$
$$384(2nu+1)^4 nu^3\Bigg(119/(2\nu)^3 - 86/((2\nu)^2(2nu+1)) +$$
$$18/(2\nu(2nu+1)^2) - 1/(2nu+1)^3\Bigg).$$

Puting in (32) $t = w/\nu$, and then multiplying the result by ν^{5+3l}, we obtain the equality

(47) $$\nu^{3+2l}(w-\nu)^{2+l} = (w+\nu)^{2+l}P_l^*(\nu; w) + w^{4+2l}Q_l^*(\nu; w)$$

where $l = 0, 1, 2$. As it was established above, (37) holds for all the $\nu \in \mathbb{Z}$; therefore, in view of (37) and (47),

(48) $$(\delta+\nu)^{2+l}f_{l,k}^{\vee}(z, \nu-1) - (\delta+\nu)^{2+l}P_l^*(\nu;\delta)f_{l,k}^{\vee}(z,\nu) -$$
$$\delta^{4+2l}Q_l^*(\nu;\delta)f_{l,k}^{\vee}(z,\nu) = 0,$$

where $\nu \in \mathbb{Z}$, $l = 0, 1, 2$ with $k = 1, 2, 3$ for $l = 0$, with $k = 1, 2, 3, 5$ for $l = 1$ and with $k = 1, 2, 3, 5, 7$ for $l = 2$. Clearly,

(49)
$$z(\delta-\nu)^{2+l}(\delta+\nu+1)^{2+l} = (\delta+\nu)^{2+l}z(\delta-\nu)^{2+l},\ Q_l^*(\nu;\delta)z =$$
$$zQ_l^*(\nu;\delta+1),$$

where $\nu \in \mathbb{Z}$, $l = 0, 1, 2$ As it was stablished above, (35) holds for all the $\nu \in \mathbb{Z}$; therefore, in view of (33), (47) and (49),

(50) $$(\delta+\nu)^{2+l}(\nu^{3+2l}f_{l,k}^{\vee}(z,\nu-1) -$$
$$(P_l^*(\nu;\delta) + zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,k}^{\vee}(z,\nu)) = 0,$$

where $\nu \in \mathbb{Z}$, $l = 0, 1, 2$ with $k = 1, 2, 3$ for $l = 0$, with $k = 1, 2, 3, 5$ for $l = 1$ and with $k = 1, 2, 3, 5, 7$ for $l = 2$. As it was established above, (38) holds for all the $\nu\in\mathbb{Z}$; therefore, replacing in (50) ν by $-\nu-1$ we obtain the equality

$$(\delta-\nu-1)^{2+l}(-(\nu+1)^{3+2l}f^{\vee}_{l,k}(z,\nu+1)- \tag{51}$$

$$(P^*_l(-\nu-1;\delta)+zQ^*_l(-\nu-1;\delta+1)(\delta+\nu+1)^{2+l})f^{\vee}_{l,k}(z,\nu))=0,$$

where $\nu\in\mathbb{Z}$, $l=0,\ 1,\ 2$ with $k=1,\ 2,\ 3$ for $l=0$, with $k=1,\ 2,\ 3,\ 5$ for $l=1$ and with $k=1,\ 2,\ 3,\ 5,\ 7$ for $l=2$. We prove below that the multipliers $(\delta+\nu)^{2+l}$ and $(\delta-\nu-1)^{2+l}$ in respectively (50) and (51) can be omit. In view of (2), the function

$$\nu^{3+2l}f^{\vee}_{l,1}(z,\nu-1)-(P^*_l(\nu;\delta)-zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f^{\vee}_{l,1}(z,\nu)$$

belongs to $\mathbb{C}[z]$ for $\nu\in\mathbb{Z}$, $l=0,\ 1,\ 2$; if $\nu\in[0,+\infty)\cap\mathbb{Z}$, then null-space of the operator $(\delta+\nu)^{2+l}$ (as operaor on $\mathbb{C}[z]$) is different from zero only if $\nu=0$, and coincides with $\mathbb{C}$ in this case; moreover the equality

$$\nu^{3+2l}f^{\vee}_{l,1}(z,\nu-1)- \tag{52}$$

$$(P^*_l(\nu;\delta)+zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f^{\vee}_{l,1}(z,\nu)=0,$$

where $\nu\in\mathbb{Z}$, $l=0,\ 1,\ 2$, holds for $\nu=0$, because, according to (2), $f^{\vee}_{l,1}(z,0))=1$ for $l=0,\ 1,\ 2$, and, in wiew of (39) – (41),

$$P^*_0(0;\delta)=-12\delta^3,\ P^*_1(0;\delta)=102\delta^5,\ P^*_2(0;\delta)=-952\delta^7.$$

Thus, (50) holds for all the $\nu\in[0,+\infty)\cap\mathbb{Z}$, $l=0,\ 1,\ 2$. In view of (2), the function

$$-(\nu+1)^{3+2l}f^{\vee}_{l,1}(z,\nu+1)-$$

$$(P^*_l(-\nu-1;\delta)+zQ^*_l(-\nu-1;\delta+1)(\delta+\nu+1)^{2+l})f^{\vee}_{l,1}(z,\nu)$$

belongs to $\mathbb{C}[z]$ for $\nu\in\mathbb{Z}$, $l=0,\ 1,\ 2$; if $\nu\in[0,+\infty)\cap\mathbb{Z}$, then null-space of the operator $(\delta-\nu-1)^{2+l}$ (as operaor on $\mathbb{C}[z]$) coincides with $\mathbb{C}z^{\nu+1}$. Terefore, in order to establish the equality

$$-(\nu+1)^{3+2l}f^{\vee}_{l,1}(z,\nu+1)- \tag{53}$$

$$(P^*_l(-\nu-1;\delta)+zQ^*_l(-\nu-1;\delta+1)(\delta+\nu+1)^{2+l})f^{\vee}_{l,1}(z,\nu)=0$$

for $\nu\in\in[0,+\infty)\cap\mathbb{Z}$, $l=0,\ 1,\ 2$ it sufficient, in view of (2), (39), (40), and (41), to check the equaalities

$$-(\nu+1)^3\binom{2\nu+2}{\nu+1}^2+4(\nu+1)(2\nu+1)^2\binom{2\nu}{\nu}^2=0,$$

$$-(\nu+1)^5\binom{2\nu+2}{\nu+1}^3+8(\nu+1)^2(2\nu+1)^4\binom{2\nu}{\nu}^3=0,$$

$$-(\nu+1)^7\binom{2\nu+2}{\nu+1}^4+16(\nu+1)^3(2\nu+1)^4\binom{2\nu}{\nu}^4=0,$$

which, evidently, hold. Since (38) holds for all the $\nu \in \mathbb{Z}$, it follows from (52) and (53) that these equalities (52) and (53) also hold for all the $\nu \in \mathbb{Z}$. In view of (8), the function

$$-(\nu+1)^{3+2l} f^{\vee}_{l,2}(z,\nu+1)-$$

$$(P^*_l(-\nu-1;\delta)+zQ^*_l(-\nu-1;\delta+1)(\delta+\nu+1)^{2+l})f^{\vee}_{l,2}(z,\nu)$$

belongs to $\mathbb{C}[[z^{-1}]]$ where $\nu \in \mathbb{Z}$, $l = 0, 1, 2$, and $\mathbb{C}[[x]]$ denotes the linear space (and also ring) of all the formal power series over $\mathbb{C}$ with variable x. If $\nu \in [0,+\infty) \cap \mathbb{Z}$, then null-space of the operator $(\delta-\nu-1)^{2+l}$ (as operaor on linar over $\mathbb{C}$ space $\mathbb{C}[[z^{-1}]]$) coincides with 0. Terefore,

$$-(\nu+1)^{3+2l} f^{\vee}_{l,2}(z,\nu+1)- \tag{54}$$

$$(P^*_l(-\nu-1;\delta)+2zQ^*_l(-\nu-1;\delta+1)(\delta+\nu+1)^{2+l})f^{\vee}_{l,}(z,\nu)=0$$

for $\nu \in\in [0,+\infty) \cap \mathbb{Z}$, $l = 0, 1, 2$. In view of (8), the function

$$\nu^{3+2l} f^{\vee}_{l,2}(z,\nu-1)- \tag{55}$$

$$(P^*_l(\nu;\delta)+zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f^{\vee}_{l,2}(z,\nu)$$

belongs to $\mathbb{C}[[z^{-1}]]$ for $\nu \in \mathbb{Z}$, $l = 0, 1, 2$; if $\nu \in [0, +\infty) \cap \mathbb{Z}$, then null-space of the operator $(\delta+\nu)^{2+l}$ (as operaor on linar over $\mathbb{C}$ space $\mathbb{C}[[z^{-1}]]$) coincides with $z^{-\nu}\mathbb{C}$. Therefore in this case, according to (50), the series (55) belongs to $z^{-\nu}\mathbb{C}$. In view of (8), (43), the coefficient at $z^{-\nu}$ in the series (55) is equal to

$$\nu^{3+2l}(((\nu-1)!)^2/(2\nu-1)!)^{2+l}- \tag{56}$$

$$z^{\nu+1}Q^*_l(\nu;\delta+1)z^{-\nu-1}(-1)^l(2\nu+1)^{2+l}((\nu!)^2/(2\nu+1)!)^{2+l}=$$
$$\nu^{3+2l}(((\nu-1)!)^2/(2\nu-1)!)^{2+l}-$$
$$2^{l+2}\nu^{l+1}(2\nu+1)^{2+l}((\nu!)^2/(2\nu+1)!)^{2+l}=0.$$

Consequently,

$$\nu^{3+2l} f^{\vee}_{l,2}(z,\nu-1)- \tag{57}$$

$$(P^*_l(\nu;\delta)-zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f^{\vee}_{l,2}(z,\nu)=0$$

for $\nu \in [0, +\infty) \cap \mathbb{Z}$, $l = 0, 1, 2$. Since (38) holds for all the $\nu \in \mathbb{Z}$, it follows from (54) and (57) that these equalities (54) and (57) also hold for all the $\nu \in \mathbb{Z}$. In view of (11) – (13), the function

$$-(\nu+1)^{3+2l} f^{\vee}_{l,3}(z,\nu+1)-$$

$$(P^*_l(-\nu-1;\delta)+zQ^*_l(-\nu-1;\delta+1)(\delta+\nu+1)^{2+l})f^{\vee}_{l,3}(z,\nu)$$

belongs to the linear space $\mathbb{C}[[z^{-1}]]+(\log(z))\mathbb{C}[[z^{-1}]]$ where $\nu \in \mathbb{Z}$, $l = 0, 1, 2$; we can interpret this linear space as linear over $\mathbb{C}$ space

$$\mathbb{C}[[z^{-1}]] \oplus \mathbb{C}[[z^{-1}]]$$

with operator δ, which acts according to the formula

$$\delta\begin{pmatrix}T_1\\T_2\end{pmatrix}=\begin{pmatrix}\delta T_1+T_2\\ \delta T_2\end{pmatrix}, \tag{58}$$

where $T_k\in\mathbb{C}[[z^{-1}]]$ for $k=1,\,2$. If $\nu\in[0,+\infty)\cap\mathbb{Z}$, then null-space of the operator $(\delta-\nu-1)^{2+l}$ (as operaor on linar over $\mathbb{C}$ space $\mathbb{C}[[z^{-1}]]\oplus\mathbb{C}[[z^{-1}]]$) coincides with 0. Terefore,

$$-(\nu+1)^{3+2l}f^{\vee}_{l,3}(z,\nu+1)- \tag{59}$$

$$(P^*_l(-\nu-1;\delta)+zQ^*_l(-\nu-1;\delta+1)(\delta+\nu+1)^{2+l})f^{\vee}_{l,3}(z,\nu)=0$$

for $\nu\in\in[0,+\infty)\cap\mathbb{Z}$, $l=0,\,1,\,2$. In view of (10) – (13), the function

$$\nu^{3+2l}f^{\vee}_{l,3}(z,\nu-1)- \tag{60}$$

$$(P^*_l(\nu;\delta)-zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f^{\vee}_{l,3}(z,\nu)$$

also belongs to linear over $\mathbb{C}$ space

$$\mathbb{C}[[z^{-1}]]+(\log(z))\mathbb{C}[[z^{-1}]].$$

If $\nu\in[0,+\infty)\cap\mathbb{Z}$, then null-space of the operator $(\delta+\nu)^{2+l}$ (as operaor on linar over $\mathbb{C}$ space $\mathbb{C}[[z^{-1}]]+(\log(z))\mathbb{C}[[z^{-1}]]$), in view of (58), belongs to $z^{-\nu}(\mathbb{C}[[z^{-1}]+(\log(z))\mathbb{C}[[z^{-1}]])])$. Sine for any diferentiable $f(z)$ we have the equality

$$w^k\Big|_{w=\delta}(\log(z))f(z))=\left(\log(z))w^k\Big|_{w=\delta}+(\partial w^k/\partial w)\Big|_{w=\delta}\right)f(z),$$

it follows that for any $T(w)\in\mathbb{C}[w]$ the equality

$$T(w)\Big|_{w=\delta}(\log(z))f(z))=\left((log(z))T(w)\Big|_{w=\delta}+(\partial T(w)/\partial w)\Big|_{w=\delta}\right)f(z)$$

holds. In other words

$$T(\delta)\begin{pmatrix}T_1\\T_2\end{pmatrix}=\begin{pmatrix}T(\delta)T_1+(\partial T(w)/\partial w)\Big|_{w=\delta}T_2\\ T(\delta)T_2\end{pmatrix}, \tag{61}$$

where $T\in\mathbb{C}[w]$, $T_k\in\mathbb{C}[[z^{-1}]]$ for $k=1,\,2$. In view of (50), (10) – (13), (61), (58), (42) – (44) (57), (27) – (29)

$$\nu^{3+2l}f^{\vee}_{l,3}(z,\nu-1)-(P^*_l(\nu;\delta)+zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f^{\vee}_{l,3}(z,\nu)=$$

$$(log(z))(\nu^{3+2l}f^{\vee}_{l,2}(z,\nu-1)-(P^*_l(\nu;\delta)+zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f^{\vee}_{l,2}(z,\nu))-$$

$$\frac{\partial(P^*_l(\nu;w)+zQ^*_l(\nu;w+1)(w-\nu)^{2+l})}{\partial w}\Bigg|_{w=\delta}f^{\vee}_{l,2}(z,\nu)+$$

$$\nu^{3+2l}f^{\vee}_{l,4}(z,\nu-1)-(P^*_l(\nu;\delta)+zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f^{\vee}_{l,4}(z,\nu)=$$

$$-\frac{\partial(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l})}{\partial w}\Bigg|_{w=\delta} f_{l,2}^{\vee}(z,\nu))+$$

$$\nu^{3+2l}f_{l,4}^{\vee}(z,\nu-1)-(P_l^*(\nu;\delta)+zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,4}^{\vee}(z,\nu)=$$

$$-((\nu!)^2/(2\nu+1)!)^{2+l}\frac{\partial zQ_l^*(\nu;w+1)(w-\nu)^{2+l}}{\partial w}\Bigg|_{w=\delta} z^{-\nu-1}-$$
$$\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{1+l}S_1(\nu,\nu-1)z^{-\nu}+$$
$$(2+l)((\nu!)^2/(2\nu+1)!)^{1+l}S_1(\nu+1,\nu)\times$$
$$zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})z^{-\nu-1}=$$
$$-((\nu!)^2/(2\nu+1)!)^{2+l}4(2^l+l2^{l-1})\nu^l(4\nu+3)(2\nu+1)^{l+1}-$$
$$\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}S_1(\nu,\nu-1)z^{-\nu}+$$
$$(2+l)((\nu!)^2/(2\nu+1)!)^{2+l}2^{l+2}\nu^{l+1}(2\nu+1)^{2+l}S_2(\nu+1,\nu)z^{-\nu}=$$
$$\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$(1/nu+1/nu-1/(2\nu)-1/(2\nu+1)-(4\nu+3)/(2\nu(2\nu+1)))=0.$$

Consequently,

(62) $$\nu^{3+2l}f_{l,3}^{\vee}(z,\nu-1)-$$
$$(P_l^*(\nu;\delta)-zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,2}^{\vee}(z,\nu)=0$$

for $\nu\in[0,+\infty)\cap\mathbb{Z}$, $l=0,\,1,\,2$. Since (38) holds for all the $\nu\in\mathbb{Z}$, it follows from (59) and (62) that these equalities (59) and (62) also hold for all the $\nu\in\mathbb{Z}$. In view of (16) – (20), the function

$$-(\nu+1)^{3+2l}f_{l,5}^{\vee}(z,\nu+1)-$$
$$(P_l^*(-\nu-1;\delta)+zQ_l^*(-\nu-1;\delta+1)(\delta+\nu+1)^{2+l})f_{l,5}^{\vee}(z,\nu)$$

belongs to the linear space $\mathbb{C}[[z^{-1}]]+(\log(z))\mathbb{C}[[z^{-1}]]+((\log(z))^2/2)\mathbb{C}[[z^{-1}]]$, where $\nu\in\mathbb{Z}$, $l=1,\,2$; we can interpret this linear space as linear over $\mathbb{C}$ space

$$\mathbb{C}[[z^{-1}]]\oplus\mathbb{C}[[z^{-1}]]\oplus\mathbb{C}[[z^{-1}]]$$

with operator δ, which acts according to the formula

(63) $$\delta\begin{pmatrix}T_1\\T_2\\T_3\end{pmatrix}=\begin{pmatrix}\delta T_1+T_2\\\delta T_2+T_3\\\delta T_3\end{pmatrix},$$

where $T_k\in\mathbb{C}[[z^{-1}]]$ for $k=1,\,2,\,3$. If $\nu\in[0,+\infty)\cap\mathbb{Z}$, then null-space of the operator $(\delta-\nu-1)^{2+l}$ (as operaor on linar over $\mathbb{C}$ space $\mathbb{C}[[z^{-1}]]\oplus\mathbb{C}[[z^{-1}]]\oplus\mathbb{C}[[z^{-1}]]$) coincides with 0. Terefore,

(64) $$-(\nu+1)^{3+2l}f_{l,5}^{\vee}(z,\nu+1)-$$

$$(P_l^*(-\nu-1;\delta)+zQ_l^*(-\nu-1;\delta+1)(\delta+\nu+1)^{2+l})f_{l,5}^{\vee}(z,\nu)=0$$

for $\nu\in[0,+\infty)\cap\mathbb{Z}$, $l=1$, 2. In view of (16) – (20), the function

(65)
$$\nu^{3+2l}f_{l,5}^{\vee}(z,\nu-1)-$$
$$(P_l^*(\nu;\delta)-zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,5}^{\vee}(z,\nu)$$

also belongs to linear over $\mathbb{C}$ space

$$\mathbb{C}[[z^{-1}]]+(\log(z))\mathbb{C}[[z^{-1}]]+((\log(z))^2)/2\mathbb{C}[[z^{-1}]].$$

If $\nu\in[0,+\infty)\cap\mathbb{Z}$, then null-space of the operator $(\delta+\nu)^{2+l}$ (as operaor on linar over $\mathbb{C}$ space $\mathbb{C}[[z^{-1}]]+(\log(z))\mathbb{C}[[z^{-1}]]+((\log(z))^2/2)\mathbb{C}[[z^{-1}]]$), in view of (58), belongs to $z^{-\nu}(\mathbb{C}[[z^{-1}]+(\log(z))\mathbb{C}[[z^{-1}]]))+((\log(z))^2/2)\mathbb{C}[[z^{-1}]]))$. Since for any diferentiable $f(z)$ we have the equality

$$w^k\Big|_{w=\delta}\left(\frac{(log(z))^2}{2}f(z)\right)=$$
$$\left(\frac{(log(z))^2}{2}w^3\Big|_{w=\delta}+\frac{log(z)}{2}\left(\frac{\partial}{\partial w}\right)w^k\Big|_{w=\delta}\right)f(z)+$$
$$\frac{log(z)}{2}\left(\frac{\partial}{\partial w}\right)^2w^k\Big|_{w=\delta}f(z),$$
$$(1/6)\left(\left(\frac{\partial}{\partial w}\right)^3\Big|_{w=\delta}\right)f(z)+$$

it follows that for any $T(w)\in\mathbb{C}[w]$ the equality

$$T(w)\Big|_{w=\delta}\left(\frac{(\log(z))^2}{2}f(z)\right)=$$
$$\frac{(\log(z))^2}{2}T(w)\Big|_{w=\delta}f(z)+(\log(z))\left(\frac{\partial}{\partial w}\right)T(w)\Big|_{w=\delta}f(z)+$$
$$\frac{1}{2}\left(\frac{\partial}{\partial w}\right)^2T(w)\Big|_{w=\delta}f(z)$$

holds. Therefore

(66)
$$T(\delta)\begin{pmatrix}T_1\\T_2\\T_3\end{pmatrix}=$$
$$\begin{pmatrix}T(\delta)T_1+\frac{\partial T(w)}{\partial w}\Big|_{w=\delta}T_2+\frac{1}{2}\left(\frac{\partial}{\partial w}\right)^2T(w)\Big|_{w=\delta}T_3\\ T(\delta)T_2+\left(\frac{\partial}{\partial w}\right)T(w)\Big|_{w=\delta}T_3\\ T(\delta)T_3\end{pmatrix},$$

where $T\in\mathbb{C}[w]$, $T_k\in\mathbb{C}[[z^{-1}]]$ for $k=1$, 2, 3. Consequently, in view of (50), (16) – (20), (58), (57), (62), (66), (27) – (30),

(67)
$$\nu^{3+2l}f_{l,5}^{\vee}(z,\nu-1)-(P_l^*(\nu;\delta)+zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,5}^{\vee}(z,\nu)=$$

$$-i\pi(\nu^{3+2l}f^{\vee}_{l,3}(z,\nu-1)-(P^*_l(\nu;\delta)+zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f^{\vee}_{l,3}(z,\nu))+$$
$$\nu^{3+2l}f_{l,5}(z,\nu-1)-(P^*_l(\nu;\delta)+zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,5}(z,\nu)=$$

$$\nu^{3+2l}f_{l,5}(z,\nu-1)-(P^*_l(\nu;\delta)+zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,5}(z,\nu)=$$
$$-\frac{(\log(z))^2}{2}(\nu^{3+2l}f_{l,2}(z,\nu-1)-(P^*_l(\nu;\delta)+zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,2})(z,\nu))+$$
$$(\log(z))(\nu^{3+2l}f_{l,3}(z,\nu-1)-(P^*_l(\nu;\delta)+zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,3})(z,\nu))+$$
$$(\log(z))\left(\left(\frac{\partial}{\partial w}\right)(P^*_l(\nu;w)+zQ^*_l(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta}f^{\vee}_{l,2}(z,\nu)+$$
$$(1/2)\left(\left(\frac{\partial}{\partial w}\right)^2(P^*_l(\nu;w)+zQ^*_l(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta}f^{\vee}_{l,2}(z,\nu)-$$
$$\frac{\partial(P^*_l(\nu;w)+zQ^*_l(\nu;w+1)(w-\nu)^{2+l})}{\partial w}\Bigg|_{w=\delta}f^{\vee}_{l,3}(z,\nu)+$$
$$\nu^{3+2l}f_{l,6}(z,\nu-1)-(P^*_l(\nu;\delta)+zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,6}(z,\nu)=$$

$$(\log(z))\frac{\partial(P^*_l(\nu;w)+zQ^*_l(\nu;w+1)(w-\nu)^{2+l})}{\partial w}\Bigg|_{w=\delta}f^{\vee}_{l,2}(z,\nu)+$$

$$(1/2)\left(\left(\frac{\partial}{\partial w}\right)^2(P^*_l(\nu;w)+zQ^*_l(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta}f^{\vee}_{l,2}(z,\nu)-$$

$$\log(z)\frac{\partial(P^*_l(\nu;w)+zQ^*_l(\nu;w+1)(w-\nu)^{2+l})}{\partial w}\Bigg|_{w=\delta}f^{\vee}_{l,2}(z,\nu)-$$

$$\left(\left(\frac{\partial}{\partial w}\right)^2(P^*_l(\nu;w)+zQ^*_l(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta}f^{\vee}_{l,2}(z,\nu)-$$
$$\frac{\partial(P^*_l(\nu;w)+zQ^*_l(\nu;w+1)(w-\nu)^{2+l})}{\partial w}\Bigg|_{w=\delta}f^{\vee}_{l,4}(z,\nu)+$$
$$\nu^{3+2l}f_{l,6}(z,\nu-1)-(P^*_l(\nu;\delta)+zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,6}(z,\nu)=$$
$$-(1/2)\left(\left(\frac{\partial}{\partial w}\right)^2(zQ^*_l(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta}f^{\vee}_{l,2}(z,\nu)-$$
$$\frac{zQ^*_l(\nu;w+1)(w-\nu)^{2+l})}{\partial w}\Bigg|_{w=\delta}f^{\vee}_{l,4}(z,\nu)+$$
$$\nu^{3+2l}f_{l,6}(z,\nu-1)-(zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,6}(z,\nu).$$

In view of (27) – (30), coefficient at $z^{-\nu}$ in the series

$$\nu^{3+2l} f_{l,6}(z,\nu-1) - (P_l^*(\nu;\delta) + zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l}) f_{l,6}(z,\nu)$$

is equal to

(68) $$(1/2)\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}((2+l)\times$$
$$S_1(\nu,\nu-1)^2 + S_2(\nu,\nu-1))-$$
$$(2+l)/2(((\nu)!)^2/(2\nu+1)!)^{2+l}((2+l)S_1(\nu+1,\nu)^2 + S_2(\nu+1,\nu))\times$$
$$(zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})z^{-\nu-1} =$$
$$(1/2)\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}((2+l)S_1(\nu,\nu-1)^2 + S_2(\nu,\nu-1))-$$
$$(1/2)(((\nu)!)^2/(2\nu+1)!)^{2+l}((2+l)S_1(\nu+1,\nu)^2 + S_2(\nu+1,\nu))\times$$
$$2^{l+2}\nu^{l+1}(2\nu+1)^{2+l} =$$
$$(1/2)\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$((2+l)(H_1(\nu)(2S_1(\nu+1,\nu) + H_1(\nu)) + H_2(\nu)) =$$
$$\frac{1}{2}\nu^{3+2l}(2+l)\left(\frac{(\nu-1)!)^2}{(2\nu-1)!}\right)^{2+l}\Bigg($$
$$\left(2S_1(\nu) - \frac{3}{2\nu} + \frac{1}{2\nu+1}\right)\left(-\frac{3}{2\nu} + \frac{1}{2\nu+1}\right)(2+l)-$$
$$\left(-\frac{2}{\nu^2} + \frac{1}{4\nu^2} + \frac{1}{(2\nu+1)^2}\right)\Bigg) =$$
$$-\nu^{3+2l}(2+l)^2(((\nu-1)!)^2/(2\nu-1)!)^{2+l}S_1(\nu+1,\nu)(4\nu+3)/((2\nu)(2\nu+1))+$$
$$\frac{1}{2}\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}((2+l)(-3/(2\nu) + 1/(2\nu+1))^2-$$
$$(-2/\nu^2 + 1/(4\nu^2) + 1/(2\nu+1)^2) =$$
$$-\nu^{3+2l}(2+l)^2(((\nu-1)!)^2/(2\nu-1)!)^{2+l}S_1(\nu+1,\nu)(4\nu+3)/((2\nu)(2\nu+1))+$$
$$\frac{1}{2}\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$((25+9l)/(2\nu)^2 + (1+l)/(2\nu+1)^2 - 6(2+l)/(2\nu) + 6(2+l)/(2\nu+1)).$$

In wiew of (12), (50), (68), (44), coefficient at $z^{-\nu}$ in the series

$$-\frac{\partial(P_l^*(\nu;w) + zQ_l^*(\nu;w+1)(w-\nu)^{2+l})}{\partial w}\Bigg|_{w=\delta} f_{l,4}^{\vee}(z,\nu)+$$

$$\nu^{3+2l} f_{l,6}(z,\nu-1) - (P_l^*(\nu;\delta) + zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l}) f_{l,6}(z,\nu)$$

is equal to

$$(2+l)(((\nu)!)^2/(2\nu+1)!)^{2+l}S_1(\nu+1,\nu)\times$$
$$z^{\nu+1}\frac{Q_l^*(\nu;w+1)(w-\nu)^{2+l})}{\partial w}\Bigg|_{w=\delta} z^{-\nu-1}-$$

$$\nu^{3+2l}(2+l)^2(((\nu-1)!)^2/(2\nu-1)!)^{2+l}S_1(\nu+1,\nu)(4\nu+3)/((2\nu)(2\nu+1))+$$
$$\frac{1}{2}\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$((25+9l)/(2\nu)^2+(1+l)/(2\nu+1)^2-6(2+l)/(2\nu)+6(2+l)/(2\nu+1))=$$
$$(2+l)((\nu)!)^2/(2\nu+1)!)^{2+l}S_1(\nu)2^{l+1}(2+l)\nu^l(4\nu+3)(2\nu+1)^{l+1}-$$
$$\nu^{3+2l}(2+l)^2(((\nu-1)!)^2/(2\nu-1)!)^{2+l}S_1(\nu+1,\nu)(4\nu+3)/((2\nu)(2\nu+1))+$$
$$\frac{1}{2}\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$((25+9l)/(2\nu)^2+(1+l)/(2\nu+1)^2-6(2+l)/(2\nu)+6(2+l)/(2\nu+1))=$$
$$\nu^{3+2l}(2+l)((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$\frac{1}{2}((25+9l)/(2\nu)^2+(1+l)/(2\nu+1)^2-6(2+l)/(2\nu)+6(2+l)/(2\nu+1)).$$

In view of (45), coefficient at $z^{-\nu}$ in the series (67) is equal to

$$-\frac{1}{2}(\nu!)^2/(2\nu+1)!)^{2+l}\times$$
$$\left(\left(\frac{\partial}{\partial w}\right)^2(z^{\nu+1}Q_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta}z^{-\nu-1}+$$
$$\frac{1}{2}\nu^{3+2l}(2+l)((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$((25+9l)/(2\nu)^2+(1+l)/(2\nu+1)^2-6(2+l)/(2\nu)+6(2+l)/(2\nu+1))=$$

$$-\frac{1}{2}\nu^{2+l}(2\nu+1)^{-2-l}((\nu-1)!)^2/(2\nu-1)!)^{2+l}2^{-2-l}\times$$
$$\left(\left(\frac{\partial}{\partial w}\right)^2(z^{\nu+1}Q_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta}z^{-\nu-1}+$$
$$\frac{1}{2}\nu^{3+2l}(2+l)((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$((25+9l)/(2\nu)^2+(1+l)/(2\nu+1)^2-6(2+l)/(2\nu)+6(2+l)/(2\nu+1))=0.$$

Therefore

$$\nu^{3+2l}f^{\vee}_{l,5}(z,\nu-1)- \tag{69}$$
$$(P_l^*(\nu;\delta)+zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f^{\vee}_{l,5}(z,\nu)=0$$

for $\nu\in[0,+\infty)\cap\mathbb{Z}$, $l=1,2$. Since (38) holds for all the $\nu\in\mathbb{Z}$, it follows from (64) and (69) that these equalities (64) and (69) also hold for all the $\nu\in\mathbb{Z}$. In view of (23) – (26), (13), (16) – (20), the function

$$-(\nu+1)^{3+2l}f^{\vee}_{l,7}(z,\nu+1)-$$
$$(P_l^*(-\nu-1;\delta)+zQ_l^*(-\nu-1;\delta+1)(\delta+\nu+1)^{2+l})f^{\vee}_{l,7}(z,\nu)$$

belongs to the linear space $\mathbb{C}[[z^{-1}]]+(\log(z))\mathbb{C}[[z^{-1}]]+((\log(z))^2/2)\mathbb{C}[[z^{-1}]]+((\log(z))^3/6)\mathbb{C}[[z^{-1}]]$, where $\nu\in\mathbb{Z}$, $l=1,\,2$; we can interpret this linear space as linear over $\mathbb{C}$ space

$$\mathbb{C}[[z^{-1}]]\oplus\mathbb{C}[[z^{-1}]]\oplus\mathbb{C}[[z^{-1}]]\oplus\mathbb{C}[[z^{-1}]]$$

with operator δ, which acts according to the formula

$$\delta\begin{pmatrix}T_1\\T_2\\T_3\\T_4\end{pmatrix}=\begin{pmatrix}\delta T_1+T_2\\\delta T_2+T_3\\\delta T_3+T_4\\\delta T_4\end{pmatrix},\tag{70}$$

where $T_k\in\mathbb{C}[[z^{-1}]]$ for $k=1,\,2,\,3,4$ If $\nu\in[0,+\infty)\cap\mathbb{Z}$, then null-space of the operator $(\delta-\nu-1)^{2+l}$ (as operaor on linar over $\mathbb{C}$ space $\mathbb{C}[[z^{-1}]]\oplus\mathbb{C}[[z^{-1}]]\oplus\mathbb{C}[[z^{-1}]]\oplus\mathbb{C}[[z^{-1}]]$) coincides with 0. Terefore,

$$-(\nu+1)^{3+2l}f^{\vee}_{l,7}(z,\nu+1)-\tag{71}$$

$$(P^*_l(-\nu-1;\delta)+zQ^*_l(-\nu-1;\delta+1)(\delta+\nu+1)^{2+l})f^{\vee}_{l,7}(z,\nu)=0$$

for $\nu\in[0,+\infty)\cap\mathbb{Z}$, $l=2$. In view of (23) – (26), (13), (16) – (20), the function

$$\nu^{3+2l}f^{\vee}_{l,7}(z,\nu-1)-\tag{72}$$

$$(P^*_l(\nu;\delta)-zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f^{\vee}_{l,7}(z,\nu)$$

also belongs to linear over $\mathbb{C}$ space

$$\mathbb{C}[[z^{-1}]]+(\log(z))\mathbb{C}[[z^{-1}]]+((\log(z)^2)/2)\mathbb{C}[[z^{-1}]]+((\log(z))^3/6)\mathbb{C}[[z^{-1}]].$$

If $\nu\in[0,+\infty)\cap\mathbb{Z}$, then null-space of the operator $(\delta+\nu)^{2+l}$ (as operaor on linar over $\mathbb{C}$ space

$$\mathbb{C}[[z^{-1}]]+(\log(z))\mathbb{C}[[z^{-1}]]+$$
$$((\log(z))^2/2)\mathbb{C}[[z^{-1}]]+((\log(z))^3/6)\mathbb{C}[[z^{-1}]],$$

in view of (72), belongs to

$$z^{-\nu}\times$$
$$(\mathbb{C}[[z^{-1}]+(\log(z))\mathbb{C}[[z^{-1}]]))+((\log(z))^2/2)\mathbb{C}[[z^{-1}]]))+((\log(z))^3/6)\mathbb{C}[[z^{-1}]].$$

Since for any diferentiable $f(z)$ we have the equality

$$w^k\Big|_{w=\delta}((log(z))^3/6)f(z))=$$

$$\left(((log(z))^3/6)w^3\Big|_{w=\delta}+((log(z))^2/2)\left(\frac{\partial}{\partial w}\right)w^k\Big|_{w=\delta}\right)f(z)+$$

$$((log(z))(1/2)\left(\left(\frac{\partial}{\partial w}\right)^2w^k\Big|_{w=\delta}\right)f(z)+$$

$$(1/6)\left(\left(\frac{\partial}{\partial w}\right)^3\Big|_{w=\delta}\right)f(z)+$$

it follows that for any $T(w) \in \mathbb{C}[w]$ the equality

$$T(w)\Big|_{w=\delta}\left(\frac{(\log(z))^3}{6}f(z)\right) =$$

$$\frac{(\log(z))^3}{6}T(w)\Big|_{w=\delta}f(z) + \frac{(\log(z))^2}{2}\left(\frac{\partial}{\partial w}\right)T(w)\Big|_{w=\delta}f(z)+$$

$$\frac{\log(z)}{2}\left(\frac{\partial}{\partial w}\right)^2T(w)\Big|_{w=\delta}f(z)+$$

$$\frac{1}{6}\left(\frac{\partial}{\partial w}\right)^3T(w)\Big|_{w=\delta}f(z)$$

holds. Therefore

(73)
$$T(\delta)\begin{pmatrix}T_1\\T_2\\T_3\\T4\end{pmatrix} =$$

$$\begin{pmatrix} T(\delta)T_1 + \frac{\partial T(w)}{\partial w}\Big|_{w=\delta}T_2 + \frac{1}{2}\left(\frac{\partial}{\partial w}\right)^2T(w)\Big|_{w=\delta}T_3 + \frac{1}{6}\left(\frac{\partial}{\partial w}\right)^3T(w)\Big|_{w=\delta}T_4\\ T(\delta)T_2 + \left(\frac{\partial}{\partial w}\right)T(w)\Big|_{w=\delta}T_3 + \frac{1}{2}\left(\frac{\partial}{\partial w}\right)^2T(w)\Big|_{w=\delta}T_4\\ T(\delta)T_3 + \left(\frac{\partial}{\partial w}\right)T(w)\Big|_{w=\delta}T_4\\ T(\delta)T_4\end{pmatrix},$$

where $T \in \mathbb{C}[w]$, $T_k \in \mathbb{C}[[z^{-1}]]$ for $k = 1, 2, 3, 4$. Consequently, in view of (50), (23) – (26), (16) – (20), (58), (57), (62, (66, (27) – (31)

(74)
$$\nu^{3+2l}f^{\vee}_{l,7}(z,\nu-1) - (P^*_l(\nu;\delta) + zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f^{\vee}_{l,7}(z,\nu) =$$

$$(2\pi^2/3)(\nu^{3+2l}f^{\vee}_{l,3}(z,\nu-1) - (P^*_l(\nu;\delta) + zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f^{\vee}_{l,3}(z,\nu))+$$

$$\nu^{3+2l}f_{l,7}(z,\nu-1) - (P^*_l(\nu;\delta) + zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,7}(z,\nu) =$$

$$\nu^{3+2l}f_{l,7}(z,\nu-1) - (P^*_l(\nu;\delta) + zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,7}(z,\nu) =$$

$$\frac{(\log(z))^3}{6}(\nu^{3+2l}f_{l,2}(z,\nu-1) - (P^*_l(\nu;\delta) + zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,2})(z,\nu))+$$

$$\frac{-(\log(z))^2}{2}(\nu^{3+2l}f_{l,3}(z,\nu-1) - (P^*_l(\nu;\delta) + zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,3})(z,\nu))+$$

$$(\log(z)(\nu^{3+2l}f_{l,5}(z,\nu-1) - (P^*_l(\nu;\delta) + zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,5})(z,\nu))+$$

$$(\nu^{3+2l}f_{l,8}(z,\nu-1) - (P^*_l(\nu;\delta) + zQ^*_l(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,8})(z,\nu))+$$

$$\frac{-(\log(z))^2}{2}\left(\left(\frac{\partial}{\partial w}\right)(P^*_l(\nu;w) + zQ^*_l(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta}f_{l,2}(z,\nu)+$$

$$\frac{-\log(z)}{2}\left(\left(\frac{\partial}{\partial w}\right)^2(P^*_l(\nu;w) + zQ^*_l(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta}f_{l,2}(z,\nu)+$$

$$\frac{-1}{6}\left(\left(\frac{\partial}{\partial w}\right)^3 (P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$(\log(z))\left(\left(\frac{\partial}{\partial w}\right) (P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,3}(z,\nu)+$$

$$\frac{1}{2}\left(\left(\frac{\partial}{\partial w}\right)^2 (P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,3}(z,\nu)+$$

$$\left(-\left(\frac{\partial}{\partial w}\right) (P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,5}(z,\nu)=$$

$$(\nu^{3+2l}f_{l,8}(z,\nu-1)-(P_l^*(\nu;\delta)+zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,8})(z,\nu))+$$

$$\frac{-(\log(z))^2}{2}\left(\left(\frac{\partial}{\partial w}\right) (P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$\frac{-\log(z)}{2}\left(\left(\frac{\partial}{\partial w}\right)^2 (P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$\frac{-1}{6}\left(\left(\frac{\partial}{\partial w}\right)^3 (P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$(\log(z))\left(\left(\frac{\partial}{\partial w}\right) (P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,3}(z,\nu)+$$

$$\frac{1}{2}\left(\left(\frac{\partial}{\partial w}\right)^2 (P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,3}(z,\nu)+$$

$$\left(-\left(\frac{\partial}{\partial w}\right) (P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,5}(z,\nu)=$$

$$(\nu^{3+2l}f_{l,8}(z,\nu-1)-(P_l^*(\nu;\delta)+zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,8})(z,\nu))+$$

$$\frac{-(\log(z))^2}{2}\left(\left(\frac{\partial}{\partial w}\right) (P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$\frac{-\log(z)}{2}\left(\left(\frac{\partial}{\partial w}\right)^2 (P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$\frac{-1}{6}\left(\left(\frac{\partial}{\partial w}\right)^3 (P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$(\log(z))^2\left(\left(\frac{\partial}{\partial w}\right) (P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$(\log(z))\left(\left(\frac{\partial}{\partial w}\right)^2 (P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$(\log(z))\left(\left(\frac{\partial}{\partial w}\right)(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,4}(z,\nu)+$$

$$(\log(z))\frac{1}{2}\left(\left(\frac{\partial}{\partial w}\right)^2(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$\frac{1}{2}\left(\left(\frac{\partial}{\partial w}\right)^3(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$\frac{1}{2}\left(\left(\frac{\partial}{\partial w}\right)^2(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,4}(z,\nu)+$$

$$\frac{-(\log(z))^2}{2}\left(\left(\frac{\partial}{\partial w}\right)(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$(-(\log(z))\left(\left(\frac{\partial}{\partial w}\right)^2(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$\frac{-1}{2}\left(\left(\frac{\partial}{\partial w}\right)^3(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$(-\log(z))\left(\left(\frac{\partial}{\partial w}\right)(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,4}(z,\nu)+$$

$$\left(-\left(\frac{\partial}{\partial w}\right)^2(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,4}(z,\nu)+$$

$$-\left(\left(\frac{\partial}{\partial w}\right)(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,6}(z,\nu)=$$

$$-(1-2+1)\times$$

$$\frac{(\log(z))^2}{2}\left(\left(\frac{\partial}{\partial w}\right)(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$-(1-1-2+2)\times$$

$$\frac{\log(z)}{2}\left(\left(\frac{\partial}{\partial w}\right)^2(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,2}(z,\nu)+$$

$$(1-1)(\log(z))\left(\left(\frac{\partial}{\partial w}\right)(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,4}(z,\nu)+$$

$$(\nu^{3+2l}f_{l,8}(z,\nu-1)-(P_l^*(\nu;\delta)+zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,8})(z,\nu))+$$

$$\left(-\left(\frac{\partial}{\partial w}\right)(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,6}(z,\nu)+$$

$$\frac{1}{2}(1-2)\left(\left(\frac{\partial}{\partial w}\right)^2(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Big|_{w=\delta} f_{l,4}(z,\nu)+$$

$$(1-3+3)\times$$

$$\frac{-1}{6}\left(\left(\frac{\partial}{\partial w}\right)^3(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Big|_{w=\delta} f_{l,2}(z,\nu)=$$

$$(\nu^{3+2l}f_{l,8}(z,\nu-1)-(P_l^*(\nu;\delta)+zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,8})(z,\nu))+$$

$$\left(-\left(\frac{\partial}{\partial w}\right)(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Big|_{w=\delta} f_{l,6}(z,\nu)+$$

$$\frac{-1}{2}\left(\left(\frac{\partial}{\partial w}\right)^2(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Big|_{w=\delta} f_{l,4}(z,\nu)+$$

$$\frac{-1}{6}\left(\left(\frac{\partial}{\partial w}\right)^3(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Big|_{w=\delta} f_{l,2}(z,\nu).$$

In view of (24), (31), coefficient at $z^{-\nu}$ in the series

$$\nu^{3+2l}f_{l,8}(z,\nu-1)-(P_l^*(\nu;\delta)+zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,8}(z,\nu)$$

is equal to

$$(-1/6)\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times \tag{75}$$

$$((2+l)^2(S_1(\nu,\nu-1)^3+3(2+l)S_1(\nu,\nu-1)S_2(\nu,\nu-1))+S_3(\nu,\nu-1))-$$

$$(-1/6)(2+l)(((\nu)!)^2/(2\nu+1)!)^{2+l}\times$$

$$((2+l)^2S_1(\nu+1,\nu)^3+3(2+l)S_1(\nu+1,\nu)S_2(\nu,\nu-1))+S_3(\nu+1,\nu))\times$$

$$(zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})z^{-\nu-1}=$$

$$(-1/6)\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$

$$((2+l)^2S_1(\nu,\nu-1)^3+3(2+l)S_1(\nu,\nu-1)S_2(\nu,\nu-1))+S_3(\nu,\nu-1))-$$

$$(-1/6)(((\nu)!)^2/(2\nu+1)!)^{2+l}\times$$

$$((2+l)^2S_1(\nu+1,\nu)^2+3(2+l)^2S_1(\nu+1,\nu)S_2(\nu+1,\nu)++S_3(\nu+1,\nu))\times$$

$$2^{l+2}\nu^{l+1}(2\nu+1)^{2+l}=$$

$$(-1/6)\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$

$$\Big((2+l)^2(S_1(\nu,\nu-1)^3-S_1(\nu+1,\nu)^3)+$$

$$3(2+l)(S_1(\nu,\nu-1)S_2(\nu,\nu-1)-S_1(\nu+1,\nu)S_2(\nu+1,\nu))+$$

$$S_3(\nu,\nu-1)-S_3(\nu+1,\nu)\Big)=$$

$$(-1/6)\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$

$$\Biggl((2+l)^2(3S_1(\nu+1,\nu)^2H_1(\nu))+3S_1(\nu+1,\nu)H_1^2(\nu))+H_1^3(\nu))+$$
$$3(2+l)(H_1(\nu)S_2(\nu+1,\nu)+S_1(\nu+1,\nu)H_2(\nu)+H_1(\nu+1,\nu)H_2(\nu))+$$
$$H_3(\nu,\nu-1)\Biggr)=$$
$$(-1/6)\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$\Biggl(3(2+l)^2H_1(\nu))(S_1(\nu+1,\nu))^2+$$
$$3((2+l)^2H_1^2(\nu)+(2+l)H_2(\nu))S_1(\nu+1,\nu)+$$
$$3(2+l)(H_1(\nu)S_2(\nu+1,\nu)+$$
$$(2+l)^2H_1^3(\nu)+3(2+l)H_1(\nu)H_2(\nu)+H_3(\nu)\Biggr)=$$
$$(-1/6)\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$\Biggl(-3(2+l)^2((4\nu+3)/(2\nu(2\nu+1)))(S_1(\nu+1,\nu))^2+$$
$$3((2+l)^2H_1^2(\nu)+(2+l)H_2(\nu))S_1(\nu+1,\nu)+$$
$$3(2+l)(H_1(\nu)S_2(\nu+1,\nu)+$$
$$(2+l)^2H_1^3(\nu)+3(2+l)H_1(\nu)H_2(\nu)+H_3(\nu)\Biggr).$$

In view of (17), (30), coefficient at $z^{-\nu}$ in the series

$$\left(-\left(\frac{\partial}{\partial w}\right)(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,6}(z,\nu)$$

is equal to

$$(76)\qquad -((\nu!)^2/(2\nu+1))^{2+l}2^{l+1}(2+l)\nu^l(4\nu+3)(2\nu+1)^{l+1}\times$$
$$2^{-1}((2+l)^2S_1^2(\nu+1,\nu)+(2+l)S_2(\nu+1,\nu))=$$
$$-\nu^{3+2l}(((\nu-1)!)^2/(2\nu-1))^{2+l}(4\nu+3)(2\nu(2\nu+1))\times$$
$$2^{-1}((2+l)^2S_1^2(\nu+1,\nu)+(2+l)S_2(\nu+1,\nu)).$$

Therefore, in view of (75), (76), coefficient at $z^{-\nu}$ in the series

$$\left(-\left(\frac{\partial}{\partial w}\right)(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{l,6}(z,\nu)+$$
$$(\nu^{3+2l}f_{l,8}(z,\nu-1)-(P_l^*(\nu;\delta)+zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,8})(z,\nu))$$

is equal to

$$(77)\qquad -\nu^{3+2l}(((\nu-1)!)^2/(2\nu-1))^{2+l}(4\nu+3)(2\nu(2\nu+1))\times$$

$$2^{-1}((2+l)^2S_1^2(\nu+1,\nu)+(2+l)S_2(\nu+1,\nu))+$$
$$(-1/6)\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$\Bigg(-3(2+l)^2((4\nu+3)/(2\nu(2\nu+1)))(S_1(\nu+1,\nu))^2+$$
$$3((2+l)^2H_1^2(\nu)+(2+l)H_2(\nu))S_1(\nu+1,\nu)+$$
$$3(2+l)H_1(\nu)S_2(\nu+1,\nu)+$$
$$(2+l)^2H_1^3(\nu)+3(2+l)H_1(\nu)H_2(\nu)+H_3(\nu)\Bigg)=$$
$$-\nu^{3+2l}(((\nu-1)!)^2/(2\nu-1))^{2+l}(-H_(\nu))\times$$
$$6^{-1}(3(2+l)^2S_1^2(\nu+1,\nu)+3(2+l)S_2(\nu+1,\nu))+$$
$$(-1/6)\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$\Bigg(-3(2+l)^2((4\nu+3)/(2\nu(2\nu+1)))(S_1(\nu+1,\nu))^2+$$
$$3((2+l)^2H_1^2(\nu)+(2+l)H_2(\nu))S_1(\nu+1,\nu)+$$
$$3(2+l)H_1(\nu)S_2(\nu+1,\nu)+$$
$$(2+l)^2H_1^3(\nu)+3(2+l)H_1(\nu)H_2(\nu)+H_3(\nu)\Bigg)=$$
$$(-1/6)\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$\Bigg(3((2+l)^2H_1^2(\nu)+(2+l)H_2(\nu))S_1(\nu+1,\nu)+$$
$$(2+l)^2H_1^3(\nu)+3(2+l)H_1(\nu)H_2(\nu)+H_3(\nu)\Bigg).$$

In view of (28) and (45)

$$(2+l)^2H_1^2(\nu)+(2+l)H_2(\nu)=$$
$$(2+l)^2(9/(2\nu)^2+1/(2\nu+1)^2-6/(2\nu)+6/(2\nu+1)+$$
$$(2+l)(7/(2\nu)^2-1/(2\nu+1)^2)=$$
$$(2+l)((25+9l)/(2\nu)^2+(1+l)/(2\nu+1)^2-6(2+l)/(2\nu)+6(2+l)/(2\nu+1)),$$

(78)
$$\left(\left(\frac{\partial}{\partial w}\right)^2(z^{\nu+1}Q_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta}z^{-\nu-1}=$$
$$\nu^{1+l}(2\nu+1)^{2+l}2^{2+l}((2+l)^2H_1^2(\nu)+(2+l)H_2(\nu)).$$

In view of (12), (78), coefficient at $z^{-\nu}$ in the series

$$\frac{-1}{2}\left(\left(\frac{\partial}{\partial w}\right)^2(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta}f_{l,4}(z,\nu)$$

is equal to

(79)
$$(1/2)((\nu!)^2/(2\nu+1))^{2+l}\times$$
$$\nu^{1+l}(2\nu+1)^{2+l}2^{2+l}((2+l)^2H_1^2(\nu)+(2+l)H_2(\nu))\times$$
$$(2+l)S_1(\nu+1,\nu)=$$
$$(1/6)\nu^{3+2l}(((\nu-1)!)^2/(2\nu-1))^{2+l}(2+l)\times$$
$$3((2+l)^2H_1^2(\nu)+(2+l)H_2(\nu))S_1(\nu+1,\nu).$$

Consequently, in view of in view of (77), (79), coefficient at $z^{-\nu}$ in the series

$$\frac{-1}{2}\left(\left(\frac{\partial}{\partial w}\right)^2(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta}f_{l,4}(z,\nu)+$$
$$\left(-\left(\frac{\partial}{\partial w}\right)(P_l^*(\nu;w)+zQ_l^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta}f_{l,6}(z,\nu)+$$
$$(\nu^{3+2l}f_{l,8}(z,\nu-1)-(P_l^*(\nu;\delta)+zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,8})(z,\nu))$$

is equal to

(80)
$$(1/6)\nu^{3+2l}(((\nu-1)!)^2/(2\nu-1))^{2+l}(2+l)\times$$
$$3((2+l)^2H_1^2(\nu)+(2+l)H_2(\nu))S_1(\nu+1,\nu)+$$
$$(-1/6)\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$\Bigg(3((2+l)^2H_1^2(\nu)+(2+l)H_2(\nu))S_1(\nu+1,\nu)+$$
$$3(2+l)^2H_1^3(\nu)+3(2+l)H_1(\nu)H_2(\nu)+H_3(\nu)\Bigg)=$$
$$(-1/6)\nu^{3+2l}(2+l)(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$(2+l)^2H_1^3(\nu)+3(2+l)H_1(\nu)H_2(\nu)+H_3(\nu).$$

If $l=2$, then

$$(2+l)^2H_1^3(\nu)+3(2+l)H_1(\nu)H_2(\nu)+H_3(\nu)=$$
$$16(-3/(2nu)+1/(2\nu+1))^3-$$
$$12(-3/(2nu)+1/(2\nu+1))(-7/(2nu)^2+1/(2\nu+1)^2)+$$
$$2(-15/(2nu)^3+1/(2\nu+1)^3)=$$
$$(-432-252-30)/(2\nu)^3+(432+84)/((2\nu)^2(2\nu+1))+$$
$$(-144+36)/((2\nu)(2\nu+1)^2)+(16-12+2)/(2\nu+1)^3=$$
$$-714/(2\nu)^3+516/((2\nu)^2(2\nu+1))+(-108)/((2\nu)(2\nu+1)^2)+6/(2\nu+1)^3=$$
$$-6(119/(2\nu)^3-86/((2\nu)^2(2\nu+1))+18/((2\nu)(2\nu+1)^2)-1/(2\nu+1)^3).$$

Therefore, if $l = 2$, then the value (80) is equal

(81) $$(-1/6)\nu^{3+2l}4(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$

$$(-6)(119/(2\nu)^3 - 86/((2\nu)^2(2\nu+1)) + 18/((2\nu)(2\nu+1)^2) - 1/(2\nu+1)^3) =$$
$$\nu^{3+2l}4(((\nu-1)!)^2/(2\nu-1)!)^{2+l}\times$$
$$(119/(2\nu)^3 - 86/((2\nu)^2(2\nu+1)) + 18/((2\nu)(2\nu+1)^2) - 1/(2\nu+1)^3) =$$
$$\nu^7 4(((\nu-1)!)^2/(2\nu-1)!)^4\times$$
$$(119/(2\nu)^3 - 86/((2\nu)^2(2\nu+1)) + 18/((2\nu)(2\nu+1)^2) - 1/(2\nu+1)^3).$$

In view of (46) and (7), coefficient at $z^{-\nu}$ in the series

$$\frac{-1}{6}\left(\left(\frac{\partial}{\partial w}\right)^3 (P_l^*(\nu;w) + zQ_2^*(\nu;w+1)(w-\nu)^{2+l}\right)\Bigg|_{w=\delta} f_{2,2}(z,\nu)$$

is equal to

(82) $$(-1/6)((\nu!)^2/(2\nu+1)!)^4\times$$

$$384(2nu+1)^4nu^3\Bigg(119/(2\nu)^3 - 86/((2\nu)^2(2nu+1)) +$$

$$18/(2\nu(2nu+1)^2) - 1/(2nu+1)^3\Bigg) =$$

$$-4\nu^7(((\nu-1)!)^2/(2\nu-1)!)^4\Bigg(119/(2\nu)^3 - 86/((2\nu)^2(2nu+1)) +$$

$$18/(2\nu(2nu+1)^2) - 1/(2nu+1)^3\Bigg).$$

In view of (80), (81), (82), coefficient at $z^{-\nu}$ in the series (74) is equal to 0. Thus,

(83) $$\nu^{3+2l}f^{\vee}_{l,7}(z,\nu-1) -$$

$$(P_l^*(\nu;\delta) + zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f^{\vee}_{l,7}(z,\nu) =$$
$$(2\pi^2/3)(\nu^{3+2l}f^{\vee}_{l,3}(z,\nu-1) - (P_l^*(\nu;\delta) + zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f^{\vee}_{l,3}(z,\nu)) +$$
$$\nu^{3+2l}f_{l,7}(z,\nu-1) - (P_l^*(\nu;\delta) + zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,7}(z,\nu) =$$
$$\nu^{3+2l}f_{l,7}(z,\nu-1) - (P_l^*(\nu;\delta) + zQ_l^*(\nu;\delta+1)(\delta-\nu)^{2+l})f_{l,7}(z,\nu) = 0$$

for $\nu \in [0,+\infty)\cap\mathbb{Z}$, $l = 2$. Since (38) holds for all the $\nu \in \mathbb{Z}$, it follows from (71) and (83) that these equalities (71) and (83) also hold for all the $\nu \in \mathbb{Z}$.

References.

[1] R.Apéry, Interpolation des fractions continues
et irrationalite de certaines constantes,
Bulletin de la section des sciences du C.T.H., 1981, No 3, 37 – 53;
[2] F.Beukers, A note on the irrationality of $\zeta(2)$ and $\zeta(3)$,
Bull. London Math. Soc., 1979, 11, 268 – 272;
[3] A.van der Porten, A proof that Euler missed...Apéry's proof of the irrationality of $\zeta(3)$,
Math Intellegencer, 1979, 1, 195 – 203;
[4] W. Maier, Potenzreihen irrationalen Grenzwertes,
J.reine angew. Math., 156, 1927, 93 – 148;
[5] E.M. Nikišin, On irrationality of the values of the functions F(x,s) (in Russian),
Mat.Sb. 109 (1979), 410 – 417;
English transl. in Math. USSR Sb. 37 (1980), 381 – 388;
[6] G.V. Chudnovsky, Pade approximations to the generalized hyper-geometric functions
I,J.Math.Pures Appl., 58, 1979, 445 – 476;
[7] ______________, Transcendental numbers, Number Theory,Carbondale,
Lecture Notes in Math, Springer-Verlag, 1979, 751, 45 – 69;
[8] ______________, Approximations rationelles des logarithmes de nombres rationelles
C.R.Acad.Sc. Paris, Série A, 1979, 288, 607 – 609;
[9] ______________, Formules d'Hermite pour les approximants de Padé de logarithmes
et de fonctions binômes, et mesures d'irrationalité,
C.R.Acad.Sc. Paris, Série A, 1979, t.288, 965 – 967;
[10] ______________,Un systḿe explicite d'approximants de Padé
pour les fonctions hypérgéometriques généraliesées,
avec applications a l'arithmétique,
C.R.Acad.Sc. Paris, Série A, 1979, t.288, 1001 – 1004;
[11] ______________, Recurrenses defining Rational Approximations
to the irrational numbers, Proceedings
of the Japan Academie, Ser. A, 1982, 58, 129 – 133;
[12] ______________, On the method of Thue-Siegel,
Annals of Mathematics, 117 (1983), 325 – 382;
[13] K.Alladi and M. Robinson, Legendre polinomials and irrationality,
J. Reine Angew.Math., 1980, 318, 137 – 155;
[14] A. Dubitskas, An approximation of logarithms of some numbers,
Diophantine approximations II,Moscow, 1986, 20 – 34;
[15] ___________, On approximation of $\pi/\sqrt{3}$ by rational fractions,
Vestnik MGU, series 1, 1987, 6, 73 – 76;
[16] S.Eckmann, Über die lineare Unaqbhangigkeit der Werte gewisser Reihen,
Results in Mathematics, 11, 1987, 7 – 43;
[17] M.Hata, Legendre type polinomials and irrationality mesures,
J. Reine Angew. Math., 1990, 407, 99 – 125;
[18] A.O. Gelfond, Transcendental and algebraic numbers (in Russian),
GIT-TL, Moscow, 1952;
[19] H.Bateman and A.Erdélyi, Higher transcendental functions,1953,
New-York – Toronto – London, Mc. Grow-Hill Book Company, Inc.;
[20] E.C.Titchmarsh, The Theory of Functions, 1939, Oxford University Press;
[21] E.T.Whittaker and G.N. Watson, A course of modern analysis,
1927, Cambridge University Press;
[22] O.Perron, Über die Poincaresche Differenzengleichumg,
Journal für die reine und angewandte mathematik,
1910, 137, 6 – 64;
[23] A.O.Gelfond, Differenzenrechnung (in Russian), 1967, Nauka, Moscow.
[24] A.O.Gelfond and I.M.Kubenskaya, On the theorem of Perron
in the theory of differrence equations (in Russian),
IAN USSR, math. ser., 1953, 17, 2, 83 – 86.
[25] M.A.Evgrafov, New proof of the theorem of Perron
(in Russian),IAN USSR, math. ser., 1953, 17, 2, 77 – 82;
[26] G.A.Frejman, On theorems of of Poincaré and Perron
(in Russian), UMN, 1957, 12, 3 (75), 243 – 245;
[27] N.E.Nörlund, Differenzenrechnung, Berlin, Springer Verlag, 1924;

[28] I.M.Vinogradov, Foundtions of the Number Theory, (in Russian), 1952, GIT-TL;
[29] J. Diedonne, Foundations of modern analysis,
Institut des Hautes Études Scientifiques, Paris,
Academic Press, New York and London, 1960
[30] CH.-J. de la Vallée Poussin, Course d'analyse infinitésimale,
Russian translation by G.M.Fikhtengolts,
GT-TI, 1933;
[31] H.WEYL, Algebraic theory of numbers,
1940,
Russian translation by L.I.Kopejkina,

[32] L.A.Gutnik, On the decomposition of the difference operators of Poincaré type
(in Russian), VINITI, Moscow, 1992, 2468 – 92, 1 – 55;
[33] ___________, On the decomposition of the difference operators
of Poincaré type in Banach algebras
(in Russian), VINITI, Moscow, 1992, 3443 – 92, 1 – 36;
[34] ___________, On the difference equations of Poincaré type
(in Russian), VINITI, Moscow 1993, 443 – B93, 1 – 41;
[35] ___________, On the difference equations of Poincaré type in normed algebras
(in Russian), VINITI, Moscow, 1994, 668 – B94, 1 – 44;
[36] ___________, On the decomposition of the difference equations of Poincaré type
(in Russian), VINITI, Moscow, 1997, 2062 – B97, 1 – 41;
[37] ___________, The difference equations of Poincaré type
with characteristic polynomial having roots equal to zero
(in Russian), VINITI, Moscow, 1997, 2418 – 97, 1 – 20;
[38] ___________, On the behavior of solutions
of difference equations of Poincaré type
(in Russian), VINITI, Moscow, 1997, 3384 – B97, 1 - 41;
[39] ___________, On the variability of solutions of difference equations of Poincaré type
(in Russian), VINITI, Moscow, 1999, 361 – B99, 1 – 9;
[40] ___________, To the question of the variability of solutions
of difference equations of Poincaré type (in Russian),
VINITI, Moscow, 2000, 2416 – B00, 1 – 22;
[41] ___________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$,
Max-Plank-Institut für Mathematik,
Bonn, Preprint Series, 2000, 3, 1 – 13;
[42] ___________, On the Irrationality of Some Quantyties Containing $\zeta(3)$ (in Russian),
Uspekhi Mat. Nauk, 1979, 34, 3(207), 190;
[43] ___________, On the Irrationality of Some Quantities Containing $\zeta(3)$,
Eleven papers translated from the Russian,
American Mathematical Society, 1988, 140, 45 - 56;
[44] ___________, Linear independence over $\mathbb{Q}$ of dilogarithms at rational points
(in Russian), UMN, 37 (1982), 179-180;
english transl. in Russ. Math. surveys 37 (1982), 176-177;
[45] ___________, On a measure of the irrationality of dilogarithms at rational points
(in Russian), VINITI, 1984, 4345-84, 1 – 74;
[46] ___________, To the question of the smallness of some linear forms
(in Russian), VINITI, 1993, 2413-B93, 1 – 94;
[47] ___________, On linear forms, whose coefficients are logarithms
of algebraic numbers (in Russian),
VINITI, 1995, 135-B95, 1 – 149;
[48] ___________, On systems of vectors, whose coordinates
are linear combinations of logarithms of algebraic numbers
with algebraic coefficients (in Russian),
VINITI, 1994, 3122-B94, 1 – 158;

[49] ___________, On the linear forms, whose
coefficients are $\mathbb{A}$ - linear combinations
of logarithms of $\mathbb{A}$ - numbers,

VINITI, 1996, 1617-B96, pp. 1 – 23.

[50] ____________, On systems of linear forms, whose coefficients are $\mathbb{A}$ - linear combinations of logarithms of $\mathbb{A}$ - numbers, VINITI, 1996, 2663-B96, pp. 1 – 18.

[51] ____________, On linear forms, whose coefficients are $\mathbb{Q}$-proportional to the number $\log 2$, and the values of $\zeta(s)$ for integer s (in Russian), VINITI, 1996, 3258-B96, 1 – 70;

[52] ____________, The lower estimate for some linear forms, coefficients of which are proportional to the values of $\zeta(s)$ for integer s (in Russian), VINITI, 1997, 3072-B97, 1 – 77;

[53] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2000, 3, 1 – 13;

[54] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version,part 1), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2001, 15, 1 – 20;

[55] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version,part 2), Max-Plank-Institut für Mathematik, Bonn, Preprint Series,2001, 104, 1 – 36;

[56] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version,part 3), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2002, 57, 1 – 33;

[57] ____________, On the rank over $\mathbb{Q}$ of some real matrices (in Russian), VINITI, 1984, 5736-84; 1 – 29;

[58] ____________, On the rank over $\mathbb{Q}$ of some real matrices, Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2002, 27, 1 – 32;

[59] ____________, On the linear forms with coefficients proportional to zeta values(in Russian), VINITI, 1999, 3207-B99; 1 – 71;

[60] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version, part 4), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2002, 142, 1 – 27;

[61] ____________, On the dimension of some linear spaces over finite extension of $\mathbb{Q}$ (part 2), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2002, 107, 1 – 37;

[62] ____________, On the dimension of some linear spaces over $\mathbb{Q}$ (part 3), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2003, 16, 1 – 45.

[63] ____________,On the difference equation of Poincaré type (Part 1). Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2003, 52, 1 – 44.

[64] ____________, On the dimension of some linear spaces over $\mathbb{Q}$, (part 4) Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2003, 73, 1 – 38.

[65] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version, part 5), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2003, 83, 1 – 13.

[66] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$ (the detailed version, part 6), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2003, 99, 1 – 33.

[67] ____________,On the difference equation of Poincaré type (Part 2).

Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2003, 107, 1 – 25.
[68] ____________, On the asymptotic behavior of solutions
of difference equation (in English).
Chebyshevskij sbornik, 2003, v.4, issue 2, 142 – 153.
[69] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$,
Bonner Mathematishe Schriften Nr. 360,
Bonn, 2003, 360.
[70] ____________, On the dimension of some linear spaces over $\mathbb{Q}$, (part 5)
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2004, 46, 1 – 42.
[71] ____________,On the difference equation of Poincaré type (Part 3).
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2004, 9, 1 – 33.
[72] ____________, On linear forms with coefficients in $\mathbb{N}\zeta(1+\mathbb{N})$
(the detailed version, part 7),
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2004, 88, 1 – 27.
[73] ____________,On the difference equation of Poincaré type (Part 4).
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2004, 129, 1 – 28.
[74] ____________, On the measure of discreteness of some modues,
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2005, 32, 1 – 51.

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