EQUIVARIANT SURGERY WITH MIDDLE DIMENSIONAL SINGULAR SETS. II

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Abstract. Let G be a finite group. Let $f : X \to Y$ be a degree 1, G-framed map of simply connected, closed, oriented, smooth manifolds X and Y of dimension $2k \geqq 6$. Assuming that the dimension of the singular set of the G-space X is at most k and further that f is k-connected, we obtained the G-equivariant surgery obstruction $\sigma(f)$ in the previous article. In the present article, we show that the $\sigma(f)$ is a G-framed cobordism invariant. Hence $\sigma(f)$ can be uniquely defined even if f is not k-connected.

1. Introduction

This article is a continuation of [2] and presents an explanation of a part of [1]. Let G be a finite group and X a smooth G-manifold. The term G-equivariant surgery or simply G-surgery will refer to G-surgery on the free part of the G-space X, namely where every nontrivial element of G acts without fixed points. Thus our G-surgery does not change the G-singular set

$$\operatorname{Sing}(G, X) = \bigcup_{g \in G \setminus \{1\}} X^g,$$

where $X^g = \{x \in X \mid gx = x\}$.

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Throughout the paper, R denote the ring of integers $\mathbb{Z}$, its localization $\mathbb{Z}_{(p)}$ at a prime p, or the ring of rational numbers $\mathbb{Q}$. Associated to a degree 1, k-connected G-framed map $f : X^n \to Y^n$ with $n = 2k$ such that the dimension of the G-singular set of X is at most k, we [2] defined an element $\sigma(f)$ in an abelian group $W(G, Y; R)$ such that if $\sigma(f) = 0$ then f can be converted to a k-connected, R-homology equivalence $f' : X' \to Y$ by G-surgery on the free part of X. The group $W(G, Y; R)$ is a generalization of Wall's group $L^h_n(R[G])$ and ingredients for the generalization are quadratic and symmetric form parameters and positioning maps. Here $R[G]$ is the group ring of G with coefficients in R. In the current paper we prove that the obstruction $\sigma(f)$ is a G-framed cobordism invariant. Thus, one can define a unique element $\sigma(f)$ in $W(G, Y; R)$ for a degree 1, G-framed map f even if f is not k-connected.

Theorem 1.1. *Let G be a finite group and Y a closed, 1-connected, oriented, smooth G-manifold of even dimension $n = 2k \geqq 6$. Suppose that* (1.1.1)–(1.1.3) *below hold.*

(1.1.1) $\dim Y^g \leqq k$ *for all* $g \in G \setminus \{1\}$.

(1.1.2) *If* $\dim Y^H = k$ *for some subgroup* $H \leqq G$ *then* $|H| = 2$ *and* Y^H *is connected and oriented such that each* $g : Y^H \to Y^{gHg^{-1}}$ $(g \in G)$ *is orientation preserving.*

(1.1.3) $\dim(Y^H \cap Y^K) \leqq k - 2$ *whenever* $\dim Y^H = k$ *and* $\dim Y^K = k - 1$ $(H,\ K \leqq G)$.

Let X be a closed, connected, oriented, smooth G-manifold satisfying (1.1.1)–(1.1.3) *above, ξ and η real G-vector bundles over Y and* $\boldsymbol{f} = (f : X \to Y,\ b : T(X) \oplus f^*\eta \to f^*(\xi \oplus \eta))$ *a degree 1, G-framed map. Suppose that*

(1.1.4) *if* $H \leqq G$ *has order 2 and if* $\dim X^H \geqq k - 1$ *or* $\dim Y^H \geqq k - 1$ *then* $\dim X^H = \dim Y^H$*;*

(1.1.5) *if* $R = \mathbb{Z}$ *then* $f^H : X^H \to Y^H$ *is a $\mathbb{Z}$-homology equivalence for any hyplerelementary subgroup* $H \neq \{1\}$ *of G;*

(1.1.6) *if* $R = \mathbb{Z}_{(p)}$ *(p a prime) then* $f^P : X^P \to Y^P$ *is a* $\mathbb{Z}_{(p)}$*-homology equivalence for any p-subgroup* $P \neq \{1\}$ *of G; and*

(1.1.7) *if* $R = \mathbb{Q}$ *then* $\chi(X^g) = \chi(Y^g)$ *for any* $g \in G \setminus \{1\}$, *where* $\chi(-)$ *denotes the Euler characteristic.*

Then one can associate to $\boldsymbol{f}$ *an element* $\sigma(\boldsymbol{f})$ *in the abelian group* $W(G, Y; R)$ *of* [2, Theorem 1.1] *such that* $\sigma(\boldsymbol{f}) = 0$ *if and only if* $\boldsymbol{f}$ *is G-framed cobordant rel.* $\bigcup_{g \in G \setminus \{1\}} X^g$ *to* *a degree* 1*, k-connected, G-framed map* $\boldsymbol{f'} = (f' : X' \to Y,\ b' : T(X') \oplus f'^*\eta \to f'^*(\xi \oplus \eta))$ *such that* $f' : X' \to Y$ *is an R-homology equivalence.*

Theorem 1.1 will be deduced from a slight generalization Theorem 3.3 of it.

Remark 1.2. (1) If $\sigma(\boldsymbol{f}) = 0$ then a G-framed map $\boldsymbol{f'}$ satisfying the properties in Theorem 1.1 and a G-framed cobordism between $\boldsymbol{f}$ and $\boldsymbol{f'}$ can be obtained by G-surgery of dimension $\leqq k$ on the free part of X .

(2) If $f : X \to Y$ is k-connected then the $\sigma(\boldsymbol{f})$ in the theorem is same as in [2, Theorem 1.1].

The reader can refer to [3] for the G-framed cobordism invariance of G-surgery obstructions under the hypothesis that $\mathrm{Sing}(G, X) \leqq k - 1$.

The remainder of the paper is organized as follows. Section 2 is devoted to preparation for Sections 3 and 4. The G-framed cobordism invariance of the G-surgery obstruction is presented as Theorem 3.2. The main body of its proof is stated as Lemma 3.1. The generalization Theorem 3.3 mentioned above is obtained as a corollary to Theorem 3.2. Section 3 consists of these three results. We prove Lemma 3.1 in Section 4 and a key method of the proof is equivariant handle subtraction.

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2. Preliminary

In the current article, $n = 2k$ is an even integer $\geqq 6$ and a manifold (resp. an action on a manifold) means a smooth manifold (resp. a smooth action) unless otherwise specified. Let ((proj)) (resp. ((s-free)), ((free))) denote the category of all finitely generated projective (resp. stably free, free) $R[G]$-modules. Let $\mathcal{C}$ denote one of the three categories.

Let X be a G-CW complex (e.g. G-manifold, G-simplicial complex). For a point $x \in X$ and a subgroup H of G, let H_x denote the isotropy subgroup at x in the H-space $\mathrm{res}^G_H X$. Set

$$\begin{aligned}
&X^H = \mathrm{Fix}(H, X) = \{x \in X \mid G_x \supseteqq H\},\\
&X^{>H} = \mathrm{Fix}_G(> H, X) = \{x \in X \mid G_x \supsetneqq H\},\\
&X^{=H} = \mathrm{Fix}_G(= H, X) = \{x \in X \mid G_x = H\},\\
&\mathrm{Free}(H, X) = \{x \in X \mid H_x = 1\} \quad \text{and}\\
&\mathrm{Sing}(H, X) = \{x \in X \mid H_x \neq \{1\}\}.
\end{aligned}$$

If $g \in G$ then set $X^g = X^{\langle g \rangle}$. For a subspace $Z \subseteqq X$, we define the subgroup

$$\rho_X^G(Z) := \bigcap_{x \in Z} G_x.$$

Let $\pi_0(X^H)$ denote the set of all connected components γ of X^H. The underlying space of γ is denoted by X_γ. If we like to emphasize the group H then we use X_γ^H instead of X_γ. For a nonnegative integer ℓ, let $\pi_0(X^H, \ell)$ denote the subset of $\pi_0(X^H)$ consisting of all γ with $\dim X_\gamma = \ell$. We set

$$\begin{aligned}
\Pi(G, X) &= \coprod_{H \leqq G} \pi_0(X^H) \quad \text{and}\\
\Pi(G, X, \ell) &= \coprod_{H \leqq G} \pi_0(X^H, \ell).
\end{aligned}$$

For $\gamma \in \Pi = \Pi(G, X)$ such that $\gamma \in \pi_0(X^H)$, define $\rho(\gamma) = \rho_\Pi(\gamma) := H$. Clearly, $\rho_\Pi(\gamma) \leqq \rho_X^G(X_\gamma)$ holds. The set Π has the canonical G-action such that $X_{g\gamma} = gX_\gamma$ and $\rho(g\gamma) = g\rho(\gamma)g^{-1}$, and $\Pi(G, X, \ell)$ is a G-invariant subset of Π. Let $\Phi : \Pi \to \Pi$ denotes the map $\gamma \mapsto \beta$ such that $\rho(\beta) = \rho_X^G(X_\gamma)$ and $X_\beta = X_\gamma$. Define

$$\begin{aligned}
&\Theta(G, X) := \mathrm{Im}(\Phi) \quad \text{and}\\
&\Theta(G, X, \ell) := \Theta(G, X) \cap \Pi(G, X, \ell).
\end{aligned}$$

Let $\boldsymbol{Mnf}^n_{\mathrm{cp}}(G)_\ell$ (resp. $\boldsymbol{Mnf}^n_{\mathrm{cl}}(G)_\ell$) denotes the family of all compact (resp. closed), ℓ-connected, oriented G-manifolds of dimension n. Each G-manifold $X \in \boldsymbol{Mnf}^n_{\mathrm{cp}}(G)_0$ has the orientation homomorphism $w_X : G \to \{\pm 1\}$ such that $w_X(g) = 1$ (resp. -1) for orientation preserving (resp. reversing) $g \in G$.

Let $G(2)$ denote the set of all elements of order 2 in G. (The identity element does not belong to $G(2)$.) In the case $n = 2k$, let $\boldsymbol{Mnf}^n_{\mathrm{cp,sg}}(G)_\ell$ (resp. $\boldsymbol{Mnf}^n_{\mathrm{cl,sg}}(G)_\ell$), denote the family of all $X \in \boldsymbol{Mnf}^n_{\mathrm{cp}}(G)_\ell$ (resp. $\boldsymbol{Mnf}^n_{\mathrm{cl}}(G)_\ell$) satisfying the following properties.

(2.1.1) $\dim \operatorname{Sing}(G, X) \leqq k$.

(2.1.2) $|\pi_0(X^H, k)| \leqq 1$ for any $H \leqq G$.

(2.1.3) If $\gamma \in \Theta(G, X, k)$ then $|\rho(\gamma) \cap G(2)| = 1$.

(2.1.4) If $\gamma \in \Theta(G, X, k)$ and $\delta \in \Theta(G, X, k-1)$ then $\dim(X_\gamma \cap X_\delta) \leqq k-2$.

(2.1.5) All submanifolds X^H_γ ($\gamma \in \Theta(G, X, k)$ and $H = \rho(\gamma)$) are oriented such that each $g \in G$ acts as an orientation preserving diffeomorphism $X^H_\gamma \to X^{gHg^{-1}}_{g\gamma}$.

If $X \in \boldsymbol{Mnf}^n_{cp,sg}(G)_0$ with $n = 2k$ then set

$$\begin{aligned} &\lambda = \lambda_X := (-1)^k, \\ &Q = Q(G, X) := \{g \in G(2) \mid \dim X^g = k-1\} \ \text{ and} \\ &S = S(G, X) := \{g \in G(2) \mid \dim X^g = k\}. \end{aligned}$$

The set S is identified with $\Theta(G, X, k)$ via the map

$$\Theta(G, X, k) \to S(G, X);\ \gamma \longmapsto g \ \text{ such that } g \in \rho(\gamma).$$

Depending on X and the choice of $R = \mathbb{Z}$, $\mathbb{Z}_{(p)}$ (p a prime) or $\mathbb{Q}$, let $\boldsymbol{A}$ denote the tupple

$$(R, G, Q, S, \lambda, w_X).$$

The group ring $R[G]$ is equipped with the anti-involution $-$ such that

$$\overline{\sum_{g\in G} c_g g} = \sum_{g\in G} w_X(g) c_g g^{-1} \quad (c_g \in R).$$

If $a \in G$ then let ε_a denote the map $R[G] \to R$ or $R/2R[G] \to R/2R$ defined by

$$\varepsilon_a(\sum_{g\in G} c_g g) = c_a \quad (c_g \in R \text{ or } R/2R).$$

The choice of R or $R/2R$ will be clear form the context. For a subset T of G, $R[T]$ denotes the R-submodule of $R[G]$ which is generated by all elements in T. Let $\Lambda = \Lambda(G, X; R)$ denote the quadratic form parameter defined by

$$\Lambda = \min{}^\lambda(R[G]) + R[Q], \quad \text{where } \ \min{}^\lambda(R[G]) = \langle x - \lambda\overline{x} \mid x \in R[G]\rangle_R.$$

A special quadratic $(\boldsymbol{A}, \Theta(G, X, k))$-module $\boldsymbol{M}$ is a tupple (M, B, q, α) of

M a finitely generated, projective $R[G]$-module,

$B : M \times M \to R[G]$ a nonsingular λ-Hermitian form,

$q : M \to R[G \setminus S]/\Lambda$ an $\boldsymbol{A}$-quadratic form and

$\alpha : \Theta(G, X, [k]) \to M$ a G-map

satisfying the conditions:

(2.2.1) B is biadditive,

(2.2.2) $B(ax, by) = bB(x, y)\overline{a}$,

(2.2.3) $B(x, y) = \lambda\overline{B(y, x)}$,

(2.2.4) $q(gx) = gq(x)\overline{g}$ and $q(rx) = r^2q(x)$ in $R[G \setminus S]/\Lambda = R[G]/(\Lambda + R[S])$,

(2.2.5) $q(x + y) - q(x) - q(y) = B(x, y)$ in $R[G \setminus S]/\Lambda = R[G]/(\Lambda + R[S])$,

(2.2.6) $\widetilde{q(x)} + \lambda\overline{\widetilde{q(x)}} = B(x, x)$ in $R[G \setminus S] = R[G]/R[S]$ and

(2.2.7) $[\varepsilon_1(B(\alpha(s) - x, sx))] = 0$ in $R/2R$,

for all x, $y \in M$, a, $b \in R[G]$, $g \in G$, $r \in R$ and $s \in S$ (see [2, pp.278–279]), where $\widetilde{q(x)}$ is a lifting of $q(x)$.

If $\boldsymbol{M} = (M, B, q, \alpha)$ is a special quadratic $(\boldsymbol{A}, \Theta(G, X, k))$-module then let $-\boldsymbol{M}$ stand for the module $(M, -B, -q, -\alpha)$. Let $\boldsymbol{SQ}(\boldsymbol{A}, \Theta(G, X, k))_{\mathcal{C}}$, denote the category of all special quadratic $(\boldsymbol{A}, \Theta(G, X, k))$-modules whose underlying modules are in $\mathcal{C}$.

For a G-map $f : X \to Y$, we set

$$K_\ell(X; R) = K_\ell(f; R) := \mathrm{Ker}[f_* : H_\ell(X; R) \to H_\ell(Y; R)].$$

It is quite important to see whether the $R[G]$-module $K_\ell(X; R)$ is stably free or projective. At this end, let us define the following conditions on a G-map $f : X \to Y$.

$\mathcal{F}_1$-Condition: $\chi(X^g) = \chi(Y^g)$ for any $G \setminus \{1\}$. (Here $\chi(-)$ denotes the Euler characteristic.)

$\mathcal{F}_p$-Condition: $f^P : X^P \to Y^P$ is a $\mathbb{Z}_{(p)}$-homology equivalence for any p-subgroup $P \neq \{1\}$ of G.

$\mathcal{P}_0$-Condition: $f^P : X^P \to Y^P$ is a $\mathbb{Z}_{(p)}$-homology equivalence for any prime p dividing $|G|$ and any p-subgroup $P \neq \{1\}$ of G.

$\mathcal{F}_0$-Condition: $f^H : X^H \to Y^H$ is a $\mathbb{Z}$-homology equivalence for any hyperelementary subgroup $H \neq \{1\}$ of G.

Remark 2.3. Let $R = \mathbb{Z}_{(m)}$ for $m = 0$, 1 or a prime, where $\mathbb{Z}_{(0)} = \mathbb{Z}$ and $\mathbb{Z}_{(1)} = \mathbb{Q}$. Let X and Y be finite G-CW complexes (e.g. compact G-manifolds), let ℓ be a positive integer, and let $f : X \to Y$ be an ℓ-connected G-map such that $f_* : H_s(X; R) \to H_s(Y; R)$ is an isomorphism for every $s \neq \ell$ and $K_\ell(X; R)$ is $\mathbb{Z}$-torsion free. Then the following hold.

(2.3.1) If f satisfies $\mathcal{F}_m$-Condition then $K_\ell(X; R)$ is stably free over $R[G]$.

(2.3.2) If $R = \mathbb{Z}$ and if f satisfies $\mathcal{P}_0$-Condition then $K_\ell(X; \mathbb{Z})$ is projective over $\mathbb{Z}[G]$. If furthermore $\widetilde{K}_0(\mathbb{Z}[G]) = 0$ then $K_\ell(X; \mathbb{Z})$ is stably free over $\mathbb{Z}[G]$.

Let X, $X' \in \boldsymbol{Mnf}^n_{\mathrm{cl,sg}}(G)_0$ and $Y \in \boldsymbol{Mnf}^n_{\mathrm{cl}}(G)_1$. Let $\boldsymbol{f} = (f : X \to Y,\ b : T(X) \oplus f^*\eta \to f^*(\xi \oplus \eta))$ and $\boldsymbol{f}' = (f' : X' \to Y,\ b' : T(X') \oplus f'^*\eta \to f'^*(\xi \oplus \eta))$ be degree 1, G-framed maps for G-vector bundles ξ and η over Y such that $\eta \supseteqq \varepsilon_Y(\mathbb{R}^{n+1})$ (cf. [2, p.286]). Let I denote the closed unit interval $[0,1]$ and $p_Y : I \times Y \to Y$ the canonical projection. Let $\boldsymbol{F} = (F : (W, \partial W) \to (I \times Y, \partial I \times Y),\ B : T(W) \oplus (p_Y \circ F)^*\eta \to (p_Y \circ F)^*(\varepsilon_Y(\mathbb{R}) \oplus \xi \oplus \eta))$ be a G-framed cobordism between $\boldsymbol{f}$ and $\boldsymbol{f}'$. In particular, W is an oriented cobordism between X and X', i.e. $\partial W = (-X) \amalg X'$.

Definition 2.4. A G-framed cobordism $\boldsymbol{F}$ between $\boldsymbol{f}$ and $\boldsymbol{f}'$ is said to be *admissible* if the following conditions (2.4.1)–(2.4.3) are fulfilled.

(2.4.1) $\dim \mathrm{Sing}(G, W) \leqq k + 1$.

(2.4.2) The inclusion $j : X \to W$ (resp. $j' : X' \to W$) induces bijections $\Pi(G, X, k) \to \Pi(G, W, k+1)$ and $\Pi(G, X, k-1) \to \Pi(G, W, k)$ (resp. $\Pi(G, X', k) \to \Pi(G, W, k+1)$, and $\Pi(G, X', k-1) \to \Pi(G, W, k)$).

We use W_γ for $W_{j_\# \gamma}$ (resp. also for $W_{j'_\# \gamma}$) if $\ell = k - 1$ or k and $\gamma \in \Pi(G, X, \ell)$ (resp. $\Pi(G, X', \ell)$). Moreover we usually identify $\Theta(G, X', \ell)$ with $\Theta(G, X, \ell)$ via the induced maps when $\ell = k - 1$ or k.

(2.4.3) For each $\gamma \in \Theta(G, X, k) = \Theta(G, X', k)$, W_γ is an oriented cobordism between X_γ and X'_γ.

Let $\boldsymbol{Cob}(G; \boldsymbol{f}, \boldsymbol{f}')$ denote the set of all admissible, G-framed cobordisms between $\boldsymbol{f}$ and $\boldsymbol{f}'$.

Lemma 2.5. *Let $\boldsymbol{f}$, $\boldsymbol{f}'$ and $\boldsymbol{F} \in \boldsymbol{Cob}(G; \boldsymbol{f}, \boldsymbol{f}')$ be as above. Then $\boldsymbol{F}$ is G-framed cobordant rel. $\partial W \cup \mathrm{Sing}(G, W)$ (by G-surgery of dimension $\leqq k-1$ on $\mathrm{Free}(G, W) \setminus \partial W$) to a k-connected cobordism $\boldsymbol{F}' \in \boldsymbol{Cob}(G; \boldsymbol{f}, \boldsymbol{f}')$.*

Proof. This is clear form the fact that $\mathrm{Sing}(G, W) + (k-1) < \dim W$. □

Suppose that $\boldsymbol{f}$, $\boldsymbol{f}'$ and $\boldsymbol{F}$ are k-connected. Then the next sequence for the coefficient ring $\mathbb{Z}$ is exact:

$$0 \to K_{k+1}(W) \to K_{k+1}(W, \partial W) \xrightarrow{\partial} K_k(\partial W) \to K_k(W) \to K_k(W, \partial W) \to 0.$$

By the universal coefficient theorem, the sequence for the coefficient ring R is also exact. Furthermore suppose $K_k(W, \partial W; R) = 0 = K_{k+1}(W; R)$ and regard $K_{k+1}(W, \partial W; R)$ as a submodule of $K_k(\partial W; R)$ via the connecting homomorphism ∂. The restriction $\partial\boldsymbol{F}$ of $\boldsymbol{F}$ to the boundary determines a special quadratic $(\boldsymbol{A}, \Theta(G, X, k))$-module

$$\overline{\boldsymbol{M}}_{\partial \boldsymbol{F}} = (K_k(\partial W; R), B_{\partial F}, q_{\partial \boldsymbol{F}}, \overline{\alpha}_{\partial F})$$

as follows. The map

$$B_{\partial F} : K_k(\partial W; R) \times K_k(\partial W; R) \to R[G]$$

is the equivariant intersection form and the map

$$q_{\partial \boldsymbol{F}} : K_k(\partial W; R) \to R[G \setminus S(G, X)]$$

is the equivariant selfintersection form. Let

$$\mathrm{InterSec}_{\partial W} : K_k(\partial W; R) \times K_k(\partial W; R) \to R$$

denote the ordinary intersection form. Then, by definition

$$B_{\partial F}(x, y) = \sum_{g \in G} \mathrm{InterSec}_{\partial W}(x, g^{-1}y)\ g \quad (x,\ y \in K_k(\partial W; R)).$$

Immediately $B_{\partial F} = (-B_f) \oplus B_{f'}$ holds. Similarly, $q_{\partial \boldsymbol{F}} = (-q_{\boldsymbol{f}}) \oplus q_{\boldsymbol{f}'}$ also holds. The G-map $\overline{\alpha}_{\partial \boldsymbol{F}} : \Theta(G, X, k) \to K_k(\partial W; R)$ is defined by

$$\overline{\alpha}_{\partial F}(\gamma) = \pi_{\partial F}({j_{\partial W_\gamma}}_*(\partial[W_\gamma])) \quad (\gamma \in \Theta(G, X, k)),$$

where $[W_\gamma] \in H_{k+1}(W_\gamma, \partial W_\gamma; R)$ is the orientation class, $j_{\partial W_\gamma}$ is the inclusion map $\partial W_\gamma \to \partial W$, and $\pi_{\partial F} : H_k(\partial W; R) \to K_k(\partial W; R)$ is the canonical projection. Thus we obtain the commutative diagram:

$$\begin{tikzcd}
\Theta(G, X, \ell) \arrow[r, "\overline{\alpha}_{\partial F}"] \arrow[d, "\text{diagonal}"'] & K_k(\partial W; R) \arrow[d, "\text{canonical iso.}"] \\
\Theta(G, X, \ell) \times \Theta(G, X', \ell) \arrow[r, "-\alpha_f \times \alpha_{f'}"] & K_k(X; R) \oplus K_k(X'; R)
\end{tikzcd} .$$

Remark 2.6. In [2, Lemma 7.1] we invoked $Y \in \boldsymbol{Mnf}^n_{\mathrm{cl,sg}}(G)_1$. But it is superfluous and the condition $Y \in \boldsymbol{Mnf}^n_{\mathrm{cl}}(G)_1$ suffices to obtain the conclusion in [2, Lemma 7.1], namely to guarantee $\boldsymbol{M_f} \in \boldsymbol{SQ}(\boldsymbol{A}, \Theta(G, X, k))_{\mathcal{C}}$ (for the definition of $\boldsymbol{M_f}$, see [2, pp. 286–288]).

Lemma 2.7. *Let* $\boldsymbol{f}$ *and* $\boldsymbol{f}'$ *be as above. Suppose that* $f : X \to Y$ *and* $f' : X' \to Y$ *are* k*-connected. Let* $\boldsymbol{F} \in \boldsymbol{Cob}(G; \boldsymbol{f}, \boldsymbol{f}')$ *be as above. Suppose that* $F : W \to I \times Y$ *is* k*-connected, and*

$$K_k(W, \partial W; R) = 0 = K_{k+1}(W; R).$$

Then the following hold.

(2.7.1) $\overline{\boldsymbol{M}}_{\boldsymbol{\partial F}} = -\boldsymbol{M_f} \oplus \boldsymbol{M_{f'}}$.

(2.7.2) *The submodule* $K_{k+1}(W, \partial W; R)$ *of* $K_k(\partial W; R)$ *includes* $\mathrm{Im}(\overline{\alpha}_{\partial F})$.

(2.7.3) *The submodule* $K_{k+1}(W, \partial W; R)$ *is totally isotropic in* $\overline{\boldsymbol{M}}_{\boldsymbol{\partial F}}$*, namely* $B_{\partial F}$ *and* $q_{\boldsymbol{\partial F}}$ *are trivial on* $K_{k+1}(W, \partial W; R)$.

Proof. (2.7.1): The conclusion immediately follows from $\partial W = (-X) \amalg X$ and the definition of $B_{\partial F}$, $q_{\boldsymbol{\partial F}}$ and $\overline{\alpha}_{\partial W}$.

(2.7.2): The conclusion is obtained from observation of the following commutative diagram.

$$\begin{tikzcd}
H_{k+1}(W_\gamma, \partial W_\gamma; R) \arrow[r, "{j_{W_\gamma}}_*"] \arrow[d, "\partial"] & H_{k+1}(W, \partial W; R) \arrow[r, "\pi_F"] \arrow[d, "\partial"] & K_{k+1}(W, \partial W; R) \arrow[d, "\partial"] \\
H_k(\partial W_\gamma; R) \arrow[r, "{j_{\partial W_\gamma}}_*"] \arrow[d, equal] & H_k(\partial W; R) \arrow[r, "\pi_{\partial F}"] & K_k(\partial W; R) \\
H_k(-X_\gamma; R) \oplus H_k(X'_\gamma; R) & & .
\end{tikzcd}$$

(2.7.3): Now we prove the last conclusion. Here the coefficient ring of homology groups is R. It is well known that the ordinary intersection form satisfies

$$\mathrm{InterSec}_{\partial W}(K_{k+1}(W, \partial W), K_{k+1}(W, \partial W)) = 0.$$

Since $K_{k+1}(W, \partial W)$ is G-invariant, it follows that

$$B_{\partial F}(K_{k+1}(W, \partial W), K_{k+1}(W, \partial W)) = 0.$$

Let us decompose G to a disjoint union

$$G = \{1\} \amalg G(2) \amalg G(\geqq 3)_h \amalg {G(\geqq 3)_h}^{-1}$$

such that $G(\geqq 3)_h$ is a subset of G consisting of elements of order $\geqq 3$, and ${G(\geqq 3)_h}^{-1} = \{g^{-1} \mid g \in G(\geqq 3)_h\}$. Let

$$G^\lambda(2) = \{g \in G(2) \mid w_X(g) = -\lambda\} \text{ and}$$
$$G^{-\lambda}(2) = \{g \in G(2) \mid w_X(g) = \lambda\}.$$

Then we can regard the map $q_{\partial \boldsymbol{F}} : K_k(\partial W) \to R[G \setminus S]/\Lambda$ as a map

$$K_k(\partial W) \to R/(1-\lambda)[\{1\}] \oplus R/2R[G^\lambda(2) \setminus Q] \oplus R[G^{-\lambda}(2) \setminus S] \oplus R[G(\geqq 3)_h].$$

(cf. [2, p.274]). In the case where $g \in G(\geqq 3)_h$, $\varepsilon_g(q_{\partial W}(x)) = 0$ for any $x \in K_{k+1}(W, \partial W)$ because

$$\varepsilon_g(q_{\partial W}(x)) = \mathrm{InterSec}_{\partial W}(x, g^{-1}x).$$

Let $g \in \{1\} \cup (G^\lambda(2) \setminus Q) \cup (G^{-\lambda}(2) \setminus S)$. Then one has

$$\varepsilon_g(q_{\partial \boldsymbol{F}}(x)) = \varepsilon_g(q_{\partial(\mathrm{res}^G_{\langle g \rangle} \boldsymbol{F})}(x))$$

for any $x \in K_k(\partial W) = K_k(\mathrm{res}^G_{\langle g\rangle}\partial W)$. However, we have

$$\begin{aligned}\dim(\mathrm{Sing}_{\langle g\rangle}(\mathrm{res}^G_{\langle g\rangle}\partial W)) + (k+1) &\leqq \{(k-2)+1\} + (k+1)\\ &= 2k\\ &< \dim(\mathrm{res}^G_{\langle g\rangle}\partial W).\end{aligned}$$

(In other words, $\mathrm{res}^G_{\langle g\rangle}\partial W$ satisfies the strong gap condition.) In this case, a similar argument to Wall [4, p.53, lines 7–11] proves that $q_{\mathrm{res}^G_{\langle g\rangle}\partial \boldsymbol{F}}(x) = 0$ for all $x \in K_{k+1}(\mathrm{res}^G_{\langle g\rangle}W, \partial(\mathrm{res}^G_{\langle g\rangle}W))$. Consequently we get $q_{\partial\boldsymbol{F}}(x) = 0$ for all $x \in K_{k+1}(W, \partial W)$. □

3. Main results

Throughout this section let $\mathcal{C}$ denote the category ((proj)), ((s-free)) or ((free)). We begin the section with a lemma which is a key to Theorems 3.2 and 3.3.

Lemma 3.1 *Let* X, $X' \in \boldsymbol{Mnf}^n_{\mathrm{cl,sg}}(G)_0$ *and* $Y \in \boldsymbol{Mnf}^n_{\mathrm{cl}}(G)_1$ *such that* $n = 2k \geqq 6$ *is even and let* $\boldsymbol{f} = (f : X \to Y,\ b : T(X) \oplus f^*\eta \to f^*(\xi \oplus \eta))$ *and* $\boldsymbol{f}' = (f' : X' \to Y,\ b' : T(X') \oplus f'^*\eta \to f'^*(\xi \oplus \eta))$ *be degree* 1*, k-connected, G-framed maps such that* $K_k(X; R)$ *and* $K_k(X'; R)$ *are modules in* $\mathcal{C}$*. Further let* $\boldsymbol{F} = (F : (W\partial W) \to (I \times Y, \partial I \times Y),\ B : T(W) \oplus (p_Y \circ F)^*\eta \to (p_Y \circ F)^*(\varepsilon_Y(\mathbb{R}) \oplus \xi \oplus \eta))$ *be an admissible, G-framed cobordism between* $\boldsymbol{f}$ *and* $\boldsymbol{f}'$*. Then* $\boldsymbol{F}$ *is G-framed cobordant rel.* $X \cup \mathrm{Sing}(G, W)$ *(by G-surgery of dimension* $\leqq k-1$ *and k-dimensional handle subtraction in* $\mathrm{Free}(G, W)$*) to a k-connected, admissible, G-framed cobordism*

$$\begin{aligned}\boldsymbol{F}' = (F' :&(W', \partial W') \to (I \times Y, \partial I \times Y),\\ &B' : T(W) \oplus (p_Y \circ F')^*\eta \to (p_Y \circ F')^*(\varepsilon_Y(\mathbb{R}) \oplus \xi \oplus \eta))\end{aligned}$$

such that for the degree 1*, G-framed map* $\boldsymbol{f}'' = (f'' : X'' \to Y,\ b'' : T(X'') \oplus f''^*\eta \to f''^*(\xi \oplus \eta))$ *obtained by*

$$\partial \boldsymbol{F}' = (-\boldsymbol{f}) \coprod \boldsymbol{f}'',$$

the following hold:

(3.1.1) $f'' : X'' \to Y$ *is k-connected,*

(3.1.2) $K_k(W', \partial W'; \mathbb{Z}) = 0 = K_{k+1}(W'; \mathbb{Z})$ *and*

(3.1.3) $\boldsymbol{M}_{f''} \cong \boldsymbol{M}_{f'} \oplus \mathbb{M}(x_1, y_1, \beta_1) \oplus \cdots \oplus \mathbb{M}(x_\ell, y_\ell, \beta_\ell)$ *in* $\boldsymbol{SQ}^\lambda(\boldsymbol{A}, \Theta(G, X, k))_{\mathcal{C}}$ *for some nonnegative integer* ℓ *and* $\beta_i = (c_i, r_i)$, $i = 1, \ldots, \ell$, *consisting of* $c_i : \Theta(G, X, k) \to \mathbb{Z}$ *and*

$r_i : S(G,X) \to \mathbb{Z}$ such that $c_i(\gamma) \equiv r_i(g) \mod 2\mathbb{Z}$ for $\gamma \in \Theta(G,X,k)$ and $g \in S(G,X)$ with $g \in \rho(\gamma)$.

In the lemma, $\Theta(G,X,k)$ is identified with $S(G,X)$ as in Section 2, and $\mathbb{M}(x_i, y_i, \beta_i)$ are the metabolic planes define in [2, Definition 5.1] (cf. [2, Lemma 5.5]). The lemma will be proven in Section 4.

Theorem 3.2. *Let $R = \mathbb{Z}_{(m)}$ ($m = 0$, 1 or a prime), let X, $X' \in \boldsymbol{Mnf}^n_{\mathrm{cl,sg}}(G)_0$ and $Y \in \boldsymbol{Mnf}^n_{\mathrm{cl}}(G)_1$ such that $n = 2k \geqq 6$ is even, and let $\boldsymbol{f} = (f : X \to Y,\ b : T(X) \oplus f^*\eta \to f^*(\xi \oplus \eta))$ and $\boldsymbol{f}' = (f' : X' \to Y,\ b' : T(X') \oplus f'^*\eta \to f'^*(\xi \oplus \eta))$ be degree 1, k-connected, G-framed maps such that $K_k(X;R)$ and $K_k(X';R)$ are modules in $\mathcal{C}$. If there exists admissible, G-framed cobordism $\boldsymbol{F} = (F, B)$ between $\boldsymbol{f}$ and $\boldsymbol{f}'$ such that $F : W \to I \times Y$ satisfies $\mathcal{F}_m$-Condition (or $\mathcal{P}_0$-Condition and $\widetilde{K}_0(\mathbb{Z}[G]) = 0$) then $\sigma(\boldsymbol{f}) = \sigma(\boldsymbol{f}')$ in $W_n(\boldsymbol{A}, \Theta(G,X,k))_{\mathcal{C}}$.*

For the definition of $W_n(\boldsymbol{A}, \Theta(G,X,k))_{\mathcal{C}}$, see [2, p.280, (4.4)].

Proof. By Lemma 3.1, there exist a degree 1, G-framed map $\boldsymbol{f}''$ and a cobordism $\boldsymbol{F}' \in \boldsymbol{Cob}(G; \boldsymbol{f}, \boldsymbol{f}'')$ satisfying Conditions (3.1.1)–(3.1.3). The equality $\sigma(\boldsymbol{f}') = \sigma(\boldsymbol{f}'')$ in $W_n(\boldsymbol{A}, \Theta(G,X,k))_{\mathcal{C}}$ is clear from (3.1.3). Since $K_k(W';R)$ is stably free over $R[G]$ (cf. Remark 2.3), the universal coefficient theorem and the Poincaré-Lefschetz duality theorem guarantee that $K_{k+1}(W', \partial W'; R)$ is also stably free over $R[G]$. In addition, it follows that

$$K_k(\partial W';R) \cong K_k(W';R) \oplus K_{k+1}(W', \partial W'; R) \text{ and}$$
$$\mathrm{rank}_{R[G]} K_{k+1}(W', \partial W'; R) = \mathrm{rank}_{R[G]} K_k(W';R).$$

Thus, we can deduce form (2.7.2) and (2.7.3) that $K_{k+1}(W', \partial W'; R)$ is a stably free Lagrangian of $\overline{\boldsymbol{M}}_{\partial \boldsymbol{F}'}$, hence

$$[\overline{\boldsymbol{M}}_{\partial \boldsymbol{F}'}] = 0 \in W_n(\boldsymbol{A}, \Theta(G,X,k))_{((\mathrm{s-free}))} = W_n(\boldsymbol{A}, \Theta(G,X,k))_{((\mathrm{free}))}.$$

By (2.7.1), we get $\sigma(\boldsymbol{f}) = \sigma(\boldsymbol{f}'')$ in $W_n(\boldsymbol{A}, \Theta(G,X,k))_{\mathcal{C}}$. □

Let $R = \mathbb{Z}_{(m)}$, let X, $X' \in \boldsymbol{Mnf}^n_{\mathrm{cl,sg}}(G)_0$ and $Y \in \boldsymbol{Mnf}^n_{\mathrm{cl}}(G)_1$, where $n = 2k \geqq 6$, and let $\boldsymbol{f} = (f, b)$ be a degree 1, G-framed map from X to Y. Suppose that $f : X \to Y$ satisfies $\mathcal{F}_m$-Condition (or $\mathcal{P}_0$-Condition and $\widetilde{K}_0(\mathbb{Z}[G]) = 0$). Since $\mathrm{Sing}(G,X) \leqq k$, $\boldsymbol{f}$ is G-framed cobordant rel. $\mathrm{Sing}(G,X)$ to $\boldsymbol{f}' = (f', b')$ such that $f' : X' \to Y$ is k-connected. (Such a

G-framed cobordism can be obtained by G-surgery of dimension $\leqq k-1$ on $\mathrm{Free}(G,X)$.) Then by $\mathcal{F}_m$-Condition (or $\mathcal{P}_0$-Condition and $\widetilde{K}_0(\mathbb{Z}[G])=0$), $K_k(X';R)$ is stably free over $R[G]$. By [2, Lemma 7.1], we obtain the element

$$\sigma(\boldsymbol{f}') = [\boldsymbol{M}_{\boldsymbol{f}'}] \in W_n(\boldsymbol{A}, \Theta(G,X,k))_{((\mathrm{free}))}.$$

Now define

$$\sigma(\boldsymbol{f}) := \sigma(\boldsymbol{f}') \in W_n(\boldsymbol{A}, \Theta(G,X,k))_{((\mathrm{free}))}.$$

Theorem 3.2 guarantees that $\sigma(\boldsymbol{f})$ is defined independently of the choice of $\boldsymbol{f}'$.

Theorem 3.3. *Let $R=\mathbb{Z}_{(m)}$, where $m=0$, 1 or a prime and let X, $X' \in \boldsymbol{Mnf}^n_{\mathrm{cl,sg}}(G)_0$ and $Y \in \boldsymbol{Mnf}^n_{\mathrm{cl}}(G)_1$ such that $n=2k \geqq 6$ is even, and let $\boldsymbol{f}=(f: X \to Y,\ b: T(X)\oplus f^*\eta \to f^*(\xi\oplus\eta))$ be a degree 1, G-framed map such that $f: X\to Y$ satisfies $\mathcal{F}_m$-Condition (or $\mathcal{P}_0$-Condition and $\widetilde{K}_0(\mathbb{Z}[G])=0$) (cf. Remark 2.3). Then $\sigma(\boldsymbol{f})=0$ in $W_n(\boldsymbol{A},\Theta(G,X,k))_{((\mathrm{free}))}$ if and only if $\boldsymbol{f}$ can be converted by G-surgery of isotropy type $\{1\}$ and of dimension $\leqq k$ to a degree 1, k-connected, G-framed map $\boldsymbol{f}'=(f': X'\to Y, b': T(X')\oplus f'^*\eta \to f'^*(\xi\oplus\eta))$ (thus $\boldsymbol{f}\sim\boldsymbol{f}'$ rel. $\mathrm{Sing}(G,X)$) such that $f': X'\to Y$ is an R-homology equivalence.*

Proof. The ‘if part’ of the claim is obvious from the definition of $\sigma(\boldsymbol{f})$. We have to prove the ‘only if part’, namely $\sigma(\boldsymbol{f})=0 \Longrightarrow \boldsymbol{f}\sim\boldsymbol{f}'$ rel. $\mathrm{Sing}(G,X)$ as in the claim. Since $\mathrm{Sing}(G,X)\leqq k$, there exists a k-connected, degree 1, G-framed map $\boldsymbol{f}''=(f'': X''\to Y,\ b'': T(X'')\oplus f''^*\eta\to f''^*(\xi\oplus\eta))$ such that $\boldsymbol{f}\sim\boldsymbol{f}''$ rel. $\mathrm{Sing}(G,X)$ by G-surgery of dimension $\leqq k-1$. Then apply Theorem 7.3 of [2] to $\boldsymbol{f}''$ and obtain a desired $\boldsymbol{f}'$. □

4. Proof of Lemma 3.1

In this section we prove Lemma 3.1 using equivariant handle subtraction.

Firstly, using Lemma 2.5, we adjust $\boldsymbol{F}$ so that $F: W\to I\times Y$ is k-connected. Fix a small, closed, n-dimensional disk D_0 in general position of $\mathrm{Free}(G,X')$. Thus we may assume that $D_0\cap gD_0=\varnothing$ if $g\in G\setminus\{1\}$. For each $\gamma\in\Theta(G,X,k)=\Theta(G,X',k)$, let us take a point $x_\gamma\in X'_\gamma$ such that $G_{x_\gamma}=\rho^G_{X'}(X'_\gamma)$ $(=\rho(\gamma))$ and small closed, n-dimensional, G_{x_γ}-invariant disks D_γ with center x_γ in $X'\setminus GD_0$ so that $gD_\gamma\cap g'D_{\gamma'}=\emptyset$ unless $\gamma=\gamma'$ and $gG_{x_\gamma}=g'G_{x_{\gamma'}}$ (i.e. $gx_\gamma=g'x_{\gamma'}$), where γ and γ' range over $\Theta(G,X,k)$ and g and g'

range over $\in G$. Here GD_γ becomes a G-equivariant tubular neighborhood of Gx_γ. Take connecting tubes T_γ from ∂D_0 to ∂D_γ in general position of

$$\mathrm{Free}(G, X') \setminus G(\mathrm{Int}(D_0)) \setminus \bigcup_{\gamma \in \Theta(G,X,k)} G(\mathrm{Int}(D_\gamma)).$$

Then,

$$D := D_0 \cup \bigcup_{\gamma \in \Theta(G,X,k)} (D_\gamma \cup T_\gamma)$$

is homeomorphic to the n-dimensional disk D^n. Since $K_k(W; \mathbb{Z}) \to K_k(W, \partial W; \mathbb{Z})$ is surjective, we can take finitely many embeddings

$$h_i : S^k \to \mathrm{Int}(W),$$

$i = 1, \cdots, \ell$ say, which together generate $K_k(W, \partial W; \mathbb{Z})$. Without loss of generality, we can assume that gh_i and $g'h_j$ are disjoint unless $i = j$. For each i, take a path

$$p_i : [0,1] \hookrightarrow \mathrm{Free}(G, W)$$

in general position, such that for $t \in [0,1]$, $p_i(t) \in X' \Leftrightarrow t = 0$ and $p_i(t) \in \mathrm{Im}(h_i) \Leftrightarrow t = 1$, and $p_i(0) \in \mathrm{Int}D_0$. Choose a thin band $u_i : [0,1] \times D^k \hookrightarrow \mathrm{Free}(G, W)$ along the path p_i (i.e., $u_i(t, 0) = p_i(t)$ for all $t \in [0,1]$) such that for $t \in [0,1]$ and $z \in D^k$, $u_i(t, z) \in X' \Leftrightarrow t = 0$ and $u_i(t, z) \in \mathrm{Im}(h_i) \Leftrightarrow t = 1$, and $u_i(0, D^k) \subset \mathrm{Int}(D_0)$. Then,

$$u_i([0,1] \times S^{k-1}) \cup \mathrm{Im}(h_i) \setminus u_i(1, \mathrm{Int}(D^k))$$

gives an embedding

$$h_i' : (D^k, S^{k-1}) \to (W, \mathrm{Free}(G, X')).$$