THE ALGEBRA OF DISCRETE TORSION

RALPH M. KAUFMANN*
OKLAHOMA STATE UNIVERSITY, STILLWATER, OK
AND MAX–PLANCK INSTITUT FÜR MATHEMATIK, BONN, GERMANY

Abstract. We analyze the algebraic structures of G–Frobenius algebras which are the algebras associated to global group quotient objects. Here G is any finite group. These algebras turn out to be modules over the Drinfeld double of the group ring $k[G]$. We furthermore prove that discrete torsion is a universal group action of $H^2(G, k^*)$ on G–Frobenius algebras by isomorphisms. These morphisms are realized explicitly by taking the tensor product with twisted group rings. This gives an algebraic realization of discrete torsion and allows for a treatment analogous to the theory of projective representations of groups, group extensions and twisted group ring modules. Lastly, we identify another set of discrete universal transformations among G–Frobenius algebras pertaining to their super–structure and classified by $\mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$.

Introduction

G–Frobenius algebras were introduced by the author in [K1] to explain the algebraic structure replacing Frobenius algebras when one is dealing with global group quotients in theories such as Cohomology, K–theory, local rings of singularities etc.. The main feature is that they are group graded non–commutative algebras with a group action and a controlled non–commutativity. Previously, one had only regarded commutative algebra structures on the invariants under the group action of the G–module, which leads to a direct sum decomposition indexed by conjugacy classes. The larger non–commutative algebras however reflect the geometry and general properties much better. For instance in the present treatise it is the full G–Frobenius structure which allows us to decode discrete torsion in all its aspects. Our G–Frobenius algebras have been proven to be exactly the right structure to describe the cohomology of global quotients [FG] and the geometry of symmetric products [K2,LS] whose structure is closely related to Hilbert schemes [LS]. Furthermore they are suited to describe mirror phenomena for

Date: September 2, 2002.
* Partially supported by NSF grant #0070681.

Frobenius structures having their origin in singularity theory [K1,K3]. One fundamentally new feature of these algebras is that given any one of them one expects the existence of several cousins of it, which are all related to each other by discrete torsion. There are several definitions of discrete torsion [V,VW,D,A,R,S,K1,K2]. The purpose of this note is to show that all definitions can be understood in terms of the fact that there is a universal group action of $H^2(G, k^*)$ on G–Frobenius algebras, by algebra isomorphisms induced by taking tensor products with twisted group rings.

The algebraic structures of a G–Frobenius algebra are as follows: it is naturally a left $k[G]$–module algebra as well as a right $k[G]$–co–module algebra. Moreover it satisfies the Yetter–Drinfeld (YD) condition for bi–modules and is thus a module over $D(k[G])$, the Drinfeld double of $k[G]$.

We show that given a G–Frobenius algebra there is an action of $H^2(G, k^*)$ on it by tensoring with the respective twisted group ring. This tensor product exhibits all desiderata of discrete torsion. In fact this action is an action of universal projective re-scalings. Moreover the action transforms the partition sums in the right way and defines a bi–character (usually denoted by $\epsilon(g, h)$) on commuting elements with the expected properties. It not only satisfies the necessary algebraic relations, but it also appears naturally as factors in front of the summands of the partition function.

The reason for the beautiful picture is the inclusion of twisted sectors for all group elements and not just for conjugacy classes, which has been discussed in detail in [K1,K2]. This also allows to understand the bi–character in terms of a 2 cocycle $\alpha \in H^2(G, k^*)$. The bi–character ϵ is then derived from α and is defined for all elements. It can actually be shown to be a one cocycle in $\epsilon \in H^1(G, k^*[G])$ where $k^*[G] \subset k[G]$ are sums with invertible coefficients and the G–module structure is given by the adjoint action.

We furthermore comment on the generic super–structures one can impose on a given G–Frobenius algebra and realize that these are also given by tensor product with superized versions of $k[G]$. These are a second type of discrete deformation, which is actually different from the one of discrete torsion.

This action of discrete torsion using twisted group rings should be viewed as a statement which is analogous to the relations between projective representations of a group, modules over the twisted group algebra and extensions of the group.

In fact we are able to establish the counterpart to the classical picture in the world of G–Frobenius algebras.

The paper is organized as follows:

In the first paragraph, we recall the notion of G–Frobenius algebras, and show that group rings and twisted group rings are G–Frobenius algebras.

In the second paragraph, we give the algebraic properties of G–Frobenius algebras. The main Theorem is that a G–Frobenius algebra has the natural algebraic structures stated above. We also characterize G–Frobenius algebras which are Galois over their identity sector as $k[G]$–comodule algebras.

The third section contains the identification of discrete torsion as an action of $H^2(G, k^*)$ on G–Frobenius algebras. This is done by analyzing universal twists of G–Frobenius algebras and showing that the universal twists are in 1–1 correspondence with $H^2(G, k^*)$. Furthermore we show that these twists can be realized by tensoring with the respective twisted group ring.

We furthermore also study the generic super–structures ($\mathbb{Z}/2\mathbb{Z}$) which one can impose on a given G–Frobenius algebra.

In the fourth paragraph we develop our understanding of these facts by introducing a theory in analogy with projective representations of a group and their relation to modules over the twisted group algebra and extensions of the group. Here the final result is that given any Abelian group H and a cocycle $[\alpha'] \in H^2(G, H)$ then for any central extension G^α of G by H with class $[\alpha']$ and any G–Frobenius algebra A there is a natural $G^{\alpha'}$–Frobenius algebra $A^{\alpha'}$ to which the Frobenius algebra A_α (A twisted by α) can be lifted. Here $[\alpha] \in H^2(G, k^*)$ is the image under the transgression map associated to $[\alpha']$ of a $\chi \in \mathrm{Hom}(H, k^*)$. Vice versa, the above χ gives a push down map, which maps A^α onto A_α. Furthermore there is a universal setup of this kind if there is a representation group for G.

Acknowledgements

I would like to thank the Max–Planck–Institut for its kind hospitality and also gratefully acknowledge the support from the NSF. Special thanks goes to Takashi Kimura, since it was a discussion with him that was the initial spark for this paper. I would also like to thank Susan Montgomery whose questions about the algebraic setting of orbifolding are the reason for the existence of the second section, which also owes a great deal to her book on "Hopf algebras and their actions on rings".

1. G–Frobenius algebras

We fix a finite group G and denote its unit element by e. We furthermore fix a ground field k of characteristic zero for simplicity. With the usual precautions the characteristic of the field does not play an important role and furthermore the group really only needs to be completely disconnected.

1.1. **Definition.** A *G–twisted Frobenius algebra* —or G–Frobenius algebra for short— over a field k of characteristic 0 is $< G, A, \circ, 1, \eta, \varphi, \chi >$, where

- G finite group
- A finite dim G-graded k–vector space
  $A = \oplus_{g\in G} A_g$
  A_e is called the untwisted sector and
  the A_g for $g \neq e$ are called the twisted sectors.
- $\circ$ a multiplication on A which respects the grading:
  $\circ : A_g \otimes A_h \to A_{gh}$
- 1 a fixed element in A_e–the unit
- η non-degenerate bilinear form
  which respects grading i.e. $g|_{A_g\otimes A_h} = 0$ unless $gh = e$.
- φ an action of G on A (which will be by algebra automorphisms),
  $\varphi \in \mathrm{Hom}(G, \mathrm{Aut}(A))$, s.t. $\varphi_g(A_h) \subset A_{ghg^{-1}}$
- χ a character $\chi \in \mathrm{Hom}(G, k^*)$

Satisfying the following axioms:
Notation: We use a subscript on an element of A to signify that it has homogeneous group degree –e.g. a_g means $a_g \in A_g$– and we write $\varphi_g := \varphi(g)$ and $\chi_g := \chi(g)$.

a) *Associativity*
$(a_g \circ a_h) \circ a_k = a_g \circ (a_h \circ a_k)$

b) *Twisted commutativity*
$a_g \circ a_h = \varphi_g(a_h) \circ a_g$

c) *G Invariant Unit*:
$1 \circ a_g = a_g \circ 1 = a_g$
and
$\varphi_g(1) = 1$

d) *Invariance of the metric*:
$\eta(a_g, a_h \circ a_k) = \eta(a_g \circ a_h, a_k)$

i) *Projective self–invariance of the twisted sectors*
$\varphi_g|A_g = \chi_g^{-1} id$

ii) *G–Invariance of the multiplication*
$\varphi_k(a_g \circ a_h) = \varphi_k(a_g) \circ \varphi_k(a_h)$

iii) *Projective G–invariance of the metric*
$\varphi_g^*(\eta) = \chi_g^{-2}\eta$
iv) *Projective trace axiom*
$\forall c \in A_{[g,h]}$ and l_c left multiplication by c:
$\chi_h \mathrm{Tr}(l_c\varphi_h|_{A_g}) = \chi_{g^{-1}}\mathrm{Tr}(\varphi_{g^{-1}} l_c|_{A_h})$

1.1.1. **Special G–Frobenius algebras.** We briefly review special G–Frobenius algebras. For details see [K1,K2].

We call a G-Frobenius algebra special if all A_g are cyclic A_e modules via the multiplication $A_e \otimes A_g \to A_g$. Fixing a cyclic generator $1_g \in A_g$ the algebra is completely characterized by two compatible cocycles, namely $\gamma \in \bar{Z}^2(G, A_e)$ and $\varphi \in Z^1(G, k^*[G])$ where $\bar{Z}$ are graded cocyles (see [K1]) and $k^*[G]$ is the group ring restricted to invertible coefficients with G–module structure induced by the adjoint action:

$$\phi(g)\cdot(\sum \mu_h h) = \sum_h \mu_h g h g^{-1}$$

We set $\varphi(g) = \sum_h \varphi_{g,h} g h g^{-1}$ and $\gamma_{g,h} = \gamma(g,h)$.

The multiplication and G–action are determined by

$$1_g 1_h = \gamma_{g,h} 1_{gh} \quad \varphi_g(1_h) = \varphi_{g,h} 1_{ghg^{-1}}$$

There are two compatibility equations:

$$\varphi_{g,h}\gamma_{ghg^{-1},g} = \gamma_{g,h} \tag{1.1}$$

and

$$\varphi_{k,g}\varphi_{k,h}\gamma_{kgk^{-1},khk^{-1}} = \varphi_k(\gamma_{g,h})\varphi_{k,gh} \tag{1.2}$$

Notice that if $\gamma_{g,h}$ is non–zero i.e. $A_g A_h \neq 0$ then (1.1) determines $\varphi_{g,h}$. We also would like to remark that (1.2) is automatically satisfied if $A_g A_h A_k \neq 0$ (cf. [K1]).

1.2. **The group ring $k[G]$.** Let $k[G]$ denote the group ring of G.

1.2.1. **The Hopf structure of $k[G]$.** Recall that $k[G]$ is a Hopf algebra with the natural multiplication, the co–multiplication induced by $\Delta(g) = g \otimes g$, co–unit $\epsilon(g) = 1$ and antipode $S(g) = g^{-1}$.

1.2.2. **The G–Frobenius structure of $k[G]$.** When considering $k[G]$ as a G–Frobenius algebra we will consider $k[G]$ as a left $k[G]$–module with respect to conjugation, i.e. the map $k[G] \otimes k[G] \to k[G]$ given by

$$\sum_g \nu_g g \otimes \sum_h \mu_h h \mapsto \sum_{g,h} \nu_h \mu_g g h g^{-1}$$

The other structures are the naturally G graded natural multiplication on $k[G]$ with the unit e, the metric $\eta(g,h) = \delta_{gh,e}$ and $\chi_g \equiv 1$. It is trivial to check all axioms.

If we were to choose a grading $\tilde{}\in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$ then $\chi_g = (-1)^{\tilde{g}}$ and $\varphi_g(h) = (-1)^{\tilde{g}\tilde{h}}$.

1.3. **The twisted group ring $k^\alpha[G]$.** Recall that given an element $\alpha \in Z^2(G, k^*)$ one defines the twisted group ring $k^\alpha[G]$ to be given by the same linear structure with multiplication given by the linear extension of

$$g \otimes h \mapsto \alpha(g,h)gh \tag{1.3}$$

with 1 remaining the unit element. To avoid confusion we will denote elements of $k^\alpha[G]$ by $\hat{g}$ and the multiplication with $\cdot$ Thus

$$\hat{g} \cdot \hat{h} = \alpha(g,h)\widehat{gh}$$

For α the following equations hold:

$$\alpha(g,e) = \alpha(e,g), \qquad \alpha(g,g^{-1}) = \alpha(g^{-1},g) \tag{1.4}$$

Furthermore

$$\hat{g}^{-1} = \frac{1}{\alpha(g,g^{-1})}\widehat{g^{-1}}$$

1.3.1. **Remark.** Given a two co–cycle α and possibly extending the field by square roots we can find a co–cycle $\tilde{\alpha}$ in the same cohomology class which also satisfies

$$\tilde{\alpha}(g,g^{-1}) = 1 \tag{1.5}$$

If one wishes to consider $\mathbb{C}$ as a ground field, one can work with such cocycles.

1.3.2. **Lemma.** Set $\epsilon(g,h) = \frac{\alpha(g,h)}{\alpha(ghg^{-1},g)}$, then the left adjoint action of $k^\alpha[G]$ on $k^\alpha[G]$ is given by

$$g \otimes h \overset{ad}{\mapsto} \epsilon(g,h)\widehat{ghg^{-1}} \tag{1.6}$$

Proof. By the definition of multiplication in $k^\alpha[G]$

$$\hat{g} \cdot \hat{h} \cdot \hat{g}^{-1} = \frac{\alpha(g,h)\alpha(gh,g^{-1})}{\alpha(g,g^{-1})}\widehat{ghg^{-1}}$$

Now by associativity

$$\alpha(gh,g^{-1})\alpha(ghg^{-1},g) = \alpha(gh,e)\alpha(g^{-1},g) = \alpha(g,g^{-1}).$$

so

$$\frac{\alpha(g,h)\alpha(gh,g^{-1})}{\alpha(g,g^{-1})} = \frac{\alpha(g,h)}{\alpha(ghg^{-1},g)}$$

1.3.3. **The G–Frobenius Algebra structure of $k^\alpha[G]$.** Recall from [K1,K2] the following structures which turn $k^\alpha[G]$ into a special G–Frobenius algebra:

$$\begin{aligned}
\gamma_{g,h} &= \alpha(g,h) \\
\varphi_{g,h} &= (-1)^{\tilde{g}\tilde{h}} \frac{\alpha(g,h)}{\alpha(ghg^{-1},g)} =: \epsilon(g,h) \\
\eta(\hat{g},\widehat{g^{-1}}) &= \alpha(g,g^{-1}) \\
\chi_g &= (-1)^{\tilde{g}}
\end{aligned} \tag{1.7}$$

Here the second line induces the third via

$$\hat{g}\cdot\hat{h} = \alpha(g,h)\hat{gh}, \quad \widehat{ghg^{-1}}\cdot\hat{g} = \alpha(ghg^{-1},g)\hat{gh}$$

We recall that if k^* is two divisible we could scale s.t. $\eta(g,g^{-1}) = 1$ and ϵ would indeed yield the adjoint action. The last equation follows from the special case of the trace axiom since the dimension of all sectors is one.

It is an exercise to check all axioms. All compatibility equations follow automatically, since $\alpha(g,h) \neq 0$. The only axiom which is not straightforward is the trace axiom, but see [K2] for a proof.

1.3.4. **Remark.** By the general theory (see above), $\epsilon \in H^1(G, k^*[G])$ where $k^*[G]$ is the group ring restricted to invertible coefficients with G–module structure induced by the adjoint action:

$$\phi(g)\cdot(\sum \mu_h h) = \sum_h \mu_h ghg^{-1}$$

1.3.5. **Relations.** The $\epsilon(g,h)$ satisfy the equations:

$$\epsilon(g,e) = \epsilon(g,g) = 1 \tag{1.8}$$

$$\epsilon(g_1g_2,h) = \epsilon(g_1,g_2hg_2^{-1})\epsilon(g_2,h) \tag{1.9}$$

$$\epsilon(g,h)^{-1} = \epsilon(g^{-1},ghg^{-1}) \tag{1.10}$$

$$\epsilon(k,gh) = \epsilon(k,g)\epsilon(k,h)\frac{\alpha(kgk^{-1},khk^{-1})}{\alpha(g,h)} \tag{1.11}$$

where (1.9) is the statement that $\varphi \in \mathrm{Hom}(G, Aut(A))$, (1.10) is a consequence of (1.9) and (1.11) is the compatibility equation which also ensures the invariance of the metric.

Furthermore the trace axiom holds [K2] which is equivalent to the equation

$$\alpha([g,h],hgh^{-1})\epsilon(h,g)=\alpha([g,h],h)\epsilon(g^{-1},ghg^{-1}) \tag{1.12}$$

or

$$\epsilon(h,g)=\epsilon(g^{-1},ghg^{-1})\frac{\alpha([g,h],h)}{\alpha([g,h],hgh^{-1})} \tag{1.13}$$

In the case that the group elements in the equations commute we obtain the famous conditions of discrete torsion which make ϵ into a bi–character on commuting elements.

For *commuting elements*:

$$\begin{aligned}
\epsilon(g,e) &= \epsilon(g,g)=1\\
\epsilon(g_1g_2,h) &= \epsilon(g_1,h)\epsilon(g_2,h)\\
\epsilon(g,h)^{-1} &= \epsilon(g^{-1},h)\\
\epsilon(g,h) &= \epsilon(h^{-1},g)=\epsilon(h,g)^{-1}\\
\epsilon(h,g_1g_2) &= \epsilon(h,g_1)\epsilon(h,g_2) \qquad (1.14)
\end{aligned}$$

where the last equation is now a consequence of the second and the fourth and the third equation follows from the second.

1.3.6. **Fact.** One can show [K2] that the twisted group algebras $k^\alpha[G]$ are the only G–Frobenius algebras with the property that all A_g are one–dimensional. To be completely precise there is an additional freedom of choosing a super (i.e. $\mathbb{Z}/2\mathbb{Z}$) structure determined by a homomorphism $\sigma\in \mathrm{Hom}(G,\mathbb{Z}/2\mathbb{Z})$ (see [K1] and 3.4 below).

1.3.7. **Geometry of $k^\alpha[G]$.** From the point of view of Jacobian Frobenius algebras [K1] it is natural to say that $k[G]$ is the Frobenius algebra naturally associated to *point*$/G$. The existence of the twisted algebras suggests that there are several equivalent ways of taking the group quotient. This is made precise by Theorem 3.3.2 below.

2. Algebraic structures of a G–Frobenius algebra

We fix a G–Frobenius algebra $\langle G,A,\circ,1,\eta,\varphi,\chi\rangle$.

2.1. **Theorem.** A G–Frobenius algebra is naturally a left $k[G]$–module algebra as well as a right $k[G]$–co–module algebra. Moreover it satisfies the Yetter–Drinfeld (YD) condition for bi–modules and is thus a module over $D(k[G])$, the Drinfeld double of $k[G]$. Where the YD condition reads

$$\sum h_1 \cdot m_0 \otimes h_2 m_1 = \sum (h_2 \cdot m)_0 \otimes (h_2 \cdot m)_1 h_1 \tag{2.1}$$

Here we used the usual notation for co–algebras and right co–modules. I.e. if $\Delta : H \to H \otimes H$ is the co–multiplication of H then for $h \in H$ we write $\Delta(h) = \sum h_1 \otimes h_2$ and if $\check{\rho} : M \to H$ is a right co–module map, then for $m \in M$ we write $\check{\rho}(m) = \sum m_0 \otimes m_1$.

Proof. The Theorem follows from the collection of facts below and the general statement that any H bi–module satisfying the YD–condition is a a module over $D(H)$ (see e.g. [M]).

2.1.1. **Remark.** In this particular case the YD condition states that the co-module structure is $k[G]$ equivariant with respect to the adjoint action of $k[G]$ on itself, viz. as a tensor product of G–Frobenius algebras of left $k[G]$ modules. See below.

2.1.2. **The $k[G]$–module structure.** Since A is a k algebra, the G–action φ turns A into a right $k[G]$ module. More precisely for $a \in A \sum_g \nu_g g \in k[G]$

$$(\sum \nu_g g) \otimes a \mapsto \sum_g \nu_g \varphi_g(a) \tag{2.2}$$

Since $\varphi \in \mathrm{Hom}(G, Aut(A))$ this is a module structure.

2.1.3. **The $k[G]$–co-module structure.** Since A is a G graded algebra it is naturally a $k[G]$ –co-module.

More precisely for $a \in A, a = \oplus_g a_g$ the $k[G]$ co-module structure $\rho : A \to A \otimes k[G]$ is given by

$$a \mapsto \sum_g (a_g \otimes g) \tag{2.3}$$

which obviously yields a co-module.

2.1.4. **Lemma.** A G–Frobenius algebra is a $k[G]$–module algebra and a $k[G]$ co-module algebra or equivalently a $k[G]^*$–module algebra.

Proof. For the module algebra structure notice that:

1) A is a left $k[G]$ module as noticed before.

2) The $k[G]$–action induced by φ is by definition by algebra automorphisms, and $\Delta(g) = g \otimes g$ thus

$$\varphi_g(ab) = \varphi_g(a)\varphi_g(b)$$

3) Since the unit is invariant:

$$\varphi_g(1) = 1 = \epsilon(g)1$$

The structure of co-module algebra follows from the fact that

$$\varphi(a_g b_h) = a_g b_h \otimes gh$$

which, as is well known, is nothing but the condition of A being a G graded algebra

$$A_g A_h \subset A_{gh} \tag{2.4}$$

2.1.5. **Remark.** Notice that the condition (2.4) is usually given by a strict inclusion, so that it is usually not $k[G]$–Galois – which is equivalent to $A_g A_h = A_{gh}$ (cf. [M]). In case it is, the structure of the algebra is particularly transparent. We will come back to this later.

2.1.6. **The compatibility.** We will view $k[G]$ as a left $k[G]$ module using the adjoint action. Then $A \otimes k[G]$ turns into a left $k[G]$–module by using the diagonal action. This is the natural left $k[G]$-module structure on the tensor product of left Hopf modules.

$$(\sum_h \mu_h h)(\sum_g a_g \otimes \nu g) = \sum_{h,g} \mu_h \nu \varphi(h)(a_g) \otimes ghg^{-1}$$

2.1.7. **Lemma.** The co-module structure is $k[G]$–equivariant and thus the co-module map is a map of left $k[G]$ modules where we use the left adjoint action of $k[G]$ on itself as the left $k[G]$ action.

$$\begin{aligned}
\rho((\sum_h \mu_h h)(a)) &= \rho(\sum_h \mu_h \varphi_h(a)) \\
&= \sum_{h,g} \mu_h \varphi_h(a_g) \otimes hgh^{-1} = (\sum_h \mu_h h) \cdot (\sum_g a_g \otimes g) \\
&= (\sum_h \mu_h h) \cdot \rho(a) \qquad (2.5)
\end{aligned}$$

2.1.8. **The YD condition.** Plugging in the co–product and action yields

$$\varphi_g(a_h) \otimes gh = \varphi_g(a_h) \otimes (ghg^{-1})g \tag{2.6}$$

which verifies the YD condition for A.

2.1.9. **Proposition.** If A is a G–Frobenius algebra that is $k[G]$–Galois over A_e as a k[G]–comodule algebra then A is special and $\gamma \in Z^2(G, A^*)$ where A^* are the units of A. So in particular γ determines φ uniquely.

Moreover if A_e is one–dimensional then $A = k^\alpha[G]$, for some $\alpha \in H^2(G, k^*)$ with a choice of parity $\tilde{} \in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$.

Proof. Since $A_{g^{-1}}A_g = A_e$ there are elements $a_g \in A_g, b_{g^{-1}} \in A_{g^{-1}}$ s.t. $b_{g^{-1}}a_g = 1$ then a_g is a cyclic generator since $\forall c_g \in A_G\ c_g = c_g(b_{g^{-1}}a_g) = (c_g b_{g^{-1}})a_g$ and $c_g b_{g^{-1}} \in A_e$. Choosing generators 1_g in this way it is easy to check that the cocycles need to be invertible and thus the φ are fixed by (1.1). Furthermore notice that the multiplication map induces an isomorphism of A_e modules between A_e and A_g via $a \mapsto a1_g$ where A_e is a cyclic A_e module over itself via left multiplication. This follows by associativity from $a = a(1_g 1_{g^{-1}}) = (a1_g)1_{g^{-1}}$ and thus $a1_g \neq 0$ and the map $A_e \to A_g$ is also injective. Thus the restriction maps are all isomorphisms and graded cocycles coincide with the usual ones.

3. The action of Discrete Torsion

3.1. **Twisting G–Frobenius algebras.** Given a G–Frobenius algebra A we can re-scale the multiplication and G–action by a scalar. More precisely let $\lambda : G \times G \to k^*$ be a function. For $a = \oplus_g a_g \in A$ we define

$$\varphi^\lambda(g)(a) = \oplus_h \lambda(g, h)\varphi(g)(a_h)$$

Given another function $\mu : G \times G \to k^*$ we can also define a new multiplication $\circ^\mu$

$$a_g \circ^\mu b_h = \mu(g, h) a_g \circ b_h$$

3.1.1. **Remark.** These twists arise from a projectivization of the G–structures induced on a module over A as for instance the associated Ramond–space (cf. [K1]). In physics terms this means that each twisted sector will have a projective vacuum, so that fixing their lifts in different ways induces the twist. Mathematically this means that g twisted sector is considered to be a Verma module over A_g based on this vacuum.

3.1.2. **Induced shift on the metric.** Due to the invariance of the metric, the twist in the multiplication results in a twisted metric

$$\eta^\mu(a_g, b_g^{-1}) := \mu(g, g^{-1})\eta(a_g, b_g^{-1})$$

3.1.3. **Definition.** We define $s(\mu,\lambda)(A)$ to be the collection $\langle G, A, \circ^{\mu}, 1, \eta^{\lambda}, \varphi^{\lambda}, \chi\rangle$.

3.1.4. **Proposition.** $s(\mu,\lambda)(A)$ is G–Frobenius algebra if and only if the following equations hold for μ,λ:

$$\mu(e,g) = \mu(g,e) = 1 \tag{3.1}$$

furthermore $\forall g,h,k \in G$ s.t. $A_gA_hA_k \neq 0$:

$$\mu(g,h)\mu(gh,k) = \mu(h,k)\mu(g,hk) \tag{3.2}$$

and if $A_gA_h \neq 0$ then

$$\lambda(g,h) = \frac{\mu(g,h)}{\mu(ghg^{-1},g)} \tag{3.3}$$

If $A_gA_h \neq 0$ as well as $A_gA_hA_k = 0$

$$\lambda(g,hk)\mu(h,k) = \lambda(g,h)\lambda(g,k)\mu(ghg^{-1},gkg^{-1}) \tag{3.4}$$

Furthermore

$$\begin{aligned} \lambda(gh,k) &= \lambda(h,k)\lambda(g,hkh^{-1}) \\ \lambda(e,g) &= \lambda(g,g) = 1 \\ \mu([g,h],hgh^{-1})\lambda(h,g) &= \mu([g,h],h)\lambda(g^{-1},ghg^{-1}) \end{aligned} \tag{3.5}$$

where the third equation has to hold for all pairs g,h s.t. $\exists c \in A_{[g,h]}$ s.t. $\chi_h \mathrm{Tr}(l_c\varphi_h|_{A_g}) \neq 0$, where l_c is the left multiplication by c. In particular it must hold for all pairs g,h with $[g,h]=e$.

Proof. The first equation (3.1) expresses that 1 is still the unit for the algebra. The statement (3.2) for μ is the obvious form of associativity. The statement (3.3) comes from the compatibility equation of the group action with the multiplication.

The equation (3.4) ensures the equivariance of the multiplication. It is automatic if $A_gA_hA_k \neq 0$ and also if $A_gA_h = 0$.

The first equation (3.5) for λ is equivalent to the fact that φ^{λ} is still a G action.

Notice that $\lambda(e,g) = 1$ since $A_eA_g = A_g$ and thus

$$\lambda(e,g) = \frac{\mu(e,g)}{\mu(g,e)} = 1$$

and so the identity remains invariant.

Also notice that there is no twist to the character!

$$\chi_g^{\lambda} = (-1)^{\tilde{g}\dim A_g}\mathrm{Str}^{-1}(\varphi_g^{\lambda)}|_{A_e}) = \chi_g\lambda(g,e) = \chi_g$$

This in turn implies the second statement in the second line by projective self–invariance

$$\chi_g^{-1} id|A_g = \varphi_g^\lambda|A_g = \lambda(g,g)\varphi_g|A_g = \lambda(g,g)\chi_g^{-1} id|A_g$$

$$1 = \lambda(e,k) = \lambda(g^{-1}g,k) = \lambda(g,k)\lambda(g^{-1},gkg^{-1})$$

so

$$\lambda(g,h) = \lambda(g^{-1},ghg^{-1})^{-1}$$

The third equation follows from the projective trace axiom.

$\forall c \in A_{[g,h]}$ and l_c left multiplication by c:

$$\chi_h \mathrm{Tr}(l_c\varphi_h|_{A_g}) = \chi_{g^{-1}} \mathrm{Tr}(\varphi_{g^{-1}} l_c|_{A_h}) \tag{3.6}$$

Thus we must have $\chi_h \mathrm{Tr}(l_c\varphi_h^\lambda|_{A_g}) = \chi_{g^{-1}} \mathrm{Tr}(\varphi_{g^{-1}}^\lambda l_c|_{A_h})$ but this is equivalent to the third equation in view of (3.6).

Now we check the other axioms.

The invariance of the metric follows from associativity.

$$\begin{aligned}
\eta^\mu(a_g, b_h \circ^\mu c_{h^{-1}g^{-1}}) &= \mu(g,g^{-1})\mu(h,h^{-1}g^{-1})\eta(a_g, b_h \circ c_{h^{-1}g^{-1}}) \\
&= \mu(g,h)\mu(gh,h^{-1}g^{-1})\eta(a_g \circ b_h, c_{h^{-1}g^{-1}}) \\
&= \eta^\mu(a_g \circ^\mu b_h, c_{h^{-1}g^{-1}})
\end{aligned}$$

The projective invariance of the metric reads as

$$\lambda(g,k)\lambda(g,k^{-1})\mu(gkg^{-1},gk^{-1}g^{-1}) = \mu(k,k^{-1})$$

which is automatic since $A_gA_kA_{k^{-1}} \neq 0$.

3.1.5. **Definition.** We call a twist $s(\lambda,\mu)$ *universal* if it transforms *any* G–Frobenius algebra into a G–Frobenius algebra. We call two twists $s(\lambda,\mu)$ and $s(\lambda',\mu')$ isomorphic if for any G–Frobenius algebra A the algebras $s(\lambda,\mu)(A)$ and $s(\lambda',\mu')(A)$ are isomorphic.

3.1.6. **Theorem.** The universal twists are in 1–1 correspondence with elements $\alpha \in Z^2(G,k^*)$ and the isomorphism classes of universal twists are given by $H^2(G,k^*)$.

Proof. If the twist is universal then there are no restrictions on the equations. In particular $\mu \in Z^2(G,k^*)$ and λ is completely determined by μ via (3.3). All the other properties are then automatic. The claim about isomorphism classes is obvious by noticing that if α and α' are cohomologous and $\alpha/\alpha' = d\beta$ for some $\beta \in Z^1(G,k^*)$ then a diagonal rescaling of the generators of $k^\alpha[G]$ by β yields $k^{\alpha'}[G]$ so the result follows from the charcterization of universal twists as taking tensor produce with twisted group rings below.

3.2. **Discrete torsion.** In this subsection we prove that universal twists are exactly given by twisting with discrete torsion.

3.2.1. **Reminder.** Given two G–Frobenius algebras $\langle G, A, \circ, 1, \eta, \varphi, \chi\rangle$ and $\langle G, A', \circ', 1', \eta', \varphi', \chi'\rangle$ we defined [K1] their tensor product as G–Frobenius algebras to be the G–Frobenius algebra
$\langle G, \bigoplus_{g\in G}(A_g \otimes A'_g), \circ\otimes\circ', 1\otimes 1', \eta\otimes\eta', \varphi\otimes\varphi', \chi\otimes\chi'\rangle$.
We will use the short hand notation $A\hat{\otimes}B$ for this product.

3.2.2. **Definition.** Given a G–Frobenius algebra A and an element $\alpha \in H^2(G,k)$ we define the α–twist of A to be the G–Frobenius algebra $A^\alpha := A\hat{\otimes}k^\alpha[G]$.

Notice that

$$A^\alpha_g = A_g \otimes k \simeq A_g \tag{3.7}$$

Using this identification the G–Frobenius structures are given by:

3.3. **Lemma.** The induced structures under the isomorphism (3.7) are

$$\begin{aligned} \circ^\alpha|_{A^\alpha_g\otimes A^\alpha_h} &= \alpha(g,h)\circ \\ \varphi^\alpha_g|_{A^\alpha_h} &= \epsilon(g,h)\varphi_g \\ \eta^\alpha|_{A^\alpha_g\otimes A^\alpha_{g^{-1}}} &= \alpha(g,g^{-1})\eta \end{aligned}$$

$$\chi_g = \chi_g \tag{3.8}$$

Proof. We notice that the two algebras have the same linear structure $A_{\alpha,g} = A_g \otimes kg \simeq A_g$ with the isomorphism given by $a_g\otimes g \mapsto a_g$. Now the multiplication is given by

$$(a_g\otimes g)\otimes(a_h\otimes h) \mapsto a_ga_h\otimes\alpha(g,h)gh = \alpha(g,h)a_ga_h\otimes gh$$

which yields the twisted multiplication.

The twist for the G action is computed to be

$$\varphi_{\alpha,h}(a_g\otimes g) = \epsilon(g,h)\varphi_h(g)\otimes hgh^{-1}$$

This leads us to the Proposition

3.3.1. **Proposition.** $A_\alpha \simeq s(\alpha,\epsilon)A$.

3.3.2. **Theorem.** The set of universal twists are described by tensoring with twisted group algebras which identifies this operation with twisting by discrete torsion.

In other words given a generic G–Frobenius algebra A there are exactly $H^2(G,k)$ twists of it by discrete torsion.

3.3.3. **Discrete torsion as phases for the partition sum.** Notice that for any $c \in A^{\alpha}_{[g,h]} \simeq A_{[g,h]}$

$$\chi_h \mathrm{Str}(l_c \varphi_h|_{A^{\alpha}_g}) = \epsilon(h,g)\chi_h \mathrm{Str}(l_c \varphi_h|_{A_g}) \tag{3.9}$$

This is the original freedom of choice of a phase for the summands of the partition function postulated by physicists. In this context, we should regard $g,h : [g,h] = e$ and set $c = e$. More precisely set

$$Z(A) = \sum_{g,h \in G:[g,h]=e} \mathrm{Str}(\chi_g \varphi_g|_{A^{\alpha}_h}) := \sum_{g,h \in G:[g,h]=e} Z_{g,h} \tag{3.10}$$

$$Z(A_\alpha) = \sum_{g,h \in G:[g,h]=e} \epsilon(g,h) Z_{g,h} \tag{3.11}$$

We could omit the factors χ_g, but from the point of view of physics we should take the trace in the Ramond space (cf. [K1]) where the $k[G]$ module structure is twisted by χ.

3.4. **Supergrading.** In this subsection we wish to address questions of super–grading. There is a general theory of super–graded G–Frobenius algebras and special G–Frobenius algebras. We will expose the structures here for the group ring.

3.4.1. **Super G–Frobenius algebras.** If the underlying algebra of a G–Frobenius algebra has a supergrading $\tilde{}$ then the axioms of a G–Frobenius algebra have to be changed to

b$^\sigma$) *Twisted super–commutativity*
$a_g \circ a_h = (-1)^{\tilde{a}_g \tilde{a}_h} \varphi_g(a_h) \circ a_g$

iv$^\sigma$) *Projective super–trace axiom*
$\forall c \in A_{[g,h]}$ and l_c left multiplication by c:
$\chi_h \mathrm{STr}(l_c \varphi_h|_{A_g}) = \chi_{g^{-1}} \mathrm{STr}(\varphi_{g^{-1}} l_c|_{A_h})$

where STr is the super–trace.

For details on the super structure as well as the role of the super structure for special G–Frobenius algebras we refer to [K1].

3.4.2. **Supergraded twisted group rings.** Fix $\alpha \in H^2(G,k^*), \sigma \in \mathrm{Hom}(G,\mathbb{Z}/2\mathbb{Z})$ then there is a twisted super–version of the group ring where now the relations read

$$\hat{g}\hat{h} = \alpha(g,h)\widehat{gh} \tag{3.12}$$

and the twisted commutativity is

$$\hat{g}\hat{h} = (-1)^{\sigma(g)\sigma(h)} \varphi_g(\hat{h})\hat{g} \tag{3.13}$$

and thus

$$\varphi_g(\hat{h}) = (-1)^{\sigma(g)\sigma(h)}\alpha(g,h)\alpha(gh,g^{-1})\widehat{ghg^{-1}} =: \varphi_{g,h}\widehat{ghg^{-1}} \tag{3.14}$$

and thus

$$\epsilon(g,h) := \varphi_{g,h} = (-1)^{\sigma(g)\sigma(h)}\frac{\alpha(g,h)}{\alpha(ghg^{-1},g)} \tag{3.15}$$

We would just like to remark that the axiom iv$^\sigma$ shows the difference between super twists and discrete torsion.

3.4.3. **Definition.** We denote the α-twisted group ring with super–structure σ by $k^{\alpha,\sigma}[G]$. We still denote $k^{\alpha,0}[G]$ by $k^\alpha[G]$ where 0 is the zero map and we denote $k^{0,\sigma}[G]$ just by $k^\sigma[G]$ where 0 is the unit of the group $H^2(G,k^*)$.

A straightforward calculation shows

3.4.4. **Lemma.** $k^{\alpha,\sigma}[G] = k^\alpha[G] \otimes k^\sigma[G]$ and more generally

3.4.5. **Lemma.** Let A be the G–Frobenius algebra or more generally super Frobenius algebra with super grading $\tilde{}$ $\langle G, A, \circ, 1, \eta, \varphi, \chi\rangle$ then $A\otimes k^\sigma[G]$ is isomorphic to the super G–Frobenius algebra $\langle G, A, \circ, 1, \eta, \varphi^\sigma, \chi^\sigma\rangle$ with super grading $\tilde{\sigma}$, where

$$\varphi^\sigma_{g,h} = (-1)^{\sigma(g)\sigma(h)}\varphi_{g,h}, \quad \chi^\sigma = (-1)^{\sigma(g)}\chi_g, \quad \tilde{a}^\sigma_g = \tilde{a}_g + \sigma(g) \tag{3.16}$$

Using arguments and definitions for universal twists as for discrete torsion we can obtain the following Proposition. Here universal means that there is no assumption on the particular structure of the G–Frobenius algebra, in other words it pertains to generic G–Frobenius algebras.

3.4.6. **Proposition.** Given a (super) G–Frobenius algebra A the universal super G–Frobenius algebra re–gradings are in 1-1 correspondence with $\mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$ and these structures can be realized by tensoring with $k^\sigma[G]$ for $\sigma \in \mathrm{Hom}(G, \mathbb{Z}/2\mathbb{Z})$.

4. Projective representations, Extensions and twisted group algebras

In this section we first assemble classical facts about groups which will be extended to G–Frobenius algebras. As an intermediate step we analyze twisted group algebras, which belong to both worlds.

4.1. **Part I: Groups.**

4.1.1. **Projective representations.** A projective representation ρ of a group is a map $\rho : G \to GL(V)$, V being a k–vector space, which satisfies

$$\rho(g)\rho(h) = \alpha(g,h)\rho(gh), \rho(e) = id \tag{4.1}$$

It is easy to check that $\alpha(g,h) \in Z^2(G,k^*)$. Moreover with a natural notion of projective isomorphy two projective representations are isomorphic if their classes are cohomologous (cf. e.g. [CR, Kar]).

4.1.2. **Extensions.** Given a central extension

$$1 \mapsto A \to G^* \stackrel{\pi}{\to} G \to 1 \tag{4.2}$$

fix a section s of π and define $\alpha : G\times G \to A$ by $s(g)s(h) = \alpha(g,h)s(gh)$. It is easy to see that indeed $\alpha \in Z^2(G,A)$ and furthermore changing the section or changing the extension by an isomorphism preserves the cohomology class of α.

Vice versa a cycle in $\alpha \in Z^2(G,A)$ were A is an Abelian group gives rise to a central group extension of G.

$$1 \mapsto A \to G^\alpha \stackrel{\pi}{\to} G \to 1 \tag{4.3}$$

where $G^\alpha = A \rtimes G$. The maps are given by $a \mapsto (A, e_G)$, $(a,g) \mapsto g$ and the multiplication is given by $(a,g)(a',g') = (aa'\alpha(g,g'),gg')$.

4.1.3. **The transgression map.** Given a cycle $\alpha \in H^2(G,A)$ there is a natural map

$$Tra_\alpha : \mathrm{Hom}(A,k^*) \to H^2(G,k^*) \tag{4.4}$$

which sends $\chi \in \mathrm{Hom}(A,k^*)$ to the cocycles defined by $(g,h) \mapsto \chi\alpha(g,h)$. Actually this map maps into the cohomology group with values in the torsion subgroup of k^* which we call $tors(k^*)$.

$$Tra_\alpha : \mathrm{Hom}(A,k^*) \to H^2(G,tors(k^*)) \tag{4.5}$$

4.1.4. **Facts.** We briefly give the facts linking group cohomology, projective representations and twisted group algebras. For a detailed account see [Kar].

1) The classes of central extensions of a group G by an Abelian group A are in 1–1 correspondence with $H^2(G,A)$.
2) Any projective α–representation is a module over the α–twisted group algebra $k^\alpha[G]$. (This is in fact an equivalence of categories).

3) Every projective representation with cycle α is projectively equivalent to one that can be lifted to linear representation on $G^{\hat{\alpha}}$ if $[\alpha]$ is in the image of the transgression map associated to $[\hat{\alpha}]$.
4) If $H^2(G,k^*) = H^2(G,(tors(k^*))$ then:
   a) any projective representation can be lifted to a suitable group, and
   b) there is a universal extension
   $$1 \to A \to G^* \to G \to 1$$
   such that *any* projective representation lifts to G^* and moreover the group $A \simeq H^2(G,k^*)$.

ASSUMPTION:
For the remainder of the section we will assume that k has the property that $H^2(G,k^*) = H^2(G,(tors(k^*))$. This is the case e.g. if is algebraically closed or $k = \mathbb{R}$, see eg. [Kar].

4.2. **Part II: The twisted group algebra revisited.** Fix $[\alpha'] \in H^2(G,A)$, an element $[\alpha] \in \mathrm{Im}(Tra_{[\alpha']})$ and a pre–image character $\chi \in \mathrm{Hom}(A,k^*)$.

This yields a central extension:

$$1 \mapsto A \to G^{\alpha'} \xrightarrow{\pi} G \to 1 \tag{4.6}$$

with a section s of π s.t. the cocycle corresponding to s is α'. The map χ induces map

$$\chi : k[G^\alpha] \to k[G] : ag \mapsto \chi(a)g \tag{4.7}$$

while the section s induces a map

$$s : k[G] \to k[G^\alpha] : g \mapsto 1_A g \tag{4.8}$$

4.2.1. **Projective algebra.** Using the maps s, χ we can also characterize the multiplication μ^α in $k^\alpha[G]$ as follows: It is the map which makes the following diagram commutative:

$$\begin{array}{ccc} k[G^\alpha] \otimes k[G^\alpha] & \xrightarrow{\mu} & k[G^\alpha] \\ {\scriptstyle s\otimes s}\uparrow & & \downarrow{\scriptstyle \chi} \\ k[G] \otimes k[G] & \xrightarrow{\mu^\alpha} & k[G] \end{array}$$

We already know that μ^α induces the structure of an algebra. This diagram captures the statement about lifts of projective representations of G to linear representations of G^α.

This is essentially 4.1.4 1.

4.2.2. **Projective co–algebra.** Using the diagram above as we define a co–multiplication by commutativity of:

$$\begin{tikzcd} k[G^\alpha] \arrow[r, "\Delta"] & k[G^\alpha]\otimes k[G^\alpha] \arrow[d, "\chi\otimes\chi"] \\ k[G] \arrow[u, "s"] \arrow[r, "\Delta^\alpha"] & k[G]\otimes k[G] \end{tikzcd}$$

The co–algebra structure we induce in this way on $k[G]$ is actually the old co–algebra structure, but $k^\alpha[G]$ ceases to be a bi–algebra.

4.2.3. **Remark: Braiding.** If one would like a bi–algebra structure on the group ring $k^\alpha[G]$ then one has to consider braided objects, where the braiding is inverse to the twist. It should be possible to find analogous statements to the ones presented in this article by considering structures over $k^\alpha[G]$ in braided categories.

4.2.4. **Adjoint action.** Let ad denote the adjoint action $k[G^\alpha]$. Then there is an induced action on $k[G]$

$$\begin{tikzcd} k[G^\alpha]\otimes k[G^\alpha] \arrow[r, "ad"] & k[G^\alpha] \arrow[d, "\chi"] \\ k[G]\otimes k[G] \arrow[u, "s\otimes s"] \arrow[r, "ad^\epsilon"] & k[G] \end{tikzcd}$$

According to 1.3.2 this action is given by

$$ad^\epsilon(g)(h) := \epsilon(g,h)ghg^{-1}$$

4.3. **Part III: G–Frobenius algebras.** We now apply the logic of part II to general G–Frobenius algebras.

Let H be an Abelian group. Fix $[\alpha'] \in H^2(G,H)$, an element $[\alpha] \in \mathrm{Im}(Tra_{[\alpha']})$ and a pre-image character $\chi \in \mathrm{Hom}(H,k^*)$ and a central extension:

$$1 \mapsto H \to G^{\alpha'} \xrightarrow{\pi} G \to 1 \tag{4.9}$$

with a section s of π s.t. the cocycle corresponding to s is α'.

4.3.1. **Definition.** Let A^α be a G^α Frobenius algebra. We say that a G–Frobenius algebra F can be lifted to A^α if there are maps $i : A \to A^\alpha$ and $res : A^\alpha \to A$ such that the structural maps fit into the commutative diagrams

$$
\begin{array}{ccc}
A^\alpha & \xrightarrow{\rho^\alpha} & A^\alpha \otimes k[G^\alpha] \\
{\scriptstyle i}\uparrow & & \downarrow{\scriptstyle res\otimes\chi} \\
A & \xrightarrow{\rho} & A \otimes k[G]
\end{array}
$$

and

$$
\begin{array}{ccc}
A^\alpha \otimes A^\alpha & \xrightarrow{\mu^\alpha} & A^\alpha \\
{\scriptstyle i\otimes i}\uparrow & & \downarrow{\scriptstyle res} \\
A \otimes A & \xrightarrow{\mu} & A
\end{array}
\qquad
\begin{array}{ccc}
k[G^\alpha] \otimes A^\alpha & \xrightarrow{\varphi^\alpha} & A^\alpha \\
{\scriptstyle s\otimes s}\uparrow & & \downarrow{\scriptstyle res} \\
k[G] \otimes A & \xrightarrow{\varphi} & A
\end{array}
$$

and all algebraic structures are compatible.

4.3.2. **Definition.** We say that an H–Frobenius algebra B is H homogeneous if it is endowed with an additional left H–action τ which shifts group degree and is equivariant w.r.t. multiplication. More precisely the following two equations hold:

$$\tau(h)(A_{h'}) \subset A_{h,h'}, \qquad \tau(h)(ab) = a\tau(h)(b) \tag{4.10}$$

It is standard to see that

4.3.3. **Remark.** With the notation as above, the left action τ of H on B is necessarily by isomorphisms and thus B is a special H–Frobenius algebra whose components are all isomorphic. Moreover B is Galois as a $k[H]$–comodule over B_e.

4.3.4. **Definition.** Given a G–Frobenius algebra A, an H–homogeneous H–Frobenius algebra B and a cocycle $\alpha \in Z^2(G,H)$ we define the *crossed product* of A and B to be the G^α–Frobenius algebra

$$A\#_\alpha B := \langle G^\alpha, A\otimes B, \circ\#_\alpha\circ', 1\otimes 1, \eta\otimes\eta', \varphi\#_\epsilon\varphi', \chi\otimes\chi'\rangle \tag{4.11}$$

Where

$$(a_g\otimes b_h)\circ\#_\alpha\circ'(c_{g'}\otimes d_{h'}) = a_g c_{g'}\otimes\tau(\alpha(g,g'))b_h d_{h'} \tag{4.12}$$

and

$$\varphi\#_\epsilon\varphi'(g,h)(a_{g'}\otimes b_{h'}) = \varphi_g(a_{g'})\otimes\tau(\alpha(g,g')\alpha(gg',g^{-1}))\varphi_h(b_{h'}) \tag{4.13}$$

We leave it to the reader to verify all axioms, since it is analogous to previous calculations.

4.3.5. **Quantum symmetry group.** The postulated second left action by translation τ can be viewed as the quantum symmetry group postulated by physicists. Notice that it acts freely. The invariants are linearly isomorphic to $\bigoplus A_g\otimes B_{e_H}$ where e_H is the unit element of H.

4.3.6. **Lemma.** The linear map above induces an isomorphism

$$^{H_\tau}(A\#_\alpha B) \simeq \bigoplus_{g\in G}(A_g \otimes B_{e_H})$$

as G–Frobenius algebras with trivial action on the second factor.

Here we denoted the invariants under the action of H by τ by $^{H_\tau}$.

4.3.7. **Definition.** Fix $\chi \in \mathrm{Hom}(H, k^*)$ then there is a natural map from B to B_e given by $\tau(h)b_e \mapsto \chi(h)b_e$. This map induces a map

$$A\#_\alpha B \to \bigoplus_{g\in G}(A_g \otimes B_{e_H})$$

which induces a structure of G–Frobenius algebra on $\bigoplus_{g\in G}(A_g \otimes B_{e_H})$, the G–action on the second factor being trivial. We define

$$(A\#_\alpha B)^\chi$$

to be this G–Frobenius algebra.

It is easy to check that the following holds:

4.3.8. **Lemma.** Keeping the notation above, let $[\alpha'] = Tra_{[\alpha]}(\chi)$ and more precisely on the level of cocycles let $\alpha'(g, g') = \chi\alpha(g, g')$.

Then

$$(A\#_\alpha B)^\chi \simeq \Big(\bigoplus(A_g \otimes B_{e_H})\Big)_{\alpha'}$$

4.3.9. **Definition.** Given a cocycle $\alpha \in Z^2(G, H)$, a central extension G^α of G by H and a G–Frobenius algebra A we define A^α to be the G^α–Frobenius algebra

$$A^\alpha := A\#_\alpha k[H]$$

4.3.10. **Theorem.** Given a G–Frobenius algebra A and cocycles $\alpha \in Z^2(G, k^*), \alpha' \in Z^2(G, H)$ which are related by $\chi \in \mathrm{Hom}(G, k^*)$ via $\alpha(g, g') = \chi(\alpha'(g, g'))$.

Then the twist A_α of A lifts to the $G^{\alpha'}$–Frobenius algebra $A^{\alpha'}$ and moreover

$$(A^\alpha)^\chi \simeq A_\alpha$$

Finally if G^* is the universal extension of G whose cocycle is $\beta \in H^2(G, H^2(G, k^*))$ then any twist A_α of a G–Frobenius algebra A lifts to A^β.

Proof. Choose a section s of the extension yielding α. We denote the unit element of H by e_H and denote $s(g)$ by $e_H g$. We let $i : A_g \to A^{\alpha'}_{e_H g}$ be the map given by $A_g \to A_g \otimes ke_H : a_g \mapsto a_g \otimes e_H$ and define $res : A^{\alpha'}_{hg} \simeq A_g \otimes ke_H \mapsto A_g$ to be the map $a_g \otimes e_H \mapsto \chi(h)a_g$.

Then

$$(res \otimes \chi)(\rho^{\alpha'}(i(a_g))) = (res \otimes \chi)(\rho^{\alpha'}(a_g \otimes e_H))$$
$$= (res \otimes \chi)(a_g \otimes e_H) \otimes (e_H g) = a_g \otimes g$$

which assures the comodule algebra structure.

$$\chi(\mu^\alpha((i \otimes i)(a_g \otimes b_{g'}))) = \chi(a_g b_{g'} \otimes \alpha(g,g')gg') = \alpha(g,h)a_g b_{g'}$$

since $A_{e_H g}A_{e_H g'} \subset A_{\alpha(g,g')gg'}$ which assures the algebra structure.

$$\chi \circ \varphi \circ (s \otimes s)(g \otimes a_h) = \chi \circ \varphi((e_H g \otimes A_{e_H g'})) = \epsilon(g,g')\varphi_g(a_{g'})$$

which assures the module algebra structure, since $\varphi_{e_H g}(A_{e_H h}) \subset A_{\epsilon'(g,h)gh}$, where we set

$$\epsilon'(g,g') = \frac{\alpha'(g,h)}{\alpha'(ghg^{-1},g)}$$

to be the cocycle of the adjoint action. Then by 1.3.2

$$\chi(\epsilon'(g,g')) = \epsilon(g,g')$$

For the last statement notice that (cf. e.g. [Kar])

$$k[G^*] = \prod_{\alpha \in T} k[G^\alpha]$$

where T is a transversal for $B^2(G,k^*)$ in $Z^2(G,k^*)$. So $A^\beta \simeq \bigoplus_{\alpha \in T} A^\alpha$ and we can lift to the appropriate component.

References

[A] P. A. Aspinwall *A Note on the Equivalence of Vafa's and Douglas's Picture of Discrete Torsion* JHEP 0012 (2000) 029.

[CR] C. W. Curtis and I. Reiner. *Methods of representation theory. Vol. I. With applications to finite groups and orders.* Pure and Applied Mathematics. John Wiley & Sons, Inc., New York, 1988.

[D] M. R. Douglas. *D-branes and Discrete Torsion* Preprint hep-th/9807235.

[FG] B. Fantechi and L. Goettsche. *Orbifold cohomology for global quotients.* Preprint. math.AG/0104207

[Kar] G. Karpilovsky. *Projective Representations of Finite Groups.* Dekker, 1985.

[K1] R. Kaufmann. *Orbifolding Frobenius algebras.* Preprint. MPI 2001-56, IHES M/01/37, math.AG/0107163 and R. Kaufmann. *Orbifolding Frobenius algebras.* Talk at WAGP2000 conference at SISSA Trieste, October 2000

[K2] R. Kaufmann. *Second quantized Frobenius algebras.* Preprint. IHES M/02/46, math.AG/020613.

[K3] R. Kaufmann. *Singularities with symmetries, orbifold Frobenius algebras and mirror symmetry.* (in preparation)

[LS] M. Lehn and C. Soerger. *The cup product of the Hilbert scheme for K3 surfaces* Preprint math.AG/0012166.

[M] S. Montgomery. *Hopf algebras and their actions on rings.* CBMS Regional Conference Series in Mathematics, 82. American Mathematical Society, Providence, RI, 1993.

[R] Y. Ruan *Discrete torsion and twisted orbifold cohomology.* Preprint math.AG/0005299

[S] E. Sharpe. *Discrete Torsion and Gerbes I+II* Preprints hep-th/9909120 and hep-th/9909108

[V] C. Vafa *Modular Invariance and Discrete Torsion on Orbifolds.* Nucl. Phys. B273 (1986) 592–606.

[VW] C. Vafa and E. Witten. *On orbifolds with discrete torsion.* J. Geom. Phys. 15 (1995), 189–214.

E-mail address: kaufmann@math.usc.edu

University of Southern California, Los Angeles, USA and Max–Planck Institut für Mathematik, Bonn, Germany