Hypergeometric function F_1 and automorphic functions

III. Case with some integer parameters

by

Toshiaki Terada

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

West Germany

Shiga University of
Medical Science
Department of Mathematics
Seta Tsukinowa-cho, Otsu
Shiga, 520-21

Japan

MPI/87-50

Hypergeometric function F_1 and automorphic functions

III. Case with some integer parameters

by

Toshiaki Terada

Introduction

We consider the hypergeometric system of partial differential equations

$$(F_1) \ : \ D_{ij}F = 0 \quad (i \leqq i,j \leqq n) \ ,$$

$$D_{ii} := x_i(x_i - 1)\partial_i^2 + [x_i - (x_i - 1)\sum_{\alpha=1,\alpha\neq i}^{n} (1 - \lambda_\alpha)/(x_i - x_\alpha) + \lambda_0 + \lambda_i - 2$$

$$+ (4-\lambda_0-2\lambda_2-\lambda_{n+1})]\partial_i + (\lambda_i - 1)\sum_{\alpha=1,\alpha\neq i}^{n} [x_\alpha(x_\alpha-1)/(x_i-x_\alpha)]\partial_\alpha + \lambda_\infty(1-\lambda_i) \ ,$$

$$D_{ij} := (x_i-x_j)\partial_i\partial_j + (\lambda_j - 1)\partial_i - (\lambda_i - 1)\partial_j \quad (i \neq j)$$

of n variables $x_1, x_2, \ldots, x_n$ where $\partial_i = \partial/\partial x_i$ and λ_i $(i = 0,1,\ldots n+1,\infty)$ are complex parameters satisfying $\sum_{\alpha=0}^{\infty} \lambda_\alpha = n+1$. Gauss' hypergeometric series $F(\alpha,\beta,\gamma,x)$ $(n = 1)$, Appell's $F_1(\alpha,\beta_1,\beta_2,\gamma,x_1,x_2)$ $(n = 2)$ or Lauricella's $F_D(\alpha,\beta_1,\ldots,\beta_n,x_1,\ldots,x_n)$ $(n \geq 3)$ is one of its solutions, where $\alpha = \lambda_\infty$, $\beta_i = 1 - \lambda_i$ and $\gamma = \lambda_\infty + \lambda_{n+1}$. (F_1) is completely integrable and has $n+1$ linearly independent solutions locally holomorphic on the domain

$$D := \{x \in \mathbb{C}^n | x_i \neq 0,1,x_j (j \neq i)\} .$$

If none of λ_i are integers, (F_1) has an integral representation of Euler-Picard type:

$$\omega_i = \int_0^{x_i} u^{\lambda_0 - 1} (u-x_1)^{\lambda_1 - 1} \ldots (u-x_n)^{\lambda_n - 1} (u-1)^{\lambda_{n+1} - 1} du \quad (1 \leq i \leq n+1)$$

from a base of solutions.

The Wronskian determinant vanishing never on D , a base of solutions of (F_1) determines a locally biholomorphic mapping ω to the n-dimensional projective space $W \cong P_n(\mathbb{C})$.

<u>Definition</u>. Given a mapping ω from D to W as above, we will say that the inverse ω^{-1} is uniformizable if there exist a domain $B \subset W$ (or $B \subset$ (a modification of W)), a compactification Y of D , an analytic subset $S_0 \subset Y$ and a covering manifold Z over $Y_0 := Y - S_0$ which ramifies only on $Y_0 - D$ such that ω^{-1} can be extended to a biholomorphic mapping from B to Z .

If ω^{-1} is uniformizable, then it defines a field a automorphic functions on the domain B ; the group is induced by the monodromy group of (F_1) and the fundamental domain is biholomorphic to Y_0 .

<u>Definition</u>. We will say that the parameters λ_i satisfy Picard-Schwarz condition for all $I = \{i_0, i_1, \ldots, i_p\}$ $(1 \leq p \leq n,$

$i_\alpha = 0,1,\dots,n+1$, $i_\alpha \neq i_\beta$ $(\alpha \neq \beta)$) we have

$$\lambda_I := \lambda_{i_0} + \lambda_{i_1} + \dots + \lambda_{i_p} - p \in \mathbb{Z}^{-1} := \{0\} \cup \{1/m \mid m \in \mathbb{Z}\} .$$

In [5], the author obtained the

Theorem. Given a system (F_1), if λ_i satisfy Picard-Schwarz condition and $0 < \lambda_i < 1$ $(0 \leq i \leq \infty)$, then ω^{-1} is uniformizable.

Historically, Schwarz [3] proved it without the condition $0 < \lambda_i < 1$ but with some additional condition. Picard tried to prove and Le Vavasseur [2] found all sets of λ_i which satisfy Picard-Schwarz condition. Deligne-Mostow [1] also proved it using tools of algebraic geometry.

Now the purpose of this paper is to generalize this theorem for non-general cases (we will call general case if $0 < \lambda_i < 1$ are satisfied) in order to complete the work. Deligne-Mostow [1] has already discussed about two cases.

In § 1, we collect some basic notations, definitions and results already obtained. § 2 is devoted to some local properties of a base of solutions on singular loci. The proof and the explications of the main theorem are found in § 3.

This work was done during the author's stay at the Max-Planck-Institut für Mathematik in Bonn. He expresses his gratitude to the institute for the hospitality and comfortable condition for work.

§ 1. Preliminaries

1.1. Notations and definitions

$X := \mathbb{P}^n(x_1, x_2, \ldots, x_{n+1})$. But except for defining S_I below, we put always $x_{n+1} = 1$ and consider $x_i (1 \leq i \leq n)$ as inhomogeneous coordinates; moreover put $x_0 = 0$ and $x_\infty = \infty$.

$I = \{i_0, i_1, \ldots, i_p\}$ given,

$\lambda_I := \lambda_{i_0} + \lambda_{i_1} + \ldots + \lambda_{i_p}$, $\# I := p+1$, $\mu_I := \exp(2\pi\sqrt{-1}\lambda_I)$,

$S_I := \{x \in X \mid x_{i_0} = \ldots = x_{i_p}\}$

$S_I^0 := \{x \in S_I \mid x_{i_0} \neq x_j \text{ if } j \notin I\}$

$\hat{X}$: the compactification of D that is defined by the sequence

$$\hat{X} := X_1 \xrightarrow{\sigma_2} X_2 \xrightarrow{\sigma_3} \ldots \xrightarrow{\sigma_n} X_n = X ,$$

where X_{i-1} is obtained from X_i through Hopf's σ-process along every $S_{i,I}$ (:= the closure of $(\sigma_n \circ \sigma_{n-1} \circ \ldots \circ \sigma_{i+1})^{-1}(S_I^0)$) such that $\# I = i+1$.

$$\hat{S}_I := S_{1,I} , \quad \hat{S}_I^0 := \hat{S}_I - \bigcup_{J \neq I} \hat{S}_J .$$

1.2. Fundamental group of D

On the Riemann sphere U of the variable u, take $n+3$ distinct points $u_0, u_1, \ldots, u_{n+1}, u_\infty$. Two sets $(u_0, \ldots, u_\infty)$ and $(u'_0, \ldots, u'_\infty)$ will be called equivalent if $(u_i, u_{n+1}; u_0, u_\infty) = (u'_i, u'_{n+1}; u'_0, u'_\infty)$ hold for all i $(1 \leq i \leq n)$ where

$$(u_i, u_{n+1}; u_0, u_\infty) = \frac{u_i - u_0}{u_{n+1} - u_0} \Big/ \frac{u_i - u_\infty}{u_{n+1} - u_\infty}$$

is the anharmonic ratio. Put

$$x_i = (u_i, u_{n+1}; u_0, u_\infty) \quad (1 \leq i \leq n) .$$

Then a point of D and an equivalence class of such points is of one-to-one correspondence.

Again take $n+3$ points on the real axis of U such that

$$a_0 < a_1 < \ldots < a_n < a_\infty$$

and let

$$C_{ij} : u = u_{ij}(t) \quad (0 \leq t \leq 1) \quad (0 \leq i,j \leq \infty, i \neq j)$$

be a loop around $u = a_j$ with reference point $u = a_i$ which passes only the upper half plane and a small neighborhood of a_j, l_{ij} be the curve on D defined by

$$x_\alpha = (u_{\alpha j}(t), u_{\alpha n+1}(t); u_{\alpha 0}(t), u_{\alpha\infty}(t)) \quad (u_{\alpha j}(t) \equiv a_\alpha (\alpha \neq i)(1 \leq \alpha \leq n)$$

and A_{ij} be the homotopy class of l_{ij}. Then $A_{ij} = A_{ji}$ hold and $A_{ij}(0 \leq i \leq j \leq n+1, (i,j) \neq (0,n+1))$ generate the fundamental group of D. Put

$$A_{i_p};{}_{i_0 i_1 \ldots i_{p-1}} := A_{i_p i_0} A_{i_p i_1} \ldots A_{i_p i_{p-1}}$$

and $A_I := A_{i_0 i_1 \ldots i_p} := A_{i_1; i_0} A_{i_2; i_0 i_1} \ldots A_{i_p; i_0 i_1 \ldots i_{p-1}}$.

1.3. Base of solutions and monodromy

Given a point $x \in D$, take $u_i (0 \leq i \leq \infty)$ such that

$$x_i = (u_i, u_{n+1}; u_0, u_\infty) \quad (1 \leq i \leq n),$$

and put

$$w_{ij}(x) = \left[\left(\frac{u_{n+1}-u_0}{u_\infty - u_{n+1}} \frac{1}{u_\infty - u_0}\right)^{\lambda_\infty} \prod_{\alpha=0}^{n+1} (u_\infty - u_\alpha)^{1-\lambda_\alpha}\right] \int \prod_{\alpha=0}^{\infty} (u-u_\alpha)^{\lambda_\alpha - 1} du$$

where the path is a double loop with respect to u_i and u_j. It does not depend on the choice of u_i. Using this expression, we can calculate explicitely monodromy matrices for a base

$$\omega_i(x) := \frac{1}{(1-\mu_0)(1-\mu_i)} w_{0i}(x) \quad \text{(if none of } \lambda_\alpha \text{ are equal to 1).}$$

For example,

$$B_{01\ldots p} = \begin{pmatrix} \mu_I & \ddots & & & 0 & \\ 0 & & \mu_I & & & \\ -\tilde{\mu}_1 & -\tilde{\mu}_2 & \cdots & -\tilde{\mu}_p & 1 & \\ \cdots & \cdots & \cdots & \cdots & & \ddots \\ & & & & & 1 \\ -\tilde{\mu}_1 & -\tilde{\mu}_2 & \cdots & -\tilde{\mu}_p & 0 & \end{pmatrix} \quad (\tilde{\mu}_i = \mu_{01\ldots i-1}(1-\mu_i))$$

that is the matrice corresponding to $A_I = A_{01\ldots p}$, which represents a loop around $\hat{S}_I$.

<u>Lemma 1</u> ([6] Corollary to Theorem 5). Even if some λ_i are integers if neither λ_l nor $1-\lambda_k$ is integer, then

$$w_{ki}/(1-\mu_k)(1-\mu_i)\Gamma(\lambda_i) \quad (0 \leq i \leq \infty,\ i \neq k,l)$$

forms a base of solutions. If $k = 0$ and $l = \infty$, the monodromy matrix for A_I $(I = \{0,1,\ldots,p\})$ is given by

$$\Gamma^{-1}B_I\Gamma \quad (\Gamma = \mathrm{diag}(\Gamma(\lambda_1),\Gamma(\lambda_2),\ldots,\Gamma(\lambda_{n+1})) \ .$$

<u>Lemma 2</u>. If all λ_i are real, then there exists a Hermitian matrix

$$A = (a_{ij}) = M^* \begin{pmatrix} a_1 & a & \cdots & a \\ & a_2 & a & \cdots\ a \\ & & \ddots & \cdots \\ \bar{a} & \cdots & \bar{a} & a_{n+1} \end{pmatrix} M$$

where $M = \mathrm{diag}(\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_{n+1})$, $a = \exp(\pi\sqrt{-1}\lambda_\infty)$ and $a_i = (a - \bar{a}\mu_i)/(1-\mu_i)$, such that

$$\sum a_{ij}\omega_i\bar{\omega}_j$$

is invariant (i.e. single-valued on D).

§ 2. Local state of a base of solutions of (F_1) at a singular point

<u>Definition</u>. $I = \{i_0, i_1, \ldots, i_p\}$ $(0 \leq i_\alpha \leq n+1)$ given, it will be called of exponential type with respect to the system (F_1) if, at least, one of the following conditions is satisfied:

(1) for all $i \in I$, λ_i are positive integers,

(2) for all $i \in I$, $1-\lambda_i$ are positive integers,

(3) for all $j \notin I$ $(0 \leq j \leq \infty)$, λ_j are positive integers,

(4) for all $j \notin I$ $(0 \leq j \leq \infty)$, $1-\lambda_j$ are positive integers,

(5) λ_I is not an integer.

Otherwise it will be called of logarithmic type with respect to (F_1).

<u>Theorem 1</u>. Let ξ be a point of $\hat{S}_I^0$ $(I = \{i_0, i_1, \ldots, i_p\})$ and x_I be a part of local coordinates at ξ such that $\{x_I = 0\} = \hat{S}_I^0 \cap \{\text{a neighborhood of } \xi\}$. Then there exists a base of solutions on a small neighborhood V of ξ that consists of functions given below, where f_i are holomorphic and single-valued on V.

(I) If I is of exponential type with respect to (F_1), then

$$x_I^{\lambda_I} f_1, \ldots, x_I^{\lambda_I} f_p, f_{p+1}, \ldots, f_{n+1} .$$

(II) If I is of logarithmic type with respect to (F_1), then

(a) $x_I^{\lambda_I} f_1, \dots, x_I^{\lambda_I} f_p,\ x_I^{\lambda_I} f_p \log x_I + f_{p+1}, f_{p+2}, \dots, f_{n+1}$ $(\lambda_I \geq 0)$;

(b) $x_I^{\lambda_I} f_1, \dots, x_I^{\lambda_I} f_{p-1}, f_p, f_p \log x_I + x_I^{\lambda_I} f_{p+1}, f_{p+2}, \dots, f_{n+1}$ $(\lambda_I < 0)$.

However, if $I \supset \{0,n+1\}$, then it is necessary to multiply $x_I^{\lambda_\infty}$ to all terms.

Demonstration. This theorem is already anounced in [6] without proof but, if none of λ_i are integers, then it is proved in [5].

Let $\begin{pmatrix} 0 & 1 & \dots & n+1 & \infty \\ i_0 & i_1 & \dots & i_{n+1} & i_\infty \end{pmatrix}$ be a permutation and repeat the same discussion in § 2, replacing every α by i_α : put $x'_{i_\alpha} = (u'_{i_\alpha}, u'_{i_{n+1}}; u'_{i_0}, u'_{i_\infty})$, take a'_{i_α} such that $a'_{i_0} < a'_{i_1} < \dots$ etc. Then we can reduce the problem to $I = \{0,1,\dots,p\}$. For example the monodromy matrix $B'_{i_\alpha i_\beta}$ is obtained from $B_{\alpha\beta}$ by replacing λ_j by λ_{i_j} $(0 \leq j \leq \infty)$ and multiplying a constant factor $b_{\alpha\beta}$, which arises from the factor

$$\left\{ \frac{u_{n+1}-u_0}{u_\infty - u_{n+1}} \; \frac{1}{u_\infty - u_0} \right\}^{\lambda_\infty} \prod_{\alpha=0}^{n+1} (u_\infty - u_\infty)^{1-\lambda_\alpha}$$ in the integral representation. If $0 \leq \alpha, \beta \leq p < n+1$, then $b_{\alpha\beta} = 1 ((i_\alpha, i_\beta) \neq (0,n+1))$ and $-\mu_\infty ((i_\alpha, i_\beta) = (0,n+1))$; it is not so difficult to know $b_{\alpha\beta}$ for general cases, but we will not do it, because it is tedeous and not necessary at present.

Now we suppose $I = \{0,1,\dots,p\}$ without restricting the generality. If neither $1-\lambda_0$ nor λ_∞ is positive integer, by Lemma 1,

$$w_i(x) = w_{01}(x)/(1-\mu_0)(1-\mu_i)\Gamma(\lambda_i) \quad (1 \leq i \leq n+1)$$

form a base. As A_I represents a loop around $\hat{S}_I$, we have only to examine the matrix B_I. By explicite calculation, we see that the Jordan canonical form of $\Gamma B_I \Gamma^{-1}$ is $\operatorname{diag}(\mu_I, \ldots, \overset{p}{\dot{\mu}_I}, 1, \ldots, 1)$ or respectively the direct sum of $\operatorname{diag}(1, \ldots, 1)$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ according that I is of exponential or respectively logarithmic type with respect to (F_1). Consequently our theorem is true for some f_i $(1 \leq i \leq n+1)$ which are single-valued and holomorphic on $V-\hat{S}_I$ and so meromorphic according to the expression by integral. Choose $\xi_1, \ldots, \xi_{p-1}, x_p, x_{p+1}, \ldots, x_n$ as local coordinates at ξ, where $x_1/\xi_1 = \ldots = x_{p-1}/\xi_{p-1} = x_p$. Replacing u by $x_p v$ in the integral given in the introduction, we see $w_i/x_p^{\lambda_I}$ $(i = 1, \ldots, p)$ are holomorphic at ξ.

For some l $(p < l \leq \infty)$ and all i $(p < i \leq l,\ i \neq l)$, define

$$w'_{il} := w_i - \frac{\Gamma(\lambda_l)}{\Gamma(\lambda_i)} w_l = \frac{1}{\Gamma(\lambda_i)} (\omega_i - \omega_l)$$

as following way: choose a l such that $1 - \lambda_l$ is not a positive integer if it exists; if all $\lambda_i = m_i$ $(p < i \leq \infty)$ are non-positive integers, then choose an arbitrary l, put $\lambda_i = m_i + t$ and take the limit, t tending to zero. All w_{il} are holomorphic at ξ . Since

$$\sum_{i=1}^{\infty} \tilde{\mu}_i \omega_i = 0$$

holds ([5], p. 456), we have

$$(2.1) \qquad \sum_{i=1}^{\infty} \tilde{\mu}_i \Gamma(\lambda_i) \omega_i = 0$$

and

$$(2.2) \qquad \sum_{i=1}^{p} \tilde{\mu}_i \Gamma(\lambda_i) \omega_i + \sum_{j=p+1, j \neq l}^{\infty} \tilde{\mu}_j \Gamma(\lambda_j) \omega_{jl} + (1-\mu_I) \Gamma(\lambda_l) \omega_l = 0 .$$

Therefore, if $\mu_I \neq 1$, we see, by taking a limit if necessary, that

$$w_i (1 \leq i \leq p),\ w_{jl} (p+1 \leq j \leq \infty,\ j \neq l)$$

are linearly independent, which completes the proof for the case $\mu_I \neq 1$, for, by permutation, we can suppose neither $1-\lambda_0$ nor λ_∞ is not a positive integer.

If $\mu_I = 1$, then

$$f_p := \sum_{i=1}^{p} \tilde{\mu}_i \Gamma(\lambda_i) w_i$$

is holomorphic by (2.2) and the matrice B_I shows that ω_j $(p+1 \leq j \leq \infty)$ goes to $\omega_j + f_p$ by the analytic continuation along a loop around $\hat{S}_I$.

Now it remains the case that all $1-\lambda_i$ $(0 \leq 1 \leq p)$ or all $\lambda_j (p < j \leq \infty)$ are positive integers. We can assume, by permutation, λ_p nor $1-\lambda_{p+1}$ is not a positive integer. Then, putting $q = n-p+1$,

$$w_i' = \begin{cases} w_{p+1\ i+p+1} / \left[(1-\mu_{p+1})(1-\mu_{i+p+1})\Gamma(\lambda_{i+p+1})\right] & (1 \leqq i \leqq q) \\ w_{p+1\ i-q-1} / \left[(1-\mu_{p+1})(1-\mu_{i-q-1})\Gamma(\lambda_{i-q-1})\right] & (q < i \leqq n+1) \end{cases}$$

form a base of solutions. The monodromy matrix for this base is obtained from $B_{q+1,\ldots,n+1\infty}$ by replacing λ_i with λ_{p+1-i} (λ_∞ with λ_p), because the real axis on the Riemann sphere is a circle. So the problem reduces to calculate B_J $J = \{q+1,\ldots,n+1\infty\}$. However it is easy to see $A_{01\ldots q}$ and A_J represents a same curve; so $B_{01\ldots q} = B_{q+1\ldots\infty}$ which completes the proof.

§ 3. Main theorem

3.1. Necessary condition

Let ω be a mapping (multivalued) defined by a base of solutions of (F_1), which we consider without the condition $0 < \lambda_i < 1$.

Proposition. In order that the inverse ω^{-1} may be uniformizable, not only Picard-Schwarz condition, but also the supplementary condition:

If $\lambda_I = \pm 1$ for some I, then I is of exponential type. In fact this is a consequence of Theorem 1, Lemma 9 of [5] and the explicit form of the Wronskian ([6], Theorem 4)

3.2. Solutions of Picard-Schwarz condition

In [2], Le Vavasseur obtained, for $n=2$, all the solutions of Picard-Schwarz condition; there exist, other than 27 cases already treated, only 10 solutions (one of them contains an integer parameter) up to permutations among λ_i $(0 \leq i \leq \infty)$:

	λ_0	λ_1	λ_2	λ_3	λ_∞
(1)	3/4	3/4	3/4	3/4	0
(2)	1/2	1/2	1/2	1/2	1
(3)	1/3	1	1	1/3	1/3
(4)	1/2	1	1	1/3	1/6
(5)	1/2	1	1	1/4	1/4
(6)	1/2	1	1	1/2	0

(7)	1/m	1	1	-1/m	1	($m \in \mathbb{Z}$ or $m = \infty$)
(8)	3/2	1/2	1/2	1/2	0	
(9)	5/3	1/3	1/3	1/3	1/3	
(10)	7/6	5/6	1/3	1/3	1/3	

For $n \geq 3$, there exist some but none which are essentially new and satisfy the supplementary condition, except the case $0 < \lambda_i < 1$. For, if $\lambda_0 + \lambda_1 - 1 = 1$, for example, $\lambda_0 = \lambda_1 = 1$ must hold and $\lambda_i + \lambda_j - 1 = 0$ never occurs except that all $\lambda_k = 1$ $(k \neq i,j)$. Therefore every solution is obtained by adding some 1 to one of the cases (2) ~ (7).

Theorem 2. For (1) ~ (7), the inverse ω^{-1} is uniformizable and there exist none which are essentially new if $n \geq 3$. For (1), the domain B is biholomorphic to $\mathbb{C}^2$, the variety Y is biholomorphic to the projective space X, the subanalytic set S_0 is empty and the fundamental domain which is biholomorphic to $Y_0 = Y - S_0$ is compact. For (2), $B \cong (\text{disk}) \times \mathbb{C}^1$, $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $Y_0 \cong \mathbb{C}^1 \times \{\mathbb{C}^1 - \{0,1\}\}$.

Demonstrations. We attribute the proof to Lemma 13 in [5], so we have only to find a complete invariant metric on B and the variety Y_0, and to show that ω can be extended to a locally biholomorphic mapping to B from some variety Z over Y_0.

As to (1), in order to simplify the situation, put $\lambda_0 = \lambda_1 = \lambda_2 = \alpha_\infty = 3/4$ and $\lambda_3 = 0$. Then,

$$w_1,\ w_2 \ \text{ and } \ w_3 = \text{const.}(1-x_1)^{-1/4}(1-x_2)^{-1/4}$$

form a base. $|w_3|$ is evidently single-valued on D . Given a solution ω of (F_1) , let w_i^0 the projection of ω to the space generated by $w_i (i = 1,2)$. Then the form

$$(\bar{w}_1^0, \bar{w}_2^0) A^0 \, {}^t(w_1^0, w_2^0)$$

is invariant with respect to monodromy where A^0 is obtained from the invariant Hermitian matrix A by eliminating the third row and column. A^0 being positive definite, there exist linear combinations g_1 and g_2 of ω_1 and ω_2 such that

$$|d(g_1/w_3)|^2 + |d(g_2/w_3)|^2$$

defines a complete invariant metric on $B = \mathbb{C}^2$; we see (range of ω) $\subset B$ by Theorem 1. The last condition is assured by Theorem 1 and Lemma 10 in [5] (Case (a), $e_{j_0} = e_{j_1} = e_{j_2} = 1/2$ for $Y = Y_0 = X$, there existing no I such that $\lambda_I = 0$.

As to case (2), put $\lambda_1 = 1$ and $\lambda_i = 1/2 \ (i \neq 1)$, so $\omega_i \ (i = 1,2,3)$ form a base. ω_2 and ω_3 depend only on x_2 and the space generated by ω_2 and ω_3 is invariant under the monodromy group. And the form

$$(\omega, \omega) := (\bar{\omega}_2, \bar{\omega}_3) A_0 \, {}^t(\omega_2, \omega_3)$$

is invariant, where A_0 is obtained from A by eliminating the first row and column and is of signature (1,1) . Therefore,

$$[(\omega,\omega)(d\omega,d\omega) - |(\omega,d\omega)|^2/|(\omega,\omega)|^2 + L ,$$

where L is the image of $\left|\frac{\partial}{\partial x_1}\left(\frac{\omega_1}{\omega_3}\right)dx_1\right|^2$ by ω , defines a complete invariant metric on $B = (\text{disk}) \times \mathbb{C}^1$, which is seen by the explicit form of monodromy matrices. The further process of the proof is quite similar.

For the cases (3) ~ (7), we can prove by similar way to above. But these are reduced to one variable case. Among the equations of the system (F_1) , $D_{12}F = 0$ comes to naught and $D_{ii}F = 0$ $(i = 1,2)$ is Euler's hypergeometric differential equation of the variable x_i . So the problem is reduced to Schwarz' work; all domains B and Y_0 and groups are direct product.

Similarly, if $n \geq 3$, the problem reduces to one or two variable cases.

<u>Remark</u>. If $n = 1$, then we can always assume $0 \leqq \lambda_i \leq 1$ $(0 \leq i \leq 2)$, using a permutation and the transformation: $\lambda_i \longmapsto 1-\lambda_i$ $(0 \leq i \leq \infty)$. In this situation, Schwarz added the condition: $\lambda_\infty \in \mathbb{Z}^{-1}$ other than Picard-Schwarz condition. From our point of view, this is equivalent to the supplementary condition.

References

[1] Deligne, P. and Mostow, G.D., Monodromy of hypergeometric functions and non-lattice integral monodromy, Publ. Math. IHES, 63 (1986), 1-89.

[2] Le Vavasseur, R., Sur le système d'équations aux dérivées partielles simultanées auxquelles satisfait la serie hypergéométrique à deux variables, Thèses, Fac. des Sci. Univ. de Paris, 1893.

[3] Schwarz, H.A., Über diejenigen Fälle, in welchen die Gaussische hypergeometrische Reihe eine algebraische Funktion ihres vierten Elementes darstellt, J. Reine Angew. Math. (J. de Crelle), 54 (1973), 292-335.

[4] Terada, T., Problème de Riemann et fonctions automorphes provenant des fonctions hypergéométrique de plusieurs variables, J. Math. Kyoto Univ., 13 (1973), 557-578.

[5] ———— , Fonctions hypergéométriques F_1 et fonctions automorphes I . J. Math. Soc. Japan, 35 (1983), 452-475.

[6] ———— , On hypergeometric function F_1 , to appear.

[7] Barthel, G., Hirzebruch, F. and Höfer, T., Geradenkonfigurationen und algebraische Flächen, in preparation.