Braids, Permutations, Polynomials - I

Vladimir Ya. Lin

Department of Mathematics
Technion-Israel Institute of
Technology
32000 Haifa

Israel

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

MPI 96-118

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VLADIMIR YA. LIN

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Abstract

Continuing Artin's investigations on representations of braids by permutations, we obtain the following results. The image Im ψ of a homomorphism ψ from the Artin braid group $\mathbf{B}(k)$ on k strings into symmetric group $\mathbf{S}(n)$ of degree n must be a cyclic group whenever either $(*)$ $n < k \neq 4$ or $(**)$ $6 < k < n < 2k$ and ψ is irreducible (i. e. Im ψ is a transitive permutation group). For $k > 8$ there exist, up to conjugation, exactly 3 irreducible non-Abelian representations $\mathbf{B}(k) \to \mathbf{S}(2k)$, and each of them is imprimitive. For $n < k \neq 4$ the image of any braid homomorphism φ: $\mathbf{B}(k) \to \mathbf{B}(n)$ is an Abelian group, whereas any endomorphism φ of $\mathbf{B}(k)$ with non-Abelian image sends the pure braid group $\mathbf{I}(k)$ into itself. Moreover, for $k > 4$ the intersection $\mathbf{I}(k) \cap \mathbf{B}'(k)$ of $\mathbf{I}(k)$ with the commutator subgroup $\mathbf{B}'(k) = [\mathbf{B}(n), \mathbf{B}(n)]$ is a completely characteristic subgroup of $\mathbf{B}'(k)$.

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§0. Introduction

In the middle 1970's I announced some results on homomorphisms of braids and on representations of braids by permutations [L2,L5,L7]. The proofs were never published for the following two reasons. First, some of them were based on a straightforward modification of Artin's methods [Ar3] and on the fact that for $k > 4$ the commutator subgroup $\mathbf{B}'(k)$ of the braid group $\mathbf{B}(k)$ is a perfect group [GL1] (seemingly, Artin did not know this property). On the other hand, some other proofs contained too long combinatorial computations, and I felt that they might be simplified. Recently I found that this in fact can be done using a cohomology approach, which leads also to some new results. The new proofs are still lengthy, but involve less combinatorics and seem more suitable for publication.[1]

Some motivations: braid homomorphisms and polynomials. The principal motivation for our study of braid homomorphisms was the fact that they are closely related to algebraic equations, algebraic functions, and, particularly, to the 13th Hilbert problem for algebraic functions. Some of these relations may be described as follows.

Take a point $z = (z_1, \dots, z_n) \in \mathbb{C}$ and consider the polynomial

$$p_n(t, z) = t^n + z_1 t^{n-1} + \dots + z_n$$

in one variable t. Let $d_n(z)$ be the discriminant of $p_n(\cdot, z)$; consider the domain

$$\mathbf{G}_n = \{z \in \mathbb{C} \mid d_n(z) \neq 0\};$$

we call $\mathbf{G}_n$ the *space of separable polynomials of degree n*. Translating all the roots of p_n, we can "kill" the coefficient z_1; this leads to a natural isomorphism $\mathbf{G}_n \cong \mathbb{C} \times \mathbf{G}_n^\circ$, where $\mathbf{G}_n^\circ = \{z \in \mathbf{G}_n \mid z_1 = 0\}$. The discriminant defines the holomorphic bundle $d_n\colon \mathbf{G}_n^\circ \to \mathbb{C}^* = \mathbb{C} - \{0\}$ with the standard fiber

$$\mathbf{SG}_n = \{z \mid z_1 = 0,\ d_n(z) = 1\},$$

which is a nonsingular algebraic hypersurface in $\mathbb{C}^{n-1}$. The spaces $\mathbf{G}_n$ and $\mathbf{SG}_n$ are Eilenberg-MacLane $K(\pi, 1)$-spaces for the braid group $\mathbf{B}(n)$ and its commutator subgroup $\mathbf{B}'(n)$, respectively.

Consider a separable algebraic equation $f(t) = t^n + a_1 t^{n-1} + \dots + a_n = 0$ over the algebra $C(X)$ of all complex continuous functions on a "nice" topological space X (so, for any $x \in X$ the polynomial $f(t, x)$ has no multiple roots). Then we can define the continuous mapping $X \ni x \mapsto f(t, x) \in \mathbf{G}_n$, which, in turn, induces the homomorphism of the fundamental groups

$$f_*\colon \pi_1(X) \to \pi_1(\mathbf{G}_n) \cong \mathbf{B}(n).$$

Moreover, since $\mathbf{G}_n$ is a $K(\pi, 1)$-space, *every* homomorphism $\varphi\colon \pi_1(X) \to \mathbf{B}(n)$ may be obtained from some separable polynomial of degree n over $C(X)$. Any knowledge on the behavior of homomorphisms $\pi_1(X) \to \mathbf{B}(n)$ may provide us with some essential information about separable algebraic equations over $C(X)$ (see, for instance, [GL1]).

[1]This Preprint contains the first part of the paper; the second part will be devoted to some applications.

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Consider an entire algebraic function $t = F(w)$ of several complex variables $w = (w_1, ..., w_k) \in \mathbb{C}^k$ with the defining polynomial $P_F(t, w) = t^n + a_1(w)t^{n-1} + ... + a_n(w)$ ($a_i \in \mathbb{C}[w]$). Let $D_F(w)$ be the discriminant of $P_F(t, w)$ and $G_F = \mathbb{C}^k - \Sigma_F$ be the complement of the branch locus $\Sigma_F = \{w \in \mathbb{C}^k \mid D_{P_F}(w) = 0\}$ of the function $F(w)$. Then we have the polynomial mapping

$$\mathbf{a}\colon\ G_F \ni w \mapsto (a_1(w), ..., a_n(w)) \in \mathbf{G}_n$$

and the corresponding homomorphism $\mathbf{a}_*\colon\ \pi_1(G_F) \to \mathbf{B}(n)$.

In some settings of the 13th Hilbert Problem (see e. g. [A1, A2, L3, L4, L6, L7, L9]) one should investigate an entire algebraic (or algebroidal) function $t = F(w)$ of k complex variables $w = (w_1, ..., w_k) \in \mathbb{C}^k$ with the branch locus Σ_F that coincides with the branch locus $\mathbf{\Sigma}_k = \{w \in \mathbb{C}^k \mid d_k(w) = 0\}$ of the *"universal"* algebraic function $t = u(w)$ defined by the equation

$$t^k + w_1 t^{k-1} + ... + w_k = 0.$$

In this case we deal with the corresponding polynomial (or more generally, holomorphic) mapping $f\colon\ \mathbf{G}_k \to \mathbf{G}_n$ and with the induced homomorphism

$$f_*\colon\ \mathbf{B}(k) \cong \pi_1(\mathbf{G}_k) \to \pi_1(\mathbf{G}_n) \cong \mathbf{B}(n).$$

The behavior of the homomorphism f_* affects strongly the behavior of the mapping f. For instance, if the homomorphism f_* is Abelian (that is, its image is an Abelian subgroup of $\mathbf{B}(n)$), then f is homotopic to a composition $h \circ d_k\colon\ \mathbf{G}_k \xrightarrow{d_k} \mathbb{C}^* \xrightarrow{h} \mathbf{G}_n$ of the canonical discriminant mapping d_k and a continuous mapping $h\colon\ \mathbb{C}^* \to \mathbf{G}_n$; we call such a mapping f *splittable.* We prove that *any homomorphism* $\mathbf{B}(k) \to \mathbf{B}(n)$ *is Abelian whenever* $k > 4$ *and* $n < k$; thus, for such k and n, *every mapping* $\mathbf{G}_k \to \mathbf{G}_n$ *is splittable.* Further, let $\mathbf{E}_k \to \mathbf{G}_k$ and $\mathbf{SE}_k \to \mathbf{SG}_k$ be the coverings corresponding to the pure braid group $\mathbf{I}(k) \subset \mathbf{B}(k)$ and to the *pure commutator subgroup* $\mathbf{J}(k) = \mathbf{I}(k) \cap \mathbf{B}'(k)$, respectively. We prove that *for* $k > 4$ *the pure braid group* $\mathbf{I}(k)$ *is invariant under any non-Abelian endomorphism of* $\mathbf{B}(k)$, and *the pure commutator subgroup* $\mathbf{J}(k)$ *is a completely characteristic subgroup in* $\mathbf{B}'(k)$. These algebraic results imply that *any nonsplittable self-mapping of* $\mathbf{G}_k$ *and any self-mapping of* $\mathbf{SG}_k$ *can be lifted to self-mappings of the coverings* $\mathbf{E}_k$ *and* $\mathbf{SE}_k$, *respectively.* This enables us to obtain a complete explicit description of holomorphic self-mappings of the spaces $\mathbf{G}_k$ and $\mathbf{SG}_k$. Notice that not every homomorphism $\mathbf{B}(k) \to \mathbf{B}(n)$ is induced by a *holomorphic* mapping $\mathbf{G}_k \to \mathbf{G}_n$; the homomorphisms that are induced by holomorphic mappings are contained in the narrow class of *special* homomorphisms (§0.6).

Any homomorphism $\varphi\colon\ \mathbf{B}(k) \to \mathbf{B}(n)$ produces the corresponding homomorphism $\psi\colon\ \mathbf{B}(k) \to \mathbf{S}(n)$ into the symmetric group $\mathbf{S}(n)$ (the composition of φ with the canonical projection $\mathbf{B}(n) \to \mathbf{S}(n)$). The behavior of the homomorphism ψ influences strongly the behavior of the original homomorphism φ. This is one of the reasons of studying also homomorphisms $\mathbf{B}(k) \to \mathbf{S}(n)$. Another reason is that the latter homomorphisms are in a natural correspondence with the finite coverings of the space $\mathbf{G}_k$, which, in turn, are of great interest. Our algebraic results imply, for instance, that any (connected) n-covering over $\mathbf{G}_k$ is cyclic whenever $k > \max\{n, 4\}$ or $6 < k < n < 2k$; we prove also that for $k > 8$ there exist only three different (i. e., non-equivalent) noncyclic connected $2k$-coverings over $\mathbf{G}_k$.

0.0. Notation and some definitions. For the readers' convenience, we start with some notation and definitions used throughout the paper.

0.0.1. Sets, groups, homomorphisms. The cardinality of a set Γ is denoted by $\#\,\Gamma$. The order of a nonunit element g of a group G is denoted by $\operatorname{ord} g$. If two elements $g, h \in G$ are conjugate, we write $g \sim h$.

We denote by $\mathbb{F}_m$ the *free group of rank* m ($m \in \mathbb{Z}_+$ or $m = \infty = \#\mathbb{N}$); particularly, $\mathbb{F}_1 \cong \mathbb{Z}$; if the value of m is not essential, we write simply $\mathbb{F}$.

Commutator subgroup; perfect groups; residually finite groups. For any group G, we denote by G' the commutator subgroup of G. A group G is called *perfect* if $G = G'$. *A quotient group of a perfect group is a perfect group; a perfect group does not possess nontrivial homomorphisms into any Abelian group.*

A group G is called *residually finite* if homomorphisms into finite groups separate elements of G. The latter property is equivalent to the following one: for any element $g \in G$, $g \neq 1$, there is a subgroup $H \subset G$ of finite index such that $g \notin H$. *Any free group is residually finite.* The following theorem is due to A. I. Maltsev [Ma]:

Maltsev Theorem. *Any semidirect product of finitely generated residually finite groups is a residually finite group.*

Hopfian groups. A group G is called *Hopfian* if any surjective endomorphism $G \to G$ is an automorphism. *Any finitely generated residually finite group is Hopfian* (see, for instance, [Ne, 41.44, p. 151]).

Braid-like couples. A couple of elements g, h in a group G is called *braid-like* if $gh \neq hg$ and $ghg = hgh$; in this case we write $g\infty h$. Clearly, $g\infty h$ implies $g \sim h$.

Conjugate homomorphisms. Two homomorphisms of groups $\phi, \psi\colon G \to H$ are said to be *conjugate* if there is an element $h \in H$ such that $\psi(g) = h\phi(g)h^{-1}$ for all $g \in G$; in this case we write $\phi \sim \psi$; "$\sim$" is an equivalence relation on the set $\operatorname{Hom}(G, H)$ of all homomorphisms $G \to H$.

Abelian, cyclic and integral homomorphisms. A group homomorphism $\phi\colon G \to H$ is said to be *Abelian* (respectively, *cyclic, integral*), if its image $\operatorname{Im}\psi = \phi(G)$ is an Abelian (respectively, cyclic, torsion free cyclic) subgroup of the group H (we include the trivial homomorphism in each of these three classes).

Remark 0.1. *If a group G and its commutator subgroup G' are finitely generated, then any homomorphism $\phi\colon G \to \mathbb{F}$ is integral.* (The image $H = \operatorname{Im}\phi \subseteq \mathbb{F}$ is a free group of finite rank $r \leq 1$; indeed, if $r > 1$, then $\phi(G') = H' \cong (\mathbb{F}_r)' \cong \mathbb{F}_\infty$, which is impossible, since G' is finitely generated.) ○

0.0.2. Symmetric groups. The permutations of a set Γ form the *symmetric group* $\mathbf{S}(\Gamma)$; We regard this group as acting from the left on the set Γ.

Let $H \subseteq \mathbf{S}(\Gamma)$. A subset $\Sigma \subseteq \Gamma$ is *H-invariant*, if $S(\Sigma) = \Sigma$ for every $S \in H$; we denote by $\operatorname{Inv} H$ the family of all *nontrivial* (i. e., $\neq \varnothing$ and $\neq \Gamma$) H-invariant subsets of Γ. For a natural $r < \#\Gamma$ we denote by $\operatorname{Inv}_r H$ the family of all H-invariant subsets of cardinality r (if H consists of a single permutation S, we write $\operatorname{Inv} S$ and $\operatorname{Inv}_r S$ instead of $\operatorname{Inv}\{S\}$ and

$\mathrm{Inv}_r\{S\}$, respectively). The restriction of S to a set $\Sigma \in \mathrm{Inv}\, S$ is denoted by $S \mid \Sigma$; we regard $S \mid \Sigma$ as an element of $\mathbf{S}(\Sigma)$.

An element $\gamma \in \Gamma$ is a *fixed point* of $S \in \mathbf{S}(\Gamma)$ if $S(\gamma) = \gamma$; we denote by $\mathrm{Fix}\, S$ the set of all fixed points of S. The *support* $\mathrm{supp}\ S$ of $S \in \mathbf{S}(\Gamma)$ is the complement $\Gamma - \mathrm{Fix}\ S$. For any set $\Sigma \subseteq \Gamma$ we identify $\mathbf{S}(\Sigma)$ with the subgroup in $\mathbf{S}(\Gamma)$ consisting of all the permutations S with $\mathrm{supp}\ S \subseteq \Sigma$. Two permutations S, S' are *disjoint* if $\mathrm{supp}\ S \cap \mathrm{supp}\ S' = \varnothing$. If $n \in \mathbb{N}$ and $\mathbf{\Delta}_n = \{1, 2, \dots, n\}$, we write $\mathbf{S}(n)$ instead of $\mathbf{S}(\mathbf{\Delta}_n)$; $\mathbf{S}(n)$ is the *symmetric group of degree n*. The *alternating subgroup* $\mathbf{A}(n) \subset \mathbf{S}(n)$ consists of all even permutations $S \in \mathbf{S}(n)$ and coincides with the commutator subgroup $\mathbf{S}'(n)$; for $n > 4$ the group $\mathbf{A}(n)$ is perfect.○

Cyclic types, r-components. For $A, B \in \mathbf{S}(n)$ we write $A \preccurlyeq B$ if and only if each cycle entering in the cyclic decomposition of A is contained in the cyclic decomposition of B. Let $A = C_1 \cdots C_q$ be the cyclic decomposition of $A \in \mathbf{S}(n)$ and $r_i \geq 2$ be the length of the cycle C_i $(1 \leq i \leq q)$; the *unordered* q-tuple of the natural numbers $[r_1, \dots, r_q]$ is called the *cyclic type* of A and is denoted by $[A]$ (any r_i occurs in $[A]$ as many times as r_i-cycles enter in the cyclic decomposition of A). Clearly, $\mathrm{ord}\, A = \mathrm{LCM}(r_1, \dots r_q)$ (the least common multiple of $r_1, \dots r_q$).

For any $A \in \mathbf{S}(n)$ and any natural number $r \geq 2$ we denote by $\mathfrak{C}_r(A)$ the set of all the r-cycles in the cyclic decomposition of A; we call this set the *r-component* of A. The set $\mathrm{Fix}\ A$ is called the *degenerate component* of A. ○

Transitive and primitive homomorphisms. We suppose that the reader is familiar with the notions of transitive and primitive groups of permutations (see, for instance, [Ha]). A homomorphism ψ: $G \to \mathbf{S}(n)$ is said to be *transitive* (respectively, *intransitive, primitive, imprimitive*), if its image $\psi(G)$ is a transitive (respectively, intransitive, primitive, imprimitive) subgroup of the symmetric group $\mathbf{S}(n)$.

Disjoint products. Reductions of homomorphisms $G \to \mathbf{S}(n)$**.** Given some decomposition $\mathbf{\Delta}_n = D_1 \cup \cdots \cup D_q$, $\#D_j = n_j$, we have the corresponding embedding $\mathbf{S}(D_1) \times \cdots \times \mathbf{S}(D_q) \hookrightarrow \mathbf{S}(n)$. For group homomorphisms ψ_j: $G \to \mathbf{S}(D_j) \cong \mathbf{S}(n_j)$, we define the *disjoint product* $\psi = \psi_1 \times \cdots \times \psi_q$: $G \to \mathbf{S}(D_1) \times \cdots \times \mathbf{S}(D_q) \hookrightarrow \mathbf{S}(n)$ by $\psi(g) = \psi_1(g) \cdots \psi_q(g) \in \mathbf{S}(n)$, $g \in G$.

Let ψ: $G \to \mathbf{S}(n)$ be a group homomorphism, $H = \mathrm{Im}\ \psi$, and let $\Sigma \subseteq \mathbf{\Delta}_n$ be some H-invariant subset (for instance, Σ may be an H-orbit). Consider the homomorphism ϕ_Σ: $H \ni S \mapsto S \mid \Sigma \in \mathbf{S}(\Sigma)$; the composition

$$\psi_\Sigma = \phi_\Sigma \circ \psi\colon\ G \xrightarrow{\psi} H \xrightarrow{\phi_\Sigma} \mathbf{S}(\Sigma)$$

is called the *reduction of* ψ *to the* $(\mathrm{Im}\ \psi)$*-invariant subset* Σ. The homomorphism ψ_Σ is transitive if and only if Σ is an $(\mathrm{Im}\ \psi)$-orbit. Any homomorphism ψ is the disjoint product of its reductions to all the $(\mathrm{Im}\ \psi)$-orbits (this is just the decomposition of the representation ψ in the direct sum of irreducible representations.) The following simple observation is used throughout the paper:

Observation. *A group homomorphism ψ: $G \to \mathbf{S}(n)$ is Abelian if and only if its reduction to each* (Im ψ)*-orbit is Abelian.*

For natural numbers q, r, we denote by (q,r) their greatest common divisor, and by $|q|_r$ the residue of q modulo r $(0 \le |q|_r \le r-1)$.

0.1. Canonical presentation of the braid group $\mathbf{B}(k)$. The *braid group* $\mathbf{B}(k)$ *on k strings* is defined by the presentation with $k-1$ generators $\sigma_1, .., \sigma_{k-1}$ and the defining system of relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad (|i-j| \ge 2) \tag{0.1}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad (1 \le i \le k-2). \tag{0.2}$$

The generators $\sigma_1, .., \sigma_{k-1}$ and the presentation (0.1),(0.2) are called *canonical.*

Torsion. The following theorem was first proven by Fadell and Neuwirth, [FaN], using a topological argument; an algebraic proof was suggested by .... [??].

Fadell-Neuwirth Theorem. *The braid group* $\mathbf{B}(n)$ *is torsion free.*

It follows from (0.1),(0.2) that $\mathbf{B}(k)/\mathbf{B}'(k) \cong \mathbb{Z}$. This fact and Fadell-Neuwirth Theorem imply:

Abelian and cyclic homomorphisms of $\mathbf{B}(k)$. *Any Abelian homomorphism of* $\mathbf{B}(k)$ *is cyclic. Any Abelian homomorphism* $\mathbf{B}(k) \to \mathbf{B}(n)$ *is integral. If a homomorphism* ϕ: $\mathbf{B}(k) \to H$ *is cyclic, then* $\phi(\sigma_1) = \phi(\sigma_2) = \cdots = \phi(\sigma_{k-1})$.

0.2. Special presentation of $\mathbf{B}(k)$. For $1 \le i < j \le k$, we put

$$\alpha_{ij} = \sigma_i\sigma_{i+1}\cdots\sigma_{j-1}, \quad \beta_{ij} = \alpha_{ij}\sigma_i, \qquad \alpha = \alpha_{1k} = \sigma_1\sigma_2\cdots\sigma_{k-1}, \quad \beta = \alpha\sigma_1. \tag{0.3}$$

It is easily checked that

$$\sigma_{i+1} = \alpha\sigma_i\alpha^{-1} \qquad (1 \le i \le k-2) \tag{0.4}$$

$$\sigma_i = \alpha^{i-1}\sigma_1\alpha^{-(i-1)} \qquad (1 \le i \le k-1) \tag{0.5}$$

$$\begin{aligned} \alpha_{ij}\sigma_m &= \sigma_m\alpha_{ij} && \text{for } m < i-1 \text{ or } m > j, \\ \alpha_{ij}\sigma_m &= \sigma_{m+1}\alpha_{ij} && \text{for } i \le m \le j-2, \\ \alpha_{ij}^q\sigma_m &= \sigma_{m+q}\alpha_{ij}^q && \text{for } i \le m \le m+q \le j-1. \end{aligned} \tag{0.6}$$

Relations (0.3), (0.6) imply that for $1 \le q \le j-i$

$$\beta_{ij}^q = (\alpha_{ij}\sigma_i)\cdot\underbrace{(\alpha_{ij}\sigma_i)\cdots(\alpha_{ij}\sigma_i)}_{q-1 \text{ times}} = \alpha_{ij}\sigma_i\cdot\sigma_{i+1}\cdots\sigma_{i+q-1}\alpha_{ij}^{q-1};$$

for $q = j-i$ this shows that $\beta_{ij}^{j-i} = \alpha_{ij}\sigma_i\sigma_{i+1}\cdots\sigma_{j-1}\alpha_{ij}^{j-i-1} = \alpha_{ij}^{j-i+1}$. Moreover, for $m = i$ relations (0.6) may be written as $\sigma_{i+q} = \alpha_{ij}^q\sigma_i\alpha_{ij}^{-q} = \alpha_{ij}^{q-1}\beta_{ij}\alpha_{ij}^{-q}$ $(i \le i+q \le j-1)$. Therefore, we have:

$$\begin{aligned} \alpha_{ij}^{j-i+1} &= \beta_{ij}^{j-i} && \text{for } 1 \le i < j \le k, \\ \sigma_{i+q} &= \alpha_{ij}^{q-1}\beta_{ij}\alpha_{ij}^{-q} && \text{for } 0 \le q \le j-i-1. \end{aligned} \tag{0.7}$$

Particularly, these relations show that *the element* $\alpha_{ij}^{j-i+1} = \beta_{ij}^{j-i}$ *commutes with all the elements* $\sigma_i, \ldots, \sigma_{j-1}$.

For $i = 1$ and $j = k$, relations (0.7) take the form

$$\begin{aligned} \alpha^k &= \beta^{k-1}, \\ \sigma_{1+q} &= \alpha^{q-1}\beta\alpha^{-q} \qquad \text{for } 0 \le q \le k-2, \end{aligned} \tag{0.8}$$

which shows that *the elements α, β generate the whole group $\mathbf{B}(k)$, and the element $\alpha^k = \beta^{k-1}$ is central in $\mathbf{B}(k)$.* The defining system of relations for the generators α, β is as follows:

$$\beta\alpha^{i-1}\beta = \alpha^i\beta\alpha^{-(i+1)}\beta\alpha^i \qquad (2 \le i \le [k/2]), \tag{0.9}$$

$$\alpha^k = \beta^{k-1}. \tag{0.10}$$

The presentation of the group $\mathbf{B}(k)$ given by (0.9),(0.10) is called *special.* A pair of elements $a, b \in \mathbf{B}(k)$ is said to be a *special system of generators* in $\mathbf{B}(k)$ if there exists an automorphism ψ of $\mathbf{B}(k)$ such that $\psi(\alpha) = a$ and $\psi(\beta) = b$. If $\{a, b\}$ is such a system of generators, then the elements

$$s_i = \psi(\sigma_i) = a^{i-2}ba^{-(i-1)} \qquad (1 \le i \le k-1) \tag{0.11}$$

also form a system of generators of $\mathbf{B}(k)$ that satisfy relations (0.1), (0.2); we call such a system of generators *standard.* The elements σ_1, α also generate the whole group $\mathbf{B}(k)$ (since $\beta = \alpha\sigma_1$).

0.3. Pure braid group. The *canonical projection* $\mu = \mu_k$: $\mathbf{B}(k) \to \mathbf{S}(k)$ is defined by $\mu(\sigma_i) = (i, i+1) \in \mathbf{S}(k)$ $(1 \le i \le k-1)$. The kernel Ker $\mu = \mathbf{I}(k) \subset \mathbf{B}(k)$ of the epimorphism μ is the normal subgroup in $\mathbf{B}(k)$ generated (as a normal subgroup) by the elements $\sigma_1^2, \ldots, \sigma_{k-1}^2$; $\mathbf{I}(k)$ is called the *pure braid group.*

A presentation of the pure braid group $\mathbf{I}(k)$ was first found by W. Burau [Bu] (see also [Mr,Bi]). We need some properties of the group $\mathbf{I}(k)$ proven in [Mr].

The group $\mathbf{I}(k)$ is generated by the elements $s_{i,j} \in \mathbf{B}(k)$ $(1 \le i < j \le k)$ defined by the recurrent relations

$$s_{i,i+1} = \sigma_i^2 \qquad \text{and} \qquad s_{i,j+1} = \sigma_j s_{i,j}\sigma_j^{-1} \text{ for } 1 \le i < j < k.$$

The elements $s_{i,j}$ are called the *canonical generators* of $\mathbf{I}(k)$. Assume that $1 < r < k$ and denote by $s'_{i,j}$, $1 \le i < j \le r$, the canonical generators of $\mathbf{I}(r)$. The mapping of the generators

$$\begin{aligned} \xi_{k,r}(s_{i,j}) &= 1 \quad \text{if } 1 \le i < j \text{ and } r < j \le k, \\ \xi_{k,r}(s_{i,j}) &= s'_{i,j} \quad \text{if } 1 \le i < j \le r, \end{aligned} \tag{0.12}$$

defines an epimorphism $\xi_{k,r}$: $\mathbf{I}(k) \to \mathbf{I}(r)$. The kernel of this epimorphism coincides with the subgroup $\mathbf{I}^r(k) \subset \mathbf{I}(k)$ generated by all the elements $s_{i,j}$ with $j > r$. The following important theorem was proven by A. A. Markov [Mr]:

Markov Theorem. *The normal subgroups $\mathbf{I}^r(k) \subseteq \mathbf{I}(k)$ fit into a normal series*

$$\{1\} = \mathbf{I}^k(k) \subset \mathbf{I}^{k-1}(k) \subset \cdots \subset \mathbf{I}^2(k) \subset \mathbf{I}^1(k) = \mathbf{I}(k)$$

such that $\mathbf{I}^r(k)/\mathbf{I}^{r+1}(k) \cong \mathbb{F}_r$ (the free group of rank r, $1 \le r \le k-1$).

Each group $\mathbf{I}^r(k)$ is finitely generated; Markov Theorem implies the following two corollaries.

Corollary 0.1. *A perfect group does not possess nontrivial homomorphisms into the pure braid group* $\mathbf{I}(k)$.

Proof. It suffices to show that any nontrivial subgroup $H \subseteq \mathbf{I}(k)$ has a nontrivial homomorphism into an Abelian group. For some r, $1 \le r \le k-1$, we have $H \subseteq \mathbf{I}^r(k)$ and $H \not\subseteq \mathbf{I}^{r+1}(k)$. Projecting H into the quotient group $\mathbf{I}^r(k)/\mathbf{I}^{r+1}(k) \cong \mathbb{F}_r$, we obtain a nontrivial free subgroup $\widetilde{H} \subseteq \mathbf{I}^r(k)/\mathbf{I}^{r+1}(k)$, which certainly has nontrivial homomorphisms into Abelian groups; hence, the subgroup H itself has such homomorphisms. $\square$

Corollary 0.2. *The group* $\mathbf{B}(k)$ *is residually finite; any finitely generated subgroup of* $\mathbf{B}(k)$ *is Hopfian.*

Proof. By Markov Theorem, every $\mathbf{I}^r(k)$ is a semidirect product of the finitely generated groups $\mathbf{I}^{r+1}(k)$ and $\mathbb{F}_r$. Maltsev Theorem implies (by induction) that the group $\mathbf{I}(k)$ is residually finite. Any subgroup $H \subseteq \mathbf{I}(k)$ of finite index in $\mathbf{I}(k)$ is also a subgroup of finite index in $\mathbf{B}(k)$; hence, $\mathbf{B}(k)$ is residually finite and any finitely generated subgroup of $\mathbf{B}(k)$ is Hopfian. $\square$

0.4. Center. Denote by $\mathbf{C}(k)$ $(k \ge 2)$ the infinite cyclic subgroup in $\mathbf{B}(k)$ generated by the element $A_k = \alpha^k = (\sigma_1\sigma_2\cdots\sigma_{k-1})^k$. Since $\mu(\alpha) = (1,2,...,k) \in \mathbf{S}(k)$, we have $\mu(A_k) = 1$; hence, $\mathbf{C}(k) \subseteq \mathbf{I}(k)$. Clearly, $\mathbf{C}(2) = \mathbf{I}(2)$. Chow [Ch] proved that for $k \ge 3$ the subgroup $\mathbf{C}(k)$ coincides with the center of the braid group $\mathbf{B}(k)$ (see also [Bo]).

0.5. Transitive homomorphisms $\mathbf{B}(k) \to \mathbf{S}(k)$. Any transitive Abelian homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ is cyclic and conjugate to the homomorphism ψ_0 defined by

$$\psi_0(\sigma_1) = \psi_0(\sigma_2) = \ldots = \psi_0(\sigma_{k-1}) = (1,2,\ldots,n);$$

particularly, $[\psi(\sigma_i)] = [n]$ for all $i = 1,...,k-1$. The following classical theorem of E. Artin [Ar3] describes all noncyclic transitive homomorphisms of the group $\mathbf{B}(k)$ into the symmetric group $\mathbf{S}(k)$. (See also Remark 2.2.)

Artin Theorem. *Let* ψ: $\mathbf{B}(k) \to \mathbf{S}(k)$ *be a noncyclic transitive homomorphism.*

a) If $k \ne 4$ *and* $k \ne 6$*, then* ψ *is conjugate to the canonical projection* μ.

b) If $k = 6$*, then* ψ *is either conjugate to* μ *or conjugate to the homomorphism* ν_6 *defined by*

$$\nu_6(\sigma_1) = (1,2)(3,4)(5,6), \qquad \nu_6(\alpha) = (1,2,3)(4,5).$$

c) If $k = 4$*, then* ψ *is either conjugate to* μ *or conjugate to one of the following three homomorphisms* $\nu_{4,1}$, $\nu_{4,2}$, $\nu_{4,3}$:

$$\begin{aligned}
&\nu_{4,1}(\sigma_1) = (1,2,3,4), && \nu_{4,1}(\alpha) = (1,2); && [\nu_{4,1}(\sigma_3) = \nu_{4,1}(\sigma_1)]\\
&\nu_{4,2}(\sigma_1) = (1,3,2,4), && \nu_{4,2}(\alpha) = (1,2,3,4); && [\nu_{4,2}(\sigma_3) = \nu_{4,2}(\sigma_1^{-1})]\\
&\nu_{4,3}(\sigma_1) = (1,2,3), && \nu_{4,3}(\alpha) = (1,2)(3,4); && [\nu_{4,3}(\sigma_3) = \nu_{4,3}(\sigma_1)].
\end{aligned}$$

d) Except of the case when $k = 4$ *and* $\psi \sim \nu_{4,3}$*, the homomorphism* ψ *is surjective. In the exceptional case when* $\psi \sim \nu_{4,3}$*, the image of* ψ *coincides with the alternating subgroup* $\mathbf{A}(4) \subset \mathbf{S}(4)$. $\square$

0.6. Commutator subgroup $\mathbf{B}'(k)$; canonical integral projection. We have already noted that $\mathbf{B}(k)/\mathbf{B}'(k) \cong \mathbb{Z}$; the homomorphism χ: $\mathbf{B}(k) \to \mathbb{Z}$ defined by

$$\chi(\sigma_1) = \ldots = \chi(\sigma_{k-1}) = 1 \in \mathbb{Z}$$

is called the *canonical integral projection* of the group $\mathbf{B}(k)$. Clearly, Ker $\chi = \mathbf{B}'(k)$.

Remark 0.2. *If G is a torsion free group and ϕ: $\mathbf{B}(k) \to G$ is a nontrivial Abelian homomorphism, then* Ker $\phi = \mathbf{B}'(k)$. (Clearly, $\mathbf{B}'(k) \subseteq$ Ker ϕ; this inclusion cannot be strict, for otherwise Im $\phi \cong \mathbf{B}(k)/$ Ker ϕ would be a nontrivial proper quotient group of the group $\mathbf{B}(k)/\mathbf{B}'(k) \cong \mathbb{Z}$, which is impossible since G is torsion free.) ○

$\mathbf{B}'(2) = \{1\}$; the following theorem (see [GL1] for the proof) contains some information about the groups $\mathbf{B}'(k)$ for $k \geq 3$. In the formulation of this theorem we regard the group $\mathbf{B}'(k)$ as naturally embedded into the group $\mathbf{B}(k)$ and write generators of $\mathbf{B}'(k)$ as words in the canonical generators of $\mathbf{B}(k)$.

Gorin-Lin Theorem. *a)* $\mathbf{B}'(3)$ *is a free group of rank* 2 *with the free base*

$$u = \sigma_2\sigma_1^{-1}, \quad v = \sigma_1\sigma_2\sigma_1^{-2}.$$

b) For $k > 3$ the group $\mathbf{B}'(k)$ has a finite presentation with the generators

$$\begin{aligned} &u = \sigma_2\sigma_1^{-1}, && v = \sigma_1\sigma_2\sigma_1^{-2}, \\ &w = \sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1}, && c_i = \sigma_{i+2}\sigma_1^{-1} \qquad (1 \leq i \leq k-3) \end{aligned} \tag{0.13}$$

and with the following defining system of relations:[2]

$$uc_1u^{-1} = w, \tag{0.14}$$

$$uwu^{-1} = w^2c_1^{-1}w, \tag{0.15}$$

$$vc_1v^{-1} = c_1^{-1}w, \tag{0.16}$$

$$vwv^{-1} = (c_1^{-1}w)^3c_1^{-2}w, \tag{0.17}$$

$$uc_i = c_iv \qquad (2 \leq i \leq k-3), \tag{0.18}$$

$$vc_i = c_iu^{-1}v \qquad (2 \leq i \leq k-3), \tag{0.19}$$

$$c_ic_j = c_jc_i \qquad (1 \leq i < j-1 \leq k-4), \tag{0.20}$$

$$c_ic_{i+1}c_i = c_{i+1}c_ic_{i+1} \qquad (1 \leq i \leq k-4). \tag{0.21}$$

c) The subgroup $\mathbf{T}$ of the group $\mathbf{B}'(4)$ generated by the elements w and c_1 is a free group of rank 2; this subgroup coincides with the intersection of the lower central series of the group $\mathbf{B}'(4)$; the quotient group $\mathbf{B}'(4)/\mathbf{T} \cong \mathbb{F}_2$.

d) For $k > 4$ the group $\mathbf{B}'(k)$ is perfect. □

Since $\mathbf{B}(k)/\mathbf{B}'(k) \cong \mathbb{Z}$, statements (a) and (c) of this theorem imply:

[2]For $k = 4$ the generators c_i with $i \geq 2$ and relations (0.18) - (0.21) do not appear.

Corollary 0.3. *The groups* $\mathbf{B}(3)$ *and* $\mathbf{B}(4)$ *admit finite normal series with free quotient groups. Hence, these braid groups contain no perfect subgroups. For* $k \leq 4$ *any nontrivial subgroup* $G \subseteq \mathbf{B}(k)$ *possesses nontrivial homomorphisms* $G \to \mathbb{Z}$. □

Remark 0.3. The fact that the groups $\mathbf{B}'(k)$, $k > 4$, are perfect, was first proven in [GL1] using the presentation given by (0.13)-(0.21). During a few years after the publication of [GL1], it was a challenge to find a simpler proof of this very important property of the braid groups. At the end, E. A. Gorin has discovered a simple and very beautiful relation which holds for any $k \geq 4$:

$$\sigma_3\sigma_1^{-1} = (\sigma_1\sigma_2)^{-1} \cdot \left[\sigma_3\sigma_1^{-1}, \sigma_1\sigma_2^{-1}\right] \cdot (\sigma_1\sigma_2),$$

where $\left[\sigma_3\sigma_1^{-1}, \sigma_1\sigma_2^{-1}\right] = \left(\sigma_3\sigma_1^{-1}\right)^{-1} \cdot \left(\sigma_1\sigma_2^{-1}\right)^{-1} \cdot \left(\sigma_3\sigma_1^{-1}\right) \cdot \left(\sigma_1\sigma_2^{-1}\right)$ is the commutator of the elements $g_1 = \sigma_3\sigma_1^{-1}$ and $g_2 = \sigma_1\sigma_2^{-1}$. Evidently, these elements g_1, g_2 belong to the commutator subgroup $\mathbf{B}'(k)$, and Gorin's relation shows that the element $\sigma_3\sigma_1^{-1}$ belongs to the second commutator subgroup $\mathbf{B}''(k) = (\mathbf{B}'(k))'$. Hence, the whole normal subgroup N of the group $\mathbf{B}(k)$ generated (as a normal subgroup) by the element $\sigma_3\sigma_1^{-1}$ is contained in $\mathbf{B}''(k)$. However, if $k > 4$, it follows readily from relations (0.1), (0.2) that this normal subgroup N contains the whole commutator subgroup $\mathbf{B}'(k)$ (for $k > 4$, the relations (0.1),(0.2) joined with the additional relation $\sigma_3\sigma_1^{-1} = 1$ give a presentation of the group $\mathbb{Z}$; see, for instance, Lemma 1.14). This implies that $\mathbf{B}'(k) = \mathbf{B}''(k)$. ○

Remark 0.4. Assume that $k \geq 4$ and denote the canonical generators in $\mathbf{B}(k-2)$ and $\mathbf{B}(k)$ by s_i and σ_j, respectively. Since σ_1 commutes with $\sigma_3, \ldots, \sigma_{k-1}$, for any integer m we can define a homomorphism $\lambda_{k,m}\colon \mathbf{B}(k-2) \to \mathbf{B}(k)$ by $\lambda_{k,m}(s_i) = \sigma_{i+2}\sigma_1^{-m}$, $1 \leq i \leq k-3$. It is well known that $\lambda_{k,m}$ *is an embedding* (from the geometrical point of view, this is evident).

Further, relations (0.20), (0.21) for the generators $c_i = \sigma_{i+2}\sigma_1^{-1}$ in $\mathbf{B}'(k)$ show that we can define a homomorphism

$$\lambda'_k\colon B(k-2) \to \mathbf{B}'(k)$$

by $s_i \mapsto c_i$, $1 \leq i \leq k-3$. The composition of λ'_k with the natural embedding of $\mathbf{B}'(k)$ into $\mathbf{B}(k)$ coincides with $\lambda_{k,1}$; hence, λ'_k *is an embedding*. ○

Canonical homomorphism $\mathbf{B}'(k) \to \mathbf{S}(k)$***; pure commutator subgroup.*** The presentation of $\mathbf{B}'(k)$ given by Gorin-Lin Theorem is called *canonical*. The restriction μ' of the canonical projection μ to the commutator subgroup $\mathbf{B}'(k) \subset \mathbf{B}(k)$,

$$\mu' = \mu \mid \mathbf{B}'(k)\colon\ \mathbf{B}'(k) \to \mathbf{S}(k),$$

is called *the canonical homomorphism* of $\mathbf{B}'(k)$ into $\mathbf{S}(k)$. Its image coincides with the alternating subgroup $\mathbf{A}(k) \subset \mathbf{S}(k)$, and its kernel coincides with the normal subgroup $\mathbf{J}(k) = \mathbf{I}(k) \cap \mathbf{B}'(k)$ of the group $\mathbf{B}(k)$. The normal subgroup $\mathbf{J}(k) = \mathbf{I}(k) \cap \mathbf{B}'(k)$ is called the *pure commutator subgroup* of the braid group $\mathbf{B}(k)$. It is easily checked that

$$\begin{aligned} &\mu'(u) = (1,3,2), \quad \mu'(v) = (1,2,3), \quad \mu'(w) = (1,3)(2,4), \\ &\mu'(c_i) = (1,2)(i+2, i+3) \qquad (1 \leq i \leq k-1). \end{aligned} \tag{0.22}$$

0.7. What we prove. Here we formulate the main algebraic results of the paper.

Theorem A. *Assume that* $k > 4$ *and* $n < k$. *Then*

a) any homomorphism $\mathbf{B}(k) \to \mathbf{S}(n)$ *is cyclic;*

b) any homomorphism $\mathbf{B}(k) \to \mathbf{B}(n)$ *is integral;*

c) the commutator subgroup $\mathbf{B}'(k)$ *of the group* $\mathbf{B}(k)$ *does not possess nontrivial homomorphisms into the groups* $\mathbf{S}(n)$ *and* $\mathbf{B}(n)$.

Theorem B. *If* $k \neq 4$, *then* $\phi(\mathbf{I}(k)) \subseteq \mathbf{I}(k)$, $\phi^{-1}(\mathbf{I}(k)) = \mathbf{I}(k)$ *and* $\operatorname{Ker} \phi \subseteq \mathbf{J}(k)$ *for any nonintegral endomorphism* ϕ: $\mathbf{B}(k) \to \mathbf{B}(k)$.

Let ν_6' denote the restriction of Artin's homomorphism ν_6: $\mathbf{B}(6) \to \mathbf{S}(6)$ to the commutator subgroup $\mathbf{B}'(6) \subset \mathbf{B}(6)$.

Theorem C. *Assume that* $k > 4$. *Let* ψ: $\mathbf{B}'(k) \to \mathbf{S}(k)$ *be a nontrivial homomorphism. Then either* $\psi \sim \mu_k'$ *(which may happen for any* k*) or* $k = 6$ *and* $\psi \sim \nu_6'$. *In particular,*

$$\operatorname{Im} \psi = \mathbf{A}(k) \qquad \text{and} \qquad \operatorname{Ker} \psi = \mathbf{J}(k) = \mathbf{I}(k) \cap \mathbf{B}'(k).$$

Theorem D. *Suppose* $k > 4$. *The pure commutator subgroup* $\mathbf{J}(k) = \mathbf{I}(k) \cap \mathbf{B}'(k)$ *is a completely characteristic subgroup of the group* $\mathbf{B}'(k)$, *that is,* $\phi(\mathbf{J}(k)) \subseteq \mathbf{J}(k)$ *for any endomorphism* ϕ: $\mathbf{B}'(k) \to \mathbf{B}'(k)$. *Moreover,* $\phi^{-1}(\mathbf{J}(k)) = \mathbf{J}(k)$ *for every nontrivial endomorphism* ϕ.

Theorem E. *a) If* $k > 5$, *then any transitive homomorphism* $\mathbf{B}(k) \to \mathbf{S}(k+1)$ *is cyclic.*

b) For $k > 4$ *any transitive homomorphism* $\mathbf{B}(k) \to \mathbf{S}(k+2)$ *cyclic.*

Theorem F. *a) Assume that* $6 < k < n < 2k$. *Then any transitive homomorphism* $\mathbf{B}(k) \to \mathbf{S}(n)$ *is cyclic.*

b) If $k > 8$, *then any noncyclic transitive homomorphism* ψ: $\mathbf{B}(k) \to \mathbf{S}(2k)$ *is conjugate to one of the following three homomorphisms* φ_j*:*

$$\varphi_1(\sigma_i) = \underbrace{(2i-1, 2i+2, 2i, 2i+1)}_{\text{4-cycle}};$$

$$\varphi_2(\sigma_i) = (1,2)\cdots(2i-3, 2i-2)\underbrace{(2i-1, 2i+1)(2i, 2i+2)}_{\text{two transpositions}}(2i+3, 2i+4)\cdots(2k-1, 2k);$$

$$\varphi_3(\sigma_i) = (1,2)\cdots(2i-3, 2i-2)\underbrace{(2i-1, 2i+2, 2i, 2i+1)}_{\text{4-cycle}}(2i+3, 2i+4)\cdots(2k-1, 2k);$$

(in each of the above formulas $i = 1, \dots, k-1$).

Theorem G. *Assume that* $k > 4$ *and* $n < 2k$. *Then*

a) any transitive imprimitive homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ *is cyclic;*

b) any transitive homomorphism ψ': $\mathbf{B}'(k) \to \mathbf{S}(n)$ *is primitive.*

To formulate the next theorem, it is convenient to introduce the following definitions.

Special homomorphisms. Let $\{a, b\}$ be a special system of generators in $\mathbf{B}(m)$. Let $\mathcal{H}_m(a, b)$ be the subset in $\mathbf{B}(m)$ consisting of all the elements $g^{-1}a^q g$ and $g^{-1}b^q g$, where g runs over $\mathbf{B}(m)$ and p runs over $\mathbb{Z}$. We say that a homomorphism ϕ: $\mathbf{B}(k) \to \mathbf{B}(n)$ is *special* if $\phi(\mathcal{H}_k(a, b)) \subseteq \mathcal{H}_n(a', b')$ for some choice of special systems of generators $a, b \in \mathbf{B}(k)$ and $a', b' \in \mathbf{B}(n)$.

Murasugi Theorem ([Mu]). *A braid h belongs to $\mathcal{H}_m(a, b)$ if and only if h is an element of finite order modulo the center $\mathbf{C}(m)$ of the group $\mathbf{B}(m)$.*

This theorem implies that *the set $\mathcal{H}_m(a, b)$ does not depend on a choice of a special system of generators $a, b \in \mathbf{B}(m)$, and a homomorphism ϕ: $\mathbf{B}(k) \to \mathbf{B}(n)$ is special if and only if for any element $g \in \mathbf{B}(k)$ of finite order modulo $\mathbf{C}(k)$ its image $\phi(g) \in \mathbf{B}(n)$ is an element of finite order modulo $\mathbf{C}(n)$* (actually, we do not use this result in this paper).

It was stated in [L1] and proven in [L7] that *for any holomorphic mapping f: $\mathbf{G}_k \to \mathbf{G}_n$ the induced homomorphism of the fundamental groups f_*: $\mathbf{B}(k) \to \mathbf{B}(n)$ is special* (see also §8). This is a reason to study such homomorphisms.

Notation 0.1. Let $P(k)$ be the union of the four increasing infinite arithmetic progressions $P^{k,i} = \left\{p_j^{k,i} = p_1^{k,i} + (j-1)d(k) \mid j \in \mathbb{N}\right\}$ $(1 \le i \le 4)$ having the same difference $d(k) = k(k-1)$ and starting, respectively, with the following initial terms:

$$p_1^{k,1} = k, \quad p_1^{k,2} = k(k-1), \quad p_1^{k,3} = k(k-1)+1, \quad \text{and} \quad p_1^{k,4} = (k-1)^2 . \qquad \bigcirc$$

Theorem H. *a) Assume that $k \ne 4$ and $n \notin P(k)$. Then any special homomorphism $\mathbf{B}(k) \to \mathbf{B}(n)$ is integral.*

b) For any $k \ge 3$ and any $n \in P^{k,1} \cup P^{k,2}$ there exists a non-Abelian special homomorphism $\mathbf{B}(k) \to \mathbf{B}(n)$.

0.8. How and where the main theorems are proven. Let us start with some comments on results concerning homomorphisms $\mathbf{B}(k) \to \mathbf{S}(n)$. For any k, n, the set $\mathrm{Hom}(\mathbf{B}(k), \mathbf{S}(n))$ of all such homomorphisms is finite; in fact, using the special presentation of $\mathbf{B}(k)$, we see that $\#\mathrm{Hom}(\mathbf{B}(k), \mathbf{S}(n)) \le h(n) = (n!-1)n!$. To find all the homomorphisms, one may use straightforward combinatorial computations. However, for large n this approach is almost useless (say $h(10) \approx 1.3 \cdot 10^{15}$).

Any homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ is a disjoint product of transitive homomorphisms ψ_j: $\mathbf{B}(k) \to \mathbf{S}(n_j)$, $\sum n_j = n$. If ψ is noncyclic, at least one of the homomorphisms ψ_j must be noncyclic. Taking into account this fact, we try to describe all possible cyclic types of the permutation $\widehat{\sigma}_1 = \psi(\sigma_1) \in \mathbf{S}(n)$ for a transitive (or noncyclic and transitive) homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$. (Note that all $\widehat{\sigma}_i = \psi(\sigma_i)$ are conjugate to each other, and hence are of the same cyclic type.)

FIXED POINTS AND PRIMES. TRANSITIVE HOMOMORPHISMS $\mathbf{B}(k) \to \mathbf{S}(k)$ AND $\mathbf{B}(k) \to \mathbf{S}(k+1)$. The following remarkable result is due to Artin:

Artin Lemma. *If $k > 4$ and there is a prime $p > 2$ such that $n/2 < p \le k-2$, then for any noncyclic transitive homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ the permutation $\widehat{\sigma}_1$ must have at least $k-2$ fixed points.*

Artin treated the case $n = k$, but his proof does not depend on the latter assumption (see Lemma 1.22).

The famous theorem of P. L. Chebyshev implies the existence of a prime p with the required properties whenever $k > 4$ and $n < k$ or $6 \neq k > 4$ and $n \leq k$. If $6 \neq k = n > 4$, Artin Lemma shows that *all the permutations* $\widehat{\sigma}_i$ *must be transpositions*, and the proof of Artin Theorem (a) can be completed in a few words. Moreover, for any $k > 6$ there is a prime p such that $(k+1)/2 < p \leq k-2$. Hence, the inequality $\#\,\mathrm{Fix}\,\widehat{\sigma}_1 \geq k-2$ holds true for any $k > 6$ and any noncyclic transitive homomorphism $\psi\colon \mathbf{B}(k) \to \mathbf{S}(k+1)$, which yields Theorem E$(a)$ whenever $k > 6$ (in Theorem 6.3(a) the case $k = 6$ is treated as well).

HOMOMORPHISMS $\psi\colon \mathbf{B}(k) \to \mathbf{S}(n),\ \ k > n$. AN IMPROVEMENT OF ARTIN THEOREM. We represent ψ as a disjoint product of transitive homomorphisms ψ_j. If some ψ_j is noncyclic, then, by Artin Lemma, all $\psi_j(\sigma_i)$ are transpositions. This leads to a contradiction, which proves Theorem A(a) (see Theorem 2.1(a)). Using this theorem and Artin Theorem, we show that *for* $k \neq 4$ *any nonsurjective homomorphism* $\mathbf{B}(k) \to \mathbf{S}(k)$ *is cyclic* (Lemma 2.7). This implies that *for* $k \neq 4, 6$ *any noncyclic homomorphism* $\mathbf{B}(k) \to \mathbf{S}(k)$ *is conjugate to the canonical projection*, which is a useful improvement of Artin Theorem (Remark 2.2).

HOMOMORPHISMS $\psi\colon \mathbf{B}'(k) \to \mathbf{S}(n)$. Take the restriction ϕ of a homomorphism ψ to the subgroup $\mathbf{B} \cong \mathbf{B}(k-2)$ of $\mathbf{B}'(k)$ generated by the elements $c_i = \sigma_{i+2}\sigma_1^{-1}$ (§0.6). Using Theorem A(a) and Lemma 2.7, we show that for $k > 4$ and $n < k$ the homomorphism ϕ must be cyclic. On the other hand, using relations (0.14)-(0.21), we show that the original homomorphism ψ is trivial whenever ϕ is cyclic (Lemma 6.4). This proves the statement of Theorem A(c) concerning homomorphisms $\mathbf{B}'(k) \to \mathbf{S}(n)$. Using this result (which is an essential strengthening of Theorem A(a)), we prove Theorem G. The primitivity of transitive homomorphisms is a very helpful property: it allows us to apply Jordan Theorem about primitive permutation groups. (See Theorem 6.6, Lemma 6.7, and Proposition 6.8.)

SMALL SUPPORTS. Artin proved that the cyclic decomposition of $\widehat{\sigma}_1$ cannot contain a cycle of length $> n/2$ whenever $k > 4$ and ψ is noncyclic and transitive (Lemma 1.19(a)). On the other hand, in Lemma 1.21 we show that $\#\,\mathrm{supp}\,\widehat{\sigma}_1 \leq n/E(k/2)$ whenever all the cycles $C \preccurlyeq \widehat{\sigma}_1$ are of pairwise distinct lengths ($E(x)$ is the integral part of x). This means that the support of $\widehat{\sigma}_1$ is relatively small; say for $k \geq 6$ and $n \leq 2k$ we have $\#\,\mathrm{supp}\,\widehat{\sigma}_1 \leq 4$. Such homomorphisms can be studied without great difficulties. Actually, for $6 < k < n \leq 2k$ Lemma 6.9 provides an explicit description of all noncyclic transitive homomorphisms $\psi\colon \mathbf{B}(k) \to \mathbf{S}(n)$ that satisfy $\#\,\mathrm{supp}\,\widehat{\sigma}_1 \leq 5$. In particular, it is proven that for k, n as above such a homomorphism does exist only if $n = 2k$ and $\widehat{\sigma}_1$ is a 4-cycle.

WHAT TO DO IF n IS CLOSE TO $2k$? The above methods do not go too far from original Artin's ideas (the main innovation is that Gorin-Lin Theorem applies systematically). With some exceptions, Artin Lemma still works if $n > k+1$ but n is close to k. Say, if $n = k+2$ and $8, 12 \neq k \geq 7$ or $n = k+3$ and $11, 12 \neq k \geq 9$, then it follows from the Finsler inequality $\pi(2m) - \pi(m) > m/(3\log(2m))$ (see [Fi, Tr]) that there is a prime p on the interval $(n/2, k-2]$. However, if n is near $2k$, these methods hardly work, since there is no hope to find a prime on a rather short interval $(n/2, k-2]$. Actually, for such n, even if we were lucky to get $\#\,\mathrm{Fix}\,\widehat{\sigma}_1 \geq k-2$, the support of $\widehat{\sigma}_1$ may still contain about $k+2$ points, and we cannot come to any immediate conclusion. We must look for new ideas.

HOMOMORPHISM Ω; COHOMOLOGY. By the reasons explained above, we should mainly handle a noncyclic transitive homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ such that the cyclic decomposition of $\widehat{\sigma}_1$ contains a few cycles of the same length. A simple idea described below occurs crucial (see §§4, 5).

Fix some $r \geq 2$ and assume that the r-component $\mathfrak{C}_r$ of $\widehat{\sigma}_1$ consists of $t \geq 2$ r-cycles $C_1, ..., C_t$. Since $\widehat{\sigma}_3, ..., \widehat{\sigma}_{k-1}$ commute with $\widehat{\sigma}_1$, the conjugation by any $\widehat{\sigma}_i$ $(3 \leq i \leq k-1)$ induces a permutation of the r-cycles $C_1, ..., C_t$. This gives rise to a homomorphism

$$\Omega\colon\ \mathbf{B}(k-2) \to \mathbf{S}(\mathfrak{C}_r) \cong \mathbf{S}(t),$$

where $\mathbf{B}(k-2)$ is the subgroup of $\mathbf{B}(k)$ generated by $\sigma_3, ..., \sigma_{k-1}$.

Further, for any $i = 3, ..., k-1$, the support $\Sigma = \Sigma(\mathfrak{C}_r) = \bigcup_{i=1}^{t} \operatorname{supp} C_i$ of the r-component $\mathfrak{C}_r$ is a $\widehat{\sigma}_i$-invariant subset of $\mathbf{\Delta}_n$. Take the restriction $\Psi = \psi \mid \mathbf{B}(k-2)$ of the original homomorphism ψ to the above subgroup $\mathbf{B}(k-2) \subset \mathbf{B}(k)$, and then consider the reduction of Ψ

$$\Psi_{\Sigma}\colon\ \mathbf{B}(k-2) \to \mathbf{S}(\Sigma) \cong \mathbf{S}(rt) \subseteq \mathbf{S}(n)$$

to the above $(\operatorname{Im} \Psi)$-invariant subset $\Sigma \subseteq \mathbf{\Delta}_n$. It is easily seen that the homomorphisms Ω and Ψ_{Σ} fit in a commutative diagram of the form

$$\begin{tikzcd}
 & & \mathbf{B}(k-2) \arrow[r, equal] \arrow[d, "\Psi_\Sigma"'] & \mathbf{B}(k-2) \arrow[d, "\Omega"] & \\
1 \arrow[r] & H \arrow[r] & G \arrow[r, "\pi"] & \mathbf{S}(\mathfrak{C}_r) \arrow[r] & 1.
\end{tikzcd}$$

Here $H \cong (\mathbb{Z}/r\mathbb{Z})^t$ is the Abelian subgroup of $\mathbf{S}(\Sigma)$ generated by the (disjoint) r-cycles $C_1, ..., C_t$, G is the centralizer of the element $C = C_1 \cdots C_t$ in $\mathbf{S}(\Sigma)$, and the second horizontal line of the above diagram is an exact sequence with a fixed splitting ρ: $\mathbf{S}(\mathfrak{C}_r) \cong \mathbf{S}(t) \to G$. The latter exact sequence is, in fact, universal, meaning that we have "the same" sequence for all the homomorphisms ψ having an r-component of length t.

The complementary set $\Sigma' = \mathbf{\Delta}_n - \Sigma$ is also $(\operatorname{Im} \Psi)$-invariant, and we can take the reduction $\Psi_{\Sigma'}$ of Ψ to Σ'. The perfectness of the commutator subgroup implies the following two properties of the homomorphism Ω (see Lemma 4.4 and Theorem 5.10(a)):

Suppose $k > 6$. If ψ is noncyclic and $\Psi_{\Sigma'}$ is Abelian, then Ω must be noncyclic. Moreover, the same conclusion holds true whenever Ψ_{Σ} is noncyclic.

Since $k - 2 < k$ and $t \leq n/r \leq n/2$, we may hope to handle Ω. Assuming that Ω is already known, we try to recover (as far as possible) the homomorphism Ψ_{Σ}, which keeps important information about the original homomorphism ψ. Clearly, this is a homological problem. Namely, the homomorphism Ω and the splitting ρ give rise to a $\mathbf{B}(k-2)$-action T on the Abelian normal subgroup $H \subset G$. We show (Proposition 4.6) that *there is a natural bijection between the cohomology group $H^1_T(\mathbf{B}(k-2), H)$ and the set of all the classes of H-conjugate homomorphisms φ: $\mathbf{B}(k-2) \to G$ that satisfy the commutativity relation $\pi \circ \varphi = \Omega$.* The action T and the corresponding cohomology $H^1_T(\mathbf{B}(k-2), H)$ can be computed explicitly in many cases that we are interested in (see §§5,6). As a result, we obtain a description of all possible homomorphisms Ψ_{Σ} (up to conjugation). This description leads to some very restrictive conditions on the original homomorphism ψ.

On the other hand, $\widehat{\sigma}_{k-1}$ is conjugate to $\widehat{\sigma}_1$, the r-component $\mathfrak{C}_r^*$ of $\widehat{\sigma}_{k-1}$ is also of length t. Actually, the conjugation by the element $\widehat{\alpha}^{k-2} = \psi(\sigma_1 \cdots \sigma_{k-1})^{k-2}$ induces natural isomorphisms $\mathfrak{C}_r \xrightarrow{\cong} \mathfrak{C}_r^*$ and $\mathbf{S}(\mathfrak{C}_r) \xrightarrow{\cong} \mathbf{S}(\mathfrak{C}_r^*)$. All the permutations $\widehat{\sigma}_1, ..., \widehat{\sigma}_{k-3}$ commute with $\widehat{\sigma}_{k-1}$, and we can apply the above construction to obtain a homomorphism

$$\Omega^*\colon\ \mathbf{B}^*(k-2) \to \mathbf{S}(\mathfrak{C}_r^*) \cong \mathbf{S}(t),$$

where $\mathbf{B}^*(k-2) \cong \mathbf{B}(k-2)$ is the subgroup of $\mathbf{B}(k)$ generated by $\sigma_1, ..., \sigma_{k-3}$. The following simple observation is very useful: *under the identification* $\mathbf{B}(k-2) \cong \mathbf{B}^*(k-2)$ *given by* $\sigma_{i+2} \mapsto \sigma_i$, $1 \leq i \leq k-3$, *the homomorphism* Ω^* *coincides with the above homomorphism* Ω (Lemma 4.3).

Down with long components! Fixed points without primes. Using the cohomology approach (and also Lemma 4.3 mentioned above), we prove that *for a noncyclic homomorphism* $\psi\colon \mathbf{B}(k) \to \mathbf{S}(n)$ *the permutation* $\widehat{\sigma}_1$ *cannot have an* r*-component of length* $t > k-3$ *whenever* $6 < k < n < 2k$ (see Lemma 6.11, which is one of the crucial technical results). On the other hand, using Theorem A(a), we prove in Lemma 6.10: *if* $k > 6$ *and all the components of* $\widehat{\sigma}_1$ *(including the degenerate component* Fix $\widehat{\sigma}_1$*) are of length at most* $k-3$, *then the homomorphism* ψ *must be cyclic*. Combining these two results, we prove the following analog of Artin Lemma: *if* $6 < k < n < 2k$, *then* $\#\,\mathrm{Fix}\,\widehat{\sigma}_1 \geq k-2$ *for any noncyclic homomorphism* ψ (Corollary 6.12). This leads to Theorem E(b) (at least for $k > 6$; the cases $k = 5, 6$ may be treated as well; see Theorem 6.15(a)).

Homomorphisms $\mathbf{B}(k) \to \mathbf{S}(n)$, $k < n \leq 2k$. Theorem F(a) is proven by induction on k. First, to get a base of induction, we study the cases $k = 7, 8$ (Lemma 6.17 and Lemma 6.19). Further, assuming that $6 < k < n < 2k$ and $\psi\colon \mathbf{B}(k) \to \mathbf{S}(n)$ is a noncyclic transitive homomorphism, we take the restriction $\phi\colon \mathbf{B}(k-2) \to \mathbf{S}(n)$ of ψ to the subgroup $\mathbf{B}(k-2) \subset \mathbf{B}(k)$ generated by $\sigma_3, ..., \sigma_{k-2}$ and the reductions $\varphi\colon \mathbf{B}(k-2) \to \mathbf{S}(\Sigma)$ and $\varphi'\colon \mathbf{B}(k-2) \to \mathbf{S}(\Sigma')$ of ϕ to the mutually complementary $\psi(\mathbf{B}(k-2))$-invariant sets $\Sigma = \mathrm{supp}\,\widehat{\sigma}_1$ and $\Sigma' = \mathbf{\Delta}_n - \Sigma$, respectively. So, ϕ is the disjoint product of φ and φ'. In Lemma 6.16, which is the main technical tool of induction, we show that φ *is trivial*, φ' *is noncyclic and* $\phi = \varphi'$. The proof of this lemma involves the homomorphisms Ω, Ω^* (corresponding to each nondegenerate component of $\widehat{\sigma}_1$) and Theorem 5.10(a) mentioned above. Finally, assuming existence of a natural m such that any transitive homomorphism $\psi\colon \mathbf{B}(k) \to \mathbf{S}(n)$ is cyclic whenever k, n satisfy $6 < k \leq m$ and $k < n < 2k$ (the induction hypothesis), we prove that the same conclusion must be true whenever $6 < k \leq m+2$ and $k < n < 2k$. The justification of this induction step involves Lemma 6.16, Corollary 6.12, Artin Theorem, Theorem G, and Jordan Theorem on primitive permutation groups. (See Theorem 6.20.).

The case $n = 2k$ is more sophisticated, since for a noncyclic transitive homomorphism $\psi\colon \mathbf{B}(k) \to \mathbf{S}(2k)$ the permutation $\widehat{\sigma}_1 = \psi(\sigma_1)$ may possess a 2-component of length $t > k-3$. However, using the corresponding cohomology and Theorem F(a), we show that in the latter case either $t = k$ and $\psi \sim \varphi_2$ or $t = k-2$ and $\psi \sim \varphi_3$ (see Lemma 6.21). Combining this property with Theorem A(a), Artin Theorem, and Theorem F(a), we prove Theorem F(b) (see Theorem 6.23).

HOMOMORPHISMS ϕ: $\mathbf{B}(k) \to \mathbf{B}(n)$. Assume that $k > 4$ and consider the composition $\psi = \mu \circ \phi$: $\mathbf{B}(k) \to \mathbf{S}(n)$ of ϕ with the canonical projection μ: $\mathbf{B}(n) \to \mathbf{S}(n)$. If ψ is cyclic, then $\psi(\mathbf{B}'(k)) \subseteq \operatorname{Ker} \mu = \mathbf{I}(k)$. Since $\mathbf{B}'(k)$ is perfect, Markov Theorem implies that $\psi(\mathbf{B}'(k)) = \{1\}$ and ϕ is integral. Combining this simple observation with Theorem A(a), we obtain Theorem A(b). In the same way, we complete the proof of Theorem A(c) (we already have commented on the absence of nontrivial homomorphisms $\mathbf{B}'(k) \to \mathbf{S}(n)$ for $k > 4$ and $n < k$).

To prove Theorem B, we show (using Lemma 2.7 cited above) that if $k \neq 4$, then for a nonintegral endomorphism ϕ of $\mathbf{B}(k)$ the composition $\psi = \mu \circ \phi$ must be surjective and, therefore, noncyclic and transitive. By Artin Theorem, Ker $\psi = \mathbf{I}(k)$, which implies $\phi^{-1}(\mathbf{I}(k)) = \mathbf{I}(k)$; the other assertions of the theorem follow readily from this fact.

HOMOMORPHISMS ψ: $\mathbf{B}'(k) \to \mathbf{S}(k)$. ENDOMORPHISMS OF $\mathbf{B}'(k)$. Suppose $6 \neq k > 4$. Restricting a nontrivial homomorphism ψ to the subgroup $\mathbf{B}(k-2) \subset \mathbf{B}'(k)$ generated by all the elements $c_i = \sigma_{i+2}\sigma_1^{-1}$, $i \leq k-3$, we obtain a homomorphism ϕ: $\mathbf{B}(k-2) \to \mathbf{S}(k)$. Using Lemma 6.4, Theorem A(a), and Theorem E, we show that ψ is *tame*, meaning that the permutation group $\phi(\mathbf{B}(k-2)) \subset \mathbf{S}(k)$ has an orbit $Q \subset \mathbf{\Delta}_k$ of length $k-2$ (Lemma 7.3). By Artin Theorem, the reduction ϕ_Q of ϕ to Q is conjugate to the canonical projection $\mathbf{B}(k-2) \to \mathbf{S}(Q) \cong \mathbf{S}(k-2)$; hence, without loss of generality, we may assume that $\widehat{c}_i = \psi(c_i) = (1,2)(i+2, i+3)$ for all $i = 1, ..., k-3$. Using this property and the defining relations (0.14)-(0.21), we show (by a straightforward computation) that $\psi \sim \mu'_k$; this proves Theorem C (see Theorem 7.5, where the case $k = 6$ is treated as well). In view of Markov Theorem and the perfectness of the group $\mathbf{B}'(k)$, Theorem D follows immediately from Theorem C (see Theorem 7.7).

SPECIAL HOMOMORPHISMS ϕ: $\mathbf{B}(k) \to \mathbf{B}(n)$. The proof of Theorem H is mainly based on Lemma 1.17, which provides us with some "arithmetical" properties of noncyclic homomorphisms of braid groups (see Theorem 8.1).

0.9. Some open problems. We present here a list of some questions and open problems on homomorphisms of braids and related topics. Some of these problems certainly can be solved by the methods described in the paper, but some other problems seem more difficult.

0.9.1. HOMOMORPHISMS ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$. *a*) Describe all noncyclic transitive homomorphisms ψ in the following cases: $k = 4$ and $n = 8$; $k = 5$ and $n = 8, 9, 10$; $k = 6$ and $n = 10, 11, 12$; $k = 7$ and $n = 14$; $k = 8$ and $n = 16$.

b) It seems that our methods make it possible to classify all noncyclic transitive homomorphisms ψ for $n \leq 3k$ (at least for k large enough). However, for $n > 3k$ one needs some new ideas to simplify computations.

I do not know any counterexample to the following conjecture:

Conjecture. *For any natural r there exists $K(r)$ such that any transitive homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ is cyclic whenever $k > K(r)$ and $(r-1)k < n < rk$.*

Theorem A(a) and Theorem F(a) show that Conjecture is true for $r = 1, 2$ (with $K(1) = 4$ and $K(2) = 6$, respectively). On the other hand, even if k is large and does not divide n, a noncyclic transitive homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ can exist. Indeed, take any prime

p and consider the family $\mathfrak{P}_{2p,p}$ of all partitions of the set $\mathbf{\Delta}_{2p}$ into two complementary subsets Σ, Σ' with $\#\Sigma = \#\Sigma' = p$. Clearly, the number

$$n(2p) = \#\mathfrak{P}_{2p,p} = \frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1} = \frac{(2p-1)(2p-2)\cdots(p+1)}{(p-1)!}$$

is not divisible by p. However, $\mathbf{S}(2p)$ acts transitively on $\mathfrak{P}_{2p,p}$, and we have a noncyclic transitive homomorphism $\mathbf{B}(2p) \xrightarrow{\mu_{2p}} \mathbf{S}(2p) \to \mathbf{S}(\mathfrak{P}_{2p,p}) \cong \mathbf{S}(n(2p))$. (If p is large, then the ratio $n/k = n(2p)/2p$ is certainly very large.)

0.9.2. Endomorphisms of $\mathbf{B}(k)$ and $\mathbf{B}'(k)$. Torsion. Any non-Abelian endomorphism of the group $\mathbf{B}'(3) \cong \mathbb{F}_2$ is injective, but the group $\mathbf{B}'(4) \cong \mathbb{F}_2 \rtimes \mathbb{F}_2$ admits non-Abelian noninjective endomorphisms. Any nonintegral endomorphism of $\mathbf{B}(3)$ is injective (Theorem 2.12). However, $\mathbf{B}(4)$ admits nonintegral noninjective endomorphisms ϕ (say the composition of the well-known epimorphism $\mathbf{B}(4) \to \mathbf{B}(3)$ with the natural embedding $\mathbf{B}(3) \hookrightarrow \mathbf{B}(4)$); actually, Ker $\phi = \mathbf{T}$ for any such ϕ (Theorem 2.15).

Suppose $k > 4$. *a*) Does exist a nonintegral noninjective endomorphism ϕ of $\mathbf{B}(k)$? (By Lemma 2.11, Ker $\phi \subseteq \mathbf{B}'(k)$ for any nontrivial endomorphism ϕ.)

Does exist a nontrivial endomorphism φ of $\mathbf{B}'(k)$ such that *b*) the kernel of φ is nontrivial? *c*) φ is not an automorphism? *d*) φ does not extend to an endomorphism of the whole group $\mathbf{B}(k)$?

e) Is it true that any automorphism of $\mathbf{B}'(k)$ extends to an automorphism of $\mathbf{B}(k)$? (A complete description of automorphisms of $\mathbf{B}(k)$ was obtained by J. Dyer and E. Grossman in 1982).

Does exist *f*) a proper torsion free quotient group of $\mathbf{B}'(k)$? *g*) a proper torsion free non-Abelian quotient group of $\mathbf{B}(k)$? (See Remark 7.3.)

0.9.3. Special homomorphisms and holomorphic mappings. *a*) Suppose $k > 4$. Is it true that for $n \notin P^{k,1} \cup P^{k,2}$ any special homomorphism ϕ: $\mathbf{B}(k) \to \mathbf{B}(n)$ is cyclic? (See Theorem H and §8.)

b) Is it true that for $n \neq k > 4$ any holomorphic mapping $\mathbf{G}_k \to \mathbf{G}_n$ is splittable? Theorem H implies that this is the case if $n \notin P(k)$.

§1. Auxiliary results

To facilitate the exposition of the main proofs, we have collected here many auxiliary results. Some of them are used just occasionally; however, some other are involved throughout the paper. We recommend to read them when and if it is needed.

1.0. Three algebraic lemmas. Recall that a word in the variables $a_1, \ldots, a_s$ and $a_1^{-1}, \ldots, a_s^{-1}$ is said to be *reduced* if it does not contain a part of the form $a_i a_i^{-1}$ or $a_i^{-1} a_i$ $(1 \le i \le s)$.

Lemma 1.1. *Let $f(x, x^{-1}, y, y^{-1})$ be a nonempty reduced word in variables x, y and x^{-1}, y^{-1}. If the elements u and v of a group G satisfy $f(u, u^{-1}, v, v^{-1}) = 1$, then for any homomorphism ϕ: $G \to \mathbb{F}$ into a free group the elements $\widehat{u} = \phi(u)$ and $\widehat{v} = \phi(v)$ commute.*

Proof. The subgroup $H \subseteq \mathbb{F}$ generated by the elements $\widehat{u}, \widehat{v}$ is a free group of rank $r \le 2$. In fact, $r \le 1$; for otherwise $\{\widehat{u}, \widehat{v}\}$ would be a free base of H, which is impossible since $f(\widehat{u}, \widehat{u}^{-1}, \widehat{v}, \widehat{v}^{-1}) = 1$. Hence the group H is commutative. □

Lemma 1.2. *Let H be a group and $K \subseteq H$ be a subgroup generated by two elements u, v. Assume that there exist an element $\sigma \in H$ and elements x, y in the commutator subgroup K' of K such that*

$$\sigma u \sigma^{-1} = u^p v^q x, \qquad \sigma v \sigma^{-1} = u^s v^t y, \tag{1.1}$$

where the integral exponents p, q, s, t form the matrix $M = \begin{pmatrix} p & q \\ s & t \end{pmatrix}$ with $\det M = \pm 1$ and without eigenvalues ± 1. Then any integral group homomorphism ψ: $K \to G$ admitting an extension Ψ: $H \to G$ to the whole group H is trivial.

Proof. Put $\widehat{u} = \psi(u) = \Psi(u)$, $\widehat{v} = \psi(v) = \Psi(v)$, $\widehat{\sigma} = \Psi(\sigma)$. It suffices to show that $\widehat{u} = \widehat{v} = 1$. Since ψ is integral, the subgroup $\widehat{K} = \operatorname{Im} \psi = \Psi(K) \subseteq G$ generated by these two elements is either trivial or isomorphic to $\mathbb{Z}$. In the first case there is nothing to prove. So, we may assume that $\widehat{K} \cong \mathbb{Z}$. Since $x, y \in K'$ and the group $\Psi(K) = \widehat{K}$ is commutative, we have $\Psi(x) = \Psi(y) = 1$. Therefore, it follows from relations (1.1) that

$$\widehat{\sigma}\widehat{u}\widehat{\sigma}^{-1} = \widehat{u}^p \widehat{v}^q, \qquad \widehat{\sigma}\widehat{v}\widehat{\sigma}^{-1} = \widehat{u}^s \widehat{v}^t. \tag{1.2}$$

Since $\det M = \pm 1$, these relations imply that the conjugation by the element $\widehat{\sigma} \in G$ defines an automorphism S of the subgroup $\widehat{K} \cong \mathbb{Z}$. Any automorphism of $\mathbb{Z}$ is involutive; that is, $S^2 = \mathrm{id}$. Combining this fact with relations (1.2) and passing to the additive notations (which is natural since we deal with the elements $\widehat{u}, \widehat{v} \in \widehat{K} \cong \mathbb{Z}$), we obtain the following system of linear equations for $\widehat{u}, \widehat{v}$:

$$\begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix} = S^2 \begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix} = M^2 \begin{pmatrix} \widehat{u} \\ \widehat{v} \end{pmatrix}.$$

By our assumptions, this system has only trivial solution, which concludes the proof. □

Example 1.1. Let $H = G$ be the group with two generators a, σ and one defining relation $\sigma a \sigma^{-1} = a^2$. The subgroup $K \subset H$ generated by the elements $u = a^2$ and $v = a^3$ is isomorphic to $\mathbb{Z}$. The identical embedding ψ: $K \hookrightarrow H = G$ is a nontrivial integral homomorphism that extends to H. Clearly, $\sigma u \sigma^{-1} = u^2$, $\sigma v \sigma^{-1} = v^2$; the corresponding matrix $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ has $\det M = 4$ (and has no eigenvalues ± 1). This example shows that the condition $\det M = \pm 1$ in Lemma 1.2 is essential. ○

Lemma 1.3. *Let $q \geq 2$ be a natural number and let $\nu = (q+1, 4)$. Assume that a braid-like couple a, b in a group G satisfy also the condition $b \infty a^q$. Then*

$$a^{\nu(q-1)} = b^{\nu(q-1)} = 1.$$

Proof. The conditions $b \infty a$ and $b \infty a^q$ imply that

$$a^{q-1} b = b^q a^{q-1}, \tag{1.3}$$

$$b a^{q-1} = a^{q-1} b^q, \tag{1.4}$$

and also

$$a b^{2(q-1)} a^{-1} = b^{-1} a^{2(q-1)} b = b^{-1} a^{q-1} a^{q-1} b = b^{-1} a^{q-1} b^q a^{q-1} = b^{-1} b a^{q-1} a^{q-1} = a^{2(q-1)}.$$

Hence, $b^{2(q-1)} = a^{2(q-1)}$ and a commutes with $b^{2(q-1)}$. Further, it follows from (1.3),(1.4) that $b^{q+1} a^{q-1} = b a^{q-1} b = a^{q-1} b^{q+1}$, i. e., a^{q-1} commutes with b^{q+1}; therefore, the element $c = a^{q-1}$ commutes with b^{q+1} and $b^{2(q-1)}$. This implies that the element c commutes also with b^d, where $d = (q+1, 2(q-1))$ (since $d = (q+1)m + 2(q-1)n$ for suitable m, n). But $(q+1, 2(q-1)) = (q+1, 4) = \nu$, and thus

$$a^{q-1} \text{ commuts with } b^\nu. \tag{1.5}$$

Since ν divides $q+1$, we have $q = r\nu + (\nu - 1)$ for some nonnegative integer r. Taking into account (1.5) and using one time (1.4) and $\nu - 1$ times (1.3), we obtain

$$b a^{q-1} = a^{q-1} b^q = a^{q-1} b^{r\nu} b^{\nu-1} = b^{r\nu} a^{q-1} b^{\nu-1} = b^{r\nu} b^{(\nu-1)q} a^{q-1} = b^{r\nu+(\nu-1)q} a^{q-1}.$$

Hence, $b^{\nu(q-1)} = b^{r\nu+(\nu-1)q-1} = 1$. Because of the condition $a \infty b$, the latter relation implies that $a^{\nu(q-1)} = 1$. □

Example 1.2. Let $G = \mathbf{S}(8)$, $a = (1,2,3,4,5,6,7,8)$, $b = (1,7,6,8,5,3,2,4)$ and $q = 3$. Then $b \infty a$, $b \infty a^3$, $\nu = (q+1, 4) = 4$ and

$$\nu(q-1) = 8 = \operatorname{ord} a = \operatorname{ord} b.$$

This shows that the result of Lemma 1.3 is sharp. □

1.1. Some properties of permutations. Here we prove some simple lemmas on permutations.

Lemma 1.4 (cf. [Ar3]). *Suppose $A, B \in \mathbf{S}(n)$ and $AB = BA$. Then:*

a) the set supp A *is B-invariant;*

b) if for some r, $2 \leq r \leq n$, the r-component of A consists of a single r-cycle C, then the set supp C *is B-invariant, and $B \mid \operatorname{supp} C = C^q$ for some integer q, $0 \leq q < r$.*

Proof. *a*) Let $i \in \operatorname{supp} A$; then $A(i) \neq i$ and $AB(i) = BA(i) \neq B(i)$; therefore, $B(i) \in \operatorname{supp} A$.

b) Let $A = CD_1 \cdots D_t$ be the cyclic decomposition of A. Then

$$CD_1 \cdots D_t = A = B^{-1}AB = B^{-1}CD_1 \cdots D_tB = B^{-1}CB \cdot B^{-1}D_1B \cdots B^{-1}D_tB.$$

Since C is the only r-cycle in the cyclic decomposition of A, we have $C = B^{-1}CB$, and (a) implies that the set supp C is B-invariant. Let $C = (i_0, i_1, ..., i_{r-1})$. Then $B(i_0) = i_q$ for some q, $0 \leq q \leq r-1$. Let us prove that $B \mid \operatorname{supp} C = C^q$. Since $C^q(i_s) = i_{|s+q|_r}$, we should show that $B(i_s) = i_{|s+q|_r}$ for all $s = 0, 1, ..., r-1$. The proof is by induction over s with the case $s = 0$ clear ($0 \leq q \leq r-1$ and $modoq_r = q$). Assume that for some k, $1 \leq k \leq r-1$, we have $B(i_s) = i_{|s+q|_r}$ for all $s = 0, ..., k-1$. Then

$$B(i_k) = BA(i_{k-1}) = AB(i_{k-1}) = A(i_{|k-1+q|_r}) = i_{||k-1+q|_r+1|_r} = i_{|k+q|_r}. \qquad \square$$

Lemma 1.5. *Assume that $A \in \mathbf{S}(n)$ and for some natural r $(1 \leq r < n)$ the family* $\operatorname{Inv}_r A$ *consists of a single set Σ.*

a) If $C \in \mathbf{S}(n)$ and $D = CAC^{-1}$, then $\operatorname{Inv}_r D$ *consists of the single set $C(\Sigma)$.*

b) If $B \in \mathbf{S}(n)$ and $AB = BA$, then the set Σ is B-invariant.

c) If a permutation $B \in \mathbf{S}(n)$ commutes with A and is also conjugate to A, then $\operatorname{Inv}_r B$ *consists of the single set Σ.*

Proof. *a*) Since $DC(\Sigma) = CA(\Sigma) = C(\Sigma)$, the set $C(\Sigma) \in \operatorname{Inv}_r D$. If $\Sigma' \in \operatorname{Inv}_r D$, then $AC^{-1}(\Sigma') = C^{-1}D(\Sigma') = C^{-1}(\Sigma')$, so that $C^{-1}(\Sigma') \in \operatorname{Inv}_r A$; hence, $C^{-1}(\Sigma') = \Sigma$.

b) $A = BAB^{-1}$, and (a) implies that $B(\Sigma) \in \operatorname{Inv}_r A$, so that $B(\Sigma) = \Sigma$.

c) In this case we have $AB = BA$ and $B = CAC^{-1}$ for some $C \in \mathbf{S}(n)$. By (a), $\operatorname{Inv}_r B$ consists of the single set $C(\Sigma)$. By (b), we have $\Sigma \in \operatorname{Inv}_r B$ and $C(\Sigma) = \Sigma$. $\square$

Lemma 1.6. $\#(\operatorname{supp} A \cap \operatorname{supp} B) = 2$ *for any braid-like couple of 3-cycles $A, B \in \mathbf{S}(n)$.*

Proof. Since $AB \neq BA$, we have $\operatorname{supp} A \cap \operatorname{supp} B \neq \varnothing$. Moreover, $\operatorname{supp} A \neq \operatorname{supp} B$ (every 3-cycles with the same support commute). Finally, if $\#(\operatorname{supp} A \cap \operatorname{supp} B) = 1$, a simple computation shows that $ABA \neq BAB$. $\square$

The following lemma is evident.

Lemma 1.7. *If $[A] = [C] = [3]$ and* $\operatorname{supp} A \cap \operatorname{supp} C \neq \varnothing$, *then* $\operatorname{supp} A = \operatorname{supp} C$ *and $C = A^q$, where either $q = 1$ or $q = 2$.* $\square$

Lemma 1.8. *Assume that* $[A] = [B] = [C] = [3]$, $ABA = BAB$, $BCB = CBC$, *and* $AC = CA$. *Then* $A = C$.

Proof. If B commutes with A or with C, the assumed relations imply that $A = B = C$. So, we may assume that $B \infty A$ and $B \infty C$. Then Lemma 1.6 implies that supp B has exactly two common points with each of the sets supp A and supp C. Hence, the commuting 3-cycles A and C are not disjoint and, by Lemma 1.7, we have $C = A^q$ with $q = 1$ or $q = 2$. If $q = 2$, then $(q+1, 4) = 1$, $q - 1 = 1$, $B \infty A$, $B \infty A^q$, and Lemma 1.3 implies $A = B = 1$, which contradicts our assumptions. Thus, $q = 1$ and $C = A$. □

Lemma 1.9. *Let p be a prime number. Assume that* $A, D \in \mathbf{S}(2p)$, $AD = DA$, *and* $A = BC$, *where* $B = (b_0, b_1, \ldots, b_{p-1})$ *and* $C = (c_0, c_1, \ldots, c_{p-1})$ *are disjoint p-cycles. Then the following three cases may only occur:*

i) $D = B^m C^n$, $0 \le m, n < p$;

ii) *D is the product of p disjoint transpositions* $D_i = (b_i, c_{|i+r|_p})$, $0 \le i \le p-1$, *where an integer r satisfies* $0 \le r < p$ *and does not depend on i;*

iii) *D is a $2p$-cycle of the form*

$$(b_0, c_r, b_{|r+s|_p}, c_{|r+(r+s)|_p}, b_{|2(r+s)|_p}, c_{|r+2(r+s)|_p}, \ldots, b_{|(p-1)(r+s)|_p}, c_{|r+(p-1)(r+s)|_p})$$

and $A = D^{2q}$, *where* $0 \le r, s < p$, $|r+s|_p \neq 0$, *and the number q is defined by the conditions* $1 \le q < p$, $|q(r+s)|_p = 1$.

Proof. Since D commutes with $A = BC$, we have $DBD^{-1} \cdot DCD^{-1} = BC$. Clearly, DBD^{-1} and DCD^{-1} are disjoint p-cycles. So, either $DBD^{-1} = B$ and $DCD^{-1} = C$, or $DBD^{-1} = C$ and $DCD^{-1} = B$. In the first case Lemma 1.4(b) implies that $D = B^m C^n$, where $0 \le m, n < p$. In the second case there are uniquely determined integers r, s such that $D(b_0) = c_r$, $D(c_0) = b_s$, and $0 \le r, s < p$. Then, for all i, j, $0 \le i, j < p$, relations $D(b_i) = c_{|i+r|_p}$ and $D(c_j) = b_{|j+s|_p}$ are held. If $|r+s|_p = 0$, we have $D^2(b_i) = D(c_{|i+r|_p}) = b_{|i+r+s|_p} = b_i$, so that D is the product or p disjoint transpositions $D_i = (b_i, c_{|i+r|_p})$. Finally, if $|r+s|_p \neq 0$, then $|t(r+s)|_p \neq 0$ for any t with $1 \le t < p$ (since p is prime), and therefore D must be the $2p$-cycle exhibited in the formulation of the lemma. The other assertions related to this case are evident. □

The following three lemmas may be proved by a direct checking.

Lemma 1.10. *Let $A \neq B$ be two commuting nondisjoint permutations such that* $[A] = [B] = [2, 2]$. *Then either* supp A = supp B *and the cyclic decompositions of A and B contain no common transpositions, or* supp A *and* supp B *have exactly two common points and the transposition of these points is contained both in A and B.* □

Lemma 1.11. *Let A, B be a braid-like couple of permutations of cyclic type* $[2, 2]$. *Then either* supp A *and* supp B *have exactly three common points and a transposition of two of them is contained both in A and B, or* supp A *and* supp B *have exactly two common points and the transposition of these points is neither contained in A nor in B.* □

Lemma 1.12. *Let A, B be a braid-like couple of 4-cycles. Then either* supp A = supp B *and B may be obtained from A by a transposition of two neighboring (in A) symbols, or* supp A *and* supp B *have exactly two common symbols which are neither neighboring in A nor in B.* □

1.2. Some elementary properties of braid homomorphisms. Statements 1.13-1.17 concern a group homomorphism ψ: $\mathbf{B}(k) \to H$, $k \geq 3$. Lemmas 1.13-1.15 are contained in [Ar3] (the latter one in a slightly weaker form), but for the completeness of the exposition we give the proofs.

Lemma 1.13. *Assume that for some i, $1 \leq i \leq k-2$, the elements $\psi(\sigma_i)$ and $\psi(\sigma_{i+1})$ commute. Then the homomorphism ψ is cyclic.*

Proof. Relation (0.2) implies $\psi(\sigma_i) = \psi(\sigma_{i+1})$. In view of (0.1), it follows that $\psi(\sigma_{i-1})$ commutes with $\psi(\sigma_i)$, and $\psi(\sigma_{i+1})$ commutes with $\psi(\sigma_{i+2})$. Hence, we have $\psi(\sigma_{i-1}) = \psi(\sigma_i) = \psi(\sigma_{i+1}) = \psi(\sigma_{i+2})$. Proceeding with this process, we obtain that all the elements $\psi(\sigma_i)$, $1 \leq i \leq k-1$, coincide. □

Lemma 1.14. *Assume that $\psi(\sigma_i) = \psi(\sigma_j)$ for some i, j, $1 \leq i < j \leq k-1$. If $j \neq i+2$ or $k \neq 4$, then ψ is cyclic.*

Proof. If $k = 3$ or $j = i+1$, then ψ is cyclic by Lemma 1.13. So, we may assume that $k > 3$ and $j \geq i+2$. If $j > i+2$, then (0.1) and the assumption $\psi(\sigma_i) = \psi(\sigma_j)$ show that $\psi(\sigma_i)$ commutes with $\psi(\sigma_{i+1})$, and ψ is cyclic (Lemma 1.13). Finally, if $k > 4$ and $\psi(\sigma_i) = \psi(\sigma_{i+2})$, then either $i > 1$ or $i = 1$ and $i+2 = 3 < k-1$; in the first case $\psi(\sigma_{i-1})$ commutes with $\psi(\sigma_i)$; in the second case $\psi(\sigma_3)$ commutes with $\psi(\sigma_4)$; by Lemma 1.13, ψ is cyclic. □

Lemma 1.15. *Assume that for some i, j $(1 \leq i < j-1 \leq k-1)$ there exists a natural number r such that $|r|_{j-i+1} \neq 0$, but $\psi(\alpha_{ij}^r)$ commutes with $\psi(\sigma_i)$. Then $\psi(\sigma_1) = \psi(\sigma_3)$; particularly, if $k \neq 4$, then the homomorphism ψ is cyclic.*

Proof. Put $q = |r|_{j-i+1}$, so that $1 \leq q \leq j-i$. Relations (0.7) imply that the element α_{ij}^{j-i+1} commutes with all the elements $\sigma_i, \sigma_{i+1}, \dots, \sigma_{j-1}$; therefore, it follows from our assumptions that

$$\psi(\alpha_{ij}^q)\psi(\sigma_i) = \psi(\sigma_i)\psi(\alpha_{ij}^q). \tag{1.6}$$

If $q = j-i$, then $q \equiv -1 \pmod{j-i+1}$, and (1.4) shows that $\psi(\alpha_{ij}^{-1})$ commutes with $\psi(\sigma_i)$. But then $\psi(\alpha_{ij})$ commutes with $\psi(\sigma_i)$, and (0.6) implies $\psi(\sigma_i) = \psi(\sigma_{i+1})$; by Lemma 1.14, ψ is cyclic .

Assume now that $q \leq j-i-1$. Then $i+q-1 \leq j-2$ and (0.6) implies that $\alpha_{ij}^q\sigma_i = \sigma_{i+q}\alpha_{ij}^q$. Combining this with (1.6), we obtain $\psi(\sigma_i) = \psi(\sigma_{i+q})$. If $q \neq 2$ or $k \neq 4$, the homomorphism ψ is cyclic (Lemma 1.14); if $q = 2$ and $k = 4$, then $i = 1$ and $\psi(\sigma_1) = \psi(\sigma_3)$. □

Lemma 1.13, Lemma 1.15, and relations (0.2) imply the following corollary:

Corollary 1.16. *Assume that ψ: $\mathbf{B}(k) \to H$ is a noncyclic homomorphism. Then*

$$\psi(\sigma_i)\infty\psi(\sigma_{i+1}) \quad \textit{for} \quad 1 \leq i \leq k-2.$$

If $1 \leq i < j \leq k$, then $\psi(\alpha_{ij}) \neq 1$. If $1 \leq i < j-1 \leq k-1 \neq 3$ (or $k = 4$, $1 \leq i \leq 2$, $j = i+2$) and the group H is finite, then $\operatorname{ord}\psi(\alpha_{ij}) \equiv 0 \pmod{j-i+1}$. □

Lemma 1.17. *Let ψ: $\mathbf{B}(k) \to H$ be a noncyclic group homomorphism and m be a natural number.*

a) Assume that $\psi(\alpha^m)$ commutes with $\psi(\beta)$. Then either k divides m or $k = 4$, $(m, 4) = 2$, and $\psi(\sigma_1) = \psi(\sigma_3)$.

b) If $\psi(\beta^m)$ commutes with $\psi(\alpha)$, then $k - 1$ divides m.

Proof. Set $\widehat{\alpha} = \psi(\alpha)$, $\widehat{\beta} = \psi(\beta)$. Since $\alpha^k = \beta^{k-1}$, the element $\widehat{\alpha}^k = \widehat{\beta}^{k-1}$ commutes with $\widehat{\alpha}$ and $\widehat{\beta}$. Note that $k > 2$ and $\widehat{\alpha}$ does not commute with $\widehat{\beta}$ (for ψ is noncyclic).

a) Assume that k does not divide m; then $\nu \overset{\text{def}}{=} (m, k) < k$. Since $\widehat{\alpha}^m$ and $\widehat{\alpha}^k$ commute with $\widehat{\beta}$, the element $\widehat{\alpha}^\nu$ also commutes with $\widehat{\beta}$, and therefore $\nu \geq 2$. Moreover, $\nu \leq k - 2$, since $k > 2$. Now, relations (0.3), (0.8) show that

$$\psi(\sigma_{\nu+1}) = \psi(\alpha^{\nu-1}\beta\alpha^{-\nu}) = \widehat{\alpha}^{\nu-1}\widehat{\beta}\widehat{\alpha}^{-\nu} = \widehat{\alpha}^{-1}\widehat{\beta} = \psi(\alpha^{-1}\beta) = \psi(\sigma_1),$$

and Lemma 1.14 implies that $k = 4$, $\nu = 2$, and $\psi(\sigma_3) = \psi(\sigma_1)$.

b) Assume that $k - 1$ does not divide m and set $\mu = (m, k - 1)$. Then

$$2 \leq \mu \leq k - 2 \tag{1.7}$$

(since $\widehat{\beta}^\mu$ commutes with $\widehat{\alpha}$ and ψ is noncyclic). It follows from relations (0.6) that

$$\sigma_1\sigma_2\cdots\sigma_\mu \cdot \sigma_1 = \sigma_2 \cdot \sigma_1\sigma_2\cdots\sigma_\mu.$$

Using the expressions of $\sigma_1, ..., \sigma_\mu$ in terms of α, β given by (0.8), we can rewrite the latter relation in the form

$$\begin{aligned}(\alpha^{-1}\beta \cdot \beta\alpha^{-1} \cdot \alpha^1\beta\alpha^{-2} \cdot \alpha^2\beta\alpha^{-3} \cdots \alpha^{\mu-2}\beta\alpha^{-\mu+1}) \cdot \alpha^{-1}\beta \\ = \beta\alpha^{-1} \cdot (\alpha^{-1}\beta \cdot \beta\alpha^{-1} \cdot \alpha^1\beta\alpha^{-2} \cdot \alpha^2\beta\alpha^{-3} \cdots \alpha^{\mu-2}\beta\alpha^{-\mu+1}),\end{aligned}$$

which leads to the relation $\alpha^{-1}\beta^\mu\alpha^{-\mu}\beta = \beta\alpha^{-2}\beta^\mu\alpha^{-\mu+1}$. Hence,

$$\widehat{\alpha}^{-1}\widehat{\beta}^\mu\widehat{\alpha}^{-\mu}\widehat{\beta} = \widehat{\beta}\widehat{\alpha}^{-2}\widehat{\beta}^\mu\widehat{\alpha}^{-\mu+1}.$$

Since $\widehat{\beta}^\mu$ commutes with $\widehat{\alpha}$, it follows from the latter relation that $\widehat{\alpha}^{-(\mu+1)}\widehat{\beta} = \widehat{\beta}\widehat{\alpha}^{-(\mu+1)}$, that is, $\psi(\alpha^{\mu+1})$ commutes with $\psi(\beta)$. Because of (1.7), k cannot divide $\mu + 1$; it follows from statement (a) that $k = 4$ and $(\mu + 1, 4) = 2$, so that μ must be odd. However, (1.7) implies $\mu = 2$, and we obtain a contradiction. □

Remark 1.1. For any homomorphism ψ: $\mathbf{B}(k) \to H$, it follows from relations (0.5) that $\psi(\sigma_i) \sim \psi(\sigma_j)$ $(1 \leq i, j < k)$. Particularly, if $H = \mathbf{S}(n)$ then all the permutations $\psi(\sigma_i)$ have the same cyclic type, that is, $[\psi(\sigma_1)] = \ldots = [\psi(\sigma_{k-1})]$. ○

In the following lemmas 1.18-1.20 we assume that $k \geq 3$ and consider a homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$.

Lemma 1.18. *If $k > 4$ and for some r, $1 \le r < n$, the family* $\mathrm{Inv}_r(\widehat{\sigma}_1)$ *consists of a single set Σ, then the homomorphism ψ is intransitive.*

Proof. If $i \ne 2$, then $\widehat{\sigma}_i$ commutes with $\widehat{\sigma}_1$ and is also conjugate to $\widehat{\sigma}_1$; by Lemma 1.5(c), Σ is the only $\widehat{\sigma}_i$-invariant set of cardinality r. Particularly, it is so for $i = 4$; since $\widehat{\sigma}_2$ commutes with $\widehat{\sigma}_4$, we obtain $\Sigma \in \mathrm{Inv}(\widehat{\sigma}_2)$. Thus, $\Sigma \in \mathrm{Inv}(\widehat{\sigma}_i)$ for all $i = 1, \dots, k-1$ and ψ is intransitive. □

Assertions (a) and (b) of the following lemma were proved in [Ar3] (in the slightly less general case $n = k$; the general case may be treated as well).

Lemma 1.19. *Let $k > 4$ and let ψ be transitive. Assume that the cyclic decomposition of $\widehat{\sigma}_1$ contains an r-cycle C.*

a) If $r > n/2$, then $r = n$ and ψ is cyclic.

b) If n is even and $r = n/2$, then the cyclic decomposition of $\widehat{\sigma}_1$ is of the form $\widehat{\sigma}_1 = B_1 \cdots B_s C$, where all the cycles B_i are of the same length t, $2 \le t \le r$, and $r = st$.

c) If $n = 2p$, where p is a prime number, then either $r < p$ or $r = 2p = n$, and in the latter case ψ is cyclic.

Proof. a) Let $\Sigma = \mathrm{supp}\, C$, so that $\#\Sigma = r$. It follows from the assumption $r > n/2$ that Σ is the only $\widehat{\sigma}_1$-invariant set of cardinality r; since ψ is transitive, Lemma 1.17 implies that $\Sigma = \mathbf{\Delta}_n$, $r = n$, and $\widehat{\sigma}_1 = C$. The permutations $\widehat{\sigma}_3$ and $\widehat{\sigma}_4$ commute with $\widehat{\sigma}_1$; hence, each of them is a power of the cycle C and ψ is cyclic (Lemma 1.13).

b) If $\widehat{\sigma}_1 = BC$, where B is an r-cycle, then assertion (b) is true (with $s = 1$, $t = r$). So, we may assume that C is the only r-cycle in the cyclic decomposition of $\widehat{\sigma}_1$. In this case for each $i \ne 2$ the set $\Sigma = \mathrm{supp}\, C$ is $\widehat{\sigma}_i$-invariant and $\widehat{\sigma}_i \mid \Sigma = C^{q_i}$ for some integer q_i, $0 \le q_i < r$ (Lemma 1.4(b)). Set $s = (q_4, r)$; let us show that $1 < s < r$. If $s = 1$, then C^{q_4} is the only r-cycle in the cyclic decomposition of $\widehat{\sigma}_4$; since $\widehat{\sigma}_2$ commutes with $\widehat{\sigma}_4$, the set Σ is also $\widehat{\sigma}_2$-invariant, which contradicts the transitivity of ψ. If $s = r$, then $q_4 = 0$ and $\widehat{\sigma}_4 \mid \Sigma = C^{q_4} = \mathrm{id}_\Sigma$, so that $\Sigma \subseteq \mathrm{Fix}\,(\widehat{\sigma}_4)$. However, the cyclic decomposition of $\widehat{\sigma}_4$ must contain some r-cycle (for $\widehat{\sigma}_4 \sim \widehat{\sigma}_1$). Consequently, $\Sigma = \mathrm{Fix}\,(\widehat{\sigma}_4)$ and therefore $\Sigma \in \mathrm{Inv}(\widehat{\sigma}_2)$, which contradicts the transitivity of ψ. Thus, $1 < s < r$, $r = st$, $2 \le t < r$, and C^{q_4} is a product of s disjoint t-cycles. Since $\widehat{\sigma}_1 \sim \widehat{\sigma}_4$ and $C \preccurlyeq \widehat{\sigma}_1$, we obtain the desired representation of $\widehat{\sigma}_1$.

c) In view of (a), we should only show that $r \ne p$. Assume, on the contrary, that $r = p$; since p is prime, (b) implies that $\widehat{\sigma}_1 = BC$, where B and C are disjoint p-cycles.

Note that $p \ne 2$. For otherwise $n = 4$ and $[\widehat{\sigma}_1] = [\widehat{\sigma}_2] = [2, 2]$; however, in $\mathbf{S}(4)$ any two permutations of cyclic type $[2, 2]$ commute and, by Lemma 1.13, the homomorphism ψ is cyclic. Since ψ is transitive, $\widehat{\sigma}_1$ must be a 4-cycle (§0.4), and we obtain a contradiction.

So, $p \ge 3$. It follows from Lemma 1.9 that for each $i \ne 2$ there exist natural numbers m_i, n_i $(1 \le m_i, n_i < p)$ such that $\widehat{\sigma}_i = B^{m_i} C^{n_i}$. (Cases ($ii$), ($iii$) described in Lemma 1.9 cannot occur here, since $[\widehat{\sigma}_i] = [\widehat{\sigma}_1] = [p, p]$ and $p \ge 3$.) Applying Lemma 1.9 to the permutations $\widehat{\sigma}_2$ and $\widehat{\sigma}_4$, we conclude that $\widehat{\sigma}_2$ is also of the form $B^s C^t$, which contradicts the transitivity of ψ. □

Lemma 1.20. *a) If $k < n$ and $[\widehat{\sigma}_1] = [2]$, then ψ is intransitive.*

b) If $k > 3$ and $[\widehat{\sigma}_1] = [3]$, then $\widehat{\sigma}_1 = \widehat{\sigma}_3$; consequently, if $k > 4$, then ψ is cyclic.

c) If $4 < k < n$ and $\#\operatorname{Fix}\widehat{\sigma}_1 > n - 4$, then ψ is intransitive.

Proof. *a)* Clearly, $[\widehat{\sigma}_i] = [2]$ for any i; it is readily seen that in this case either ψ is cyclic and $\#(\cup_{i=1}^{k-1}\operatorname{supp}\widehat{\sigma}_i) = 2 < k < n$ or ψ is noncyclic, $\widehat{\sigma}_i\infty\widehat{\sigma}_{i+1}$, and $\#(\cup_{i=1}^{k-1}\operatorname{supp}\widehat{\sigma}_i) \leq k < n$. Anyway, ψ is intransitive.

b) Since $[\widehat{\sigma}_i] = [3]$ for every i, Lemma 1.8 implies that $\widehat{\sigma}_1 = \widehat{\sigma}_3$; if $k > 4$, then ψ is cyclic by Lemma 1.14.

c) Let $m = \#\operatorname{Fix}\widehat{\sigma}_1$; so, either $m = n$, or $m = n - 2$, or $m = n - 3$. If $m = n$, then $\widehat{\sigma}_1 = \mathrm{id}$ and ψ is trivial. If $m = n - 2$, then $[\widehat{\sigma}_1] = [2]$ and ψ is intransitive by (a). Finally, if $m = n - 3$, then $[\widehat{\sigma}_1] = [3]$ and, by (b), ψ is cyclic; in this case all $\widehat{\sigma}_i$ coincide with the same 3-cycle, and ψ is intransitive, since $n > k > 4$. □

In the following lemma we show that the support of the permutation $\widehat{\sigma}_1 = \psi(\sigma_1)$ mast be relatively small whenever ψ is transitive and all the cycles in the cyclic decomposition of $\widehat{\sigma}_1$ are of pairwise distinct lengths. For any real $x \geq 0$, we denote the integral part of x by $E(x)$.

Lemma 1.21. *Suppose $k > 4$. Let ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ be a transitive homomorphism. Assume that $[\widehat{\sigma}_1] = [r_1, ..., r_\mu]$, where all the integers r_ν, $1 \leq \nu \leq \mu$, are distinct and satisfy $1 < r_\nu < n$. Then the permutations $\widehat{\sigma}_i$ and $\widehat{\sigma}_j$ are disjoint whenever $|j - i| \geq 2$; in particular,*

$$\sum_{\nu=1}^{\mu} r_\nu \leq n/E(k/2).$$

Proof. Each $\widehat{\sigma}_i$, $1 \leq i \leq k - 1$, is a disjoint product of μ cycles of pairwise distinct lengths $r_1, ..., r_\mu$. Let us fix some i and consider the cyclic decomposition $\widehat{\sigma}_i = C_1 \cdots C_\mu$, $[C_\nu] = [r_\nu]$. Set $\Sigma_\nu = \operatorname{supp} C_\nu$ and $\Sigma = \operatorname{supp}\widehat{\sigma}_i = \cup_{\nu=1}^{\mu}\Sigma_\nu$. Clearly, any set Σ_ν is $\widehat{\sigma}_i$-invariant. For every j that satisfies $1 \leq j \leq k - 1$ and $|j - i| \geq 2$, the permutation $\widehat{\sigma}_j$ commutes with $\widehat{\sigma}_i$; since all the numbers r_ν are distinct, any set Σ_ν is $\widehat{\sigma}_j$-invariant, and

$$\widehat{\sigma}_j \mid \Sigma = C_1^{q_{j,1}} \cdots C_\mu^{q_{j,\mu}}$$

with some integers $q_{j,\nu}$, $0 \leq q_{j,\nu} < r_\nu$ (Lemma 1.4(b)).

Let us show that all $q_{j,\nu} = 0$ whenever $|j - i| > 2$. Assume, on the contrary, that for some $j_\circ$ with $|j_\circ - i| > 2$ there is a nonzero $q_{j_\circ,\nu}$. Then $D = C_\nu^{q_{j_\circ,\nu}}$ is an r_ν-cycle (for otherwise, the permutation $D \preccurlyeq \widehat{\sigma}_{j_\circ}$ would be a product of a few cycles *of the same length*, which is impossible). Clearly, $D = C_\nu^{q_{j_\circ,\nu}}$ is the only r_ν-cycle in the cyclic decomposition of $\widehat{\sigma}_{j_\circ}$. Since $|j_\circ - i| > 2$, the permutations $\widehat{\sigma}_s$ with $|s - i| = 1$ commute with $\widehat{\sigma}_{j_\circ}$; so, the set $\operatorname{supp} D = \Sigma_\nu$ is $\widehat{\sigma}_s$-invariant. However, this set Σ_ν is $\widehat{\sigma}_i$-invariant and also $\widehat{\sigma}_j$-invariant for every j with $|j - i| \geq 2$. It follows that Σ_ν is an $(\operatorname{Im}\psi)$-invariant set. Since $2 \leq r_\nu < n$, this contradicts the transitivity of ψ.

Thus, for $|j - i| > 2$ all $q_{j,\nu} = 0$, so that $\widehat{\sigma}_j \mid \Sigma = \mathrm{id}_\Sigma$. This means that the permutations $\widehat{\sigma}_i$ and $\widehat{\sigma}_j$ are disjoint.

We are left with showing that $\widehat{\sigma}_i$ and $\widehat{\sigma}_j$ are also disjoint for $|j-i|=2$. Since $k>4$, there exists an index t, $1\le t\le k-1$, neighboring to one of the indices i,j and non-neighboring to another; say $|t-j|=1$ and $|t-i|\ge 2$. Since $|j-i|=2$, it follows that in fact $|t-i|>2$. Therefore, as we have already proved, for any $m\in\Sigma$, we have $\widehat{\sigma}_t(m)=m$ and $\widehat{\sigma}_j(m)\in\Sigma$; particularly, $\widehat{\sigma}_t(\widehat{\sigma}_j(m))=\widehat{\sigma}_j(m)=\widehat{\sigma}_j(\widehat{\sigma}_t(m))$. Hence, taking into account that the indices t and j are neighboring, for any $m\in\Sigma$ we obtain

$$\begin{aligned}\widehat{\sigma}_j(m)=\widehat{\sigma}_t(\widehat{\sigma}_j(m))&=\widehat{\sigma}_t((\widehat{\sigma}_j(\widehat{\sigma}_t(m)))=(\widehat{\sigma}_t\widehat{\sigma}_j\widehat{\sigma}_t)(m)\\&=(\widehat{\sigma}_j\widehat{\sigma}_t\widehat{\sigma}_j)(m)=\widehat{\sigma}_j(\widehat{\sigma}_t(\widehat{\sigma}_j(m)))=\widehat{\sigma}_j(\widehat{\sigma}_j(m))=m.\end{aligned}$$

This shows that $\widehat{\sigma}_j(m)\mid\Sigma=\mathrm{id}_\Sigma$ and the permutations $\widehat{\sigma}_i$ and $\widehat{\sigma}_j$ are disjoint. □

1.3. Transitive homomorphisms $\mathbf{B}(k)\to\mathbf{S}(n)$ and prime numbers. The following lemma is the heart of Artin's methods developed in [Ar3]. Actually, it was not formulated explicitly, but it was proven in the course of the proof of Lemma 6 in the cited paper (for the case $k=n$). For completeness of the exposition, we present the proof of this very important lemma.

Lemma 1.22 (E. Artin). *Let $k>4$ and n be natural numbers such that there is a prime $p>2$ satisfying*

$$n/2<p\le k-2. \tag{1.8}$$

Then for every noncyclic transitive homomorphism $\psi\colon\mathbf{B}(k)\to\mathbf{S}(n)$ the permutation $\widehat{\sigma}_1=\psi(\sigma_1)$ has at least $k-2$ fixed points (particularly, $n\ge k$).

Proof. Set $\widehat{\alpha}=\psi(\alpha)$; since the elements α, σ_1 generate the whole group $\mathbf{B}(k)$, the permutations $\widehat{\alpha}$, $\widehat{\sigma}_1$ generate the whole subgroup $\operatorname{Im}\psi\subseteq\mathbf{S}(n)$.

For each i, $3\le i\le k-p+1$, put $T_i=\widehat{\alpha}_{i,p+i-1}=\psi(\alpha_{i,p+i-1})$. Relations (0.3) and (0.4) imply that $T_{i+1}=\widehat{\alpha}T_i\widehat{\alpha}^{-1}$. Thus, the permutations $T_3,\dots,T_{k-p+1}$ are conjugate to each other and have the same cyclic type.

Corollary 1.16 implies that $T_i\ne 1$ and $\operatorname{ord}T_i\equiv 0 \pmod p$; since p is prime, the length l_i of some cycle $C_i\preccurlyeq T_i$ is divisible by p. Since $p>n/2$, we have $l_i=p$, and C_i is the only p-cycle in the cyclic decomposition of T_i. Let $\Sigma_i=\operatorname{supp}C_i$, $\#\Sigma_i=p$. The permutation $\widehat{\sigma}_1$ commutes with all T_i, $3\le i\le k-p+1$. By Lemma 1.4(b), $\widehat{\sigma}_1\mid\Sigma_i=C_i^{q_i}$ for some q_i, $0\le q_i<p$. In fact, each $q_i=0$ (for otherwise, $(q_i,p)=1$ and $C_i^{q_i}\preccurlyeq\widehat{\sigma}_1$ is a p-cycle of length $>n/2$, which contradicts Lemma 1.19(a)). Hence, $\widehat{\sigma}_1\mid\Sigma_i=1$ and $\Sigma_i\subseteq\operatorname{Fix}\widehat{\sigma}_1$ for $i=3,\dots,k-p+1$.

Consider the sets $S_r=\bigcup_{i=3}^{r}\Sigma_i$, $3\le r\le k-p+1$. Clearly, $S_3\subseteq S_4\subseteq\cdots\subseteq S_{k-p+1}$. We shall show that all these inclusions are strict; if so, then $\#\operatorname{Fix}\widehat{\sigma}_1\ge\#S_{k-p+1}\ge\#\Sigma_3+k-p-2=k-2$, and we are done.

Every S_r is a nontrivial $\widehat{\sigma}_1$-invariant set. Indeed, $S_r\subseteq\operatorname{Fix}\widehat{\sigma}_1$, $\#S_r\ge p$, and $S_r\ne\mathbf{\Delta}_n$ (for otherwise, $\mathbf{\Delta}_n=S_r\subseteq\operatorname{Fix}\widehat{\sigma}_1$ and $\widehat{\sigma}_1=1$).

Suppose that $S_r=S_{r+1}$ for some r, $3\le r<k-p+1$. Since C_j is the only p-cycle in the cyclic decomposition of T_j, we have $\widehat{\alpha}(\Sigma_i)=\Sigma_{i+1}$. If $3\le i<r$, then $\Sigma_{i+1}\subseteq S_r$; and if $i=r$, then $\Sigma_{i+1}\subseteq S_{r+1}=S_r$. Thus, $\widehat{\alpha}(\Sigma_i)=\Sigma_{i+1}\subseteq S_r$ for all $i\le r$. Hence, $\widehat{\alpha}(S_r)=S_r$ and the set S_r is $\widehat{\alpha}$-invariant. However, S_r is also $\widehat{\sigma}_1$-invariant. Therefore, it is a nontrivial $(\operatorname{Im}\psi)$-invariant set, which contradicts the transitivity of ψ. □

Remark 1.2. The mapping $\sigma_1 \mapsto (1,2)(3,4)(5,6)$, $\alpha \mapsto (1,2,3,4,5)$, extends to a non-cyclic transitive homomorphism $\psi_{5,6}$: $\mathbf{B}(5) \to \mathbf{S}(6)$. This shows that the assertion of Lemma 1.22 becomes false if we replace the inequalities (1.8) by the slightly weaker inequalities $n/2 \leq p \leq k-2$.

Nevertheless, *the conclusion of Artin Lemma holds true whenever there is a prime $p > 3$ that satisfies $n/2 \leq p \leq k-3$.* For $p > n/2$ this follows directly from Lemma 1.22. Thus, to justify our assertion, we need only to consider the case when $n = 2p$ and $3 < p \leq k-3$. Define the elements T_i as in the proof of Lemma 1.22; as before, the length l_i of some cycle $C_i \preccurlyeq T_i$ must be divisible by p. Hence, either *i*) T_i contains the only cycle of length p, or *ii*) $[T_i] = [p,p]$, or *iii*) $[T_i] = [2p]$.

In case (*i*) all the arguments used in the proof of Lemma 1.22 work as well, with only one exception: to prove that $q_i = 0$, we should refer to Lemma 1.19(*c*) instead of Lemma 1.19(*a*). Let us show that cases (*ii*) and (*iii*) cannot occur. Note that $k - p + 1 \geq 4$; therefore, we can deal with the permutation $A = T_4$.

In case (*ii*), $A = T_4 = BC$, where B and C are disjoint p-cycles. Let $B = (b_0, \dots, b_{p-1})$, $C = (c_0, \dots, c_{p-1})$. Since the permutations $D = \widehat{\sigma}_1$ and $D' = \widehat{\sigma}_2$ commute with $A = T_4 = BC$, we can apply Lemma 1.9 to the couples A, D and A, D', respectively. Note that D, D' are conjugate, and thus they have the same cyclic type. Clearly, D and D' are nontrivial. So, if one of these permutations is of the form (*i*) described in Lemma 1.9, then it contains a p-cycle; however, this contradicts Lemma 1.19(*c*). The same lemma shows also that $[D] = [D'] \neq [2p]$. So D, D' must be of the form (*ii*) described in Lemma 1.9, that is, $D = \widehat{\sigma}_1 = D_0 \cdots D_{p-1}$ and $D' = \widehat{\sigma}_2 = D'_0 \cdots D'_{p-1}$, where $D_i = (b_i, c_{|i+r|_p})$, $D'_j = (b_j, c_{|j+r'|_p})$, r and r' do not depend on i, j, and $0 \leq r, r' < p$. Clearly, $(\widehat{\sigma}_1\widehat{\sigma}_2\widehat{\sigma}_1)(b_0) = (\widehat{\sigma}_2\widehat{\sigma}_1\widehat{\sigma}_2)(b_0)$. It is readily seen that the left hand side of the latter relation equals $c_{|2r-r'|_p}$, and the right hand side equals $c_{|2r'-r|_p}$. Consequently, $2r - r' \equiv 2r' - r \pmod{p}$, that is, $3(r - r') \equiv 0 \pmod{p}$. Since p is prime and $p \neq 3$, we have $r = r'$; therefore, $D_i = D'_i$ for all i. Thus, $\widehat{\sigma}_1 = \widehat{\sigma}_2$ and the homomorphism ψ is cyclic, which contradicts our assumption.

In case (*iii*), the permutation T_4 is a $2p$-cycle. Since $2p = n$ and the permutations $\widehat{\sigma}_1$, $\widehat{\sigma}_2$ commute with T_4, they commute with each other (Lemma 1.4(*b*)), and we again obtain a contradiction.

The homomorphism ν_6: $\mathbf{B}(6) \to \mathbf{S}(6)$ shows that the condition $p > 3$ is essential. ○

Remark 1.3. A prime number $p \in ((k+l)/2, k-2]$ does exist in each of the following cases: *a*) $6 \neq k \geq 5$, $l = 0$; *b*) $k \geq 7$, $l = 1$; *c*) $8, 12 \neq k \geq 7$, $l = 2$; *d*) $11, 12 \neq k \geq 9$, $l = 3$.

Case (*a*) is known as "Bertrand Postulate"; it was proven by P. L. Chebyshev in the last century. In fact, all cases may be treated using the following inequality due to P. Finsler [Fi] (see also [Tr, p. 60, Satz 32]): $\pi(2) - \pi(m) > m/(3\log(2m))$ for any natural $m > 1$. (Here $\pi(x)$ denotes the number of all primes $p \leq x$.) ○

§2. Homomorphisms $\mathbf{B}(k) \to \mathbf{S}(n)$ and $\mathbf{B}(k) \to \mathbf{B}(n)$, $n \le k$

In this section we prove Theorem A(a, b) and Theorem B (see Theorem 2.1 and Theorem 2.12, respectively). To prove Theorem 2.1, we follow the methods of E. Artin, but the important new point is that we make use of the fact that for $k > 4$ the group $\mathbf{B}'(k)$ is perfect (see §0.6). Using Theorem 2.1, we obtain also an improvement of Artin Theorem on homomorphisms $\mathbf{B}(k) \to \mathbf{S}(k)$ (see Remark 2.2 below).

Theorem 2.1. *Assume that $k \ne 4$ and $n < k$. Then*
a) any homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ is cyclic;
b) any homomorphism ϕ: $\mathbf{B}(k) \to \mathbf{B}(n)$ is integral.

Proof. For $k < 3$ the statements are trivial, since $\mathbf{B}(2) \cong \mathbb{Z}$. Suppose therefore that $k > 4$.

a) Assume that the homomorphism ψ is noncyclic. Put $H = \operatorname{Im} \psi \subseteq \mathbf{S}(n)$. Then there is at least one H-orbit $Q \subseteq \boldsymbol{\Delta}_n$ of some length $m = \#Q \le n < k$ such that the reduction ψ_Q: $\mathbf{B}(k) \to \mathbf{S}(Q) \cong \mathbf{S}(m)$ of the homomorphism ψ to Q is a noncyclic transitive homomorphism (see Observation in §0.0.2). Since $k > 4$ and $m < k$, it follows from Chebyshev Theorem (Remark 1.3(a)) that there exists a prime number $p > 2$ such that $m/2 < p \le k-2$. Lemma 1.22 implies that the permutation $\psi_Q(\sigma_1) \in \mathbf{S}(m)$ has at least $k-2 > m-2$ fixed points; hence, $\psi_Q(\sigma_1) = \mathrm{id}_Q$ and ψ_Q is trivial, which contradicts the choice of Q.

b) Consider the composition

$$\psi = \mu \circ \phi \colon\ \mathbf{B}(k) \xrightarrow{\phi} \mathbf{B}(n) \xrightarrow{\mu} \mathbf{S}(n)$$

of ϕ with the canonical projection μ: $\mathbf{B}(n) \to \mathbf{S}(n)$. By (a), ψ is cyclic; therefore, its restriction to the commutator subgroup $\mathbf{B}'(k) \subset \mathbf{B}(k)$ is trivial. Thus, $\phi(\mathbf{B}'(k)) \subseteq \operatorname{Ker} \mu = \mathbf{I}(n)$. Since the group $\mathbf{B}'(k)$ is perfect, it does not possess nontrivial homomorphisms into the pure braid group $\mathbf{I}(n)$ (Corollary 0.1). Hence, the restriction of ϕ to $\mathbf{B}'(k)$ is trivial and the homomorphism ϕ is integral. □

Remark 2.1. The condition $k \ne 4$ in Theorem 2.1 is essential. To see this, take the canonical systems of generators $\{\sigma_1, \sigma_2, \sigma_3\}$ in $\mathbf{B}(4)$ and $\{\sigma_1', \sigma_2'\}$ in $\mathbf{B}(3)$, and consider the surjective homomorphism π: $\mathbf{B}(4) \to \mathbf{B}(3)$ defined by

$$\pi(\sigma_1) = \pi(\sigma_3) = \sigma_1', \qquad \pi(\sigma_2) = \sigma_2'.$$

This example shows that for $k = 4$ statement (b) of Theorem 2.1 is false; statement (a) is also false, since the composition $\mu \circ \pi$: $\mathbf{B}(4) \to \mathbf{S}(3)$ of π with the canonical projection μ: $\mathbf{B}(3) \to \mathbf{S}(3)$ is surjective. We call π: $\mathbf{B}(4) \to \mathbf{B}(3)$ the *canonical epimorphism.* It is easily seen that the kernel of π coincides with the normal subgroup $\mathbf{T} \subset \mathbf{B}(4)$ described in §0.6. With the special generators $\alpha, \beta \in \mathbf{B}(4)$ and $\alpha', \beta' \in \mathbf{B}(3)$, the canonical epimorphism π looks as follows: π: $\alpha \mapsto \beta'$, $\beta \mapsto (\beta')^{-1}(\alpha')^2\beta'$. ○

Our goal now is to prove Theorem B. To this end, we need some preparations. We start with some additional properties of the pure braid group $\mathbf{I}(k)$. In what follows, we use the notation introduced in §0.1-0.4. Particularly, $\sigma_1, \cdots \sigma_{k-1}$ are the canonical generators of $\mathbf{B}(k)$, and the elements α_{ij}, $1 \le i < j \le k$, are defined by (0.3).

Lemma 2.2. *Assume that* $1 < t \le k-1$. *Then*

$$(\alpha_{1t})^t \cdot \sigma_t\sigma_{t-1}\cdots\sigma_1 = (\alpha_{1,t+1})^t. \tag{2.1}$$

Proof. Let q, r, s be integers such that $0 \le r \le s \le t$, $1 \le q \le t$, and $q + r \le t$. We prove that

$$(\alpha_{1t})^s \cdot \sigma_t\sigma_{t-1}\cdots\sigma_q = (\alpha_{1t})^{s-r} \cdot \sigma_t\sigma_{t-1}\cdots\sigma_{q+r}(\alpha_{1,t+1})^r. \tag{2.2}$$

The proof is by induction on r with the case $r = 0$ trivial. Suppose that (2.2) is true for some $r \ge 0$ such that $r + 1 \le s$ and $q + r + 1 \le t$. Note that

$$\alpha_{1t}\sigma_t = \sigma_1\cdots\sigma_{t-1}\cdot\sigma_t = \alpha_{1,t+1}. \tag{2.3}$$

Therefore, using relations (0.6), we obtain

$$\begin{aligned}
&(\alpha_{1t})^s \cdot \sigma_t\sigma_{t-1}\cdots\sigma_q = (\alpha_{1t})^{s-r}\cdot\sigma_t\sigma_{t-1}\cdots\sigma_{q+r}\cdot(\alpha_{1,t+1})^r \\
&= (\alpha_{1t})^{s-r-1}\alpha_{1t}\cdot\sigma_t\sigma_{t-1}\cdots\sigma_{q+r}\cdot(\alpha_{1,t+1})^r = (\alpha_{1t})^{s-r-1}\alpha_{1,t+1}\cdot\sigma_{t-1}\cdots\sigma_{q+r}\cdot(\alpha_{1,t+1})^r \\
&= (\alpha_{1t})^{s-r-1}\cdot\sigma_t\cdots\sigma_{q+r+1}\alpha_{1,t+1}\cdot(\alpha_{1,t+1})^r = (\alpha_{1t})^{s-(r+1)}\cdot\sigma_t\cdots\sigma_{q+(r+1)}\cdot(\alpha_{1,t+1})^{r+1};
\end{aligned}$$

this completes the step of induction and proves (2.2). For $q = 1$, $r = t-1$, and $s = t$, relation (2.2) takes the form

$$(\alpha_{1t})^t\cdot\sigma_t\sigma_{t-1}\cdots\sigma_1 = \alpha_{1t}\cdot\sigma_t\cdot(\alpha_{1,t+1})^{t-1}.$$

In view of (2.3), the latter relation coincides with (2.1). □

Assume now that $1 < r < k$. Let $s_{i,j} \in \mathbf{B}(k)$ $(1 \le i < j \le k)$ and $s'_{i,j}$ $(1 \le i < j \le r)$ be the canonical generators in the groups $\mathbf{I}(k)$ and $\mathbf{I}(r)$, respectively. Consider the group epimorphism $\xi_{k,r}\colon \mathbf{I}(k) \to \mathbf{I}(r)$ defined by (0.12) and set

$$R_t = s_{1,t}s_{2,t}\cdots s_{t-1,t} \quad (2 \le t \le k), \qquad R'_t = s'_{1,t}s'_{2,t}\cdots s'_{t-1,t} \quad (2 \le t \le r).$$

Lemma 2.3. *For every integer t that satisfies $2 \le t \le k$ the following relations hold:*

$$R_t = \sigma_{t-1}\sigma_{t-2}\cdots\sigma_2\sigma_1^2\sigma_2\cdots\sigma_{t-2}\sigma_{t-1}, \qquad R_2R_3\cdots R_t = (\sigma_1\sigma_2\cdots\sigma_{t-1})^t. \tag{2.4}$$

Proof. The proof is by induction on t with the case $t = 2$ trivial ($R_2 = s_{1,2}$ and $s_{1,2} = \sigma_1^2$). Assume that relations (2.4) hold for some t, $2 \le t < k$. Then

$$\begin{aligned}
R_{t+1} &= s_{1,t+1}s_{2,t+1}\cdots s_{t-1,t+1}s_{t,t+1} = \sigma_t s_{1,t}\sigma_t^{-1}\cdot\sigma_t s_{2,t}\sigma_t^{-1}\cdots\sigma_t s_{t-1,t}\sigma_t^{-1}\cdot\sigma_t^2 \\
&= \sigma_t s_{1,t}s_{2,t}\cdots s_{t-1,t}\sigma_t = \sigma_t R_t\sigma_t = \sigma_t\sigma_{t-1}\sigma_{t-2}\cdots\sigma_2\sigma_1^2\sigma_2\cdots\sigma_{t-2}\sigma_{t-1}\sigma_t
\end{aligned}$$

and, according to (2.1),

$$\begin{aligned}
R_2R_3\cdots R_tR_{t+1} &= (\sigma_1\sigma_2\cdots\sigma_{t-1})^t\cdot\sigma_t\sigma_{t-1}\cdots\sigma_2\sigma_1^2\sigma_2\cdots\sigma_{t-1}\sigma_t \\
&= \left[(\alpha_{1t})^t\cdot\sigma_t\sigma_{t-1}\cdots\sigma_2\sigma_1\right]\cdot\sigma_1\sigma_2\cdots\sigma_{t-1}\sigma_t \\
&= (\alpha_{1,t+1})^t\cdot\sigma_1\sigma_2\cdots\sigma_{t-1}\sigma_t = (\alpha_{1,t+1})^{t+1};
\end{aligned}$$

this completes the step of induction and proves the lemma. □

Recall that $\mathbf{C}(m) \subseteq \mathbf{I}(m)$ is the cyclic subgroup of $\mathbf{B}(m)$ generated by the element A_m. This subgroup coincides with the center of $\mathbf{B}(m)$ whenever $m > 2$; moreover, $\mathbf{C}(2) = \mathbf{I}(2)$.

Lemma 2.4. *Assume that* $1 < r < k$. *Then*

$$\xi_{k,r}(A_k) = A_r; \tag{2.5}$$

in particular, $\xi_{k,r} \mid \mathbf{C}(k)$: $\mathbf{C}(k) \to \mathbf{C}(r)$ *is a group isomorphism.*

Proof. According to Lemma 2.3,

$$\xi_{k,r}(A_k) = \xi_{k,r}\left((\sigma_1 \cdots \sigma_{k-1})^k\right) = \xi_{k,r}(R_2 \cdots R_k). \tag{2.6}$$

Clearly, $\xi_{k,r}(R_j) = R'_j$ for $j \le r$ and $\xi_{k,r}(R_j) = 1$ for $j > r$. Using these relations and taking into account relations (2.6), (2.4) (the latter one for the elements R''s and σ''s), we obtain $\xi_{k,r}(A_k) = R'_2 \cdots R'_r = (\sigma'_1 \cdots \sigma'_{r-1})^r = A_r$. □

Corollary 2.5. *For any* $k \ge 3$, *the pure braid group* $\mathbf{I}(k)$ *is the direct product of its subgroups* $\mathbf{I}^2(k)$ *and* $\mathbf{C}(k)$.

Proof. We noted in §0.3 that the kernel of the epimorphism $\xi_{k,2}$: $\mathbf{I}(k) \to \mathbf{I}(2) = \mathbf{C}(2)$ coincides with $\mathbf{I}^2(k)$; so, we have the exact sequence $1 \to \mathbf{I}^2(k) \to \mathbf{I}(k) \xrightarrow{\xi_{k,2}} \mathbf{C}(2) \to 1$. By Lemma 2.4, $\xi_{k,2}$ maps the subgroup $\mathbf{C}(k)$ onto $\mathbf{C}(2)$ isomorphically; since $\mathbf{C}(k)$ is the center of $\mathbf{B}(k)$, this proves the lemma. □

Lemma 2.6. *Suppose* $k \ne 4$. *If* G *is the kernel of a cyclic homomorphism* ψ: $\mathbf{B}(k) \to \mathbf{S}(k)$, *then every homomorphism* ϕ: $G \to \mathbf{I}(k)$ *is integral.*

Proof. Assume first that $k > 4$. Since ψ is cyclic, $G = \operatorname{Ker} \psi \supseteq \mathbf{B}'(k)$. Taking into account that $\mathbf{B}'(k)$ is perfect, we have $\mathbf{B}'(k) \supseteq G' \supseteq (\mathbf{B}'(k))' = \mathbf{B}'(k)$; hence, $G' = \mathbf{B}'(k)$ and G' is perfect. By Corollary 0.1, $\phi(G') = \{1\}$ and the homomorphism ϕ is Abelian. Furthermore, $G/G' = G/\mathbf{B}'(k)$ is a subgroup of $\mathbf{B}(k)/\mathbf{B}'(k) \cong \mathbb{Z}$; thus G/G' is cyclic. Consequently, the homomorphism ϕ is cyclic. Since $\mathbf{I}(k)$ is torsion free, ϕ is integral.

Now consider the case $k = 3$. By Corollary 2.5, $\mathbf{I}(3) \cong \mathbf{I}^2(3) \times \mathbf{C}(3)$. Let π_1 and π_2 be the projections of $\mathbf{I}(3)$ onto the first and the second factor, respectively. We know also that $\mathbf{C}(3) \cong \mathbb{Z}$ and $\mathbf{I}^2(3) \cong \mathbb{F}_2$ (Markov Theorem).

If ψ is trivial, then $G = \mathbf{B}(3)$ and the homomorphisms $\pi_1 \circ \phi$: $\mathbf{B}(3) \to \mathbf{I}^2(3)$ and $\pi_2 \circ \phi$: $\mathbf{B}(3) \to \mathbf{C}(3)$ are integral (Remark 0.1); consequently, the homomorphism ϕ is Abelian and, therefore, integral.

Assume that ψ is nontrivial. Then either (a) $[\psi(\sigma_1)] = [2]$ or (b) $[\psi(\sigma_1)] = [3]$. Using Reidemeister-Schreier process, it is easy to show that in case (a) the group $G = \operatorname{Ker} \psi$ is generated by the elements $u = \sigma_2\sigma_1^{-1}$ and $v = \sigma_1^2$ that satisfy the single defining relation $(uvu)^2 = vuv$. In case (b) the group G is generated by the elements $u = \sigma_2\sigma_1^{-1}$, $x = \sigma_1\sigma_2\sigma_1^{-2}$, and $y = \sigma_1^2\sigma_2$ that satisfy the defining relations $yuy^{-1} = u^{-1}$, $yxy^{-1} = x^{-1}$. In case (a), applying Lemma 1.1 to the elements u, v and to the homomorphisms $\pi_1 \circ \phi$, $\pi_2 \circ \phi$, we obtain that the elements $\widehat{u} = \phi(u)$ and $\widehat{v} = \phi(v)$ commute. Since they satisfy the relation $(\widehat{u}\widehat{v}\widehat{u})^2 = \widehat{v}\widehat{u}\widehat{v}$, we have $\widehat{u}^3 = 1$. But the group $\mathbf{I}(3)$ is torsion free; hence, $\widehat{u} = 1$. Consequently, the subgroup $\operatorname{Im} \phi \subset \mathbf{I}(3)$ is generated by the single element $\widehat{v}$; therefore, the homomorphism ϕ is integral. Similarly, in case (b) we obtain that the element $\widehat{y} = \phi(y)$ commutes with the elements $\widehat{u} = \phi(u)$ and $\widehat{x} = \phi(x)$. Since $\widehat{y}\widehat{u}\widehat{y}^{-1} = \widehat{u}^{-1}$, $\widehat{y}\widehat{x}\widehat{y}^{-1} = \widehat{x}^{-1}$, and the group $\mathbf{I}(3)$ is torsion free, we obtain $\widehat{u} = \widehat{x} = 1$. Thus, the group $\operatorname{Im} \phi$ is generated by the single element $\widehat{y}$ and the homomorphism ϕ is integral. □

Lemma 2.7. *For $k \neq 4$, any nonsurjective homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(k)$ is cyclic.*

Proof. If ψ is transitive, then the statement of the lemma follows directly from Artin Theorem. So, we may assume that ψ is intransitive; then $\#Q < n$ for any (Im ψ)-orbit $Q \subset \mathbf{\Delta}_n$. Theorem 2.1 implies that the reduction of ψ to any such orbit is cyclic; therefore, ψ itself is cyclic. □

Remark 2.2. Lemma 2.7 shows that any noncyclic homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(k)$ is transitive provided $k \neq 4$. This implies the following useful improvement of Artin Theorem:

statements (a), (b) (and also (d) for $k \neq 4$) of Artin Theorem hold true for any noncyclic homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(k)$ (even if we omit the additional assumption that ψ is transitive). *So, for $k \neq 4, 6$ any noncyclic homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(k)$ is conjugate to μ_k, and any noncyclic homomorphism ψ: $\mathbf{B}(6) \to \mathbf{S}(6)$ is conjugate either to μ_6 or to ν_6.* ○

Lemma 2.8. *For $k \neq 4$, any homomorphism ϕ: $\mathbf{B}(k) \to \mathbf{B}'(k)$ is integral.*

Proof. If $k = 3$, the lemma follows from Remark 0.1, since $\mathbf{B}'(3) \cong \mathbb{F}_2$. Assume that $k > 4$. Let μ': $\mathbf{B}'(k) \to \mathbf{S}(k)$ be the canonical homomorphism. Consider the composition

$$\psi = \mu' \circ \phi\colon \mathbf{B}(k) \xrightarrow{\phi} \mathbf{B}'(k) \xrightarrow{\mu'} \mathbf{S}(k).$$

Clearly, Im $\psi \subseteq$ Im $\mu' = \mathbf{A}(k)$; hence, ψ is nonsurjective and, by Lemma 2.7, cyclic. Consequently, $(\mu' \circ \phi)(\mathbf{B}'(k)) = \psi(\mathbf{B}'(k)) = \{1\}$ and $\phi(\mathbf{B}'(k)) \subseteq \operatorname{Ker} \mu' \subseteq \mathbf{I}(k)$. Since $\mathbf{B}'(k)$ is perfect, $\phi(\mathbf{B}'(k)) = \{1\}$ and ϕ is cyclic. □

Remark 2.3. If $k > 4$ and the restriction ϕ' of a homomorphism ϕ: $\mathbf{B}(k) \to G$ to $\mathbf{B}'(k)$ is Abelian, then $\phi(\mathbf{B}'(k)) = \{1\}$ (since $\mathbf{B}'(k)$ is perfect) and ϕ is cyclic. For $k = 3$ (and also for $k = 4$) this is not the case; for instance, the image of $\mathbf{B}'(3)$ under the canonical projection μ: $\mathbf{B}(3) \to \mathbf{S}(3)$ is the cyclic group $\mathbf{A}(3) \cong \mathbb{Z}/3\mathbb{Z}$. Moreover, the natural projection ϕ: $\mathbf{B}(3) \to \mathbf{B}(3)/(\mathbf{B}'(3))'$ is non-Abelian and $\phi(\mathbf{B}'(3)) \cong \mathbb{Z} \oplus \mathbb{Z}$; so, even if we assume that G is torsion free, this will not save the situation. Nevertheless, the following statement is true. ○

Lemma 2.9. *Assume that $3 \leq k \leq 4$. Let ϕ: $\mathbf{B}(k) \to G$ be a group homomorphism, and ϕ': $\mathbf{B}'(k) \to G$ be the restriction of ϕ to $\mathbf{B}'(k)$. If the homomorphism ϕ' is integral, then it is trivial and ϕ is integral.*

Proof. Assume first that $k = 3$. Let $u = \sigma_2\sigma_1^{-1}$ and $v = \sigma_1\sigma_2\sigma_1^{-2}$ be the canonical generators of the group $\mathbf{B}'(3)$ (§0.6). It is easily seen that relations $\sigma_1 u \sigma_1^{-1} = v$ and $\sigma_1 v \sigma_1^{-1} = u^{-1}v$ hold; so, Lemma 1.2 applies to the homomorphism $\psi = \phi'$ (the matrix $M = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$ has the eigenvalues $(1 \pm i\sqrt{3})/2$ and $\det M = 1$). Therefore, $\phi' = 1$ and ϕ is integral.

Now consider the case $k = 4$. Let π: $\mathbf{B}(4) \to \mathbf{B}(3)$ be the canonical epimorphism (see Remark 2.1), $\mathbf{T} = \operatorname{Ker} \pi$. Since ϕ' is integral, $\operatorname{Ker} \phi \supseteq \operatorname{Ker} \phi' \supseteq (\mathbf{B}''(4))$ (the second commutator subgroup of the group $\mathbf{B}(4)$). By Gorin-Lin Theorem (*c*), the normal subgroup $\mathbf{T}$ coincides with the intersection of the lower central series of the group $\mathbf{B}'(4)$. Hence, $\mathbf{T} \subset \mathbf{B}''(4)) \subseteq \operatorname{Ker} \phi$. Therefore, the homomorphism ϕ may be represented as the composition of the homomorphisms

$$\phi = \psi \circ \pi\colon\ \mathbf{B}(4) \xrightarrow{\pi} \mathbf{B}(4)/\mathbf{T} \cong \mathbf{B}(3) \xrightarrow{\psi} G.$$

Let ψ' be the restriction of ψ to $\mathbf{B}'(3)$. Then $\psi'(\mathbf{B}'(3)) = \psi(\pi(\mathbf{B}'(4))) = \phi'(\mathbf{B}'(4))$. Hence, the homomorphism ψ' is integral. Since for $k = 3$ the lemma is already proven, ψ is integral; therefore, the original homomorphism $\phi = \psi \circ \pi$ is integral. $\square$

Lemma 2.10. *Assume that $k \neq 4$. Let ϕ be an endomorphism of the group $\mathbf{B}(k)$ such that the composition*

$$\psi = \mu \circ \phi\colon\ \mathbf{B}(k) \xrightarrow{\phi} \mathbf{B}(k) \xrightarrow{\mu} \mathbf{S}(k)$$

of ϕ with the canonical projection μ is cyclic. Then ϕ is integral.

Proof. Let $G = \operatorname{Ker} \psi$. Then $(\mu \circ \phi)(G) = \psi(G) = \{1\}$ and $\phi(G) \subseteq \operatorname{Ker} \mu = \mathbf{I}(k)$. By Lemma 2.6, the homomorphism $\phi\,|_G\colon G \to \mathbf{I}(k)$ is integral. Since ψ is cyclic, $G = \operatorname{Ker} \psi \supseteq \mathbf{B}'(k)$; hence, the restriction ϕ' of ϕ to $\mathbf{B}'(k)$ is integral. If $k > 4$, the group $\mathbf{B}'(k)$ is perfect, which implies that ϕ' is trivial and ϕ is integral. For $k = 3$ the same conclusion follows from Lemma 2.9. $\square$

Recall that $\chi\colon \mathbf{B}(k) \to \mathbb{Z}$ denotes the canonical integral projection (§0.6).

Lemma 2.11. $\operatorname{Ker} \phi \subseteq \mathbf{B}'(k)$ *for any nontrivial endomorphism $\phi\colon \mathbf{B}(k) \to \mathbf{B}(k)$. Moreover, if $k \neq 4$ and ϕ is nonintegral, then $\phi^{-1}(\mathbf{B}'(k)) = \mathbf{B}'(k)$.*

Proof. If ϕ is integral, then $\operatorname{Ker} \phi = \mathbf{B}'(k)$ (for $\mathbf{B}(k)$ is torsion free). Assume that ϕ is nonintegral.

If $k = 4$, then there is a nontrivial homomorphism $\eta\colon \operatorname{Im} \phi \to \mathbb{Z}$ (Corollary 0.3). The composition

$$\xi = \eta \circ \phi\colon\ \mathbf{B}(4) \xrightarrow{\phi} \mathbf{B}(4) \xrightarrow{\eta} \mathbb{Z}$$

is also nontrivial; so, $\operatorname{Ker} \xi = \mathbf{B}'(4)$ and $\operatorname{Ker} \phi \subseteq \operatorname{Ker} \xi = \mathbf{B}'(4)$.

Finally, if $k \neq 4$, then Lemma 2.8 implies that the subgroup $\operatorname{Im} \phi \subseteq \mathbf{B}(k)$ is not contained in $\mathbf{B}'(k)$; therefore, the composition

$$\zeta = \chi \circ \phi\colon\ \mathbf{B}(k) \xrightarrow{\phi} \mathbf{B}(k) \xrightarrow{\chi} \mathbb{Z}$$

is nontrivial. Hence, $\mathbf{B}'(k) = \operatorname{Ker} \zeta = \operatorname{Ker}(\chi \circ \phi) = \phi^{-1}(\operatorname{Ker} \chi) = \phi^{-1}(\mathbf{B}'(k))$ and $\operatorname{Ker} \phi \subseteq \operatorname{Ker} \zeta = \mathbf{B}'(k)$. $\square$

E. Artin [Ar3] proved that *the pure braid group $\mathbf{I}(k)$ is a characteristic subgroup of the braid group $\mathbf{B}(k)$*, that is, $\phi(\mathbf{I}(k)) = \mathbf{I}(k)$ *for any automorphism ϕ of the group $\mathbf{B}(k)$*. The following theorem shows that for $k \neq 4$ the subgroup $\mathbf{I}(k)$ possesses in fact some stronger invariance properties.

Theorem 2.12. *If $k \neq 4$, then $\phi[\mathbf{I}(k)] \subseteq \mathbf{I}(k)$, $\phi^{-1}(\mathbf{I}(k)) = \mathbf{I}(k)$ and* Ker $\phi \subseteq \mathbf{J}(k)$ *for any nonintegral endomorphism ϕ: $\mathbf{B}(k) \to \mathbf{B}(k)$.*

Proof. According to Lemma 2.10, the composition

$$\psi = \mu \circ \phi\colon \mathbf{B}(k) \xrightarrow{\phi} \mathbf{B}(k) \xrightarrow{\mu} \mathbf{S}(k)$$

is noncyclic. By Lemma 2.7, ψ is surjective and hence transitive. It follows from Artin Theorem that Ker $\psi = \mathbf{I}(k)$. Thus,

$$\mathbf{I}(k) = \text{Ker } \psi = \phi^{-1}(\text{Ker } \mu) = \phi^{-1}(\mathbf{I}(k))$$

and $\phi(\mathbf{I}(k)) = \phi(\phi^{-1}(\mathbf{I}(k))) \subseteq \mathbf{I}(k)$. Moreover, Ker $\phi \subseteq$ Ker $(\mu \circ \phi) =$ Ker $\psi = \mathbf{I}(k)$. Finally, by Lemma 2.11, Ker $\phi \subseteq \mathbf{B}'(k)$, and thus Ker $\phi \subseteq \mathbf{I}(k) \cap \mathbf{B}'(k) = \mathbf{J}(k)$. □

Remark 2.4. It follows from relations (0.6), (0.1), (0.2) that

$$\begin{aligned}(\sigma_1\sigma_2) \cdot (\sigma_3\sigma_2) \cdot (\sigma_1\sigma_2) &= (\sigma_1\sigma_2\sigma_3) \cdot \sigma_2\sigma_1\sigma_2 = \sigma_3\sigma_2\sigma_3 \cdot (\sigma_1\sigma_2\sigma_3) = (\sigma_3\sigma_2) \cdot \sigma_1 \cdot (\sigma_3\sigma_2\sigma_3) \\ &= (\sigma_3\sigma_2) \cdot \sigma_1 \cdot (\sigma_2\sigma_3\sigma_2) = (\sigma_3\sigma_2) \cdot (\sigma_1\sigma_2) \cdot (\sigma_3\sigma_2).\end{aligned}$$

Therefore, we can define an endomorphism ϕ of the group $\mathbf{B}(4)$ by

$$\phi(\sigma_1) = \phi(\sigma_3) = \sigma_1\sigma_2, \qquad \phi(\sigma_2) = \sigma_3\sigma_2.$$

This endomorphism is non-Abelian (for $\phi(\sigma_1) \neq \phi(\sigma_2)$), but $\phi(\sigma_1^2) = (\sigma_1\sigma_2)^2 \notin \mathbf{I}(4)$; thus $\phi(\mathbf{I}(4)) \not\subseteq \mathbf{I}(4)$. Besides, Ker $\phi = \mathbf{T} \not\subseteq \mathbf{J}(4)$. Moreover, $\phi(\sigma_1^3) = (\sigma_1\sigma_2)^3 \in \mathbf{I}(4)$, which shows that $\sigma_1^3 \in \phi^{-1}(\mathbf{I}(4))$; but $\sigma_1^3 \notin \mathbf{I}(4)$, and therefore $\phi^{-1}(\mathbf{I}(4)) \not\subseteq \mathbf{I}(4)$. This example shows that the condition $k \neq 4$ in Theorem 2.12 is essential.

For any $k \geq 3$, there is an *integral* endomorphism ϕ: $\mathbf{B}(k) \to \mathbf{B}(k)$ with $\phi(\mathbf{I}(k)) \not\subseteq \mathbf{I}(k)$. Indeed, take $c \in \mathbf{B}(k)$ such that $c \notin \mathbf{I}(k)$ and define ϕ by $\phi(\sigma_1) = \ldots = \phi(\sigma_{k-1}) = c$. ○

The rest of this section is devoted to some results on endomorphisms of the groups $\mathbf{B}(3)$, $\mathbf{B}(4)$ and on homomorphisms from $\mathbf{B}(4)$ into $\mathbf{B}(3)$ and $\mathbf{S}(3)$.

Theorem 2.13. *Any nonintegral endomorphism ϕ of $\mathbf{B}(3)$ is an embedding.*

Proof. Since $\phi(\mathbf{B}'(3)) \subseteq \mathbf{B}'(3)$, the restriction ϕ' of ϕ to $\mathbf{B}'(3)$ may be regarded as an endomorphism of the group $\mathbf{B}'(3) \cong \mathbb{F}_2$. The image $G =$ Im ϕ' is a free group of rank $r \leq 2$. Since ϕ is nonintegral, Lemma 2.9 implies that ϕ' is nonintegral; hence, $G \cong \mathbb{F}_2$ and ϕ': $\mathbf{B}'(3) \to G$ is an isomorphism. By Lemma 2.11, Ker $\phi \subset \mathbf{B}'(3)$. Thus, Ker $\phi =$ Ker $\phi' = \{1\}$. □

Remark 2.5. For any $k \geq 2$, there exist *proper embeddings* $\mathbf{B}(k) \hookrightarrow \mathbf{B}(k)$. For $k = 2$ this is evident. If $k > 2$, take any $m \in \mathbb{Z}$ and define the endomorphism $\phi_{k,m}$ by

$$\phi_{k,m}\colon \mathbf{B}(k) \ni g \mapsto (A_k)^{m\chi(g)} \cdot g \in \mathbf{B}(k),$$

where χ: $\mathbf{B}(k) \to \mathbb{Z}$ is the canonical integral projection. For any $g \in \mathbf{B}'(k)$, we have $\chi(g) = 0$ and $\phi_{k,m}(g) = g$; particularly, $\phi_{k,m}$ is nonintegral. By Lemma 2.11, Ker $\phi_{k,m} \subseteq \mathbf{B}'(k)$; hence $\phi_{k,m}$ is an embedding. If $h = \phi_{k,m}(g) \in \operatorname{Im} \phi_{k,m}$, then

$$\chi(h) = \chi\left(\phi_{k,m}(g)\right) = \chi\left((A_k)^{m\chi(g)} \cdot g\right) = (mk(k-1)+1)\chi(g);$$

consequently, $\chi(h)$ is divisible by the number $s(m) = mk(k-1)+1$. On the other hand, if $h \in \mathbf{B}(k)$ and $\chi(h) = ts(m)$ for some $t \in \mathbb{Z}$, take $g = (A_k)^{-tm} \cdot h$; then

$$\chi(g) = -tmk(k-1) + t(mk(k-1)+1) = t,$$

and hence

$$\phi_{k,m}(g) = (A_k)^{m\chi(g)} \cdot g = (A_k)^{mt} \cdot (A_k)^{-tm} \cdot h = h.$$

It follows that the image of $\phi_{k,m}$ coincides with the normal subgroup $\chi^{-1}(s(m)\mathbb{Z}) \subseteq \mathbf{B}(k)$. If $m \neq 0$, then $s(m) \neq \pm 1$, Im $\phi_{k,m} = \chi^{-1}(s(m)\mathbb{Z}) \neq \mathbf{B}(k)$, and $\phi_{k,m}$ is a proper embedding. ○

Let π: $\mathbf{B}(4) \to \mathbf{B}(3)$ and μ: $\mathbf{B}(3) \to \mathbf{S}(3)$ be the canonical projections.

Theorem 2.14. *a) Any noncyclic homomorphism* ψ: $\mathbf{B}(4) \to \mathbf{S}(3)$ *is conjugate to the composition* $\mu \circ \pi$: $\mathbf{B}(4) \xrightarrow{\pi} \mathbf{B}(3) \xrightarrow{\mu} \mathbf{S}(3)$.

b) Let ϕ: $\mathbf{B}(4) \to \mathbf{B}(3)$ *be a nonintegral homomorphism. Then there exists a monomorphism* ξ: $\mathbf{B}(3) \to \mathbf{B}(3)$ *such that*

$$\phi = \xi \circ \pi\colon\ \mathbf{B}(4) \xrightarrow{\pi} \mathbf{B}(3) \xrightarrow{\xi} \mathbf{B}(3).$$

Particularly, Ker $\phi = \mathbf{T}$. *Moreover, if* ϕ *is surjective, then* ξ *is an automorphism of the group* $\mathbf{B}(3)$.

Proof. a) Clearly, $\psi(\mathbf{B}'(4)) \subseteq \mathbf{S}'(3) = \mathbf{A}(3) \cong \mathbb{Z}/3\mathbb{Z}$. Consequently,

$$\operatorname{Ker} \psi \supseteq (\mathbf{B}'(4))' \supset \mathbf{T} = \operatorname{Ker} \pi.$$

Therefore, there exists a homomorphism ϕ: $\mathbf{B}(3) \to \mathbf{S}(3)$ such that $\psi = \phi \circ \pi$. Since ψ is noncyclic, ϕ is noncyclic too; by Artin Theorem, ϕ is conjugate to μ. Hence, ψ is conjugate to the composition $\mu \circ \pi$.

b) Since $\phi(\mathbf{B}'(4)) \subseteq \mathbf{B}'(3)$ and $\mathbf{T} = \operatorname{Ker} \pi$ is the intersection of the lower central series of the group $\mathbf{B}'(4)$, the image $\phi(\mathbf{T})$ is contained in the intersection H of the lower central series of the group $\mathbf{B}'(3)$. But $\mathbf{B}'(3) \cong \mathbb{F}_2$, and thus $H = \{1\}$. Consequently, $\phi(\mathbf{T}) = \{1\}$ and Ker $\pi = \mathbf{T} \subseteq$ Ker ϕ. Therefore, there exists an endomorphism ξ of the group $\mathbf{B}(3)$ such that $\phi = \xi \circ \pi$. Since ϕ is nonintegral, ξ is also nonintegral; according to Theorem 2.13, ξ is injective. Hence, Ker $\phi = \pi^{-1}(\operatorname{Ker} \xi) = \pi^{-1}(\{1\}) = \operatorname{Ker} \pi = \mathbf{T}$.

Finally, if ϕ is surjective, ξ is surjective, too. Hence, ξ is an automorphism of $\mathbf{B}(3)$. □

Theorem 2.15. Ker $\phi = \mathbf{T}$ *for any nonintegral noninjective endomorphism* ϕ *of* $\mathbf{B}(4)$.

Proof. Let $\psi = \pi \circ \phi$, where $\pi\colon \mathbf{B}(4) \to \mathbf{B}(3)$ is the canonical epimorphism. The homomorphism ψ is nontrivial (for otherwise Im $\phi \subseteq$ Ker $\pi = \mathbf{T} \cong \mathbb{F}_2$ and, by Remark 0.1, ϕ is integral). Consider the following two cases: *a*) ψ is integral, and *b*) ψ is nonintegral.

a) In this case $\phi(\mathbf{B}'(4)) \subseteq$ Ker $\pi = \mathbf{T} \cong \mathbb{F}_2$, and (as in the proof of Theorem 2.14(*b*)) we obtain a homomorphism $\xi\colon \mathbf{B}(3) \to \mathbf{B}(4)$ such that $\phi = \xi \circ \pi$. Since ϕ is nonintegral, ξ is nonintegral. By Lemma 2.9, the restriction ξ' of ξ to $\mathbf{B}'(3)$ is nonintegral. It is easily seen that $\xi'(\mathbf{B}'(3)) \subseteq \mathbf{T}$. Consequently, Im $\xi' \cong \mathbb{F}_r$, where $r \leq 2$. Since ξ' is nonintegral, $r = 2$. Thus, we obtain a surjective endomorphism $\mathbb{F}_2 \cong \mathbf{B}'(3) \to$ Im $\xi' \cong \mathbb{F}_2$ of the Hopfian group $\mathbb{F}_2$; hence ξ' must be injective. On the other hand, it is easy to check that Ker $\xi \subseteq \mathbf{B}'(3)$. Thus, Ker $\xi =$ Ker ξ' and ξ is injective. Therefore,

$$\operatorname{Ker} \phi = \operatorname{Ker} (\xi \circ \pi) = \pi^{-1}(\operatorname{Ker} \xi) = \pi^{-1}(\{1\}) = \operatorname{Ker} \pi = \mathbf{T}.$$

b) $\mathbf{T}$ is a completely characteristic subgroup of the group $\mathbf{B}(4)$. Hence, $\phi(\mathbf{T}) \subseteq \mathbf{T}$. Let $\widetilde{\phi}\colon \mathbf{T} \to \mathbf{T}$ be the restriction of ϕ to $\mathbf{T}$. Since ψ is nonintegral, Theorem 2.14(*b*) implies that Ker $\psi = \mathbf{T}$; hence, Ker $\phi \subseteq \mathbf{T}$ and Ker $\phi =$ Ker $\widetilde{\phi}$. Clearly, $\widetilde{\phi}(\mathbf{T}) \cong \mathbb{F}_r$ where $r \leq 2$. If $r = 2$, then $\widetilde{\phi}$ is injective (since $\mathbf{T} \cong \mathbb{F}_2$ is Hopfian), and ϕ is injective too. Finally, if $r < 2$, then $\widetilde{\phi}$ is integral. In this case it follows from relations (0.14), (0.15) and Lemma 1.2 that the homomorphism $\widetilde{\phi}$ is trivial; hence Ker $\phi =$ Ker $\widetilde{\phi} = \mathbf{T}$. □

§3. Transitive homomorphisms $\mathbf{B}(k) \to \mathbf{S}(n)$ for small k and n

For k large enough, transitive homomorphisms $\mathbf{B}(k) \to \mathbf{S}(n)$, $k < n \le 2k$, can be studied using some general methods based mainly on Lemma 1.22, Theorem 2.1, Gorin-Lin Theorem and the techniques developed in §§4,5. However, for small k these methods do not work. For this reason, we consider the case of small k in this section.

Given a group homomorphism $\psi\colon \mathbf{B}(k) \to H$, we denote the ψ-images of the canonical generators $\sigma_1, \ldots, \sigma_{k-1}$ and of the corresponding special generators $\alpha = \sigma_1 \cdots \sigma_{k-1}$, $\beta = \alpha\sigma_1$ by $\widehat{\sigma}_1, \ldots, \widehat{\sigma}_{k-1}$ and $\widehat{\alpha}, \widehat{\beta}$, respectively. Assume that the group H is finite and the homomorphism ψ is noncyclic. Then it follows from Lemma 1.17 that $\operatorname{ord}\widehat{\beta}$ is divisible by $k-1$. Moreover, $\operatorname{ord}\widehat{\alpha}$ is divisible by k whenever $k \ne 4$; if $k = 4$, then either $\operatorname{ord}\widehat{\alpha}$ is divisible by 4 or $\widehat{\sigma}_1 = \widehat{\sigma}_3$ and $\operatorname{ord}\widehat{\alpha}$ is divisible by 2 (but not by 4). The following proposition follows immediately from these remarks.

Proposition 3.1. *Suppose $4 \le n \le 7$. Then any noncyclic transitive homomorphism $\psi\colon \mathbf{B}(3) \to \mathbf{S}(n)$ is conjugate to one of the following homomorphisms $\psi_{3,n}^{(i)}$:*

a) $n = 4$: $\psi_{3,4}^{(1)}\colon \alpha \mapsto (1,2,3)$, $\beta \mapsto (1,4)$; $\psi_{3,4}^{(2)}\colon \alpha \mapsto (1,2,3)$, $\beta \mapsto (1,2)(3,4)$ $(\operatorname{Im}\psi_{3,4}^{(1)} = \mathbf{S}(4)$, $\operatorname{Im}\psi_{3,4}^{(2)} = \mathbf{A}(4))$.

b) $n = 5$: $\psi_{3,5}\colon \alpha \mapsto (1,2,3)$, $\beta \mapsto (1,4)(2,5)$ $(\operatorname{Im}\psi_{3,5} = \mathbf{A}(5))$.

c) $n = 6$:

$$\begin{array}{llll}
\psi_{3,6}^{(1)}\colon & \alpha \mapsto (1,2,3)(4,5,6), & \beta \mapsto (1,2)(3,4)(5,6), & \sigma_1 \mapsto (2,3,6,4);\\
\psi_{3,6}^{(2)}\colon & \alpha \mapsto (1,2,3)(4,5,6), & \beta \mapsto (1,4)(2,6)(3,5), & \sigma_1 \mapsto (1,6)(2,5)(3,4);\\
\psi_{3,6}^{(3)}\colon & \alpha \mapsto (1,2,3)(4,5,6), & \beta \mapsto (1,2)(3,4), & \sigma_1 \mapsto (2,3,6,5,4);\\
\psi_{3,6}^{(4)}\colon & \alpha \mapsto (1,2,3)(4,5,6), & \beta \mapsto (1,4)(2,5), & \sigma_1 \mapsto (1,6,5)(2,4,3);\\
\psi_{3,6}^{(5)}\colon & \alpha \mapsto (1,2,3)(4,5,6), & \beta \mapsto (1,2), & \sigma_1 \mapsto (2,3)(4,6,5);\\
\psi_{3,6}^{(6)}\colon & \alpha \mapsto (1,2,3)(4,5,6), & \beta \mapsto (1,4), & \sigma_1 \mapsto (1,6,5,4,3,2);\\
\psi_{3,6}^{(7)}\colon & \alpha \mapsto (1,2,3), & \beta \mapsto (1,4)(2,5)(3,6), & \sigma_1 \mapsto (1,4,3,6,2,5).
\end{array}$$

d) $n = 7$:

$$\begin{array}{llll}
\psi_{3,7}^{(1)}\colon & \alpha \mapsto (1,2,3)(4,5,6), & \beta \mapsto (1,4)(2,7), & \sigma_1 \mapsto (1,6,5,4,3,2,7);\\
\psi_{3,7}^{(2)}\colon & \alpha \mapsto (1,2,3)(4,5,6), & \beta \mapsto (1,2)(3,4)(5,7), & \sigma_1 \mapsto (2,3,6,5,7,4);\\
\psi_{3,7}^{(3)}\colon & \alpha \mapsto (1,2,3)(4,5,6), & \beta \mapsto (1,4)(2,5)(3,7), & \sigma_1 \mapsto (1,6,5)(2,4,3,7).
\end{array}$$

□

Remark 3.1. One of the 7 homomorphisms $\psi_{3,6}^{(i)}\colon \mathbf{B}(3) \to \mathbf{S}(6)$ listed above, namely, $\psi_{3,6}^{(1)}$, appears in a way, which deserves some comments.

Take any $z = (z_1, z_2, z_3) \in \mathbb{C}^3$; let $\lambda_1, \lambda_2, \lambda_3$ be the roots of the polynomial $p_3(t, z) = t^3 + z_1 t^2 + z_2 t + z_3$. Let $P(t, w) = t^6 + w_1 t^5 + w_2 t^4 + w_3 t^3 + w_4 t^2 + w_5 t + w_6$ be the monic polynomial in t of degree 6 with the roots $\mu_1^\pm, \mu_2^\pm, \mu_3^\pm$ defined by the quadratic equations

$$\begin{aligned}(\mu_1^\pm - \lambda_1)^2 &= (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3),\\ (\mu_2^\pm - \lambda_2)^2 &= (\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1),\\ (\mu_3^\pm - \lambda_3)^2 &= (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2).\end{aligned} \tag{3.1}$$

The set of the 6 numbers μ's (taking into account possible multiplicities) is invariant under any permutation of the roots λ_i. Since the coefficients w_i are the elementary symmetric polynomials in μ's, they are polynomials in z_1, z_2, z_3. Thereby, we obtain a polynomial mapping $f\colon \mathbb{C}^3 \ni z \mapsto w \in \mathbb{C}^6$. It is easy to compute the coordinate functions of this mapping:

$$\begin{aligned}f_1(z) &= 2z_1; & f_4(z) &= 20z_1z_3 - 5z_2^2;\\ f_2(z) &= 5z_2; & f_5(z) &= 8z_1^2z_3 - 2z_1z_2^2 - 4z_2z_3;\\ f_3(z) &= 20z_3; & f_6(z) &= 4z_1z_2z_3 - z_2^3 - 8z_3^2.\end{aligned} \tag{3.2}$$

The formulas for f_1, f_2, f_3 show that f is an embedding. Computing the discriminants $D_P(w)$ and $D_{p_3}(z) = d_3(z)$ of the polynomials $P(t, w)$ and $p_3(t, z)$, respectively, we obtain the relation $D_P(w) = -4^9 \cdot [d_3(z)]^5$. Particularly, *if the polynomial $p_3(t, z)$ has no multiple roots, then the polynomial $P(t, w) = p_6(t, f(z))$ has no multiple roots.* Hence, the restriction of f to the domain $\mathbf{G}_3 = \{z \in \mathbb{C}^3 \mid d_3(z) \neq 0\}$ (see §0) defines the polynomial mapping

$$f\colon \mathbf{G}_3 \ni z \mapsto w = f(z) \in \mathbf{G}_6 = \{w \in \mathbb{C}^6 \mid d_6(w) \neq 0\}.$$

Moreover, formulas (3.1) show that *for any $z \in \mathbf{G}_3$ the polynomials $p_3(t, z)$ and $P(t, w) = p_6(t, f(z))$ have no common roots.*

On the other hand, it was proven in [L9] that *for any $k > 3$, any natural n, and any holomorphic mapping $F\colon \mathbf{G}_k \to \mathbf{G}_n$ there must be a point $z^\circ \in \mathbf{G}_k$ such that the polynomials $p_k(t, z^\circ)$ and $p_n(t, F(z^\circ))$ have common roots.* This means that the mapping $f\colon \mathbf{G}_3 \to \mathbf{G}_6$ constructed above is very exceptional.

Take $z^\circ \in \mathbf{G}_3$ and fix an isomorphism $\tau\colon \mathbf{B}(3) \xrightarrow{\cong} \pi_1(\mathbf{G}_3, z^\circ)$. Any element $s \in \mathbf{B}(3)$ produces the permutation $\widehat{s}$ of the 6 roots of the polynomial $P(t, f(z))$ along the loop in $\mathbf{G}_3$ (based at z°) representing the 3-braid s. This gives rise to a homomorphism $\mathbf{B}(3) \to \mathbf{S}(6)$; up to conjugation, this homomorphism does not depend on $z_\circ$, τ and coincides with $\psi_{3,6}^{(1)}$. Since $\psi_{3,6}^{(1)}$ is noncyclic, *the mapping f is unsplittable.* This mapping is related to a holomorphic section of the universal Teichmüller family $\mathbf{V}'(0, 4) \to \mathbf{T}(0, 4)$ over the Teichmüller space $\mathbf{T}(0, 4)$ and to elliptic functions. In fact, this is the way how f was found; however, now it is written down explicitly, and one can ask whether it may be found in a shorter way (say in some paper of the last century!). Let me also mention that the points μ's lie on the bisectors of the triangle Δ with the vertices λ's (these bisectors are well defined, even if Δ degenerates to a segment with a marked interior point), and the distance between λ_i and $\mu_i^\pm$ is the geometric mean of the corresponding legs of the triangle Δ (the latter observation is due to E. Gorin). ○

Lemma 3.2. *$\psi(\sigma_1) = \psi(\sigma_3)$ for any transitive homomorphism ψ: $\mathbf{B}(4) \to \mathbf{S}(5)$.*

Proof. Suppose $\psi(\sigma_1) \neq \psi(\sigma_3)$. Then ψ is noncyclic, 4 divides ord $\widehat{\alpha}$, and 3 divides ord $\widehat{\beta}$. Hence, $[\widehat{\alpha}] = [4]$ and $[\widehat{\beta}] = [3]$. Regarding $\mathbf{S}(5)$ as $\mathbf{S}(\{0,1,2,3,4\})$, we may assume that $\widehat{\alpha} = (0,1,2,3)$. Since ψ is transitive, $4 \in \operatorname{supp} \widehat{\beta}$; consequently, $\widehat{\beta} = (p,q,4)$, where $p,q \in \{0,1,2,3\}$ and $p \neq q$. Put $A = \widehat{\beta}\widehat{\alpha}\widehat{\beta}$ and $B = \widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}^{-3}\widehat{\beta}\widehat{\alpha}^2$. It follows from (0.9) (with $i = 2$) that $A = B$; particularly, $A(4) = B(4)$ and $A(p) = B(p)$. $A(4) = B(4)$ implies that $q = |p+1|_4$. Combining this with $A(p) = B(p)$, we obtain that $|p+2|_4 = |p+3|_4$, which is impossible. □

Proposition 3.3. *Any noncyclic transitive homomorphism ψ: $\mathbf{B}(4) \to \mathbf{S}(5)$ is conjugate to the homomorphism $\psi_{4,5}$: $\alpha \mapsto (1,4)(2,5)$, $\beta \mapsto (3,5,4)$ $(\operatorname{Im} \psi_{4,5} = \mathbf{A}(5))$. Moreover, $\psi(\sigma_1) = \psi(\sigma_3)$.*

Proof. It follows from Lemma 3.2 that the homomorphism ψ may be represented as a composition of the canonical epimorphism π: $\mathbf{B}(4) \to \mathbf{B}(3)$ (Remark 2.1) with a noncyclic transitive homomorphism $\mathbf{B}(3) \to \mathbf{S}(5)$; Proposition 3.1($b$) completes the proof. □

Our next goal is to describe all transitive homomorphisms ψ: $\mathbf{B}(4) \to \mathbf{S}(6)$ that satisfy the condition $\psi(\sigma_1) \neq \psi(\sigma_3)$. We start with some examples of such homomorphisms.

Recall that $\mathbf{S}(6)$ is the only symmetric group having outer automorphisms. Any such automorphism is conjugate to the automorphism $\varkappa$ defined by

$$\varkappa\colon \widetilde{\sigma}_1 \mapsto (1,2)(3,4)(5,6), \qquad \widetilde{\alpha} \mapsto (1,2,3)(4,5), \tag{3.3}$$

where $\widetilde{\sigma}_1 = (1,2)$ and $\widetilde{\alpha} = (1,2,3,4,5,6)$ (for instance, this can be proven using Artin Theorem).

Define the following two embeddings $\xi, \eta\colon \mathbf{S}(4) \hookrightarrow \mathbf{S}(6)$. The embedding ξ is just induced by the natural inclusion $\mathbf{\Delta}_4 = \{1,2,3,4\} \hookrightarrow \{1,2,3,4,5,6\} = \mathbf{\Delta}_6$. Further, $\mathbf{S}(4)$ may be regarded as the group of all isometries of the tetrahedron; thereby, $\mathbf{S}(4)$ acts naturally on the set $E \cong \mathbf{\Delta}_6$ consisting of the 6 edges of the tetrahedron, which defines the embedding $\eta\colon \mathbf{S}(4) \hookrightarrow \mathbf{S}(6)$. With the canonical generators $\widetilde{\sigma}_i = (i, i+1) \in \mathbf{S}(4)$, $1 \le i \le 3$, the embedding η looks as follows:

$$\eta(\widetilde{\sigma}_1) = (1,2)(3,4), \qquad \eta(\widetilde{\sigma}_2) = (2,5)(4,6), \qquad \eta(\widetilde{\sigma}_3) = (1,4)(2,3).$$

Let $\mu_4\colon \mathbf{B}(4) \to \mathbf{S}(4)$ be the canonical projection and $\nu_{4,2}$: $\mathbf{B}(4) \to \mathbf{S}(4)$ be the homomorphism described in Artin Theorem. It is easy to check that each of the compositions

$$\psi_{4,6}^{(1)} = \varkappa \circ \xi \circ \mu_4, \qquad \psi_{4,6}^{(2)} = \varkappa \circ \xi \circ \nu_{4,2}, \qquad \psi_{4,6}^{(3)} = \eta \circ \mu_4, \qquad \psi_{4,6}^{(4)} = \eta \circ \nu_{4,2}$$

defines a noncyclic transitive homomorphism $\mathbf{B}(4) \to \mathbf{S}(6)$ such that $\psi_{4,6}^{(i)}(\sigma_1) \neq \psi_{4,6}^{(i)}(\sigma_3)$ for each $i = 1,2,3,4$. These homomorphisms act on the canonical generators $\sigma_1, \sigma_2, \sigma_3 \in \mathbf{B}(4)$ as follows:

$$\begin{aligned}
&\psi_{4,6}^{(1)}\colon \sigma_1 \mapsto (1,2)(3,4)(5,6), && \sigma_2 \mapsto (1,5)(2,3)(4,6), && \sigma_3 \mapsto (1,3)(2,4)(5,6);\\
&\psi_{4,6}^{(2)}\colon \sigma_1 \mapsto (1,2,4,3), && \sigma_2 \mapsto (1,5,4,6), && \sigma_3 \mapsto (3,4,2,1);\\
&\psi_{4,6}^{(3)}\colon \sigma_1 \mapsto (1,2)(3,4), && \sigma_2 \mapsto (2,5)(4,6), && \sigma_3 \mapsto (1,4)(2,3);\\
&\psi_{4,6}^{(4)}\colon \sigma_1 \mapsto (4,3,2,1)(5,6), && \sigma_2 \mapsto (4,6,2,5)(1,3), && \sigma_3 \mapsto (1,2,3,4)(5,6).
\end{aligned} \tag{3.4}$$

We shall show that any transitive homomorphism ψ: $\mathbf{B}(4) \to \mathbf{S}(6)$ that satisfies the condition $\psi(\sigma_1) \neq \psi(\sigma_3)$ is conjugate to one of the homomorphisms $\psi_{4,6}^{(i)}$, $1 \le i \le 4$.

Lemma 3.4. *Let* ψ: $\mathbf{B}(4) \to \mathbf{S}(6)$ *be a homomorphism that satisfies* $\widehat{\sigma}_1 \neq \widehat{\sigma}_3$. *Then a)* $[\widehat{\sigma}_1] \neq [2,3]$; *b)* $[\widehat{\sigma}_1] \neq [5]$; *c)* $[\widehat{\sigma}_1] \neq [6]$; *d)* $[\widehat{\sigma}_1] \neq [3,3]$.

Proof. *a)* Assume that $\widehat{\sigma}_1 = C_2C_3$, where $\operatorname{supp} C_2 \cap \operatorname{supp} C_3 = \varnothing$, $[C_2] = [2]$, and $[C_3] = [3]$. Since $[\widehat{\sigma}_3] = [2,3]$ and $\widehat{\sigma}_1\widehat{\sigma}_3 = \widehat{\sigma}_3\widehat{\sigma}_1$, Lemma 1.4 implies that

$$\widehat{\sigma}_3 = C_2C_3^2 = (C_2C_3)^5 = \widehat{\sigma}_1^5.$$

Since ψ is noncyclic, the latter relation shows that $\widehat{\sigma}_2$ forms braid-like couples with $\widehat{\sigma}_1$ and $\widehat{\sigma}_1^5$. By Lemma 1.3 (with $q = 5$, $\nu = (q+1, 4) = 2$, and $\nu(q-1) = 2 \cdot 4 = 8$), we have $\widehat{\sigma}_1^8 = 1$, which contradicts the property $C_3 \preccurlyeq \widehat{\sigma}_1$.

b) For $[\widehat{\sigma}_1] = [5]$ Lemma 1.4 shows that $\widehat{\sigma}_3 = \widehat{\sigma}_1^q$, where $q = 3$ or $q = 4$. Hence, $\widehat{\sigma}_2$ forms braid-like couples with $\widehat{\sigma}_1$ and $\widehat{\sigma}_1^q$. By Lemma 1.3, either $\widehat{\sigma}_1^8 = 1$ or $\widehat{\sigma}_1^3 = 1$, respectively; but this is impossible.

c) Similarly, for $[\widehat{\sigma}_1] = [6]$ we obtain $\widehat{\sigma}_3 = \widehat{\sigma}_1^5$ and $\widehat{\sigma}_1^8 = 1$, which is impossible.

d) Assume that $\widehat{\sigma}_1 = BC$, where B, C are disjoint 3-cycles. Then Lemma 1.9 implies that $\widehat{\sigma}_3 = BC^2$ (cases *(ii)*, *(iii)* described in this lemma cannot occur here, because of $[\widehat{\sigma}_3] = [3,3]$). Since $\psi(\sigma_1) \neq \psi(\sigma_3)$, Lemma 1.17(*a*) shows that 4 divides $\operatorname{ord}\widehat{\alpha}$. Therefore, either $[\widehat{\alpha}] = [4]$ or $[\widehat{\alpha}] = [4,2]$. In any case, $[\widehat{\alpha}^2] = [2,2]$. It follows from relation (0.5) that $\widehat{\sigma}_3 = \widehat{\alpha}^2\widehat{\sigma}_1\widehat{\alpha}^{-2}$. So, either

$$B = \widehat{\alpha}^2B\widehat{\alpha}^{-2} \ \text{ and } \ C^2 = \widehat{\alpha}^2C\widehat{\alpha}^{-2}, \quad \text{or} \quad B = \widehat{\alpha}^2C\widehat{\alpha}^{-2} \ \text{ and } \ C^2 = \widehat{\alpha}^2B\widehat{\alpha}^{-2}.$$

However, it easy to see that this contradicts the condition $[\widehat{\alpha}^2] = [2,2]$. □

Proposition 3.5. *Any transitive homomorphism* ψ: $\mathbf{B}(4) \to \mathbf{S}(6)$ *that satisfies* $\psi(\sigma_1) \neq \psi(\sigma_3)$ *is conjugate to one of the homomorphisms* $\psi_{4,6}^{(i)}$ $(1 \le i \le 4)$ *defined by* (3.4).

Proof. Lemma 1.20 and Lemma 3.4 show that $\widehat{\sigma}_1$ has one of the following cyclic types: *a)* $[\widehat{\sigma}_1] = [2,2,2]$; *b)* $[\widehat{\sigma}_1] = [4]$; *c)* $[\widehat{\sigma}_1] = [2,2]$; *d)* $[\widehat{\sigma}_1] = [4,2]$.

a) We may assume that $\widehat{\sigma}_1 = (1,2)(3,4)(5,6)$. This permutation is odd; hence, $\widehat{\alpha} = \widehat{\sigma}_1\widehat{\sigma}_2\widehat{\sigma}_3$ is also odd. Since 4 divides $\operatorname{ord}\widehat{\alpha}$, we have $[\widehat{\alpha}] = [4]$; so, $\widehat{\alpha}$ has precisely two fixed points. These points cannot be in the same transposition entering in $\widehat{\sigma}_1$ (since $\widehat{\sigma}_1$, $\widehat{\alpha}$ generate $\operatorname{Im}\psi$ and ψ is transitive). Thus, up to a $\widehat{\sigma}_1$-admissible conjugation (i. e., a conjugation that does not change the above form of $\widehat{\sigma}_1$), we have $\operatorname{Fix}\widehat{\alpha} = \{1,4\}$. Since $\widehat{\sigma}_3 = \widehat{\alpha}^2\widehat{\sigma}_1\widehat{\alpha}^{-2}$ commutes with $\widehat{\sigma}_1$ and $\widehat{\sigma}_3 \neq \widehat{\sigma}_1$, we obtain that $\widehat{\alpha}^2 = (2,3)(5,6)$. It follows that (up to a $\widehat{\sigma}_1$-admissible conjugation) $\widehat{\alpha} = (2,5,3,6)$; so, $\widehat{\sigma}_2 = \widehat{\alpha}\widehat{\sigma}_1\widehat{\alpha}^{-1} = (1,5)(2,3)(4,6)$, $\widehat{\sigma}_3 = \widehat{\alpha}^2\widehat{\sigma}_1\widehat{\alpha}^{-2} = (1,3)(2,4)(5,6)$, and $\psi \sim \psi_{4,6}^{(1)}$.

b) Since $\widehat{\sigma}_1, \widehat{\sigma}_3$ commute but do not coincide, it follows from Lemma 1.4 that $\widehat{\sigma}_3 = (\widehat{\sigma}_1)^3$. Therefore, up to conjugation, $\widehat{\sigma}_1 = (1,2,4,3)$ and $\widehat{\sigma}_3 = (3,4,2,1)$. Since ψ is transitive, we have $\{5,6\} \subset \operatorname{supp}\widehat{\sigma}_2$. It follows from Lemma 1.12 that (up to a $\widehat{\sigma}_1, \widehat{\sigma}_3$-admissible conjugation) $\widehat{\sigma}_2 = (1,5,4,6)$. Thus, $\psi \sim \psi_{4,6}^{(2)}$.

c) We may assume that $\widehat{\sigma}_1 = (1,2)(3,4)$. It follows from Lemma 1.11 that (up to a $\widehat{\sigma}_1$-admissible conjugation) either $\widehat{\sigma}_2 = (1,2)(4,5)$ or $\widehat{\sigma}_2 = (2,5)(4,6)$. Using Lemma

1.10, Lemma 1.11, and taking into account that $\widehat{\sigma}_3 \neq \widehat{\sigma}_1$, we obtain that in the first case $\widehat{\sigma}_3 = (1,2)(5,6)$ (which contradicts the transitivity of ψ), and in the second case $\widehat{\sigma}_3 = (1,4)(2,3)$. Hence, $\psi \sim \psi_{4,6}^{(3)}$.

d) We may assume that $\widehat{\sigma}_1 = (4,3,2,1)(5,6)$. Since $\widehat{\sigma}_1, \widehat{\sigma}_3$ commute but do not coincide, Lemma 1.4 implies that $\widehat{\sigma}_3 = (1,2,3,4)(5,6)$. All $\widehat{\sigma}_i$ $(1 \leq i \leq 3)$ are even; so $\widehat{\alpha}$ is even too; since 4 divides $\operatorname{ord}\widehat{\alpha}$, we see that $[\widehat{\alpha}] = [4,2]$; hence, $[\widehat{\alpha}^2] = [2,2]$. Since ψ is transitive, the transposition $T \preccurlyeq \widehat{\alpha}$ cannot coincide with $(5,6)$. Therefore, it follows from the relation $\widehat{\sigma}_3 = \widehat{\alpha}^2\widehat{\sigma}_1\widehat{\alpha}^{-2}$ that $\widehat{\alpha}^2 \mid \{5,6\} = (5,6)$ and (up to a $\widehat{\sigma}_1, \widehat{\sigma}_3$-admissible conjugation) $\widehat{\alpha}^2 \mid \{1,2,3,4\} = (1,3)$. So, $\widehat{\alpha}^2 = (1,3)(5,6)$. Since $[\widehat{\alpha}] = [4,2]$, it follows that $\widehat{\alpha} = (2,4)(5,1,6,3)$ (up to conjugation of the above type). Thereby, $\widehat{\sigma}_2 = \widehat{\alpha}\widehat{\sigma}_1\widehat{\alpha}^{-1} = (2,5,4,6)(1,3)$ and $\psi \sim \psi_{4,6}^{(4)}$. □

Proposition 3.6. *Any noncyclic transitive homomorphism* $\psi\colon \mathbf{B}(4) \to \mathbf{S}(6)$ *is either conjugate to one of the homomorphisms* $\psi_{4,6}^{(i)}$ $(1 \leq i \leq 4)$ *defined by* (3.4) *or conjugate to one of the compositions*

$$\psi_{3,6}^{(i)} \circ \pi\colon \mathbf{B}(4) \xrightarrow{\pi} \mathbf{B}(3) \xrightarrow{\psi_{3,6}^{(i)}} \mathbf{S}(6),$$

where $\pi\colon \mathbf{B}(4) \to \mathbf{B}(3)$ *is the canonical epimorphism and* $\psi_{3,6}^{(i)}$ $(1 \leq i \leq 7)$ *are the homomorphisms exhibited in* Proposition 3.1(*c*).

Proof. Proposition 3.5 covers the case when $\psi(\sigma_1) \neq \psi(\sigma_3)$. If $\psi(\sigma_1) = \psi(\sigma_3)$, then $hat\sigma_3\widehat{\sigma}_1^{-1} = 1$, so that the element $c_1 = \sigma_3\sigma_1^{-1} \in \operatorname{Ker}\psi$ (see (0.13)). Hence, the element $w = uc_1u^{-1}$ (see (0.14)) also is in $\operatorname{Ker}\psi$. Since the kernel $\mathbf{T}$ of π is generated by c_1 and w (see Gorin-Lin Theorem (*c*) and Remark 2.1), it follows that $\operatorname{Ker}\pi = \mathbf{T} \subseteq \operatorname{Ker}\psi$. Therefore, there exists a homomorphism $\psi_{3,6}\colon \mathbf{B}(3) \to \mathbf{S}(6)$ such that $\psi = \psi_{3,6} \circ \pi$. Clearly, $\psi_{3,6}$ must be noncyclic and transitive; Proposition 3.1(*c*) completes the proof. □

Remark 3.2. The homomorphism $\psi_{4,6}^{(3)}$ is conjugate (by $(1,3,2)(5,6)$) to the homomorphism $\widetilde{\nu}_6'$ that is defined as follows. Let ν_6' be the restriction of Artin's homomorphism $\nu_6\colon \mathbf{B}(6) \to \mathbf{S}(6)$ to $\mathbf{B}'(6)$. The mapping of the generators $\sigma_i \mapsto c_i = \sigma_{i+2}\sigma_1^{-1} \in \mathbf{B}'(6)$, $i = 1,2,3$, extends to an embedding $\lambda_6'\colon \mathbf{B}(4) \hookrightarrow \mathbf{B}'(6)$ (Remark 0.4). The homomorphism $\widetilde{\nu}_6'$ is the composition of λ_6' with ν_6'. ○

Remark 3.3. The trivial embedding $\mathbf{S}(5) \hookrightarrow \mathbf{S}(6)$ is of little moment. However, its composition with the outer automorphism $\varkappa$ of $\mathbf{S}(6)$ is more interesting. This composition can be also described as follows. It is well known that $\mathbf{A}(5)$ may be regarded as the group of all rotations of the icosahedron. Particularly, $\mathbf{A}(5)$ acts on the set $LD \cong \mathbf{\Delta}_6$ of all the 6 "long diagonals" of the icosahedron. This action of $\mathbf{A}(5)$ on $\mathbf{\Delta}_6$ extends to an action of $\mathbf{S}(5)$, which leads to an embedding $\phi_{5,6}\colon \mathbf{S}(5) \to \mathbf{S}(6)$. In terms of the generators $\widetilde{\sigma}_i = (i,i+1) \in \mathbf{S}(5)$, $1 \leq i \leq 4$, it looks as follows:

$$\phi_{5,6}\colon \begin{cases} \widetilde{\sigma}_1 \mapsto (1,2)(3,4)(5,6), & \widetilde{\sigma}_2 \mapsto (1,5)(2,3)(4,6), \\ \widetilde{\sigma}_3 \mapsto (1,3)(2,4)(5,6), & \widetilde{\sigma}_4 \mapsto (1,2)(3,5)(4,6). \end{cases} \tag{3.5}$$

Of course, it is easy to check directly that these formulas indeed define a group homomorphism, which is certainly transitive and non-Abelian, and therefore faithful (since $\mathbf{S}(5)$ does not possesses proper non-Abelian quotient groups). Hence, the composition

$$\psi_{5,6} = \phi_{5,6} \circ \mu_5\colon \mathbf{B}(5) \xrightarrow{\mu_5} \mathbf{S}(5) \xrightarrow{\phi_{5,6}} \mathbf{S}(6) \tag{3.6}$$

is a noncyclic transitive homomorphism with $\operatorname{Ker} \psi_{5,6} = \mathbf{I}(5)$ and $\operatorname{Im} \psi_{5,6} \cong \mathbf{S}(5)$. It is easily seen that $\psi_{5,6}$ coincides with the composition

$$\psi_{5,6} = \nu_6 \circ j_6^5\colon \mathbf{B}(5) \overset{j_6^5}{\hookrightarrow} \mathbf{B}(6) \xrightarrow{\nu_6} \mathbf{S}(6), \tag{3.6$'$}$$

where $j_6^5\colon \mathbf{B}(5) \ni \sigma_i \mapsto \sigma_i \in \mathbf{B}(6),\ 1 \le i \le 4$, and ν_6 is Artin's homomorphism. ○

Proposition 3.9. *Any noncyclic transitive homomorphism* $\psi\colon \mathbf{B}(5) \to \mathbf{S}(6)$ *is conjugate to the homomorphism* $\psi_{5,6}$ *defined by* (3,5), (3.6) *(or by* (3.6$'$)*, which is the same). Particularly,* $\operatorname{Ker} \psi = \mathbf{I}(5)$ *and* $\operatorname{Im} \psi \cong \mathbf{S}(5)$.

Proof. Since $3 = 6/2$ is prime, Lemma 1.19(c) shows that the cyclic decomposition of $\widehat{\sigma}_1$ cannot contain a cycle of length ≥ 3. Lemma 1.20(a) excludes the case $[\widehat{\sigma}_1] = [2]$. The noncommuting permutations $\widehat{\sigma}_3$, $\widehat{\sigma}_4$ commute with $\widehat{\sigma}_1$. On the other hand, any two permutations of cyclic type $[2,2]$ supported on the same 4 points commute; therefore, Lemma 1.9 excludes the case $[\widehat{\sigma}_1] = [2,2]$. Thus, the only possible case is $[\widehat{\sigma}_1] = [2,2,2]$. Two distinct permutations of this cyclic type in $\mathbf{S}(6)$ commute if and only if they have precisely one common transposition. Therefore, without loss of generality, we may assume that

$$\widehat{\sigma}_1 = (1,2)(3,4)(5,6), \qquad \widehat{\sigma}_3 = (1,3)(2,4)(5,6). \tag{3.7}$$

By the same reason, $\widehat{\sigma}_4$ has one common transposition with $\widehat{\sigma}_1$ but not with $\widehat{\sigma}_3$; this common transposition may be either $(1,2)$ or $(3,4)$. The renumbering of the symbols $1 \rightleftarrows 3,\ 2 \rightleftarrows 4$ takes one of these cases into another and does not change the forms (3.7); so, we may assume that this common transposition is $(1,2)$. Since $\widehat{\sigma}_4$ has no common transpositions with $\widehat{\sigma}_3$, we have either $\widehat{\sigma}_4 = (1,2)(3,5)(4,6)$ or $\widehat{\sigma}_4 = (1,2)(3,6)(4,5)$. The second case can be obtain from the first one by $5 \rightleftarrows 6$, which does not change the forms (3.7); hence, we may assume that

$$\widehat{\sigma}_4 = (1,2)(3,5)(4,6). \tag{3.8}$$

An argument of the same kind shows that $\widehat{\sigma}_2$ must contain a single common transposition with $\widehat{\sigma}_4$, but not with $\widehat{\sigma}_1$ and $\widehat{\sigma}_3$. Each of the transpositions $(1,2)$, $(3,4)$, $(5,6)$, $(1,3)$, $(2,4)$ is contained in $\widehat{\sigma}_1$ or in $\widehat{\sigma}_3$; hence, either $\widehat{\sigma}_2 = (1,5)(2,3)(4,6)$ or $\widehat{\sigma}_2 = (1,4)(2,6)(3,5)$. However, the second case may be obtained from the first one by $1 \rightleftarrows 2,\ 3 \rightleftarrows 4,\ 5 \rightleftarrows 6$, which does not change the forms (3.7),(3.8). This shows that $\psi \sim \psi_{5,6}$. □

§4. Retractions of homomorphisms $\mathbf{B}(k) \to \mathbf{S}(n)$; homomorphisms and cohomology

We are interested to study homomorphisms ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ up to conjugation. In this section we develop an approach to this problem. In general terms, this approach may be described as follows.

Consider the permutations $\widehat{\sigma}_i = \psi(\sigma_i)$, $i \le i \le k-1$. All $\widehat{\sigma}_i$ with $3 \le i \le k-1$ commute with $\widehat{\sigma}_1$. Hence, for any r-cycle $C \le \widehat{\sigma}_1$ $(2 \le r \le n)$, the r-cycles $C_i' = \widehat{\sigma}_i C \widehat{\sigma}_i^{-1}$, $3 \le i \le k-1$, also enter into the cyclic decomposition of $\widehat{\sigma}_1$. Thereby, we obtain an action Ω_ψ of the braid group $\mathbf{B}(k-2)$ onto the set $\mathfrak{C}_r$ of all the r-cycles entering in the cyclic decomposition of $\widehat{\sigma}_1$, or (which is the same) the representation Ω_ψ: $\mathbf{B}(k-2) \to \mathbf{S}(\mathfrak{C}_r)$. Since $\#\mathfrak{C}_r \le n/r \le n/2$, the homomorphism Ω_ψ is "simpler" than the original homomorphism ψ. If we are lucky, we can study Ω_ψ and then obtain some information on ψ.

4.0. Components and corresponding exact sequences. We denote by $\{\psi\}$ the conjugation class of a homomorphism $\psi \in \mathrm{Hom}(\mathbf{B}(k), \mathbf{S}(n))$; that is, $\psi' \in \{\psi\}$ if and only if $\psi' \sim \psi$. Recall that for any $r \ge 2$ the r-component $\mathfrak{C} = \mathfrak{C}_r(A)$ of a permutation $A \in \mathbf{S}(n)$ is the set of all the r-cycles entering in the cyclic decomposition of A (§0.0.2). For natural numbers r, t $(2 \le r \le n,\ t \le n/r)$, we denote by $\mathrm{Hom}_{r,t}(\mathbf{B}(k), \mathbf{S}(n))$ the subset of $\mathrm{Hom}(\mathbf{B}(k), \mathbf{S}(n))$ consisting of all homomorphisms ψ that satisfy the following condition:

(!) *the permutation $\widehat{\sigma}_1 = \psi(\sigma_1) \in \mathbf{S}(n)$ has an r-component $\mathfrak{C}$ of length t.*

Let $\psi \in \mathrm{Hom}_{r,t}(\mathbf{B}(k), \mathbf{S}(n))$ and let $\mathfrak{C} = \{C_1, ..., C_t\}$ be the r-component of the permutation $\widehat{\sigma}_1 = \psi(\sigma_1)$ (so, $C_1, ..., C_t$ are disjoint r-cycles). The union

$$\Sigma = \Sigma(\mathfrak{C}) = \bigcup_{m=1}^{t} \operatorname{supp} C_m \subseteq \operatorname{supp} \widehat{\sigma}_1 \subseteq \mathbf{\Delta}_n = \{1, \dots, n\}$$

of the supports $\Sigma_m = \operatorname{supp} C_m$ of all the cycles C_m is called the *support of the r-component* $\mathfrak{C}$; we denote this set Σ also by $\operatorname{supp} \mathfrak{C}$.

A homomorphism $\psi \in \mathrm{Hom}_{r,t}(\mathbf{B}(k), \mathbf{S}(n))$ is said to be *normalized* if

$$\text{(!!)}\qquad \begin{aligned} &\Sigma = \operatorname{supp} \mathfrak{C} = \{1, 2, \dots, tr\}, \quad \mathfrak{C} = \{C_1, ..., C_t\}, \\ &C_m = ((m-1)r+1, (m-1)r+2, \dots, mr), \quad m = 1, \dots, t, \quad \text{and} \\ &\psi(\sigma_1) \mid \Sigma = C_1 \cdots C_t. \end{aligned}$$

The following two statements are evident:

Claim 1. *If $\psi \in \mathrm{Hom}_{r,t}(\mathbf{B}(k), \mathbf{S}(n))$, then $\{\psi\} \subseteq \mathrm{Hom}_{r,t}(\mathbf{B}(k), \mathbf{S}(n))$ and the class $\{\psi\}$ contains at least one normalized homomorphism. Two normalized homomorphisms $\psi, \widetilde{\psi} \in \mathrm{Hom}_{r,t}(\mathbf{B}(k), \mathbf{S}(n))$ are conjugate if and only if there is a permutation $\widetilde{s} \in \mathbf{S}(n)$ such that the set $\Sigma = \operatorname{supp} \mathfrak{C} = \{1, 2, \dots, tr\}$ is $\widetilde{s}$-invariant and*

$$\begin{aligned} &\widetilde{\psi}(b) = \widetilde{s}\psi(b)\widetilde{s}^{-1} \quad \textit{for all } b \in \mathbf{B}(k) \qquad \textit{and} \\ &\widetilde{s} \cdot C_1 \cdots C_t \cdot \widetilde{s}^{-1} = \widetilde{s} \cdot \psi(\sigma_1) \mid \Sigma \cdot \widetilde{s}^{-1} = \widetilde{\psi}(\sigma_1) \mid \Sigma = C_1 \cdots C_t. \end{aligned} \tag{4.1}$$

Claim 2. *To study homomorphisms in* $\mathrm{Hom}_{r,t}(\mathbf{B}(k), \mathbf{S}(n))$ *up to conjugation, it is sufficient to study normalized homomorphisms up to conjugation by permutations* $\widetilde{s}$ *that satisfy* (4.1). *All such permutations* $\widetilde{s}$ *form the subgroup* $\widetilde{G} \subset \mathbf{S}(n)$ *that coincides with the centralizer* $\mathbf{C}(C, \mathbf{S}(n))$ *of the element* $C = C_1 \cdots C_t$ *in* $\mathbf{S}(n)$. *This subgroup* $\widetilde{G}$ *is naturally isomorphic to the direct product* $G \times \mathbf{S}(\Sigma')$, *where* $G = \mathbf{C}(C, \mathbf{S}(\Sigma))$ *is the centralizer of the element* C *in the symmetric group* $\mathbf{S}(\Sigma) \cong \mathbf{S}(rt)$, *and* $\mathbf{S}(\Sigma')$ *is the symmetric group of all permutations of the complement* $\Sigma' = \mathbf{\Delta}_n - \Sigma$.

We denote by $H \cong (\mathbb{Z}/r\mathbb{Z})^t$ the Abelian subgroup of the symmetric group $\mathbf{S}(\Sigma)$ generated by all the r-cycles $C_1, ..., C_t$ defined in (!!); particularly, H contains the product $C = C_1 \cdots C_t$, and therefore $H \subset G$. Clearly, *H is an Abelian normal subgroup in G, and the quotient group G/H is isomorphic to the symmetric group $\mathbf{S}(\mathfrak{C}) \cong \mathbf{S}(t)$ of all permutations of the cycles $C_1, ..., C_t$.* Thereby, we obtain the exact sequence

$$1 \to H \to G \xrightarrow{\pi} \mathbf{S}(t) \to 1, \tag{4.2}$$

where π is the natural projection onto the quotient group; in fact, this projection π may be described explicitly as follows. Since any element $g \in G$ commutes with the product $C = C_1 \cdots C_t$, we have

$$C_1 \cdots C_t = g \cdot C_1 \cdots C_t \cdot g^{-1} = gC_1g^{-1} \cdots gC_tg^{-1}.$$

Since $C_1, \ldots, C_t$ and also $gC_1g^{-1}, \ldots, gC_tg^{-1}$ are disjoint r-cycles, there is a unique permutation $s = s_g \in \mathbf{S}(t)$ such that $gC_mg^{-1} = C_{s(m)}$ for all m; clearly, $\pi(g) = s_g$.

Lemma 4.1. *The exact sequence* (4.2) *splits, that is, there exists a homomorphism* ρ: $\mathbf{S}(t) \to G$ *such that* $\pi \circ \rho = \mathrm{id}_{\mathbf{S}(t)}$. *In fact, the group* G *is the semi-direct product of the groups* H *and* $\mathbf{S}(t)$ *defined by the natural action of the symmetric group* $\mathbf{S}(t)$ *on the direct product* $(\mathbb{Z}/r\mathbb{Z})^t$.

Proof. For any element $s \in \mathbf{S}(t)$, define the permutation $\rho(s) \in \mathbf{S}(\Sigma)$ by

$$\rho(s)((m-1)r + q) = (s(m) - 1)r + q \qquad (1 \le m \le t, \ \ 1 \le q \le r); \tag{4.3}$$

thereby, we obtain the homomorphism

$$\rho\colon \ \mathbf{S}(t) \ni s \mapsto \rho(s) \in \mathbf{S}(\Sigma).$$

Using the explicit forms of the cycles $C_1, ..., C_t$ (see (!!)), it is easy to check that for any $m = 1, \ldots, t$ and any $s \in \mathbf{S}(t)$

$$\rho(s)C_m\rho(s)^{-1} = C_{s(m)}, \tag{4.4}$$

which implies $\rho(s)C\rho(s)^{-1} = C$. So, $\rho(s) \in G$, and hence ρ may be considered as a homomorphism from $\mathbf{S}(t)$ to G. It follows from (4.4) and the above description of π that $\pi \circ \rho = \mathrm{id}_{\mathbf{S}(t)}$. □

From now on, *we fix the splitting homomorphism* ρ: $\mathbf{S}(t) \to G \subset \mathbf{S}(\Sigma)$ *defined by* (4.3).

4.1. Retractions of homomorphisms to components. In what follows, we fix a normalized homomorphism $\psi \in \operatorname{Hom}_{r,t}(\mathbf{B}(k), \mathbf{S}(n))$. and put $\widehat{\sigma}_i = \psi(\sigma_i)$, $1 \leq i \leq k-1$. We work with the r-component $\mathfrak{C} = \{C_1, ..., C_t\}$ of the permutation $\widehat{\sigma}_1$ keeping in mind the particular forms of the r-cycles $C_1, ..., C_t$ exhibited in condition (!!).

Relations (0.5) show that $\widehat{\sigma}_j = \widehat{\alpha}^{j-1}\widehat{\sigma}_1\widehat{\alpha}^{-(j-1)}$ for any $j = 1, \ldots, k-1$ (as usual, $\widehat{\alpha} = \widehat{\sigma}_1 \cdots \widehat{\sigma}_{k-1}$). For any natural numbers q, m such that $1 \leq q \leq k-1$ and $1 \leq m \leq t$, we put

$$C_m^{(q)} = \widehat{\alpha}^{q-1} C_m \alpha^{-(q-1)}, \qquad \mathfrak{C}^{(q)} = \{C_1^{(q)}, \ldots, C_t^{(q)}\}. \tag{4.5}$$

Clearly, $C_m^{(1)} = C_m$ for each $m = 1, \ldots t$, and $\mathfrak{C}^{(1)} = \mathfrak{C}$; moreover, the set $\mathfrak{C}^{(q)} = \{C_1^{(q)}, \ldots, C_t^{(q)}\}$ coincides with the r-component $\mathfrak{C}_r(\widehat{\sigma}_q)$ of the permutation $\widehat{\sigma}_q$, and formulas (4.5) provide the *marked identifications*

$$\mathfrak{J}_q\colon\ \mathfrak{C}_r(\widehat{\sigma}_q) = \mathfrak{C}^{(q)} \cong \mathbf{\Delta}_t \ \text{ and } \ \mathfrak{J}_q\colon\ \mathbf{S}(\mathfrak{C}_r(\widehat{\sigma}_q)) = \mathbf{S}(\mathfrak{C}^{(q)}) \cong \mathbf{S}(t). \tag{4.6}$$

In what follows, *we always have in mind these identifications.*

Put

$$\Sigma_m^{(q)} = \operatorname{supp} C_m^{(q)} = \widehat{\alpha}^{q-1}(\operatorname{supp} C_m);$$

clearly,

$$\operatorname{supp} \mathfrak{C}_r(\widehat{\sigma}_q) = \operatorname{supp} \mathfrak{C}^{(q)} = \Sigma^{(q)} = \Sigma(\mathfrak{C}^{(q)}) = \widehat{\alpha}^{q-1}(\operatorname{supp} \mathfrak{C}) = \bigcup_{m=1}^{t} \Sigma_m^{(q)} \subseteq \mathbf{\Delta}_t.$$

The set $\Sigma^{(q)}$ is invariant under all the permutations $\widehat{\sigma}_j$, $j \neq q-1, q+1$ (for each of them commutes with $\widehat{\sigma}_q$). By the same reason, for any $j \neq q-1, q+1$ and any $m = 1, \ldots, t$, the r-cycle

$$\widetilde{C}_m^{(q)} = \widehat{\sigma}_j \cdot C_m^{(q)} \cdot \widehat{\sigma}_j^{-1}$$

belongs to the cyclic decomposition of $\widehat{\sigma}_q$, and therefore $\widetilde{C}_m^{(q)}$ coincides with one of the cycles $C_1^{(q)}, \ldots, C_t^{(q)}$. Thereby, for each $j \in \{1, \ldots, k-1\}$, $j \neq q-1, q+1$, the correspondence

$$C_m^{(q)} \mapsto \widetilde{C}_m^{(q)} = \widehat{\sigma}_j \cdot C_m^{(q)} \cdot \widehat{\sigma}_j^{-1}, \qquad m = 1, \ldots, t,$$

gives rise to the permutation $\mathfrak{g}_j^{(q)} \in \mathbf{S}(\mathfrak{C}^{(q)}) \cong \mathbf{S}(t)$, and we obtain the correspondence

$$\widehat{\sigma}_j \mapsto \mathfrak{g}_j^{(q)} \in \mathbf{S}\left(\mathfrak{C}^{(q)}\right) \overset{\mathfrak{J}_q}{\cong} \mathbf{S}(t), \qquad j \in \{1, \ldots, k-1\}, \qquad j \neq q-1, q+1, \tag{4.7}$$

such that

$$\widehat{\sigma}_j \cdot C_m^{(q)} \cdot \widehat{\sigma}_j^{-1} = \mathfrak{g}_j^q \left(C_m^{(q)}\right) = C_s^{(q)}, \qquad s = \mathfrak{g}_j^q(m). \tag{4.8}$$

It is convenient to introduce some special notations for some of the above objects corresponding to the values $q = 1$ and $q = k-1$. Namely, we put

$$\begin{array}{llll} g_i = \mathfrak{g}_{i+2}^{(1)} & \text{and} \quad g_i^* = \mathfrak{g}_i^{(k-1)} & \text{for} & i = 1, \ldots, k-3, \\ C_m^* = C_m^{(k-1)} & \text{and} \quad \Sigma_m^{(k-1)} = \Sigma_m^* & \text{for} & m = 1, \ldots, t, \end{array} \tag{4.9}$$

and

$$\mathfrak{C}^* = \mathfrak{C}^{(k-1)} = \{C_1^*, ..., C_t^*\}, \quad C^* = C_1^* \cdots C_t^*, \quad \Sigma^* = \Sigma^{(k-1)} = \text{supp } \mathfrak{C}^*. \tag{4.10}$$

We should also keep in mind that

$$\begin{aligned} &C_m = C_m^{(1)}, \qquad C = C_1 \cdots C_t, \qquad \mathfrak{C} = \{C_1, ..., C_t\} = \mathfrak{C}^{(1)}, \\ &\Sigma_m = \Sigma_m^{(1)} = \text{supp } C_m, \qquad \Sigma = \Sigma^{(1)} = \text{supp } \mathfrak{C}. \end{aligned}$$

The construction of §4.0 applies also to the r-cycles $C_1^*, \ldots, C_t^*$. Namely, we denote by G^* the centralizer of the element $C^* = C_1^* \cdots C_t^*$ in $\mathbf{S}(\Sigma^*)$, and denote by $H^* \cong (\mathbb{Z}/r\mathbb{Z})^t$ the Abelian normal subgroup in G^* generated by all the r-cycles $C_1^*, ..., C_t^*$. Then $G^*/H^* \cong \mathbf{S}(\mathfrak{C}^*) \cong \mathbf{S}(t)$, and we obtain the exact sequence

$$1 \to H^* \to G^* \xrightarrow{\pi^*} \mathbf{S}(t) \to 1. \tag{4.2*}$$

The projection π^* may be described as follows. Any element $g \in G^*$ commutes with the product $C^* = C_1^* \cdots C_t^*$, and thus

$$C_1^* \cdots C_t^* = g \cdot C_1^* \cdots C_t^* \cdot g^{-1} = gC_1^*g^{-1} \cdots gC_t^*g^{-1}.$$

Since $C_1^*, \ldots, C_t^*$ and also $gC_1^*g^{-1}, \ldots, gC_t^*g^{-1}$ are disjoint r-cycles, there is a unique permutation $s^* = s_g^* \in \mathbf{S}(t)$ such that $gC_m^*g^{-1} = C_{s^*(m)}^*$ for all m; we put $\pi^*(g) = s_g^*$. The following statement follows immediately from our definitions:

Claim 3. *The conjugation by the element* $\widehat{\alpha}^{k-2} = \psi(\alpha^{k-2})$,

$$c_\psi\colon\ \mathbf{S}(n) \ni A \mapsto \widehat{\alpha}^{k-2} \cdot A \cdot \widehat{\alpha}^{-(k-2)} \in \mathbf{S}(n),$$

provides the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H & \longrightarrow & G & \xrightarrow{\pi} & \mathbf{S}(t) & \longrightarrow & 1 \\ & & \cong\Big\downarrow c_\psi & & \cong\Big\downarrow c_\psi & & \Big\downarrow \text{id} & & \\ 1 & \longrightarrow & H^* & \longrightarrow & G^* & \xrightarrow{\pi^*} & \mathbf{S}(t) & \longrightarrow & 1 \end{array} \tag{$CD[r,t;\psi]$}$$

The first line of this diagram (the exact sequence (4.2)*) is universal for all normalized homomorphisms* $\psi \in \text{Hom}_{r,t}(\mathbf{B}(k), \mathbf{S}(n))$. *However, the second line (the exact sequence* (4.2*)*) and the vertical isomorphisms* c_ψ *may depend on* ψ.

As we know, α^k is a central element in $\mathbf{B}(k)$ (see (0.8) or §0.4); this implies some useful relations between the permutations $\mathfrak{g}_j^{(q)}$ (with various q, j) defined by (4.7),(4.8).

Lemma 4.2. *a) If* $1 \le q \le k-3$ *then*

$$\mathfrak{g}_j^{(q)} = \mathfrak{g}_{j-q-1}^{(k-1)} = g_{j-q-1}^* \qquad \textit{for } q+2 \le j \le k-1. \tag{4.11}$$

b) If $3 \le q \le k-1$ *then*

$$\mathfrak{g}_j^{(q)} = \mathfrak{g}_{j+k-q+1}^{(1)} = g_{j+k-q-1} \qquad \textit{for } 1 \le j \le q-2. \tag{4.11*}$$

Particularly,

$$g_j^* = \mathfrak{g}_j^{(k-1)} = \mathfrak{g}_{j+2}^{(1)} = g_j \qquad \textit{for } 1 \le j \le k-3. \tag{4.12}$$

Proof. *a)* Take any $m \in \{1, ..., t\}$ and any q, j such that $1 \le q \le k-3$, $q+2 \le j \le k-1$, and put $s = \mathfrak{g}_j^{(q)}(m)$. It follows from (4.5) and (4.8) that

$$\begin{aligned} C_s^{(q)} = \mathfrak{g}_j^q \left(C_m^{(q)}\right) &= \widehat{\sigma}_j \cdot C_m^{(q)} \cdot \widehat{\sigma}_j^{-1} \\ &= \widehat{\sigma}_j \cdot \widehat{\alpha}^{q-1} C_m \widehat{\alpha}^{-(q-1)} \cdot \widehat{\sigma}_j^{-1} = \widehat{\sigma}_j \widehat{\alpha}^{q-1} \cdot C_m \cdot (\widehat{\sigma}_j \widehat{\alpha}^{q-1})^{-1}. \end{aligned} \tag{4.13}$$

Since $\widehat{\alpha}^k$ commutes with any element in Im ψ, we have

$$\widehat{\sigma}_j = \widehat{\alpha}^{q+1} \widehat{\sigma}_{j-q-1} \widehat{\alpha}^{-(q+1)} = \widehat{\alpha}^{-(k-q-1)} \widehat{\sigma}_{j-q-1} \widehat{\alpha}^{k-q-1},$$

and thus

$$\widehat{\sigma}_j \widehat{\alpha}^{q-1} = \widehat{\alpha}^{-(k-q-1)} \widehat{\sigma}_{j-q-1} \cdot \widehat{\alpha}^{k-2}. \tag{4.14}$$

Relations (4.13),(4.14) and (4.5) (the latter one with $q = k-1$) show that

$$\begin{aligned} C_s^{(q)} &= \widehat{\alpha}^{-(k-q-1)} \widehat{\sigma}_{j-q-1} \widehat{\alpha}^{k-2} \cdot C_m \cdot \left(\widehat{\alpha}^{-(k-q-1)} \widehat{\sigma}_{j-q-1} \widehat{\alpha}^{k-2}\right)^{-1} \\ &= \widehat{\alpha}^{-(k-q-1)} \widehat{\sigma}_{j-q-1} \cdot \widehat{\alpha}^{k-2} \cdot C_m \cdot \widehat{\alpha}^{-(k-2)} \cdot \left(\widehat{\alpha}^{-(k-q-1)} \widehat{\sigma}_{j-q-1}\right)^{-1} \\ &= \widehat{\alpha}^{-(k-q-1)} \cdot \widehat{\sigma}_{j-q-1} \cdot C_m^{(k-1)} \cdot \widehat{\sigma}_{j-q-1}^{-1} \cdot \widehat{\alpha}^{k-q-1} \end{aligned} \tag{4.15}$$

According to (4.8) (with $q = k-1$ and $j-q-1$ instead of j), we have

$$\widehat{\sigma}_{j-q-1} \cdot C_m^{(k-1)} \cdot \widehat{\sigma}_{j-q-1}^{-1} = \mathfrak{g}_{j-q-1}^{(k-1)} \left(C_m^{(k-1)}\right) = C_{s'}^{(k-1)}, \quad \text{where } s' = \mathfrak{g}_{j-q-1}^{(k-1)}(m);$$

thus, (4.15) can be written as

$$\begin{aligned} C_s^{(q)} &= \widehat{\alpha}^{-(k-q-1)} \cdot C_{s'}^{(k-1)} \cdot \widehat{\alpha}^{k-q-1} \\ &= \widehat{\alpha}^{-(k-q-1)} \cdot \widehat{\alpha}^{k-2} C_{s'} \widehat{\alpha}^{-(k-2)} \cdot \widehat{\alpha}^{k-q-1} = \widehat{\alpha}^{q-1} \cdot C_{s'} \cdot \widehat{\alpha}^{-(q-1)} = C_{s'}^{(q)}, \end{aligned} \tag{4.16}$$

where

$$s = \mathfrak{g}_j^{(q)}(m), \qquad s' = \mathfrak{g}_{j-q-1}^{(k-1)}(m).$$

It follows from (4.16) that $s = s'$, and thus $\mathfrak{g}_j^{(q)}(m) = \mathfrak{g}_{j-q-1}^{(k-1)}(m)$. The latter relation holds for any $m \in \{1, ..., t\}$, which means that $\mathfrak{g}_j^{(q)} = \mathfrak{g}_{j-q-1}^{(k-1)} = g_{j-q-1}^*$.

b) For any $q = 3, \dots, k-1$, any $j = 1, \dots, q-2$ and every $m = 1, \dots, t$ we have:

$$
\begin{aligned}
C_s^{(q)} &= \widehat{\sigma}_j \cdot C_m^{(q)} \cdot \widehat{\sigma}_j^{-1} = \widehat{\sigma}_j \widehat{\alpha}^{q-1} \cdot C_m \cdot \widehat{\alpha}^{-(q-1)} \widehat{\sigma}_j^{-1} \\
&= \widehat{\alpha}^{q-1} \widehat{\alpha}^{k-q+1} \widehat{\sigma}_j \widehat{\alpha}^{-(k-q+1)} \cdot C_m \cdot \widehat{\alpha}^{k-q+1} \widehat{\sigma}_j^{-1} \widehat{\alpha}^{-(k-q+1)} \widehat{\alpha}^{-(q-1)} \\
&= \widehat{\alpha}^{q-1} \widehat{\sigma}_{j+k-q+1} \cdot C_m \cdot \widehat{\sigma}_{j+k-q+1}^{-1} \widehat{\alpha}^{-(q-1)} = \widehat{\alpha}^{q-1} \cdot C_{s'} \cdot \widehat{\alpha}^{-(q-1)} = C_{s'}^{(q)},
\end{aligned}
$$

where $s = \mathfrak{g}_j^{(q)}(m)$ and $s' = \mathfrak{g}_{j+k-q+1}^{(1)}(m)$. Consequently, $\mathfrak{g}_j^{(q)} = \mathfrak{g}_{j+k-q+1}^{(1)} = g_{j+k-q+1}$. Using the latter relations for $q = k-1$, we obtain (4.12). □

CONSTRUCTION OF THE HOMOMORPHISM Ω. Assume that $k > 3$ and denote by $s_1, ..., s_{k-3}$ the canonical generators of the braid group $\mathbf{B}(k-2)$. Consider the homomorphisms

$$
\begin{aligned}
\Psi\colon\ & \mathbf{B}(k-2) \to \mathbf{S}(n), && \Psi(s_i) = \psi(\sigma_{i+2}) = \widehat{\sigma}_{i+2}, \\
\Psi^*\colon\ & \mathbf{B}(k-2) \to \mathbf{S}(n), && \Psi^*(s_i) = \psi(\sigma_i) = \widehat{\sigma}_i,
\end{aligned}
\qquad (i = 1, ..., k-3). \tag{4.17}
$$

According to (4.7),(4.8), we have:

$$
\begin{aligned}
\Psi(s_i) \cdot C_m \cdot \Psi(s_i)^{-1} &= \widehat{\sigma}_{i+2} \cdot C_m \cdot \widehat{\sigma}_{i+2}^{-1} = \mathfrak{g}_{i+2}^{(1)} \left(C_m^{(1)} \right) \\
&= C_s^{(1)} = C_{g_i(m)}, \qquad s = \mathfrak{g}_{i+2}^{(1)}(m) = g_i(m),
\end{aligned} \tag{4.18}
$$

$$
\begin{aligned}
\Psi^*(s_i) \cdot C_m^* \cdot \Psi^*(s_i)^{-1} &= \widehat{\sigma}_i \cdot C_m^* \cdot \widehat{\sigma}_i^{-1} = \mathfrak{g}_i^{(k-1)} \left(C_m^{(k-1)} \right) \\
&= C_s^{(k-1)} = C_{g_i^*(m)}^*, \qquad s = \mathfrak{g}_i^{(k-1)}(m) = g_i^*(m),
\end{aligned} \tag{4.18*}
$$

for any $i = 1, \dots, k-3$.

The image of Ψ is generated by the permutations $\widehat{\sigma}_j$, $3 \le j \le k-1$; since any such $\widehat{\sigma}_j$ commutes with $\widehat{\sigma}_1$, Lemma 1.4 implies that the set $\Sigma = \operatorname{supp} \mathfrak{C}$ is invariant under the subgroup $\operatorname{Im} \Psi \subseteq \mathbf{S}(n)$. Similarly, $\operatorname{Im} \Psi^*$ is generated by the permutations $\widehat{\sigma}_j$, $1 \le j \le k-3$, and the set $\Sigma^* = \operatorname{supp} \mathfrak{C}^*$ is invariant under the subgroup $\operatorname{Im} \Psi^* \subseteq \mathbf{S}(n)$.

Let

$$
\Psi_\Sigma\colon \mathbf{B}(k-2) \to \mathbf{S}(\Sigma) \quad \text{and} \quad \Psi_{\Sigma^*}^*\colon \mathbf{B}(k-2) \to \mathbf{S}(\Sigma^*) \tag{4.19}
$$

be the reductions of the homomorphisms Ψ and Ψ^* to the invariant sets Σ and Σ^*, respectively (see §0.0.2). That is,

$$
\begin{aligned}
\Psi_\Sigma(s_i) &= \Psi(s_i) \mid \Sigma = \psi(\sigma_{i+2}) \mid \Sigma = \widehat{\sigma}_{i+2} \mid \Sigma, \\
\Psi_{\Sigma^*}^*(s_i) &= \Psi^*(s_i) \mid \Sigma^* = \psi(\sigma_i) \mid \Sigma^* = \widehat{\sigma}_i \mid \Sigma^*
\end{aligned}
\qquad (i = 1, ..., k-3). \tag{4.20}
$$

It follows from (4.18),(4.18*) that

$$
\Psi_\Sigma(s_i) \cdot C_m \cdot \Psi_\Sigma(s_i)^{-1} = C_{g_i(m)} \tag{4.21}
$$

and

$$\Psi^*_{\Sigma^*}(s_i) \cdot C^*_m \cdot \Psi^*_{\Sigma^*}(s_i)^{-1} = C^*_{g^*_i(m)} \tag{4.21*}$$

for all $m = 1, \ldots, t$ and all $i = 1, \ldots, k-3$, and thus

$$\Psi_\Sigma(s_i) \cdot \mathcal{C} \cdot \Psi_\Sigma(s_i)^{-1} = \mathcal{C}, \qquad \Psi^*_{\Sigma^*}(s_i) \cdot \mathcal{C}^* \cdot \Psi^*_{\Sigma^*}(s_i)^{-1} = \mathcal{C}^*.$$

Therefore,

$$\operatorname{Im} \Psi_\Sigma \subseteq G, \qquad \operatorname{Im} \Psi^*_{\Sigma^*} \subseteq G^*,$$

which means that Ψ_Σ *and* $\Psi^*_{\Sigma^*}$ *may be regarded as homomorphisms from* $\mathbf{B}(k-2)$ *into the groups* G *and* G^*, *respectively.*

Relations (4.21), (4.21*) and the definitions of the projection π, π^* show that

$$\pi(\Psi_\Sigma(s_i)) = g_i \quad \text{and} \quad \pi^*(\Psi^*_{\Sigma^*}(s_i)) = g^*_i \quad \text{for all } i = 1, \ldots, k-3. \tag{4.22}$$

Consider the compositions

$$\Omega = \pi \circ \Psi_\Sigma\colon \mathbf{B}(k-2) \xrightarrow{\Psi_\Sigma} G \xrightarrow{\pi} \mathbf{S}(t) \tag{4.26}$$

and

$$\Omega^* = \pi^* \circ \Psi^*_{\Sigma^*}\colon \mathbf{B}(k-2) \xrightarrow{\Psi^*_{\Sigma^*}} G^* \xrightarrow{\pi^*} \mathbf{S}(t). \tag{4.26*}$$

The following simple lemma is, in fact, important for us.

Lemma 4.3. *a) The homomorphisms* $\Omega\colon \mathbf{B}(k-2) \to \mathbf{S}(t)$ *and* $\Omega^*\colon \mathbf{B}(k-2) \to \mathbf{S}(t)$ *coincide.*

b) All the permutations $\mathfrak{g}^{(q)}_j$ $(1 \le q \le k-1;\ j \ne q-1,\, q,\, q+1)$ *are conjugate to each other.*

Proof. *a)* Formulas (4.22), (4.22*) show that $\Omega(s_i) = g_i$ and $\Omega^*(s_i) = g^*_i$ for all $i = 1, \ldots, k-3$. According to Lemma 4.2(*b*) (see (4.12)), $g_i = g^*_i$ for all such i. Consequently, $\Omega = \Omega^*$.

b) Lemma 4.2 implies that for $1 \le q \le k-1$ and $j \ne q-1, q, q+1$ the permutation $\mathfrak{g}^{(q)}_j$ coincides either with some g_i or with some g^*_i. Since the latter permutations coincide with $\Omega(s_i)$ and the canonical generators s_i are conjugate to each other, all the permutations $\mathfrak{g}^{(q)}_j$ are pairwise conjugate. □

Definition 4.1. The homomorphism $\Omega\colon \mathbf{B}(k-2) \to \mathbf{S}(t)$ defined by (4.26) is called the *retraction* of the original normalized homomorphism $\psi \in \operatorname{Hom}_{r,t}(\mathbf{B}(k), \mathbf{S}(n))$ (to an r-component $\mathfrak{C}$ of $\widehat{\sigma}_1$). According to Lemma 4.3, Ω coincides with the homomorphism Ω^* defined by (4.26*); Ω^* is called the *co-retraction* of ψ. ○

Since the set $\Sigma = \operatorname{supp} \mathfrak{C}$ is $(\operatorname{Im} \Psi)$-invariant, its complement $\Sigma' = \mathbf{\Delta}_n - \Sigma$ is also $(\operatorname{Im} \Psi)$-invariant, and we can consider the reduction $\Psi_{\Sigma'}\colon \mathbf{B}(k-2) \to \mathbf{S}(\Sigma')$ of the homomorphism Ψ to Σ':

$$\Psi_{\Sigma'}(s_i) = \Psi(s_i) \mid \Sigma' = \psi(\sigma_{i+2}) \mid \Sigma' = \widehat{\sigma}_{i+2} \mid \Sigma', \qquad i = 1, \ldots, k-3. \tag{4.20$'$}$$

Lemma 4.4. *Assume that $k > 6$ and that the homomorphism ψ is noncyclic. If the homomorphism $\Psi_{\Sigma'}$ is Abelian, then the homomorphisms Ψ, Ψ_{Σ} and Ω are non-Abelian.*

Proof. If Ψ is Abelian, then it is cyclic and $\Psi(s_3) = \Psi(s_4)$ $(k-2 > 4)$. So $\psi(\sigma_5) = \psi(\sigma_6)$, which contradicts the assumption that ψ is noncyclic.

Since Σ and Σ' are disjoint, Ψ is the disjoint product of the reductions Ψ_{Σ} and $\Psi_{\Sigma'}$. Since $\Psi_{\Sigma'}$ is Abelian and we have already proved that Ψ is non-Abelian, Ψ_{Σ} must be non-Abelian.

Finally, assume that the homomorphism $\Omega = \pi \circ \Psi_{\Sigma}$ is Abelian. Then

$$(\pi \circ \Psi_{\Sigma})(\mathbf{B}'(k-2)) = \{1\}, \qquad \text{that is,} \qquad \Psi_{\Sigma}(\mathbf{B}'(k-2)) \subseteq \operatorname{Ker} \pi = H. \tag{4.27}$$

Since $k-2 > 4$, the group $\mathbf{B}'(k-2)$ is perfect. On the other hand, the group H is Abelian. Hence, (4.27) implies that $\Psi_{\Sigma}(\mathbf{B}'(k-2)) = \{1\}$; this means that the homomorphism Ψ_{Σ} is Abelian, which contradicts the statement proven above. □

The construction described above provides us with the *universal* exact sequence (4.2) with the fixed splitting ρ. This sequence and the homomorphisms Ψ_{Σ} and Ω defined by (4.20),(4.27) form the commutative diagram

$$\begin{tikzcd}
 & & \mathbf{B}(k-2) \arrow[r, equal] \arrow[d, "\Psi_{\Sigma}"'] & \mathbf{B}(k-2) \arrow[d, "\Omega"] & \\
1 \arrow[r] & H \arrow[r] & G \arrow[r, "\pi"] & \mathbf{S}(t) \arrow[r] & 1
\end{tikzcd} \tag{4.28}$$

The homomorphism Ψ_{Σ} defined by $\Psi_{\Sigma}(s_i) = \psi(\sigma_{i+2}) \mid \Sigma,\ 1 \le i \le k-3$ (see (4.17), (4.20)), keeps a lot of information on the original normalized homomorphism ψ. Hence, it seems reasonable to find out to which extent we can recover the homomorphism Ψ_{Σ} if we know the homomorphism Ω.

Remark 4.1. Let us clarify the actual nature of this problem.

We are interested to classify (as far as possible) homomorphisms $\mathbf{B}(k) \to \mathbf{S}(n)$ up to conjugation. If the permutation $\widehat{\sigma}_1$ corresponding to such a homomorphism ψ has an r-component of length t, then, without loss of generality, we may assume that ψ is normalized. So, we have diagram (4.28) corresponding to this ψ. Suppose that we can somehow find out what is the homomorphism Ψ_{Σ}. Then we know all the restrictions $\psi(\sigma_3) \mid \Sigma, \dots, \psi(\sigma_{k-1}) \mid \Sigma$. This would provide us with an essential (and in some cases even sufficient) information to determine the homomorphism ψ itself. The knowledge of all these restrictions is certainly the best possible result, which we may hope to get by studying diagram (4.28).

Unfortunately, if ψ is unknown to us, then we know neither Ω nor Ψ_{Σ} in diagram (4.28).

A reassuring circumstance is, however, that $k-2 < k$ and $t \le n/r < n$. Hence, we may suppose that we succeeded in classifying the homomorphisms $\mathbf{B}(k-2) \to \mathbf{S}(t)$ up to conjugation, meaning that we have a finite list of pairwise nonconjugate homomorphisms Ω_p: $\mathbf{B}(k-2) \to \mathbf{S}(t)$ $(p = 1, \dots, N)$ such that any $\Omega \in \operatorname{Hom}(\mathbf{B}(k-2), \mathbf{S}(t))$ is conjugate to one of Ω_p's. Moreover, suppose that for each Ω_p we have classified up to conjugation the homomorphisms $\varphi\colon B(k-2) \to G$ satisfying the commutativity condition $\pi \circ \varphi = \Omega_p$. If so, then for any $p = 1, \ldots, N$ we have a finite list $\{\varphi_{p,q_p} \mid 1 \le q_p \le M_p\}$ of the pairwise nonconjugate representatives, and any $\varphi \in \mathrm{Hom}(B(k-2), G)$ that satisfies $\pi \circ \varphi = \Omega_p$ is conjugate to one of φ_{p,q_p}.

Further, let Ψ_Σ and Ω be the homomorphisms related to our (unknown) normalized homomorphism $\psi \in \mathrm{Hom}_{r,t}(\mathbf{B}(k), \mathbf{S}(n))$. Then $\Omega = s\Omega_p s^{-1}$ for some p and some $s \in \mathbf{S}(t)$. Using the splitting ρ, define the homomorphism $\varphi\colon B(k-2) \to G$ by $\varphi = \rho(s^{-1})\Psi_\Sigma\rho(s)$. It is easily seen that there are an element $g \in G$ and an index q $(1 \le q \le M_p)$ such that $\varphi = g\varphi_{p,q}g^{-1}$. Since the element $\widetilde{g} = \rho(s)g \in G \subset \mathbf{S}(\Sigma) \subseteq \mathbf{S}(n)$, we can define the homomorphism

$$\widetilde{\psi}\colon \mathbf{B}(k) \to \mathbf{S}(n), \qquad \widetilde{\psi} = \widetilde{g}^{-1}\psi\widetilde{g} = g^{-1}\rho(s^{-1}) \cdot \psi \cdot \rho(s)g.$$

The condition $\widetilde{g} \in G$ means that $\widetilde{g}C\widetilde{g}^{-1} = C$; therefore,

$\widetilde{\psi}$ is a normalized homomorphism in $\mathrm{Hom}_{r,t}(\mathbf{B}(k), \mathbf{S}(n))$ conjugate to our original homomorphism ψ.

Let $\widetilde{\Psi}_\Sigma$ and $\widetilde{\Omega}$ be the homomorphisms related to this homomorphism $\widetilde{\psi}$; then $\pi \circ \widetilde{\Psi}_\Sigma = \widetilde{\Omega}$. The set $\Sigma = \mathrm{supp}\; C$ is $\widetilde{\psi}(\sigma_{i+2})$-invariant (for any i, $1 \le i \le k-2$), and (by definition) the permutation $\widetilde{\Psi}_\Sigma(s_i)$ coincides with the permutation

$$\begin{aligned}\widetilde{\psi}(\sigma_{i+2}) \mid \Sigma = \widetilde{g}^{-1} \cdot (\psi(\sigma_{i+2}) \mid \Sigma) \cdot \widetilde{g} &= g^{-1}\rho(s^{-1}) \cdot \Psi_\Sigma(s_i) \cdot \rho(s)g \\ &= g^{-1} \cdot \varphi(s_i) \cdot g = g^{-1}g \cdot \varphi_{p,q}(s_i) \cdot g^{-1}g = \varphi_{p,q}(s_i),\end{aligned}$$

which means that $\widetilde{\psi}(\sigma_{i+2}) \mid \Sigma = \varphi_{p,q}(s_i)$ and $\widetilde{\Psi}_\Sigma = \varphi_{p,q}$. These observations lead to the following

Declaration. *Suppose that we solved the above mentioned classification problems for homomorphisms $\mathbf{B}(k-2) \to \mathbf{S}(t)$ and $\mathbf{B}(k-2) \to G$. Hence, we have the list of representatives $\{\varphi_{p,q}\}$. Then, without loss of generality, we may assume that the homomorphism ψ (which we want to identify up to conjugation), besides the normalization condition (!!), satisfies for some p, q the condition*

$$\text{(!!!)} \qquad \psi(\sigma_{i+2}) \mid \Sigma = \varphi_{p,q}(s_i) \qquad \textit{for all } i,\ 1 \le i \le k-3. \qquad \bigcirc$$

We have almost nothing to say about the first classification problem. If fact, this is the same problem which we started with, but rather easier (since $t \le n/2$); in some cases it can be solved, indeed. For instance, if $k \ne 6$ and $n < 2k-4$ then $t < k-2$ and any homomorphism $\Omega\colon \mathbf{B}(k-2) \to \mathbf{S}(t)$ is cyclic (Theorem 2.1(a)); this puts a strict restriction to the original homomorphism ψ.

As to the second problem, it is as follows:

Problem. *Given exact sequence* (4.2) *and a homomorphism*

$$\Omega\colon \mathbf{B}(k-2) \to \mathbf{S}(t),$$

find (up to conjugation) all the homomorphisms $\varphi\colon \mathbf{B}(k-2) \to G$ that satisfy the commutativity relation $\pi \circ \varphi = \Omega$.

We postpone the study of this problem to §5, since we need first to develop an adequate tool; the next subsection is devoted to this task.

4.2. Homomorphisms and cohomology. In this section we consider a diagram of the form

$$\begin{tikzcd} & & B \arrow[r, equal] & B \arrow[d, "\Omega"] & \\ 1 \arrow[r] & H \arrow[r] & G \arrow[r, "\pi"] & S \arrow[r] & 1 \end{tikzcd} \tag{4.29}$$

where all the groups and all the homomorphisms are given, and the second horizontal line is an exact sequence with some fixed splitting homomorphism

$$\rho\colon\ S \to G, \qquad \pi \circ \rho = \mathrm{id}_S\,. \tag{4.30}$$

Moreover, we assume that H *is an Abelian group* and identify this group with its image under the given embedding $H \hookrightarrow G$.

Definition 4.2. A homomorphism $\varphi\colon\ B \to G$ is said to be an *Ω-homomorphism*, if $\pi \circ \varphi = \Omega$. The set of all Ω-homomorphisms is denoted by $\mathrm{Hom}_\Omega(B, G)$. ○

We consider the composition

$$\varepsilon = \varepsilon_\Omega = \rho \circ \Omega\colon\ B \xrightarrow{\Omega} S \xrightarrow{\rho} G, \qquad \pi \circ \varepsilon = \Omega, \tag{4.31}$$

and define the left actions τ and $T = T_\Omega$ of the groups S and B, respectively, on the group H by

$$\tau_s(h) = \rho(s) \cdot h \cdot \rho(s), \qquad T_b(h) = \tau_{\Omega(b)}(h) = \varepsilon(b) \cdot h \cdot \varepsilon(b^{-1}). \tag{4.32}$$

Note that if the exact sequence

$$1 \to H \xrightarrow{\pi} S \to 1$$

(with the fixed splitting homomorphism ρ) in diagram (4.29) is given, then the homomorphism $\varepsilon = \varepsilon_\Omega$ and the action $T = T_\Omega$ defined by (4.31),(4.32) are determined by the homomorphism Ω. Therefore, in our notation of the groups and homomorphisms related to the corresponding cohomology, we use the sign of the homomorphism Ω instead of the traditional usage of the sign of an action.

A mapping $z\colon\ B \to H$ with $z(1) = 1$ is called a 1-*cochain on B with values in H*. A 1-cochain z is a 1-*cocycle* if its 1-*coboundary* $\delta^1_\Omega z\colon\ B \times B \to H$ is trivial, that is, if

$$\left(\delta^1_\Omega z\right)(b_1, b_2) \stackrel{\mathrm{def}}{=} [T_{b_1} z(b_2)] \cdot [z(b_1 b_2)]^{-1} \cdot z(b_1) = 1 \qquad \text{for all } \ b_1, b_2 \in B. \tag{4.33}$$

The group of all 1-cocycles is denoted by $\mathcal{Z}^1_\Omega(B, H)$. The subgroup $\mathcal{B}^1_\Omega(B, H) \subseteq \mathcal{Z}^1_\Omega(B, H)$ consists of all 0-*coboundaries*, that is, a 1-cocycle $z\colon\ B \to H$ belongs to $\mathcal{B}^1_\Omega(B, H)$ if and only if

$$\textit{there is an element } h \in H \textit{ such that } z(b) = \left(\delta^0_\Omega h\right)(b) \stackrel{\mathrm{def}}{=} (T_b h) \cdot h^{-1}\ \forall\ b \in B. \tag{4.34}$$

The cohomology group $H^1_\Omega(B, H)$ is defined by

$$H^1_\Omega(B, H) = \mathcal{Z}^1_\Omega(B, H)/\mathcal{B}^1_\Omega(B, H).$$

For any Ω-homomorphism $\varphi\colon B \to G$, define the mapping

$$z_\varphi\colon B \to G, \qquad z_\varphi(b) = \varphi(b)\varepsilon(b^{-1}), \tag{4.35}$$

and vice versa, for any 1-cocycle $z \in \mathcal{Z}^1_\Omega(B,H)$, define the mapping

$$\varphi_z\colon B \to G, \qquad \varphi_z(b) = z(b)\varepsilon(b). \tag{4.36}$$

The following simple lemma seems very well known; however, I could not find it in standard textbooks in homological algebra.

Lemma 4.5. *a) The mapping $z_\varphi\colon B \to G$ defined by* (4.35) *is, in fact, a 1-cocycle of the group B with values in H.*

b) The mapping $\varphi_z\colon B \to G$ defined by (4.36) *is an Ω-homomorphism, and, besides, the 1-cocycle z_φ corresponding to this Ω-homomorphism $\varphi = \varphi_z$ (via statement* (a)*) coincides with the original 1-cocycle z.*

Thereby, formulas (4.35), (4.36) *define the two (mutually inverse) one-to-one correspondences*

$$\mathcal{Z}^1_\Omega(B,H) \ni z \mapsto \varphi_z \in \operatorname{Hom}_\Omega(B,G), \quad \operatorname{Hom}_\Omega(B,G) \ni \varphi \mapsto z_\varphi \in \mathcal{Z}^1_\Omega(B,H). \tag{4.37}$$

Proof. *a)* Since $\pi\circ\varphi = \Omega = \pi\circ\varepsilon$, we have $\pi\left(z_\varphi(b)\right) = \pi(\varphi(b))\,(\pi\circ\varepsilon)\left(b^{-1}\right) = \Omega(b)\Omega\left(b^{-1}\right) = 1$, and thus $z_\varphi(b) \in \operatorname{Ker}\pi = H$; moreover, $z_\varphi(1) = \varphi(1)\varepsilon(1) = 1$. So, we can regard z_φ as a 1-cochain of the group B with values in H. Further,

$$\begin{aligned}
\left(\delta^1_\Omega z_\varphi\right)(b_1,\, b_2) &= \left[T_{b_1} z_\varphi\left(b_2\right)\right]\cdot\left[z_\varphi\left(b_1 b_2\right)\right]^{-1}\cdot z_\varphi(b_1)\\
&= T_{b_1}\left[\varphi(b_2)\cdot\varepsilon\left(b_2^{-1}\right)\right]\times\left[\varphi\left(b_1 b_2\right)\cdot\varepsilon\left(\left(b_1 b_2\right)^{-1}\right)\right]^{-1}\times\left[\varphi\left(b_1\right)\cdot\varepsilon\left(b_1^{-1}\right)\right]\\
&= \varepsilon\left(b_1\right)\cdot\varphi\left(b_2\right)\cdot\varepsilon\left(b_2^{-1}\right)\cdot\varepsilon\left(b_1^{-1}\right)\times\left[\varphi\left(b_1 b_2\right)\cdot\varepsilon\left(\left(b_1 b_2\right)^{-1}\right)\right]^{-1}\times\left[\varphi(b_1)\cdot\varepsilon\left(b_1^{-1}\right)\right]\\
&= \varepsilon(b_1)\cdot\varphi(b_2)\cdot\varepsilon\left(b_2^{-1}\right)\cdot\varepsilon\left(b_1^{-1}\right)\cdot\varepsilon(b_1)\cdot\varepsilon(b_2)\cdot\varphi\left(b_2^{-1}\right)\cdot\varphi\left(b_1^{-1}\right)\cdot\varphi(b_1)\cdot\varepsilon\left(b_1^{-1}\right) = 1,
\end{aligned}$$

which shows that z_φ is a 1-cocycle.

b) Since z is a 1-cocycle, $\left(\delta^1_\Omega z\right)(b_1,\, b_2) = 1$ for all $b_1, b_2 \in B$, which means that $\varepsilon(b_1)z(b_2)\varepsilon\left(b_1^{-1}\right)\cdot\left[z(b_1 b_2)\right]^{-1}\cdot z(b_1) = 1$. Since H is Abelian, the latter relation may be written as

$$z(b_1 b_2) = z(b_1)\varepsilon(b_1)z(b_2)\varepsilon\left(b_1^{-1}\right);$$

hence,

$$\begin{aligned}
\varphi_z(b_1 b_2) &= z(b_1 b_2)\varepsilon(b_1 b_2)\\
&= z(b_1)\varepsilon(b_1)z(b_2)\varepsilon\left(b_1^{-1}\right)\varepsilon(b_1 b_2) = z(b_1)\varepsilon(b_1)z(b_2)\varepsilon(b_2) = \varphi_z(b_1)\varphi_z(b_2),
\end{aligned}$$

which shows that $\varphi_z\colon B \to G$ is a group homomorphism. Moreover, $z(b) \in H = \operatorname{Ker} \pi$ for any $b \in B$, and $\pi \circ \varepsilon = \Omega$; thus, $\pi(\varphi_z(b)) = \pi(z(b)\varepsilon(b)) = \pi(z(b))\pi(\varepsilon(b)) = \Omega(b)$ and φ_z is an Ω-homomorphism. Finally, applying (4.35) to the Ω-homomorphism $\varphi = \varphi_z$ and using (4.36), we have

$$z_\varphi(b) = \varphi(b)\varepsilon\left(b^{-1}\right) = \varphi_z(b)\varepsilon\left(b^{-1}\right) = z(b)\varepsilon(b) \cdot \varepsilon\left(b^{-1}\right) = z(b),$$

which concludes the proof. $\square$

Our immediate goal is to study Ω-homomorphisms $B \to G$ up to conjugation. In view of the previous lemma, it is useful to find out the binary relation in $\mathcal{Z}^1_\Omega(B,H)$ corresponding to the conjugacy relation "$\sim$" for Ω-homomorphisms. The optimistic expectation that the equivalent cycles must be in the same cohomology class is not very far from the truth. Actually, it is so under some simple and soft additional restriction on Ω.

Definition 4.3. Two Ω-homomorphisms $\varphi_1, \varphi_2\colon B \to G$ are called *H-conjugate*, if there exists an element $h \in H$ such that

$$\varphi_2(b) = h \cdot \varphi_1(b) \cdot h^{-1} \tag{4.38}$$

for all $b \in B$. If the latter condition holds, we write $\varphi_1 \approx \varphi_2$. $\bigcirc$

Clearly, $\approx$ *is an equivalence relation on the set* $\operatorname{Hom}_\Omega(B,G)$, *which is stronger than the usual conjugacy relation* $\sim$ (that is, $\varphi_1 \approx \varphi_2$ implies $\varphi_1 \sim \varphi_2$). There is the following evident inclusion involving the centralizers: $\pi\{\mathbf{C}(\pi^{-1}(\operatorname{Im}\Omega), G)\} \subseteq \mathbf{C}(\operatorname{Im}\Omega, S)$. In general, this inclusion may be strict; however, if

(i) $\mathbf{C}(\operatorname{Im}\Omega, S) = \{1\}$, that is, *the centralizer of the subgroup* $\Omega(B) \subseteq S$ *in* S *is trivial* (for instance, Ω *is surjective, and the center of* S *is trivial*)

or the group G is Abelian, then we have

$$\pi\{\mathbf{C}(\pi^{-1}(\operatorname{Im}\Omega), G)\} = \mathbf{C}(\operatorname{Im}\Omega, S). \tag{4.39}$$

Proposition 4.6. *Let $\varphi_1, \varphi_2\colon B \to G$ be two Ω-homomorphisms, and let $z_1 = z_{\varphi_1}$, $z_2 = z_{\varphi_2}$ be the corresponding 1-cocycles.*

a) The relation $\varphi_1 \approx \varphi_2$ holds if and only if $z_1 z_2^{-1} \in \mathcal{B}^1_\Omega(B,H)$. Thus, the set of the $\approx$-equivalence classes of Ω-homomorphisms is in natural one-to-one correspondence with the cohomology group $H^1_\Omega(B,H)$.

b) Assume that condition (4.39) is held. Then the relations $\varphi_1 \sim \varphi_2$ and $\varphi_1 \approx \varphi_2$ are equivalent, and the set of the classes of conjugate Ω-homomorphisms $B \to G$ is in natural one-to-one correspondence with the cohomology group $H^1_\Omega(B,H)$.

Proof. a) First, assume that $\varphi_1 \approx \varphi_2$; so, for some $h \in H$, we have

$$\varphi_2(b) = h \cdot \varphi_1(b) \cdot h^{-1} \qquad \text{for all } b \in B.$$

According to (4.35), $z_1(b) = \varphi_1(b)\varepsilon(b^{-1})$, $z_2(b) = h \cdot \varphi_1(b) \cdot h^{-1} \cdot \varepsilon(b^{-1})$. Consequently,

$$\begin{aligned} z_1(b)(z_2(b))^{-1} &= \left[\varphi_1(b) \cdot \varepsilon\left(b^{-1}\right)\right] \cdot \left[h \cdot \varphi_1(b) \cdot h^{-1} \cdot \varepsilon\left(b^{-1}\right)\right]^{-1} \\ &= \left[\varphi_1(b) \cdot \varepsilon\left(b^{-1}\right)\right] \cdot \left[\varepsilon(b) \cdot h \cdot \varphi_1\left(b^{-1}\right) \cdot h^{-1}\right] \\ &= \left[\varphi_1(b) \cdot \varepsilon\left(b^{-1}\right)\right] \cdot \left[\varepsilon(b) \cdot h \cdot \varepsilon\left(b^{-1}\right)\right] \cdot \left[\varepsilon(b) \cdot \varphi_1\left(b^{-1}\right)\right] \cdot \left[h^{-1}\right]. \end{aligned} \tag{4.40}$$

The four expressions in the brackets in the third line of (4.40) belong to the *Abelian* normal subgroup $H \subset G$, and the first and the third of them are mutually inverse; hence,

$$z_1(b)(z_2(b))^{-1} = \left[\varepsilon(b) \cdot h \cdot \varepsilon\left(b^{-1}\right)\right] \cdot h^{-1} = (T_b h) \cdot h^{-1} = \left(\delta_\Omega^0 h\right)(b) \qquad \text{for all } b \in B,$$

and $z_1 z_2^{-1} \in \mathcal{B}_\Omega^1(B, H)$.

Now, let $z_1 z_2^{-1} \in \mathcal{B}_\Omega^1(B, H)$. Then there is an element $h \in H$ such that for all $b \in B$

$$z_1(b) \cdot [z_2(b)]^{-1} = \left(\delta_\Omega^0 h\right)(b) = (T_b h) \cdot h^{-1} = \left[\varepsilon(b) \cdot h \cdot \varepsilon\left(b^{-1}\right)\right] \cdot h^{-1}. \tag{4.41}$$

By (4.36), $\varphi_j(b) = z_j(b)\varepsilon(b)$, $j = 1, 2$. Using (4.41) and commutativity of H, we have

$$\begin{aligned} \varphi_2(b) \cdot \left[h \cdot \varphi_1\left(b^{-1}\right) \cdot h^{-1}\right] &= [z_2(b)\varepsilon(b)] \cdot \left[h \cdot \varepsilon\left(b^{-1}\right)(z_1(b))^{-1} \cdot h^{-1}\right] \\ &= z_2(b) \cdot \left[\varepsilon(b) \cdot h \cdot \varepsilon\left(b^{-1}\right)\right] \cdot (z_1(b))^{-1} \cdot h^{-1} \\ &= \left[z_2(b)\,(z_1(b))^{-1}\right] \cdot \left[\varepsilon(b) \cdot h \cdot \varepsilon\left(b^{-1}\right) \cdot h^{-1}\right] \\ &= z_2(b)(z_1(b))^{-1} \cdot z_1(b)(z_2(b))^{-1} = 1; \end{aligned}$$

so, $\varphi_2(b) = h \cdot \varphi_1(b) \cdot h^{-1}$ for all $b \in B$ and $\varphi_1 \approx \varphi_2$.

b) In view of (*a*), we should only prove that (under condition (4.39)) $\varphi_1 \sim \varphi_2$ implies $\varphi_1 \approx \varphi_2$. The relation $\varphi_1 \sim \varphi_2$ means that there exists an element $g \in G$ such that $\varphi_2(b) = g \cdot \varphi_1(b) \cdot g^{-1}$ for all $b \in B$. Since φ_1 and φ_2 are Ω-homomorphisms, we have

$$\Omega(b) = (\pi \circ \varphi_2)(b) = \pi\left[g \cdot \varphi_1(b) \cdot g^{-1}\right] = \pi(g) \cdot (\pi \circ \varphi_1)(b) \cdot \pi(g^{-1}) = \pi(g) \cdot \Omega(b) \cdot \pi(g^{-1})$$

for all $b \in B$; thus, $\pi(g) \in \mathbf{C}(\mathrm{Im}\,\Omega, S)$. It follows from (4.39) that there is an element $\widetilde{g} \in \mathbf{C}(\pi^{-1}(\mathrm{Im}\,\Omega), G)$ such that $\pi(\widetilde{g}) = \pi(g)$; clearly, the element $h = g\widetilde{g}^{-1}$ is in H. The element $\widetilde{g}$ commutes with any element of the subgroup $\pi^{-1}(\mathrm{Im}\,\Omega)$. This subgroup contains the image of any Ω-homomorphism $B \to G$; hence, $\widetilde{g}$ commutes with all the elements $\varphi_1(b)$, $b \in B$, and

$$\varphi_2(b) = g \cdot \varphi_1(b) \cdot g^{-1} = h\widetilde{g} \cdot \varphi_1(b) \cdot (h\widetilde{g})^{-1} = h \cdot \varphi_1(b) \cdot h^{-1}.$$

This shows that $\varphi_1 \approx \varphi_2$. □

Remark 4.2. If we replace Ω by a conjugate homomorphism Ω', $\Omega'(b) = s\Omega(b)s^{-1}$, and define the corresponding $\varepsilon' = \varepsilon_{\Omega'}$ and $T' = T_{\Omega'}$ according to (4.31),(4.32), then we have the bijection

$$\mathcal{Z}^1_\Omega(B, H) \ni z \mapsto z' \in \mathcal{Z}^1_{\Omega'}(B, H), \qquad z'(b) = \rho(s)z(b)\rho(s^{-1}),$$

which induces an isomorphism of the cohomology groups $H^1_\Omega(B, H) \xrightarrow{\cong} H^1_{\Omega'}(B, H)$. We have also the bijection

$$\mathrm{Hom}_\Omega(B, G) \ni \varphi \mapsto \varphi' \in \mathrm{Hom}_{\Omega'}(B, G), \qquad \varphi'(b) = \rho(s)\varphi(b)\rho(s^{-1}),$$

which is compatible with the equivalence relations $\approx$, $\approx'$. The matching between cocycles and (Ω- or Ω'-) homomorphisms defined by (4.35),(4.36) is also compatible with the above bijections. Moreover, if Ω satisfies (4.39), then Ω' does as well. Combined with Remark 4.1, this shows that in our problem we may freely pass from a homomorphism Ω to a conjugate one. ○

Remark 4.3. Even if condition (4.39) is not held, we may compute the cohomology group $H^1_\Omega(B, H)$, choose some 1-cocycle $z_{\mathcal{H}}$ in each cohomology class $\mathcal{H}$, and then take the corresponding Ω-homomorphisms $\varphi_{\mathcal{H}} = \varphi_{z_{\mathcal{H}}}$. The homomorphisms $\varphi_{\mathcal{H}}$ corresponding to distinct cohomology classes $\mathcal{H}$ cannot be H-conjugate; but some of them can be conjugate (by means of an element in G). Even if this happens, the homomorphisms $\varphi_{\mathcal{H}}$, $\mathcal{H} \in H^1_\Omega(B, H)$, form a *complete system* of Ω-homomorphisms $B \to G$, meaning that any Ω-homomorphisms φ: $B \to G$ is conjugate to some of $\varphi_{\mathcal{H}}$. Such a system $\{\varphi_{\mathcal{H}}\}$ provides us with a solution of our problem (maybe, not with "the best" one; see Remark 4.1). Anyway, this procedure reduces our nonlinear classification problem to computation of cohomology, which is, in a sense, a linear algebra problem. ○

§5. Ω-HOMOMORPHISMS AND COHOMOLOGY: SOME COMPUTATIONS

In this section, we study some particular diagrams of the form (4.29) and compute the corresponding cohomology and Ω-homomorphisms. Namely, we consider a diagram

$$\begin{array}{ccccccccc} & & & & \mathbf{B}(n) & = & \mathbf{B}(n) & & \\ & & & & & & \downarrow{\scriptstyle\Omega} & & \\ 1 & \longrightarrow & H & \longrightarrow & G & \xrightarrow{\pi} & \mathbf{S}(t) & \longrightarrow & 1, \end{array} \tag{5.1}$$

where Ω: $\mathbf{B}(n) \to \mathbf{S}(t)$ is some given homomorphism of the braid group $\mathbf{B}(n)$ into the symmetric group $\mathbf{S}(t)$ $(n, t \geq 2)$. We fix some Abelian group A, which is written as *additive*, and assume that H is the direct sum of t copies of A:

$$H = A^{\oplus t} = \bigoplus_{j=1}^{t} A. \tag{5.2}$$

We denote elements of the group H by bold letters (say $\mathbf{h}$) and regard them as "vectors" with t "coordinates" in A: $\mathbf{h} = (a^1, ..., a^t) \in H$, $a^1, ..., a^t \in A$. We consider the *standard left action* τ of the symmetric group $\mathbf{S}(t)$ on this group H. Namely, for any element $\mathbf{h} = (a^1, ..., a^t) \in H$ and any $s \in \mathbf{S}(t)$, we put

$$\tau_s \mathbf{h} = (a^{s^{-1}(1)}, \dots, a^{s^{-1}(t)}). \tag{5.3}$$

The group G is assumed to be the *semidirect product* $H \rtimes_\tau \mathbf{S}(t)$ of the groups H and $\mathbf{S}(t)$ corresponding to the action τ. That is, G is the set of all pairs $(\mathbf{h}, s)$, $\mathbf{h} \in H$, $s \in \mathbf{S}(t)$, with the multiplication

$$(\mathbf{h}, s) \cdot (\mathbf{h}', s') = (\mathbf{h} + \tau_s \mathbf{h}', s \cdot s'). \tag{5.4}$$

The injection $\mathbf{j}$: $H \hookrightarrow G$, the projection π: $G \to \mathbf{S}(t)$, the splitting homomorphism ρ: $\mathbf{S}(t) \to G$, and the homomorphism ε: $\mathbf{B}(n) \to G$ are defined as follows:

$$\mathbf{j}(\mathbf{h}) = (\mathbf{h}, 1), \quad \pi(\mathbf{h}, s) = s, \quad \rho(s) = (\mathbf{0}, s), \quad \varepsilon(b) = (\rho \circ \Omega)(b) = (\mathbf{0}, \Omega(b)). \tag{5.5}$$

We identify any element $\mathbf{h} \in H$ with its image $\mathbf{j}(\mathbf{h}) = (\mathbf{h}, 1) \in G$ (however, we must remember that the group H is additive, and G is multiplicative). The left action T of the group $\mathbf{B}(n)$ on the group H is defined by the given homomorphism Ω and the action τ:

$$T_b \mathbf{h} = \varepsilon(b)(\mathbf{h}, 1)\varepsilon(b^{-1}) = (\mathbf{0}, \Omega(b))(\mathbf{h}, 1)(\mathbf{0}, \Omega(b^{-1})) = (\tau_{\Omega(b)}\mathbf{h}, 1) = \tau_{\Omega(b)}\mathbf{h}. \tag{5.6}$$

For a cocycle $\mathbf{z} \in \mathcal{Z}^1_T(\mathbf{B}(n), H)$, we have $\mathbf{z}(1) = \mathbf{0}$ and

$$(\delta^1_T \mathbf{z})(b_1, b_2) = T_{b_1}\mathbf{z}(b_2) - \mathbf{z}(b_1 b_2) + \mathbf{z}(b_1) = \mathbf{0} \qquad \text{for all } b_1, b_2 \in \mathbf{B}(n)$$

(this is the additive version of (4.33)); thus,

$$\mathbf{z}(b_1 b_2) = \mathbf{z}(b_1) + T_{b_1}\mathbf{z}(b_2). \tag{5.7}$$

Particularly, setting $b_1 = b$ and $b_2 = b^{-1}$, we obtain

$$\mathbf{z}(b^{-1}) = -T_{b^{-1}}\mathbf{z}(b). \tag{5.8}$$

It follows from (5.7),(5.8) that *any 1-cocycle* $\mathbf{z}$ *is completely determined by its values*

$$\mathbf{h}_i = \mathbf{z}(s_i) = (z_i^1, ..., z_i^t) \in H, \qquad z_i^j \in A, \tag{5.9}$$

on the canonical generators $s_1, ..., s_{n-1}$ *of the group* $\mathbf{B}(n)$.

Assume that some elements

$$\mathbf{h}_i = (a_i^1, ..., a_i^t) \in H, \qquad a_i^j \in A, \qquad 1 \le i \le n-1, \tag{5.10}$$

are given, and we are looking for a cocycle $\mathbf{z} \in \mathcal{Z}_T^1(\mathbf{B}(n), H)$ with the values

$$\mathbf{z}(s_i) = \mathbf{h}_i, \qquad 1 \le i \le n-1. \tag{5.11}$$

Since $s_p s_q = s_q s_p$ whenever $1 \le p, q \le n-1$ and $|p-q| \ge 2$, we have for any such p, q the relation $\mathbf{z}(s_p s_q) = \mathbf{z}(s_q s_p)$. In view of (5.7), the latter relations may be written as

$$T_{s_p}\mathbf{z}(s_q) + \mathbf{z}(s_p) = T_{s_q}\mathbf{z}(s_p) + \mathbf{z}(s_q), \tag{5.12}$$

which shows that the elements (5.10) must satisfy the relations

$$\mathbf{h}_q - T_{s_p}\mathbf{h}_q = \mathbf{h}_p - T_{s_q}\mathbf{h}_p, \qquad 1 \le p, q \le n-1, \ \ |p-q| \ge 2. \tag{5.13}$$

We should also take into account the braid relations $s_p s_{p+1} s_p = s_{p+1} s_p s_{p+1}$, $1 \le p < n-1$, which leads to the conditions

$$T_{s_p s_{p+1}}\mathbf{z}(s_p) + T_{s_p}\mathbf{z}(s_{p+1}) + \mathbf{z}(s_p) = T_{s_{p+1}s_p}\mathbf{z}(s_{p+1}) + T_{s_{p+1}}\mathbf{z}(s_p) + \mathbf{z}(s_{p+1}) \tag{5.12$'$}$$

and

$$\mathbf{h}_p - T_{s_{p+1}}\mathbf{h}_p + T_{s_p s_{p+1}}\mathbf{h}_p = \mathbf{h}_{p+1} - T_{s_p}\mathbf{h}_{p+1} + T_{s_{p+1}s_p}\mathbf{h}_{p+1}, \ \ 1 \le p \le n-2. \tag{5.13$'$}$$

The following lemma is evident.

Lemma 5.1. *A cocycle* $\mathbf{z} \in \mathcal{Z}_T^1(\mathbf{B}(n), H)$ *with the values* $\mathbf{z}(s_i) = \mathbf{h}_i$ $(1 \le i \le n-1)$ *does exist if and only if the elements* $\mathbf{h}_i$ *satisfy relations (5.13),* (5.13$'$). *If these relations hold, then the corresponding cocycle* $\mathbf{z}$ *is uniquely determined by the elements* $\mathbf{h}_i$. *Moreover, this cocycle* $\mathbf{z}$ *is a coboundary if and only if there exists an element* $\mathbf{h} \in H$ *such that*

$$\mathbf{h}_p = T_{s_p}\mathbf{h} - \mathbf{h} \qquad \textit{for all} \ \ p = 1, ..., n-1. \tag{5.14}$$

The cohomology group $H_T^1(\mathbf{B}(n), H)$ *is isomorphic to the quotient group* $\mathcal{Z}/\mathcal{B}$, *where* $\mathcal{Z}$ *consists of all the solutions* $[\mathbf{h}_1, ..., \mathbf{h}_{n-1}]$ *of the linear system* (5.13), (5.13$'$), *and* $\mathcal{B} \subseteq \mathcal{Z}$ *is the subgroup consisting of all the solutions* $[\mathbf{h}_1, ..., \mathbf{h}_{n-1}]$ *such that there is an element* $\mathbf{h} \in H$ *that satisfies* (5.14). □

If the action T is given explicitly, the computation of the quotient group $\mathcal{Z}/\mathcal{B}$ is a routine (however, it can be very long).

In the sequel we use the coordinate representation (5.10) of the vectors $\mathbf{h}_i \in H$.

Lemma 5.2. *Assume that $t = n$ and that $\Omega = \mu$: $\mathbf{B}(n) \to \mathbf{S}(n) = \mathbf{S}(t)$ is the canonical projection. Then system of equations* (5.13) *is equivalent to the system*

$$a_q^p = a_q^{p+1}, \qquad 1 \le p, q \le n-1, \quad |p-q| \ge 2, \tag{5.15}$$

and system (5.16′) *is equivalent to*

$$\begin{aligned} &a_q^p = a_{q+1}^p, && 1 \le q \le n-2, \ \ 1 \le p \le n, \ \ p \ne q, q+1, q+2; \\ &a_q^{q+2} = a_{q+1}^q, && 1 \le q \le n-2; \\ &a_q^q + a_q^{q+1} = a_{q+1}^{q+1} + a_{q+1}^{q+2}, && 1 \le q \le n-2. \end{aligned} \tag{5.15′}$$

Proof. Take any element $\mathbf{h} = (a^1, \dots, a^n) \in H$. Using the definitions (5.3), (5.6) of the actions τ and T, and taking into account that for $\Omega = \mu$ we have

$$\Omega(s_p) = (p, p+1), \quad \Omega(s_p s_{p+1}) = (p, p+1, p+2), \quad \text{and} \quad \Omega(s_{p+1}s_p) = (p+2, p+1, p),$$

we can readily compute that

$$\begin{aligned} T_{s_p}\mathbf{h} \quad &= (a^1, \dots, a^{p-1}, \underbrace{a^{p+1}, a^p}, a^{p+2}, a^{p+3}, \dots, a^n), \\ T_{s_p s_{p+1}}\mathbf{h} &= (a^1, \dots, a^{p-1}, \underbrace{a^{p+2}, a^p, a^{p+1}}, a^{p+3}, \dots, a^n), \\ T_{s_{p+1} s_p}\mathbf{h} &= (a^1, \dots, a^{p-1}, \underbrace{a^{p+1}, a^{p+2}, a^p}, a^{p+3}, \dots, a^n) \end{aligned} \tag{5.16}$$

(we underbrace the "nonregular" permuted parts). Using these formulas, we can write relations (5.13), (5.13′) in the coordinates; after evident cancellations, this leads to (5.15) and (5.15′), respectively. □

Now we can compute certain cohomology and homomorphisms.

Remark 5.1. Assume that $A = \mathbb{Z}/r\mathbb{Z}$ and take the following t disjoint r-cycles

$$C_m = ((m-1)r+1, (m-1)r+2, \dots, mr) \in \mathbf{S}(rt), \quad m = 1, \dots, t.$$

Identify any $\mathbf{h} = (a^1, ..., a^t) \in (\mathbb{Z}/r\mathbb{Z})^{\oplus t}$ with the product $C_1^{a^1} \cdots C_t^{a^t} \in \mathbf{S}(rt)$. Thereby, we obtain an embedding $(\mathbb{Z}/r\mathbb{Z})^{\oplus t} \hookrightarrow \mathbf{S}(rt)$. Using this embedding and Lemma 4.1, we may identify the second horizontal line of diagram (5.1) with the exact sequence (4.2). This identification is compatible with the actions, splittings, etc. This means that for the group $A = \mathbb{Z}/r\mathbb{Z}$ any Ω-homomorphism in diagram (5.1) may be regarded as a homomorphism $\mathbf{B}(n) \to G \subset \mathbf{S}(rt)$.

When A is a commutative ring with unit 1 (say $A = \mathbb{Z}/r\mathbb{Z}$), we set

$$\mathbf{e}_i = (\underbrace{0, \dots, 0}_{i-1 \text{ times}}, 1, \underbrace{0, \dots, 0}_{t-i \text{ times}}) \in H, \quad 1 \le i \le t. \tag{5.17}$$

Clearly, in this case H is a free A-module with the free base $\mathbf{e}_1, \dots, \mathbf{e}_t$, and the action T on H is compatible with the A-module structure of H. Hence, the cohomology group is also an A-module. ○

Theorem 5.3. *Assume that $t = n$ and that $\Omega = \mu$: $\mathbf{B}(n) \to \mathbf{S}(n) = \mathbf{S}(t)$ is the canonical projection. Then*

$$H^1_\mu(\mathbf{B}(n), A^{\oplus n}) \cong A \oplus A.$$

If A is a ring with unit, then the A-module $H^1_\mu(\mathbf{B}(n), A^{\oplus n})$ is generated by the cohomology classes of the following two cocycles $\mathbf{z}_1$, $\mathbf{z}_2$:

$$\begin{aligned} \mathbf{z}_1(s_i) &= \mathbf{e}_{i+1}, \\ \mathbf{z}_2(s_i) &= \mathbf{e}_1 + \ldots + \mathbf{e}_{i-1} + \mathbf{e}_{i+2} + \ldots + \mathbf{e}_n. \end{aligned} \qquad (1 \le i \le n-1) \tag{5.18}$$

Proof. *a*) Any solution $\mathbf{h}_1, ..., \mathbf{h}_{n-1}$ of the system of equations (5.15),(5.15′) is of the form

$$\mathbf{h}_i = (\underbrace{b, ..., b}_{i-1 \text{ times}}, c_i, a - c_i, \underbrace{b, ..., b}_{n-i-1 \text{ times}}), \qquad i = 1, ..., n-1,$$

where a, b and $c_1, ..., c_{n-1}$ are arbitrary elements of the group A (the elements a and b do not depend on i). Hence,

$$\mathcal{Z} = \mathcal{Z}^1_\mu(\mathbf{B}(n), H) \cong A^{\oplus n+1} = \bigoplus_{i=1}^{n+1} A.$$

Such a solution satisfies the system of equations (5.14) for some $\mathbf{h} \in H$ *if and only if* $a = b = 0$. That is, the subgroup

$$\mathcal{B} = \mathcal{B}^1_\mu(\mathbf{B}(n), H) \subseteq \mathcal{Z}^1_\mu(\mathbf{B}(n), H)$$

consists of all $[\mathbf{h}_1, ..., \mathbf{h}_{n-1}]$ of the form

$$\mathbf{h}_i = (\underbrace{0, ..., 0}_{i-1 \text{ times}}, c_i, -c_i, \underbrace{0, ..., 0}_{n-i-1 \text{ times}}), \quad i = 1, ..., n-1.$$

This implies that any cohomology class in $H^1_\mu(\mathbf{B}(n), H)$ contains a unique cocycle $\mathbf{z} \in \mathcal{Z}^1_\mu(\mathbf{B}(n), H)$ that takes on the generators $s_1, ..., s_{n-1}$ the values of the form

$$\mathbf{z}(s_i) = \mathbf{h}_i = (\underbrace{b, ..., b}_{i-1 \text{ times}}, 0, a, \underbrace{b, ..., b}_{n-i-1 \text{ times}}), \quad i = 1, ..., n-1, \tag{5.19}$$

where elements $a, b \in A$ do not depend on i. Therefore, $H^1_\mu(\mathbf{B}(n), A^{\oplus n}) \cong A \oplus A$.

Now, if A is a ring (with unit), any cocycle of the form (5.19) may be represented as $\mathbf{z} = a \cdot \mathbf{z}_1 + b \cdot \mathbf{z}_2$, where $\mathbf{z}_1, \mathbf{z}_2$ are defined by (5.18). □

Remark 5.2. Remarks 4.2 and 2.2 show that for $n \neq 4, 6$ Theorem 5.3 applies, in fact, to any *noncyclic* homomorphism Ω: $\mathbf{B}(n) \to \mathbf{S}(n)$. Namely, for such a homomorphism Ω, we have $H^1_\Omega(\mathbf{B}(n), A^{\oplus n}) \cong H^1_\mu(\mathbf{B}(n), A^{\oplus n})$. □

Now we should compute the $A^{\oplus 6}$-cohomology for the two exceptional homomorphisms $\Omega = \psi_{5,6}$: $\mathbf{B}(5) \to \mathbf{S}(6)$ and $\Omega = \nu_6$: $\mathbf{B}(6) \to \mathbf{S}(6)$, where $\psi_{5,6}$ is defined by (3.6′) and ν_6 is Artin's homomorphism (§0.5). We skip some completely elementary details, since the computations are very long.

Notation. For an additive Abelian group A, we denote by A_2 the subgroup in A consisting of the zero element and all elements of order 2: $A_2 = \{a \in A \mid 2a = 0\}$. ○

Theorem 5.4. *Let $n = 5$, $t = 6$ and let $\Omega = \psi_{5,6}$: $\mathbf{B}(5) \to \mathbf{S}(6)$ be the homomorphism defined by (3.6), (3.6′). Then*

$$H^1_{\psi_{5,6}}(\mathbf{B}(5), A^{\oplus 6}) \cong A_2 \oplus A.$$

Moreover, any cohomology class contains a cocycle $\mathbf{z}$ that takes on the canonical generators $s_i \in \mathbf{B}(5)$ the values $\mathbf{z}(s_i) = \mathbf{h}_i$ of the form

$$\begin{array}{llllll} \mathbf{h}_1 = (\ 0, & x+2y, & 0, & x+2y, & 0, & x+2y) \\ \mathbf{h}_2 = (\ 0, & 0, & x+2y, & x+2y, & x+2y, & 0\ \ ) \\ \mathbf{h}_3 = (-y, & -y, & x+3y, & x+3y, & x, & 2y\ \ ) \\ \mathbf{h}_4 = (\ x, & 2y, & 2y, & 2y, & x, & x\ \ ) \end{array} \tag{5.20}$$

where $x \in A_2$ and $y \in A$.

Proof. The action $T = T_{\psi_{5,6}}$ of $\mathbf{B}(5)$ on the group $A^{\oplus 6}$ corresponding to the homomorphism $\psi_{5,6}$ is given by

$$\begin{aligned} T_{s_1}(a^1,a^2,a^3,a^4,a^5,a^6) &= (a^2,a^1,a^4,a^3,a^6,a^5), \\ T_{s_2}(a^1,a^2,a^3,a^4,a^5,a^6) &= (a^5,a^3,a^2,a^6,a^1,a^4), \\ T_{s_3}(a^1,a^2,a^3,a^4,a^5,a^6) &= (a^3,a^4,a^1,a^2,a^6,a^5), \\ T_{s_4}(a^1,a^2,a^3,a^4,a^5,a^6) &= (a^2,a^1,a^5,a^6,a^3,a^4). \end{aligned}$$

Let $\mathbf{z}$ be a cocycle with the values $\mathbf{z}(s_i) = \mathbf{h}_i = (a_i^1, ..., a_i^6) \in A^{\oplus 6}$, $i = 1,2,3,4$. Then the system of equations corresponding to the commutativity relations $s_i \rightleftarrows s_j$ ($|i-j| > 1$) and the braid relations $s_i \infty s_{i+1}$ between the generators s_i looks as follows:

$$\begin{array}{lll} a_1^1 - a_1^3 = a_3^1 - a_3^2; & a_1^2 - a_1^4 = a_3^2 - a_3^1; & a_1^3 - a_1^1 = a_3^3 - a_3^4; \\ a_1^4 - a_1^2 = a_3^4 - a_3^3; & a_1^5 - a_1^6 = a_3^5 - a_3^6; & a_1^6 - a_1^5 = a_3^6 - a_3^5; \end{array} \qquad (s_1 \rightleftarrows s_3)$$

$$\begin{array}{lll} a_1^1 - a_1^2 = a_4^1 - a_4^2; & a_1^2 - a_1^1 = a_4^2 - a_4^1; & a_1^3 - a_1^5 = a_4^3 - a_4^4; \\ a_1^4 - a_1^6 = a_4^4 - a_4^3; & a_1^5 - a_1^3 = a_4^5 - a_4^6; & a_1^6 - a_1^4 = a_4^6 - a_4^5; \end{array} \qquad (s_1 \rightleftarrows s_4)$$

$$\begin{array}{lll} a_2^1 - a_2^2 = a_4^1 - a_4^5; & a_2^2 - a_2^1 = a_4^2 - a_4^3; & a_2^3 - a_2^5 = a_4^3 - a_4^2; \\ a_2^4 - a_2^6 = a_4^4 - a_4^6; & a_2^5 - a_2^3 = a_4^5 - a_4^1; & a_2^6 - a_2^4 = a_4^6 - a_4^4; \end{array} \qquad (s_2 \rightleftarrows s_4)$$

$(s_1 \infty s_2)$:

$$\begin{array}{lll} a_1^1 - a_1^5 + a_1^3 = a_2^1 - a_2^2 + a_2^6; & a_1^2 - a_1^3 + a_1^3 = a_2^2 - a_2^1 + a_2^4; & a_1^3 - a_1^2 + a_1^6 = a_2^3 - a_2^4 + a_2^1; \\ a_1^4 - a_1^6 + a_1^2 = a_2^4 - a_2^3 + a_2^5; & a_1^5 - a_1^1 + a_1^4 = a_2^5 - a_2^6 + a_2^2; & a_1^6 - a_1^4 + a_1^1 = a_2^6 - a_2^5 + a_2^3; \end{array}$$

$(s_2 \infty s_3)$:

$$a_2^1 - a_2^3 + a_2^6 = a_3^1 - a_3^5 + a_3^2;\quad a_2^2 - a_2^4 + a_2^1 = a_3^2 - a_3^3 + a_3^6;\quad a_2^3 - a_2^1 + a_2^4 = a_3^3 - a_3^2 + a_3^5;$$
$$a_2^4 - a_2^2 + a_2^5 = a_3^4 - a_3^6 + a_3^3;\quad a_2^5 - a_2^6 + a_2^3 = a_3^5 - a_3^1 + a_3^4;\quad a_2^6 - a_2^5 + a_2^2 = a_3^6 - a_3^4 + a_3^1;$$

$(s_3 \infty s_4)$:

$$a_3^1 - a_3^2 + a_3^5 = a_4^1 - a_4^3 + a_4^4;\quad a_3^2 - a_3^1 + a_3^6 = a_4^2 - a_4^4 + a_4^3;\quad a_3^3 - a_3^5 + a_3^2 = a_4^3 - a_4^1 + a_4^6;$$
$$a_3^4 - a_3^6 + a_3^1 = a_4^4 - a_4^2 + a_4^5;\quad a_3^5 - a_3^3 + a_3^4 = a_4^5 - a_4^6 + a_4^1;\quad a_3^6 - a_3^4 + a_3^3 = a_4^6 - a_4^5 + a_4^2.$$

Straightforward computations show that the general solution $\mathbf{h}_i = (a_i^1, ..., a_i^6)$ $(1 \le i \le 4)$ of this system depends linearly (over $\mathbb{Z}$) on seven parameters $x_1, \dots, x_7 \in A$ that must satisfy the only relation

$$2x_1 = 0. \tag{5.21}$$

Explicitly, the general solution is of the form

$$\begin{aligned} a_1^1 &= x_3 & a_1^4 &= x_1 + 2x_2 - x_4 \\ a_1^2 &= x_1 + 2x_2 - x_3 & a_1^5 &= x_5 \\ a_1^3 &= x_4 & a_1^6 &= x_1 + 2x_2 - x_5 \end{aligned} \tag{$\mathbf{h}_1$}$$

$$\begin{aligned} a_2^1 &= x_6 & a_2^4 &= x_1 + 2x_2 - x_3 - x_4 + x_5 + x_6 - x_7 \\ a_2^2 &= x_7 & a_2^5 &= x_1 + 2x_2 - x_6 \\ a_2^3 &= x_1 + 2x_2 - x_7 & a_2^6 &= x_3 + x_4 - x_5 - x_6 + x_7 \end{aligned} \tag{$\mathbf{h}_2$}$$

$$\begin{aligned} a_3^1 &= -x_2 + x_3 + x_7 & a_3^4 &= x_1 + 3x_2 - x_4 - x_7 \\ a_3^2 &= -x_2 + x_4 + x_7 & a_3^5 &= x_1 + x_5 \\ a_3^3 &= x_1 + 3x_2 - x_3 - x_7 & a_3^6 &= 2x_2 - x_5 \end{aligned} \tag{$\mathbf{h}_3$}$$

$$\begin{aligned} a_4^1 &= x_1 + x_3 & a_4^4 &= 2x_2 - x_3 - x_4 + x_5 + x_6 - x_7 \\ a_4^2 &= 2x_2 - x_3 & a_4^5 &= x_1 + x_3 - x_6 + x_7 \\ a_4^3 &= 2x_2 - x_3 + x_6 - x_7 & a_4^6 &= x_1 + x_3 + x_4 - x_5 - x_6 + x_7 \end{aligned} \tag{$\mathbf{h}_4$}$$

To select the solutions corresponding to coboundaries, we must find out all the solutions $\mathbf{h}_i = (a_i^1, ..., a_i^6)$ $(1 \le i \le 4)$ such that the system of equations

$$\mathbf{h}_i = T_{s_i}\mathbf{h} - \mathbf{h}, \qquad 1 \le i \le 4,$$

has a solution $\mathbf{h} = (u^1, ..., u^6) \in A^{\oplus 6}$. In coordinates, this system of equations looks as follows:

$$\begin{array}{ll} a_1^1 = x_3 = u^2 - u^1; & a_1^4 = x_1 + 2x_2 - x_4 = u^3 - u^4; \\ a_1^2 = x_1 + 2x_2 - x_3 = u^1 - u^2; & a_1^5 = x_5 = u^6 - u^5; \\ a_1^3 = x_4 = u^4 - u^3; & a_1^6 = x_1 + 2x_2 - x_5 = u^5 - u^6; \end{array} \qquad (\mathbf{h}_1')$$

$$\begin{array}{ll} a_2^1 = x_6 = u^5 - u^1; & a_2^4 = x_1 + 2x_2 - x_3 - x_4 + x_5 + x_6 - x_7 = u^6 - u^4; \\ a_2^2 = x_7 = u^3 - u^2; & a_2^5 = x_1 + 2x_2 - x_6 = u^1 - u^5; \\ a_2^3 = x_1 + 2x_2 - x_7 = u^2 - u^3; & a_2^6 = x_3 + x_4 - x_5 - x_6 + x_7 = u^4 - u^6; \end{array} \qquad (\mathbf{h}_2')$$

$$\begin{array}{ll} a_3^1 = -x_2 + x_3 + x_7 = u^3 - u^1; & a_3^4 = x_1 + 3x_2 - x_4 - x_7 = u^2 - u^4; \\ a_3^2 = -x_2 + x_4 + x_7 = u^4 - u^2; & a_3^5 = x_1 + x_5 = u^6 - u^5; \\ a_3^3 = x_1 + 3x_2 - x_3 - x_7 = u^1 - u^3; & a_3^6 = 2x_2 - x_5 = u^5 - u^6; \end{array} \qquad (\mathbf{h}_3')$$

$$\begin{array}{ll} a_4^1 = x_1 + x_3 = u^2 - u^1; & a_4^4 = 2x_2 - x_3 - x_4 + x_5 + x_6 - x_7 = u^6 - u^4; \\ a_4^2 = 2x_2 - x_3 = u^1 - u^2; & a_4^5 = x_1 + x_3 - x_6 + x_7 = u^3 - u^5; \\ a_4^3 = 2x_2 - x_3 + x_6 - x_7 = u^5 - u^3; & a_4^6 = x_1 + x_3 + x_4 - x_5 - x_6 + x_7 = u^4 - u^6. \end{array} \qquad (\mathbf{h}_4')$$

It has a solution $(u^1, ..., u^6) \in A^{\oplus 6}$ if and only if the parameters x_i satisfy the relations

$$x_1 = x_2 = 0. \qquad (5.22)$$

Hence, the group $\mathcal{Z}^1_{\psi_{5,6}}(\mathbf{B}(5), A^{\oplus 6})$ of all cocycles is isomorphic to the direct sum

$$\mathcal{Z} = A_2 \oplus A^{\oplus 6} = \{(x_1, ..., x_7) \in A^{\oplus 7} \mid 2x_1 = 0\},$$

and the group $\mathcal{B}^1_{\psi_{5,6}}(\mathbf{B}(5), A^{\oplus 6})$ of all coboundaries is isomorphic to the subgroup $\mathcal{B} \subset \mathcal{Z}$

$$\mathcal{B} = A^{\oplus 5} = \{(x_1, ..., x_7) \in A^{\oplus 7} \mid x_1 = x_2 = 0\}.$$

This means that

$$H^1_{\psi_{5,6}}(\mathbf{B}(5), A^{\oplus 6}) \cong \mathcal{Z}/\mathcal{B} \cong A_2 \oplus A.$$

Clearly, any cohomology class in $H^1_{\psi_{5,6}}(\mathbf{B}(5), A^{\oplus 6})$ contains a cocycle of the form $(\mathbf{h}_1)-(\mathbf{h}_4)$ with $x_3 = x_4 = x_5 = x_6 = x_7 = 0$; this proves (5.20) (with $x = x_1 \in A_2$, $y = x_2 \in A$). $\square$

Remark 5.3. In the next theorem we use some details of the proof of Theorem 5.4. To simplify our notations, we denote the system of equations $(s_1 \rightleftarrows s_3) - (s_3 \infty s_4)$ by $(\mathcal{S})$, and formulas $(\mathbf{h}_1) - (\mathbf{h}_4)$ by $(\mathcal{H})$. We denote by $(\mathcal{H}_0)$ the formulas for a_i^j given by $(\mathbf{h}_1) - (\mathbf{h}_4)$ *with the particular value* $x_1 = 0$. Finally, we denote by $(\mathcal{U}_0)$ the system of equations $(\mathbf{h}_1') - (\mathbf{h}_4')$ with the same particular value $x_1 = 0$. It follows from the proof of Theorem 5.4 that *system* $(\mathcal{U}_0)$ *has a solution* $(u^1, ..., u^6) \in A^{\oplus 6}$ *if and only if* $x_2 = 0$. $\bigcirc$

Theorem 5.5. *Let $t = n = 6$, and let $\Omega = \nu_6$: $\mathbf{B}(6) \to \mathbf{S}(6)$ be Artin's homomorphism. Then*

$$H^1_{\nu_6}(\mathbf{B}(6), A^{\oplus 6}) \cong A.$$

Any cohomology class may be represented by a cocycle that takes on the canonical generators $s_i \in \mathbf{B}(6)$ the values $\mathbf{z}(s_i) = \mathbf{h}_i$ of the form

$$\begin{aligned}
\mathbf{z}(s_1) = \mathbf{h}_1 &= (\ \ 0,\ 2y,\ \ 0,\ \ 2y,\ \ 0,\ 2y)\\
\mathbf{z}(s_2) = \mathbf{h}_2 &= (\ \ 0,\ \ 0,\ \ 2y,\ 2y,\ 2y,\ 0\ )\\
\mathbf{z}(s_3) = \mathbf{h}_3 &= (\ -y,\ -y,\ 3y,\ 3y,\ \ 0,\ 2y)\\
\mathbf{z}(s_4) = \mathbf{h}_4 &= (\ \ 0,\ 2y,\ 2y,\ 2y,\ \ 0,\ \ 0\ )\\
\mathbf{z}(s_5) = \mathbf{h}_5 &= (-2y,\ 0,\ \ 2y,\ 4y,\ \ 0,\ 2y)
\end{aligned} \tag{5.23}$$

Proof. According to (3.6′), we may regard the homomorphism $\psi_{5,6}$: $\mathbf{B}(5) \to \mathbf{S}(6)$ as the restriction of Artin's homomorphism ν_6: $\mathbf{B}(6) \to \mathbf{S}(6)$ to the subgroup $\mathbf{B} \cong \mathbf{B}(5)$ in $\mathbf{B}(6)$ generated by the first four canonical generators s_1, s_2, s_3, s_4. Since $\varepsilon_{\nu_6} = \rho \circ \nu_6$ and $\varepsilon_{\psi_{5,6}} = \rho \circ \psi_{5,6}$, we have $\varepsilon_{\nu_6} \mid \mathbf{B}(5) = \varepsilon_{\psi_{5,6}}$, and thus the $\mathbf{B}(5)$-action $T_{\psi_{5,6}}$ coincides with the restriction of the $\mathbf{B}(6)$-action T_{ν_6} to $\mathbf{B} = \mathbf{B}(5)$. It follows that *the restriction $\mathbf{z}_{\mathbf{B}(5)}$ of any 1-cocycle $\mathbf{z} \in \mathcal{Z}^1_{\nu_6}(\mathbf{B}(6), A^{\oplus 6})$ to the subgroup $\mathbf{B}(5) = \mathbf{B} \subset \mathbf{B}(6)$ belongs to $\mathcal{Z}^1_{\psi_{5,6}}(\mathbf{B}(5), A^{\oplus 6})$*. Moreover, *if such a cocycle $\mathbf{z}$ is a coboundary, then its restriction $\mathbf{z}_{\mathbf{B}(5)}$ is also a coboundary.* So, in order to compute $H^1_{\nu_6}(\mathbf{B}(6), A^{\oplus 6})$, we may use some computations already made in the proof of Theorem 5.4.

Let $\mathbf{z} \in Z^1_{\nu_6}(\mathbf{B}(6), A^{\oplus 6}$ be a cocycle with the values $\mathbf{z}(s_i) = \mathbf{h}_i = (a^1_i, ..., a^6_i) \in A^{\oplus 6}$, $1 \le i \le 5$. Then the elements a^j_i with $1 \le i \le 4$ must satisfy the system of equations $(\mathcal{S})$. According to the proof of Theorem 5.4, they are of the form $(\mathcal{H})$ with some $x_1, ..., x_7 \in A$. The elements a^j_i $(1 \le i \le 4)$ together with the elements $a^1_5, ..., a^6_5$ must satisfy the system of equations $(s_i \rightleftarrows s_5)$, $(s_4 \infty s_5)$ corresponding to the relations $s_i s_5 = s_5 s_i$ $(1 \le i \le 3)$ and $s_4 s_5 s_4 = s_5 s_4 s_5$. Using the formula

$$T_{s_5}(a^1, a^2, a^3, a^4, a^5, a^6) = (a^4, a^3, a^2, a^1, a^6, a^5)$$

for the transformation $T_{s_5} = (T_{\nu_6})_{s_5}$, we can write down the equations $(s_i \rightleftarrows s_5)$,$(s_4 \infty s_5)$ explicitly:

$$\begin{aligned}
&a^1_1 - a^4_1 = a^1_5 - a^2_5; \quad && a^2_1 - a^3_1 = a^2_5 - a^1_5; \quad && a^3_1 - a^2_1 = a^3_5 - a^4_5;\\
&a^4_1 - a^1_1 = a^4_5 - a^3_5; && a^5_1 - a^6_1 = a^5_5 - a^6_5; && a^6_1 - a^5_1 = a^6_5 - a^5_5;
\end{aligned} \qquad (s_1 \rightleftarrows s_5)$$

$$\begin{aligned}
&a^1_2 - a^4_2 = a^1_5 - a^5_5; \quad && a^2_2 - a^3_2 = a^2_5 - a^3_5; \quad && a^3_2 - a^2_2 = a^3_5 - a^2_5;\\
&a^4_2 - a^1_2 = a^4_5 - a^6_5; && a^5_2 - a^6_2 = a^5_5 - a^1_5; && a^6_2 - a^5_2 = a^6_5 - a^4_5;
\end{aligned} \qquad (s_2 \rightleftarrows s_5)$$

$$\begin{aligned}
&a^1_3 - a^4_3 = a^1_5 - a^3_5; \quad && a^2_3 - a^3_3 = a^2_5 - a^4_5; \quad && a^3_3 - a^2_3 = a^3_5 - a^1_5;\\
&a^4_3 - a^1_3 = a^4_5 - a^2_5; && a^5_3 - a^6_3 = a^5_5 - a^6_5; && a^6_3 - a^5_3 = a^6_5 - a^5_5;
\end{aligned} \qquad (s_3 \rightleftarrows s_5)$$

$(s_4 \infty s_5)$:

$$a_4^1 - a_4^4 + a_4^3 = a_5^1 - a_5^2 + a_5^6;\quad a_4^2 - a_4^3 + a_4^4 = a_5^2 - a_5^1 + a_5^5;\quad a_4^3 - a_4^2 + a_4^6 = a_5^3 - a_5^5 + a_5^1;$$
$$a_4^4 - a_4^1 + a_4^5 = a_5^4 - a_5^6 + a_5^2;\quad a_4^5 - a_4^6 + a_4^2 = a_5^5 - a_5^3 + a_5^4;\quad a_4^6 - a_4^5 + a_4^1 = a_5^6 - a_5^4 + a_5^3.$$

Let us denote the system of equations $(s_i \rightleftarrows s_5)$ $(1 \le i \le 3)$ and $(s_4 \infty s_5)$ by $(\mathcal{S}_{new})$.

By substitution of the expressions $(\mathcal{H})$ for a_i^j $(1 \le i \le 4)$ in terms of the parameters x_i into the equations $(\mathcal{S}_{new})$, we obtain the system of *nonhomogeneous* equations $(\widetilde{\mathcal{S}_{new}})$ for the remainder elements $a_5^j \in A$, $1 \le j \le 6$. This (very unpleasant!) procedure leads to the following result:

Claim. *System of equations* $(\widetilde{\mathcal{S}_{new}})$ *has a solution* $a_5^j \in A$ $(1 \le j \le 6)$ *if and only if* $x_1 = 0$. *Clearly, the latter condition means that the elements* a_i^j $(1 \le i \le 4)$ *must be chosen according to formulas* $(\mathcal{H}_0)$. *If this is done, the solution* $(a_5^1, \ldots, a_5^6) \in A^{\oplus 6}$ *of* $(\widetilde{\mathcal{S}_{new}})$ *is unique and reads as follows:*

$$\begin{aligned} a_5^1 &= -2x_2 + x_3 + x_4 + x_7, & a_5^4 &= \ \ 4x_2 - x_3 - x_4 - x_7,\\ a_5^2 &= \ \ x_7, & a_5^5 &= \ \ x_5,\\ a_5^3 &= \ \ 2x_2 - x_7, & a_5^6 &= \ \ 2x_2 - x_5. \end{aligned} \tag{$\mathbf{h}_5$}$$

Combined with formulas $(\mathcal{H}_0)$, this shows that

$$\mathcal{Z} = \mathcal{Z}^1_{\nu_6}(\mathbf{B}(6), A^{\oplus 6}) \cong A^{\oplus 6} = \{(x_1, \ldots, x_7) \in A^{\oplus 7} \mid x_1 = 0\}.$$

Now we must select the solutions corresponding to coboundaries. To this end, we add the following new equations $(\mathbf{h}'_5)$

$$\begin{aligned} a_5^1 &= -2x_2 + x_3 + x_4 + x_7 = u_4 - u_1, & a_5^4 &= 4x_2 - x_3 - x_4 - x_7 = u_1 - u_4,\\ a_5^2 &= x_7 = u_3 - u_2, & a_5^5 &= x_5 = u_6 - u_5,\\ a_5^3 &= 2x_2 - x_7 = u_2 - u_3, & a_5^6 &= 2x_2 - x_5 = u_5 - u_6, \end{aligned} \tag{$\mathbf{h}'_5$}$$

to the equations $(\mathcal{U}_0)$; then we need to find out when the resulting system of equations $(\mathcal{U}_0)$, $(\mathbf{h}'_5)$ has a solution $(u^1, \ldots, u^6) \in A^{\oplus 6}$. We must certainly assume that $x_2 = 0$ (this is necessary for solvability of the equations $(\mathcal{U}_0)$; see Remark 5.3). A straightforward computation shows that, in fact, $x_2 = 0$ is *the only* condition for solvability of the system of equations $(\mathcal{U}_0)$, $(\mathbf{h}'_5)$. That is, *a cocycle* $\mathbf{z}$ *of the form* $(\mathcal{H}_0)$,$(\mathbf{h}_5)$ *is a coboundary if and only if* $x_2 = 0$. Hence,

$$\mathcal{B} = \mathcal{B}^1_{\nu_6}(\mathbf{B}(6), A^{\oplus 6}) \cong A^{\oplus 5} = \{(x_2, \ldots, x_7) \in A^{\oplus 6} \mid x_2 = 0\},$$

and thus

$$H^1_{\nu_6}(\mathbf{B}(6), A^{\oplus 6}) = \mathcal{Z}/\mathcal{B} \cong A.$$

Since the parameters $(x_3, \dots, x_7) \in A^{\oplus 5}$ are completely free both in cocycles and coboundaries, any cohomology class may be represented by a cocycle of the form $(\mathcal{H}_0), (\mathbf{h}_5)$ with $x_3 = \ldots = x_7 = 0$, which gives formulas (5.23) $(y = x_2 \in A)$. □

Remark 5.4. The homomorphisms $\Omega = \mu$: $\mathbf{B}(n) \to \mathbf{S}(n)$ and $\Omega = \nu_6$: $\mathbf{B}(6) \to \mathbf{S}(6)$ are surjective, and the image of the homomorphism $\Omega = \psi_{5,6}$: $\mathbf{B}(5) \to \mathbf{S}(6)$ is isomorphic to $\mathbf{S}(5)$. In any of these cases, the centralizer of the image $\operatorname{Im}\Omega$ in the corresponding symmetric group is trivial, and thus the two equivalence relations $\approx$ and $\sim$ on the set of all Ω-homomorphisms coincide (see Proposition 4.6(b)). Hence, to compute all the Ω-homomorphisms up to conjugation, it is sufficient to choose one cocycle $\mathbf{z}$ in each cohomology class (which indeed was done in Theorems 5.3-5.5), and then to compute the corresponding Ω-homomorphisms $\varphi_{\mathbf{z}}$ defined by (4.36). For the latter step, we must also compute the homomorphism $\varepsilon = \rho \circ \Omega$; this is not a problem at all, as far as the splitting ρ and the homomorphism Ω are given explicitly. ○

In the following three corollaries we consider only the case when $A = \mathbb{Z}/2\mathbb{Z}$, which is important for some applications (see §6). We skip the proofs since they follow immediately from the results stated in Theorems 5.3-5.5 (see also Remarks 5.1 and 5.4).

Corollary 5.6. *The cohomology group* $H^1_\mu(\mathbf{B}(n), (\mathbb{Z}/2\mathbb{Z})^{\oplus n}) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ *consists of the cohomology classes of the following four cocycles:*

$$\left.\begin{aligned}
&\mathbf{z}_0(s_i) = \mathbf{0} \qquad \text{(zero cocycle)},\\
&\mathbf{z}_1(s_i) = \mathbf{e}_{i+1},\\
&\mathbf{z}_2(s_i) = \mathbf{e}_1 + \cdots + \mathbf{e}_{i-1} + \mathbf{e}_{i+2} + \cdots + \mathbf{e}_n,\\
&\mathbf{z}_3(s_i) = \mathbf{z}_1(s_i) + \mathbf{z}_2(s_i) = \mathbf{e}_1 + \cdots + \mathbf{e}_{i-1} + \mathbf{e}_{i+1} + \cdots + \mathbf{e}_n
\end{aligned}\right\} \quad (1 \le i \le n-1). \tag{5.24}$$

Any μ_n*-homomorphism* Ψ: $\mathbf{B}(n) \to G \subset \mathbf{S}(2n)$ *is* H*-conjugate to one of the following four* μ*-homomorphisms* φ_j, $j = 0, 1, 2, 3$ (*in each formula* $1 \le i \le n-1$)*:*

$$\begin{aligned}
&\varphi_0 = \rho \circ \mu_n = \varepsilon_{\mu_n} \sim \mu_n \times \mu_n : \qquad \varphi_0(s_i) = (2i-1, 2i+1)(2i, 2i+2);\\
&\varphi_1(s_i) = \underbrace{(2i-1, 2i+2, 2i, 2i+1)}_{\text{4-cycle}};\\
&\varphi_2(s_i) = (1,2)\cdots(2i-3, 2i-2)\underbrace{(2i-1, 2i+1)(2i, 2i+2)}_{\text{two transpositions}}(2i+3, 2i+4)\cdots(2n-1, 2n);\\
&\varphi_3(s_i) = (1,2)\cdots(2i-3, 2i-2)\underbrace{(2i-1, 2i+2, 2i, 2i+1)}_{\text{4-cycle}}(2i+3, 2i+4)\cdots(2n-1, 2n).
\end{aligned}$$

Corollary 5.7. *The cohomology group* $H^1_{\psi_{5,6}}(\mathbf{B}(5), (\mathbb{Z}/2\mathbb{Z})^{\oplus 6}) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ *consists of the cohomology classes of the four cocycles* $\mathbf{z}_{(x,y)}$, $(x,y) \in (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$ *that take on the generators* s_1, s_2, s_3, s_4 *the values*

$$\begin{aligned}
&\mathbf{z}_{(x,y)}(s_1) = (0,\ x,\ 0,\ x,\ 0,\ x), && \mathbf{z}_{(x,y)}(s_2) = (0,\ 0,\ x,\ x,\ x,\ 0),\\
&\mathbf{z}_{(x,y)}(s_3) = (y,\ y,\ x+y,\ x+y,\ x,\ 0), && \mathbf{z}_{(x,y)}(s_4) = (x,\ 0,\ 0,\ 0,\ x,\ x).
\end{aligned}$$

Any $\psi_{5,6}$-homomorphism Ψ: $\mathbf{B}(5) \to G \subset \mathbf{S}(12)$ is H-conjugate to one of the following four $\psi_{5,6}$-homomorphisms $\eta_i = \phi_{5,6;(x,y)}$, $i = 0,1,2,3$, $(x,y) \in (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$:

$\eta_0 = \phi_{5,6;(0,0)} = \varepsilon_{\psi_{5,6}}$:
$\eta_0(s_1) = (1,3)(2,4)(5,7)(6,8)(9,11)(10,12)$, $\eta_0(s_2) = (1,9)(2,10)(3,5)(4,6)(7,11)(8,12)$,
$\eta_0(s_3) = (1,5)(2,6)(3,7)(4,8)(9,11)(10,12)$, $\eta_0(s_4) = (1,3)(2,4)(5,9)(6,10)(7,11)(8,12)$;

$\eta_1 = \phi_{5,6;(1,0)}$:
$\eta_1(s_1) = (1,4,2,3)(5,8,6,7)(9,12,10,11)$, $\eta_1(s_2) = (1,10,2,9)(3,6,4,5)(7,11,8,12)$,
$\eta_1(s_3) = (1,6,2,5)(3,8,4,7)(9,11,10,12)$, $\eta_1(s_4) = (1,3,2,4)(5,10,6,9)(7,12,8,11)$;

$\eta_2 = \phi_{5,6;(0,1)}$:
$\eta_2(s_1) = (1,3)(2,4)(5,7)(6,8)(9,11)(10,12)$, $\eta_2(s_2) = (1,9)(2,10)(3,5)(4,6)(7,11)(8,12)$,
$\eta_2(s_3) = (1,6)(2,5)(3,8)(4,7)(9,11)(10,12)$, $\eta_2(s_4) = (1,3)(2,4)(5,9)(6,10)(7,11)(8,12)$;

$\eta_3 = \phi_{5,6;(1,1)}$:
$\eta_3(s_1) = (1,4,2,3)(5,8,6,7)(9,12,10,11)$, $\eta_3(s_2) = (1,10,2,9)(3,6,4,5)(7,11,8,12)$,
$\eta_3(s_3) = (1,5,2,6)(3,7,4,8)(9,11,10,12)$, $\eta_3(s_4) = (1,3,2,4)(5,10,6,9)(7,12,8,11)$.

Corollary 5.8. *The group $H^1_{\nu_6}(\mathbf{B}(6), (\mathbb{Z}/2\mathbb{Z})^{\oplus 6}) \cong \mathbb{Z}/2\mathbb{Z}$ consists of the cohomology classes of the two cocycles $\mathbf{z}_y$ $(y \in \mathbb{Z}/2\mathbb{Z})$ that take on the generators $s_1, ..., s_5$ the values*

$$\mathbf{z}_y(s_1) = (0,0,0,0,0,0),\quad \mathbf{z}_y(s_2) = (0,0,0,0,0,0),\quad \mathbf{z}_y(s_3) = (y,y,y,y,0,0),$$
$$\mathbf{z}_y(s_4) = (0,0,0,0,0,0),\quad \mathbf{z}_y(s_5) = (0,0,0,0,0,0).$$

Any ν_6-homomorphism Ψ: $\mathbf{B}(6) \to G \subset \mathbf{S}(12)$ is H-conjugate to one of the following two ν_6-homomorphisms ϕ_y, $y \in \mathbb{Z}/2\mathbb{Z}$:

$\phi_0 = \rho \circ \nu_6 = \varepsilon_{\nu_6}$,
$\phi_0(s_1) = (1,3)(2,4)(5,7)(6,8)(9,11)(10,12)$, $\phi_0(s_2) = (1,9)(2,10)(3,5)(4,6)(7,11)(8,12)$,
$\phi_0(s_3) = (1,5)(2,6)(3,7)(4,8)(9,11)(10,12)$, $\phi_0(s_4) = (1,3)(2,4)(5,9)(6,10)(7,11)(8,12)$,
$$\phi_0(s_5) = (1,7)(2,8)(3,5)(4,6)(9,11)(10,12);$$

$\phi_1(s_1) = (1,3)(2,4)(5,7)(6,8)(9,11)(10,12)$, $\phi_1(s_2) = (1,9)(2,10)(3,5)(4,6)(7,11)(8,12)$,
$\phi_1(s_3) = (1,6)(2,5)(3,8)(4,7)(9,11)(10,12)$, $\phi_1(s_4) = (1,3)(2,4)(5,9)(6,10)(7,11)(8,12)$,
$$\phi_1(s_5) = (1,7)(2,8)(3,5)(4,6)(9,11)(10,12).$$

Remark 5.5. For any natural m and any group G, we denote by $\mathbf{1}_m$ the trivial homomorphism $G \to \mathbf{S}(m)$. $\{\mathbf{2}\}$ denotes the unique nontrivial homomorphism $\mathbf{B}(n) \to \mathbf{S}(2)$; so for every $i = 1, ..., n-1$, $\{\mathbf{2}\}(s_i)$ is the unique transposition in $\mathbf{S}(2)$. Given a homomorphism φ: $\mathbf{B}(n) \to \mathbf{S}(N)$, we regard the disjoint products $\varphi \times \mathbf{1}_m$, $\varphi \times \{\mathbf{2}\}$ as homomorphisms in the groups $\mathbf{S}(N+m)$, $\mathbf{S}(N+2)$, respectively (§0.0.2). For instance, the homomorphism $\mu_n \times \mathbf{1}_1$: $\mathbf{B}(n) \to \mathbf{S}(n+1)$ is defined by $(\mu_n \times \mathbf{1}_1)(s_i) = (i, i+1)$, $1 \le i \le n-1$. ○

We skip the (trivial) proof of the next corollary; φ_j: $\mathbf{B}(n) \to G \subset \mathbf{S}(2n)$ are the μ_n-homomorphisms exhibited in Corollary 5.6.

Corollary 5.9. *Any $(\mu_n \times \mathbf{1}_1)$-homomorphism Ψ: $\mathbf{B}(n) \to G \subset \mathbf{S}(2(n+1))$ is conjugate to one of the eight homomorphisms $\varphi_j \times \mathbf{1}_2$, $\varphi_j \times \{\mathbf{2}\}$, $j = 0, 1, 2, 3$.* □

Remark 5.6. Take any $n \geq 3$ and any $r \geq 2$. Remark 5.1 and Theorem 5.3 give rise to some noncyclic homomorphisms $\mathbf{B}(n) \to \mathbf{S}(rn)$. To simplify the form of the final result, we identify the group $\mathbf{S}(n)$ with the group $\mathbf{S}(\mathbb{Z}/n\mathbb{Z})$, and regard the group $\mathbf{S}(rn)$ as the symmetric group of the direct product $\mathcal{D}(r,n) = (\mathbb{Z}/r\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z})$ via the identification

$$\begin{aligned} &\mathbf{\Delta}_{rn} \ni a \mapsto (R(a), N(a)) \in (\mathbb{Z}/r\mathbb{Z}) \times (\mathbb{Z}/n\mathbb{Z}), \\ &\text{where } R(a) = |a-1|_r \in \mathbb{Z}/r\mathbb{Z} \ \text{ and } \ N(a) = |(a-1-R(a))/r|_n \in \mathbb{Z}/n\mathbb{Z} \end{aligned} \tag{5.25}$$

($|\cdot|_r$ and $|\cdot|_n$ denote the r- and n-residues, respectively). Then the subgroup $H \cong (\mathbb{Z}/r\mathbb{Z})^{\oplus n}$ generated by the r-cycles

$$C_m = ((m-1)r+1, (m-1)r+2, \ldots, mr) \in \mathbf{S}(rn), \quad 1 \leq m \leq n, \tag{5.26}$$

acts on $\mathcal{D}(r,n)$ by translations of the first argument:

$$\begin{aligned} &H \ni \mathbf{h} = (C_1^{a^0} \cdots C_n^{a^{n-1}})\colon \mathcal{D}(r,n) \ni (R,N) \mapsto (R + a^N, N) \in \mathcal{D}(r,n), \\ &\text{where } \ a^0, \ldots, a^{n-1} \in \mathbb{Z}/r\mathbb{Z}. \end{aligned} \tag{5.27}$$

The subgroup $G \subset \mathbf{S}(rn)$ (the centralizer of the element $C = C_1 \cdots C_n$) was already identified with the semidirect product $(\mathbb{Z}/r\mathbb{Z})^{\oplus n} \leftthreetimes_\tau \mathbf{S}(n) = (\mathbb{Z}/r\mathbb{Z})^{\oplus n} \leftthreetimes_\tau \mathbf{S}(\mathbb{Z}/n\mathbb{Z})$; the latter group acts on the set $\mathcal{D}(r,n)$ by permutations as follows:

$$(\mathbf{h}, s)(R, N) = (R + b^{s(N)}, s(N)), \tag{5.28}$$

where $\mathbf{h} = (b^0, \ldots, b^{n-1}) \in (\mathbb{Z}/r\mathbb{Z})^{\oplus n}$, $s \in \mathbf{S}(\mathbb{Z}/n\mathbb{Z})$. Clearly, *this action is transitive and imprimitive* (any subset $(\mathbb{Z}/r\mathbb{Z}) \times \{N\}$ in $\mathcal{D}(r,n)$ is a set of imprimitivity).

According to Theorem 5.3, there are r^2 cocycles $\mathbf{z}_{(x,y)}$, $(x,y) \in (\mathbb{Z}/r\mathbb{Z})^{\oplus 2}$, representing all the cohomology classes. The μ-homomorphism

$$\varphi_{(x,y)}\colon \mathbf{B}(n) \to (\mathbb{Z}/r\mathbb{Z})^{\oplus n} \leftthreetimes_\tau \mathbf{S}(n) \subset \mathbf{S}(\mathcal{D}(r,n))$$

corresponding to the cocycle $\mathbf{z}_{(x,y)}$ is defined by its values $\widehat{s}_{(x,y);i} = \varphi_{(x,y)}(s_i) \in \mathbf{S}(\mathcal{D}(r,n))$ on the canonical generators $s_i \in \mathbf{B}(n)$. The permutations $\widehat{s}_i = \widehat{s}_{(x,y);i}$ act on the elements $(R,N) \in \mathcal{D}(r,n)$ as follows:

$$\widehat{s}_i(R,N) = \begin{cases} (R+y, N) & \text{if } \ N \neq i-1, i; \\ (R, N+1) & \text{if } \ N = i-1; \\ (R+x, N-1) & \text{if } \ N = i; \end{cases} \qquad (1 \leq i \leq n-1). \tag{5.29}$$

It is easy to show that *the homomorphism $\varphi_{(x,y)}$ defined by* (5.29) *is transitive if and only if the elements $x, y \in \mathbb{Z}/r\mathbb{Z}$ generate the whole group $\mathbb{Z}/r\mathbb{Z}$,* or (which is the same), *if and only if x and y are co-prime.* However, *the homomorphism $\varphi_{(x,y)}$ never can be primitive* (since its image is contained in the imprimitive permutation group $G = (\mathbb{Z}/r\mathbb{Z})^{\oplus n} \leftthreetimes_\tau \mathbf{S}(n)$).

Using the same approach and Theorems 5.4, 5.5, one can construct "exceptional" noncyclic homomorphisms $\mathbf{B}(5) \to \mathbf{S}(6r)$ and $\mathbf{B}(6) \to \mathbf{S}(6r)$, $r \geq 2$. ○

Remark 5.7. Assume that the homomorphism Ω in diagram (5.1) is the disjoint product

$$\Omega = \Omega' \times \Omega''\colon\ \mathbf{B}(n) \to \mathbf{S}(t') \times \mathbf{S}(t'') \subset \mathbf{S}(t), \qquad t' + t'' = t.$$

Then we have the decomposition $H = A^{\oplus t} = A^{\oplus t'} \oplus A^{\oplus t''}$, the actions τ', τ'' of the groups $\mathbf{S}(t')$ and $\mathbf{S}(t'')$ on $A^{\oplus t'}$ and $A^{\oplus t''}$, respectively, and the corresponding semidirect products $G' = A^{\oplus t'} \leftthreetimes_{\tau'} \mathbf{S}(t') \subset G$ and $G'' = A^{\oplus t''} \leftthreetimes_{\tau''} \mathbf{S}(t'') \subset G$. Any two elements $g' \in G'$, $g'' \in G''$ commute in G and $\pi(g'g'') \in \mathbf{S}(t') \times \mathbf{S}(t'')$. It is readily seen that the image of any Ω-homomorphism $\Psi\colon \mathbf{B}(n) \to G$ is contained in the subgroup $G' \cdot G'' \cong G' \times G''$. Hence, Ψ is the direct product of the two homomorphisms

$$\Psi'\colon \mathbf{B}(n) \to G' \qquad \text{and} \qquad \Psi''\colon \mathbf{B}(n) \to G''.$$

Each of the latter homomorphisms fits in its own commutative diagram of the form (5.1) and may be studied separately. ○

Let us compute Ω-cohomology for a *cyclic* homomorphism $\Omega\colon \mathbf{B}(n) \to \mathbf{S}(t)$. In this case there is a permutation $S \in \mathbf{S}(t)$ such that $\Omega(s_i) = S$ for all $i = 1, \dots, n-1$. Actually, in view of Remark 5.7, it is sufficient to consider the following two cases: *i)* $t = 1$; *ii)* $t \geq 2$ and S is a t-cycle.

Theorem 5.10. *Suppose $n > 4$. Let $\Omega\colon \mathbf{B}(n) \to \mathbf{S}(t)$ be a cyclic homomorphism.*

a) Any Ω-homomorphism $\Psi\colon \mathbf{B}(n) \to G$ is cyclic.

b) Assume that either i) $t = 1$ or ii) $t \geq 2$ and S is a t-cycle. Then $H^1_\Omega(\mathbf{B}(n), A^{\oplus t}) \cong A$. In case (ii), any cohomology class contains a unique cocycle $\mathbf{z}$ of the form

$$\mathbf{z}(s_i) = (a, \underbrace{0, \dots, 0}_{t-1 \text{ times}}), \qquad a \in A, \quad 1 \leq i \leq n-1. \tag{5.30}$$

Proof. *a)* Since the homomorphism $\pi \circ \varphi = \Omega$ is cyclic, we have $\pi[\Psi(\mathbf{B}'(n))] = \{1\}$; hence, $\Psi(\mathbf{B}'(n))$ is contained in the Abelian group $\operatorname{Ker} \pi = H$. Therefore, $\Psi(\mathbf{B}'(n)) = \{1\}$ and Ψ is cyclic.

b) In case (i), the $\mathbf{B}(n)$-action on the group $H = A$ is trivial, and hence

$$H^1_\Omega(\mathbf{B}(n), A) \cong \operatorname{Hom}(\mathbf{B}(n), A) \cong A.$$

Consider case (ii). By Lemma 5.1, the elements $\mathbf{h}_i = (a_i^1, \dots, a_i^t) \in A^{\oplus t}$ are the values $\mathbf{z}(s_i)$ of a cocycle $\mathbf{z}$ if and only if they satisfy relations (5.13), (5.13′), which may be written as

$$\mathbf{h}_i - T_S\mathbf{h}_i = \mathbf{h}_j - T_S\mathbf{h}_j, \qquad 1 \leq i, j \leq n-1, \quad |i-j| \geq 2, \tag{5.31}$$

and

$$\mathbf{h}_i - T_S\mathbf{h}_i + T_{S^2}\mathbf{h}_i = \mathbf{h}_{i+1} - T_S\mathbf{h}_{i+1} + T_{S^2}\mathbf{h}_{i+1}, \qquad 1 \leq i \leq n-2, \tag{5.31′}$$

respectively. Since $n > 4$, system (5.31) contains the equations

$$\begin{aligned} \mathbf{h}_1 - T_S\mathbf{h}_1 &= \mathbf{h}_j - T_S\mathbf{h}_j, \quad 3 \leq j \leq n-1, \\ \mathbf{h}_2 - T_S\mathbf{h}_2 &= \mathbf{h}_j - T_S\mathbf{h}_j, \quad 4 \leq j \leq n-1; \end{aligned}$$

thereby,

$$\mathbf{h}_1 - T_S\mathbf{h}_1 = \mathbf{h}_2 - T_S\mathbf{h}_2 = \ldots = \mathbf{h}_{n-1} - T_S\mathbf{h}_{n-1}\,.$$

Combined with (5.31′), this shows that $T_{S^2}\mathbf{h}_1 = T_{S^2}\mathbf{h}_2 = \ldots = T_{S^2}\mathbf{h}_{n-1}$. However, S^2 is just a permutation of coordinates, and thus

$$\mathbf{h}_1 = \mathbf{h}_2 = \ldots = \mathbf{h}_{n-1}. \tag{5.32}$$

In turn, (5.32) implies both (5.31) and (5.31′), which shows that

$$\mathcal{Z} = \mathcal{Z}^1_\Omega(\mathbf{B}(n), A^{\oplus t}) \cong A^{\oplus t}. \tag{5.33}$$

Further, a cocycle $\mathbf{z}$ with the values $\mathbf{h}_1 = \mathbf{h}_2 = \ldots = \mathbf{h}_{n-1} = \mathbf{v} = (v^1, ..., v^t) \in A^{\oplus t}$ is a coboundary if and only if there exists $\mathbf{h} = (u^1, ..., u^t) \in A^{\oplus t}$ such that

$$\mathbf{v} = T_S\mathbf{h} - \mathbf{h}. \tag{5.34}$$

Since S is a t-cycle, it is readily seen that for a given $\mathbf{v}$ the equation (5.34) has a solution $\mathbf{h}$ if and only if

$$\sum_{i=1}^{t} v^i = 0. \tag{5.35}$$

Thus, $\mathcal{B} \subset \mathcal{Z} = A^{\oplus t}$ consists of all the elements $\mathbf{v} = (v^1, ..., v^t) \in A^{\oplus t}$ that satisfy (5.35). Joined with (5.33), this completes the proof of the statement (b). $\square$

§6. Homomorphisms $\mathbf{B}'(k) \to \mathbf{S}(n)$, $\mathbf{B}'(k) \to \mathbf{B}(n)$ $(n < k)$, and $\mathbf{B}(k) \to \mathbf{S}(n)$ $(n \leq 2k)$

Here we prove Theorems A(c), E, F, and G. Our first goal is Theorem E(a).

6.0. Homomorphisms $\mathbf{B}(k) \to \mathbf{S}(k+1)$. We start with the following obvious property of retractions Ω.

Lemma 6.1. *Let ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ be a homomorphism, and let $\mathfrak{C} = \{C_1, ..., C_t\}$ be the r-component of the permutation $\widehat{\sigma}_1$. Assume that either $t < k-2 \neq 4$ or $t \leq 2$. Then the homomorphism $\Omega = \Omega_{\mathfrak{C}}$ is cyclic, i. e. there exists a permutation $g \in \mathbf{S}(\mathfrak{C}) \cong \mathbf{S}(t)$ such that*

$$\widehat{\sigma}_{i+2} C_m \widehat{\sigma}_{i+2}^{-1} = g(C_m) = C_{g(m)} \tag{6.1}$$

whenever $1 \leq i \leq k-3$ and $1 \leq m \leq t$.

Proof. The case $t \leq 2$ is trivial. If $t < k-2 \neq 4$, Ω is cyclic by Theorem 2.1(a). □

The next lemma might be proven by a straightforward (but rather long) computation. Instead, we use the cohomology approach in order to show how it works in the simplest case.

Lemma 6.2. *Assume that $3 \leq k \neq 4$. Let ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ be a noncyclic homomorphism such that $\widehat{\sigma}_1 = [2,2]$. Then $n \geq k+2$. Moreover,*

a) if $n < 2k$, then the homomorphism ψ is conjugate to the homomorphism

$$\phi^{(1)}_{k,n}\colon\ \sigma_i \mapsto (1,2)(i+2, i+3),\quad 1 \leq i \leq k-1,\quad \phi^{(1)}_{k,n} \sim \{\mathbf{2}\} \times \mu_k \times \mathbf{1}_{n-k-2};$$

b) if $n \geq 2k$, then ψ is either conjugate to $\phi^{(1)}_{k,n}$ or conjugate to the homomorphism

$$\phi^{(2)}_{k,n}\colon\ \sigma_i \mapsto (2i-1, 2i+1)(2i, 2i+2),\quad 1 \leq i \leq k-1,\quad \phi^{(2)}_{k,n} \sim \mu_k \times \mu_k \times \mathbf{1}_{n-2k};$$

c) in any case the homomorphism ψ is intransitive.

Proof. For $k = 3$, all the assertions follow from Lemma 1.9. Suppose $k > 4$. Let $\widehat{\sigma}_1 = C_1 C_2$, where $C_1 = (1,3)$, $C_2 = (2,4)$[3]; so, $\mathfrak{C} = \{C_1, C_2\}$ is the only nondegenerate component of $\widehat{\sigma}_1$, with $\Sigma = \operatorname{supp} \mathfrak{C} = \{1,2,3,4\}$. The corresponding retraction Ω: $\mathbf{B}(k-2) \to \mathbf{S}(2)$ is cyclic; hence, either Ω is trivial or $\Omega = \{\mathbf{2}\}$.

Suppose first $\Omega = \{\mathbf{2}\}$. Then Theorem 5.10 shows that the cocycles $z_0(s_i) = (0,0) \in (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ and $z_1(s_i) = (1,0) \in (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ $(1 \leq i \leq k-3)$ represent all the cohomology classes. It follows from Lemma 4.5 that the Ω-homomorphism Ψ_Σ: $\mathbf{B}(k-2) \to G \subset \mathbf{S}(4)$ (up to H-conjugation) coincides with the homomorphism $\varepsilon = \rho \circ \Omega$, $\varepsilon(s_i) = (1,2)(3,4)$ for all i (the second possibility, namely, $\Psi_\Sigma(s_i) = C_1(1,2)(3,4) = (1,2,3,4)$ for all $i \leq k-3$, cannot occur here, since $[\widehat{\sigma}_{i+2}] = [2,2]$). This means that $\widehat{\sigma}_{i+2} \mid \Sigma = (1,3)(2,4)$, and the condition $[\widehat{\sigma}_{i+2}] = [2,2]$ implies that $\widehat{\sigma}_{i+2} = (1,3)(2,4)$ for all $i \leq k-3$. Hence, ψ is cyclic, which contradicts our assumption.

[3]Here we choose this normalization of ψ instead of the usual $C_1 = (1,2)$, $C_2 = (3,4)$.

So, Ω is trivial; any Ω-homomorphism is just a homomorphism ϕ: $\mathbf{B}(k-2) \to H$. There are precisely four of them, namely, the homomorphisms defined by

$$\phi_0(s_i) = 1; \quad \phi_1(s_i) = C_1; \quad \phi_2(s_i) = C_2; \quad \phi_3(s_i) = C_1C_2; \quad (1 \le i \le k-3).$$

Therefore, we may assume that the homomorphism Ψ_Σ coincides with one of the homomorphisms ψ_j, $j = 0, 1, 2, 3$.

If $\Psi_\Sigma = \phi_0$, then $\widehat{\sigma}_{i+2} \mid \Sigma = \Psi_\Sigma(s_i) = \phi_0(s_i) = 1$. Hence, all the permutations $\widehat{\sigma}_3, ..., \widehat{\sigma}_{k-1}$ are disjoint with $\widehat{\sigma}_1$, and we may assume that $\widehat{\sigma}_3 = (5,7)(6,8)$. The relations $\widehat{\sigma}_2 \infty \widehat{\sigma}_1$, $\widehat{\sigma}_2 \infty \widehat{\sigma}_3$ and Lemma 1.9 imply that (up to a $\widehat{\sigma}_1$- and $\widehat{\sigma}_3$-admissible conjugation) $\widehat{\sigma}_2 = (3,5)(4,6)$. Since $\operatorname{supp} \widehat{\sigma}_4 \cap \{1,2,3,4\} = \varnothing$ and $\widehat{\sigma}_4\widehat{\sigma}_2 = \widehat{\sigma}_2\widehat{\sigma}_4$, Lemma 1.8 shows that $\operatorname{supp} \widehat{\sigma}_4 \cap \operatorname{supp} \widehat{\sigma}_2 = \varnothing$; in particular, $5, 6 \notin \operatorname{supp} \widehat{\sigma}_4$. Since $\widehat{\sigma}_4 \infty \widehat{\sigma}_3$, it follows from Lemma 1.9 that $\widehat{\sigma}_4 = (7,9)(8,10)$ (up to conjugation that is $\widehat{\sigma}_i$-admissible for all $i \le 3$). By induction, we obtain that $n \ge 2k$ and $\psi \sim \phi_{k,n}^{(2)}$.

The homomorphisms ϕ_1, ϕ_2 are not H-conjugate; however, they are G-conjugate; so, it is sufficient to handle the case $\Psi_\Sigma = \phi_1$. In this case $C_1 \preccurlyeq \widehat{\sigma}_i$ for all $i \ne 2$; hence, $\widehat{\sigma}_i = C_1D_i$, where every D_i is a transposition disjoint with C_1 and C_2. Since $\widehat{\sigma}_2\widehat{\sigma}_4 = \widehat{\sigma}_4\widehat{\sigma}_2$, we have $\widehat{\sigma}_2C_1D_4\widehat{\sigma}_2^{-1} = C_1D_4$. The relation $\widehat{\sigma}_1 \infty \widehat{\sigma}_2$ implies that $\widehat{\sigma}_2C_1\widehat{\sigma}_2^{-1} = C_1$. (For otherwise, $\widehat{\sigma}_2C_1\widehat{\sigma}_2^{-1} = D_4$, and the supports of $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ have exactly two common symbols belonging to the transposition C_1; however, this contradicts Lemma 1.9.) Hence, the set $\Sigma_1 = \operatorname{supp} \widehat{\sigma}_1$ is $\widehat{\sigma}_2$-invariant. The relations $\widehat{\sigma}_2 \infty \widehat{\sigma}_1$ and $\widehat{\sigma}_3 \mid \Sigma_1 = C_1$ imply that $\widehat{\sigma}_2 \mid \Sigma_1 = C_1$. Thus, $C_1 \preccurlyeq \widehat{\sigma}_2$. Taking into account that $k > 4$, it is easy to see that $n \ge k+2$ and $\psi \sim \phi_{k,n}^{(1)}$.

Finally, if $\Psi_\Sigma = \phi_3$, we have $\widehat{\sigma}_{i+2} \mid \Sigma = \Psi_\Sigma(s_i) = \phi_3(s_i) = (1,3)(2,4)$ for all $i \le k-3$. But $[\widehat{\sigma}_{i+2}] = [2,2]$; hence, $\widehat{\sigma}_{i+2} = (1,3)(2,4)$ for all $i \le k-3$ and ψ is cyclic, which contradicts our assumption. Thereby, statements (a) and (b) of the lemma are proven. The statement (c) is a trivial corollary of (a) and (b). □

Remark 6.1. *Let ψ: $\mathbf{B}(4) \to \mathbf{S}(n)$ be a noncyclic homomorphism such that $\widehat{\sigma}_1 = [2,2]$. Then, besides the possibilities described in statements (a) and (b) of Lemma 6.2, only the following four cases may occur:*

4c) $n \ge 5$ and

$$\psi \sim \phi_{4,n}^{(3)}: \begin{cases} \sigma_1, \sigma_3 \mapsto (1,2)(3,4), \\ \sigma_2 \quad \mapsto (1,2)(4,5); \end{cases}$$

4d) $n \ge 6$ and

$$\psi \sim \phi_{4,n}^{(4)}: \begin{cases} \sigma_1, \sigma_3 \mapsto (1,2)(3,4), \\ \sigma_2 \quad \mapsto (2,5)(4,6); \end{cases}$$

4e) $n \ge 6$ and

$$\psi \sim \phi_{4,n}^{(5)}: \begin{cases} \sigma_1 \mapsto (1,2)(3,4), \\ \sigma_2 \mapsto (2,5)(4,6), \\ \sigma_3 \mapsto (1,4)(2,3); \end{cases}$$

4f) $n \ge 7$ and

$$\psi \sim \phi_{4,n}^{(6)}: \begin{cases} \sigma_1 \mapsto (1,2)(3,4), \\ \sigma_2 \mapsto (2,5)(4,6), \\ \sigma_3 \mapsto (1,2)(6,7). \end{cases}$$

All these homomorphisms except $\phi_{4,6}^{(5)} = \psi_{4,6}^{(3)}$ (see Proposition 3.8) *are intransitive.* □

Theorem 6.3. *Assume that $k > 5$. Then:*

a) any transitive homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(k+1)$ is cyclic;

b) any noncyclic homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(k+1)$ is either conjugate to the homomorphisms

$$\mu_{k+1}^k = \mu_k \times \mathbf{1}_1\colon\ \mathbf{B}(k) \to \mathbf{S}(k+1),$$

or (which may happen for $k = 6$ only) conjugate to the homomorphism

$$\nu_7^6 = \nu_6 \times \mathbf{1}_1\colon\ \mathbf{B}(6) \to \mathbf{S}(7),$$

where ν_6 is Artin's homomorphism.

Proof. *a*) Assume first that for some $k \geq 7$ there is a noncyclic transitive homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(k+1)$. By Lemma 1.24(*b*), there is a prime p that satisfies $4 \leq (k+1)/2 < p \leq k-2$, and Lemma 1.22 implies that $\widehat{\sigma}_1$ has at least $k-2$ fixed points. Therefore, $\#\operatorname{supp} \widehat{\sigma}_1 \leq 3$, and thus either $[\widehat{\sigma}_1] = [2]$ or $[\widehat{\sigma}_1] = [3]$. However, this contradicts Lemma 1.20.

Assume now that there is a noncyclic transitive homomorphism ψ: $\mathbf{B}(6) \to \mathbf{S}(7)$. Let $T = \widehat{\alpha}_{3,6} = \widehat{\sigma}_3\widehat{\sigma}_4\widehat{\sigma}_5 \in \mathbf{S}(7)$. We apply Corollary 1.16 with $i = 3$, $j = 6$ (so, $j-i+1 = 4$) and obtain that 4 divides $\operatorname{ord} T$. Hence, the cyclic decomposition of T contains precisely one 4-cycle C. $\widehat{\sigma}_1$ commutes with T (see (0.6)), and Lemma 1.4(*b*) implies that $\widehat{\sigma}_1 \mid \operatorname{supp} C = C^q$ for some integer q, $0 \leq q \leq 3$. Let us consider all these possibilities for q.

If $q = 0$, then $\operatorname{supp} C \subseteq \operatorname{Fix} \widehat{\sigma}_1$ and $\#\operatorname{supp} \widehat{\sigma}_1 \leq 3$, which contradicts Lemma 1.20.

If $q = 1$ or $q = 3$, then $[C^q] = [4]$ and $C^q \preccurlyeq \widehat{\sigma}_1$, which contradicts Lemma 2.19(*a*) (with $k = 6$, $n = 7$, $r = 4 > 7/2 = n/2$).

Finally, let $q = 2$. Then $[C^q] = [C^2] = [2,2]$ and $C^2 \preccurlyeq \widehat{\sigma}_1$. If $\widehat{\sigma}_1 \neq C^2$, then either $[\widehat{\sigma}_1] = [2,2,2]$ and $\widehat{\sigma}_1$ has the only fixed point, or $[\widehat{\sigma}_1] = [2,2,3]$ and $\widehat{\sigma}_1$ has the only invariant set of length 3 (the support of the 3-cycle); however, this contradicts Lemma 1.18. Hence, $\widehat{\sigma}_1 = C^2$ and $[\widehat{\sigma}_1] = [2,2]$, which contradicts Lemma 6.2(*c*).

b) Since ψ is noncyclic and (by the statement (*a*)) intransitive, Theorem 2.1(*a*) shows that the group $G = \operatorname{Im} \psi \subset \mathbf{S}(k+1)$ has exactly one orbit Q of length k and one fixed point. Hence, ψ is the composition of its reduction ψ_Q: $\mathbf{B}(k) \to \mathbf{S}(Q) \cong \mathbf{S}(k)$ and the natural embedding $\mathbf{S}(Q) \hookrightarrow \mathbf{S}(k+1)$. Clearly, ψ_Q is a noncyclic transitive homomorphism, and Artin Theorem shows that (up to conjugation of ψ) either $\psi_Q = \mu_k$ or $k = 6$ and $\psi_Q = \nu_6$. This gives the desired result. □

6.1. Some homomorphisms of the commutator subgroup $\mathbf{B}'(k)$. Our next goal is Theorem A(*c*). For any $k \geq 4$, we have the embedding λ_k': $B(k-2) \to \mathbf{B}'(k)$ defined by $s_i \mapsto c_i = \sigma_{i+2}\sigma_1^{-1}$, $1 \leq i \leq k-3$ (Remark 0.4). Recall also that the *multiple commutator subgroups* $H^{(n)}$ of a group H are defined by $H^{(0)} = H$ and $H^{(n)} = (H^{(n-1)})'$ for $n \geq 1$.

Lemma 6.4. *Suppose $k > 4$. Given a group homomorphism ψ: $\mathbf{B}'(k) \to H$, consider the composition $\phi = \psi \circ \lambda_k'$: $\mathbf{B}(k-2) \xrightarrow{\lambda_k'} \mathbf{B}'(k) \xrightarrow{\psi} H$.*

a) $\operatorname{Im} \phi \subseteq \operatorname{Im} \psi \subseteq H^{(n)}$ for any $n \geq 0$.

b) Assume that either i) $\phi(s_1) = \phi(s_q)$ for some q that satisfies $2 \leq q \leq k-3$, or ii) $\phi(s_1^{-1}) = \phi(s_3)$. Then ψ is trivial. Particularly, if ϕ is cyclic, then ψ is trivial.

Proof. *a*) Since $k > 4$, the group $\mathbf{B}'(k)$ is perfect; hence, $\mathbf{B}'(k) = (\mathbf{B}'(k))^{(n)}$ for any $n \geq 0$. Therefore, $\operatorname{Im}\,\phi \subseteq \operatorname{Im}\,\psi = \psi(\mathbf{B}'(k)) = \psi((\mathbf{B}'(k))^{(n)}) \subseteq H^{(n)}$ for any $n \geq 0$.

b) Clearly, $\phi(s_i) = \psi(\lambda'_k(s_i)) = \psi(c_i)$; thus, (i) means $\psi(c_1) = \psi(c_q)$ and (ii) means $\psi(c_1^{-1}) = \psi(c_3)$. Hence, to prove the lemma, it is sufficient to show that *the system of relations* (0.14) – (0.21) *joined with one of the relations* (i_q) $c_1 = c_q$, (ii_3) $c_1^{-1} = c_3$ *defines a presentation of the trivial group.* This is a simple exercise, and we only sketch the proof.

Assume that (i_q) is fulfilled. Then (0.19) implies $vc_1v^{-1} = c_1u^{-1}$; by (0.16), this shows that $c_1^2 = wu$. Now (0.14) implies $uc_1^2u^{-1} = w^2$, which leads to $w^2 = uv$ and $u = w$. Using (0.14) once again, we obtain $uc_1u^{-1} = w = u$; hence, $c_1 = u = w$. In view of (0.15), this means that $c_1 = u = w = 1$. Braid relations (0.20), (0.21) and the relation $c_1 = 1$ imply that $c_i = 1$ for all i. Finally, (0.18) shows that $u = 1$.

Assume that $c_1^{-1} = c_3$. Then (0.19) implies $vc_1v^{-1} = uc_1$; taking into account (0.16), we obtain $uc_1 = c_1^{-1}w$. Then (0.14) leads to $uwu^{-1} = wuw$. Using this and (0.15), we obtain $c_1 = u^{-1}w$, and (0.14) shows that $u = 1$ and $w = c_1$. These relations and (0.15) imply $c_1 = w = 1$, and relations (0.20), (0.21) show that $c_i = 1$ for all i. Finally, from (0.18) we obtain $v = 1$. □

Lemma 6.5. *Consider a homomorphism* $\psi\colon \mathbf{B}'(6) \to \mathbf{S}(5)$ *and assume that the composition* $\phi = \psi \circ \lambda'_6\colon \mathbf{B}(4) \hookrightarrow \mathbf{B}'(6) \to \mathbf{S}(5)$ *is intransitive. Then* ψ *is trivial.*

Proof. In view of Lemma 6.4(b), it is sufficient to show that $\phi(s_1) = \phi(s_3)$. Set $G = \operatorname{Im}\,\phi \subseteq \mathbf{S}(5)$ and consider all G-orbits $\Sigma \subseteq \mathbf{\Delta}_5$ and all the reductions

$$\phi_\Sigma\colon \mathbf{B}(4) \to \mathbf{S}(\Sigma)$$

of ϕ to these orbits. By our assumption, $\#\Sigma \leq 4$ for any G-orbit Σ. If all the reductions are cyclic, we are done. Assume that there is a G-orbit Σ with the noncyclic reduction ϕ_Σ; then Σ is the only orbit with this property and $\#\Sigma \geq 3$. If $\#\Sigma = 3$, then Theorem 2.14(a) implies that $\phi_\Sigma(s_1) = \phi_\Sigma(s_3)$; in fact, we have $\phi(s_1) = \phi(s_3)$ (since the reduction to any other G-orbit is cyclic). Finally, assume that $\#\Sigma = 4$. By Lemma 6.4(a), $G = \operatorname{Im}\,\phi \subseteq \mathbf{S}'(5) = \mathbf{A}(5)$; hence, G contains only even permutations. The set $\mathbf{\Delta}_5 - \Sigma$ consists of a single point that is a fixed point of G. It follows that *the image of the noncyclic transitive homomorphism* $\phi_\Sigma\colon \mathbf{B}(4) \to \mathbf{S}(\Sigma) \cong \mathbf{S}(4)$ *contains only even permutations.* This property and the sentence (c) of Artin Theorem imply that ϕ_Σ is conjugate to the homomorphism $\nu_{4,3}$; thus, $\phi_\Sigma(s_1) = \phi_\Sigma(s_3)$ and $\phi(s_1) = \phi(s_3)$. □

Theorem 6.6. *If* $k > 4$ *and* $n < k$, *then the group* $\mathbf{B}'(k)$ *does not possess nontrivial homomorphisms into the groups* $\mathbf{S}(n)$ *and* $\mathbf{B}(n)$.

Proof. Consider first a homomorphism $\psi\colon \mathbf{B}'(k) \to \mathbf{S}(n)$. By Lemma 6.4($a$), we have $\operatorname{Im}\,\psi \subseteq \mathbf{S}'(n) = \mathbf{A}(n)$; particularly, the homomorphism ψ cannot be surjective. The case $n < 5$ is trivial, since for such n the alternating group $\mathbf{A}(n)$ is solvable and the group $\mathbf{B}'(k)$ is perfect. So, we may assume that $4 < n < k$ and $k > 5$.

Consider the composition $\phi = \psi \circ \lambda'_k\colon \mathbf{B}(k-2) \xrightarrow{\lambda'_k} \mathbf{B}'(k) \xrightarrow{\psi} \mathbf{S}(n)$. By Lemma 6.4($b$), it is sufficient to prove that ϕ is cyclic or at least satisfies the condition $\phi(s_1) = \phi(s_3)$. If $k > 6$ and $n < k - 2$, then ϕ is cyclic by Theorem 2.1(a). Hence, must only consider the following three cases: i) $k = 6$ and $n = 5$; ii) $k > 6$ and $n = k - 2$; iii) $k > 6$ and $n = k - 1$.

i) In this case we deal with the homomorphism ϕ: $\mathbf{B}(4) \to \mathbf{S}(5)$. If ϕ is intransitive, then the conclusion follows by Lemma 6.5. If ϕ is transitive, then Lemma 3.2 shows that $\phi(s_1) = \phi(s_3)$.

ii) In this case we deal with the homomorphism ϕ: $\mathbf{B}(k-2) \to \mathbf{S}(k-2)$. As we noted above, the homomorphism ψ cannot be surjective; hence, ϕ is nonsurjective and Lemma 2.7 implies that ϕ is cyclic.

iii) In this case we deal with the homomorphism ϕ: $\mathbf{B}(k-2) \to \mathbf{S}(k-1)$. We shall consider the following two cases: iii_1) the homomorphism ϕ is intransitive; iii_2) the homomorphism ϕ is transitive.

iii_1) In this case we may also assume that the image $G = \operatorname{Im} \phi \subset \mathbf{S}(k-1)$ has at least one G-orbit $\Sigma \subset \mathbf{\Delta}_{k-1}$ such that the reduction ϕ_Σ is noncyclic. Since ϕ is intransitive, it follows from Theorem 2.1(a) that $\#\Sigma = k-2$; certainly, Σ is the only orbit of such length. By Lemma 6.4(a), $G = \operatorname{Im} \phi \subseteq \mathbf{S}'(k-1) = \mathbf{A}(k-1)$; hence, G contains only even permutations. The set $\mathbf{\Delta}_{k-1} - \Sigma$ consists of a single point that is a fixed point of G. This implies that the image of the noncyclic homomorphism ϕ_Σ: $\mathbf{B}(k-2) \to \mathbf{S}(\Sigma) \cong \mathbf{S}(k-2)$ contains only even permutations. However, this contradicts Lemma 2.7.

iii_2) If $k > 7$, then $k-2 > 5$ and ϕ is cyclic by Theorem 6.3(a). Finally, if $k = 7$, then $n = k-1 = 6$ and we deal with the transitive noncyclic homomorphism ϕ: $\mathbf{B}(5) \to \mathbf{S}(6)$. By Proposition 3.9, ϕ must be conjugate to the homomorphism $\psi_{5,6}$. However, this is impossible, since the $\psi_{5,6}$ is surjective and $\operatorname{Im} \phi \subseteq \mathbf{A}(6)$. This concludes the proof for homomorphisms $\mathbf{B}'(k) \to \mathbf{S}(n)$.

Consider now a homomorphism φ: $\mathbf{B}'(k) \to \mathbf{B}(n)$. As we have already proved, the composition $\psi = \mu \circ \varphi$: $\mathbf{B}'(k) \xrightarrow{\varphi} \mathbf{B}(n) \xrightarrow{\mu} \mathbf{S}(n)$ of the homomorphism φ with the canonical projection μ must be trivial. Therefore, $\varphi(\mathbf{B}'(k)) \subseteq \operatorname{Ker} \mu = \mathbf{I}(n)$. By Corollary 0.1, the perfect group $\mathbf{B}'(k)$ does not possess nontrivial homomorphisms into the pure braid group $\mathbf{I}(n)$; hence the homomorphism φ is trivial. □

Remark 6.2. The groups $\mathbf{B}'(3)$ and $\mathbf{B}'(4)$ have many nontrivial homomorphisms into any (nontrivial) group. Moreover, for any $k \geq 3$ and any $n \geq k$ there exist nontrivial homomorphisms $\mathbf{B}'(k) \to \mathbf{S}(n)$ and $\mathbf{B}'(k) \to \mathbf{B}(n)$. This shows that the conditions $k > 4$ and $n < k$ in Theorem 6.6 are, in a sense, sharp. Theorem 6.6 implies Theorem 2.1 for $k > 4$. However, we could not skip Theorem 2.1 since it was used essentially (and many times) in the proof of Theorem 6.6. Note that Theorem 2.1 would hardly help to prove the very useful proposition 6.8, while Theorem 6.6 works perfectly. ○

Now we are almost ready to prove Theorem G (see Proposition 6.8 below).

Lemma 6.7. *Suppose $n < 2k$. Let ψ: $B \to \mathbf{S}(n)$ be a transitive group homomorphism.*

a) Assume that for any $m < k$ the commutator subgroup B' of B does not admit nontrivial homomorphisms into $\mathbf{S}(m)$. If ψ is imprimitive, then it is Abelian.

b) Assume that for any $m < k$ the group B itself does not admit nontrivial homomorphisms into $\mathbf{S}(m)$. Then the homomorphism ψ is primitive.

Proof. The statement (a) *presumes* that ψ is imprimitive; to treat (b) simultaneously with (a), we *assume* that ψ is imprimitive and show that this leads to a contradiction.

Let $\mathbf{\Delta}_n = Q_1 \cup \cdots \cup Q_t$, $t \geq 2$, be some decomposition of $\mathbf{\Delta}_n$ into imprimitivity sets of the group $G = \operatorname{Im} \psi \subset \mathbf{S}(n)$. Since ψ is transitive, $\#Q_1 = \cdots = \#Q_t = r$, where $r \geq 2$ and $rt = n$. Clearly,

$$2 \leq t < k \qquad \text{and} \qquad 2 \leq r < k. \tag{6.2}$$

Consider the normal subgroup $H \lhd G$ consisting of all elements $h \in G$ such that every set Q_i is h-invariant. Thus, we have the exact sequence

$$1 \to H \to G \xrightarrow{\pi} \widetilde{G} \to 1, \tag{6.3}$$

where π is the natural projection onto the quotient group $\widetilde{G} = G/H$. This quotient group, in turn, possesses the natural embedding $\widetilde{G} \hookrightarrow \mathbf{S}(\{Q_1, \ldots, Q_t\}) \cong \mathbf{S}(t)$. Consider the composition

$$\widetilde{\psi} = \pi \circ \psi \colon B \xrightarrow{\psi} G \xrightarrow{\pi} \widetilde{G}, \qquad \widetilde{G} \subseteq \mathbf{S}(t).$$

Clearly, $\widetilde{\psi}$ is surjective.

Under the assumptions made in the statement (b), the homomorphism $\widetilde{\psi}$ must be trivial; this means that the quotient group $\widetilde{G}$ is trivial, and hence $H = G$. It follows that every set Q_i is G-invariant. However, this contradicts transitivity of the homomorphism ψ.

Under the assumptions made in the statement (a), the restriction of $\widetilde{\psi}$ to the commutator subgroup B' must be trivial; hence, $\widetilde{\psi}\colon B \to \widetilde{G}$ is a surjective Abelian homomorphism, and the group $\widetilde{G}$ is Abelian. Thereby, the exact sequence (6.3) shows that the commutator subgroup G' of G is contained in H. Particularly, we have

$$\psi(B') \subseteq G' \subseteq H. \tag{6.4}$$

Every Q_i is H-invariant, and (6.4) shows that the restriction $\psi' = \psi \mid B' \to G' \subseteq H$ may be regarded as the disjoint product of the reductions

$$\psi'_{Q_i}\colon B' \to \mathbf{S}(Q_i) \cong \mathbf{S}(r), \quad 1 \leq i \leq t, \qquad \psi'_{Q_i}(b) = (\psi'(b)) \mid Q_i \quad \text{for all} \quad b \in B'.$$

In view of (6.2), each homomorphism ψ'_{Q_i} is trivial. Hence, the homomorphism $\psi' = \psi \mid B'$ is trivial, and our original homomorphism ψ is Abelian. □

Proposition 6.8. *Assume that $k > 4$ and $n < 2k$. Then*

a) Any transitive imprimitive homomorphism $\psi\colon \mathbf{B}(k) \to \mathbf{S}(n)$ is cyclic.

b) Any transitive homomorphism $\psi'\colon \mathbf{B}'(k) \to \mathbf{S}(n)$ is primitive.

Proof. The statement (a) follows immediately from Theorem 6.6 and Lemma 6.7(a) (for the group $B = \mathbf{B}(k)$), and (b) is a trivial corollary of Theorem 6.6 and Lemma 6.7(b) (for the group $B = \mathbf{B}'(k)$). □

6.2. Homomorphisms $\mathbf{B}(k) \to \mathbf{S}(k+2)$. Here we prove Theorem E(b) (see Theorem 6.15 below). To this end, we need some preparation. In the following lemma, φ_1 is the homomorphism exhibited in Corollary 5.6 (with $n = k$).

Lemma 6.9. *Assume that $6 < k < n \leq 2k$. Let $\psi\colon \mathbf{B}(k) \to \mathbf{S}(n)$ be a transitive noncyclic homomorphism. If $\#\operatorname{supp} \widehat{\sigma}_1 < 6$, then $n = 2k$, $\widehat{\sigma}_1$ is a 4-cycle, and the homomorphism ψ is conjugate to the homomorphism φ_1.*

Proof. A priori, we have the following 6 possibilities for the cyclic type of $\widehat{\sigma}_1$:

$$[\widehat{\sigma}_1] = [2];\ [2,2];\ [2,3];\ [3];\ [4];\ [5].$$

The types [2], [3], and [2, 2] are forbidden by Lemma 1.20 and Lemma 6.2(c), respectively. Let us note that $E(n/E(k/2)) \le E(2k/E(k/2)) \le 4$ for any $k \ge 6$; hence, the inequality of Lemma 1.21 eliminates the types [2, 3] and [5].

So, we are left with the type $[\widehat{\sigma}_1] = [4]$; in this case $[\widehat{\sigma}_i] = [4]$ for all i. Put $\Sigma_i = \operatorname{supp} \widehat{\sigma}_i$. The first statement of Lemma 1.21 says that $\Sigma_i \cap \Sigma_j = \varnothing$ for $|i-j| \ge 2$; particularly, $\Sigma_i \cap \Sigma_{i+2} = \varnothing$ for $1 \le i \le k-3$. Since ψ is noncyclic, we have $\widehat{\sigma}_{i+1} \infty \widehat{\sigma}_i$, $\widehat{\sigma}_{i+1} \infty \widehat{\sigma}_{i+2}$, and all three permutations are 4-cycles. The set Σ_{i+1} cannot coincide with one of the sets Σ_i, Σ_{i+2} (for otherwise, $\widehat{\sigma}_{i+1}$ would be disjoint with one of the permutations $\widehat{\sigma}_i$, $\widehat{\sigma}_{i+2}$). Hence, Lemma 1.12 implies that $\#(\Sigma_i \cap \Sigma_{i+1}) = 2$ and $\#(\Sigma_i \cap \Sigma_{i+2}) = 2$. It follows immediately that for any $m \le k-1$ the union $\Sigma_1 \cup \Sigma_2 \cup \ldots \cup \Sigma_m \subseteq \mathbf{\Delta}_n$ consists of $4 + 2(m-1)$ points. Particularly, for $m = k-1$ this union consists of $4 + 2(k-2) = 2k \ge n$ points; so, $n = 2k$. Without loss of generality, we may assume that $\widehat{\sigma}_1 = (1,4,2,3)$. It follows from Lemma 1.12 and from what has been proven above that we may assume $\widehat{\sigma}_2 = (3,6,4,5)$ (any of the other possibilities can be reduced to this case by a $\widehat{\sigma}_1$-admissible conjugation of ψ). Taking into account the above arguments and the property $\Sigma_1 \cap \Sigma_3 = \varnothing$, we obtain that (up to an admissible conjugation) $\widehat{\sigma}_3 = (5,8,6,7)$, and so on. Hence, $\psi \sim \varphi_1$. □

Lemma 6.10. *Suppose $k > 6$. Let ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ be a homomorphism such that all the components of $\widehat{\sigma}_1$ (including the degenerate component* Fix $\widehat{\sigma}_1$*) are of lengths at most $k-3$. Then ψ is cyclic.*

Proof. If $\widehat{\sigma}_1 = \mathrm{id}$, then ψ is trivial. So, we may assume that for some $r \ge 2$ the permutation $\widehat{\sigma}_1$ has the r-component $\mathfrak{C}$ of some length $t \ge 1$. Put $\Sigma_{\mathfrak{C}} = \operatorname{supp} \mathfrak{C}$ and consider the retraction $\Omega_{\mathfrak{C}}$: $\mathbf{B}(k-2) \to \mathbf{S}(\mathfrak{C}) \cong \mathbf{S}(t)$ of ψ to $\mathfrak{C}$ (see §4). According to our assumptions, we have $k-2 > 4$ and $t < k-2$; by Lemma 6.1, $\Omega_{\mathfrak{C}}$ is cyclic. It follows from Theorem 5.10(a) that the $\Omega_{\mathfrak{C}}$-homomorphism $\Psi_{\Sigma_{\mathfrak{C}}}$: $\mathbf{B}(k-2) \to G_{\mathfrak{C}} \subset \mathbf{S}(rt)$ is also cyclic. This means that $\Psi_{\Sigma_{\mathfrak{C}}}(s_1) = \ldots = \Psi_{\Sigma_{\mathfrak{C}}}(s_{k-3})$, and thus

$$\widehat{\sigma}_3 \mid \Sigma_{\mathfrak{C}} = \ldots = \widehat{\sigma}_{k-1} \mid \Sigma_{\mathfrak{C}}. \tag{6.5}$$

Put $\Sigma = \bigcup_{\mathfrak{C}} \Sigma_{\mathfrak{C}}$, where $\mathfrak{C}$ runs over all the nondegenerate components of $\widehat{\sigma}_1$; clearly, $\Sigma = \operatorname{supp} \widehat{\sigma}_1$. The sets Σ and $\Sigma' = \operatorname{Fix} \widehat{\sigma}_1 = \mathbf{\Delta}_n - \Sigma$ are invariant under all the permutations $\widehat{\sigma}_3, \ldots, \widehat{\sigma}_{k-1}$. Since (6.5) holds for *every* nondegenerate component $\mathfrak{C}$ of $\widehat{\sigma}_1$, it follows that there is a permutation $S \in \mathbf{S}(\Sigma)$ such that $\widehat{\sigma}_3 \mid \Sigma = \ldots = \widehat{\sigma}_{k-1} \mid \Sigma = S$. By our assumption, the degenerate component $\Sigma' = \operatorname{Fix} \widehat{\sigma}_1$ contains at most $k-3$ points:

$$\#\Sigma' = \#\operatorname{Fix} \widehat{\sigma}_1 \le k-3. \tag{6.6}$$

Set $S_i' = \widehat{\sigma}_i \mid \Sigma'$, $i = 3, \ldots, k-1$. Clearly,

$$\widehat{\sigma}_i = S \cdot S_i' \qquad \text{for all } i = 3, \ldots, k-1. \tag{6.7}$$

For any $i = 3, \ldots, k-1$, we have $\operatorname{supp} S \cap \operatorname{supp} S_i' = \varnothing$; hence, it follows from (6.7) that the permutation $S_3', \ldots, S_{k-1}'$ satisfy the standard braid relations $S_i'S_j' = S_j'S_i'$ for $|i-j| > 1$

and $S'_iS'_{i+1}S'_i = S'_{i+1}S'_iS'_{i+1}$ for $3 \le i < k-1$. This means that we can define a group homomorphism ϕ: $\mathbf{B}(k-2) \to \mathbf{S}(\Sigma')$ by $\phi(s_i) = S'_{i+2}$, $i = 1, ..., k-3$. By Theorem 2.1(a), condition (6.6) implies that this homomorphism ϕ is cyclic. Hence, $S'_3 = \ldots = S'_{k-1}$. In view of (6.7), this shows that $\widehat{\sigma}_3 = \ldots = \widehat{\sigma}_{k-1}$ and the homomorphism ψ is cyclic. □

The following lemma supplies the upper bound $t \le k-2$ for the length t of any nondegenerate component of every permutation $\widehat{\sigma}_i$ (provided $6 < k < n < 2k$ and ψ is noncyclic). Actually, Theorem 6.20 shows that $t \le (k+1)/2$; however, we are not ready to prove the latter statement now.

Lemma 6.11. *Suppose $6 < k < n < 2k$. Let ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ be a homomorphism such that the permutation $\widehat{\sigma}_1$ has a nondegenerate component $\mathfrak{C}$ of length $t > k-3$. Then ψ is cyclic.*

Proof. Suppose, on the contrary, that ψ is noncyclic. The assumptions $n < 2k$ and $t > k-3$ imply that $\mathfrak{C}$ is the 2-component of $\widehat{\sigma}_1$. Put $\Sigma = \operatorname{supp} \mathfrak{C}$ and $\Sigma' = \mathbf{\Delta}_n - \Sigma$; so $\#\Sigma = 2t$ and $\#\Sigma' = n - 2t < 2k - 2(k-2) = 4$. Since $k > 6$, any homomorphism $\mathbf{B}(k-2) \to S(\Sigma')$ is cyclic (Theorem 2.1(a)). Particularly, the homomorphism $\Psi_{\Sigma'}$ is cyclic, and Lemma 4.4 implies that *the homomorphisms Ω: $\mathbf{B}(k-2) \to \mathbf{S}(t)$ and Ψ_Σ: $\mathbf{B}(k-2) \to G \subset \mathbf{S}(2t)$ must by noncyclic.*

We may assume that the homomorphism ψ is normalized; this means that

$$\Sigma = \{1, 2, \ldots, 2t\}, \quad \widehat{\sigma}_1 \mid \Sigma = C_1 \cdots C_t, \text{ where } C_m = (2m-1, 2m) \text{ for } m = 1, \ldots, t.$$

Clearly, $t \le k-1$; so, either $t = k-2$ or $t = k-1$. Therefore, we must consider the following three cases:

i) $t = k-2$ and $\widehat{\sigma}_1$ is a disjoint product of $k-2$ transpositions;
ii) $t = k-2$ and $\widehat{\sigma}_1$ is a disjoint product of $k-2$ transpositions and a 3-cycle;
iii) $t = k-1$ and $\widehat{\sigma}_1$ is a disjoint product of $k-1$ transpositions.

i) In this case we deal with the noncyclic homomorphisms Ω: $\mathbf{B}(k-2) \to \mathbf{S}(k-2)$ and Ψ_Σ: $\mathbf{B}(k-2) \to G \subset \mathbf{S}(2k-4)$. By Artin Theorem and Remark 2.2, either

i_a) *Ω is conjugate to the canonical projection μ: $\mathbf{B}(k-2) \to \mathbf{S}(k-2)$*

or

i_b) *$k = 8$ and Ω is conjugate to Artin's homomorphism ν_6: $\mathbf{B}(6) \to \mathbf{S}(6)$.*

We shall show that these cases are impossible.

i_a) In this case, without loss of generality we may assume that $\Omega = \mu$ (see Remark 4.1). Then the homomorphism Ψ_Σ must be H-conjugate to one of the four μ-homomorphisms φ_j: $\mathbf{B}(k-2) \to G \subseteq \mathbf{S}(2k-4)$, $j = 0, 1, 2, 3$, listed in Corollary 5.6 (with $n = k-2$).

The homomorphisms $\varphi_0, \varphi_1, \varphi_3$ may be eliminated by trivial reasons. Indeed, if Ψ_Σ is conjugate to one of φ_0, φ_1, then the support of $\widehat{\sigma}_{i+2} \mid \Sigma = \Psi_\Sigma(s_i)$ consists of 4 points; however, $\#\operatorname{supp} \widehat{\sigma}_{i+2} = \#\operatorname{supp} \widehat{\sigma}_1 = 2k-4$, and the rest $2k-8$ points of $\operatorname{supp} \widehat{\sigma}_{i+2}$ must be situated in the set Σ' containing at most 3 points, which is impossible. If $\Psi_\Sigma \sim \varphi_3$, then the cyclic decomposition of $\widehat{\sigma}_{i+2} \mid \Sigma = \Psi_\Sigma(s_i)$ contains a 4-cycle; but this is not the case, since $\widehat{\sigma}_{i+2} \sim \widehat{\sigma}_1$.

By condition (!!!) (see Declaration in §4.1), we may assume that Ψ_Σ *coincides* with φ_2. Then any permutation $\widehat{\sigma}_{i+2} \mid \Sigma = \Psi_\Sigma(s_i) = \varphi_2(s_i)$ is a product of $k-2$ disjoint transpositions. However, $\widehat{\sigma}_{i+2}$ itself is such a product; therefore, $\widehat{\sigma}_{i+2} = \varphi_2(s_i)$. Particularly, for $i=1$ and $i=k-3$, we obtain, respectively,

$$\begin{aligned}\widehat{\sigma}_3 &= \underbrace{(1,3)(2,4)}(5,6)\cdots(2k-9,2k-8)(2k-7,2k-6)(2k-5,2k-4),\\ \widehat{\sigma}_{k-1} &= (1,2)(3,4)(5,6)\cdots(2k-9,2k-8)\underbrace{(2k-7,2k-5)(2k-6,2k-4)}.\end{aligned}$$

Using these formulas and computing the permutations $\widehat{\sigma}_3 \cdot C \cdot \widehat{\sigma}_3^{-1}$ for some particular transpositions $C \preccurlyeq \widehat{\sigma}_{k-1}$, we obtain:

$$\begin{aligned}&\widehat{\sigma}_3(1,2)\widehat{\sigma}_3^{-1} = (3,4), &&\widehat{\sigma}_3(2k-7,2k-5)\widehat{\sigma}_3^{-1} = (2k-6,2k-4),\\ &\widehat{\sigma}_3(3,4)\widehat{\sigma}_3^{-1} = (1,2), &&\widehat{\sigma}_3(2k-6,2k-4)\widehat{\sigma}_3^{-1} = (2k-7,2k-5).\end{aligned}$$

In view of our definition of the homomorphism Ω^*, these formulas show that the cyclic decomposition of the permutation $\Omega^*(s_1) \in \mathbf{S}(\mathfrak{C}^*) \cong \mathbf{S}(k-2)$ must contain at least two disjoint transpositions. On the other hand, since $\Omega \sim \mu$, the permutation $\Omega(s_1) \sim \mu(s_1)$ is a transposition; hence, $\Omega^*(s_1) \neq \Omega(s_1)$, which contradicts Lemma 4.3(a).

i_b) We may assume that $\Omega = \nu_6$ (see Remark 4.1). Then the homomorphism Ψ_Σ must be H-conjugate to one of the two ν_6-homomorphisms ϕ_j: $\mathbf{B}(6) \to G \subseteq \mathbf{S}(12)$, $j=0,1$, listed in Corollary 5.8. We may assume that Ψ_Σ *coincides* with one of the homomorphisms ϕ_j. Then any permutation $\widehat{\sigma}_{i+2} \mid \Sigma = \Psi_\Sigma(s_i)$, $i = 1,\dots,5$, is a product of 6 disjoint transpositions. Since $\widehat{\sigma}_{i+2}$ itself is such a product, this means that either $\widehat{\sigma}_{i+2} = \phi_0(s_i)$ or $\widehat{\sigma}_{i+2} = \phi_1(s_i)$ for all $i=1,\dots,5$.

In each of these two cases the permutations $\phi_j(s_1)$ and $\phi_j(s_5)$ contain two common transpositions, namely, $D_5 = (9,11)$ and $D_6 = (10,12)$. Hence, the conjugation of the six transpositions $D_m \preccurlyeq \widehat{\sigma}_7 = \phi_j(s_5)$ by the permutation $\widehat{\sigma}_3 = \phi_j(s_1)$ does not change these transpositions D_5 and D_6. Consequently, the permutation $\Omega^*(s_1) \in \mathbf{S}(\mathfrak{C}^*) \cong \mathbf{S}(6)$ (defined by this conjugation) has at least two fixed points. However, this contradicts Lemma 4.3(a), since the permutation $\Omega(s_1) = \nu_6(s_1) = (C_1,C_2)(C_3,C_4)(C_5,C_6) \in \mathbf{S}(\mathfrak{C}) \cong \mathbf{S}(6)$ has no fixed points. This concludes the proof in case (i).

ii) In this case $2k > n \geq \#\operatorname{supp}\widehat{\sigma}_1 = 2(k-2)+3 = 2k-1$; so, $n = 2k-1$ and $\widehat{\sigma}_1$ has no fixed points. Hence, Σ' (the support of the 3-cycle) is the only $\widehat{\sigma}_1$-invariant set of length 3 and, by Lemma 1.18, ψ is intransitive. Clearly, ψ is the disjoint product of its reductions ψ_Σ and $\psi_{\Sigma'}$ to the (Im ψ)-invariant sets Σ and Σ', respectively.

The homomorphism $\psi_{\Sigma'}$ is cyclic ($k > \#\Sigma'$). As to the homomorphism

$$\widetilde{\psi} = \psi_\Sigma\colon\ \mathbf{B}(k) \to \mathbf{S}(\Sigma) \cong \mathbf{S}(2k-4),$$

it must be noncyclic (since ψ is noncyclic). Clearly, the permutation

$$\widetilde{\psi}(\sigma_1) = \psi_\Sigma(\sigma_1) = \psi(\sigma_1) \mid \Sigma$$

is a product of $k-2$ disjoint transposition. However, it has been already proven that this is impossible (see case (i)).

$iii)$ In this case either $n = 2k-2$ or $n = 2k-1$, and the set Σ' contains at most 1 point. We deal with the noncyclic homomorphism $\Omega\colon \mathbf{B}(k-2) \to \mathbf{S}(k-1)$. It follows from Theorem 6.3 and Proposition 3.9 that either

$iii_a)$ *Ω is conjugate to the homomorphism*

$$\mu_{k-1}^{k-2} = \mu_{k-2} \times \mathbf{1}_1\colon \mathbf{B}(k-2) \to \mathbf{S}(k-1),$$

or

$iii_b)$ *$k = 7$ and Ω is conjugate to the homomorphism $\psi_{5,6}\colon \mathbf{B}(5) \to \mathbf{S}(6)$.*

Let us show that these cases are impossible.

$iii_a)$ Again, we may assume that $\Omega = \mu_{k-1}^{k-2}$. Then the homomorphism Ψ_Σ must be H-conjugate to one of the eight homomorphisms $\psi_{0;j},\ \psi_{1;j}$,

$$\psi_{0;j} = \varphi_j \times \mathbf{1}_2, \quad \psi_{1;j} = \varphi_j \times \{\mathbf{2}\}, \quad j = 0,1,2,3,$$

listed in Corollary 5.9 (with $n = k-2$).

The homomorphisms $\psi_{x;0},\ \psi_{x;1},\ \psi_{x;3}$ $(x = 0,1)$ may be eliminated as in case (i_a). Indeed, if Ψ_Σ is conjugate to one of $\psi_{x;1},\ \psi_{x;3}$, then the cyclic decomposition of $\widehat{\sigma}_{i+2} \mid \Sigma = \Psi_\Sigma(s_i)$ contains a 4-cycle, which is impossible (for $\widehat{\sigma}_{i+2} \sim \widehat{\sigma}_1$). If $\Psi_\Sigma \sim \psi_{x;0}$, then either $[\widehat{\sigma}_{i+2} \mid \Sigma] = [2,2]$ or $[\widehat{\sigma}_{i+2} \mid \Sigma] = [2,2,2]$. Since $\widehat{\sigma}_{i+2}$ is a disjoint product of $k-1$ transpositions, at least $(k-1)-3 = k-4$ of them must be situated on the set Σ' containing at most 1 point, which is impossible.

So, we may assume that the homomorphism Ψ_Σ coincides with one of the homomorphisms $\psi_{x;2},\ x = 0,1$. In any of these two cases

$$\widehat{\sigma}_{i+2} \mid \Sigma = \Psi_\Sigma(s_i) = \psi_{x;2}(s_i) = \varphi_2(s_i) \cdot T^x,$$

where $T = (2k-3, 2k-2)$. Hence, $\widehat{\sigma}_{i+2} = \varphi_2(s_i) \cdot T^x$ for all $i = 1, \dots, k-2$. For $i = 1$ and $i = k-3$, we have, respectively,

$$\widehat{\sigma}_3 \quad = \underbrace{(1,3)(2,4)}(5,6)\cdots(2k-9,2k-8)(2k-7,2k-6)(2k-5,2k-4)\cdot T^x,$$

$$\widehat{\sigma}_{k-1} = (1,2)(3,4)(5,6)\cdots(2k-9,2k-8)\underbrace{(2k-7,2k-5)(2k-6,2k-4)}\cdot T^x.$$

As in case (i_a),

$$\begin{aligned} \widehat{\sigma}_3(1,2)\widehat{\sigma}_3^{-1} &= (3,4), \qquad & \widehat{\sigma}_3(2k-7,2k-5)\widehat{\sigma}_3^{-1} &= (2k-6,2k-4), \\ \widehat{\sigma}_3(3,4)\widehat{\sigma}_3^{-1} &= (1,2), \qquad & \widehat{\sigma}_3(2k-6,2k-4)\widehat{\sigma}_3^{-1} &= (2k-7,2k-5). \end{aligned}$$

This implies that the cyclic decomposition of $\Omega^*(s_1)$ contains at least two disjoint transpositions, which is impossible, since $\Omega(s_1) = \mu_{k-1}^{k-2}(s_1) = (1,2)$.

iii_b) In this case, we can assume that $\Omega = \psi_{5,6}$: $\mathbf{B}(5) \to \mathbf{S}(6)$ and Ψ_Σ coincides with one of the four homomorphisms η_j: $\mathbf{B}(5) \to S(12)$, $j = 0, 1, 2, 3$, exhibited in Corollary 5.7. We may certainly exclude the homomorphisms η_1, η_3 because of the 4-cycles presence. If $\Psi_\Sigma = \eta_j$, $j = 0$ or $j = 2$, then, in fact, $\widehat{\sigma}_{i+2} = \eta_j(s_i)$ for all $i = 1, 2, 3, 4$. In both cases the permutations $\eta_j(s_1)$ and $\eta_j(s_4)$ contain the two common transpositions $D_1 = (1,3)$ and $D_2 = (2,4)$; as in case (i_b), it follows that the permutation $\Omega^*(s_1) \in \mathbf{S}(\mathfrak{C}^*) \cong \mathbf{S}(6)$ has at least two fixed points. However, this contradicts Lemma 4.3(a), since the permutation $\Omega(s_1) = \psi_{5,6}(s_1) = (C_1, C_2)(C_3, C_4)(C_5, C_6) \in \mathbf{S}(\mathfrak{C}) \cong \mathbf{S}(6)$ has no fixed points. □

Corollary 6.12. *Suppose* $6 < k < n < 2k$. *Let* ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ *be a noncyclic homomorphism. Then the permutation* $\widehat{\sigma}_1$ *has at least* $k - 2$ *fixed points.*

Proof. Lemma 6.11 implies that $\widehat{\sigma}_1$ does not possess nondegenerate components of length $t > k - 3$; combined with Lemma 6.10, this shows that $\#\operatorname{Fix} \widehat{\sigma}_1 \geq k - 2$. □

Remark 6.3. Corollary 6.12 is a useful and powerful "partner" of Artin Lemma 1.22. If $6 < k < n < 2k$ and n is far from k, we have no reasonable hope to find a prime number p between $n/2$ and $k-2$; hence, Lemma 1.22 does not work. However, this lemma played important part in the proof of Corollary 6.12 (via Artin Theorem, Theorem 2.1(a), Theorem 6.3, Lemma 6.10 and Lemma 6.11). ○

In order to study homomorphisms $\mathbf{B}(k) \to \mathbf{S}(k+2)$, we start with the "exceptional" cases $k = 5$ and $k = 6$.

Remark 6.4. Note that for $k > 4$ there are the following evident noncyclic homomorphisms ψ: $\mathbf{B}(k) \to \mathbf{S}(k+2)$:

$$\psi^k_{k+2} = \mu_k \times \mathbf{1}_2\colon\ \mathbf{B}(k) \to \mathbf{S}(k+2); \qquad \widetilde{\psi}^k_{k+2} = \mu_k \times \{\mathbf{2}\}\colon\ \mathbf{B}(k) \to \mathbf{S}(k+2);$$
$$\phi^6_8 \ = \nu_6 \times \mathbf{1}_2\colon\ \mathbf{B}(6) \to \mathbf{S}(8); \qquad \widetilde{\phi}^6_8 \ = \nu_6 \times \{\mathbf{2}\}\colon\ \mathbf{B}(6) \to \mathbf{S}(8);$$
$$\psi^5_7 = \psi_{5,6} \times \mathbf{1}_1\colon\ \mathbf{B}(5) \to \mathbf{S}(7),$$

where $\psi_{5,6}$ is defined by (3.6) or (3.6′), and ν_6 is Artin's homomorphism. All these homomorphisms have the following property: *the image of any generator* σ_i *is a product of disjoint transpositions.* ○

Proposition 6.13. *Let* ψ: $\mathbf{B}(5) \to \mathbf{S}(7)$ *be a noncyclic homomorphism. Then* ψ *is intransitive and conjugate to one of the homomorphism* ψ^5_7, $\widetilde{\psi}^5_7$, ϕ^5_7. *In particular, every permutation* $\widehat{\sigma}_i$, $1 \leq i \leq 4$, *is a product of disjoint transpositions.*

Proof. Suppose, on the contrary, that ψ is transitive. Then Lemma 1.21 implies that if all the cycles $C_\nu \preccurlyeq \widehat{\sigma}_1$ are of the distinct lengths r_ν, then $\sum r_\nu \leq 3$. This eliminates all the cyclic types for $\widehat{\sigma}_1$ but [2], [3], [2, 2], [3, 3], [2, 2, 2], [2, 2, 3]. However, Lemma 1.20 excludes [2], [3], Lemma 6.2(c) excludes [2, 2], Lemma 1.18 excludes the rest types, and we obtain a contradiction.

Further, since ψ is noncyclic and (as we already have proved) intransitive, the group $G = \operatorname{Im} \psi \subset \mathbf{S}(7)$ has exactly one orbit Q of length L, $5 \leq L \leq 6$; put $Q' = \mathbf{\Delta}_7 - Q$. Clearly, either Q' is a G-orbit of length $7 - L$ or Q' consists of $7 - L$ fixed points. The homomorphism ψ is the disjoint product of its reductions ψ_Q and $\psi_{Q'}$. The reduction ψ_Q is a noncyclic transitive homomorphism $\mathbf{B}(5) \to \mathbf{S}(Q) \cong \mathbf{S}(L)$. By Artin Theorem and Proposition 3.9, we obtain that (up to conjugation of ψ) either $L = 5$ and $\psi_Q = \mu_5$ or $L = 6$ and $\psi_Q = \psi_{5,6}$. The reduction $\psi_{Q'}$ is either trivial or takes all σ_i to the same transposition. This concludes the proof. □

In the sequel, we use the following theorem of C. Jordan (see [Wi, Theorem 13.3, Theorem 13.9], or [Ha, Theorem 5.6.2, Theorem 5.7.2]):

JORDAN THEOREM. *Let $G \subseteq \mathbf{S}(n)$ be a primitive group of permutations. If G contains a transposition, then $G = \mathbf{S}(n)$; if G contains a 3-cycle, then either $G = \mathbf{S}(n)$ or $G = \mathbf{A}(n)$. Moreover, if $n = p + r$, where p is prime, $r \geq 3$, and G contains a p-cycle, then either $G = \mathbf{S}(n)$ or $G = \mathbf{A}(n)$.*

Remark 6.5. The following fact (certainly, well known to experts in permutation groups) follows trivially from Jordan Theorem:

Let $G \subseteq \mathbf{S}(n)$ be a primitive permutation group. Assume that either i) $k < n$ and G contains a p-cycle of length $p \leq 3$, or ii) $k = 6$ and $8 \leq n \leq 9$, or iii) $k = 7$ and $10 \leq n \leq 13$. Then the group G cannot be isomorphic to $\mathbf{S}(k)$.

Indeed, suppose on the contrary that $G \cong S(k)$; then $\#G = k! < n!/2$. Let us show that either $G = \mathbf{S}(n)$ or $G = \mathbf{A}(n)$ (clearly, this will contradict the above inequality). In case (i) our statement follows immediately from Jordan Theorem. In all other cases the assumption $G \cong S(k)$ implies that there is an element $g \in G$ of order p, where $p = 5$ in case (ii) and $p = 7$ in case (iii). It follows from the constraints on n that g is a p-cycle and $r = n - p \geq 3$. Hence, the last sentence of Jordan Theorem shows that either $G = \mathbf{S}(n)$ or $G = \mathbf{A}(n)$. ○

Proposition 6.14. *Any transitive homomorphism ψ: $\mathbf{B}(6) \to \mathbf{S}(8)$ is cyclic.*

Proof. Suppose that ψ is noncyclic. We claim that then *any cycle $C \preccurlyeq \widehat{\sigma}_1$ must be a transposition, that is, $\widehat{\sigma}_1^2 = 1$.* If this is the case, then $\widehat{\sigma}_i^2 = 1$ for all i, and hence $\mathbf{I}(6) \subseteq \operatorname{Ker} \psi$. Therefore, there is a homomorphism ϕ: $\mathbf{S}(6) \to \mathbf{S}(8)$ such that $\psi = \phi \circ \mu_6$. Since μ_6 is surjective, we have $G = \operatorname{Im} \psi = \operatorname{Im} \phi \subset \mathbf{S}(8)$. The group G is transitive and non-Abelian, and the group $\mathbf{S}(6)$ has no proper non-Abelian quotient groups; hence, $G \cong \mathbf{S}(6)$. Moreover, by Proposition 6.8(a), the group G is primitive. These properties contradict Jordan Theorem (see Remark 6.5).

To justify the above claim, assume, on the contrary, that $\widehat{\sigma}_1^2 \neq 1$, and let us find out the possible cyclic types of $\widehat{\sigma}_1$. Certainly, $\widehat{\sigma}_1$ cannot contain a cycle of length > 4 (say by Lemma 1.19(a)). Since $E(8/E(6/2)) = 2$, Lemma 1.21 shows that if all the cycles $\preccurlyeq \widehat{\sigma}_1$ are of distinct lengths, then $\widehat{\sigma}_1$ is a transposition (in fact, this is forbidden by Lemma 1.20); so, we may assume that (*) *$\widehat{\sigma}_1$ contains at least two cycles of the same length.* Under this assumption, Lemma 1.19(b) shows that if $\widehat{\sigma}_1$ contains a 4-cycle, then either $[\widehat{\sigma}_1] = [4,4]$ or $[\widehat{\sigma}_1] = [4,2,2]$. Moreover, (*) and Lemma 1.18 exclude all the cyclic types with a 3-cycle. This shows that (**) *either $[\widehat{\sigma}_1] = [4,4]$ or $[\widehat{\sigma}_1] = [4,2,2]$.* Consider now the permutation $A = \widehat{\alpha}_{3,5} = \widehat{\sigma}_3\widehat{\sigma}_4$. By Corollary 1.16 (with $i = 3$ and $j = 5$), $\operatorname{ord} A$ is divisible by 3; hence, only the following cyclic types of A may occur:

$$[3],\ [3,2],\ [3,2,2],\ [3,3],\ [3,3,2],\ [3,4],\ [3,5],\ [6],\ [6,2].$$

Since $\widehat{\sigma}_1$ satisfies (**) and commutes with A, Lemma 1.4(b) excludes all the types but $[3,3]$, $[3,3,2]$ (for A of any other type, $\widehat{\sigma}_1$ would contain either a power of a 3-cycle or a power of a 6-cycle; however, this contradicts (**)). The same argument exclude the case $[\widehat{\sigma}_1] = [4,2,2]$ (a permutation of cyclic type $[3,3]$ or $[3,3,2]$ cannot contain a power of a 4-cycle). Finally, the type $[\widehat{\sigma}_1] = [4,4]$ is also impossible; indeed, A has only one invariant set of length 2 (two fixed points for $[A] = [3,3]$ and the support of the transposition for $[A] = [3,3,2]$); this set must be $\widehat{\sigma}_1$-invariant, which cannot happen if $[\widehat{\sigma}_1] = [4,4]$). This completes the proof. □

Theorem 6.15. *a) Any transitive homomorphism* ψ: $\mathbf{B}(k) \to \mathbf{S}(k+2)$ *is cyclic whenever* $k > 4$.

b) Suppose $k > 4$. *Let* ψ: $\mathbf{B}(k) \to \mathbf{S}(k+2)$ *be a noncyclic homomorphism. Then either* ψ *is conjugate to one of the homomorphism* ψ^k_{k+2}, $\widetilde{\psi}^k_{k+2}$ *(this may happen for any* k*) or* $k = 5$ *and* ψ *is conjugate to the homomorphism* ϕ^5_7, *or, finally,* $k = 6$ *and* ψ *is conjugate to one of the homomorphisms* ϕ^6_8, $\widetilde{\phi}^6_8$.

Proof. a) The cases $k = 5$ and $k = 6$ are already considered in Propositions 6.13 and 6.14; assume now that $k > 6$ and that the homomorphism ψ is noncyclic. By Lemma 6.12(a), the permutation $\widehat{\sigma}_1$ has at least $k-2$ fixed points, and thus $\#\operatorname{supp} \widehat{\sigma}_1 \leq (k+2) - (k-2) = 4$. Hence, only the following cyclic types of $\widehat{\sigma}_1$ may occur: $[2]$, $[2,2]$, $[3]$, $[4]$. However, $[2]$, $[3]$ are forbidden by Lemma 1.20, $[2,2]$ is forbidden by Lemma 6.2(c), and $[4]$ is forbidden by Lemma 1.21, since the numbers $R_k = (k+2)/E(k/2)$ satisfy $E(R_7) = 3$ for $k > 6$ and $E(R_k) = 2$ for $k > 7$.

b) Since ψ is noncyclic and (by the statement (a)) intransitive, Theorem 2.1(a) shows that the group $G = \operatorname{Im} \psi \subset \mathbf{S}(k+2)$ has exactly one orbit Q of length L, $k \leq L \leq k+1$; set $Q' = \mathbf{\Delta}_{k+2} - Q$. Clearly, either Q' is a G-orbit of length $k+2-L$ or Q' consists of $k+2-L$ fixed points. The homomorphism ψ is the disjoint product of its reductions ψ_Q and $\psi_{Q'}$. The reduction ψ_Q is a noncyclic transitive homomorphism $\mathbf{B}(k) \to \mathbf{S}(Q) \cong \mathbf{S}(L)$. By Artin Theorem, Theorem 6.3, and Proposition 3.9, we obtain that (up to conjugation of ψ) either $\psi_Q = \mu_k$, or $k = 5$ and $\psi_Q = \psi_{5,6}$, or, finally, $k = 6$ and $\psi_Q = \nu_6$. The reduction $\psi_{Q'}$ is either trivial or takes all σ_i to the same transposition. This concludes the proof. □

6.3. Homomorphisms $\mathbf{B}(k) \to \mathbf{B}(n)$, $6 < k < n < 2k$. Our main goal now is to prove Theorem F(a) (see Theorem 6.20 below). We prove this theorem by induction on k. Lemma 6.11 and Lemma 6.16 enable us to pass from k to $k+2$ (the step of induction); Lemma 6.17 and Lemma 6.19 provide a base of induction.

Convention. Given a homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$, we use the following notation. We set

$$\begin{aligned} &\widehat{\sigma}_i = \psi(\sigma_i),\ 1 \leq i \leq k-1; \qquad \Sigma_i = \operatorname{supp} \widehat{\sigma}_i; \qquad \Sigma'_i = \operatorname{Fix} \widehat{\sigma}_i; \\ &N = \#\Sigma_i; \qquad N' = \#\Sigma'_i; \qquad G = \operatorname{Im} \psi \subseteq \mathbf{S}(n). \end{aligned} \tag{6.8}$$

Obviously, N, N' *do not depend on* i, *and* $N + N' = n$. Moreover, *if* $k > 4$, $n < 2k$, *and the homomorphism* ψ *is noncyclic, then* G *is a non-Abelian primitive permutation group* *of degree* n (see Proposition 6.8). We denote by

$$\phi\colon \mathbf{B}(k-2) \xrightarrow{\cong} \mathbf{B} \xrightarrow{\psi|\mathbf{B}} \mathbf{S}(n) \tag{6.9}$$

the restriction of ψ to the subgroup $\mathbf{B} \cong \mathbf{B}(k-2) \subset \mathbf{B}(k)$ generated by $\sigma_3, \ldots, \sigma_{k-1}$, and set

$$H = \operatorname{Im} \phi \subseteq G \subseteq \mathbf{S}(n).$$

Since $\widehat{\sigma}_1$ commutes with $\widehat{\sigma}_3, \ldots, \widehat{\sigma}_{k-1}$, *the sets* Σ_1 *and* Σ_1' *are* H*-invariant, and the homomorphism* ϕ *is the disjoint product* $\varphi \times \varphi'$ *of its reductions*

$$\varphi = \phi_{\Sigma_1}\colon \mathbf{B}(k-2) \to \mathbf{S}(\Sigma_1) \cong \mathbf{S}(N) \tag{6.10}$$

and

$$\varphi' = \phi_{\Sigma_1'}\colon \mathbf{B}(k-2) \to \mathbf{S}(\Sigma_1') \cong \mathbf{S}(N') \tag{6.11}$$

to the sets Σ_1 *and* Σ_1', *respectively.* We consider also the restriction

$$\widetilde{\phi}\colon \mathbf{B}(k-2) \xrightarrow{\cong} \widetilde{\mathbf{B}} \xrightarrow{\psi|\widetilde{\mathbf{B}}} \mathbf{S}(n) \tag{$\widetilde{6.9}$}$$

of the homomorphism ψ to the subgroup $\widetilde{\mathbf{B}} \cong \mathbf{B}(k-2) \subset \mathbf{B}(k)$ generated by $\sigma_1, \ldots, \sigma_{k-3}$. *The homomorphism* $\widetilde{\phi}$ *is the disjoint product* $\widetilde{\varphi} \times \widetilde{\varphi}'$ *of its reductions*

$$\widetilde{\varphi} = \widetilde{\phi}_{\Sigma_{k-1}}\colon \mathbf{B}(k-2) \to \mathbf{S}(\Sigma_{k-1}) \cong \mathbf{S}(N) \tag{$\widetilde{6.10}$}$$

and

$$\widetilde{\varphi}' = \widetilde{\phi}_{\Sigma_{k-1}'}\colon \mathbf{B}(k-2) \to \mathbf{S}(\Sigma_{k-1}') \cong \mathbf{S}(N') \tag{$\widetilde{6.11}$}$$

to the sets Σ_{k-1} *and* Σ_{k-1}', *respectively.* □

Lemma 6.16. *Suppose* $k > 6$. *Let* $\psi\colon \mathbf{B}(k) \to \mathbf{S}(n)$ *be a noncyclic homomorphism. Assume that every nondegenerate component of the permutation* $\widehat{\sigma}_1$ *is of length at most* $k-3$ (certainly, this is the case if $6 < k < n < 2k$; see Lemma 6.11). *Then the following statements hold true:*

a) *The homomorphism* $\varphi = \phi_{\Sigma_1}$ *is cyclic, and the homomorphism* $\varphi' = \phi_{\Sigma_1'}$ *is noncyclic. In particular,*

$$\widehat{\sigma}_i = SS_i' \qquad \textit{for all } \ i = 3, \ldots, k-1, \tag{6.12}$$

where the permutation

$$S = \varphi(\sigma_i) = \widehat{\sigma}_i \mid \Sigma_1 \in \mathbf{S}(\Sigma_1) \cong \mathbf{S}(N) \tag{6.13}$$

does not depend on i, *and all the permutations*

$$S_i' = \varphi'(\sigma_i) = \widehat{\sigma}_i \mid \Sigma_1' \in \mathbf{S}(\Sigma_1') \cong \mathbf{S}(N') \tag{6.14}$$

are disjoint with S *and* $\widehat{\sigma}_1$.

b) The homomorphism $\widetilde{\varphi} = \widetilde{\phi}_{\Sigma_{k-1}}$ *is cyclic, and the homomorphism* $\widetilde{\varphi}' = \widetilde{\phi}_{\Sigma'_{k-1}}$ *is noncyclic. In particular,*

$$\widehat{\sigma}_i = \widetilde{S}\widetilde{S}'_i \qquad \text{for all } i = 1, ..., k-3, \tag{$\widetilde{6.12}$}$$

where the permutation

$$\widetilde{S} = \widetilde{\varphi}(\sigma_i) = \widehat{\sigma}_i \mid \Sigma_{k-1} \in \mathbf{S}(\Sigma_{k-1}) \cong \mathbf{S}(N) \tag{$\widetilde{6.13}$}$$

does not depend on i*, and all the permutations*

$$\widetilde{S}'_i = \widetilde{\varphi}'(\sigma_i) = \widehat{\sigma}_i \mid \Sigma'_{k-1} \in \mathbf{S}(\Sigma'_{k-1}) \cong \mathbf{S}(N') \tag{$\widetilde{6.14}$}$$

are disjoint with $\widetilde{S}$ *and* $\widehat{\sigma}_{k-1}$*.*

c) Actually, $S = \widetilde{S}$ *and* $S'_i = \widetilde{S}'_i$ *for all* $i = 3, ..., k-3$*. Moreover, if* ψ *is transitive, then* $S = \widetilde{S} = 1$*,* $\widehat{\sigma}_i = S'_i$ *for* $i \geq 3$ *(so every such* $\widehat{\sigma}_i$ *is disjoint with* $\widehat{\sigma}_1$*), the homomorphism* $\varphi = \phi_{\Sigma_1}$ *is trivial, and the homomorphism* $\varphi' = \phi_{\Sigma'_1}$ *coincides with the restriction* ϕ *of* ψ *to the subgroup* $\mathbf{B}(k-2) \subset \mathbf{B}(k)$ *generated by* $\sigma_3, ..., \sigma_{k-1}$*.*

Proof. a) Lemma 6.1 shows that for any nondegenerate component $\mathfrak{C} = \{C_1, ..., C_t\}$ (of some length t, $1 \leq t \leq k-3$) the retraction

$$\Omega = \Omega_{\mathfrak{C}}\colon\ B(k-2) \to \mathbf{S}(\mathfrak{C}) \cong \mathbf{S}(t)$$

is cyclic. By Theorem 5.10(a), any Ω-homomorphism $\mathbf{B}(k-2) \to G_{\mathfrak{C}} \subseteq \mathbf{S}(\text{supp } \mathfrak{C})$ is cyclic; particularly, the homomorphism $\Psi_{\text{supp } \mathfrak{C}}\colon \mathbf{B}(k-2) \to G_{\mathfrak{C}} \subseteq \mathbf{S}(\text{supp } \mathfrak{C})$ is cyclic (recall that $G_{\mathfrak{C}}$ is the centralizer of the element $C = C_1 \cdots C_t$ in $\mathbf{S}(\text{supp } \mathfrak{C})$). Thereby,

$$\Psi_{\text{supp } \mathfrak{C}}(s_1) = \ldots = \Psi_{\text{supp } \mathfrak{C}}(s_{k-3}).$$

By the definition of $\Psi_{\text{supp } \mathfrak{C}}$, we have

$$\widehat{\sigma}_{i+2} \mid \text{supp } \mathfrak{C} = \psi(\sigma_{i+2}) \mid \text{supp } \mathfrak{C} = \Psi_{\text{supp } \mathfrak{C}}(s_i) = \Psi_{\text{supp } \mathfrak{C}}(s_1), \qquad i = 1, ..., k-3;$$

hence, the reduction $\phi_{\text{supp } \mathfrak{C}}\colon \mathbf{B}(k-2) \to \mathbf{S}(\text{supp } \mathfrak{C})$ of ϕ to the H-invariant set supp $\mathfrak{C}$ is a cyclic homomorphism. The homomorphism $\varphi = \phi_{\Sigma_1}$ is the disjoint product of all the reductions $\phi_{\text{supp } \mathfrak{C}}$, where $\mathfrak{C}$ runs over all the nondegenerate components of $\widehat{\sigma}_1$. Hence, φ is cyclic and the permutation S defined by (6.13) does not depend on i. Clearly, the permutations S'_i ($i = 3, ..., k-1$) defined by (6.14) are disjoint with S and $\widehat{\sigma}_1$. Finally, the homomorphism φ' must be noncyclic (for otherwise, $S'_3 = ... = S'_{k-1}$ and (6.12) shows that ψ is cyclic).

The statement (b) follows by the same argument (one has just to work with the permutation $\widehat{\sigma}_{k-1}$ and the sets Σ_{k-1}, Σ'_{k-1} instead of $\widehat{\sigma}_1$, Σ_1 and Σ'_1, respectively).

c) The proof of this statement is an exercise in elementary set theory. By (a) and (b), we have

$$\begin{aligned} &\text{supp } S \subseteq \Sigma_1 &&= \text{supp } \widehat{\sigma}_1, &&\text{supp } S'_i \subseteq \Sigma'_1 &&= \text{Fix } \widehat{\sigma}_1, &&3 \leq i \leq k-1,\\ &\text{supp } \widetilde{S} \subseteq \Sigma_{k-1} &&= \text{supp } \widehat{\sigma}_{k-1}, &&\text{supp } \widetilde{S}'_i \subseteq \Sigma'_{k-1} &&= \text{Fix } \widehat{\sigma}_{k-1}, &&1 \leq i \leq k-3.\end{aligned}$$

Taking into account the representation for $\widehat{\sigma}_1$ given by $\widetilde{(6.12)}$, we see that

$$\text{supp } S \subseteq \Sigma_1 = \text{supp } \widehat{\sigma}_1 = (\text{supp } \widetilde{S}) \cup (\text{supp } \widetilde{S_1'})$$

and

$$\text{supp } S_i' \subseteq \Sigma_1' = \mathbf{\Delta}_n - \text{supp } \widehat{\sigma}_1 = \mathbf{\Delta}_n - ((\text{supp } \widetilde{S}) \cup (\text{supp } \widetilde{S_1'})), \quad 3 \le i \le k-1.$$

Hence,

$$(\text{supp } S_i') \cap (\text{supp } \widetilde{S}) = \varnothing \qquad \text{for } \; 3 \le i \le k-1. \tag{$*$}$$

Completely analogously, using the representation for $\widehat{\sigma}_{k-1}$ given by (6.12), we get

$$(\text{supp } \widetilde{S}_i') \cap (\text{supp } S) = \varnothing \qquad \text{for } \; 1 \le i \le k-3. \tag{$\widetilde{*}$}$$

There are at least two i's such that $3 \le i \le k-3$ (since $k > 6$); for any such i, formulas (6.12), ($\widetilde{(6.12)}$) provide us with the two representations of $\widehat{\sigma}_i$ in the form of disjoint products:

$$SS_i' = \widehat{\sigma}_i = \widetilde{S}\widetilde{S}_i'. \tag{$**$}$$

Obviously, $(*)$, $(\widetilde{*})$, and $(**)$ imply $S = \widetilde{S}$ and $S_i' = \widetilde{S}_i'$ for $3 \le i \le k-3$. In view of (6.12), $\widetilde{(6.12)}$, the set $Q = \text{supp } S = \text{supp } \widetilde{S}$ is invariant under all the permutations $\widehat{\sigma}_1, ..., \widehat{\sigma}_{k-1}$. Clearly, $Q \ne \mathbf{\Delta}_n$ (for $\widehat{\sigma}_1$ has at least $k-2$ fixed points). If ψ is transitive, the set Q must be empty, and we get $S = \widetilde{S} = 1$. Hence, $\widehat{\sigma}_i = S_i$ for $3 \le i \le k-1$, $\phi = \varphi'$, and $\widehat{\sigma}_1$ is disjoint with $\widehat{\sigma}_3, ..., \widehat{\sigma}_{k-1}$. □

To get a base for induction, we study some homomorphisms of $\mathbf{B}(7)$ and $\mathbf{B}(8)$.

Lemma 6.17. *Suppose* $7 < n < 14$. *Then any transitive homomorphism* ψ: $\mathbf{B}(7) \to \mathbf{S}(n)$ *is cyclic.*

Proof. Suppose, on the contrary, that ψ is noncyclic. By Lemma 6.9, we have $N = \#\Sigma_1 = \#\text{supp } \widehat{\sigma}_1 \ge 6$. Corollary 6.12 shows that $N' = \#\Sigma_1' = \#\text{Fix } \widehat{\sigma}_1 \ge 5$. Hence, $11 \le N + N' = n \le 13$ and $5 \le N' \le 7$.

By Lemma 6.16(a, c), all the permutations $\widehat{\sigma}_3, ..., \widehat{\sigma}_6$ are supported in the set Σ_1', and the noncyclic homomorphism ϕ (that is, the restriction of ψ to the subgroup in $\mathbf{B}(7)$ generated by $\sigma_3, ..., \sigma_6$) coincides with its reduction $\varphi' = \phi_{\Sigma_1'}$: $\mathbf{B}(5) \to \mathbf{S}(\Sigma_1') \cong \mathbf{S}(N')$.

Claim. *Every permutation* $\widehat{\sigma}_i$, $1 \le i \le 6$, *is a product of disjoint transpositions.*

It is sufficient to prove this for $i \ge 3$; let us deal with such i's. We consider the following three cases: $N' = 5$, $N' = 6$ and $N' = 7$. If $N' = 5$, then φ' must be transitive (otherwise, all orbits are of length < 5 and $\phi = \varphi'$ is cyclic); by Artin Theorem, any $\widehat{\sigma}_i$ is a transposition. If $N' = 7$, Claim follows from Proposition 6.13. Suppose $N' = 6$. If φ' is intransitive, then its image in $\mathbf{S}(\Sigma_1') \cong \mathbf{S}(6)$ has exactly one orbit Q of length 5 and one fixed point; Artin Theorem applies to the reduction of φ' to Q, and we see that any $\widehat{\sigma}_i$ is a transposition. Finally, if φ' is transitive, then, by Proposition 3.9, $\varphi' \sim \psi_{5,6}$ and every $\widehat{\sigma}_i$ is a product of 3 disjoint transpositions.

To conclude the proof of the lemma, we use the approach that was already used in the proof of Proposition 6.14. Claim shows that $\widehat{\sigma}_i^2 = 1$ for all i, and hence the non-Abelian primitive group $G = \text{Im } \psi \subset \mathbf{S}(n)$ $(7 < n < 14)$ is isomorphic to $\mathbf{S}(7)$; in view of Remark 6.5, this contradicts Jordan Theorem. □

To treat homomorphisms $\mathbf{B}(8) \to \mathbf{S}(n)$, $8 < n < 16$, we need the following fact.

Proposition 6.18. *a) Any transitive homomorphism* ψ: $\mathbf{B}(6) \to \mathbf{S}(9)$ *is cyclic.*

b) Any noncyclic homomorphism ψ: $\mathbf{B}(6) \to \mathbf{S}(9)$ *is conjugate to a disjoint product* $\psi_1 \times \psi_2$, *where* ψ_1: $\mathbf{B}(6) \to \mathbf{S}(6)$ *is either* μ_6 *or* ν_6, *and* ψ_2: $\mathbf{B}(6) \to \mathbf{S}(3)$ *is a cyclic homomorphism. In particular, either every* $\widehat{\sigma}_i$ *is a disjoint product of transpositions or every* $\widehat{\sigma}_i$ *is a disjoint product of transpositions and a* 3*-cycle that does not depend on* i.

Proof. $a)$ Suppose, on the contrary, that ψ is noncyclic. By Proposition 6.8(a) and Remark 6.5, $\widehat{\sigma}_1^2 \neq 1$. Using this and Lemmas 1.18, 1.19, 1.21, we can exclude all the cyclic types of $\widehat{\sigma}_1$ but the following three: $i)$ $[2,2,3]$; $ii)$ $[3,3]$; $iii)$ $[3,3,3]$. Let us consider these cases.

$i)$ For $i \geq 3$, let $\widehat{\sigma}_i'$ be the restriction of $\widehat{\sigma}_i$ to the support of the 3-cycle C in $\widehat{\sigma}_1$; then, by Lemma 1.4, $\widehat{\sigma}_i = C^{q_i}$, $0 \leq q_i \leq 2$. Clearly, $q_i \neq 0$ (for $\widehat{\sigma}_i$ has only 2 fixed points); hence, all C^{q_i} are 3-cycles with the same support supp C. Now, C^{q_5} is the *only* 3-cycle in the cyclic decomposition of $\widehat{\sigma}_5$, and $\widehat{\sigma}_2$ commutes with $\widehat{\sigma}_5$. Hence, the set supp C is (Im ψ)-invariant, which contradicts the transitivity of ψ.

$ii)$ In this case the only nondegenerate component of $\widehat{\sigma}_1$ is the 3-component $\mathfrak{C} = \{C_0, C_1\}$, and we have the corresponding retraction $\Omega = \Omega_{\mathfrak{C}}$: $\mathbf{B}(4) \to \mathbf{S}(2)$. Clearly, either Ω is trivial or all $\Omega(s_i)$ coincide with the transposition (C_0, C_1). In any case $\Omega(s_i^2) = 1$, which means that $\widehat{\sigma}_{i+2}^2 C_j \widehat{\sigma}_{i+2}^{-2} = C_j$ whenever $j = 0,1$ and $i = 1,2,3$; thus, $\widehat{\sigma}_{i+2}^2 \mid \text{supp } \mathfrak{C} = C_0^{q_{0,i}} C_1^{q_{0,i}}$ with some $q_{j,i}$, $0 \leq q_{j,i} \leq 2$. Because of $[\widehat{\sigma}_{i+2}] = [3,3]$, this implies that for some $p_{j,i}$, $0 \leq p_{j,i} \leq 2$, the permutations $\widehat{\sigma}_{i+2}$ themselves satisfy

$$\widehat{\sigma}_{i+2} \mid \text{supp } \mathfrak{C} = C_0^{p_{0,i}} C_1^{p_{1,i}}, \qquad i = 1,2,3. \tag{6.15}$$

Since $\#(\boldsymbol{\Delta}_9 - \text{supp } \mathfrak{C}) = 3$, the conditions $[\widehat{\sigma}_{i+2}] = [3,3]$ and (6.15) show that the permutations $\widehat{\sigma}_{i+2}$, $i = 1,2,3$, commute with each other, which is impossible.

$iii)$ In this case the only nondegenerate component of $\widehat{\sigma}_1$ is the 3-component $\mathfrak{C} = \{C_0, C_1, C_2\}$, and we have the retraction $\Omega = \Omega_{\mathfrak{C}}$: $\mathbf{B}(4) \to \mathbf{S}(3)$. We consider the following two cases: $iii_1)$ Ω is noncyclic; $iii_2)$ Ω is cyclic.

$iii_1)$ In this case, by Theorem 2.14, $\Omega \sim \mu_3 \circ \pi$, where π: $\mathbf{B}(4) \to \mathbf{B}(3)$ is the canonical epimorphism. This means that all $\Omega(s_i)$ are transpositions; hence, also $\Omega(s_i^3)$ are transpositions. Therefore, there is a value j, $j = 0,1,2$, such that $\widehat{\sigma}_3^3 C_j \widehat{\sigma}_3^{-3} \neq C_j$. However, this contradicts the relation $\widehat{\sigma}_3^3 = 1$.

$iii_2)$ In this case all $\Omega(s_i) = A$, where $A \in \mathbf{S}(3)$ does not depend on i. If $A^2 = 1$, then $\widehat{\sigma}_{i+2}^2 C_j \widehat{\sigma}_{i+2}^{-2} = C_j$ for $i = 1,2,3$, $j = 0,1,2$; combined with the condition $[\widehat{\sigma}_{i+2}] = [3,3,3]$, this shows that the permutations $\widehat{\sigma}_{i+2}$, $i = 1,2,3$, commute with each other, which is impossible. Finally, if $A^2 \neq 1$, then A is a 3-cycle, and we may assume that

$$\widehat{\sigma}_i C_j \widehat{\sigma}_i^{-1} = C_{|j+1|_3}, \qquad i = 1,2,3, \quad j = 0,1,2, \tag{6.16}$$

where $|\cdot|_3 \in \mathbb{Z}/3\mathbb{Z}$. Let $C_j = (a_j^0, a_j^1, a_j^2)$, $j = 0,1,2$. It follows from (6.16) that there exist $t(j,i) \in \mathbb{Z}/3\mathbb{Z}$ such that

$$\widehat{\sigma}_i(c_j^k) = c_{j+1}^{k+t(j,i)}, \quad j,k \in \mathbb{Z}/3\mathbb{Z}, \quad i = 3,4,5. \tag{6.17}$$

The condition $\widehat{\sigma}_i^3 = 1$ implies that

$$t(0,i) + t(1,i) + t(2,i) = 0, \qquad i = 3,4,5 \tag{6.18}$$

(here and below all the equalities are in $\mathbb{Z}/3\mathbb{Z}$). Using $\widehat{\sigma}_3 \infty \widehat{\sigma}_4$, we obtain that

$$\begin{aligned} t(0,3) + t(1,4) + t(2,3) &= t(0,4) + t(1,3) + t(2,4), \\ t(0,3) + t(2,4) + t(1,3) &= t(0,4) + t(2,3) + t(1,4). \end{aligned} \tag{6.19}$$

Relations (6.18), (6.19) show that $t(j,3) = t(j,4)$ for all $j = 0,1,2$; it follows that

$$\widehat{\sigma}_3(c_j^k) = c_{j+1}^{k+t(j,3)} = c_{j+1}^{k+t(j,4)} = \widehat{\sigma}_4(c_j^k)$$

and $\widehat{\sigma}_3 = \widehat{\sigma}_4$. This contradiction concludes the proof of the statement (a). The proof of (b) follows immediately from (a), Theorem 6.3, and Theorem 6.15. □

Remark 6.6. By Theorem 6.3, Proposition 6.14, and Lemma 6.18, any transitive homomorphism $\mathbf{B}(6) \to \mathbf{S}(n)$ is cyclic whenever $7 \le n \le 9$. However, there is a noncyclic transitive homomorphism $\mathbf{B}(6) \to \mathbf{S}(10)$. To see this, consider all the 10 partitions of $\boldsymbol{\Delta}_6$ into two (disjoint) subsets consisting of 3 points. The group $\mathbf{S}(6)$ acts transitively on the family $\mathfrak{P} \cong \boldsymbol{\Delta}_{10}$ of all these partitions; this action defines the transitive homomorphism $\mathbf{S}(6) \to \mathbf{S}(10)$; the composition of the canonical projection μ_6 with this homomorphism is a noncyclic transitive homomorphism $\mathbf{B}(6) \to \mathbf{S}(10)$. Under suitable notations, this homomorphism looks as follows:

$$\begin{gathered} \widehat{\sigma}_1 = (1,2)(3,4)(5,6); \;\; \widehat{\sigma}_2 = (1,7)(3,8)(5,9); \;\; \widehat{\sigma}_3 = (3,6)(4,5)(7,10); \\ \widehat{\sigma}_4 = (1,3)(2,4)(7,8); \qquad \widehat{\sigma}_5 = (3,5)(4,6)(8,9). \end{gathered}$$

Instead of the canonical projection μ_6, one could use Artin's homomorphism ν_6. ○

Lemma 6.19. *Assume that* $8 < n < 16$. *Then any transitive homomorphism* ψ: $\mathbf{B}(8) \to \mathbf{S}(n)$ *is cyclic.*

Proof. Suppose, on the contrary, that ψ is noncyclic. By Lemma 6.9, $N = \#\Sigma_1 = \#\operatorname{supp}\widehat{\sigma}_1 \ge 6$. Corollary 6.12 shows that $N' = \Sigma_1' = \#\operatorname{Fix}\widehat{\sigma}_1 \ge 6$. Hence, $12 \le N + N' = n \le 15$ and $6 \le N' \le 9$.

By Lemma 6.16(a,c), all the permutations $\widehat{\sigma}_3, \ldots, \widehat{\sigma}_7$ are supported in Σ_1', and the restriction

$$\phi = \varphi'\colon \mathbf{B}(6) \to \mathbf{S}(\Sigma_1') \cong \mathbf{S}(N')$$

of ψ to the subgroup in $\mathbf{B}(8)$ generated by $\sigma_3, \ldots, \sigma_7$ is a noncyclic homomorphism. As usual, we set $H = \operatorname{Im}\phi \subseteq \mathbf{S}(\Sigma_1') \cong \mathbf{S}(N')$.

Claim. *There is exactly one H-orbit $Q \subseteq \Sigma_1'$ of length 6. The reduction*

$$\phi_Q\colon \mathbf{B}(6) \to \mathbf{S}(Q) \cong \mathbf{S}(6)$$

is conjugate to one of the homomorphisms μ_6, ν_6. The complement $Q' = \Sigma' - Q$ contains at most 3 points, and there is a permutation $A \in \mathbf{S}(Q')$ such that every $\widehat{\sigma}_i$ $(i = 3, \ldots, 7)$ is a disjoint product of some transpositions and this permutation A.

Indeed, since $\#\Sigma_1' = N' \leq 9$ and the homomorphism ϕ is noncyclic, theorems 2.1(a), 6.3, 6.15, and Proposition 6.18 show that there is only one H-orbit Q of length 6. In view of Artin Theorem and Theorem 2.1(a), the other statements of Claim follow immediately from this fact.

We have the following cases: i) $N' = 6$; ii) $N' = 7$; iii) $N' = 8$; iv) $N' = 9$.

Let us show that in all these cases the primitive group $G = \operatorname{Im} \psi \subseteq \mathbf{S}(n)$ is isomorphic to $\mathbf{S}(8)$ and, besides, contains a 3-cycle; this will contradict Jordan Theorem.

i) In this case $\Sigma_1' = Q$; hence, either $\phi = \phi_Q \sim \mu_6$ or to $\phi = \phi_Q \sim \nu_6$. Therefore, $\widehat{\sigma}_i^2 = 1$ for all i and $G \cong \mathbf{S}(8)$.

If $\phi \sim \mu_6$, then any $\widehat{\sigma}_i$ is a transposition, and the product $(\widehat{\sigma}_3\widehat{\sigma}_4)^2$ is a 3-cycle in G (in fact, the element $\widehat{\sigma}_3\widehat{\sigma}_4$ itself is a 3-cycle; we take its square only to unify the proofs for all cases $(i) - (iv)$).

If $\phi \sim \nu_6$, then the permutation $(\widehat{\sigma}_3\widehat{\sigma}_4 \cdots \widehat{\sigma}_7)^2$ is a 3-cycle in G (here the square is essential, since $\widehat{\sigma}_3\widehat{\sigma}_4 \cdots \widehat{\sigma}_7$ is of cyclic type $[3, 2]$).

ii) In this case Q' consists of one point that is a fixed point of H. Applying the same arguments as in case (i), we obtain the desired result.

iii) The only difference with the previous cases is that all the permutations $\widehat{\sigma}_i$, $i \geq 3$, may contain one additional disjoint transposition A. However, this does not change anything (the square kills this transposition).

iv) Here $\#Q' = 3$. Hence, either $A = 1$, or $[A] = [2]$, or, finally, $[A] = [3]$. In the first two cases we follow the same arguments as above. Let us show that the third case cannot occur. Indeed, $N' = 9$ and $N \geq 6$; thus, $N = 6$ (for $N + N' = n \leq 15$). That is, the support of any permutation $\widehat{\sigma}_i$ consists of 6 points. If A is a 3-cycle, it takes 3 points from the 6, and the rest three places cannot be filled by transpositions. This concludes the proof. □

Now we are ready to prove Theorem F(a). Actually, the proof is simple, since the main work was already done.

Theorem 6.20. *Assume that* $6 < k < n < 2k$. *Then*

a) any transitive homomorphism $\mathbf{B}(k) \to \mathbf{S}(n)$ *is cyclic;*

b) any noncyclic homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ *is conjugate to a homomorphism of the form* $\mu_k \times \widetilde{\psi}$, *where* $\widetilde{\psi}$: $\mathbf{B}(k) \to \mathbf{S}(n-k)$ *is a cyclic homomorphism.*

Proof. a) Let us call $\Gamma(m)$ the following conjecture:

Conjecture $\Gamma(m)$. *Every transitive homomorphism* ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$ *is cyclic whenever* $6 < k \leq m$ *and* $6 < k < n < 2k$.

We have already proved $\Gamma(m)$ for $m = 7$ and $m = 8$ (Lemma 6.17, Lemma 6.19). Suppose that $\Gamma(m)$ is fulfilled for some $m \geq 7$. We shall show that then $\Gamma(m+2)$ is fulfilled. By Induction Principle, this will prove the statement (a).

Suppose, on the contrary, that $\Gamma(m+2)$ is wrong. That is, for some k and n that satisfy $k \leq m+2$ and $6 < k < n < 2k$ there exists a transitive noncyclic homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(n)$. It follows from Lemma 6.17 and Lemma 6.19 that $k > 8$; hence, $6 < k-2 \leq m$ and $\Gamma(k-2)$ is fulfilled.

By Lemmas 6.9 and Corollary 6.12, we have $N \geq 6$ and $N' \geq k-2$; thus

$$6 < k-2 \leq N' \leq n-6 < 2k-6 < 2(k-2). \tag{6.20}$$

By Lemma 6.16(a, c), the restriction ϕ: $\mathbf{B}(k-2) \to \mathbf{S}(n)$ of ψ to the subgroup $\mathbf{B}(k-2) \subset \mathbf{B}(k)$ generated by $\sigma_3, ..., \sigma_{k-1}$ coincides with its noncyclic reduction

$$\varphi' = \phi_{\Sigma_1'}\colon \mathbf{B}(k-2) \to \mathbf{S}(\Sigma_1') \cong \mathbf{S}(N')$$

to the H-invariant set $\Sigma_1' = \operatorname{Fix} \widehat{\sigma}_1$. To conclude the proof of the statement (a), it is sufficient to prove the following

Claim. $\widehat{\sigma}_i^2 = 1$ for all i, and hence $G = \operatorname{Im} \psi \cong \mathbf{S}(k)$. Moreover, the primitive permutation group $G \subsetneq \mathbf{S}(n)$ contains a 3-cycle.

Indeed, as we know, these properties are incompatible; hence, the assumption that $\Gamma(m+2)$ is wrong leads to a contradiction.

To justify Claim, assume first that $N' = k-2$. Since φ' is noncyclic, Theorem 2.1(a) shows that φ' is transitive; by Artin Theorem, $\phi = \varphi' \sim \mu_{k-2}$ (for $k-2 > 6$). Hence, any $\widehat{\sigma}_i$ is a transposition and $\widehat{\sigma}_1\widehat{\sigma}_2$ is a 3-cycle containing in G.

Assume now that $N' > k-2$. Since $\Gamma(k-2)$ is fulfilled, any transitive homomorphism $\mathbf{B}(k-2) \to \mathbf{S}(N')$ is cyclic; therefore, *the homomorphism $\phi = \varphi'$ must be intransitive.* The reduction ϕ_Q of the noncyclic intransitive homomorphism $\phi = \varphi'$ to any H-orbit $Q \subset \Sigma_1'$ is a transitive homomorphism $\mathbf{B}(k-2) \to \mathbf{S}(Q)$. Clearly, $\#Q < N' < 2(k-2)$. If $\#Q \neq k-2$, then ϕ_Q is cyclic (this follows from Theorem 2.1(a) whenever $\#Q < k-2$; if $\#Q > k-2$, then ϕ_Q must be cyclic by our assumption that $\Gamma(k-2)$ is fulfilled). Hence, there exists a unique H-orbit Q of length $k-2$, and the reduction ϕ_Q of ϕ to this orbit is noncyclic and transitive. Since $k-2 > 6$, Artin Theorem shows that $\phi_Q \sim \mu_{k-2}$. Let $Q' = \Sigma' - Q$; clearly, ϕ is the disjoint product of the reductions ϕ_Q and $\phi_{Q'}$, and $\phi_{Q'}$ is cyclic. This means that there is a permutation $A \in \mathbf{S}(Q')$ such that for every i, $3 \leq i \leq k-1$, the permutation $\widehat{\sigma}_i$ is the disjoint product of A and the transposition $A_i = \phi_Q(\sigma_i)$.

Let us show that $A^2 = 1$. Indeed, if this is not the case, then for some $r > 2$ the cyclic decomposition of A contains an r-cycle. Let $\mathfrak{C}_r(A)$ be the r-component of A. Since any $\widehat{\sigma}_i$, $3 \leq i \leq k-1$, is the disjoint product of A and the transposition A_i, we obtain that $\mathfrak{C}_r(A)$ is, actually, the r-component of every $\widehat{\sigma}_i$, $3 \leq i \leq k-1$. Thereby, the support $\Sigma_{\mathfrak{C}_r(A)}$ of this component $\mathfrak{C}_r(A)$ is invariant under all the permutations $\widehat{\sigma}_3, ..., \widehat{\sigma}_{k-1}$. Moreover, the permutations $\widehat{\sigma}_1$, $\widehat{\sigma}_2$ commute with $\widehat{\sigma}_{k-1}$, and hence the set $\Sigma_{\mathfrak{C}_r(A)}$ is invariant under $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$. However, this contradicts the transitivity of ψ.

Since $\widehat{\sigma}_i = A_iA$ for $i \geq 3$ and all A_i are transpositions, the property $A^2 = 1$ implies that $\widehat{\sigma}_i^2 = 1$ for $i \geq 3$, and hence for all i. Moreover, $\widehat{\sigma}_3\widehat{\sigma}_4 = A_3AA_4A = A_3A_4$ is a 3-cycle containing in G. This concludes the proof of Claim and proves the statement (a). In view of Theorem 2.1(a) and Artin Theorem, (a) implies (b). □

Remark 6.7. There is a noncyclic transitive homomorphism $\mathbf{B}(6) \to \mathbf{S}(10)$ (see Remark 6.6). On the other hand, for any $k \geq 3$, Corollary 5.6 provides us with the four noncyclic homomorphisms φ_j: $\mathbf{B}(k) \to \mathbf{S}(2k)$, $j = 0, 1, 2, 3$. The homomorphism φ_0 is intransitive, and the homomorphisms φ_j, $j = 1, 2, 3$, are transitive (see also Remark 5.6 and §6.4). These remarks show that the conditions $6 < k < n < 2k$ of Theorem 6.20(a) are, in a sense, sharp. ○

6.4. Homomorphisms $\mathbf{B}(k) \to \mathbf{S}(2k)$. Here we prove Theorem F(b). In the following lemma (which is similar to Lemma 6.11) we use the homomorphisms φ_j: $\mathbf{B}(k) \to \mathbf{S}(2k)$ exhibited in Corollary 5.6 (with $n = k$).

Lemma 6.21. *Assume that $k > 6$. Let ψ: $\mathbf{B}(k) \to \mathbf{S}(2k)$ be a noncyclic homomorphism such that the permutation $\widehat{\sigma}_1$ has a nondegenerate component $\mathfrak{C}$ of length $t > k - 3$. Then either $t = k - 2$ and $\psi \sim \varphi_3$ or $t = k$ and $\psi \sim \varphi_2$.*

Proof. We follow the proof of Lemma 6.11. Clearly, $\mathfrak{C}$ must be the 2-component of $\widehat{\sigma}_1$, and $t \leq k$; hence, either $t = k - 2$ or $t = k - 1$ or $t = k$.

Set

$$\begin{aligned} &\Sigma = \operatorname{supp} \mathfrak{C}, \quad \Sigma' = \mathbf{\Delta}_{2k} - \Sigma, \qquad Q = \operatorname{supp} \widehat{\sigma}_1, \quad Q' = \mathbf{\Delta}_{2k} - Q, \\ &\#\Sigma = 2t, \qquad \#\Sigma' = 2k - 2t \leq 2k - 2(k-2) = 4. \end{aligned} \tag{6.21}$$

Since $k > 6$, any homomorphism $\mathbf{B}(k-2) \to \mathbf{S}(\Sigma')$ is cyclic (Theorem 2.1(a)). Particularly, the homomorphism $\Psi_{\Sigma'}$ is cyclic, and Lemma 4.4 implies that *the homomorphisms Ω: $\mathbf{B}(k-2) \to \mathbf{S}(\mathfrak{C}) \cong \mathbf{S}(t)$ and Ψ_Σ: $\mathbf{B}(k-2) \to G \subset \mathbf{S}(2t)$ must by noncyclic.*

We may assume that the homomorphism ψ is normalized; this means that

$$\Sigma = \{1, 2, \ldots, 2t\}, \quad \widehat{\sigma}_1 \mid \Sigma = C_1 \cdots C_t, \text{ where } C_m = (2m-1, 2m) \text{ for } m = 1, \ldots, t.$$

We must consider the following five cases:

i) $t = k - 2$ and $\widehat{\sigma}_1$ is a disjoint product of $k - 2$ transpositions;
ii) $t = k - 2$ and $\widehat{\sigma}_1$ is a disjoint product of $k - 2$ transpositions and a 3-cycle;
ii′) $t = k - 2$ and $\widehat{\sigma}_1$ is a disjoint product of $k - 2$ transpositions and a 4-cycle F_1;
iii) $t = k - 1$ and $\widehat{\sigma}_1$ is a disjoint product of $k - 1$ transpositions;
iv) $t = k$ and $\widehat{\sigma}_1$ is a disjoint product of k transpositions.

First we prove that cases (i), (ii), (iii) are impossible.

Case (i) may be eliminated by the same argument that were used in the proof of Lemma 6.11 (the only difference is that now Σ' consists of four points; actually, this does not change anything).

ii) In this case $\widehat{\sigma}_1$ has exactly one fixed point. Clearly, this point is also the only fixed point of any $\widehat{\sigma}_i$ (see, for instance, Lemma 1.18). Hence, ψ is the disjoint product $\psi_Q \times \mathbf{1}_1$, where ψ_Q: $\mathbf{B}(k) \to \mathbf{S}(Q) \cong \mathbf{S}(2k-1)$ is the reduction of ψ to the ($\operatorname{Im} \psi$)-invariant set $Q = \operatorname{supp} \widehat{\sigma}_1$. Since ψ is noncyclic, ψ_Q is noncyclic too, and the permutation $\psi_Q(\sigma_1) = \widehat{\sigma}_1$ has a 2-component of length $k - 2$. However, this contradicts Lemma 6.11.

iii) In this case $\Sigma = Q$, $\Sigma' = Q'$, and we deal with the noncyclic homomorphisms Ψ_Σ: $\mathbf{B}(k-2) \to G \subset \mathbf{S}(2k-2)$ and Ω: $\mathbf{B}(k-2) \to \mathbf{S}(k-1)$. By Remark 4.1, Proposition 3.9, and Theorem 6.3, we must consider the following two cases: *iii$_a$*) $\Omega = \mu_{k-2} \times \mathbf{1}_1$; *iii$_b$*) $k = 7$ and $\Omega = \psi_{5,6}$: $\mathbf{B}(5) \to \mathbf{S}(6)$.

In case (iii_a), as in Lemma 6.11, Ψ_{Σ} must be conjugate to one of the eight homomorphisms $\psi_{x;j}$ listed in Corollary 5.9 (with $n = k - 2$). All these cases may be eliminated in the same way as in Lemma 6.11 (the only difference is that now the set Σ' consists of two points, which does not change anything). Case (iii_b) is impossible by the same reasons as in Lemma 6.11.

Now we must handle cases (ii') and (iv).

$ii')$ We prove that in this case $\psi \sim \varphi_3$. We may assume that

$$\Sigma = \{1, ..., 2k-4\}, \qquad \Sigma' = \{2k-3, 2k-2, 2k-1, 2k\}.$$

We deal with the noncyclic homomorphisms

$$\Psi_{\Sigma}\colon \mathbf{B}(k-2) \to G \subset \mathbf{S}(2k-4), \qquad \Omega\colon \mathbf{B}(k-2) \to \mathbf{S}(k-2).$$

Clearly, either $ii'_a)$ $\Omega \sim \mu_{k-2}$ or $ii'_b)$ $k = 8$ and $\Omega \sim \nu_6$.

Case (ii'_b) is actually impossible; this may be proven by the argument used in Lemma 6.11 in case (i_b) (the only difference is that now the cyclic decomposition of $\widehat{\sigma}_1$ contains the additional 4-cycle $F_1 \in \mathbf{S}(\{13, 14, 15, 16\})$, and $\widehat{\sigma}_3, ..., \widehat{\sigma}_7$ contain the additional 4-cycle $F = F_1^{\pm 1}$; however, this does not change anything).

$ii'_a)$ We may assume that $\Omega = \mu_{k-2}$. Then, by Corollary 5.6, Ψ_{Σ} is conjugate to one of the homomorphisms φ_j, $j = 0, 1, 2, 3$ (with $n = k - 2$). For $j = 0, 1$ we have $\widehat{\sigma}_{i+2} \mid \Sigma = \Psi_{\Sigma}(s_i) = \varphi_j(s_i)$; hence, $\#\operatorname{supp}(\widehat{\sigma}_{i+2} \mid \Sigma) = 4$ and there is no room in Σ' for the rest $2k - 4$ points of $\operatorname{supp} \widehat{\sigma}_{i+2}$. For $j = 2$ we have $\widehat{\sigma}_{i+2} \mid \Sigma = \varphi_2(s_i)$, and hence $\widehat{\sigma}_{i+2} = \varphi_2(s_i)F$ for all $i \geq 1$, where $F \in \mathbf{S}(\{2k-3, 2k-2, 2k-1, 2k\})$ is a 4-cycle; the argument used in Lemma 6.11 for case (i_a) show that this is impossible. So, we are left with the case $j = 3$, i. e. $\Psi_{\Sigma} = \varphi_3\colon \mathbf{B}(k-2) \to \mathbf{S}(2k-4)$. Without loss of generality, we may assume that $F_1 = (2k-3, 2k, 2k-2, 2k-1)$; to simplify notation, put $a = 2k - 3$, $b = 2k - 2$, $c = 2k - 1$, $d = 2k$. Since ψ is normalized and $\widehat{\sigma}_{i+2} \mid \Sigma = \varphi_3(s_i)$, we have

$$\widehat{\sigma}_1 = (1,2)(3,4)(5,6)\cdots(2k-9, 2k-8)(2k-7, 2k-6)(2k-5, 2k-4)(a, d, b, c) \tag{6.22}$$

and

$$\begin{aligned} \widehat{\sigma}_i = (1,2)(3,4)\cdots(2i-7, 2i-6)\underbrace{(2i-5, 2i-2, 2i-4, 2i-3)}_{\text{4-cycle}}&(2i-1, 2i) \\ \times \cdots \times (2k-5, 2k-4)&(a,b)(c,d) \end{aligned} \tag{6.23}$$

for $3 \leq i \leq k - 1$. To recover the homomorphism ψ, we need to compute $\widehat{\sigma}_2$. This permutation commutes with $\widehat{\sigma}_i$, $4 \leq i \leq k - 1$; the cyclic decomposition of $\widehat{\sigma}_i$ contains only one 4-cycle, namely, $F_i = (2i-5, 2i-2, 2i-4, 2i-3)$; therefore, each of the sets $\{2i-5, 2i-2, 2i-4, 2i-3\}$, $4 \leq i \leq k-1$, must be $\widehat{\sigma}_2$-invariant. It follows that each of the sets $\{3,4\}, \{5,6\}, ..., \{2k-5, 2k-4\}$ is $\widehat{\sigma}_2$-invariant. Since the permutation $\widehat{\sigma}_2 \sim \widehat{\sigma}_1$ has no fixed points, its cyclic decomposition contains the product

$$A = (3,4)(5,6)\cdots(2k-5, 2k-4)$$

of $k-3$ disjoint transpositions; it must also contain one more transposition T and some 4-cycle F_2. By Lemma 4.3(a), $\Omega^* = \Omega = \mu_{k-2}$; it follows that exactly $k-4$ transpositions from the $k-2$ transpositions $(1,2)$, $(3,4)$,..., $(2k-9, 2k-8)$, (a,b), (c,d) entering in the cyclic decomposition of $\widehat{\sigma}_{k-1}$ must be $\Omega^*(s_2)$-invariant (that is, invariant under conjugation by $\widehat{\sigma}_2$), and the rest two transpositions must mutually interchange. Evidently, the $k-3$ transpositions $(3,4), (5,6), \dots, (2k-5, 2k-4)$ are the fixed points of $\Omega^*(s_2)$; hence, exactly one of the transpositions $(1,2)$, (a,b), (c,d) must be a fixed point of $\Omega^*(s_2)$; denote this transposition by T.

Let us show that $T \neq (1,2)$. Indeed, if $T = (1,2)$, then the cyclic decomposition of $\widehat{\sigma}_2$ contains the product $P = (1,2)A$ and a 4-cycle F_2 supported on $\{a,b,c,d\}$. Since $\widehat{\sigma}_{k-1}$ commutes with $\widehat{\sigma}_2$, and the product $(a,b)(c,d)$ is contained in $\widehat{\sigma}_{k-1}$ (see (6.22)), we have $(a,b)(c,d) = F_2^2$; thereby $F_2 = (a,c,b,d)^{\pm 1}$. It is easy to check that in this case $\widehat{\sigma}_2$, $\widehat{\sigma}_1$ cannot be a braid-like couple.

So, either $T = (a,b)$ or $T = (c,d)$. If $T = (a,b)$, then either $F_2 = (1,c,2,d)$ or $F_2 = (1,d,2,c)$; however, conjugation by $(a,b)(c,d)$ does not change $\widehat{\sigma}_1, \widehat{\sigma}_3, \dots, \widehat{\sigma}_{k-1}, T$ and transforms $(1,d,2,c)$ into $(1,c,2,d)$. Similarly, if $T = (c,d)$, then either $F_2 = (1,a,2,b)$ or $F_2 = (1,b,2,a)$, and the same conjugation transforms $(1,b,2,a)$ into $(1,a,2,b)$. Moreover, conjugation by (a,d,b,c) does not change $\widehat{\sigma}_1, \widehat{\sigma}_3, \dots, \widehat{\sigma}_{k-1}$ and transforms (c,d) into (a,b) and $(1,a,2,b)$ into $(1,d,2,c)$. Hence, without loss of generality, we may assume that $T = (a,b)$ and $F_2 = (1,c,2,d)$, and thus

$$\widehat{\sigma}_2 = (1,c,2,d)(3,4)(5,6)\cdots(2k-5, 2k-4)(a,b).$$

Finally, we conjugate the original homomorphism ψ by the permutation

$$B = \begin{pmatrix} 1 & 2 & 3 & \dots & 2k-4 & a & b & c & d \\ 5 & 6 & 7 & \dots & 2k & 1 & 2 & 3 & 4 \end{pmatrix} \tag{6.24}$$

and obtain the homomorphism $\widetilde{\psi}$: $\mathbf{B}(k) \to \mathbf{S}(2k)$,

$$\widetilde{\psi}(\sigma_i) = B\psi(\sigma_i)B^{-1} = B\widehat{\sigma}_i B^{-1}, \qquad 1 \le i \le k-1,$$

that coincides with φ_3. This concludes the proof in case (ii').

iv) We prove that in this case $\psi \sim \varphi_2$. We deal with the noncyclic homomorphisms Ω: $\mathbf{B}(k-2) \to \mathbf{S}(k)$ and $\Psi = \mathbf{B}(k-2) \to \mathbf{S}(2k)$ (we write Ψ instead of Ψ_Σ, for $\Sigma = \mathbf{\Delta}_{2k}$). According to Theorem 6.15, we must consider the following cases:

iv_a) $k = 7$ and $\Omega = \psi_{5,6} \times \mathbf{1}_1$;
iv_b) $k = 8$ and $\Omega = \nu_6 \times \omega$;
iv_c) $\Omega = \mu_{k-2} \times \omega$;

in cases (iv_b), (iv_c) ω: $\mathbf{B}(6) \to \mathbf{S}(2)$ is some (cyclic) homomorphism.

Cases (iv_a), (iv_b) cannot actually occur. To see this, we use Corollary 5.7, Corollary 5.8, and Theorem 5.10.

Since $[\widehat{\sigma}_i] = [\underbrace{2, \ldots, 2}_{k}]$, in case (iv_a) we may assume that Ψ coincides with one of the four homomorphisms $\eta_j \times [\mathbf{2}]\colon \mathbf{B}(5) \to \mathbf{S}(12) \times \mathbf{S}(2) \subset \mathbf{S}(14)$, $j = 0, 1, 2, 3$ (see Corollary 5.7), where $[\mathbf{2}]\colon \mathbf{B}(5) \to \mathbf{S}(2)$ is the cyclic homomorphism sending each generator s_i into the transposition $(13, 14)$. The cases $j = 1, 3$ cannot occur because of 4-cycles. For $j = 0, 2$ the permutation $A = \widehat{\sigma}_4\widehat{\sigma}_5$ is a product of four disjoint 3-cycles supported on $\mathbf{\Delta}_{12} = \{1, \ldots, 12\}$. Since $\widehat{\sigma}_2$ commutes with A, $\mathbf{\Delta}_{12}$ is $\widehat{\sigma}_2$-invariant. Hence, $\mathbf{\Delta}_{12}$ is $(\mathrm{Im}\ \psi)$-invariant, and the reduction $\psi_{\mathbf{\Delta}_{12}}\colon \mathbf{B}(7) \to \mathbf{S}(12)$ is a noncyclic homomorphism; it follows from Theorem 6.20 that $\mathrm{Im}\ \psi_{\mathbf{\Delta}_{12}}$ must have an invariant set $E \subset \mathbf{\Delta}_{12}$ of cardinality $12 - 7 = 5$. Particularly, E must be invariant under all the permutations $\widehat{\sigma}_{i+2} \mid \mathbf{\Delta}_{12} = \eta_j(s_i)$, $1 \le i \le 4$, which is not the case.

In case (iv_b) we may assume that Ψ is of the form

$$\Psi = \phi_j \times \varphi_\omega\colon \mathbf{B}(6) \to \mathbf{S}(12) \times \mathbf{S}(4) \subset \mathbf{S}(16)$$

(see Corollary 5.8). Here $\varphi_\omega\colon \mathbf{B}(6) \to \mathbf{S}(4)$ is a cyclic homomorphism defined by the following conditions: if the homomorphism $\omega\colon \mathbf{B}(6) \to \mathbf{S}(2)$ is trivial, then $\varphi_\omega(s_i) = (13, 14)(15, 16)$, $i = 1, \ldots, 5$; and $\varphi_\omega(s_i) = (13, 15)(14, 16)$, $i = 1, \ldots, 5$, for the only nontrivial ω. As in case (iv_a), the permutation $A = \widehat{\sigma}_4\widehat{\sigma}_5$ is a product of four disjoint 3-cycles supported on $\mathbf{\Delta}_{12}$, the set $\mathbf{\Delta}_{12}$ is $(\mathrm{Im}\ \psi)$-invariant, and the reduction $\psi_{\mathbf{\Delta}_{12}}\colon \mathbf{B}(8) \to \mathbf{S}(12)$ is a noncyclic homomorphism. It follows from Theorem 6.20 that there is an $(\mathrm{Im}\ \psi)$-invariant set $E \subset \mathbf{\Delta}_{12}$ of cardinality 4. However, the formulas for ϕ_0, ϕ_1 show that even the permutations $\widehat{\sigma}_{i+2} \mid \mathbf{\Delta}_{12} = \phi_j(s_i)$, $1 \le i \le 5$, do not have a common invariant set of such cardinality (in fact, ϕ_1 is transitive and $\mathrm{Im}\ \phi_0$ has in $\mathbf{\Delta}_{12}$ exactly two orbits, each of length 6). Hence, case (iv_b) is impossible.

We are left with case (iv_c). By Corollary 5.6 and Theorem 5.10, we may assume that Ψ is of the form

$$\Psi = \varphi_2 \times \varphi_\omega\colon \mathbf{B}(k-2) \to \mathbf{S}(2k-4) \times \mathbf{S}(4) \subset \mathbf{S}(2k) \tag{6.25}$$

(see Corollary 5.8); here $\varphi_\omega\colon \mathbf{B}(k-2) \to \mathbf{S}(4)$ is a cyclic homomorphism defined as follows: if the homomorphism $\omega\colon \mathbf{B}(k-2) \to \mathbf{S}(2)$ is trivial, then $\varphi_\omega(s_i) = (2k-3, 2k-2)(2k-1, 2k)$, $i = 1, \ldots, k-3$; and $\varphi_\omega(s_i) = (2k-3, 2k-1)(2k-2, 2k)$, $i = 1, \ldots, k-3$, for the only nontrivial ω. (In fact, there is one more possibility, namely, $\varphi_\omega(s_i) = (2k-3, 2k)(2k-2, 2k-1)$, $i = 1, \ldots, k-3$; if so, we conjugate ψ by the transposition $(2k-1, 2k)$ and reduce this case to the previous one).

First, we note that the set $R = \{2k-3, 2k-2, 2k-1, 2k\}$ cannot be $\widehat{\sigma}_2$-invariant. Otherwise, R would be $(\mathrm{Im}\ \psi)$-invariant and the reduction ψ_S of ψ to the complement $S = \mathbf{\Delta}_{2k} - R$ would be a *noncyclic transitive* homomorphism $\mathbf{B}(k) \to \mathbf{S}(2k-4)$, which contradicts Theorem 6.20 (ψ_S must be transitive, since φ_2 is so).

Now we show that ω must be nontrivial. Indeed, if ω is trivial, then $\Omega = \mu_{k-2} \times \mathbf{1}_2$ and the action of $\Omega(s_2)$ on the 2-component $\mathfrak{C}$ of $\widehat{\sigma}_1$ (that is, the conjugation by $\widehat{\sigma}_4$) interchanges some two transpositions and does not move the rest $k-2$. By Lemma 4.3, the action of $\Omega^*(s_2)$ on the 2-component $\mathfrak{C}^*$ of $\widehat{\sigma}_{k-1}$ (that is, the conjugation by $\widehat{\sigma}_2$) is of the same type. This means that $\widehat{\sigma}_2$ and $\widehat{\sigma}_{k-1}$ have exactly $k-2$ common transpositions. Any of these common transpositions is neither $(2k-3, 2k-2)$ nor $(2k-1, 2k)$, since otherwise the reduction of ψ to at least one of the complements $\mathbf{\Delta}_{2k} - \{2k-3, 2k-2\}$, $\mathbf{\Delta}_{2k} - \{2k-1, 2k\}$, $\mathbf{\Delta}_{2k} - \{2k-3, 2k-2, 2k-1, 2k\}$ would be a noncyclic transitive homomorphism

$$\mathbf{B}(k) \to \mathbf{S}(n), \qquad 6 < k < n, \quad n = 2k-2 \ \text{ or } \ n = 2k-4,$$

which contradicts Theorem 6.20. Hence, the conjugation by $\widehat{\sigma}_2$ interchanges the transpositions $(2k-3, 2k-2)$, $(2k-1, 2k)$. It follows that the set $\{2k-3, 2k-2, 2k-1, 2k\}$ is $\widehat{\sigma}_2$-invariant; however, we have already proved that this is impossible.

Taking into account that ψ is normalized and using (6.25) and what has been proven above, we see that

$$\widehat{\sigma}_1 = (1,2)(3,4)(5,6)\cdots(2k-5, 2k-4)(a,b)(c,d)$$

and for $3 \le i \le k-1$

$$\begin{aligned}\widehat{\sigma}_i = (1,2)(3,4)\cdots(2i-7, 2i-6)\,&\underbrace{(2i-5, 2i-3)(2i-4, 2i-2)}(2i-1, 2i)\\ &\times\cdots\times(2k-5, 2k-4)(a,c)(b,d),\end{aligned}$$

where $a = 2k-3$, $b = 2k-2$, $c = 2k-1$, $d = 2k$. Now it is convenient to conjugate the original homomorphism ψ by the permutation

$$C = \begin{pmatrix} 1 & 2 & 3 & \dots & 2k-4 & a & b & c & d \\ 5 & 6 & 7 & \dots & 2k & 1 & 3 & 2 & 4 \end{pmatrix};$$

we denote the new homomorphism by $\widetilde{\psi}$, but preserve the notations $\widehat{\sigma}_i$ for all the permutations $\widetilde{\psi}(\sigma_i)$, $1 \le i \le k-1$. Clearly,

$$\widehat{\sigma}_1 = \underbrace{(1,3)(2,4)}(5,6)\cdots(2k-3, 2k-2)(2k-1, 2k) \tag{6.26}$$

and for $3 \le i \le k-1$

$$\begin{aligned}\widehat{\sigma}_i = (1,2)(3,4)\cdots(2i-3, 2i-2)\,&\underbrace{(2i-1, 2i+1)(2i, 2i+2)}(2i+3, 2i+4)\\ &\times\cdots\times(2k-3, 2k-2)(2k-1, 2k).\end{aligned} \tag{6.27}$$

Claim. *The set* $\mathbf{\Delta}_6 = \{1,2,3,4,5,6\}$ *is* $\widehat{\sigma}_2$*-invariant and the restriction of* $\widehat{\sigma}_2$ *to the complement* $\mathbf{\Delta}_{12} - \mathbf{\Delta}_6$ *coincides with* $(7,8)(9,10)\cdots(2k-1, 2k)$.

Indeed, it follows from (6.27) that $\widehat{\sigma}_i\widehat{\sigma}_{i+1} = (2i-1, 2i+2, 2i+3)(2i, 2i+1, 2i+4)$; if $4 \le i \le k-2$, this product commutes with $\widehat{\sigma}_2$; hence, for such i every set $\{2i-1, 2i, 2i+1, 2i+2, 2i+3, 2i+4\}$ is $\widehat{\sigma}_2$-invariant. In particular, the set $\mathbf{\Delta}_6 = \{1,2,3,4,5,6\}$ is $\widehat{\sigma}_2$-invariant. Moreover, if $k > 7$, then each of the sets $\{7,8\}, \{9,10\}, \dots, \{2k-1, 2k\}$ is $\widehat{\sigma}_2$-invariant; since is $\widehat{\sigma}_2$ has no fixed points, this shows that the cyclic decomposition of $\widehat{\sigma}_2$ contains the disjoint product $(7,8)(9,10)\cdots(2k-1, 2k)$.

Consider the case $k = 7$. We still have the two $\widehat{\sigma}_2$-invariant sets $\{7, 8, 9, 10, 11, 12\}$ and $\{9, 10, 11, 12, 13, 14\}$; hence, each of the sets $\{7, 8\}$, $\{9, 10, 11, 12\}$, $\{13, 14\}$ is $\widehat{\sigma}_2$-invariant. Since $\widehat{\sigma}_2$ has no fixed points and is a product of disjoint transpositions, it must contain the transpositions $(7, 8)$, $(13, 14)$ and some two transpositions that are supported in $\{9, 10, 11, 12\}$. The product $\widehat{\sigma}_5\widehat{\sigma}_6 = (9, 12, 13)(10, 11, 14)$ commutes with $\widehat{\sigma}_2$, and we already know that $\widehat{\sigma}_2$ contains the transposition $(13, 14)$; hence, the restriction of $\widehat{\sigma}_2$ to $\{9, 10, 11, 12\}$ coincides with $(9, 10)(11, 12)$. This completes the proof of Claim.

To complete the whole proof, we consider the restrictions

$$A = \widehat{\sigma}_1 \mid \mathbf{\Delta}_6 = (1, 3)(2, 4)(5, 6), \quad B = \widehat{\sigma}_2 \mid \mathbf{\Delta}_6, \quad C = \widehat{\sigma}_4 \mid \mathbf{\Delta}_6 = (1, 2)(3, 4)(5, 6).$$

Claim shows that the restrictions of $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ to the complement $\mathbf{\Delta}_{2k} - \mathbf{\Delta}_6$ coincide; hence $A \infty B$. Clearly, $AC = CA$; we know also that B must be a product of 3 disjoint transpositions supported in $\mathbf{\Delta}_6$. There exist exactly 4 permutations B that satisfy all these conditions:

$$B_1 = (1, 2)(3, 5)(4, 6); \qquad B_2 = (1, 2)(3, 6)(4, 5);$$
$$B_3 = (1, 5)(2, 6)(3, 4); \qquad B_4 = (1, 6)(2, 5)(3, 4).$$

If $B = B_1$, then $\widetilde{\psi}$ coincides with the homomorphism φ_2 from Corollary 5.6 (with $n = k$). Any of the other three possibilities leads to a conjugate homomorphism. Indeed, the conjugation by the permutation $(5, 6)(7, 8) \cdots (2k - 1, 2k)$ interchanges B_1 with B_2, and B_3 with B_4; the conjugation by the permutation $(1, 3)(2, 4)$ interchanges B_1 with B_3. Further, these two conjugations preserve formulas (6.26), (6.27). They preserve also the form of the restriction of $\widehat{\sigma}_2$ to $\mathbf{\Delta}_{12} - \mathbf{\Delta}_6$ exhibited in Claim. This concludes the proof. □

The following statement is similar to Corollary 6.12.

Corollary 6.22. *Assume that $k > 6$. Let ψ: $\mathbf{B}(k) \to \mathbf{S}(2k)$ be a noncyclic homomorphism such that $\widehat{\sigma}_1$ has at most $k - 3$ fixed points. Then either $\psi \sim \varphi_2$ or $\psi \sim \varphi_3$.*

Proof. It follows from Lemma 6.10 that $\widehat{\sigma}_1$ must have a nondegenerate component of length at least $k - 2$; Lemma 6.21 completes the proof. □

Theorem 6.23. *For $k > 8$, any noncyclic transitive homomorphism ψ: $\mathbf{B}(k) \to \mathbf{S}(2k)$ is conjugate to one of the homomorphisms φ_1, φ_2, φ_3.*

Proof. We use the notation introduced in §6.3 (see Convention therein). Suppose, on the contrary, that there is a noncyclic transitive homomorphism ψ that is conjugate neither to φ_1 nor to φ_2 nor to φ_3. By Lemma 6.21, every nondegenerate component of $\widehat{\sigma}_1$ is of length at most $k - 3$. Hence, Lemma 6.16(a, c) applies to the homomorphism ψ; this leads to the following conclusions:

a) the reduction $\varphi = \phi_{\Sigma_1}$: $\mathbf{B}(k-2) \to \mathbf{S}(\Sigma_1) \cong \mathbf{S}(N)$ of the homomorphism $\phi = \psi \mid \mathbf{B}(k-2)$: $\mathbf{B}(k-2) \to \mathbf{S}(2k)$ to the set $\Sigma_1 = \operatorname{supp} \widehat{\sigma}_1$ is trivial (here and below $\mathbf{B}(k-2) \subset \mathbf{B}(k)$ is the subgroup generated by $\sigma_3, \ldots, \sigma_{k-1}$);

b) the reduction $\varphi' = \phi_{\Sigma_1'}$: $\mathbf{B}(k-2) \to \mathbf{S}(\Sigma_1') \cong \mathbf{S}(N')$ of ϕ to the set $\Sigma_1' = \operatorname{Fix} \widehat{\sigma}_1$ is noncyclic;

c) since $\phi = \phi_{\Sigma_1} \times \phi_{\Sigma_1'}$: $\mathbf{B}(k-2) \to \mathbf{S}(\Sigma_1) \times \mathbf{S}(\Sigma_1') \subset \mathbf{S}(2k)$ and ϕ_{Σ_1} is trivial, we see that eventually ϕ coincides with $\phi_{\Sigma_1'}$.

By Lemma 6.9, we have $N \geq 6$, and Corollary 6.22 shows that $N' \geq k-2$. It follows that $6 < k-2 \leq N' = 2k - N \leq 2k-6 < 2(k-2)$. Since the homomorphism $\phi_{\Sigma_1'}$ is noncyclic, Theorem 2.1(a), Artin Theorem and Theorem 6.20 imply that $\phi_{\Sigma_1'}$ is conjugate to a homomorphism of the form $\mu_{k-2} \times \nu$, where ν: $\mathbf{B}(k-2) \to \mathbf{S}(N'-k+2)$ is some cyclic homomorphism. Hence, without loss of generality we may assume that $\Sigma_1' = \{1, ..., N'\}$, $\Sigma_1 = \{N'+1, ..., 2k\}$ and $\widehat{\sigma}_i = (i, i+1) \cdot S$ (disjoint product) for $3 \leq i \leq k-1$, where S is some permutation not depending on i and supported on a set $Q \subseteq \Sigma_1' - \{3, ..., k\}$. We have $\#Q \leq N' - (k-2) \leq 2k-6-k+2 = k-4$ and $\#Q + 2 = \#\operatorname{supp} \widehat{\sigma}_i = N \geq 6$; hence, *the set* $Q = \operatorname{supp} S$ *is nonempty and does not coincide with the whole set* $\mathbf{\Delta}_{2k}$. Clearly, Q is $\widehat{\sigma}_i$-invariant for any $i \neq 2$ (since $Q \subseteq \operatorname{Fix} \widehat{\sigma}_1$ and $S \preccurlyeq \widehat{\sigma}_i$ for every $i \geq 3$). On the other hand, $\widehat{\sigma}_2$ commutes with any $\widehat{\sigma}_i = (i, i+1) \cdot S$, $4 \leq i \leq k-1$, and thus each of the sets $\{4,5\} \cup Q, ..., \{k-1, k\} \cup Q$ is $\widehat{\sigma}_2$-invariant. Hence, their intersection Q is $\widehat{\sigma}_2$-invariant; this contradicts the transitivity of ψ and concludes the proof. □

6.5. Some applications: n-coverings of $\mathbf{G}_k$, $n \leq 2k$. We say that an unbranched covering $\mathcal{E} = (E, q, X)$, q: $E \to X$ over a connected topological space X is an *n-covering* whenever E is connected and $\#q^{-1}(x) = n$ for any $x \in X$. Assuming that X is "good enough" (say a smooth manifold or a locally finite cell complex) and fixing a base point $x_* \in X$, we have a natural $1-1$ correspondence between the equivalence classes of n-coverings over X and the classes of conjugate transitive homomorphisms $\pi_1(X, x_*) \to \mathbf{S}(n)$. An n-covering q: $E \to X$ is said to be *cyclic* if the corresponding *monodromy homomorphism* q^*: $\pi_1(X, x_*) \to \mathbf{S}(n)$ is cyclic; in this case we may regarded q^* as an epimorphism onto the group $\mathbb{Z}/n\mathbb{Z}$. We say that two epimorphisms φ, φ': $\pi_1(X, x_*) \to \mathbb{Z}/n\mathbb{Z}$ are equivalent if $\operatorname{Ker} \varphi = \operatorname{Ker} \varphi'$; it is readily seen that the latter condition is fulfilled if and only if there is an invertible element $m \in \mathbb{Z}/n\mathbb{Z}$ such that $\varphi(\gamma) = m\varphi'(\gamma)$ for all $\gamma \in \pi_1(X, x_*)$. The equivalence classes of cyclic n-coverings are in a natural $1-1$ correspondence with the equivalence classes of epimorphisms $\pi_1(X, x_*) \to \mathbb{Z}/n\mathbb{Z}$. The set $[X, \mathbb{C}^*]$ of homotopy classes of continuous functions $X \to \mathbb{C}^* = \mathbb{C} - \{0\}$ is a group isomorphic to the cohomology group $H^1(X, \mathbb{Z})$ (Brushlinski-Eilenberg Theorem). If the homology group $H_1(X, \mathbb{Z}) \cong \pi_1(X, x_*)/\pi_1(X, x_*)'$ is finitely generated and torsion free (that is, a free Abelian group), then any n-covering over X is isomorphic to a covering of the form $Y = \{(x, \zeta) \in X \times \mathbb{C}^* \mid \zeta^n = f(x)\} \ni (x, \zeta) \mapsto x \in X$, where f: $X \to \mathbb{C}^*$ is a continuous function such that $f^{1/n}$ does not possess a global single-valued continuous branch. Two n-coverings of the above form corresponding to functions f_1, f_2: $X \to \mathbb{C}^*$ are equivalent if and only if there exist an integer m and a continuous function g: $X \to \mathbb{C}$ such that $f_2 = f_1^m \exp g$ (clearly, $(m, n) = 1$).

Let us study some n-coverings over the space $\mathbf{G}_k$ of all separable polynomials of degree k over $\mathbb{C}$ (§0). The cohomology group $H^1(\mathbf{G}_k, \mathbb{Z}) \cong [\mathbf{G}_k, \mathbb{C}^*] \cong \operatorname{Hom}(\mathbf{B}(k), \mathbb{Z}) \cong \mathbb{Z}$ is generated by the cohomology class of the *canonical discriminant mapping* d_k: $\mathbf{G}_k \ni z = (z_1, ..., z_k) \mapsto d_k(z) \in \mathbb{C}^*$, where $d_k(z)$ is the discriminant of the polynomial $p_k(t, z) = t^k + z_1 t^{k-1} + ... + z_k$. Since $\mathbf{B}(k)/\mathbf{B}'(k) \cong \mathbb{Z}$, *any cyclic n-covering q: $E \to \mathbf{G}_k$ is equivalent to the standard cyclic n-covering* $\mathcal{E}_{cycl}^{(n)}(k) = (E_{cycl}^{(n)}(k), C_k^{(n)}, \mathbf{G}_k)$, where

$$C_k^{(n)}\colon\ E_{cycl}^{(n)}(k) = \{(\lambda, z) \in \mathbb{C}^* \times \mathbf{G}_k) \mid \lambda^n = d_k(z)\} \ni (\lambda, z) \mapsto z \in \mathbf{G}_k.$$

We have also the *standard noncyclic k-covering* $\mathcal{E}_1(k) = (\mathbf{E}_1(k), M_{\mu_k}, \mathbf{G}_k)$,

$$M_{\mu_k}\colon \mathbf{E}_1(k) = \{(\lambda,\, z) \in \mathbb{C} \times \mathbf{G}_k) \mid p_k(\lambda, z) = 0\} \ni (\lambda,\, z) \mapsto z \in \mathbf{G}_k,$$

corresponding to the canonical projection $\mu_k\colon \mathbf{B}(k) \to \mathbf{S}(k)$.

There are three $2k$-coverings $\mathcal{E}_1^{(j)}(k) = (\mathbf{E}_1^{(j)}(k), q_j, \mathbf{G}_k)$ corresponding to the transitive homomorphisms $\varphi_j\colon \mathbf{B}(k) \to \mathbf{S}(2k)$, $j = 1, 2, 3$, which occur in Theorem 6.23. To describe these coverings, consider the open subset $\mathbf{G}(B_{k-1}) \subset \mathbf{G}_{k-1}$ consisting of all separable polynomials $p_{k-1}(t, w) = t^{k-1} + w_1 t^{k-2} + \ldots + w_{k-1}$ in $\mathbf{G}_{k-1}$ that satisfy the condition $w_{k-1} \neq 0$. We use the notation $\mathbf{G}(B_{k-1})$ since this set may be identified with the regular orbits space of the complex Coxeter group B_{k-1} acting naturally onto $\mathbb{C}^{k-1}$. Using the fact that $p_k(\lambda, z) = 0$ and $p_k'(\lambda, z) \neq 0$ for all $(\lambda, z) \in \mathbf{E}_1(k)$, we can define a mapping $\pi\colon \mathbf{E}_1(k) \to \mathbf{G}(B_{k-1})$ as follows: the image $\pi(\lambda, z)$ of any point $(\lambda, z) \in \mathbf{E}_1(k)$ is the separable polynomial $P(t; (\lambda, z))$ of degree $k-1$ in t defined by

$$\begin{aligned} P(t; (\lambda, z)) = \frac{1}{t} p_k(t + \lambda, z) &= \frac{1}{t} \sum_{j=0}^{k} \frac{t^{k-j}}{(k-j)!} \frac{d^{k-j} p_k}{dt^{k-j}}(\lambda, z) \\ &= t^{k-1} + \frac{t^{k-2}}{(k-1)!} \frac{d^{k-1} p_k}{dt^{k-1}}(\lambda, z) + \ldots + \frac{dp_k}{dt}(\lambda, z). \end{aligned}$$

Actually, this mapping shows that $\mathbf{E}_1(k) \cong \mathbb{C} \times \mathbf{G}(B_{k-1})$; thus, $\pi_1(\mathbf{E}_1(k)) \cong \pi_1(\mathbf{G}(B_{k-1}))$ and

$$H_1(\mathbf{E}_1(k), \mathbb{Z}) \cong H_1(\mathbf{G}(B_{k-1}), \mathbb{Z}) \cong \pi_1(\mathbf{G}(B_{k-1}))/\pi_1'(\mathbf{G}(B_{k-1})) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Moreover, the cohomology group $H^1(\mathbf{G}(B_{k-1}), \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by the cohomology classes of the two nonvanishing functions w_{k-1} and $d_{k-1}(w)$ (the discriminant); it follows that the cohomology group $H^1(\mathbf{E}_1(k), \mathbb{Z})$ is generated by the cohomology classes of the functions

$$f_1(\lambda, z) = \frac{dp_k}{dt}(\lambda, z) \qquad \text{and} \qquad f_2(\lambda, z) = d_k(z).$$

Therefore, up to equivalence, there are exactly three 2-coverings over the space $\mathbf{E}_1(k)$; these coverings are as follows:

$$\begin{aligned} \mathbf{E}_1^{(1)}(k) &= \left\{(\xi; (\lambda,\, z)) \in \mathbb{C}^* \times \mathbf{E}_1(k) \mid \xi^2 = \frac{dp_k}{dt}(\lambda,\, z)\right\}, \\ \mathbf{E}_1^{(2)}(k) &= \left\{(\xi; (\lambda,\, z)) \in \mathbb{C}^* \times \mathbf{E}_1(k) \mid \xi^2 = d_k(z)\right\}, \\ \mathbf{E}_1^{(3)}(k) &= \left\{(\xi; (\lambda,\, z)) \in \mathbb{C}^* \times \mathbf{E}_1(k) \mid \xi^2 = \frac{dp_k}{dt}(\lambda,\, z) \cdot d_k(z)\right\}, \end{aligned}$$

all the three with the same natural projection $q_j = q\colon\ (\xi; (\lambda,\, z)) \mapsto (\lambda,\, z)$. The composition of the latter projection with the projection $M_{\mu_k}\colon\ \mathbf{E}_1(k) \to \mathbf{G}_k$ defines the three $2k$-coverings $\mathbf{E}_1^{(j)}(k) \to \mathbf{G}_k$, $j = 1, 2, 3$. It is not difficult to see that *for any $j = 1, 2, 3$ the monodromy homomorphism $\mathbf{B}(k) \to \mathbf{S}(2k)$ corresponding to the covering $\mathcal{E}_1^{(j)}(k) = (\mathbf{E}_1^{(j)}(k), q_j, \mathbf{G}_k)$ is conjugate to φ_j.*

Let us say that a covering $q\colon E \to X$ *splits* if there exist two nontrivial coverings $q'\colon E \to E'$ and $q''\colon E' \to X$ such that $q = q'' \circ q'$. *An n-covering $q\colon E \to X$ splits if and only if its monodromy homomorphism $q^*\colon \pi_1(X) \to \mathbf{S}(n)$ is imprimitive.* (A cyclic n-covering splits if and only if n is non-prime.)

The following corollary (which is, actually, a topological equivalent of Artin Theorem and theorems 2.1(a), 6.20(a), and 6.23) describes all n-coverings over $\mathbf{G}_k$ for $n \le 2k$. Seemingly, no direct topological proof of this corollary is known.

Corollary 6.24. *Let $\mathcal{E} = (E, q, \mathbf{G}_k)$ be an n-covering over $\mathbf{G}_k$.*

a) Assume that either $n < k \ne 4$ or $6 < k < n < 2k$. Then $\mathcal{E}$ is equivalent to the standard cyclic n-covering $\mathcal{E}_{cycl}^{(n)}(k) = (E_{cycl}^{(n)}(k), C_k^{(n)}, \mathbf{G}_k)$.

b) If $k \ne 4, 6$ and $n = k$, then $\mathcal{E}$ is equivalent to one of the two standard k-coverings $\mathcal{E}_{cycl}^{(k)}(k) = (E_{cycl}^{(k)}(k), C_k^{(n)}, \mathbf{G}_k)$, $\mathcal{E}_1(k) = (\mathbf{E}_1(k), M_{\mu_k}, \mathbf{G}_k)$.

c) If $k > 8$ and $n = 2k$, then $\mathcal{E}$ is equivalent to one of the tree standard $2k$-coverings $\mathcal{E}_{cycl}^{(2k)}(k) = (E_{cycl}^{(2k)}(k), C_k^{(n)}, \mathbf{G}_k)$, $\mathcal{E}_1^{(j)}(k) = (\mathbf{E}_1^{(j)}(k), q_j, \mathbf{G}_k)$, $j = 1, 2, 3$.

In particular, if $8 < k \ne n \le 2k$ and an n-covering $\mathcal{E} = (E, q, \mathbf{G}_k)$ is noncyclic, then $n = 2k$ and $\mathcal{E}$ splits into a composition of a 2-covering and a k-covering. ☐

§7. Homomorphisms $\mathbf{B}'(k) \to \mathbf{S}(k)$ and $\mathbf{B}'(k) \to \mathbf{B}'(k)$

In this section we apply the results of §6 to prove Theorem C and Theorem D. We start with some preparations to the proof of Theorem C.

Assume that $k > 4$ and consider a *nontrivial* homomorphism $\psi\colon \mathbf{B}'(k) \to \mathbf{S}(k)$. Taking into account the presentation of the commutator subgroup $\mathbf{B}'(k)$ given by (0.14)-(0.21), we denote the ψ-images of the generators u, v, w, c_i by $\widehat{u}, \widehat{v}, \widehat{w}, \widehat{c}_i$, respectively. The latter permutations satisfy the system of equations

$$\widehat{u}\widehat{c}_1\widehat{u}^{-1} = \widehat{w}, \tag{7.1}$$

$$\widehat{u}\widehat{w}\widehat{u}^{-1} = \widehat{w}^2\widehat{c}_1^{-1}\widehat{w}, \tag{7.2}$$

$$\widehat{v}\widehat{c}_1\widehat{v}^{-1} = \widehat{c}_1^{-1}\widehat{w}, \tag{7.3}$$

$$\widehat{v}\widehat{w}\widehat{v}^{-1} = (\widehat{c}_1^{-1}\widehat{w})^3\widehat{c}_1^{-2}\widehat{w}, \tag{7.4}$$

$$\widehat{u}\widehat{c}_i = \widehat{c}_i\widehat{v} \qquad (2 \le i \le k-3), \tag{7.5(i)}$$

$$\widehat{v}\widehat{c}_i = \widehat{c}_i\widehat{u}^{-1}\widehat{v} \qquad (2 \le i \le k-3), \tag{7.6(i)}$$

$$\widehat{c}_i\widehat{c}_j = \widehat{c}_j\widehat{c}_i \qquad (1 \le i < j-1 \le k-4), \tag{7.7}$$

$$\widehat{c}_i\widehat{c}_{i+1}\widehat{c}_i = \widehat{c}_{i+1}\widehat{c}_i\widehat{c}_{i+1} \qquad (1 \le i \le k-4). \tag{7.8}$$

Consider the embedding

$$\lambda'_k\colon \mathbf{B}(k-2) \hookrightarrow \mathbf{B}'(k), \quad \lambda'_k(s_i) = c_i, \qquad 1 \le i \le k-3,$$

and the composition

$$\phi = \psi \circ \lambda'_k\colon \mathbf{B}(k-2) \xrightarrow{\lambda'_k} \mathbf{B}'(k) \xrightarrow{\psi} \mathbf{S}(k), \quad \phi(s_i) = \widehat{c}_i, \qquad 1 \le i \le k-3.$$

Definition 7.1. For a nontrivial homomorphism $\psi\colon \mathbf{B}'(k) \to \mathbf{S}(k)$, set $G = \operatorname{Im}\psi \subseteq \mathbf{S}(k)$ and $H = \operatorname{Im}\phi \subseteq \mathbf{S}(k)$. For any H-orbit $Q \subseteq \mathbf{\Delta}_k$ we put $Q' = \mathbf{\Delta}_k - Q$ and denote by $\phi_Q\colon B(k-2) \to \mathbf{S}(Q)$ and $\phi_{Q'}\colon B(k-2) \to \mathbf{S}(Q')$ the reductions of ϕ to the H-invariant sets Q and Q', respectively; ϕ is the disjoint product $\phi_Q \times \phi_{Q'}$. A (nontrivial) homomorphism ψ is called *tame* if there is an H-orbit $Q \subset \mathbf{\Delta}_k$ of length $k-2$. This orbit Q (if it exists) is the only H-orbit of length $\ge k-2$; we call it the *tame orbit* of ψ; evidently, $\#Q' = 2$, and ψ is the disjoint product of the noncyclic transitive homomorphism $\phi_Q\colon B(k-2) \to \mathbf{S}(Q) \cong \mathbf{S}(k-2)$ and the cyclic homomorphism $\phi_{Q'}\colon B(k-2) \to \mathbf{S}(Q') \cong \mathbf{S}(2)$ (a priori, $\phi_{Q'}$ might be trivial; however, we shall see that actually this cannot happen).

A group homomorphism $K \to \mathbf{S}(k)$ is said to be *even* if its image is contained in the alternating subgroup $\mathbf{A}(k) = \mathbf{S}'(k)$. ○

By Lemma 6.4, for any nontrivial homomorphism $\psi\colon \mathbf{B}'(k) \to \mathbf{S}(k)$ $(k > 4)$ we have:

($*$) *the homomorphisms ψ and ϕ are even, that is, $H \subseteq G \subseteq \mathbf{A}(k)$; moreover, ϕ is noncyclic, $\phi(s_1) \ne \phi(s_3)$, and $\phi(s_1^{-1}) \ne \phi(s_3)$.*

To handle the "unpleasant" cases $k = 5, 6$, we need the following simple lemma.

Lemma 7.1. *a) Assume that $A, B \in \mathbf{S}(k)$ are 3-cycles. Then at least one of the permutations AB, $A^{-1}B$ is not a 3-cycle.*

b) Assume that $A, B \in \mathbf{S}(5)$ are 5-cycles. Then at least one of the permutations AB, $A^{-1}B$, $A^{-2}B$, B^2AB is not a 5-cycle.

Proof. (*a*) is trivial. To check (*b*), suppose that $A = (a, b, c, d, e)$ and $B, AB \in \mathbf{S}(5)$ are 5-cycles. Then B must be one of the following eight 5-cycles:

$$A,\ A^2,\ A^3,\ (a,b,d,e,c),\ (a,b,e,c,d),\ (a,c,d,b,e),\ (a,d,e,b,c),\ (a,d,b,c,e).$$

The condition $[B^2AB] = [5]$ eliminates all the cycles from this list but A and A^2. Finally, for $B = A$ we have $A^{-1}B = 1$; and if $B = A^2$, then $A^{-2}B = 1$. $\square$

In the following lemma we establish some properties of the permutations $\widehat{u}$, $\widehat{v}$, $\widehat{w}$, $\widehat{c}_i$ corresponding to a nontrivial homomorphism ψ.

Lemma 7.2. *a) $[\widehat{w}] = [\widehat{c}_1^{\,-1}\widehat{w}] = [\widehat{c}_1] = \ldots = [\widehat{c}_{k-3}]$ and $[\widehat{u}] = [\widehat{v}] = [\widehat{u}^{-1}\widehat{v}]$.*

b) All the permutations $\widehat{u}, \widehat{v}, \widehat{w}, \widehat{c}_i$ are nontrivial (and even).

c) $\widehat{u}$ commutes with all the permutations $\widehat{c}_{i,j} = \widehat{c}_i\widehat{c}_j^{\,-1}$, $2 \le i, j \le k-3$.

d) If $\widehat{c}_1^{\,2} = 1$, then $\widehat{u}^3 = \widehat{v}^3 = 1$ and $\widehat{v} = \widehat{u}^{-1}$.

e) If $k = 5$, then $[\widehat{c}_1] \ne [3]$ and $[\widehat{c}_1] \ne [5]$.

Proof. a) Follows immediately from (7.1), (7.3), (7.5(2)), (7.6(2)), and (7.8), which shows that all $\widehat{c}_i$ are conjugate to each other.

b) Since ψ is nontrivial, it is sufficient to show that if one of the permutations $\widehat{u}, \widehat{v}, \widehat{w}, \widehat{c}_i$ is trivial, then all of them are trivial. If some $\widehat{c}_i = 1$ or $\widehat{w} = 1$, then it follows from (*a*) that $\widehat{w} = \widehat{c}_1 = \ldots = \widehat{c}_{k-3} = 1$, and (7.5), (7.6) imply that $\widehat{u} = \widehat{v} = 1$. If $\widehat{u} = 1$ or $\widehat{v} = 1$, then, by (*a*), we have $\widehat{u} = \widehat{v} = 1$, and (7.1), (7.2) imply $\widehat{w} = \widehat{c}_1 = \widehat{w}^2$; hence $\widehat{w} = 1$.

c) Relations (7.5) may be written in the form

$$\widehat{v} = \widehat{c}_2^{\,-1}\widehat{u}\widehat{c}_2 = \widehat{c}_3^{\,-1}\widehat{u}\widehat{c}_3 = \ldots = \widehat{c}_{k-3}^{\,-1}\widehat{u}\widehat{c}_{k-3};$$

this shows that $\widehat{u} = (\widehat{c}_i\widehat{c}_j^{\,-1}) \cdot \widehat{u} \cdot (\widehat{c}_i\widehat{c}_j^{\,-1})^{-1}$ for all $2 \le i, j \le k-3$.

d) By (*a*), the condition $\widehat{c}_1^{\,2} = 1$ implies that $\widehat{c}_i^{\,2} = 1$ for all i; hence $c_i^{-1} = c_i$, and relation (7.5(2)) can be written in the form $\widehat{u} = \widehat{c}_2\widehat{v}\widehat{c}_2^{\,-1} = \widehat{c}_2^{\,-1}\widehat{v}\widehat{c}_2$. In view of (7.6(2)), the right hand side of the latter relation is equal to $\widehat{u}^{-1}\widehat{v}$, and we get $\widehat{u} = \widehat{u}^{-1}\widehat{v}$; hence, $\widehat{v} = \widehat{u}^2$. Using the same relations (7.5(2)), (7.6(2)) and $\widehat{c}_2^{\,2} = 1$, we have

$$\widehat{v} = \widehat{c}_2\widehat{u}^{-1}\widehat{v}\widehat{c}_2^{\,-1} = (\widehat{c}_2\widehat{u}\widehat{c}_2^{\,-1})^{-1} \cdot \widehat{c}_2\widehat{v}\widehat{c}_2^{\,-1} = (\widehat{c}_2^{\,-1}\widehat{u}\widehat{c}_2)^{-1} \cdot \widehat{u} = \widehat{v}^{-1}\widehat{u},$$

and thus $\widehat{u} = \widehat{v}^2$. Thereby, $\widehat{u}^3 = \widehat{v}^3 = 1$ and $\widehat{v} = \widehat{u}^{-1}$.

e) Assume that $[\widehat{c}_1] = [p]$, where $p = 3$ or $p = 5$. Then also $[\widehat{c}_1^{\,-1}] = [p]$; by (*a*), we have $[\widehat{w}] = [\widehat{c}_1^{\,-1}\widehat{w}] = [p]$. If $p = 3$, then $(\widehat{c}_1^{\,-1}\widehat{w})^3 = 1$, $\widehat{c}_1^{\,-2} = \widehat{c}_1$, and (7.4) shows that $[\widehat{c}_1\widehat{w}] = [\widehat{c}_1^{\,-2}\widehat{w}] = [\widehat{w}] = [3]$; however, this contradicts Lemma 7.1(*a*) (with $A = \widehat{c}_1^{\,-1}$ and $B = \widehat{w}$). Consider the case $p = 5$. Then $[\widehat{c}_1] = [\widehat{c}_1^{\,-1}] = [\widehat{w}] = [\widehat{c}_1^{\,-1}\widehat{w}] = [5]$. It follows from (7.1) and (7.2) that

$$\widehat{w}^2\widehat{c}_1^{\,-1}\widehat{w} = \widehat{u}\widehat{w}\widehat{u}^{-1} \qquad \text{and} \qquad \widehat{u}\widehat{c}_1^{\,2}\widehat{w}\widehat{u}^{-1} = \widehat{w}^4\widehat{c}_1^{\,-1}\widehat{w} = \widehat{w}^{-1}\widehat{c}_1^{\,-1}\widehat{w};$$

hence, $[\widehat{w}^2\widehat{c}_1^{-1}\widehat{w}] = [5]$ and $[\widehat{c}_1^2\widehat{w}] = [5]$. Moreover, from (7.3), (7.4) we have

$$\widehat{v}\widehat{c}_1\widehat{w}\widehat{v}^{-1} = (\widehat{c}_1^{-1}\widehat{w})^4\widehat{c}_1^{-2}\widehat{w} = (\widehat{c}_1^{-1}\widehat{w})^{-1} \cdot \widehat{c}_1^{-1} \cdot (\widehat{c}_1^{-1}\widehat{w});$$

therefore, $[\widehat{c}_1\widehat{w}] = [\widehat{c}_1^{-1}] = [5]$. Taking $A = \widehat{c}_1^{-1}$, $B = \widehat{w}$, we see that A, B, AB, $A^{-1}B$, $A^{-2}B$, and B^2AB are 5-cycles in $\mathbf{S}(5)$, which contradicts Lemma 7.1(b). □

The following lemma brings us essentially closer to the desired result.

Lemma 7.3. *a) The homomorphism ψ is tame whenever $k \neq 6$.*

b) If $k = 6$ and ψ is nontame, then the homomorphism ϕ is transitive and conjugate to the homomorphism $\widetilde{\nu}_6'$: $\mathbf{B}(4) \to \mathbf{S}(6)$ defined in Remark 3.2.

c) If ψ is tame, then the reduction ϕ_Q: $B(k-2) \to \mathbf{S}(Q) \cong \mathbf{S}(k-2)$ to the tame orbit Q is conjugate to the canonical projection μ_{k-2}: $\mathbf{B}(k-2) \to \mathbf{S}(k-2)$, and the reduction $\phi_{Q'}$: $B(k-2) \to \mathbf{S}(Q') \cong \mathbf{S}(2)$ is a nontrivial homomorphism. In particular, $\widehat{c}_i = \psi(c_i) = \phi(s_i) = S_iT$, $1 \leq i \leq k-3$, where every $S_i = \phi_Q(s_i)$ is a transposition supported in Q, and T is the (only) transposition supported on Q'.

Proof. We start with the following claim, which is true for any $k > 4$ and any nontrivial homomorphism ψ:

Claim 1. *There exists* (exactly one) *H-orbit of length $q \geq k-2$.*

For $k \neq 6$ this follows immediately from the property $(*)$ and Theorem 2.1(a). For $k = 6$, we deal with the noncyclic even homomorphism ϕ: $\mathbf{B}(4) \to \mathbf{S}(6)$ that satisfies $(*)$. In this case there exists (exactly one) H-orbit of length $q \geq 4$. Indeed, let Q be an H-orbit of some length q. If $q \leq 3$ and ϕ_Q is noncyclic, then, by Theorem 2.14, $\phi_Q(s_1) = \phi_Q(s_3)$. Hence, if $\#Q \leq 3$ for all H-orbits, then the homomorphism ϕ cannot satisfy $(*)$.

Claim 2. *If ψ is nontame, then $k = 6$.*

Taking into account Claim 1, we may assume that there is an H-orbit Q with $\#Q = q > k-2$. Clearly, either $q = k-1$ or $q = k$; in any case, $\#Q' \leq 1$ and $\phi = j_Q \circ \phi_Q$, where j_Q: $\mathbf{S}(Q) \hookrightarrow \mathbf{S}(k)$ is the natural embedding. Since ϕ_Q is noncyclic and transitive, Theorem 6.3 and Theorem 6.15 show that this could happen only in one of the following five cases:

i) $k = 5$, $k-2 = 3$, $q = 4$, $\phi = j_Q \circ \phi_Q{:}\,\mathbf{B}(3) \xrightarrow{\phi_Q} \mathbf{S}(4) \overset{j_Q}{\hookrightarrow} \mathbf{S}(5)$, ϕ_Q is transitive and noncyclic, ϕ is even;

ii) $k = 6$, $k-2 = 4$, $q = 5$, $\phi = j_Q \circ \phi_Q{:}\,\mathbf{B}(4) \xrightarrow{\phi_Q} \mathbf{S}(5) \overset{j_Q}{\hookrightarrow} \mathbf{S}(6)$, ϕ_Q is transitive and noncyclic, ϕ is even;

iii) $k = 7$, $k-2 = 5$, $q = 6$, $\phi = j_Q \circ \phi_Q{:}\,\mathbf{B}(5) \xrightarrow{\phi_Q} \mathbf{S}(6) \overset{j_Q}{\hookrightarrow} \mathbf{S}(7)$, ϕ_Q is transitive and noncyclic, ϕ is even;

iv) $k = 5$, $k-2 = 3$, $q = 5$, $\phi = \phi_Q{:}\,\mathbf{B}(3) \to \mathbf{S}(5)$, ϕ is transitive, noncyclic and even;

v) $k = 6$, $k-2 = 4$, $q = 6$, $\phi = \phi_Q{:}\,\mathbf{B}(4) \to \mathbf{S}(6)$, ϕ is transitive, noncyclic and even.

However, all these cases, but (v), are impossible. Indeed, in case (i), applying Proposition 3.1(a), we see that the (even!) homomorphism ϕ_Q must be conjugate to the homomorphism $\psi_{3,4}^{(2)}$; clearly, $\widehat{c}_1 \sim \psi_{3,4}^{(2)}(s_1) = (2,3,4)$; hence, $[\widehat{c}_1] = [3]$, which contradicts Lemma 7.2(e). In case (ii), by Lemma 3.2, the homomorphism ϕ would satisfy

$\phi(s_1) = \phi_Q(s_1) = \phi_Q(s_3) = \phi(s_3)$, which contradicts property $(*)$. In case (iii), by Proposition 3.9, the homomorphism ϕ_Q must be conjugate to the homomorphism $\psi_{5,6}$ that sends any s_i into an *odd* permutation; clearly, ϕ makes the same (for $\#Q' = 1$), which is impossible (since ϕ must be even). To eliminate (iv), we use Proposition 3.1(b), which shows that the homomorphism ϕ must be conjugate to the homomorphism $\psi_{3,5}$; so, $\widehat{c}_1 \sim \psi_{3,5}(s_1) = (1,4,3,2,5)$; however, this contradicts Lemma 7.2(e). This proves Claim 2 and the statement (a) of the lemma.

$b)$ If $k = 6$ and ψ is nontame, the proof of Claim 2 shows that we are in the situation of case (v). By Proposition 3.5 and condition $(*)$, the homomorphism ϕ must be conjugate to one of the homomorphisms $\psi_{4,6}^{(i)}$ defined by (3.4). However, $\psi_{4,6}^{(1)}$ and $\psi_{4,6}^{(2)}$ are not even, and for $\psi_{4,6}^{(4)}$ we have $\psi_{4,6}^{(4)}(s_1^{-1}) = \psi_{4,6}^{(4)}(s_3)$, which is uncompatible with $(*)$; hence, $\psi \sim \psi_{4,6}^{(3)}$. By Remark 3.2, $\psi_{4,6}^{(3)}$ is conjugate to the homomorphism $\widetilde{\nu}_6'$.

$c)$ Since ψ is tame, the reduction $\phi_Q\colon \mathbf{B}(k-2) \to \mathbf{S}(Q) \cong \mathbf{S}(k-2)$ is a noncyclic transitive homomorphism. If this homomorphism is conjugate to μ_{k-2}, the other assertions of the statement (c) are evident (note that if $\phi_Q \sim \mu_{k-2}$, then the "complementary" reduction $\phi_{Q'}\colon \mathbf{B}(k-2) \to \mathbf{S}(Q') \cong \mathbf{S}(2)$ *must* be nontrivial, since the homomorphism ψ is even).

Let us assume that ϕ_Q is not conjugate to μ_{k-2}; by Artin Theorem, this may only happen if $k = 6$ or $k = 8$. The complementary reduction $\phi_{Q'}$ is either trivial or takes each s_i to the only transposition T supported on Q'; in any case, we have $\phi_{Q'}(s_1) = \phi_{Q'}(s_3)$ and $\phi_{Q'}(s_1^{-1}) = \phi_{Q'}(s_3)$. If $k = 6$, the reduction ϕ_Q must be conjugate to one of Artin's homomorphisms $\nu_{4,j}$, $1 \le j \le 3$; however, in each of these cases we have either $\psi(s_1) = \psi(s_3)$ or $\psi(s_1^{-1}) = \psi(s_3)$, which contradicts $(*)$.

Finally, we must show that the case when $k = 8$ and $\phi_Q \sim \nu_6$ is impossible. Since $\nu_6(s_1)$ is the product of three disjoint transpositions and ϕ must be even, the complementary reduction $\phi_{Q'}$ sends each s_i to the only transposition T supported on Q'. Without loss of generality, we may assume that $T = (1,2)$ and

$$\phi\colon \begin{cases} s_1 \mapsto \widehat{c}_1 = (1,2)(3,4)(5,6)(7,8), \quad s_2 \mapsto \widehat{c}_2 = (1,2)(3,7)(4,5)(6,8), \\ s_3 \mapsto \widehat{c}_3 = (1,2)(3,5)(4,6)(7,8), \quad s_4 \mapsto \widehat{c}_4 = (1,2)(3,4)(5,7)(6,8), \\ \qquad\qquad s_5 \mapsto \widehat{c}_5 = (1,2)(3,6)(4,5)(7,8). \end{cases} \tag{7.9}$$

By Lemma 7.2(d), $\widehat{u}^3 = 1$; since $\widehat{u}$ is even and nontrivial, we see that either $[\widehat{u}] = [3]$ or $[\widehat{u}] = [3,3]$. By Lemma 7.2(c), $\widehat{u}$ commutes with all the permutations $\widehat{c}_{i,j} = \widehat{c}_i\widehat{c}_j^{-1}$, $i,j \ge 2$; in particular, this is the case for $\widehat{c}_{2,3} = (3,4,8)(5,7,6)$. Since $\mathrm{Fix}\, \widehat{c}_{2,3} = \{1,2\}$, this set is $\widehat{u}$-invariant. It follows that $\{1,2\} \subseteq \mathrm{Fix}\, \widehat{u}$ (the cyclic decomposition of $\widehat{u}$ cannot contain a transposition). Hence, $\mathrm{supp}\, \widehat{u} \subseteq \{3,4,5,6,7,8\}$. Further, $\widehat{c}_{3,5} = (3,4)(5,6)$. The set $\{7,8\}$ is the fixed points set of the permutation $(3,4)(5,6)$ acting on $\{3,4,5,6,7,8\}$; therefore, it must be $\widehat{u}$-invariant; as above, this shows that $\{7,8\} \subseteq \mathrm{Fix}\, \widehat{u}$ and $\mathrm{supp}\, \widehat{u} \subseteq \{3,4,5,6\}$. Therefore, $\widehat{u}$ must be a 3-cycle supported in $\{3,4,5,6\}$; however, such a permutation cannot commute with $(3,4)(5,6)$. This contradiction concludes the proof. $\square$

Recall that we denote by μ_k' the restriction of the canonical projection

$$\mu_k\colon \mathbf{B}(k) \to \mathbf{S}(k)$$

to the commutator subgroup $\mathbf{B}'(k)$; similarly, ν_6' denotes the restriction to $\mathbf{B}'(6)$ of Artin's homomorphism ν_6. If $\psi = \mu_k'$, then

$$\begin{aligned} &\widehat{u} = (1,3,2), \qquad \widehat{v} = (1,2,3), \qquad \widehat{w} = (1,3)(2,4), \\ &\widehat{c}_i = (1,2)(i+2,i+3), \qquad 1 \le i \le k-3. \end{aligned} \tag{7.10}$$

Moreover, if $k = 6$ and $\psi = \nu_6'$, then

$$\begin{aligned} &\widehat{u} = (1,3,6)(2,5,4), \ \widehat{v} = (1,6,3)(2,4,5), \ \widehat{w} = (2,3)(5,6), \\ &\widehat{c}_1 = (1,4)(2,3), \ \widehat{c}_2 = (3,6)(4,5), \ \widehat{c}_3 = (1,3)(2,4). \end{aligned} \tag{7.11}$$

Remark 7.1. Suppose $k > 4$. In view of Lemma 7.3, in order to classify nontrivial homomorphisms ψ: $\mathbf{B}'(k) \to \mathbf{S}(k)$ up to conjugation, it is sufficient to study the following two cases:

i) The homomorphism ψ is tame, with the tame H-orbit $Q = \{3, 4, \ldots, k\}$. The reduction ϕ_Q: $\mathbf{B}(k-2) \to \mathbf{S}(k-2)$ coincides with the "shifted" canonical projection

$$\widetilde{\mu}_{k-2}\colon \mathbf{B}(k-2) \to \mathbf{S}(Q), \ \ \widetilde{\mu}_{k-2}(s_i) = (i+2, i+3), \ \ 1 \le i \le k-3.$$

$Q' = \{1,2\}$ and the complementary reduction $\phi_{Q'}$: $\mathbf{B}'(k-2) \to \mathbf{S}(Q') \cong \mathbf{S}(2)$ is of the form $\phi_{Q'}(s_i) = (1,2)$, $1 \le i \le k-3$. The homomorphism ϕ is the disjoint product $\phi_Q \times \phi_{Q'}$ and

$$\widehat{c}_i = \phi(s_i) = (1,2)(i+2,i+3) \qquad \text{for all } \ i = 1, \ldots, k-3. \tag{7.12}$$

ii) $k = 6$ and the homomorphism ψ is nontame, with the only H-orbit $Q = \mathbf{\Delta}_6$. The homomorphism ϕ: $\mathbf{B}(4) \to \mathbf{S}(6)$ coincides with the homomorphism $\widetilde{\nu}_6'$ and

$$\begin{aligned} \phi(s_1) = \widehat{c}_1 = (1,4)(2,3), \quad \phi(s_2) = \widehat{c}_2 = (3,6)(4,5), \\ \phi(s_3) = \widehat{c}_3 = (1,3)(2,4). \end{aligned} \tag{7.13}$$

Let us say that ψ is *reduced* if it is either of type (i) or of type (ii). ○

Lemma 7.4. *Let ψ be a reduced homomorphism of type* (i).

a) *If $k \ge 6$, then $\widehat{u}(\{1,2,3\}) = \{1,2,3\}$ and $\widehat{u}(\{4,5,6\}) = \{4,5,6\}$.*

b) *If $k \ge 7$, then $4,5,\ldots,k \in \operatorname{Fix}\widehat{u}$ and $\widehat{u}$ is a 3-cycle supported on $\{1,2,3\}$.*

Proof. By Lemma 7.2(c) and (7.12), $\widehat{u}$ commutes with any permutation

$$\widehat{c}_{i,i+1} = \widehat{c}_i \widehat{c}_{i+1}^{-1} = (i+2, i+3, i+4), \ \ 2 \le i \le k-4.$$

Hence, each of the sets $\{4,5,6\}$, $\{5,6,7\}$,...,$\{k-2, k-1, k\}$ is $\widehat{u}$-invariant. The union, the intersection, and the difference of two $\widehat{u}$-invariant sets are $\widehat{u}$-invariant. This implies (a). Moreover, if $k \ge 7$, we have

$$j+2, j+5 \in \operatorname{Fix}\widehat{u} \ \text{ and } \ \widehat{u}(\{j+3, j+4\}) = \{j+3, j+4\} \text{ whenever } 2 \le j \le k-5;$$

by Lemma 7.2(d), all the cycles in the cyclic decomposition of $\widehat{u}$ are of length 3; hence, $j+3, j+4 \in \operatorname{Fix}\widehat{u}$ and supp $\widehat{u} = \{1,2,3\}$. □

Theorem 7.5. *Suppose $k > 4$. Let ψ: $\mathbf{B}'(k) \to \mathbf{S}(k)$ be a nontrivial homomorphism. Then either $\psi \sim \mu'_k$ or $k = 6$ and $\psi \sim \nu'_6$. In any case* $\operatorname{Im}\psi = \mathbf{A}(k)$ *and* $\operatorname{Ker}\psi = \mathbf{J}(k) = \mathbf{I}(k) \cap \mathbf{B}'(k)$.

Proof. By Remark 7.1, we may assume that ψ is reduced. Let us start with case (i). By Lemma 7.2(a), $[\widehat{c}_1^{-1}\widehat{w}] = [\widehat{w}] = [\widehat{c}_1] = [2,2]$; hence, $\widehat{w}$ and $\widehat{c}_1$ cannot be disjoint.

Claim 1. supp $\widehat{w} = \{1,2,3,4\}$ and either $\widehat{w} = (1,3)(2,4)$ or $\widehat{w} = (1,4)(2,3)$.

Let $m = \#(\{1,2,3,4\} \cap \operatorname{supp}\widehat{w})$. We already know that $m \geq 1$. The values $m = 1$ and $m = 3$ cannot occur by trivial reasons ($m = 1$ implies $[\widehat{c}_1^{-1}\widehat{w}] = [3,2,2]$; and if $m = 3$, then either $[\widehat{c}_1^{-1}\widehat{w}] = [5]$ or $[\widehat{c}_1^{-1}\widehat{w}] = [3]$). Assume that $m = 2$, that is, supp $\widehat{w} = \{a,b,p,q\}$, where $a, b \in \{1,2,3,4\}$ and $p, q \geq 5$. Then $k \geq 6$. By (7.1) and (7.12), $\widehat{w} = \widehat{u}(1,2)(3,4)\widehat{u}^{-1}$; hence $\widehat{u}(\{1,2,3,4\}) = \operatorname{supp}\widehat{w} = \{a,b,p,q\}$. In view of Lemma 7.4$(a)$, this shows that $\{1,2,3\} = \widehat{u}(\{1,2,3\}) \subset \widehat{u}(\{1,2,3,4\}) = \{a,b,p,q\}$, which contradicts the condition $p, q \geq 5$. Thus, $\widehat{w}$ is a product of two disjoint transpositions supported on $\{1,2,3,4\}$, and the condition $[(1,2)(3,4)\cdot\widehat{w}] = [\widehat{c}_1^{-1}\widehat{w}] = [2,2]$ implies the desired result.

If $\widehat{w} = (1,4)(2,3)$, we conjugate the homomorphism ψ by the transposition $(1,2)$ and obtain a homomorphism that sends any c_i into $\widehat{c}_i$ and sends w into $(1,3)(2,4)$; therefore, without loss of generality we may assume that the original homomorphism ψ itself satisfies the condition

$$\widehat{w} = \psi(w) = (1,3)(2,4). \tag{7.14}$$

Then relation (7.1) takes the form

$$\widehat{u}(1,2)(3,4)\widehat{u}^{-1} = (1,3)(2,4); \tag{7.15}$$

in particular,

$$\widehat{u}(\{1,2,3,4\}) = \{1,2,3,4\}. \tag{7.16}$$

Taking into account (7.10), (7.12), and (7.14), we conclude the proof of the theorem in case (i) by proving the following claim:

Claim 2. $a)$ Any $i \geq 4$ is a fixed point of $\widehat{u}$, and thus $\widehat{u}$ is a 3-cycle supported on $\{1,2,3\}$. $b)$ $\widehat{u} = (1,3,2)$ and $\widehat{v} = (1,2,3)$.

In view of Lemma 7.4(b), we need to prove (a) only for $k = 5,6$. For $k = 6$, Lemma 7.4(a) shows that $\widehat{u}(\{4,5,6\}) = \{4,5,6\}$; by (7.16), we have $\widehat{u}(4) = 4$ and $\widehat{u}(\{5,6\}) = \{5,6\}$. In fact, $\{5,6\} \subset \operatorname{Fix}\widehat{u}$ (since $\widehat{u}$ cannot contain a transposition); this proves (a) for $k = 6$. If $k = 5$, (7.16) shows that $\widehat{u}(5) = 5$. Relations (7.5(2)) and Lemma 7.2(d) imply that $(\widehat{u}\widehat{c}_2)(5) = (\widehat{c}_2\widehat{u}^{-1})(5)$; since $\widehat{u}(5) = 5$ and (by (7.12)) $\widehat{c}_2(5) = 4$, this means that $\widehat{u}(4) = 4$, which concludes the proof of (a). To prove (b), we note that $\widehat{u} = (1,3,2)$ is the only 3-cycle supported on $\{1,2,3\}$ that satisfies (7.15).

Case (ii) may be treated by straightforward computations; however, they are too long, and we prefer to use a simple trick. Namely, instead of the original homomorphism ψ of type (ii), we consider its composition $\widetilde{\psi} = \varkappa \circ \psi$ with the outer automorphism $\varkappa$ of the group $\mathbf{S}(6)$. (see (3.3)). It is completely clear that $\widetilde{\psi}$ is a tame homomorphism of type (i); it follows from what has been proven above that $\widetilde{\psi} \sim \mu'_6$. The automorphism $\varkappa$ is involutive and $\nu'_6 = \varkappa \circ \mu'_6$; therefore, $\psi \sim \nu'_6$. □

Corollary 7.6. *Suppose $k > 4$. Any nontrivial homomorphism ψ: $\mathbf{B}'(k) \to \mathbf{S}(k)$ admits a unique extension Ψ: $\mathbf{B}(k) \to \mathbf{S}(k)$.*

Proof. The existence follows immediately from Theorem 7.5. The uniqueness follows from the facts that $\mu_k(\mathbf{B}'(k)) = \mathbf{A}(k)$, $\nu_6(\mathbf{B}'(6)) = \mathbf{A}(6)$ and (for any $k \geq 3$) the centralizer of $\mathbf{A}(k)$ in $\mathbf{S}(k)$ is trivial. $\square$

Remark 7.2. In view of Artin Theorem, Corollary 7.6 implies Theorem 7.5. However, I have no idea how to extend nontrivial homomorphisms ψ: $\mathbf{B}'(k) \to \mathbf{S}(k)$ to homomorphisms Ψ: $\mathbf{B}(k) \to \mathbf{S}(k)$ *without* Theorem 7.5. $\bigcirc$

Theorem 7.7. *Suppose $k > 4$. The pure commutator subgroup $\mathbf{J}(k) = \mathbf{I}(k) \cap \mathbf{B}'(k)$ is a completely characteristic subgroup of the group $\mathbf{B}'(k)$, that is, $\phi(\mathbf{J}(k)) \subseteq \mathbf{J}(k)$ for any endomorphism ϕ: $\mathbf{B}'(k) \to \mathbf{B}'(k)$. Moreover, $\phi^{-1}(\mathbf{J}(k)) = \mathbf{J}(k)$ and* Ker $\phi \subset \mathbf{J}(k)$ *for every nontrivial endomorphism ϕ.*

Proof. The case of trivial ϕ is trivial. Given a nontrivial ϕ, consider the composition

$$\psi = \mu_k' \circ \phi\colon\ \mathbf{B}'(k) \xrightarrow{\phi} \mathbf{B}'(k) \xrightarrow{\mu_k'} \mathbf{S}(k).$$

This homomorphism ψ must be nontrivial, since otherwise $\operatorname{Im}\phi \subseteq \operatorname{Ker}\mu_k' \subset \mathbf{I}(k)$ and Markov Theorem implies that ϕ is trivial. By Theorem 7.5, either $\psi \sim \mu_k'$ or $k = 6$ and $\psi \sim \nu_6'$; in any of these cases, $\operatorname{Ker}\psi = \mathbf{I}(k) \cap \mathbf{B}'(k) = \mathbf{J}(k)$ and we have

$$\mathbf{J}(k) = \operatorname{Ker}\psi = \operatorname{Ker}(\mu_k' \circ \phi) = \phi^{-1}(\operatorname{Ker}\mu_k') = \phi^{-1}(\mathbf{J}(k)).$$

Certainly, this shows also that $\phi(\mathbf{J}(k)) \subseteq \mathbf{J}(k)$ and $\operatorname{Ker}\phi \subset \mathbf{J}(k)$. $\square$

Remark 7.3. For a nontrivial endomorphism ϕ the inclusion $\operatorname{Ker}\phi \subset \mathbf{J}(k)$ must be strict, since $\mathbf{B}'(k)/\mathbf{J}(k) \cong \mathbf{A}(k)$ and $\mathbf{B}'(k)$ is torsion free. It seems that for $k > 4$ no examples of nontrivial endomorphisms $\mathbf{B}'(k) \to \mathbf{B}'(k)$ with nontrivial kernels are known. I conjectured that for $k > 4$ a proper quotient group of the commutator subgroup $\mathbf{B}'(k)$ cannot be torsion free (this would imply that any nontrivial endomorphism of $\mathbf{B}'(k)$ must be injective). I was told that D. Goldsmith's braid group (which is a proper non-Abelian quotient group of $\mathbf{B}(k)$) is torsion free. For sure, this is true if $k = 3$, but I newer saw any proof for $k > 4$. If so, this would disprove my conjecture.

E. Artin [Ar3] proved that the pure braid group $\mathbf{I}(k)$ is a characteristic subgroup of the braid group $\mathbf{B}(k)$, that is, $\phi(\mathbf{I}(k)) = \mathbf{I}(k)$ for any *automorphism* ϕ of the *whole* braid group $\mathbf{B}(k)$ (see also Theorem 2.12). Formally, for $k > 4$ Theorem 7.7 is essentially stronger than this Artin theorem (and also essentially stronger than Theorem 2.12, which, in turn, is an improvement of Artin's result). However, I do not know any nontrivial endomorphism of $\mathbf{B}'(k)$ $(k > 4)$ that is not an automorphism. Seemingly, nobody knows whether there is an automorphism of $\mathbf{B}'(k)$ that cannot be extended to an automorphism of the whole braid group $\mathbf{B}(k)$. In view of these remarks, it may actually happen that Theorem 7.7 does not say more than Artin's result says. Nevertheless, Theorem 7.7 works in some situations when Artin Theorem and Theorem 2.12 (in their present forms) are useless. $\bigcirc$

§8. Special homomorphisms $\mathbf{B}(k) \to \mathbf{B}(n)$

8.0. Necessary conditions. Here we prove Theorem H(a) that provides us with certain strict necessary conditions for the existence of nonintegral special homomorphisms $\mathbf{B}(k) \to \mathbf{B}(n)$.

Given a special system of generators $\{a, b\}$ in $\mathbf{B}(m)$, we denote by $\mathcal{H}_m(a, b)$ the subset in $\mathbf{B}(m)$ consisting of all the elements $g^{-1}a^q g$ and $g^{-1}b^q g$, where g runs over $\mathbf{B}(m)$ and p runs over $\mathbb{Z}$. By Murasugi Theorem (see §0.7), *an m-braid h belongs to $\mathcal{H}_m(a, b)$ if and only if h is an element of finite order modulo the center $\mathbf{C}(m)$ of the group $\mathbf{B}(m)$*; hence, *the subset $\mathcal{H}_m(a, b) \subset \mathbf{B}(m)$ does not depend on a choice of a special system of generators $a, b \in \mathbf{B}(m)$.*

Definition 8.1. A homomorphism φ: $\mathbf{B}(k) \to \mathbf{B}(n)$ is said to be *special* if $\varphi(\mathcal{H}_k(a, b)) \subseteq \mathcal{H}_n(a', b')$ for some (and hence for any) choice of special systems of generators $a, b \in \mathbf{B}(k)$ and $a', b' \in \mathbf{B}(n)$. ○

My interests to the special homomorphisms is motivated by the fact that *for every holomorphic mapping f: $\mathbf{G}_k \to \mathbf{G}_n$, every point $z^\circ \in \mathbf{G}_k$, and any choice of isomorphisms $\mathbf{B}(k) \cong \pi_1(\mathbf{G}_k, z^\circ)$ and $\pi_1(\mathbf{G}_n, f(z^\circ)) \cong \mathbf{B}(n)$, the induced homomorphism of braid groups*

$$f_*\colon\ \mathbf{B}(k) \cong \pi_1(\mathbf{G}_k, z^\circ) \to \pi_1(\mathbf{G}_n, f(z^\circ)) \cong \mathbf{B}(n)$$

is special (see [L7] or Part II of this paper for the proof).

Let $P(k)$ be the union of the four arithmetic progressions $P^{k,i}$ $(1 \le i \le 4)$ introduced in §0.7 (Notation 0.1).

Theorem 8.1. *Assume that for some $k \ne 4$ and some n there exists a nonintegral special homomorphism φ: $\mathbf{B}(k) \to \mathbf{B}(n)$; then $n \in P(k)$. In more details, there exist a special system of generators $\{a, b\} \in \mathbf{B}(n)$, an element $g \in \mathbf{B}(n)$, and integers l, t such that at least one of the following four conditions*[4] *is fulfilled:*

a) $\varphi(\alpha) = a^p$ *and* $\varphi(\beta) = gb^q g^{-1}$, *where* $p = t(l(k-1)+1)$, $q = t(lk+1)$, $l \ge 0$, $(t, k(k-1)) = 1$, *and* $n = k + lk(k-1) \in P_1(k)$;

b) $\varphi(\alpha) = a^p$ *and* $\varphi(\beta) = ga^q g^{-1}$, *where* $p = t(k-1)$, $q = tk$, *g commutes with* $a^{tk(k-1)}$, $l \ge 1$, *and* $n = lk(k-1) \in P_2(k)$;

c) $\varphi(\alpha) = b^p$ *and* $\varphi(\beta) = ga^q g^{-1}$, *where* $p = t(k-1)$, $q = tk$, *g commutes with* $b^{tk(k-1)}$, $l \ge 1$, *and* $n = lk(k-1) + 1 \in P_3(k)$;

d) $\varphi(\alpha) = b^p$ *and* $\varphi(\beta) = ga^q g^{-1}$, *where* $p = t(l(k-1)-1)$, $q = t(lk-1)$, $l \ge 1$, $(t, k(k-1)) = 1$, *and* $n = (k-1)(lk-1) \in P_4(k)$.

In particular, every special homomorphism φ: $\mathbf{B}(k) \to \mathbf{B}(n)$ is integral whenever $k \ne 4$ and $n \notin P(k)$.

Proof. Let us denote by $\mathcal{S}_i(k, n)$, $1 \le i \le 4$, the class of all special homomorphisms φ: $\mathbf{B}(k) \to \mathbf{B}(n)$ that satisfy (every time for an appropriate special system of generators

[4] As usual, $\alpha = \sigma_1 \cdots \sigma_{k-1}$ and $\beta = \alpha\sigma_1$ is the special system of generators in $\mathbf{B}(k)$ corresponding to the canonical generators $\sigma_1, \cdots, \sigma_{k-1}$.

$\{a, b\} \in \mathbf{B}(n)$, an element $g \in \mathbf{B}(n)$, and some integers p, q) one of the following four conditions 1) – 4), respectively:

$$1)\ \varphi(\alpha) = a^p \text{ and } \varphi(\beta) = gb^q g^{-1}; \qquad 2)\ \varphi(\alpha) = a^p \text{ and } \varphi(\beta) = ga^q g^{-1};$$
$$3)\ \varphi(\alpha) = b^p \text{ and } \varphi(\beta) = gb^q g^{-1}; \qquad 4)\ \varphi(\alpha) = b^p \text{ and } \varphi(\beta) = ga^q g^{-1}.$$

Any special homomorphism φ must certainly belong to at least one of these four classes.

There exists a homomorphism $\delta\colon \mathbf{B}(n) \to \mathbb{Z}$ such that $\delta(a) = n - 1$ and $\delta(b) = n$ (since $\mathbf{B}(n)/\mathbf{B}'(n) \cong \mathbb{Z}$ and $a^n = b^{n-1}$). Moreover, in view of $\alpha^k = \beta^{k-1}$, we have

$$k\delta(\varphi(\alpha)) = (k - 1)\delta(\varphi(\beta)). \tag{8.1}$$

The element $a^n = b^{n-1}$ is central in $\mathbf{B}(n)$. Hence, if $\varphi \in \mathcal{S}_1(k, n) \cup \mathcal{S}_2(k, n)$, then the element $\varphi(\alpha^n) = a^{pn}$ commutes with $\varphi(\beta)$; since $k \neq 4$, Lemma 1.17(a) implies that k divides n. If $\varphi \in \mathcal{S}_1(k, n) \cup \mathcal{S}_3(k, n)$, then $\varphi(\beta^{n-1}) = gb^{q(n-1)}g^{-1} = b^{q(n-1)}$ commutes with $\varphi(\alpha)$, and Lemma 1.17(b) shows that $k - 1$ divides $n - 1$. By a similar argument, $\varphi \in \mathcal{S}_2(k, n) \cup \mathcal{S}_4(k, n)$ implies that $\varphi(\beta^n) = ga^{qn}g^{-1} = a^{qn}$ commutes with $\varphi(\alpha)$ and $k - 1$ divides n, and $\varphi \in \mathcal{S}_3(k, n) \cup \mathcal{S}_4(k, n)$ implies that $\varphi(\alpha^{n-1}) = b^{p(n-1)}$ commutes with $\varphi(\beta)$ and k divides $n - 1$.

Assume that $\varphi \in \mathcal{S}_1(k, n)$, that is, $\varphi(\alpha) = a^p$, $\varphi(\beta) = gb^q g^{-1}$. It follows from the above consideration that k divides n and $k - 1$ divides $n - 1$. Hence, there exists an integer $l \geq 0$ such that $n = k + lk(k - 1)$. Relation (8.1) shows that $k(n - 1)p = (k - 1)nq$; therefore, $(lk+1)p = (l(k-1)+1)q$. Since the numbers $lk+1$, $l(k-1)+1$ are co-prime, there exists an integer t such that $p = t(l(k - 1) + 1)$ and $q = t(lk + 1)$. Let us show that $(t, k(k - 1)) = 1$. Indeed, if $m = (t, k) > 1$, then the ratios $t' = t/m$ and $k' = k/m$ are integral and $1 \leq k' < k$, $pk' = tk'(l(k - 1) + 1) = mt'k'(l(k - 1) + 1) = t'k(l(k - 1) + 1) = t'n$. Hence, the element $\varphi(\alpha^{k'}) = a^{pk'} = a^{t'n}$ commutes with $\varphi(\beta)$; by Lemma 1.17(a), k must be a divisor of k', which is impossible. Similarly, one can check that the inequality $(t, k - 1) > 1$ leads to a contradiction; this completes the proof in the case when $\varphi \in \mathcal{S}_1(k, n)$.

Assume now that $\varphi \in \mathcal{S}_2(k, n)$, that is, $\varphi(\alpha) = a^p$, $\varphi(\beta) = ga^q g^{-1}$. Then k and $k - 1$ divide n; hence, $n = lk(k - 1)$ for some integer $l \geq 1$. Relation (8.1) shows that $kp = (k - 1)q$; consequently, there is an integer t such that $p = t(k - 1)$ and $q = tk$. Taking into account the relations $kp = (k - 1)q = tk(k - 1)$ and $\alpha^k = \beta^{k-1}$, we obtain $a^{tk(k-1)} = ga^{tk(k-1)}q^{-1}$; thus, g commutes with $a^{tk(k-1)}$. This concludes the proof in the case when $\varphi \in \mathcal{S}_2(k, n)$.

We skip the proofs for the cases $\varphi \in \mathcal{S}_3(k, n)$ and $\varphi \in \mathcal{S}_4(k, n)$, which are very similar to the cases considered above. □

Remark 8.1. $\mathbf{B}(3)$ possesses nonintegral special homomorphisms $\mathbf{B}(3) \to \mathbf{B}(n)$ for every n that is not forbidden by Theorem 8.1. Moreover, the conditions $\varphi(\alpha) = b$, $\varphi(\beta) = a^2$ define a special *epimorphism* $\varphi\colon \mathbf{B}(4) \to \mathbf{B}(3)$; hence, if there exists a nonintegral special homomorphism $\mathbf{B}(3) \to \mathbf{B}(n)$, then there is also a nonintegral special homomorphism $\mathbf{B}(4) \to \mathbf{B}(n)$. For $k > 4$ and $n \in P_3(k) \cup P_4(k)$ I do not know any example of a nonintegral special homomorphism $\mathbf{B}(k) \to \mathbf{B}(n)$; however, Theorem H($b$) proven below asserts that for any k and any $n \in P_1(k) \cup P_2(k)$ such homomorphisms do exist. ○

8.1. Existence of non-Abelian special homomorphisms. Here we explain a construction which proves Theorem H(b).

Definition 8.2. A geometric braid is called *wide* if the distance between every two of its strings is at least 1. A geometric p-braid is said to be a *δ-thin p-rope* if it is contained in an open "circular tube" of diameter δ around one of its strings. We denote by $\mathbf{1}^p$ (respectively, by $\mathbf{1}^p_\delta$) the unity of $\mathbf{B}(p)$ (respectively, a δ-thin p-rope representing $\mathbf{1}^p$).○

Let $g \in \mathbf{B}(k)$ and $v \in \mathbf{B}(p)$. We represent g and v by a wide geometric k-braid $\widetilde{g}$ and a $\frac{1}{3}$-thin p-rope $\widetilde{v}$, respectively. Then, replacing each of the strings of $\widetilde{g}$ with the same thin p-rope $\widetilde{v}$, we obtain a geometric pk-braid $\widetilde{g} \otimes \widetilde{v}$; the corresponding element of the braid group $\mathbf{B}(kp)$ is said to be the *tensor product* $g \otimes v$ of g and v.

Let $\sigma_1, ..., \sigma_{k-1}$ be the canonical generators in $\mathbf{B}(k)$ and χ: $\mathbf{B}(k) \to \mathbb{Z}$, $\sigma_1, ..., \sigma_{k-1} \mapsto 1$, be the canonical integral projection. Given an element $v \in \mathbf{B}(p)$, we define a homomorphism ψ_v: $\mathbf{B}(k) \to \mathbf{B}(pk)$ as follows: $\psi_v(g) = g \otimes v^{\chi(g)}$ for any $g \in \mathbf{B}(k)$. In particular, $\psi_v(\sigma_i) = \sigma_i \otimes v$ for any $i = 1, ..., k-1$. Clearly, ψ_v is an embedding.

This construction may be modified as follows. Let $v \in \mathbf{B}(p)$. We represent v by a $\frac{1}{3}$-thin p-rope $\widetilde{v}$, and any generator $\sigma_i \in \mathbf{B}(k)$ by a wide geometric k-braid $\widetilde{\sigma_i}$. Then we replace the i'th string of $\widetilde{\sigma_i}$ with the above thin rope $\widetilde{v}$, and all the rest strings of $\widetilde{\sigma_i}$ with $\mathbf{1}^p_{\frac{1}{3}}$. The resulting geometric pk-braids defines an element of $\mathbf{B}(pk)$, which is denoted by $\sigma_i * v$. The correspondence $\sigma_i \mapsto \sigma_i * v \in \mathbf{B}(pk)$, $i = 1, ..., k-1$, determines an injective homomorphism ϕ_v: $\mathbf{B}(k) \to \mathbf{B}(pk)$.

Let, as usual, $a = a_{1,pk} = s_1 s_2 \cdots s_{pk-1}$ and $b = s_{pk-1}$ form the special system of generators in $\mathbf{B}(pk)$ corresponding to the canonical generators $s_1, \ldots, s_{pk-1}$. In the following lemma-notation we exhibit an algebraic description of the homomorphism ϕ_v (especially, see statements (c) and (d)).

Lemma-Notation 8.2. *a) We define the elements a_{ij} according to* (0.3)*, that is,*

$$a_{ii} = 1 \ \ \text{for all } i, \qquad \text{and} \qquad a_{ij} = s_i s_{i+1} \cdots s_{j-1} \ \ \text{for} \ \ 1 \le i < j \le kp. \tag{8.2}$$

For $1 \le i \le k$ set

$$\begin{aligned} &a_i = a_{p(i-1)+1,pi} = s_{p(i-1)+1} s_{p(i-1)+2} \cdots s_{p(i-1)+p-1}, \qquad b_i = a_i s_{p(i-1)+1}, \\ &\Delta_i = a_{p(i-1)+1,pi} a_{p(i-1)+1,pi-1} \cdots a_{p(i-1)+1,pi-(p-2)} \\ &= s_{p(i-1)+1} \cdots s_{pi-1} \times s_{p(i-1)+1} \cdots s_{pi-2} \times \cdots \times s_{p(i-1)+1} s_{p(i-1)} \times s_{p(i-1)+1} \,. \end{aligned} \tag{8.3}$$

Clearly,

$$a_i^p = b_i^{p-1} \qquad \text{and} \qquad a_i s_j = s_{j+1} a_i \tag{8.4}$$

whenever $1 \le i \le k$ and $p(i-1)+1 \le j \le p(i-1)+p-2$, and the element $A_i \overset{\text{def}}{=} a_i^p = b_i^{p-1}$ generates the center of the subgroup $B_i \cong \mathbf{B}(p)$ in $\mathbf{B}(kp)$ spanned by $s_{p(i-1)+1}, s_{p(i-1)+2}, \ldots, s_{p(i-1)+p-1}$.

b) For $1 \le i \le k-1$ define the elements $t_i = \sigma_i \otimes \mathbf{1}^p$. The following relations are held:

$$\begin{aligned} t_i &= s_{pi}s_{pi-1}\cdots s_{pi-(p-1)} \times s_{pi+1}s_{pi}\cdots s_{pi-(p-2)} \\ &\qquad \times \cdots \times s_{pi+(p-1)}s_{pi+(p-2)}\cdots s_{pi} \\ &= s_{pi}s_{pi+1}\cdots s_{pi+(p-1)} \times s_{pi-1}s_{pi}\cdots s_{pi+(p-2)} \\ &\qquad \times \cdots \times s_{pi-(p-1)}s_{pi-(p-2)}\cdots s_{pi} \\ &= a_{pi,(i+1)p}a_{pi-1,(i+1)p-1}\cdots a_{pi-(p-1),(i+1)p-(p-1)}\,; \end{aligned} \tag{8.5}$$

$$\begin{aligned} &t_it_j = t_jt_i \qquad \textit{whenever } i,j = 1,\dots,k-1 \textit{ and } |i-j|>1; \\ &t_it_{i+1}t_i = t_{i+1}t_it_{i+1} \qquad \textit{for } 1 \le i < k-1\,; \end{aligned} \tag{8.6}$$

$$t_it_{i+1}\cdots t_j = a_{pi,(j+1)p}a_{pi-1,(j+1)p-1}\cdots a_{p(i-1)+1,jp+1} \qquad \textit{for } \ 1 \le i \le j < k\,; \tag{8.7}$$

$$s_rt_i = t_is_{r+p} \qquad \textit{whenever } \ 1 \le i \le k-1 \ \textit{ and } \ p(i-1) < r < pi\,. \tag{8.8}$$

Moreover, for any word $\omega(y_1,\dots,y_{(j-i+1)p-1})$ in variables

$$y_1,\, y_1^{-1},\dots, y_{(j-i+1)p-1},\, y_{(j-i+1)p-1}^{-1}$$

and any $i,\ j,\ \ 1 \le i \le j \le k-1$, one has

$$\omega(s_{pi+1},\dots,s_{(j+1)p-1})\cdot t_it_{i+1}\cdots t_j = t_it_{i+1}\cdots t_j\cdot \omega(s_{p(i-1)+1},\dots,s_{jp-1})\,. \tag{8.9}$$

c) For any word $v = v(x_1,\dots,x_{p-1})$ in variables $x_1, x_1^{-1},\dots,x_{p-1},x_{p-1}^{-1}$ define the elements $v_j \in \mathbf{B}(kp)$ by

$$v_j = v(s_{(j-1)p+1}, s_{(j-1)p+2},\dots,s_{(j-1)p+p-1}), \quad 1 \le j \le k\,. \tag{8.10}$$

Then

$$\begin{aligned} &v_iv_j = v_jv_i \qquad \textit{for } i,j = 1,\dots,k\,; \\ &t_iv_j = v_jt_i \qquad \textit{whenever } j<i \ \textit{ or } \ j > i+1\,; \\ &t_iv_i = v_{i+1}t_i \ \textit{ and } \ t_iv_{i+1} = v_it_i \qquad \textit{for } 1 \le i \le k-1\,. \end{aligned} \tag{8.11}$$

d) It follows from relations (8.6) and (8.11) that the correspondence

$$\phi_v\colon\ \sigma_i \mapsto \sigma_i * v \overset{\mathrm{def}}{=} v_it_i, \quad 1 \le i \le k-1\,, \tag{8.12}$$

defines a homomorphism ϕ_v: $\mathbf{B}(k) \to \mathbf{B}(kp)$. This homomorphism ϕ_v is non-Abelian whenever $k>2$ and $v \ne 1$.

e) The following relations are held:

$$(*)\ \ \Delta_i^2 = A_i \ \textit{ for } \ 1 \le i \le k\,, \qquad (**)\ \ \Delta_it_i = t_i\Delta_{i+1} \ \textit{ for } \ 1 \le i \le k-1\,, \tag{8.13}$$

$$a^p = \Delta_1^2t_1t_2\cdots t_{k-1}\,. \tag{8.14}$$

Proof. The proofs of (a)–(d) are by geometric evidence or by a straightforward calculation; we leave it to the reader. Relation (8.13($*$)) is actually very well known[5]. Relation (8.13($**$)) is a special case of (8.9) (or the last of relations (8.11)). To prove (8.14), we represent a^p as the product of the p factors $a = s_1 \cdots s_{pk-1}$. Using (8.5), we re-arrange the factors s_j in the product a^p in order to single out the factors $\Delta_1,\ t_1, \ldots, t_{k-1},\ \Delta_{k-1}$; then we use $k-1$ times relation (8.13($**$)). The corresponding calculation looks as follows:

$$\begin{aligned}
a^p &= \underbrace{s_1 \cdots s_{pk-1}}_{\text{factor }(1)} \times \underbrace{s_1 \cdots s_{pk-1}}_{\text{factor }(2)} \times \cdots \times \underbrace{s_1 \cdots s_{pk-1}}_{\text{factor }(p)} \\
&= \underbrace{\underbrace{s_1 \cdots s_{p-1}}_{\text{from factor }(1)} \times \underbrace{s_1 \cdots s_{p-2}}_{\text{from factor }(2)} \times \cdots \times \underbrace{s_1 s_2}_{\text{from factor }(p-2)} \times \underbrace{s_1}_{\text{from factor }(p-1)}}_{\Delta_1} \times \\
&\quad \times \underbrace{\underbrace{s_p \cdots s_{2p-1}}_{\text{from factor }(1)} \times \underbrace{s_{p-1} \cdots s_{2p-2}}_{\text{from factor }(2)} \times \cdots \times \underbrace{s_1 \cdots s_p}_{\text{from factor }(p)}}_{t_1} \times \\
&\quad \times \underbrace{\underbrace{s_{2p} \cdots s_{3p-1}}_{\text{from factor }(1)} \times \underbrace{s_{2p-1} \cdots s_{3p-2}}_{\text{from factor }(2)} \times \cdots \times \underbrace{s_{p+1} \cdots s_{2p}}_{\text{from factor }(p)}}_{t_2} \times \cdots \times \\
&\quad \times \underbrace{\underbrace{s_{p(k-1)} \cdots s_{pk-1}}_{\text{from factor }(1)} \times \underbrace{s_{p(k-1)-1} \cdots s_{pk-2}}_{\text{from factor }(2)} \times \cdots \times \underbrace{s_{p(k-2)+1} \cdots s_{p(k-1)}}_{\text{from factor }(p)}}_{t_{k-1}} \times \\
&\quad \times \underbrace{\underbrace{s_{pk-1}}_{\text{from factor }(2)} \times \underbrace{s_{pk-2}s_{pk-1}}_{\text{from factor }(3)} \times \underbrace{s_{pk-3}s_{pk-2}s_{pk-1}}_{\text{from factor }(4)} \times \cdots \times \underbrace{s_{p(k-1)+1} \cdots s_{pk-1}}_{\text{from factor }(p)}}_{B} .
\end{aligned}$$

It is easily seen that $B = \Delta_k$. Indeed,

$$\begin{aligned}
B &= a_{pk-1,pk}a_{pk-2,pk}a_{pk-3,pk} \cdots a_{p(k-1)+3,pk}a_{p(k-1)+2,pk}a_{p(k-1)+1,pk} \\
&= a_{pk-1,pk}a_{pk-2,pk}a_{pk-3,pk} \cdots a_{p(k-1)+3,pk}a_{p(k-1)+1,pk}a_{p(k-1)+1,pk-1} \\
&= a_{pk-1,pk}a_{pk-2,pk}a_{pk-3,pk} \cdots a_{p(k-1)+1,pk}a_{p(k-1)+1,pk-1}a_{p(k-1)+1,pk-2} \\
&= \cdots = \\
&= a_{pk-1,pk}a_{pk-2,pk} \\
&\qquad \times a_{p(k-1)+1,pk}a_{p(k-1)+1,pk-1}a_{p(k-1)+1,pk-2} \cdots a_{p(k-1)+1,p(k-1)+4} \\
&= a_{pk-1,pk} \times a_{p(k-1)+1,pk}a_{p(k-1)+1,pk-1}a_{p(k-1)+1,pk-2} \\
&\qquad\qquad \times \cdots \times a_{p(k-1)+1,p(k-1)+4}a_{p(k-1)+1,p(k-1)+3} \\
&= a_{p(k-1)+1,pk}a_{p(k-1)+1,pk-1}a_{p(k-1)+1,pk-2} \\
&\qquad \times \cdots \times a_{p(k-1)+1,p(k-1)+4}a_{p(k-1)+1,p(k-1)+3}a_{p(k-1)+1,p(k-1)+2} = \Delta_k .
\end{aligned}$$

Hence $a^p = \Delta_1 \cdot t_1 t_2 \cdots t_{k-1} \cdot B = \Delta_1 \cdot t_1 t_2 \cdots t_{k-1} \cdot \Delta_k = \Delta_1^2 \cdot t_1 t_2 \cdots t_{k-1}$. $\square$

[5] Δ_i is the *Garside fundamental element* of the braid group $B_i \cong \mathbf{B}(p)$, see [Ga] or [De]).

By Theorem 8.1, a non-Abelian special homomorphism φ: $\mathbf{B}(k) \to \mathbf{B}(n)$ may only exist if n is contained in one of the four arithmetic progressions $P^{k,i}$ $(1 \leq i \leq 4)$. The following theorem states that for $n \in P^{k,1} \cup P^{k,2}$ such a homomorphism actually does exist; this proves Theorem H(b).

Theorem 8.3. *Suppose $k \geq 3$. The non-Abelian homomorphism ϕ_v: $\mathbf{B}(k) \to \mathbf{B}(kp)$ constructed in Lemma-Notation* 8.2 *is special whenever one of the following two conditions is fulfilled:*

a) $p = l(k-1)+1$ and $v = (x_1 \cdots x_{k-1}x_1)^l$, where $l \in \mathbb{Z}_+$;

b) $p = l(k-1)$ and $v = (x_1 \cdots x_{k-1})^l$, where $l \in \mathbb{N}$.

In the case when (a) holds, ϕ_v is conjugate to a homomorphism of the form described in Theorem 8.1(a) *with $t = 1$, $p = l(k-1)+1$, and $q = lk+1$. That is,*

$$\phi_v(\alpha) \sim a^{l(k-1)+1} \qquad \textit{and} \qquad \phi_v(\beta) \sim b^{lk+1}\,. \tag{8.15(a)}$$

In the case when (b) holds, ϕ_v is conjugate to a homomorphism of the form described in Theorem 8.1(b) *with $t = l$, $p = l(k-1)$, and $q = lk = p+l$. That is,*

$$\phi_v(\alpha) \sim a^{l(k-1)} \qquad \textit{and} \qquad \phi_v(\beta) \sim a^{lk}\,. \tag{8.15(b)}$$

Proof. By Lemma-Notation 8.2, we already know that ϕ_v is a non-Abelian homomorphism. Hence, to prove that ϕ_v is special, it is sufficient to prove relations 8.15(a) and 8.15(b), respectively. Note that for our choice of the word v we have

$$\phi_v(\sigma_i) = \sigma_i * v = v_i t_i = b_i^l t_i \ \text{ in case } (a), \quad \text{and} \quad \phi_v(\sigma_i) = \sigma_i * v = v_i t_i = a_i^l t_i \ \text{ in case } (b)\,.$$

Respectively, we have

$$\phi_v(\alpha) = b_1^l t_1 \cdots b_{k-1}^l t_{k-1} \quad \text{and} \quad \phi_v(\beta) = b_1^l t_1 \cdots b_{k-1}^l t_{k-1} b_1^l t_1 \qquad \text{in case } (a)\,,$$

and

$$\phi_v(\alpha) = a_1^l t_1 \cdots a_{k-1}^l t_{k-1} \quad \text{and} \quad \phi_v(\beta) = a_1^l t_1 \cdots a_{k-1}^l t_{k-1} a_1^l t_1 \qquad \text{in case } (b)\,.$$

In case (a), using (8.4), (8.11), and (8.14), we have:

$$\begin{cases} \phi_v(\alpha) = b_1^l t_1 \cdots b_{k-1}^l t_{k-1} = b_1^{l(k-1)} t_1 \cdots t_{k-1} \\ \qquad\quad = b_1^{p-1} t_1 \cdots t_{k-1} = a_1^p t_1 \cdots t_{k-1} = \Delta_1^2 t_1 \cdots t_{k-1} = a^p\,, \\ \phi_v(\beta) = \phi_v(\alpha\sigma_1) = a^p b_1^l t_1\,. \end{cases}$$

A straightforward (but rather long!) calculation shows that for $p = l(k-1)+1$ the element $a^p b_1^l t_1$ of $\mathbf{B}(pk)$ is conjugate to $b^{lk+1} = b^q$. This completes the proof in case (a).

In case (b)

$$\begin{cases} \phi_v(\alpha) = a_1^l t_1 \cdots a_{k-1}^l t_{k-1} = a_1^{l(k-1)} t_1 \cdots t_{k-1} = a_1^p t_1 \cdots t_{k-1} = \Delta_1^2 t_1 \cdots t_{k-1} = a^p\,, \\ \phi_v(\beta) = \phi_v(\alpha\sigma_1) = a^p a_1^l t_1\,. \end{cases}$$

It is not difficult to show that for $p = l(k-1)$ the element $a^p a_1^l t_1$ of $\mathbf{B}(pk)$ is conjugate to $a^{lk} = a^q$. This completes the proof. $\square$

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DEPARTMENT OF MATHEMATICS, TECHNION, HAIFA 32000, ISRAEL
E-mail address: vlin@techunix.technion.ac.il

PRESENT ADDRESS: MAX PLANCK INSTITUTE FOR MATHEMATICS, GOTTFRIED-CLAREN-STRASSE 26, 53225 BONN, GERMANY
E-mail address: vlin@mpim-bonn.mpg.de