CONVOLUTION THEOREM FOR NON-DEGENERATE MAPS AND COMPOSITE SINGULARITIES

Tomohide Terasoma

Department of Mathematical Science,
University of Tokyo,
Komaba, Meguro, Tokyo, 153, Japan
Max-Planck-Institut für Mathematik,
Gottfried Claren Strasse 26,
D-53225, Bonn, Germany.

§0 Introduction

Let n be a natural number greater than 1 and $f = f(x) = f(x_1, \dots, x_n)$ be a germ of holomorphic function on $\mathbf{C}^n$ at 0. Let ϵ be a sufficiently small number such that f is defined on $B(x) = \{(x_1, \dots, x_n) \in \mathbf{C}^n \mid\mid x_i \mid< \epsilon\}$. If δ' is sufficiently small, the restriction of $f : B(x) \to \mathbf{C}$ to $f^{-1}(B^0(s))$, where $B^0(s) = \{s \in \mathbf{C} \mid 0 <\mid s \mid< \delta'\}$ is a fiber bundle over $B^0(s)$. Therefore for $0 < \delta < \delta'$, $\pi_1(B^0(s)) = \pi_1(B^0(s), \delta)$ acts on $H^i(f^{-1}(\delta), \mathbf{Q})$. ¿From now on, we use the notation $f : B(x) \to B(t)$ for short in this situation, and the morphism f is called a local morphism. If f has only isolated singularities, then $H^i(f^{-1}(\mathbf{Q}), \mathbf{Q}) = 0$ if $i \neq 0, n-1$. In the same way if m is a natural number greater than 1 and g is a germ of holomorphic function on $\mathbf{C}^m = \{y = (y_1, \dots, y_m)\}$ at 0, a restriction of g defines a fiber bundle $g^{-1}(B^0(t)) \to B^0(t)$ over $B^0(t)$. If f and g has only isolated singularities, $u = f + g$ also has only isolated singularity. Let γ_f , γ_g and γ_{f+g} be the action of the canonical generator of $\pi_1(B^0(s))$, $\pi_1(B^0(t))$ and $\pi_1(B^0(u))$ respectively. Then relation between $H^{n+m-1}((f+g)^{-1}(\delta), \mathbf{Q})$, $H^{n-1}(f^{-1}(\delta), \mathbf{Q})$ and $H^{m-1}(g^{-1}(\delta), \mathbf{Q})$ is given by Thom-Sebastiani theorem [S-T]: There exists an isomorphism

$$H^{n+m-1}((f+g)^{-1}(\delta), \mathbf{Q}) \simeq H^{n-1}(f^{-1}(\delta), \mathbf{Q}) \otimes H^{m-1}(g^{-1}(\delta), \mathbf{Q}).$$

compatible with the actions of the monodromies, i.e. $\gamma_{f+g} = \gamma_f \otimes \gamma_g$.

The Hodge theoretic refinement of Thom-Sebastiani theorem is conjectured by Steenbrink [St], and proved by Varchenko [V], M.Saito [S] and Denef-Loeser [D-L] in different contexts. Let f be a germ of holomorphic function $f : B(x) \to B(t)$ such that the restriction of f to $f^{-1}(B^0(t))$ is smooth. After Steenbrink, the space $H^i(f^{-1}(\delta), \mathbf{Q})$ is equipped with the mixed Hodge structure for a sufficiently small δ and the action of γ_f is quasi-unipotent and preserves mixed Hodge structure. The minimal natural number m for which the action of γ^m is unipotent for $H^i(f^{-1}(\delta), \mathbf{Q})$ $(0 \leq i \leq 2(n-1)$ is called the exponent of f. If m is the exponent of f, then the action of $N = \frac{1}{m}\log(\gamma^m)$ satisfies $NW_k \subset W_{k-2}$. Therefore the action of γ on $Gr_k^W(H^i(f^{-1}(\delta), \mathbf{Q}))$ preserves the Hodge structure and has a finite image. Therefore $[H^i(f^{-1}(\delta),\mathbf{Q})]=\sum_k[Gr_k^W(H^i(f^{-1}(\delta),\mathbf{Q}))]$ defines an element of the Grothendieck group $K_{MH}(\mathbf{C},\mu_m)$ of mixed Hodge structures with an action of μ_m. For an element V in $K_{MH}(\mathbf{C},\mu_m)$, the invariant part and the complement of invariant part under the action of μ_m is denoted by V_1 and $V_{\neq 1}$, respectively. The action of μ_m preserves the space $H^{p,q}$, where

$$H^{p,q}=F^pGr_k^W(H^i(f^{-1}(\delta),\mathbf{C}))\cap\bar{F}^qGr_k^W(H^i(f^{-1}(\delta),\mathbf{C}))$$

and χ-part of $H^{p,q}$ is denoted by $H^{p,q}(\chi)$, for a finite character χ of $\pi_1(B^0(t),\delta)$. The set of natural numbers $\{h^{p,q}(\chi)\}$, where $h^{p,q}(\chi)=\dim H^{p,q}(\chi)$ is called the spectral pair of f and it is an important invariant for singularities. The Hodge analog of Thom-Sebastiani theorem is stated as follows. Let d_1 and d_2 be the exponent of f and g respectively and m be the l.c.m. of d_1 and d_2. On the mixed Hodge structure $V(d_1,d_2)=H^1(\{\sigma^{d_1}+\tau^{d_2}=\delta\},\mathbf{Q})$ the group $\mu_{d_1}\times\mu_{d_2}\times\pi_1(B(u)-\{0\},\delta)$ acts and the exponent is equal to $m=\mathrm{lcm}(d_1,d_2)$. Therefore $V(d_1,d_2)$ defines an element $[V(d_1,d_2)]$ in $K_{MH}(\mathbf{C},\mu_{d_1}\times\mu_{d_2}\times\mu_m)$.

Theorem 0.1 ([V], [S], [D-L]). *Under the above notation, the following equality holds in the Grothendieck group $K_{MH}(\mathbf{C},\mu_m)$.*

$$\begin{aligned}&[H^{n+m-1}((f+g)^{-1}(\delta),\mathbf{Q})]\\ =&-([V(d_1,d_2)]\otimes[H^{n-1}(f^{-1}(\delta),\mathbf{Q})]\otimes[H^{m-1}(g^{-1}(\delta),\mathbf{Q})])^{\mu_{d_1}\times\mu_{d_2}}\\ &+[H^{n-1}(f^{-1}(\delta),\mathbf{Q})(\chi_1)]_{\neq 1}\otimes[H^{m-1}(g^{-1}(\delta),\mathbf{Q})(\chi_2)]_{\neq 1}\\ &-[H^{n-1}(f^{-1}(\delta),\mathbf{Q})(\chi_1)]\otimes[H^{m-1}(g^{-1}(\delta),\mathbf{Q})(\chi_2)].\end{aligned}$$

This theorem is generalized by Nemethi-Steenbrink [N-S] as follows. Let $h(s,t)$ be a function of two variables with an isolated singularity. The composite $h\circ(f,g)$ of $(f,g):B(x)\times B(y)\to B(s)\times B(t)$ and $h:B(s)\times B(t)\to B(u)$ has singularities and it is called a composite singularity of two variables. The spectral pair of the composite singularity of two variables is computed by those of f, g and h in [N-S]. Note that if $h(s,t)=s+t$, then their theorem is nothing but Theorem 0.1. In this paper, we generalize this theorem for composite singularities of several variables. We explain the situation of our result.

Let n be a natural number greater than 1 and $g_i(y_i)=g_i(y_{i1},\dots,y_{im_i})$ be germs of holomorphic functions of m_i variables for $1\le i\le n$. Then $x_i=g_i(y_i)$ defines a local morphism $g_i:B(y_i)\to B(x_i)$. Suppose that the restriction of g_i to $g_i^{-1}(B(x_i)-\{0\})$ is smooth. Let f be a germ of holomorphic function on $B(x)=\{(x_1,\dots,x_n)\mid\mid x_i\mid<\delta(1\le i\le n)\}$ non-degenerate with respect to the Newton boundary. (See Section 1. for the definition of the non-degeneracy condition.) Roughly speaking, our main theorem (Theorem 3.6.1) gives the relation between the spectral pair of the composite singularity $f(g_1(y_1),\dots,g_n(y_n))$ and those of $g_1,\dots,g_n$ and f. This theorem is called the convolution theorem for composite singularities. Section 3 is devoted to prove this main theorem.

In the paper of [D-L] they proved the motivic version of Thom-Sebastiani theorem under the assumption of tameness and the existence of the resolution of singularities. Their theorem, Motivic Thom-Sebastiani theorem implies the l-adic analog of Thom-Sebastiani theorem. We prove the l-adic analog of the convolution theorem for composite singularities under the assumption of tameness condition in Section 2. Although the l-adic analog is not a deep theorem, it is worth giving a proof in this paper, because from this result, one can easily expect the formula for the spectral pairs for composite singularities.

§1 Non degenerate maps and reducing multiplicities

§1.1 Toric geometry

Let k be a finite field $\mathbf{F}_q$ of q elements or the complex number field $\mathbf{C}$ and $n \geq 1$. Let $f = f(x_1, \ldots, x_n) \in k[[x_1, \ldots, x_n]]$ be a formal power series on $x_1, \ldots, x_n$ over k such that $f(0) = 0$. We assume that for all $i = 1, \ldots, n$, the coefficient of $x_i^{m_i}$ is not zero for some m_i. The Newton polygon $\Delta = \Delta(f)$ of $f = \sum_{w \in \mathbf{N}^n} a_w x^w$ is defined by the convex hull of $\{w \in \mathbf{N}^n \mid a_w \neq 0\}$ and $\mathbf{R}^{(i)}_{\geq N} = \{(0, \ldots, \overset{i}{r}, \ldots, 0) \mid r \geq N\}$ for a sufficiently large number N. First we recall the notion of non-degeneracy with respect to the Newton boundary. Let σ be a compact face of Δ, L_σ be the affine linear hull of σ in $\mathbf{R}^n$ and $L_{\sigma,0}$ be the linear subspace of $\mathbf{R}^n$ (containing 0) parallel to L_σ. We define f_σ by $f_\sigma = \sum_{w \in \sigma} a_w x^w$. If we choose $w_0 \in \sigma$, then $f_{\sigma,0} = f_\sigma / x^{w_0} \in k[L_{\sigma,0} \cap \mathbf{Z}^n]$. If σ is non-compact face, there exists a non-empty subset J of $[1, n]$ such that σ is equal to $\sum_{j \in J} \mathbf{R}_+ \mathbf{e}_j$ outside of a compact subset. The scheme $Spec(k[[x_j]]_{j \in J}[x_j^{-1}]_{j \in J})$ is denoted by B^0_σ. The function f is said to be non-degenerate with respect to a compact face σ (resp. a non-compact face σ) if $\{f_{\sigma,0} = 0\}$ is a smooth variety on $Z_\sigma = Spec(k[L_{\sigma,0} \cap \mathbf{Z}^n])$ (resp. B^0_σ). The series f is said to be non-degenerate with respect to the Newton boundary if it is non-degenerate with respect to all the faces σ of Δ. The closed subschemes $\{x_i = 0\}$ and $\{f = 0\}$ in $B(x) = Spec(k[[x_1, \ldots, x_n]])$ are denoted by $Z(x_i)$ and $Z(f)$, respectively. If f is non-degenerate with respect to the Newton boundary, we can construct a partial resolution of the singularity of the hypersurface $Z(f) \cup \cup_{i=1}^n Z(x_i)$, using toric geometry. Let us define the dual fan $F(\Delta)$ of Δ as the set $\{\sigma^*(x) \mid x \in \Delta\}$, where $\sigma^*(x) = \{l \in (\mathbf{R}^n)^* \mid l(\sum_{y \in \Delta} \mathbf{R}_+(y - x)) \geq 0\}$. As in [Od], we can associate a toric variety X_F for a fan F. Note that the toric variety $X_{F(\Delta)}$ associated to $F(\Delta)$ is isomorphic to $\mathbf{P}_\Delta$ defined in [Dan]. The dual coordinate of $(\mathbf{R}^n)^*$ is written by $w^* = (w_1^*, \ldots, w_n^*)$. Let F be a simplicial refinement of $F(\Delta)$ such that

(1.1.1)

$$\text{the restriction of } F \text{ to the hyperplane } \{w_i^* = 0\} \text{ consists of } \sum_{j \in J} \mathbf{R}_+ e_j \text{ with } i \notin J \subset [1, n].$$

The natural morphism from X_F to the affine space $\mathbf{A}^n(x) = Spec(k[x_1, \ldots, x_n])$ is denoted by $\pi : X_F \to \mathbf{A}^n(x)$. The base change of π by the morphism $B(x) = Spec(k[[x_1, \ldots, x_n]]) \to \mathbf{A}^n(x)$ is denoted by $b(x) : \hat{B}(x) \to B(x)$.

Now we recall the definitions of quasi-smooth varieties and quasi-normal crossing divisors. (See [St].) An integral variety V is called quasi-smooth at a point $p \in V$ if there exists a local parameter $x_1, \ldots, x_n$ at p with the inclusion $\mathcal{O}_{V,p} \subset \mathcal{O}_{V,p}[\xi_1, \ldots, \xi_n]$, where $\xi_1^{d_i} = x_1, \ldots, \xi_1^{d_i} = x_1$ such that (1) $\mathcal{O}_{V,p}[\xi_1, \ldots, \xi_n]$ is smooth with the regular parameters $\xi_1, \ldots, \xi_n$, and (2) $\mathcal{O}_{V,p}$ is identified with the invariant ring of $\mathcal{O}_{V,p}[\xi_1, \ldots, \xi_n]$ under the action of a subgroup G of $\mu_{d_1} \times \cdots \times \mu_{d_n}$, where the action of $\mu_{d_1} \times \cdots \times \mu_{d_n}$ is given by the multiplication of coordinate $(\xi_1, \ldots, \xi_n)$. If V is quasi-smooth at all $p \in V$, V is called quasi-smooth variety. A divisor D in V is called a quasi-normal crossing divisor at $p \in D$ if it is the image of $\xi_1 \cdots \xi_s$ $0 < s \leq n$ under the projection $Spec(\mathcal{O}_{V,p}[\xi_1, \ldots, \xi_n]) \to Spec(\mathcal{O}_{V,p})$. The divisor is called quasi-normal corssing if it is quasi-normal crossing at all $p \in D$.

Then it is known that X_F and $\hat{B}(x)$ are quasi-smooth, and the reduced part of $(b(x))^{-1}(Z(f) \cup \cup_{i=1}^n Z(x_i))$ is a quasi-normal crossing divisor.

Now we consider a covering $B(\xi)$ of $B(x)$. Let $d_1, \dots, d_n$ be natural numbers such that $\mu_{d_i} \subset k^\times$. We introduce a system of coordinates $\xi_1, \dots, \xi_n$ with $\xi_1^{d_1} = x_1, \dots, \xi_n^{d_n} = x_n$. Then $B(\xi) = Spec(k[[\xi_1, \dots, \xi_n]])$ is a finite Galois covering of $B(x)$ with Galois group $G = \mu_{d_1} \times \cdots \times \mu_{d_n}$. By the base chage of $\hat{B}(x) \to B(x)$ by $B(\xi) \to B(x)$, we get a morphism $b(\xi) : \hat{B}(\xi) \to B(\xi)$. The power series f defines a closed subscheme $Z(f(\xi))$ of $B(\xi)$. Then $\hat{B}(\xi)$ is quasi-smooth and the reduced part of $(b(\xi))^{-1}(Z(f(\xi)) \cup \cup_{i=1}^n Z(\xi_i))$ is a quasi-normal crossing divisor:

$$((b(\xi))^{-1}(Z(f(\xi)) \cup \cup_{i=1}^n Z(\xi_i)))_{red} = \cup_{i=0}^{s+n} D_i,$$

where D_0 and $D_{s+1}, \dots, D_{s+n}$ are proper transforms of $Z(f(\xi))$ and $Z(\xi_1), \dots, Z(\xi_n)$ respectively and $D_1, \dots, D_s$ correspond to 1-dimensional cones $r_1, \dots r_s$ of F different from $\mathbf{R}_+ e_i^*$ $(i = 1, \dots, n)$. Let $B(t) = Spec(k[[t]])$. The morphism $B(x) \to B(t)$ (resp. $B(\xi) \to B(t)$) defined by $t = f(x_1, \dots, x_n)$ (resp. $t = f(\xi_1^{d_1}, \dots, \xi_n^{d_n})$) is denoted by $f(x)$ (resp. $f(\xi)$).

$$\begin{array}{ccc} \hat{B}(\xi) & \longrightarrow & \hat{B}(x) \\ {\scriptstyle b(\xi)}\downarrow & & \downarrow{\scriptstyle b(x)} \\ B(\xi) & \longrightarrow & B(x) \\ {\scriptstyle f(\xi)}\downarrow & & \downarrow{\scriptstyle f(x)} \\ B(t) & \xrightarrow{=} & B(t) \end{array}$$

Let $\hat{f}(\xi) = f(\xi) \circ b(\xi)$. The multiplicities of $D_1, \dots, D_s$ in $b(\xi)^{-1}(Z(f(\xi))) = (\hat{f}(\xi))^{-1}(0)$ are computed as follows. Let $L(x) = \mathbf{Z}^n$ and $L(\xi)$ be the lattice defined by $\frac{1}{d_1}\mathbf{Z} \oplus \cdots \oplus \frac{1}{d_n}\mathbf{Z}$. The primitive generator of r_i with respect to the dual lattice $L(\xi)^*$ is denoted by l_i. The multiplicity m_i of D_i in $(b(\xi))^{-1}(Z(f(\xi)))$ is equal to

$$m_i = \min\{l_i(x) \mid x \in \Delta\}.$$

Definition 1.1.1. Let p be the characteristic of k and $d_1, \dots, d_n$ be natural numbers prime to p. If there exists a simplicial refinement F of $F(\Delta)$ with the property (1.1.1) such that $m = \mathrm{lcm}(m_i)$ is prime to p, the n-tuple of natural numbers $(d_1, \dots, d_n)$ is said to be tame with respect to f. The number m is called the exponent of $(d_1, \dots, d_n)$ with respect to f. (We fix the simplicial refinement F of $F(\Delta)$ once and for all.)

§1.2 Reduction of multiplicity

We reduce the multiplicity of the special fiber of $f(\xi)$ by the base change by the morphism $B(\tau) \to B(t)$. We use the same notations of §1.1. Let m be $\mathrm{lcm}(d_i)$ and $B(\tau) = Spec(k[[\tau]])$, where $\tau^m = t$. We assume $(m, p) = 1$, i.e. $(d_1, \dots, d_n)$ is tame with respect to f. We construct a variety $\tilde{B}(\xi)$ over $B(\tau)$ with the following commutative diagram:

$$\begin{array}{ccc} \tilde{B}(\xi) & \xrightarrow{\pi} & \hat{B}(\xi) \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle \hat{f}} \\ B(\tau) & \longrightarrow & B(t) \end{array}$$

such that (1) $\tilde{f}^{-1}(0)$ is reduced and quasi normal crossing and (2) π is a finite Galois covering with a Galois group μ_m.

Let F be a simplicial refinement of $F(\Delta)$. First we construct the suspension $\tilde{F}$ of F, which is a fan in $(\mathbf{R}^{n+1})^*$. The polar dual Δ^* of the Newton polyhedron Δ is defined by

$$\Delta^* = \{x^* \in (\mathbf{R}^n)^* \mid x^*(\Delta) \geq 1\}$$

Since F is a refinement of $F(\Delta)$, F defines a decomposition $\Delta^* = \coprod_{\sigma^* \in F} \sigma^* \cap \Delta^*$ of Δ^*. We define lower cone $L(\sigma^*)$, upper cone $U(\sigma^*)$ of σ^* and boundary cone $B(\sigma^*)$ as

$$L(\sigma^*) = \mathbf{R}_+((\sigma^* \cap \Delta^*) \times [0, \frac{1}{m}]), U(\sigma^*) = (\sigma^* \times [0, \infty] - L(\sigma^*))^-,$$
$$B(\sigma^*) = \mathbf{R}_+((\partial\Delta^* \cap \sigma^*), \frac{1}{m})$$

The base cone B is defined by $B = (\mathbf{R}^n)^* - \cup_{\sigma^* \in F} U(\sigma^*)$. We define the suspension $\tilde{F}$ of F as

$$\tilde{F} = \{U(\sigma^*)\}_{\sigma^* \in F} \cup \{B(\sigma^*)\}_{\sigma^* \in F} \cup \{B\}.$$

It is easy to see that if σ^* is a simplicial cone, $U(\sigma^*)$ and $B(\sigma^*)$ are also simplicial cones. Even though if F is a simplicial fan, B may not be simplicial cone. Since the suspension $\tilde{F}(\Delta)$ of $F(\Delta)$ is the dual fan $F(\tilde{\Delta})$ of the suspension $\tilde{\Delta}$, where $\tilde{\Delta}$ is the convex hull of $0 \times \mathbf{R}_{\geq m}$ and $\Delta \times 0$. Therefore, in general, $\tilde{F}$ is a refinement of $F(\tilde{\Delta})$. Now we consider an element $f(\xi) - \tau^m = f(\xi_1^{d_1}, \ldots, \xi_n^{d_n}) - \tau^m$ in $k[[\xi_1, \ldots, \xi_n, \tau]]$. Let $B(\xi, \tau) = Spec(k[[\xi_1, \ldots, \xi_n, \tau]])$. The fiber product of $X_{\tilde{F}} \to \mathbf{A}(\xi, \tau)$ and $B(\xi, \tau) \to \mathbf{A}(\xi, \tau)$ is denoted by $\bar{B}(\xi)$. Since $\tilde{\Delta}$ is the Newton polygon of $f(\xi) - \tau^m$, $f(\xi) - \tau^m$ defines a hypersurface $\tilde{B}(\xi) = Z(f(\xi) - \tau^m)$ in $\bar{B}(\xi)$. The composite of $\tilde{B}(\xi) \to \bar{B}(\xi)$, $\bar{B}(\xi) \to B(\xi, \tau)$ and $B(\xi, \tau) \to B(\tau)$ is denoted by $\tilde{f}(\xi)$.

Proposition 1.2.1. *The variety $Z(f(\xi) - u^m)$ does not contain the point corresponding to B in $\bar{B}(\xi)$ and quasi-smooth. The (total) fiber $(\tilde{f}(\xi))^{-1}(0)$ of $\tilde{f}$ at 0 is reduced and quasi-normal crossing.*

§2 Tame l-adic sheaves and convolution theorem.

§2.1 Grothendieck groups of etale sheaves.

In this section, we assume that k is a finite field $\mathbf{F}_q$ and d be an integer such that $d \mid q - 1$. The Grothendieck group of etale $\mathbf{Q}_l$-sheaves on $B(x) = Spec(k[[x]])$, $B^0(x) = Spec(k((x)))$ and $Spec(k)$ is denoted by $K(B(x))$, $K(B^0(x))$ and $K(k)$, respectively. The geometric generic point of $B^0(x)$ is denoted by $\bar{x}$. A sheaf F on $B^0(x)$ corresponds to the continuous representation $(\rho_F, F_{\bar{x}})$ of $\mathrm{Gal}(\overline{k((x))}/k((x)))$. By Grothendieck's theorem, there exists an open subgroup H in the inertia group I of $\mathrm{Gal}(\overline{k((x))}/k((x)))$ which acts on $F_{\bar{x}}$ unipotently. If there exists a subgroup H such that the index $[I : H]$ is prime to p, it is called tame. The Grothendieck group of tame sheaves on $B^0(x)$ is denoted by $K(B^0(x))^t$. For a tame sheaf F, $\min\{[I : H] \mid ([I : H], p) = 1, \text{ and } \rho_F\mid_H \text{ is unipotent }\}$ is called the exponent of F. The Grothendieck group of tame sheaves on $B^0(x)$ whose exponents divide d is denoted by $K(B^0(x))^{t,d}$. Note that $K(B^0(x))^{t,1}$ is nothing but the Grothendieck group of etale sheaves on $B^0(x)$ whose inertia action is unipotent. Let us fix a generator γ of the tame quotient $\hat{\mathbf{Z}}(1)' = \prod_{(p,l)=1} \bar{\mathbf{Z}}_l(1)$ of I. Then the logarithm $N = \log \rho_F(\gamma)$ of $\rho_F(\gamma)$ acts on $F_{\bar{x}}$ nilpotently. The action of N on $F_{\bar{x}}$ defines a monodromy filtration $W_k(F_{\bar{x}})$ on $F_{\bar{x}}$ satisfying $NW_k(F_{\bar{x}}) \subset W_{k-2}(F_{\bar{x}})$. It is characterized by the property:

> The k-th iteration $N^k : Gr_k^W(F_{\bar{x}}) \to Gr_{-k}^W(F_{\bar{x}})$ of the homomorphism induced by N is an isomorphism.

If we define the primitive part P_k as the kernel of the homomorphism $N^{k+1} : Gr_k^W(F_{\bar{x}}) \to Gr_{-k-2}^W(F_{\bar{x}})$, then F is equal to $\sum_{k\geq 0}\sum_{0\leq i\leq k} N^i P_k$ in $K(B^0(x))^{t,1}$. Since $N_i P_k \simeq P_k \otimes \mathbf{Q}_l(i)$ $(0 \leq i \leq k)$ corresponds to an unramified representation of $\mathrm{Gal}(\overline{k((x))}/k((x)))$, we have the following Lemma.

Lemma 2.1.1. *The group $K(B^0(x))^{t,1}$ is isomorphic to the group $K(k)$*

Let $\Pi_x^t = \mathrm{Gal}((\cup_{(d,p)=1} k((x^{\frac{1}{d}}))) \otimes \bar{k}/k((x)))$ be the tame quotient of the absolute Galois group of $k((x))$. Then we have the exact sequence:

$$1 \to \hat{\mathbf{Z}}(1)' \to \Pi_x^t \to \hat{\mathbf{Z}} \to 1.$$

The Kummer extension $k((\tau))$, where $\tau^d = x$, defines a character $\Pi_x^t \to \mu_d$ of Π_x^t. The kernel of $\Pi_x^t \to \mu_d \times \hat{\mathbf{Z}}$ can be identified with $d\hat{\mathbf{Z}}(1)'$. We fix a generator γ of $d\hat{\mathbf{Z}}(1)'$. For a tame etale sheaf F, on $B^0(x)$, whose exponent is divisible by d, the action of $d\hat{\mathbf{Z}}(1)'$ is unipotent. Let N be the logarithm of $\rho_F(\gamma)$. For any element $\bar{h} \in \mu_d \times \hat{\mathbf{Z}}$ take a lifting of $h \in \Pi_x^t$. Then we have $h\gamma h^{-1} = \gamma^{c(h)}$, $c(h) \in (\hat{\mathbf{Z}}')^\times$ and as a consequence, we have $hN = c(h)Nh$. If we define a filtration $W_i' = hW_i$, then W_i' is stable under the action of N and the following diagram is commutative.

$$\begin{array}{ccc} W_i/W_{i-1} & \xrightarrow{N} & W_{i-2}/W_{i-3} \\ {\scriptstyle c(h)\cdot h}\downarrow & & \downarrow{\scriptstyle h} \\ W_i'/W_{i-1}' & \xrightarrow{N} & W_{i-2}'/W_{i-3}' \end{array}$$

By the characterization of W_i, we have $W_i' = W_i$. Therefore h induces a homomorphism on $Gr_i^W F_{\bar{x}}$. Since N acts trivially on $Gr_i^W F_{\bar{x}}$, the action of Π_x^t factors through $\mu_d \times \hat{\mathbf{Z}}$. As a consequence, $K(B^0(x))^{t,d}$ is equal to the Grothendieck group of continuous representations of $\mu_d \times \hat{\mathbf{Z}}$. Let $K(k, \mu_d)$ be the Grothendieck group of etale sheaf on $Spec(k)$ with a μ_d action. Therefore we have the following lemma.

Lemma 2.1.2. *The group $K(B^0(x))^{t,d}$ is isomorphic to the group $K(k, \mu_d)$.*

We introduce a notion of equivariant sheaf. Let $f : X \to S$ be a scheme over S and G be a finite group acting on Y over S. The action of G on Y is denoted by $\sigma : G \to \mathrm{Aut}(X/S)$. Let $\mathcal{F}$ be a sheaf on X. A descent data for $\mathcal{F}$ is a set of sheaf homomorphisms $\phi_g : g^*\mathcal{F} \to \mathcal{F}$ indexed by elements of G satisfying the 1-cocycle condition $\phi_{gh} = h^*\phi_g \circ \phi_h$. The descent data is called effective if for any $x \in X$ and $g \in G$ such that $g(x) = x$, the fiber of the descent data $\phi_g \mid_x$: $(g^*\mathcal{F})_x = \mathcal{F}_{g(x)} \to \mathcal{F}_x$ is the identity map. (Note that this terminology is different from usual one.) A sheaf on X with an effective descent data is called (G, σ)-sheaf on X. The Grothendieck group of (G,σ)-sheaf on X is denoted by $K(X/S,(G,\sigma))$. Let $\mathcal{F}$ be a (X,σ)-sheaf on X. Then the higher direct image sheaves $R^i f_*\mathcal{F}$ is a sheaf on S with the action of G. Suppose that there exists a quotient scheme $Y = X/G$ of X under the action of G. The structure morphism $Y \to S$ is denoted by g and the natural map $X \to Y$ is denoted by π. For an etale sheaf $\mathcal{G}$, the pull back $\pi^*\mathcal{G}$ of $\mathcal{G}$ by the morphism π has a natural effective descent data. It is known that $(R^i f_*(\pi^*\mathcal{G}))^G \simeq R^i g_*\mathcal{G}$.

Lemma 2.1.3. *The group $K(B^0(x))^{t,d}$ is isomorphic to the Grothendieck group $K(B^0(\tau),(\mu_d,\sigma))$ of (μ_d,σ)-sheaf on $B^0(\tau)$.*

Remark 2.1.4. This isomorphism depends on the choice of the uniformizer x of $k((x))$ which is always fixed in our context.

Let F be an etale sheaf on $B(x)$. The sheaf F is said to be tame if the restriction to $B^0(x)$ is tame. For a tame sheaf F, the exponent of F is defined by that of the restriction to $B^0(x)$. The Grothendieck group of tame sheaf and tame sheaf whose exponent divides d is denoted by $K(B(x))^t$ and $K(B(x))^{t,d}$, respectively. For an etale sheaf F on $B(x)$, the generic geometric fiber and special geometric fiber are denoted by $F_{\bar{x}}$ and $F_{\bar{0}}$ respectively. The specialization map is a $Gal(\bar{k}/k)$-equivariant map

$$sp_F : F_{\bar{0}} \to F_{\bar{x}}^I, \tag{1}$$

where I is the inertia group of $\mathrm{Gal}(\overline{k((x))}/k((x)))$. To give an etale sheaf on $B(x)$ is equivalent to give a triple $(F_{\bar{x}}, F_{\bar{0}}, sp_F)$, where $F_{\bar{x}}$ and $F_{\bar{0}}$ are $\mathrm{Gal}(\overline{k((x))}/k((x)))$-module and $\mathrm{Gal}(\bar{k}/k)$-module respectively and sp_F is a $\mathrm{Gal}(\bar{k}/k)$-equivariant homomorphism (1). By considering exact sequence,

$$0 \to j_!j^*F \to F \to i_*i^*F \to 0,$$

we have $[F] = [j_!j^*F] + [i_*i^*F]$ in $K(B(x))$. Since $(j_!j^*F)_{\bar{0}} = 0$ and $(i_*i^*F)_{\bar{x}} = 0$, we have

$$\begin{aligned}
&K(B(x)) = K(B^0(x)) \oplus K(k), K(B(x))^t = K(B^0(x))^t \oplus K(k),\\
&K(B(x))^{t,d} = K(B^0(x))^{t,d} \oplus K(k).
\end{aligned}$$

Now we consider several Grothendieck groups of etale sheaves on $B(x) = Spec(k[[x_1,\dots,x_n]])$. For a subset I of $[1,n]$, we define a closed subscheme $B_I(x) = Spec(k[[x_1,\dots,x_n]]/(x_i)_{i\in I})$ and locally closed subscheme $B_I^0(x) = B_I(x) - \cup_{J \supsetneq I} B_J(x)$. A sheaf is said to be stratified by coordinate if the restriction to B_I^0 is locally constant for all I. An etale sheaf F on $B(x)$ stratified by coordinate is called tame if the monodromy of the restriction of F to B_I^0 is tame. The Grothendieck group of tame etale sheaves on $B(x)$ stratified by coordinate is denoted by $K_c(B(x))^t$. The tame fundamental group of $B_I^0(x)$ is denoted by Π_I^t. Then we have the following exact sequence.

$$1 \to (\hat{\mathbf{Z}}(1)')^{I^c} \to \Pi_I^t \to \hat{\mathbf{Z}} \to 1$$

Let $I \subset [1,n]$ and $i \notin I$. We set $J = I \cup i$ and fix a generator e_i of geometric monodromy along the divisor $x_i = 0$. Then we have $1 \to \hat{\mathbf{Z}}(1)'e_i \to \Pi_I^t \to \Pi_J^t \to 1$.

The Grothendieck group of tame etale $\mathbf{Q}_l$-sheaves stratified by coordinate whose exponent for e_i divides d_i is denoted by $K_c(B(x))^{t,d_1,\dots,d_n}$. The generic geometric point of $B_I^0(x)$ is denoted by $\bar{x}_I$. The specialization map with respect to I and J is a Π_J^t-equivariant map $sp_{I,J} : F_{\bar{x}_J} \to F_{\bar{x}_I}^{e_i}$. Let $K(\Pi_I^t)$ be the Grothendieck group of continuous representations of Π_I^t. By the same argument as in the 1-variable case, we have

$$K_c(B(x))^t = \oplus_{I\subset[1,n]} K(\Pi_I^t).$$

If $F_1, \dots, F_n$ are tame sheaves on $B(x_1), \dots, B(x_n)$, then the exterior product $F_1 \boxtimes \cdots \boxtimes F_n = pr_1^* F_1 \otimes \cdots \otimes pr_n^* F_n$ is a tame sheaf on $B(x)$ stratified by coordinate.

Lemma 2.1.5. *By attaching $[F_1] \otimes \cdots \otimes [F_n]$ to $F_1 \boxtimes \cdots \boxtimes F_n$, we get a homomorphism*

$$\begin{aligned} &K(B(x_1))^t \otimes \cdots \otimes K(B(x_n))^t \to K_c(B(x))^t \\ &K(B(x_1))^{t,d_1} \otimes \cdots \otimes K(B(x_n))^{t,d_n} \to K_c(B(x))^{t,d_1,\dots,d_n}. \end{aligned}$$

Proof. It is enough to prove that if $0 \to F_1' \to F_1 \to F_1'' \to 0$ is an exact sequence of etale sheaves on $B(x_1)$,

$$0 \to F_1' \boxtimes F_2 \boxtimes \cdots \boxtimes F_n \to F_1 \boxtimes F_2 \boxtimes \cdots \boxtimes F_n \to F_1'' \boxtimes F_2 \boxtimes \cdots \boxtimes F_n \to 0$$

is an exact sequence of tame etale sheaves on $B(x)$. It is clear by the description of tame etale sheaves on $B(x)$ stratified by coordinate.

Proposition 2.1.6.

1. $K_c(B(\xi), (G, \sigma_\xi))^{t,1} \simeq \oplus_{I\subset[1,n]} K(B_I^0(\xi), (G_I, \sigma_I))^{t,1}$, *where G_I is the image of G in $\mathrm{Aut}(B_I^0(\xi))$ and σ_I be the action of G_I on $B_I^0(\xi)$, i.e. $G_I = \prod_{i\notin I} \mu_{d_i}$.*
2. $K_c(B_I^0(\xi), (G_I, \sigma_I))^{t,1} \simeq K(k, G_I)$.

§2.2 Vanishing cycle functors and convolution theorem

Let I be a subset of $[1, n]$. Then the closed subscheme $B_I(x)$ of $B(x)$ is defined by $B_I(x) = \{x \in B(x) \mid x_i = 0 \quad (i \in I)\}$. Note that $B_\emptyset(x) = B(x)$. Let $[F_1] \otimes \cdots \otimes [F_n]$ be elements of $K(B(x_1))^t \otimes \cdots \otimes K(B(x_n))^t$, respectively. We define a homomorphism $\phi_I : K(B(x_1))^{t,d_1} \otimes \cdots \otimes K(B(x_n))^{t,d_n} \to K(B^0(t))$ as

$$\phi_I([F_1] \otimes \cdots \otimes [F_n]) = [j_t^*(\mathbf{R}f(x)_* i_{I*} i_I^*(F_1 \boxtimes \cdots \boxtimes F_n))],$$

where $j_t : B^0(t) \to B(t)$ and $i_I : B_I(x) \to B(x)$ be the natural inclusions. Let $B(\xi_i) \to B(x_i)$ be the μ_{d_i}-covering defined in §1.1 and G be the group $\mu_{d_1} \times \cdots \mu_{d_n}$ which acts on $B(\xi)$. The action is denoted be σ_ξ. We consider the equivariant version

$$\tilde{\phi}_I : K(B(\xi_1), (\mu_{d_1}, \sigma_1))^t \otimes \cdots \otimes K(B(\xi_n), (\mu_{d_n}, \sigma_n))^t \to K(B^0(t), G),$$

of ϕ_I by

$$\tilde{\phi}_I([\tilde{F}_1] \otimes \cdots \otimes [\tilde{F}_n]) = [j_t^* \mathbf{R}f(\xi)_* i_{I*} i_I^*(\tilde{F}_1 \boxtimes \cdots \boxtimes \tilde{F}_n)],$$

for an element $[\tilde{F}_1]\otimes\cdots\otimes[\tilde{F}_n]\in K(B(\xi_1),(\mu_{d_1},\sigma_1))^t\otimes\cdots\otimes K(B(\xi_n),(\mu_{d_n},\sigma_n))^t$. Then we have the following commutative diagram.

$$
\begin{tikzcd}
K(B(x_1))^{t,d_1}\otimes\cdots\otimes K(B(x_n))^{t,d_n} \arrow[r, "\phi_I"] \arrow[d] & K(B^0(t)) \\
K(B(\xi_1),(\mu_{d_1},\sigma_1))^{t,1}\otimes\cdots\otimes K(B(\xi_n),(\mu_{d_n},\sigma_n))^{t,1} \arrow[r, "\tilde{\phi}_I"'] & K(B^0(t),G) \arrow[u, "G\text{-invariant}"']
\end{tikzcd}
\tag{2.2.1}
$$

First we prove the following proposition.

Proposition 2.2.1. *The image of ϕ_I is contained in $K(B^0(t))^{t,m}$, where m is the exponent of $(d_1,\ldots,d_n)$ with respect to f.*

Proof. We consider the following commutative diagram:

$$
\begin{tikzcd}
\tilde{B}(\xi) \arrow[r, "\nu"] \arrow[d, "\tilde{f}(\xi)"'] & \hat{B}(\xi) \arrow[r, "b"] \arrow[d, "\hat{f}(\xi)"] & B(\xi) \\
B(\tau) \arrow[r, "\nu'"'] & B(t).
\end{tikzcd}
$$

For tame etale sheaves $F_1,\ldots,F_n$ on $B(x_1),\ldots,B(x_n)$ whose exponent divides $d_1,\ldots,d_n$, the sheaf $\pi_1^*F_1\boxtimes\cdots\boxtimes\pi_n^*F_n$ are (G,σ_ξ)-sheaf on $B(\xi)$ with unipotent monodromy. Therefore there exists a filtration F^i of (G,σ_ξ)-sheaves on $\pi_1^*F_1\boxtimes\cdots\boxtimes\pi_n^*F_n$ such that F^i/F^{i+1} has trivial monodromy. Therefore the subquotient $(b\circ\nu)^*F^i/(b\circ\nu)^*F^{i+1}$ of $(b\circ\nu)^*(\pi_1^*F_1\boxtimes\cdots\boxtimes\pi_n^*F_n)$ has also trivial monodromy. Using weight spectral sequence for $\tilde{f}$, $j_\tau^*R^i\tilde{f}(\xi)_*((b\circ\nu)^*(\pi_1^*F_1\boxtimes\cdots\boxtimes\pi_n^*F_n))$ has unipotent monodromy. Since

$$j_\tau^*R^i\tilde{f}(\xi)_*((b\circ\nu)^*(\pi_1^*F_1\boxtimes\cdots\boxtimes\pi_n^*F_n))\simeq(\nu')^*j_t^*R^if(\xi)_*b^*(\pi_1^*F_1\boxtimes\cdots\boxtimes\pi_n^*F_n),$$

and the commutative diagram (2.2.1), we have

$$[j_t^*R^if(x)_*(F_1\boxtimes\cdots\boxtimes F_n)]=[j_t^*R^if(\xi)_*b^*(\pi_1^*F_1\boxtimes\cdots\boxtimes\pi_n^*F_n)]^G\in K(B^0(t))^{t,m}$$

Definition 2.2.2. Put $F=[F_1]\otimes\cdots\otimes[F_n]$.

(1)
$$\Phi_f(F)=\phi_\emptyset(F)-[(F_1\boxtimes\cdots\boxtimes F_n)_{\bar{0}}]\in K(k,\mu_m).$$

Here we use the identification, $\phi_\emptyset(F)\in K(B(t)^0)^{t,m}\simeq K(k,\mu_m)$ and $[(F_1\boxtimes\cdots\boxtimes F_n)_{\bar{0}}]\in K(k)=K(k,1)\subset K(k,\mu_m)$.

(2)
$$\begin{aligned}
&\Phi([F_i])=[j_{x_i}^*F_i]-[F_{i,\bar{0}}]\in K(k,\mu_{d_i})\text{ and}\\
&\tilde{\chi}_I(F)=\otimes_{i\notin I}[F_{i,\bar{x}_i}]\otimes\otimes_{i\in I}\Phi([F_i])\in K(k,G),\\
&\chi_I(F)=\otimes_{i\notin I}[F_{i,\bar{x}_i}]\otimes\otimes_{i\in I}\Phi([F_i])^{\mu_{d_i}}\in K(k,G_I),
\end{aligned}$$

(3)
$$\begin{aligned}
\tilde{\Phi}_I(F)=&\tilde{\phi}_I([\pi_1^*F_1]\otimes\cdots\otimes[\pi_n^*F_n])-[(F_1\boxtimes\cdots\boxtimes F_n)_{\bar{0}}]\\
&\in K(B^0(t),(G_I,\sigma_I))^{t,m}\simeq K(k,G_I\times\mu_m),
\end{aligned}$$

and $\tilde{\Phi}_I=\tilde{\Phi}_I([\mathbf{Q}_l]\otimes\cdots\otimes[\mathbf{Q}_l])$.

Theorem 2.2.3 (l-adic convolution theorem). *Let $F = [F_1] \otimes \cdots \otimes [F_n]$ in $K(B(x_1))^{t,d_1} \otimes \cdots \otimes K(B(x_n))^{t,d_n}$. Under the notation defined as above,*

$$\Phi_f(F) = \sum_{I \subset [1,n]} (-1)^{\# I} (\tilde{\Phi}_I \otimes \chi_I(F))^{G_I} \in K(\mu_m, k).$$

Proof. Since the action of G on $\tilde{\Phi}_I$ factors through G_I, it is enough to show that

$$\tilde{\Phi}_f(F) = \sum_{I \subset [1,n]} (-1)^{\# I} (\tilde{\Phi}_I \otimes \tilde{\chi}_I(F)) \in K(G \times \mu_m, k),$$

by the commutative diagram (2.2.1). Let $F_B = [\pi_1^* F_1 \boxtimes \cdots \boxtimes \pi_n^* F_n]$. Using decomposition $F_B = \sum_{I \subset [1,n]} j_{I!}(F_B \mid_{B_I^0(\xi)})$, where $j_I : B_I^0(\xi) \to B(\xi)$ is the natural inclusion. Since $F_B \mid_{B_I^0(\xi)} \in K(B_I^0(\xi), (G_I, \sigma_I))^{t,1}$, we have

$$j_{I!}(F_B \mid_{B_I^0(\xi)}) = [j_{I!}\mathbf{Q}_l] \otimes \otimes_{i \notin I} [F_{i,\bar{x}_i}] \otimes \otimes_{i \in I} [F_{i,\bar{0}}] \in K_c(B(\xi), (G, \sigma_\xi))^{t,1}.$$

Here the (G, σ_ξ)-structure on the right hand side is given by the diagonal action of (G_I, σ_I) on $\otimes_{i \notin I}[F_{i,\bar{x}_i}]$ and $[j_{I!}\mathbf{Q}_l]$. Using equality $[j_{I!}\mathbf{Q}_l] = \sum_{I \subset K} (-1)^{\#(K-I)} [i_{K!}\mathbf{Q}_l]$, where $i_K : B_K(\xi) \to B(\xi)$ is the natural inclusion, we have

$$\begin{aligned} F_B &= \sum_{K \supset I} (-1)^{\#(K-I)} [i_{K!}\mathbf{Q}_l] \otimes \otimes_{i \notin I} [F_{i,\bar{x}_i}] \otimes \otimes_{i \in I} [F_{i,\bar{0}}] \\ &= \sum_{K \subset [1,n]} (-1)^{\# K} [i_{K!}\mathbf{Q}_l] \otimes \Big(\sum_{K \supset I} (-1)^{\# I} \otimes_{i \notin I} [F_{i,\bar{x}_i}] \otimes \otimes_{i \in I} [F_{i,\bar{0}}]\Big). \end{aligned}$$

Since

$$\begin{aligned} &\sum_{K \supset I} (-1)^{\# I} \otimes_{i \notin I} [F_{i,\bar{x}_i}] \otimes \otimes_{i \in I} [F_{i,\bar{0}}] \\ &= \otimes_{i \notin K} [F_{i,\bar{x}_i}] \otimes \Big(\sum_{K \supset I} (-1)^{\# I} \otimes_{i \notin K-I} [F_{i,\bar{x}_i}] \otimes \otimes_{i \in I} [F_{i,\bar{0}}]\Big] \\ &= \otimes_{i \notin K} [F_{i,\bar{x}_i}] \otimes \otimes_{i \in K} \Phi([F_i]) \\ &= \tilde{\chi}_K(F), \end{aligned}$$

we have

$$\begin{aligned} \tilde{\Phi}_f(F) &= \sum_{K \subset [1,n]} (-1)^{\# K} \tilde{\Phi}_f([i_{K!}\mathbf{Q}_l]) \otimes \tilde{\chi}_K(F) \\ &= \sum_{K \subset [1,n]} (-1)^{\# K} \tilde{\Phi}_K \otimes \tilde{\chi}_K(F). \end{aligned}$$

This completes the proof of the theorem.

We apply Theorem 2.2.3 to composite singularities. Let $m_1, \ldots, m_n$ be positive integers, $y_i = (y_{i1}, \ldots, y_{im_i})$ sets of coordinates. We define $B(y_i) = Spec(k[[y_{i1}, \ldots, y_{im_i}]])$. Let $g_i = g_i(y_i)$ be a formal power series of $\{y_{ij}\}_j$ with no constant term. Then g_i defines a homomorphism $B(y_i) \to B(x_i)$ by $x_i = g_i(y_i)$. We assume that g_i is smooth on $B(y_i) - g_i^{-1}(0)$ and the exponent d_i of g_i divides $\# k^\times$. The fiber product of g_i defines a morphism from $B(y) = \prod_{i=1}^n B(y_i)$ to $B(x) = \prod_{i=1}^n B(x_i)$ and it is denoted by g. We assume that the exponent m of $(d_1, \ldots, d_n)$ with respect to f divides $\# k^\times$.

Lemma 2.2.4. *The composite $f \circ g : B(y) \to B(x) \to B(t)$ is smooth over $B^0(t)$.*

Let $\chi_{g,K}(\mathbf{Q}_l) = \chi_K([\mathbf{R}g_{1*}\mathbf{Q}_l] \otimes \cdots \otimes [\mathbf{R}g_{n*}\mathbf{Q}_l])$ and

$$\Phi_{f\circ g}(\mathbf{Q}_l) = \Phi_f([\mathbf{R}g_{1*}\mathbf{Q}_l] \otimes \cdots \otimes [\mathbf{R}g_{n*}\mathbf{Q}_l]) = [j_t^*\mathbf{R}(f\circ g)_*\mathbf{Q}_l] - [\mathbf{Q}_l].$$

Then the following theorem is a direct consequence of Theorem 2.2.3.

Theorem 2.2.5 (l-adic convolution theorem for composite singularities).

$$\Phi_{f\circ g}(\mathbf{Q}_l) = \sum_{I\subset[1,n]} (-1)^{\#I}(\tilde{\Phi}_I \otimes \chi_{g,I}(\mathbf{Q}_l))^{G_I}.$$

Remark 2.2.6. If we apply this theorem for the function $f(x_1, x_2) = x_1 + x_2$, it is nothing but the l-adic version of Thom-Sebastiani theorem. (See also Corollary 3.6.2.) It is proved in [L-D] assuming the existence of resolution of singularities. Here we only assume the tameness of the singularity for $g_1 = 0$ and $g_2 = 0$.

The rest of this paper is devoted to prove the Hodge analog of Theorem 2.2.5.

§3 Mixed Hodge structure of composite singularities

§3.1 Local space

Let N be an analytic space and X and Y be closed subspaces such that X is a projective algebraic variety. The triple (N, X, Y) is called a local space if

(1) Any irreducible component of Y is not contained in X.
(2) $N - (X \cup Y)$ is a smooth variety
(3) There exists a proper bimeromorphic map $b : \tilde{N} \to N$ such that b is an isomorphism on $b^{-1}(N - (X \cup Y))$ and $b^{-1}(X \cup Y)$ is a normal crossing divisor of $\tilde{N}$.
    The irreducible decomposition of the proper transform $\tilde{Y}$ of Y is written by $\tilde{Y} = \cup_{i\in I}\tilde{Y}_i$. For a subset J of I, we define $\tilde{Y}_J = \cap_{j\in J}\tilde{Y}_j$.
(4) For any subset J of I, $\tilde{Y}_J$ retracts to $\tilde{Y}_J \cap \tilde{X}$, where $\tilde{X} = b^{-1}(X)$. (In particular, $\tilde{N}$ retracts to the subvariety $\tilde{X}$.)

Note that for a triple with the properties (1)-(3), there exists a sufficiently small neighborhood N' of X such that $(N', X, Y \cap N'')$ is a local space. We introduce an equivalence relation on the set of local spaces by $(N, X, Y) \sim (N', X, Y \cap N')$, where N' is a sufficiently small neighborhood of X. ¿From now on, equivalent local spaces are always identified.

Example 3.1.1. Let $B(x) = \{(x_1, \ldots, x_n) \mid\mid x_i \mid< \epsilon\}$ and $f(x_1, \ldots, x_n)$ be a holomorphic function on $B(x)$ to $B(t) = \{t \mid\mid t \mid< \epsilon\}$. Then for a sufficiently small ϵ, $(B(x), 0, f^{-1}(0))$ is a local space.

In this section, we introduce a mixed Hodge structure on the cohomology $H^i(N - (X \cup Y), \mathbf{Q})$. To introduce a Hodge structure, we fix a resolution of singularities $\tilde{N} \to N$ of (3). Let $\tilde{Y} = \cup_{i\in I}\tilde{Y}_i$ be the irreducible decomposition of the proper transform $\tilde{Y}$ of Y and $\tilde{X} = \cup\tilde{X}_i$ be the irreducible decomposition of $b^{-1}(X)$. Put $\tilde{Y} = \cup_i\tilde{Y}_i$. Then $(\tilde{N}, \tilde{X}, \tilde{Y})$ is also a local space. First we construct a mixed Hodge structure on $H^i(\tilde{N} - (\tilde{X} \cup \tilde{Y}), \mathbf{Q}) = H^i(N - (X \cup Y), \mathbf{Q})$. Later we prove that it is independent of the choice of the resolution of singularities. The local space (N, X, Y) is called normal crossing if N is smooth and $X \cup Y$ is a normal crossing divisor in N. We assume that (N, X, Y) is a normal crossing local space and $X = \cup_{i=1}^{q} X_i$ and $Y = \cup_{i=1}^{p} Y_i$ are the irreducible decomposition. We can define the logarithmic de Rham complex $\Omega_N^\bullet(\log D)$ as in [Del]. By the result of [Del], there exist quasi-isomorphisms $\mathbf{R}j_*\mathbf{C} \to j_*\Omega_{N-D}^\bullet$ and $\Omega_N(\log D) \to j_*\Omega_{N-D}^\bullet$, where $j : N - D \to N$ is the natural inclusion. Let $i_I : X_I \to X$ be the natural inclusion. If $I \subset J$, then there exists a natural morphism $(i_I)_*\mathbf{R}j_*\mathbf{Q} \to (i_J)_*\mathbf{R}j_*\mathbf{Q}$.

Let $\mathcal{N}$ be an analytic manifold and $D = \cup_i D_i$ be a simple normal crossing divisor. For a subset J of I, we put $D_J = \cap_{j \in J} D_j$ and the ideal sheaf of D_J is denoted by $\mathcal{I}_J$. The sheaf of logarithmic differential forms $\Omega_{\mathcal{N}}^i(\log D)$ is a locally free sheaf on $\mathcal{N}$ and $\mathcal{I}_{D_J} \otimes \Omega_{\mathcal{N}}^i(\log D)$ is identified with a subsheaf of $\Omega_{\mathcal{N}}^i(\log D)$. If z_i is a local equation of D_j and $\omega \in \Omega_{\mathcal{N}}^i(\log D)$, then we have $d(z_\omega) = z_i(\frac{dz_j}{z_j} \wedge \omega) + z_j d\omega$, therefore $\mathcal{I}_{D_J} \otimes \Omega_{\mathcal{N}}^\bullet(\log D)$ is a sub-complex of $\Omega_{\mathcal{N}}^\bullet(\log D)$. Therefore the quotient sheaves $\Omega_{\mathcal{N}}^i(\log D) \otimes \mathcal{O}_{D_J}$ forms a complex of sheaves $\Omega_{\mathcal{N}}^\bullet(\log D) \otimes \mathcal{O}_{D_J}$.

Lemma 3.1.2. *Let $j : \mathcal{N} - D \to \mathcal{N}$ be the natural inclusion. Then there exists a quasi-isomorphism*

$$\mathbf{R}j_*\mathbf{C} \mid_{X_J} \simeq \Omega_{\mathcal{N}}^\bullet(\log D) \otimes \mathcal{O}_{X_J}.$$

Proof. First consider the following diagram where all the morphisms are filtered quasi isomorphisms;

$$(\mathbf{R}j_*\mathbf{C}, \tau_\bullet) \to (\mathbf{R}j_*\Omega_{\mathcal{N}-D}, \tau_\bullet) \longleftarrow (\Omega_{\mathcal{N}}(\log D), \tau_\bullet) \to (\Omega_{\mathcal{N}}(\log D), W_\bullet) \tag{3.1.1}$$

Let $W_k(\Omega_{\mathcal{N}}(\log D) \otimes \mathcal{O}_{D_J}) = \mathrm{Im}(W_k(\Omega_{\mathcal{N}}(\log D)) \to \Omega_{\mathcal{N}}(\log D) \otimes \mathcal{O}_{D_J})$. Then we have the following lemma.

Lemma 3.1.3.

(1) *The Poincare residue induces the isomorphism*

$$Gr_k^W(\Omega_{\mathcal{N}}^i(\log D) \otimes \mathcal{O}_{D_J}) \simeq \oplus_{\#K=k}\Omega_{D_{J\cup K}}^{i-k}$$

(2) *The natural morphism $\pi : (\Omega_{\mathcal{N}}^\bullet(\log D) \mid_{D_J}, W_\bullet) \to (\Omega_{\mathcal{N}}^\bullet(\log D) \otimes \mathcal{O}_{D_J}, W_\bullet)$ is a filtered quasi isomorphism*

Proof. (1) is well known and easy to check.

(2) The associated graded morphism of π is equal to

$$\oplus_{\#K=k}\Omega_{D_K}^\bullet \mid_{D_{J\cup K}} [-k] \to \oplus_{\#K=k}\Omega_{D_{J\cup K}}^\bullet[-k]$$

and they are quasi-isomorphic to $\mathbf{C}_{D_{J\cup K}}$. This proves the lemma 3.1.3.

By (3.1.1) and Lemma 3.1.3 (2), we have Lemma 3.1.2.

Now we return to the situation of a normal crossing local space (N, X, Y). We define the following complexes $K_{\mathbf{Q}} = K_{\mathbf{Q}}(N, X, Y)$ and $K_{\mathbf{C}} = K_{\mathbf{C}}(N, X, Y)$ on X.

$$K_{\mathbf{Q}} : \oplus_{\#I=1}(i_I)_*\mathbf{R}j_*\mathbf{Q} \mid_{X_I} \to \oplus_{\#I=2}(i_I)_*\mathbf{R}j_*\mathbf{Q} \mid_{X_I} \to \cdots$$

and

$$K_{\mathbf{C}} : \oplus_{\#I=1}(i_I)_*\Omega_{\mathcal{N}}^\bullet(\log(D)) \otimes \mathcal{O}_{X_I} \to \oplus_{\#I=2}(i_I)_*\Omega_{\mathcal{N}}^\bullet(\log(D)) \otimes \mathcal{O}_{X_I} \to \cdots$$

The following lemma is a direct consequence of Lemma 3.1.2.

Lemma 3.1.4. *There exists a quasi isomorphism of complexes of sheaves:*

$$\mathbf{R}j_*\mathbf{Q}\mid_X \to K_{\mathbf{Q}}, K_{\mathbf{Q}} \to K_C.$$

Definition 3.1.5. We introduce a weight filtration W on $K_{\mathbf{Q}}$, $K_{\mathbf{C}}$, and a Hodge filtration F on $K_{\mathbf{C}}$ as follows.

$$\begin{aligned} W_kK_{\mathbf{Q}} : \oplus_{\#I=1}\, \tau_k\mathbf{R}j_*\mathbf{Q}\mid_{X_I} &\to \oplus_{\#I=2}\tau_{k+1}\mathbf{R}j_*\mathbf{Q}\mid_{X_I} \\ &\to \cdots \to \oplus_{\#I=j}\tau_{k+j-1}\mathbf{R}j_{I*}\mathbf{Q} \to \cdots \end{aligned}$$

$$\begin{aligned} W_kK_{\mathbf{C}} : \oplus_{\#I=1}\, W_k\Omega^\bullet_N(\log(D))\otimes\mathcal{O}_{X_I} &\to \oplus_{\#I=2}W_{k+1}\Omega^\bullet_N(\log D)\otimes\mathcal{O}_{X_I} \\ &\to \cdots \to \oplus_{\#I=j}W_{k+j-1}\Omega^\bullet_N(\log D)\otimes\mathcal{O}_{X_I} \to \cdots \end{aligned}$$

$$\begin{aligned} F^pK_{\mathbf{C}} : \oplus_{\#I=1}\, \sigma^p\Omega^\bullet_D(\log(D))\otimes\mathcal{O}_{X_I} &\to \oplus_{\#I=2}\sigma^p\Omega^\bullet_N(\log D)\otimes\mathcal{O}_{X_I} \\ &\to \cdots \to \oplus_{\#I=j}\sigma^p\Omega^\bullet_N(\log D)\otimes\mathcal{O}_{X_I} \to \cdots, \end{aligned}$$

where $\tau_\bullet$ and $\sigma^\bullet$ are the canonical and the stupid filtration, respectively.

These filtration on the complex of sheaves induces the weight filtration W on $\mathbf{R}\Gamma(X, K_{\mathbf{Q}})$ and $\mathbf{R}\Gamma(K, K_{\mathbf{C}})$ and the Hodge filtration F on $\mathbf{R}\Gamma(X, K_{\mathbf{C}})$ up to filtered quasi-isomorphism.

Definition 3.1.6. The pair $((L_{\mathbf{Q}}, W), (L_{\mathbf{C}}, W, F))$ of filtered complex of sheaves and bi-filtered complex of sheaves on X is a cohomological mixed Hodge complex (=CMHC for short), if

(1) $(L_{\mathbf{Q}}, W)\otimes\mathbf{C}$ and $(L_{\mathbf{C}}, W)$ are filtered quasi-isomorphism.
(2) The spectral sequence for the filtered complex of sheaves $(\mathbf{R}\Gamma(X, Gr^W_k(L_{\mathbf{C}})), Gr^W_k(F))$ on X degenerates at E_1-term.
(3) The filtration $Gr^W_k(F^p)$ defines a Hodge structure of weight $m+k$ on $H^m(X, Gr^W_k(L_{\mathbf{C}}))$.

Theorem 3.1.7. *The pair $((K_{\mathbf{Q}}, W), (K_{\mathbf{C}}, W, F))$ defines a CMHC on X.*

It is sufficient to prove that $Gr^W_k(K_{\mathbf{C}})$ is a cohomological Hodge complex of weight k, where $Gr^W_k(K_{\mathbf{C}})$ is given by the double complex:

$$\oplus_{\#I=k,\#J=1}\Omega^\bullet_{Y_I\cap X_I}[-k](-k) \xrightarrow{0} \oplus_{\#I=k,\#J=2}\Omega^\bullet_{Y_I\cap X_I}[-k-1](-k-1) \xrightarrow{0}$$

Moreover, the associated graded module $Gr^p_FGr^W_k(K_{\mathbf{C}})$ for the induced filtration $Gr^W_k(F^p)$ is equal to

$$\oplus_{\#I=k,\#J=1}\Omega^{p-k}_{Y_I\cap X_I}[-p](-k) \xrightarrow{0} \oplus_{\#I=k,\#J=2}\Omega^{p-k-1}_{Y_I\cap X_I}[-p](-k-1) \xrightarrow{0}$$

Therefore the m-th hypercohomology of $Gr^p_FGr^W_k(K_{\mathbf{C}})$ is equal to

$$\oplus_{j\geq 1}\oplus_{\#I=k+j-1,\#J=j}\, H^{m-p-j+1}(\Omega^{p-k-j+1}_{Y_I\cap X_J})(-k-j+1).$$

Therefore by the classical Hodge theory the following spectral sequence degenerates at the E_1-term.

$$E_1^{p,q} = \mathbf{H}^{p+q}(X, Gr_F^p Gr_k^W(K_{\mathbf{C}})) \to E_\infty^{p+q} = \mathbf{H}^{p+q}(X, Gr_k^W(K_{\mathbf{C}})),$$

and the filtration defined by this spectral sequence defines a pure Hodge structure of weight $m+k$ on $\mathbf{H}^m(X, Gr_k^W(K_{\mathbf{C}}))$.

We prove the independence of the choice of the resolution of singularity. We use the same argument as [Del]. Let (N_1, X_1, Y_1) and (N_2, X_2, Y_2) be normal crossing local spaces, (N, X, Y) a local space and $f_i : N_i \to N$ be proper morphisms for $i = 1, 2$ such that $f^{-1}(X \cup Y) = X_i \cup Y_i$ and $f_i \mid_{N_i - (X_i \cup Y_i)}: N_i - (X_i \cup Y_i) \to N - (X \cup Y)$ are isomorphic for $i = 1, 2$. Then by taking a fiber product of f_1, f_2 and resolving it, there exists the third normal crossing local space (N_3, X_3, Y_3) and proper morphism $g_i : (N_3, X_3, Y_3) \to (N_i, X_i, Y_i)$ such that $g_i^{-1}(X_i \cup Y_i) = X_3 \cup Y_3$ and the restrictions of g_i to $N_3 - (X_3 \cup Y_3)$ are isomorphisms.

$$\begin{array}{ccc} (N_3, X_3, Y_3) & \longrightarrow & (N_1, X_1, Y_1) \\ \downarrow & & \downarrow \\ (N_2, X_2, Y_2) & \longrightarrow & (N, X, Y) \end{array}$$

The morphism g_i induces a homomorphism from the mixed Hodge structure of (N_3, X_3, Y_3) to that of (N_i, X_i, Y_i) which is bijective. Since the category of mixed Hodge structures is an abelian category, this gives rise to the isomorphism of mixed Hodge structures. In the same way, we can prove the following functoriality of the mixed Hodge structures of local spaces.

Proposition 3.1.8. *Let (N_1, X_1, Y_1) and (N_2, X_2, Y_2) be local spaces and $f : N_1 \to N_2$ be a morphism such that $f(N_1 - (X_1 \cup Y_1)) \subset N_2 - (X_2 \cup Y_2)$ and $f \mid_{N_1 - (X_1 \cup Y_1)}$ is an immersion. Then f induces a homomorphism of mixed Hodge structures of local spaces.*

Proof. We reduce the proof to the case where both (N_1, X_1, Y_1) and (N_2, X_2, Y_2) are normal crossing local spaces. In this case, the proof is exactly the same as [Del].

Remark 3.1.9. Let (N, X, Y) be a normal crossing local space and X_1 be a closed subvariety of X, N_1 be a neighborhood of X_1, and Y_1 be the closure of $(X \cup Y) \cap N_1 - X_1$. Then (N_1, X_1, Y_1) is a local space and the open immersion $N_1 \subset N$ satisfies the condition (1).

Corollary 3.1.10. *Let (N, X, Y) be a local space and W be a subvariety of X. Then $H_c^i(W, \mathbf{R}j_*\mathbf{Q})$ has a natural mixed Hodge structure.*

Proof. Let Z_1 be the closure of W in X and put $Z_2 = Z_1 - W$. The open immersion from W to Z_1 is denoted by $k : W \to Z_1$. Since

$$H_c^i(W, \mathbf{R}j_*\mathbf{Q}) \simeq H^i(Z_1, k_!(\mathbf{R}j_*\mathbf{Q} \mid_W))$$

and

$$k_!(\mathbf{R}j_*\mathbf{Q} \mid_W) \to \mathbf{R}j_* \mid_{Z_2} \overset{rest}{\to} \mathbf{R}j_* \mid_{Z_1} \overset{+1}{\to}$$

is the distinguished triangle, it is enough to prove that $\mathbf{R}j_*\mathbf{Q} \mid_{Z_1}$ and $\mathbf{R}j_*\mathbf{Q} \mid_{Z_2}$ are cohomological mixed Hodge complex and the morphism *rest* is a morphism of mixed Hodge complex. For the first statement, we may assume that W is the closed subvariety of X. By modifying, there exists a modification of local spaces $\pi : (\tilde{N}, \tilde{X}, \tilde{Y}) \to (N, X, Y)$ such that $(\tilde{N}, \tilde{X}, \tilde{Y})$ is normal crossing and $\tilde{W} = \pi^{-1}(W)$ is a divisor of $\tilde{N}$ contained in $\tilde{X}$. By using the construction of remark, we can see that $\mathbf{R}\tilde{j}_*\mathbf{Q} \mid_{\tilde{W}}$ is a cohomological mixed Hodge complex on $\tilde{W}$. Therefore,

$$\mathbf{R}\Gamma(W, \mathbf{R}j_*\mathbf{Q} \mid_W) \simeq \mathbf{R}\Gamma(W, \mathbf{R}\pi_*(\mathbf{R}\tilde{j}_*\mathbf{Q} \mid_{\tilde{W}})) \simeq \mathbf{R}\Gamma(\tilde{W}, \mathbf{R}\tilde{j}_*\mathbf{Q} \mid_{\tilde{W}})$$

is a mixed Hodge complex. We can prove that the morphism *rest* is a morphism of CMHC's in the same way.

§3.2 Mixed Hodge structures associated to hypersurface singularities

In this section, we investigate mixed Hodge structures of germs of hypersurface singularities.

Let $B(y) = \{(y_1, \ldots, y_n) \mid\mid y_i \mid < \epsilon\}$ and $f = f(y_1, \ldots, y_n)$ be a holomorphic function on $B(x)$ such that $f(0) = 0$. The function f defines a holomorphic map from $B(y)$ to $B(t) = \{t \in \mathbf{C} \mid\mid t \mid < \epsilon\}$ and put $Z(f) = \{y \in B(y) \mid f(y) = 0\}$. We assume that f is smooth outside of $Z(f)$. Let m be a positive integer. The variety $B_m(y)$ is defined by the following fiber product:

$$\begin{array}{ccc} B_m(y) & \xrightarrow{\pi} & B(y) \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B(\tau) & \longrightarrow & B(t), \end{array}$$

where $B(\tau) = \{\tau \in \mathbf{C} \mid\mid \tau \mid < \epsilon\}$ and $\tau^m = t$. Then $(B_m(y), \pi^{-1}(0), \pi^{-1}(Z(f)))$ is a local space and $H^i(B_m(y) - \pi^{-1}(Z(f)), \mathbf{Q})$ has a mixed Hodge structure defined in §2.1. The covering transformation group $\mu_m = \mathrm{Gal}(B(\tau)/B(t))$ acts on the mixed Hodge structure on $H^i(B_m(y) - \pi^{-1}(Z(f)), \mathbf{Q})$ by the functoriality. Let $K_{MH}(\mathbf{C}, \mu_m)$ be the Grothendieck group of $\mathbf{Q}$-mixed Hodge structure with a μ_m-action. Then $K_{MH}(\mathbf{C}, \mu_m)$ has a natural ring structure arising from tensor product and graded by weight. Since $[\mathbf{Q} - \mathbf{Q}(1)] = [\mathbf{Q}] - [\mathbf{Q}(1)]$ is a non zero divisor on $K_{MH}(\mathbf{C}, \mu_m)$, the natural homomorphism from $K_{MH}(\mathbf{C}, \mu_m)$ to the localization $K_{MH}(\mathbf{C}, \mu_m)_{loc}$ of $K_{MH}(\mathbf{C}, \mu_m)$ with respect to $(\mathbf{Q} - \mathbf{Q}(1))$ is injective. (See [D-L].)

The element $\Psi_{f,m}(\mathbf{Q}) = [H^*(B_m(y) - \pi^{-1}(Z(f)), \mathbf{Q})]$ in $K_{MH}(\mathbf{C}, \mu_m)$ is defined by $\sum_{i=0}^{2\dim B(x)} (-1)^i [H^i(B_m(y) - \pi^{-1}(Z(f)), \mathbf{Q})]$. We define

(3.2.1)

$$\phi_{f,m}(\mathbf{Q}) = \frac{1}{[\mathbf{Q} - \mathbf{Q}(-1)]} \Psi_{f,m}(\mathbf{Q}) \in K_{MH}(\mathbf{C}, \mu_m)_{loc}, \text{ and}$$
$$\Phi_{f,m}(\mathbf{Q}) = \phi_{f,m}(\mathbf{Q}) - [\mathbf{Q}] \in K_{MH}(\mathbf{C}, \mu_m)_{loc}.$$

Note that $\phi_{f,m}(\mathbf{Q})$ corresponds to the cohomology of the Milnor fiber if the exponent of monodromy divides m. Using the weight spectral sequence, one can show $\phi_{f,m}$ is actually an element in $K_{MH}(\mathbf{C}, \mu_m)$, but we do not use this fact in the rest of this paper.

Let $y_i=(y_{i1},\dots,y_{im_i})$ be sets of coordinates as in the end of §2.2. We define $B(y_i)=\{(y_{i1},\dots,y_{i,m_i})\in\mathbf{C}^{m_i} \mid\mid y_{ij}\mid<\epsilon \quad (1\le j\le m_i)\}$. Let $g_i=g_i(y_i)$ be a holomorphic function on $B(y_i)$ with $g_i(0)=0$. Then g_i defines a homomorphism $B(y_i)\to B(x_i)$ by $x_i=g_i(y_i)$. We assume that g_i is smooth on $B(y_i)-g_i^{-1}(0)$. The fiber product g of g_i from $B(y)=\prod_{i=1}^n B(y_i)$ to $B(x)=\prod_{i=1}^n B(x_i)$ is defined as in §2.2. The sequence of positive numbers $(d_1,\dots,d_n)$ is denoted by $\mathbf{d}$. We define $B_{\mathbf{d}}(x)$, $\tilde B_{\mathbf{d}}(x)$, $B^0_{\mathbf{d}}(x)$ and $\tilde B^0_{\mathbf{d}}(x)$ by the following cartesian squares.

$$\begin{array}{ccccccc}
B_{\mathbf{d}}(y) & \longrightarrow & B(y) & \quad & B^0_{\mathbf{d}}(y) & \longrightarrow & B^0(y)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
B(\xi) & \longrightarrow & B(x), & & B^0(\xi) & \longrightarrow & B^0(x)\\
 & & & & & & \\
\tilde B_{\mathbf{d}}(y) & \longrightarrow & B_{\mathbf{d}}(y) & & \tilde B^0_{\mathbf{d}}(y) & \longrightarrow & B^0_{\mathbf{d}}(y)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
B(\tau) & \longrightarrow & B(t), & & B(\tau) & \longrightarrow & B(t).
\end{array}$$

We introduce the equivariant version of the definition (3.2.1). The natural map $\tilde B_{\mathbf{d}}(y)$
$\to B(\tau)$ is denoted by $\tilde f(y)$. We define $\tilde\phi_{f\circ g,m}(\mathbf{Q})$ by

$$\begin{aligned}\tilde\phi_{f\circ g,m}(\mathbf{Q}) =&\frac{1}{[\mathbf{Q}-\mathbf{Q}(-1)]}[H^*(\tilde B_{\mathbf{d}}(y)-(\tilde f(y))^{-1}(0),\mathbf{Q})]-[\mathbf{Q}]\\ &\in K_{MH}(\mathbf{C},G\times\mu_m)_{loc}.\end{aligned}$$

Then it is easy to see that $\Phi_{f\circ g,m}(\mathbf{Q})=\tilde\Phi_{f\circ g,m}(\mathbf{Q})^G$. For a subset I of $[1,n]$, we define $B_I(\xi)$ by $\prod_{i\notin I}B(\xi_i)$. The fiber product of morphism $B_I(\xi)\to B(\xi)\to B(t)$ and $B(\tau)\to B(t)$ is denoted by $B_{I,m}(\xi)$. Put $G_I=\prod_{i\notin I}\mu_{d_i}$. Then $G_I\times\mu_m$ acts on $B_{I,m}(\xi)$. The natural morphism from $B_{I,m}(\xi)\to B(\tau)$ is denoted by $f_I(\xi)$. We define $\tilde\phi_{f_I,m}(\mathbf{Q})$ by

$$\begin{aligned}\tilde\phi_{f_I,m}(\mathbf{Q}) =&\frac{1}{[\mathbf{Q}-\mathbf{Q}(-1)]}[H^*(B_{I,m}(\xi)-(f_I(\xi))^{-1}(0),\mathbf{Q})]-[\mathbf{Q}]\\ &\in K_{MH}(\mathbf{C},G_I\times\mu_m)_{loc}.\end{aligned}$$

Let $\tilde B^0_{I,\mathbf{d}}=\tilde B_{I,\mathbf{d}}-\cup_{J\supsetneq I}\tilde B_{J,\mathbf{d}}$ and the k_I be the natural inclusion

$$k_I:\tilde B^0_{I,\mathbf{d}}-(\tilde f(y))^{-1}(0)\to\tilde B_{I,\mathbf{d}}-(\tilde f(y))^{-1}(0).$$

Then we use the following equality.

$$(3.2.2)\quad [H^*(\tilde B_{\mathbf{d}}(y)-(\tilde f(y))^{-1}(0),\mathbf{Q})]=\sum_{I\subset[1,n]}[H^*(\tilde B_{I,\mathbf{d}}(y)-(\tilde f(y))^{-1}(0),k_{I!}\mathbf{Q})]$$

Now we introduce several notation related to the resolution of singularities. Let $b(y_i):\hat B(y_i)\to B(y_i)$ be a resolution of singularity for $g_i^{-1}(0)\subset B(y_i)$, i.e, $b(y_i)$ is a birational projective morphism and $b(y_i)^{-1}(g^{-1}(0))_{red}$ is a simple normal crossing divisor. Let d_i be the least common multiple of the multiplicities of the components in $b(y_i)^{-1}(g^{-1}(0))$. Let $B(\xi_i) \to B(x_i)$ be the morphism defined by $\xi_i^{d_i} = x_i$. The normalization of $\hat{B}(y_i) \times_{B(x_i)} B(\xi_i)$ is denoted by $\tilde{B}(y_i)$. The second projection $\tilde{B}(y_i) \to B(\xi_i)$ is denoted by $\tilde{g}_i$. Then the morphism $\tilde{g}_i$ is μ_{d_i}-equivariant and $\tilde{B}(y_i)$ is a quasi-smooth variety with reduced quasi-normal crossing divisor $\tilde{g}_i(0)$.

$$\begin{tikzcd}
\tilde{B}(y_i) \arrow[r] \arrow[d, "\tilde{g}_i"'] & \hat{B}(y_i) \arrow[d, "\hat{g}_i"] \\
B(\xi_i) \arrow[r] & B(x_i)
\end{tikzcd}$$

Let $V = \tilde{B}(y) = \prod_{i=1}^n \tilde{B}(y_i)$ be the product of $\tilde{B}(y_i)$ and the product $V \to B(\xi) = \prod_{1=1}^n B(y_i)$ of g_i is denoted by h. The base change of h by the morphism $\hat{B}(\xi) \to B(\xi)$, $\tilde{B}(\xi) \to B(\xi)$ and $\bar{B}(\xi) \to B(\xi)$ are denoted by $\hat{h} : \hat{V} \to \hat{B}(\xi)$, $\tilde{h} : \tilde{V} \to \tilde{B}(\xi)$ and $\bar{h} : \bar{V} \to \bar{B}(\xi)$. Then we have the following diagram.

$$\begin{tikzcd}
\hat{V} \arrow[r] \arrow[d, "\hat{h}"'] & V \arrow[d, "h"] & \\
\hat{B}(\xi) \arrow[r, "b(\xi)"] & B(\xi) \arrow[r, "f(\xi)"] & B(t) \\
\tilde{B}(\xi) \arrow[r] \arrow[u] & \bar{B}(\xi) \arrow[r] & B(\tau) \arrow[u] \\
\tilde{V} \arrow[r] \arrow[u, "\tilde{h}"] & \bar{V} \arrow[u, "\bar{h}"'] &
\end{tikzcd}$$

The morphism $h : V \to B(\xi)$ is G-equivariant and $G \times \mu_m$ acts on $\tilde{V}$ as $B(\tau)$-variety. Let $B^0(\xi) = \prod_{i=1}^n B^0(\xi_i)$, $D_B = B(\xi) - B^0(\xi)$ and $D_F = (f(\xi))^{-1}$. The proper transform of D_B and D_F in $\hat{B}(\xi)$ is denoted by $\hat{D}_B$ and $\hat{D}_F$ and the exceptional divisor for $b(\xi)$ is denoted by $\hat{D}_E$. The pullbacks of $\hat{D}_B, \hat{D}_F$ and $\hat{D}_E$ are denoted by $\tilde{D}_B$, $\tilde{D}_F$ and $\tilde{D}_E$, respectively. The group $G \times \mu_m$ acts on $\tilde{D}_E$. We define divisors on $V, \hat{V}, \tilde{V}$ and $\bar{V}$ as follows:

$$\begin{aligned}
&h^{-1}(D_B) = V_B, h^{-1}(D_F) = V_F \\
&\hat{h}^{-1}(\hat{D}_B) = \hat{V}_B, \hat{h}^{-1}(\hat{D}_F) = \hat{V}_F, \hat{h}^{-1}(\hat{D}_E) = \hat{V}_E \\
&\tilde{h}^{-1}(\tilde{D}_B) = \tilde{V}_B, \tilde{h}^{-1}(\tilde{D}_F) = \tilde{V}_F, \tilde{h}^{-1}(\tilde{D}_E) = \tilde{V}_E
\end{aligned}$$

Let $\tilde{j}_V : \tilde{V} - (\tilde{V}_F \cup \tilde{V}_E) \to \tilde{V}$ be the natural immersion. Then the mixed Hodge structure of $H^*(\tilde{V} - (\tilde{V}_F \cup \tilde{V}_E), \mathbf{Q})$ is given by $\mathbf{H}^*(\tilde{V}, \mathbf{R}\tilde{j}_{V*}\mathbf{Q} \mid_{\tilde{V}_E})$. The inverse image of $(0, \ldots, 0)$ under the morphism $V \to \prod_{i=1}^n B(y_i)$ is denoted by V_c. The action of G on V induces that of V_c. The inverse image of V_c under the morphism $\tilde{V} \to V$ is denoted by $\tilde{V}_c$. Then we have $\tilde{V}_c = V_c \times \tilde{D}_E$ and the action of $G \times \mu_m$ is equal to the diagonal action. The restriction of $\tilde{h}$ to $\tilde{V}$ is denoted by $\tilde{h}_c : \tilde{V}_c \to \tilde{D}_E$. It is easy to see that $V_c \to V$ and $\tilde{V}_c \to \tilde{V}$ are retractions. The group $G \times \mu_m$ acts on the mixed Hodge structure $\mathbf{H}^i(\tilde{V}, \mathbf{R}\tilde{j}_{V*}\mathbf{Q}) \simeq \mathbf{H}^i(\tilde{V}_c, \mathbf{R}\tilde{j}_{V*}\mathbf{Q}\mid_{\tilde{V}_E})$. Therefore we have

$$[\mathbf{H}^*(\tilde{V}, \mathbf{R}\tilde{j}_{V*}\mathbf{Q})] = [H^*(\tilde{V} - (\tilde{V}_E \cup \tilde{V}_F), \mathbf{Q})] \in K_{MH}(\mathbf{C}, G \times \mu_m).$$

Let $\tilde{D}_E = \coprod_{\sigma \in F'} \tilde{Z}_\sigma^0$ be the stratification indexed by F' and $\tilde{Z}_\sigma$ be the closure of $\tilde{Z}_\sigma^0$ in $\tilde{D}_E$. Note that this stratification is stable under the action of $G \times \mu_m$. Let U_σ be a tubular neighborhood of $\tilde{Z}_\sigma$, $j_\sigma : U_\sigma - (\tilde{D}_E \cup \tilde{D}_F) \to U_\sigma$ and $\tilde{j}_\sigma : \tilde{h}^{-1}(U_\sigma - (\tilde{D}_E \cup \tilde{D}_F)) \to \tilde{h}^{-1}(U_\sigma)$ be the natural inclusion.

$$\begin{array}{ccccc}
\tilde{h}^{-1}(U_\sigma - (\tilde{D}_E \cup \tilde{D}_F)) & \xrightarrow{\tilde{j}_\sigma} & \tilde{h}^{-1}(U_\sigma) & \xleftarrow{\supset} & \tilde{h}_c(\tilde{Z}_\sigma^0) \\
\downarrow & & \downarrow & & \downarrow \\
U_\sigma - (\tilde{D}_E \cup \tilde{D}_F) & \xrightarrow{j_\sigma} & U_\sigma & \xleftarrow[\supset]{} & \tilde{Z}_\sigma^0
\end{array}$$

Let G_σ be the stabilizer of $\tilde{Z}_\sigma^0$ in $G \times \mu_m$. Then $\mathbf{H}_c^i(\tilde{h}_c^{-1}(\tilde{Z}_\sigma^0), \mathbf{R}\tilde{j}_{V*}\mathbf{Q})$ is a G_σ-module and we have

$$\begin{aligned}
[\mathbf{H}^i(\tilde{V}_c, \mathbf{R}\tilde{j}_{V*}\mathbf{Q})] &= \sum_{[\sigma] \in F'/(G \times \mu_m)} \mathrm{Ind}_{G_\sigma}^G [\mathbf{H}_c^i(\tilde{h}_c^{-1}(\tilde{Z}_\sigma^0), \mathbf{R}\tilde{j}_{V*}\mathbf{Q})] \\
&= \sum_{[\sigma] \in F'/(G \times \mu_m)} \mathrm{Ind}_{G_\sigma}^G [\mathbf{H}_c^i(\tilde{h}_c^{-1}(\tilde{Z}_\sigma^0), \mathbf{R}\tilde{j}_{\sigma*}\mathbf{Q})].
\end{aligned}$$

Put $I(\sigma) = \{i \mid \mathbf{e}_i \in \sigma\}$ and $\Psi_{V,m,\sigma}(\mathbf{Q}) = [\mathbf{H}_c^i(\tilde{h}_c^{-1}(\tilde{Z}_\sigma^0), \mathbf{R}\tilde{j}_{\sigma*}\mathbf{Q})]$. Then we have

$$[H^*(\tilde{B}_{\mathbf{d}}(y) - (f \circ g)^{-1}(0), k_{\emptyset!}\mathbf{Q})] = \sum_{\{\sigma \in F', I(\sigma) = \emptyset\}/(G \times \mu_m)} \mathrm{Ind}_{G_\sigma}^G (\Psi_{V,m,\sigma}(\mathbf{Q}))$$

in $K_{MH}(\mathbf{C}, G \times \mu_m)$. In the next section, we compute $R^i\tilde{j}_{\sigma*}\mathbf{Q}$ using toric geometry.

§3.3 The structure of $R^i\tilde{j}_{\sigma*}\mathbf{Q}$.

First we describe a stratification of $\tilde{D}_E$ in terms of toric geometry. The space $B^0(\xi) \times B^0(\tau)$ can be considered as an open set of $\bar{B}(\xi)$. The complement $\bar{D} = \bar{B}(\xi) - B^0(\xi) \times B^0(\tau)$ is a quasi normal crossing divisor whose irreducible components are indexed by the set of 1-dimensional cones r in $\tilde{F}$. Let $\{r_0, r_1, \dots, r_s, r_{s+1}, \dots, r_{s+n}\}$ be the set of 1-dimensional cones of $\tilde{F}$ such that $r_0, r_{s+1}, \dots, r_{s+n}$ correspond to the proper transform of $\{\tau = 0\}$, $\{\xi_1 = 0\}, \dots, \{\xi_n = 0\}$, respectively. The corresponding divisors are written as $\bar{D}_0, \bar{D}_1, \dots, \bar{D}_s, \bar{D}_{s+1}, \dots, \bar{D}_{s+n}$. The intersection $\bar{D}_{i_1} \cap \dots \cap \bar{D}_{i_k}$ is non-empty and as a consequence defines a stratum of $\bar{D}$ if and only $r_{i_1}, \dots, r_{i_k}$ generate a cone σ of the fan $\tilde{F}$. We denote $\bar{Z}_\sigma = \bar{D}_{i_1} \cap \dots \cap \bar{D}_{i_k}$. Since $\bar{D}_0 \cap \tilde{B}(\xi) = \tilde{D}_F$, $(\cup_{i=1}^n \bar{D}_{s+i}) \cap \tilde{B}(\xi) = \tilde{D}_B$ and $(\cup_{i=1}^s \bar{D}_i) \cap \tilde{B}(\xi) = \tilde{D}_E$, $\tilde{Z}_\sigma = \bar{Z}_\sigma \cap \tilde{B}(\xi)$, where σ is a simplicial cone of $\tilde{F}$, defines a stratification of $\tilde{D}_F \cup \tilde{D}_E \cup \tilde{D}_B$. It is easy to see that $\tilde{Z}_\sigma$ is not empty if and only if $\dim \bar{Z}_\sigma \geq 1$ i.e. $\dim \sigma < n+1$. We put $\bar{Z}_\sigma^0 = \bar{Z}_\sigma - \cup_{\tau > \sigma} \bar{Z}_\tau$. Then $\bar{Z}_\sigma^0$ is isomorphic to a torus and $\tilde{Z}_\sigma^0$ is a non-singular hypersurface in $\bar{Z}_\sigma^0$. As a consequence, we have a stratification

$$\tilde{D}_B \cup \tilde{D}_E \cup \tilde{D}_F = \cup_{\sigma:\ \text{simplicial}\ , \dim \sigma < n+1} \tilde{Z}_\sigma^0.$$

Under this stratification, $\tilde{D}_E$ corresponds to σ's whose generator contains at least one of $r_1, \dots, r_s$. Let $A_\sigma = \mathbf{C}[\hat{\sigma} \cap L(\xi, \tau)]$ and $Spec(A_\sigma) \to X_{\tilde{F}}$ be the natural map. Let U_σ be the pull back of $B(\xi)$ in $Spec(A_\sigma)$. Let $L_{\hat{\sigma}}$ be the maximal linear subspace contained in $\hat{\sigma}$ and $\bar{A}_\sigma = \mathbf{C}[L_{\hat{\sigma}} \cap L(\xi, \tau)]$. The kernel of the natural map $A_\sigma \to \bar{A}_\sigma$ is denoted by I_σ. Then the corresponding closed subvariety is identified with $\bar{Z}^0_\sigma$.

$$\begin{array}{ccccc} \bar{Z}^0_\sigma & \xrightarrow{\subset} & U_\sigma & \longrightarrow & \bar{B}(\xi) \\ & & \cap\downarrow & & \downarrow\cap \\ & & Spec(A_\sigma) & \longrightarrow & X_{\tilde{F}} \end{array}$$

By taking the intersections with $\tilde{B}(\xi)$, we have the following diagram.

$$\begin{array}{ccccc} \tilde{h}_c^{-1}(Z^0_\sigma) & \xrightarrow{\subset} & \tilde{h}^{-1}(\tilde{U}_\sigma) & \xleftarrow{\tilde{j}_\sigma} & \tilde{h}^{-1}(\tilde{U}_\sigma) - \tilde{D} \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{Z}^0_\sigma & \xrightarrow[\subset]{} & \tilde{U}_\sigma & \xleftarrow[\supset]{} & \tilde{U}_\sigma - \tilde{D} \end{array}$$

We use these varieties to compute $\mathbf{H}^*(\tilde{h}_c^{-1}(Z^0_\sigma), \mathbf{R}\tilde{j}_\sigma\mathbf{Q} \mid_{h_c^{-1}(\tilde{Z}^0_\sigma)})$

The natural morphism $\tilde{V} \to V$ and $\tilde{V} \to \tilde{B}(\xi)$ defines a closed immersion $\tilde{V} \to V \times \tilde{B}(\xi)$. This morphism defines a morphism of local spaces:

$$(\tilde{V}, \tilde{V}_c, \tilde{D}_E \cup \tilde{D}_F \cup \tilde{D}_B) \xrightarrow{\tilde{\alpha}} (V \times \tilde{B}(\xi), V_c \times \tilde{D}_E, V \times \tilde{B}(\xi) - V^0 \times B^0(\xi)),$$
$$(\bar{V}, \bar{V}_c, \bar{D}_E \cup \bar{D}_F \cup \bar{D}_B) \xrightarrow{\bar{\alpha}} (V \times \bar{B}(\xi), V_c \times \bar{D}_E, V \times \bar{B}(\xi) - V^0 \times b^0(\xi, \tau)$$

where $V^0 = h^{-1}(B^0(\xi))$. Note that $V_c \times \tilde{D}_E \simeq \tilde{V}_c$. The inclusion of divisors $\tilde{D}_E \cup \tilde{D}_F \to \tilde{D}_E \cup \tilde{D}_F \cup \tilde{D}_B$ defines the following open immersion of local spaces:

$$(\tilde{V}, \tilde{V}_c, \tilde{D}_E \cup \tilde{D}_F \cup \tilde{D}_B) \xrightarrow{\tilde{\beta}} (\tilde{V}, \tilde{V}_c, \tilde{D}_E \cup \tilde{D}_F).$$
$$(\bar{V}, \bar{V}_c, \bar{D}_E \cup \bar{D}_F \cup \bar{D}_B) \xrightarrow{\bar{\beta}} (\bar{V}, \bar{V}_c, \bar{D}_E \cup \bar{D}_F)$$

We investigate these inclusions on the open set $\tilde{h}^{-1}(\tilde{U}_\sigma) \subset \tilde{V}$ and $V \times \tilde{U}_\sigma$.

Let y be a point in $\tilde{B}(y)$ and $z_{11}, \dots, z_{1m_1}, \dots, z_{n1}, \dots, z_{nm_n}$ be a local coordinate near y such that $g_1, \dots, g_n$ can be written as

$$g_1 = z_{11}^{m_{11}} \cdots z_{1l_1}^{m_{1l_1}}, \dots, g_n = z_{n1}^{m_{n1}} \cdots z_{nl_n}^{m_{nl_n}}.$$

Then the local equation of V at a lifting $\tilde{y}$ of y can be written as

$$\xi_1^{d'_1} = z_{11}^{m'_{11}} \cdots z_{1l_1}^{m'_{1l_1}}, \dots, \xi_n^{d'_n} = z_{n1}^{m'_{n1}} \cdots z_{nl_n}^{m'_{nl_n}},$$

where

$$d'_i = \frac{1}{\gcd(d_i, m_{i,j})_j} d_i, m'_{ij} = \frac{1}{\gcd(d_i, m_{i,j})_j} m_{ij}.$$

Let $r_1, \dots, r_a$ be a generator of $\sigma \cap L(\xi,\tau)\otimes \mathbf{Q}$, such that $\sum_{i=1}^a \mathbf{N} r_i \subset (\sigma \cap L(\xi,\tau))$. Suppose that $r_1, \dots, r_k$ correspond to the components of $\bar{D}_B$ and $r_{k+1}, \dots, r_a$ correspond to the components of $\bar{D}_E \cup \bar{D}_F$. We take $r_{a+1}, \dots, r_{n+1}$ such that $\sum_{i=1}^{n+1} \mathbf{Z} r_i$ is finite index in $L(\xi,\tau)^*$. Then by changing the numbering ξ_i, $r_1, \dots, r_a$ can be written as

$$
\begin{aligned}
r_1 &= (r_{11}, 0, \dots, 0), \dots, r_k = (0, \dots, r_{kk}, 0, \dots, 0),\\
r_{k+1} &= (r_{k+1,1}, \dots, r_{k+1,n}, r_{k+1,n+1}), \dots, r_a = (r_{a1}, \dots, r_{an}, r_{a,n+1})
\end{aligned}
$$

Then $\mathbf{C}[\hat{\sigma} \cap L(\xi,\tau)]$ is a subring of $\mathbf{C}[u_1, \dots, u_a, u_{a+1}^{\pm}, \dots, u_{n+1}^{\pm}]$, where $\{u_i\}_i$ is the multiplicative expression of the dual base $\{r_i^*\}_i$ of $\{r_i\}_i$. The morphism $Spec(\mathbf{C}[u_1, \dots, u_a, u_{a+1}^{\pm}, \dots, u_{n+1}^{\pm}] \to Spec(\mathbf{C}[\xi_1, \dots, \xi_n])$ is given by

$$
\begin{aligned}
\xi_1 &= u_1^{r_{11}} \cdots u_a^{r_{a1}} \cdot (\text{ unit }), \dots, \xi_n = u_1^{r_{1n}} \cdots u_a^{r_{an}} \cdot (\text{ unit }),\\
\tau &= u_1^{r_{1,n+1}} \cdots u_a^{r_{a,n+1}} \cdot (\text{ unit })
\end{aligned}
$$

First we investigate the inclusion $\bar{\alpha}$ and $\bar{\beta}$. The local equation for $\bar{V}$ is given by

$$
\begin{aligned}
&[u_1^{r_{11}} \prod_{p=k+1}^{a} u_p^{r_{p1}}]^{d'_1} \cdot (\text{ unit }) = z_{11}^{m'_{11}} \cdots z_{1l_1}^{m'_{1l_1}}\\
&\cdots\\
&[u_k^{r_{kk}} \prod_{p=k+1}^{a} u_p^{r_{pk}}]^{d'_k} \cdot (\text{ unit }) = z_{k1}^{m'_{k1}} \cdots z_{kl_k}^{m'_{kl_k}}\\
&[\prod_{p=k+1}^{a} u_p^{r_{p,k+1}}]^{d'_{k+1}} \cdot (\text{ unit }) = z_{k+1,1}^{m'_{k+1,1}} \cdots z_{k+1,l_{k+1}}^{m'_{k+1,l_{k+1}}}\\
&\cdots\\
&[\prod_{p=k+1}^{a} u_p^{r_{pn}}]^{d'_n} \cdot (\text{ unit }) = z_{n1}^{m'_{n1}} \cdots z_{nl_n}^{n'_{nl_n}}
\end{aligned}
$$

Moreover, the open set corresponding to $\bar{V} - (\bar{V}_E \cup \bar{V}_F \cup \bar{V}_B)$ and $\bar{V} - (\bar{V}_E \cup \bar{V}_F)$ is given by

$$
\begin{gathered}
u_1 \in B(u_1)^0, \dots, u_a \in B(u_a)^0, \text{ and}\\
u_{k+1} \in B(u_{k+1})^0, \dots, u_a \in B(u_a)^0
\end{gathered}
$$

respectively. Let $\bar{j}_{EFB} : \bar{V} - (\bar{V}_E \cup \bar{V}_F \cup \bar{V}_B) \to \bar{V}$, $\bar{j}_{EF} : \bar{V} - (\bar{V}_E \cup \bar{V}_F) \to \bar{V}$ and $\bar{j}_M : V^0 \times B^0(\xi,\tau) \to V \times \bar{B}(\xi)$ be the natural inclusions. Then

$$
\begin{aligned}
R^i(\bar{j}_{EFB})_*\mathbf{Q} &= \wedge^i R^1(\bar{j}_{EFB})_*\mathbf{Q},\\
R^i(\bar{j}_{EF})_*\mathbf{Q} &= \wedge^i R^1(\bar{j}_{EF})_*\mathbf{Q},\\
R^i(\bar{j}_{M})_*\mathbf{Q} &= \wedge^i R^1(\bar{j}_{M})_*\mathbf{Q},
\end{aligned}
\tag{3.3.1}
$$

for $i \geq 1$. Using the local expression of $\bar{V} - (\bar{V}_E \cup \bar{V}_F \cup \bar{V}_B)$, $\bar{V} - (\bar{V}_E \cup \bar{V}_F)$ and $V^0 \times B(\xi,\tau)^0 \to V \times \bar{B}(\xi)$, the stalks if the natural homomorphism

$$
R^1(\bar{j}_M)_*\mathbf{Q} \xrightarrow{\bar{\alpha}^*} R^1(\bar{j}_{EFB})_*\mathbf{Q} \xleftarrow{\bar{\beta}^*} R^1(\bar{j}_{EF})_*\mathbf{Q}
$$

at $\bar{v}$ are computed as follows.

$$
\begin{aligned}
(R^1(\bar{j}_M)_*\mathbf{Q})_{\bar{v}} &= [\mathbf{Q}^a \oplus (\oplus_{i=1}^n \mathbf{Q}^{l_i} \oplus \mathbf{Q})] \otimes \mathbf{Q}(-1) \\
(R^1(\bar{j}_{EFB})_*\mathbf{Q})_{\bar{v}} &= [\mathrm{Coker}(\mathbf{Q}^{n+1} \overset{A_{EFB}}{\to} (\mathbf{Q}^a \oplus (\oplus_{i=1}^n \mathbf{Q}^{l_i} \oplus \mathbf{Q})))] \otimes \mathbf{Q}(-1) \\
(R^1(\bar{j}_{EF})_*\mathbf{Q})_{\bar{v}} &= [\mathrm{Coker}(\mathbf{Q}^{n-k+1} \overset{A_{EF}}{\to} (\mathbf{Q}^{a-k} \oplus (\oplus_{i=k+1}^n \mathbf{Q}^{l_i} \oplus \mathbf{Q})))] \otimes \mathbf{Q}(-1),
\end{aligned}
$$

where

$$
\begin{aligned}
A_{EFB}(e_i) &= d'_i(r_{1i}, \ldots, r_{ai}) \oplus (0, \ldots, 0, (m'_{i1}, \ldots, m'_{i,l_i}), 0, \ldots, 0), (1 \le i \le n) \\
A_{EFB}(e_{n+1}) &= d'_i(r_{1,n+1}, \ldots, r_{a,n+1}) \oplus (0, \ldots, 0, 1),
\end{aligned}
$$

and

$$
\begin{aligned}
A_{EF}(e_i) &= d'_i(r_{k+1i}, \ldots, r_{ai}) \oplus (0, \ldots, 0, (m'_{i1}, \ldots, m'_{i,l_i}), 0, \ldots, 0), (k+1 \le i \le n) \\
A_{EF}(e_{n+1}) &= d'_i(r_{k+1,n+1}, \ldots, r_{a,n+1}) \oplus (0, \ldots, 0, 1).
\end{aligned}
$$

The morphism $\bar{\alpha}^*$ is identified with the natural projection. Since the diagram

$$
\begin{tikzcd}
\mathbf{Q}^{n+1} \arrow[r, "A_{EFB}"] & \mathbf{Q}^a \oplus (\oplus_{i=1}^n \mathbf{Q}^{l_i} \oplus \mathbf{Q}) \\
\mathbf{Q}^{n-k+1} \arrow[u] \arrow[r, "A_{EF}"'] & \mathbf{Q}^{a-k} \oplus (\oplus_{i=k+1}^n \mathbf{Q}^{l_i} \oplus \mathbf{Q}) \arrow[u]
\end{tikzcd}
$$

is commutative, $\bar{\beta}^*$ is identified with the homomorphism induced by the natural inclusions. By (3.3.1) and the expression of $\bar{\alpha}^*$ and $\bar{\beta}^*$, we have the following proposition.

Proposition 3.3.1.

(1) *The morphism $\bar{\alpha}^*$ is injective an $\bar{\beta}^*$ is surjective.*
(2) *The morphism $R^i(\bar{j}_M)_*\mathbf{Q} \to R^i(\bar{j}_{EFB})_*\mathbf{Q}$ and $R^i(\bar{j}_{EF})_*\mathbf{Q} \to R^i(\bar{j}_{EFB})_*\mathbf{Q}$ are identified with $\wedge^i \bar{\alpha}^*$ and $\wedge^i \bar{\beta}^*$. Moreover they are surjective and injective respectively.*

Since $\tilde{B}(\xi)$ meets $\bar{D}_E \cup \bar{D}_F \cup \bar{D}_B$ transversally, we get the following proposition.

Proposition 3.3.2. *For $\bar{v} \in \tilde{V}$, we have*

$$
\begin{aligned}
(R^i(\tilde{j}_{EFB})_*\mathbf{Q})_{\bar{v}} &= (R^i(\bar{j}_{EFB})_*\mathbf{Q})_{\bar{v}}, \\
(R^i(\tilde{j}_{EF})_*\mathbf{Q})_{\bar{v}} &= (R^i(\bar{j}_{EF})_*\mathbf{Q})_{\bar{v}}, \\
(R^i(\tilde{j}_M)_*\mathbf{Q})_{\bar{v}} &= (R^i(\bar{j}_M)_*\mathbf{Q})_{\bar{v}}.
\end{aligned}
$$

As a consequence, Proposition 3.3.1 holds by replacing $\bar{j}_{EFB}$, $\bar{j}_{EF}$ and $\bar{j}_M$ by $\tilde{j}_{EFB}$, $\tilde{j}_{EF}$ and $\tilde{j}_M$.

§3.4 Supplement for mixed Hodge structures

Let X be a topological space and $K = ((K_{\mathbf{Q}}, W), (K_{\mathbf{C}}, W, F))$ and $K' = ((K'_{\mathbf{Q}}, W), (K'_{\mathbf{C}}, W, F))$ be cohomological mixed Hodge complexes (=CMHC's for short). A morphism from K to K' is defined by the pair of morphisms of filtered and bifiltered complexes $\varphi_{\mathbf{Q}} : K_{\mathbf{Q}} \to K'_{\mathbf{Q}}$ and $\varphi_{\mathbf{C}} : K_{\mathbf{C}} \to K'_{\mathbf{C}}$ where $(\varphi_{\mathbf{Q}} \otimes \mathbf{C}, W)$ and $(\varphi_{\mathbf{C}}, W)$ are filtered congruent to each other. After [Dur], we define the cone $\mathrm{Cone}(\varphi)$ of φ by

$$\begin{aligned} \mathrm{Cone}(\varphi)^p_A &= K^p_A \oplus (K')^{p-1}_A, \\ d : \mathrm{Cone}(\varphi)^p_A \to \mathrm{Cone}(\varphi)^{p+1}_A &; (x, y) \mapsto (dx, \varphi(x) + dy), \\ W_k \,\mathrm{Cone}(\varphi)^p_A &= W_k K^p_A \oplus W_{k+1}(K')^{p-1}_A, \text{ for } A = \mathbf{Q}, \mathbf{C} \text{ and} \\ F^q \,\mathrm{Cone}(\varphi)^p_{\mathbf{C}} &= F^q K^p_{\mathbf{C}} \oplus F^q (K')^{p-1}_{\mathbf{C}}. \end{aligned}$$

According to [Dur], $\mathrm{Cone}(\varphi)$ give rise to be a CMHC.

Definition 3.4.1. A morphism $\varphi : K \to K'$ of CMHC's is called weakly equivalent if the underlying morphism $K_{\mathbf{Q}} \to K'_{\mathbf{Q}}$ is a quasi-isomorphism.

Remark 3.4.2. By weak equivalence, we do not impose that $\varphi_{\mathbf{Q}}$ or $\varphi_{\mathbf{C}}$ are filtered quasi-isomorphism.

The following lemma will be useful later.

Lemma 3.4.3. *Let $f : K \to L$ and $b : L' \to L$ be morphisms of CMHC's on X. Suppose that b is weak equivalent. Then there exists a cohomological mixed Hodge complex K' and morphisms $a : K' \to K$ and $f' : K' \to L'$ of CMHC's on X such that a is a weak equivalence and $f \circ a = b \circ f'$.*

Proof. Let $K' = \mathrm{Cone}((f, b) : K \oplus L' \to L)$ and $a : K' \to K$ and $f' : K' \to L'$ be the natural morphisms. Then K' is equipped with CMHC and a and f are morphism of CMHC's which satisfy the conditions of the lemma.

We define direct sum decomposition of CMHC's.

Definition 3.4.4.

(1) Let K, L and M be CMHC's and $p_1 : K \to L$ and $p_2 : K \to M$ be morphisms of CMHC's. $p_1 \oplus p_2 : K \to L \oplus M$ is called a direct sum decomposition if it is weakly equivalent.

(2) Two direct sum decompositions $f_1 : K \to L_1 \oplus M_1$ and $f_2 : K \to L_2 \oplus M_2$ are said to be equivalent if there exists a third direct sum decomposition $p_3 : K \to L_3 \oplus M_3$ and weak equivalences $g_i : L_3 \to L_i$ and $h_i : M_3 \to M_i$ $(i = 1, 2)$ such that the following diagram commutes

$$\begin{array}{ccc} K & \xrightarrow{f_3} & L_3 \oplus M_3 \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle g_i \oplus h_i} \\ K & \xrightarrow[f_i]{} & L_i \oplus M_i \end{array}$$

Lemma 3.4.5. *Let $f : K \to L \oplus M$ be a direct sum decomposition and $K \to K'$ be a weak equivalence. Then there exist CMHC's L', M' and weak equivalences $a : L \to L'$, $b : M \to M'$ and $f' : K' \to L' \oplus M'$ such that the following diagram commutes.*

$$\begin{tikzcd} K \arrow[r, "g"] \arrow[d, "f"'] & K' \arrow[d, "f'"] \\ L \oplus M \arrow[r, "(a,b)"'] & L' \oplus M' \end{tikzcd}$$

Proof. Since f is weakly equivalent, the natural morphism $f^* : \mathrm{Cone}(K \to L) \oplus \mathrm{Cone}(K \to M) \to K$ is weakly equivalent. Therefore the composite

$$g \circ f^* : \mathrm{Cone}(K \to L) \oplus \mathrm{Cone}(K \to M) \to K'$$

is weakly equivalent. Therefore

$$f' : K' \to \mathrm{Cone}(\mathrm{Cone}(K \to L) \to K')[1] \oplus \mathrm{Cone}(\mathrm{Cone}(K \to M) \to K')[1]$$

is a direct sum decomposition. We put $L' = \mathrm{Cone}(\mathrm{Cone}(K \to L) \to K')[1]$ and $M' = \mathrm{Cone}(\mathrm{Cone}(K \to M) \to K')[1]$. Since the natural morphisms $L \to \mathrm{Cone}(\mathrm{Cone}(K \to L) \to K)$ and $M \to \mathrm{Cone}(\mathrm{Cone}(K \to M) \to K)$ are weakly equivalent, the composites

$$a : L \to \mathrm{Cone}(\mathrm{Cone}(K \to L) \to K) \to \mathrm{Cone}(\mathrm{Cone}(K \to L) \to K') = L'$$
$$b : M \to \mathrm{Cone}(\mathrm{Cone}(K \to M) \to K) \to \mathrm{Cone}(\mathrm{Cone}(K \to M) \to K') = M'$$

are weakly equivalent. Thus we have the required CMHC's and weak equivalences.

Definition 3.4.6 (Quasi-canonical filtration).

(1) Let K be a CMHC on X. A sequence $\kappa_i : K_i \to K$ and $\kappa_{i,i+1} : K_i \to K_{i+1}$ of morphism of CMHC's is called quasi-canonical filtration if (1) there exists $K_i' \to K$ such that the diagram is commutative

$$\begin{tikzcd} K_i' \arrow[r, "a"] \arrow[d, "b"'] & K_i \arrow[d] \\ \tau_i K \arrow[r] & K, \end{tikzcd}$$

where a and b are quasi-isomorphism. Note that we do not impose that either K_i' or $\tau_i K$ are equipped with structures of CMHC's, and (2) $\kappa_i = \kappa_{i+1} \circ \kappa_{i+1}$.

(2) The decomposition $\mathrm{Cone}(K_{i-1} \to K_i)[1] \to L_i \oplus M_i$ of the cone $\mathrm{Cone}(K_{i-1} \to K_i)$ is called a decomposition associated to quasi-canonical filtration $\{K_i \to K, K_{i-1} \to K_i\}$.

Definition 3.4.7 (Associated decomposition). Let $K_i \to K, K_{i-1} \to K_i$ be a quasi-canonical filtration and $\mathrm{Cone}(K_{i-1} \to K_i)[1] \to C^i \oplus I^i$ be a decomposition associated to the quasi canonical filtration. Let $L \to K$ be a morphism of CMHC's.

The decomposition is associated to the morphism $L \to K$ if there exists complexes A, B and morphisms $f : A \to B$, $a : A \to \mathcal{H}^i L$, $b : B \to \mathcal{H}^i K$ and $b' : B \to \mathrm{Cone}(K_{i-1} \to K_i)[1]$ such that

(1) The morphisms a, b and b' are quasi-isomorphism.
(2) The following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "f'"] \arrow[d, "a"'] & B \arrow[r, "b'"] \arrow[d, "b"] & \mathrm{Cone}(K_{i-1} \to K_i)[1] \\
\mathcal{H}^i L \arrow[r, "\mathcal{H}^i(f)"'] & \mathcal{H}^i K &
\end{tikzcd}$$

(3) The composite $A \to B \to \mathrm{Cone}(K_{i-1} \to K_i)[1] \to I^i$ is a quasi isomorphism.

Remark 3.4.8. We do not impose the complexes of sheaves A, B and $\mathcal{H}^i L$ are equipped with the structure of CMHC's.

Now we can prove the following proposition.

Proposition 3.4.9. *Let L, K be CMHC's, $K_i \to K, K_{i-1} \to K_i$ be a finite quasi-canonical filtration, $f_i : \mathrm{Cone}(K_{i-1} \to K_i) \to C^i \oplus I^i$ be decompositions of $\mathrm{Cone}(K_{i-1} \to K_i)$ and $\varphi : L \to K$ be a morphism of CMHC's. Suppose that the morphism φ is associated to the decompositions f_i. Then there exists a quasi-canonical filtration $\{L_i \to L, L_{i-1} \to L_i\}$ such that $\mathrm{Cone}(L_{i-1} \to L_i)$ is weakly equivalent to I^i.*

Proof. We construct L_m, $L_m \to K_m$ and $L_{m-1} \to L_m$ by descending induction. For a sufficiently large m, $K_m \to K$ is a quasi isomorphism and $K_i \to K_{i+1}$ is isomorphic for $i \geq m$. By Lemma 3.4.3, we can take a CMHC L^m such that the following diagram of CMHC's commutes

$$\begin{tikzcd}
L_m \arrow[r] \arrow[d, "i_m"'] & K_m \arrow[d] \\
L \arrow[r] & K
\end{tikzcd}$$

such that i_m is a weak equivalence. Therefore we can take $L_m \to L$ for a sufficiently large m. Suppose that L_m, $L_m \to K_m$ and $L_m \to L_{m+1}$ $(m \geq k)$ satisfying the conditions of the proposition are given by induction. Let $'I^k = \mathrm{Cone}(\mathrm{Cone}(K_{k-1} \to K_k)[1] \to C^k)$. Then the composite $'I^k \to \mathrm{Cone}(K_{k-1} \to K_k)[1] \to I^k$ is a weak equivalence and it is denoted by b. The morphism $L_k \to K_k \to \mathrm{Cone}(K_{k-1} \to K_k)[1] \to I^k$ is denoted by a. Then we have the following diagram;

$$\begin{tikzcd}
& 'I^k \arrow[r] \arrow[d, "b"] & \mathrm{Cone}(K_{k-1} \to K_k)[1] \\
L^k \arrow[r, "a"] & I_k, &
\end{tikzcd}$$

where b is weakly equivalent. Then by Lemma, there exists L_k', $a'' : L_k' \to' I^k$ and $b' : L_k' \to L_k$ such that b' is weakly equivalent. Let $L_{k-1} = \mathrm{Cone}(L_k' \to' I^k)$. Then there are natural morphisms

$$L_{k-1} = \mathrm{Cone}(L_k' \to' I^k) \to \mathrm{Cone}(K_k \to \mathrm{Cone}(K_{k-1} \to K_k)[1]) \to K_{k-1}$$

and $L_{k-1} = \mathrm{Cone}(L_k' \to' I^k) \to L_k' \to L_k$. This proceeds the steps of the induction.

Corollary 3.4.10. *Under the notations as in Proposition 3.4.9,*

$$[\mathbf{H}^*(X,L)] = \sum_i (-1)^i [\mathbf{H}^*(X,I^i)]$$

in $K_{MH}(\mathbf{C})$

Proof. Since L is finite, L_m is exact for $m << 0$ and $L_m \to L$ is weakly equivalent. for $m >> 0$. Therefore,

$$\begin{aligned}[\mathbf{H}^*(X,L)] &= \sum_i (-1)^i [\mathbf{H}^*(X,\mathrm{Cone}(L_{i-1} \to L_i))] \\ &= \sum_i (-1)^i [\mathbf{H}^*(X,I^i)]\end{aligned}$$

§3.5 Direct sum decomposition for normal crossing local spaces

Let (N,X,Y) be a normal crossing local space and $\cup_{i\in I} D_i$ be the irreducible decomposition of $D = X \cup Y$. Let us define the complex subvariety D_J of X as $\cap_{j\in J} D_j$. The complex of sheaves $\Omega^\bullet_N(\log D) \otimes \mathcal{O}_{D_j}$ is denoted by $\Omega^\bullet(J)$. We have introduced a structure of cohomological mixed Hodge complex on $\mathbf{R}j_*\mathbf{Q}\mid_{D_I}$, where $j : N - (X \cup Y) \to N$ is the natural inclusion, via the quasi-isomorphism

$$(\mathbf{R}j_*\mathbf{Q}\mid_{D_I}) \otimes \mathbf{C} \simeq \Omega^\bullet(J)$$

with the filtrations W_k and F^p on the right hand side. Let $Z_J = \cup_{i\notin J} D_{J\cup i}$ and $D_J^0 = D_J - Z_J$ and the natural inclusion from D_J^0 to D_J is denoted by $j_J : D_J^0 \to D_J$. As in §?, the structure of CMHC on $(j_J)_!(\mathbf{R}j_*\mathbf{Q}\mid_{D_J^0})$ is given by the complex

$$K_J^\bullet : \Omega^\bullet(J) \to \oplus_{\#K=1,K\cup J=\emptyset}\Omega^\bullet(J\cup K) \to \oplus_{\#K=2,K\cup J=\emptyset}\Omega^\bullet(J\cup K) \to \cdots$$

with the filtrations

$$\begin{aligned}W_kK_J^\bullet : W_k\Omega^\bullet(J) &\to \oplus_{\#K=1,K\cup J=\emptyset}W_{k+1}\Omega^\bullet(J\cup K) \\ &\to \oplus_{\#K=2,K\cup J=\emptyset}W_{k+2}\Omega^\bullet(J\cup K) \to \cdots, \text{ and}\end{aligned}$$

$$\begin{aligned}F^pK_J^\bullet : F^p\Omega^\bullet(J) &\to \oplus_{\#K=1,K\cup J=\emptyset}F^p\Omega^\bullet(J\cup K) \\ &\to \oplus_{\#K=2,K\cup J=\emptyset}F^p\Omega^\bullet(J\cup K) \to \cdots.\end{aligned}$$

We define sub-complex $K_{J,i}^\bullet$, $W_kK_{J,i}^\bullet$ and $F^pK_{J,i}^\bullet$ by

$$\begin{aligned}K_{J,i}^\bullet : W_i\Omega^\bullet(J) &\to \oplus_{\#K=1,K\cup J=\emptyset}W_i\Omega^\bullet(J\cup K) \\ &\to \oplus_{\#K=2,K\cup J=\emptyset}W_i\Omega^\bullet(J\cup K) \to \cdots,\end{aligned}$$

$W_kK_{J,i}^\bullet = K_{J,i}^\bullet \cap W_kK_J^\bullet$ and $F^pK_{J,i}^\bullet = K_{J,i}^\bullet \cap F^pK_J^\bullet$.

Proposition 3.5.1.

(1) *The sequence of morphism $K^\bullet_{J,i} \to K^\bullet_J$, $K^\bullet_{J,i-1} \to K^\bullet_{J,i}$ are quasi canonical filtration.*

(2) *The quotient complex $K_{J,i}/K_{J,i-1}$ is weakly equivalent to $L^\bullet_J \otimes \wedge^i(\mathbf{Q}^J) \simeq L^\bullet_J \otimes (\oplus_{\# L=i, L\subset J}\mathbf{Q})$, where*

$$L^\bullet_J : \Omega^\bullet_{D_J}[-i] \to \oplus_{\# K=1}\Omega_{D_{J\cup K}}[-i] \to \cdots$$

(3) *Let $\mathbf{Q}^J = M_1 \oplus M_2$ be a direct sum decomposition of $\mathbf{Q}^J$. Then the direct sum decomposition a direct sum decomposition $\wedge^i M_1 \oplus M_2 \wedge \wedge^i(\mathbf{Q}^J)$ give reise to a direct sum decomposition $L^\bullet_J \otimes \wedge^i M_1 \oplus L^\bullet_J \otimes (M_2 \wedge \wedge^i(\mathbf{Q}^J))$ of $K_{J,i}/K_{J,i-1}$.*

Proof. First we show that $K^\bullet_{J,i}$ is a CMHC. The sheaf $Gr^W_k K_{J,i}$ is given by

$$\begin{aligned}
Gr^W_i K^\bullet_{J,i} =&(Gr^W_k\Omega^\bullet(J) \xrightarrow{0} \oplus_{\# K=1, K\cup J=\emptyset} Gr^W_{k+1}\Omega^\bullet(J\cup K) \xrightarrow{0} \cdots \xrightarrow{0} \\
&\oplus_{\# K=i-k, K\cup J=\emptyset}\, Gr^W_i\Omega^\bullet(J\cup K) \to 0) \\
=&(\oplus_{\# L=k}\Omega^\bullet_{D_{L\cup J}}[-k] \xrightarrow{0} \oplus_{\# K=1, \# L=k+1, K\cup J=\emptyset}\Omega^\bullet_{D_{L\cup J\cup K}}[-k-1] \xrightarrow{0} \\
&\cdots \xrightarrow{0} \oplus_{\# K=i-k, \# L=k+(i-k), K\cup J=\emptyset}\Omega^\bullet_{D_{L\cup J\cup K}}[-i] \to 0)
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathbf{H}^m(Gr^p_F Gr^W_k K_{J,i}) = &\oplus_{\# L=k}\, H^{m-p}(\Omega^{p-k}_{D_{L\cup J}})\oplus \\
&\oplus_{\# K=1, \# L=k+1, K\cup J=\emptyset}\, H^{p-m-1}(\Omega^{p-k-1}_{D_{L\cup J\cup K}}) \\
&\oplus \cdots \oplus \oplus_{\# K=i-k, \# L=k+(i-k), K\cup J=\emptyset} H^{m-i+k-p}(\Omega^{p-i}_{D_{L\cup J\cup K}}).
\end{aligned}$$

and $Gr^W_k(K^\bullet_{J,i})$ is a cohomological Hodge complex. Since $K^\bullet_{J,i}$ for $i << 0$ and $K_{J,i} \simeq K_J$ for $i >> 0$, it is sufficient to prove that $\mathcal{H}^p(K^\bullet_{J,i}/K^\bullet_{J,i-1}) = 0$ if $p \neq i$. Since $K^\bullet_{J,i}$ is quasi isomorphic to

$$\tau_i \mathbf{R} j_*\mathbf{Q}\mid_{D_J} \to \oplus_{\# K=1, K\cup J=\emptyset}\tau_i\mathbf{R}j_*\mathbf{Q}\mid_{D_{J\cup K}} \to \cdots,$$

the quotient complex $K_{J,i}/K_{J,i-1}$ is quasi isomorphic to

$$\mathbf{R}^i j_*\mathbf{Q}\mid_{D_J}[-i] \to \oplus_{\# K=1, K\cup J=\emptyset}\mathbf{R}^i j_*\mathbf{Q}\mid_{D_{J\cup K}}[-i] \to \cdots,$$

and it is quasi isomorphic to $(j_J)_!(\mathbf{R}j_*\mathbf{Q}\mid_{D^0_J})[-i]$. This proves the proposition.

(2) The associated graded complex of sheaves $K_{J,i}/K_{J,i-1}$ is isomorphic to

$$\oplus_{\# L=i}\Omega^\bullet_{L\cup J}[-i] \to \oplus_{\# L=i, \# K=1, K\cap J=\emptyset}\Omega^\bullet_{L\cup J\cup K}[-i] \to \cdots$$

and therefore it is a direct sum of $M^\bullet_L$, $(\# L = i)$, where

$$M^\bullet_L = (\Omega^\bullet_{L\cup J}[-i] \to \oplus_{\# K=1, K\cap J=1}\Omega^\bullet_{L\cup J\cup K}[-i] \to \cdots).$$

Moreover, the direct sum is compatible with the filtration $F^\bullet$ and $W_\bullet$ induced by that of $K^\bullet_{J,i}$ and it is easy to see that these filtrations defines structure of CMHC.

This morphism defines a morphism $\mathrm{Cone}(K_{J,i-1} \to K_{J,i})[1] \to \oplus_{\#L=i} M_L^\bullet$ of CMHC which is a quasi-isomorphism, i.e. a weak equivalence of CMHC. (Note that in general if $0 \to K \to L \to M \to 0$ is an exact sequence of CMHC, then the natural map $\mathrm{Cone}(K \to L)[1] \to M$ is a weak equivalence. Therefore $H^i(X, \mathrm{Cone}(K \to L)) \simeq H^i(X, M)$ is an isomorphism of mixed Hodge structures. cf. [D], §2.5 for $X =$ a point.) We show that $M_L^\bullet$ is quasi-isomorphic to 0 if $L \not\subseteq J$. Let $a \in L - J$ and put $J' = J \cup \{a\}$. We define complexes ${}'M_L^\bullet$ and ${}''M_L^\bullet$ as

$$
\begin{aligned}
{}'M_L^\bullet &= (\Omega_{L\cup J'} \to \oplus_{\#K=1, K\cap J'=\emptyset} \Omega_{L\cup J'\cup K} \to \cdots) \\
{}''M_L^\bullet &= (0 \to \oplus_{\#K=1, K\cap J=\emptyset, a\in K} \Omega_{L\cup J'\cup K} \to \oplus_{\#K=2, K\cap J=\emptyset, a\in K} \Omega_{L\cup J'\cup K} \to \cdots)
\end{aligned}
$$

Then it is easy to see that $M_L^p =' M_L^p \oplus'' M_L^p$. The bijection between $\{K' \mid \#K' = p, K' \cap J' = \emptyset\}$ and $\{K \mid \#K = p+1, K \cap J \emptyset, a \in K\}$ given by $K' = K - \{a\}$ gives an isomorphism of complexes ${}'M_L^\bullet \to'' M_L^\bullet$. As a complex, it is easy to check that $M_L^\bullet = \mathrm{Cone}({}'M_L^\bullet \to'' M_L^\bullet)$ and this proves the exactness of $M_L^\bullet$ if $L \not\subseteq J$. Moreover, if $L \subset J$, then $M_L^\bullet \simeq L_J^\bullet$. Therefore we get the statement (2) of the proposition.

The stetement (3) is a direct consequence of (2).

§3.6 Convolution theorem for Hodge theory.

First, we compute $\Psi_{V,m,\sigma}(\mathbf{Q})$ using the result of §3.3, §3.4 and §3.5. The normal crossing divisor V_B of V defines a natural stratification and V_c is expressed as a union of this stratification: $\cup_{\tau\in K} S_\tau^0$. The morphism $R^i(\tilde{j}_M)_*\mathbf{Q} \to R^i(\tilde{j}_{EFB})_*\mathbf{Q}$ is surjective and identified with a projection for direct sum decomposition of CMHC's by the expression of $R^i(\tilde{j}_M)_*\mathbf{Q} \mid_{\tilde{Z}_\sigma^0}$ in Proposition 3.3.2 and the expression of $\tilde{\alpha}^*$ in Proposition 3.3.1.(2). Therefore $R^i(\tilde{j}_{EFB})_*\mathbf{Q} \mid_{S_\tau^0 \times \tilde{Z}_\sigma^0}$ has a structure of CMHC which is weakly equivalent to $\wedge^i(\mathbf{Q}(-1)^{\tilde{r}(\sigma,\tau)})$, as a CMHC, where $\tilde{r}(\sigma,\tau) = \dim R^1(\tilde{j}_{EFB})_*\mathbf{Q} \mid_{\bar{v}}$ for $\bar{v} \in S_\tau^0 \times \tilde{Z}_\sigma^0$ using Proposition 3.5.1.(3) and Theorem 2.4.9. We use the same argument again to compute $R^i(\tilde{j}_{EF})_*\mathbf{Q} \mid_{S_\tau^0 \times \tilde{Z}_\sigma^0}$ and we get an isomorphism $R^i(j_{EF})_*\mathbf{Q} \simeq \wedge^i\mathbf{Q}(-1)^{r(\sigma,\tau)}$ as a CMHC,s, where $r(\sigma,\tau) = \sum_{i=k+1}^n l_i + a - n$. Here we used the same notations $l_{k+1}, \dots, l_n$ and a in §3.3. Therefore, we have an isomorphism

$$\mathbf{H}_c^*(S_\tau^0 \times \tilde{Z}_\sigma^0, \mathbf{R}^i(\tilde{j}_{EF})_*\mathbf{Q}) \simeq \mathbf{H}_c^*(S_\tau^0, \mathbf{Q}) \otimes \mathbf{H}_c^*(\tilde{Z}_\sigma^0, \mathbf{Q}) \otimes \wedge^i(\mathbf{Q}(-1)^{r(\sigma,\tau)}),$$

as $G_{\sigma,\tau}$-Hodge structure, where $G_{\sigma,\tau}$ is the stabilizer of $S_\tau \times \tilde{Z}_\sigma^0$ in $G \times \mu_m$. As a consequence, we have

$$[\mathbf{H}_c^*(S_\tau^0 \times \tilde{Z}_\sigma^0, \mathbf{R}^i(\tilde{j}_{EF})_*\mathbf{Q})] = [\mathbf{H}_c^*(S_\tau^0, \mathbf{Q})] \otimes [\mathbf{H}_c^*(\tilde{Z}_\sigma^0, \mathbf{Q})] \otimes [\mathbf{Q} - \mathbf{Q}(-1)]^{r(\sigma,\tau)}.$$

in $K_{MH}(\mathbf{C}, G_{\sigma,\tau})$. Let $r'(\sigma,\tau) = \sum_{i=k+1}^n (l_i - 1)$ and $r''(\sigma,\tau) = a - k$. Then we have the following equality:

$$\sum_{[\tau]\in K/G} \mathrm{Ind}_{G_\tau}^G [\mathbf{H}_c^*(S_\tau^0, \mathbf{Q})] \otimes [\mathbf{Q} - \mathbf{Q}(-1)]^{r'(\sigma,\tau)} = \otimes_{i\in I}[\mathbf{H}_c^*(V_{c,i}, \mathbf{Q})] \otimes_{i\notin I} \phi_{g_i,d_i}(\mathbf{Q}),$$

in $K_{MH}(\mathbf{C}, G)$, where $I = I(\sigma) = \{i \mid \mathbf{e}_i \in \sigma\}$. Therefore, we have

$$\Psi_{V,m,\sigma} = \otimes_{i\in I}[\mathbf{H}_c^*(V_{c,i}, \mathbf{Q})] \otimes \otimes_{i\notin I}\phi_{g_i,d_i}(\mathbf{Q}) \otimes [\mathbf{H}_c^*(\tilde{Z}_\sigma^0, \mathbf{Q})] \otimes [\mathbf{Q} - \mathbf{Q}(-1)]^{r''(\sigma,\tau)}$$

in $K_{MH}(\mathbf{C}, G\times\mu_m)$. For any $I\subset[1,n]$, we have

$$\sum_{\{I(\sigma)=I\}/G}\mathrm{Ind}_{G_\sigma}^{G}[\mathbf{H}_c^*(\tilde{Z}_\sigma^0,\mathbf{Q})]\otimes(\mathbf{Q}-\mathbf{Q}(-1))^{r''(\sigma,\tau)}=\mathbf{H}_c^*(\tilde{Z}_I^0,\mathbf{R}\tilde{j}_*\mathbf{Q}),$$

where $\tilde{j}:\tilde{B}(\xi)-(\tilde{D}_E\cup\tilde{D}_F)\to\tilde{B}(\xi)$ and $\tilde{Z}_I^0=\cup_{I(\sigma)=I}\tilde{Z}_\sigma^0$. As a consequence, we have

$$\begin{aligned}
&[\mathbf{H}^*(\tilde{B}_{\mathbf{d}}(y)-(f(\xi)\circ g)^{-1}(0),k_{\emptyset!}\mathbf{Q})]\\
=&\sum_{K\times\{\sigma|I(\sigma)=\emptyset\}/G\times\mu_m}\mathrm{Ind}_{G_{sigma,\tau}}^{G\times\mu_m}[\mathbf{H}_c^*(S_\tau^0,\mathbf{Q})]\otimes[\mathbf{H}_c^*(\tilde{Z}_\sigma^0,\mathbf{Q})]\otimes[\mathbf{Q}-\mathbf{Q}(-1)]^{r(\sigma,\tau)}\\
=&\otimes_{i=1}^n\phi_{g_i,d_i}(\mathbf{Q})\otimes[\mathbf{H}_c^*(\tilde{Z}_\emptyset^0,\mathbf{R}\tilde{j}_*\mathbf{Q})],
\end{aligned}$$

where $k_\emptyset$ and $\tilde{B}_d^0(y)$ is defined at the end of §3.2. In the same way, for a subset I of $[1,n]$, we have

$$[\mathbf{H}^*(\tilde{B}_{I,\mathbf{d}}(y)-(f_I(\xi)\circ g)^{-1}(0),k_{I!}\mathbf{Q})]=\otimes_{i\notin I}\phi_{g_i,d_i}(\mathbf{Q})\otimes[\mathbf{H}_c^*(\tilde{Z}_I^0,\mathbf{R}\tilde{j}_*\mathbf{Q})].$$

Using the equality (3.2.2), we have

$$\begin{aligned}
&[\mathbf{H}^*(\tilde{B}_{\mathbf{d}}(y)-(f(\xi)\circ g)^{-1}(0),\mathbf{Q})]\\
=&\sum_{I\subset[1,n]}[\mathbf{H}^*(\tilde{B}_{I,\mathbf{d}}(y)-(f_I(\xi)\circ g)^{-1}(0),k_{I!}\mathbf{Q})]\\
=&\sum_{I\subset[1,n]}\otimes_{i\notin I}\phi_{g_i,d_i}(\mathbf{Q})\otimes[\mathbf{H}_c^*(\tilde{Z}_I^0,\mathbf{R}\tilde{j}_{I*}\mathbf{Q})]\\
=&\sum_{I\subset[1,n]}(-1)^{\#I}\otimes_{i\notin I}\phi_{g_i,d_i}(\mathbf{Q})\otimes\otimes_{i\in I}(\phi_{g_i,d_i}(\mathbf{Q})-[\mathbf{Q}])\otimes[\mathbf{H}_c^*(\tilde{Z}_I,\mathbf{R}\tilde{j}_{I*}\mathbf{Q})]\\
=&\sum_{I\subset[1,n]}(-1)^{\#I}\otimes_{i\notin I}\phi_{g_i,d_i}(\mathbf{Q})\otimes\otimes_{i\in I}(\phi_{g_i,d_i}(\mathbf{Q})-[\mathbf{Q}])\\
&\qquad\otimes([\mathbf{H}_c^*(\tilde{Z}_I,\mathbf{R}\tilde{j}_{I*}\mathbf{Q})]-[\mathbf{Q}-\mathbf{Q}(-1)]\\
&+[\mathbf{Q}-\mathbf{Q}(-1)]
\end{aligned}$$

we have

$$\begin{aligned}
\tilde{\Phi}_{f\circ g,m}(\mathbf{Q})&=(\frac{1}{[\mathbf{Q}-\mathbf{Q}(-1)]}[\mathbf{H}_c^*(\tilde{B}_{\mathbf{d}}(y)-(f(\xi)\circ g)^{-1}(0),\mathbf{Q})])-\mathbf{Q}\\
&=\sum_{I\subset[1,n]}(-1)^{\#I}\otimes_{i\notin I}\phi_{g_i,d_i}(\mathbf{Q})\otimes\otimes_{i\in I}\Phi_{g_i,d_i}(\mathbf{Q})\otimes\tilde{\Phi}_{f_I,m}(\mathbf{Q}).
\end{aligned}$$

By taking G-invariant part of the above equality, we have the following theorem. Note that the action of G on $\mathbf{H}_c^*(\tilde{Z}_I^0,\mathbf{R}\tilde{j}_*\mathbf{Q})$ factors through G_I.

Theorem 3.6.1 (Convolution Theorem for Hodge structures).

$$\Phi_{f\circ g,m}(\mathbf{Q})=\sum_{I\subset[1,n]}(-1)^{\#I}(\otimes_{i\notin I}\phi_{g_i,d_i}(\mathbf{Q})\otimes\tilde{\Phi}_{f_I,m}(\mathbf{Q}))^{G_I}\otimes\otimes_{i\in I}\Phi_{g_i,d_i}(\mathbf{Q})^{\mu_{d_i}}$$

By specializing to the case where $n=2$ and $f=x_1+x_2$, we have the following corollary.

Corollary 3.6.2 (Thom-Sebastiani theorem for Hodge structures). *Let g_1 and g_2 be germs of holomorphic functions on $\mathbf{C}^{n_1}$ and $\mathbf{C}^{n_2}$ at 0 with isolated singularities. Let d_1 and d_2 be the exponents of g_1 and g_2 respectively and $m = \operatorname{lcm}(d_1, d_2)$. Then we have*

$$\begin{aligned}\Phi_{g_1+g_2,m} =&(\Phi_{g_1,d_1} \otimes \Phi_{g_2,d_2} \otimes \tilde{\Phi}_{\xi_1^{d_1}+\xi_2^{d_2}})^{\mu_{d_1}\times\mu_{d_2}} \\ &- \Phi_{g_1,m,\neq 1} \otimes \Phi_{g_2,m,\neq 1} + \Phi_{g_1,m} \otimes \Phi_{g_2,m},\end{aligned}$$

where $\Phi_{g_1+g_2,m} = \Phi_{g_1+g_2,m}(\mathbf{Q})$ e.t.c.

Proof. By the Theorem 3.6.1, we have

$$\begin{aligned}\Phi_{g_1+g_2,m} =&(\phi_{g_1,d_1} \otimes \phi_{g_2,d_2} \otimes \tilde{\Phi}_{\xi_1^{d_1}+\xi_2^{d_2},m})^{\mu_{d_1}\times\mu_{d_2}} \\ &- (\phi_{g_1,d_1} \otimes \tilde{\Phi}_{\xi_1^{d_1},m})^{\mu_{d_1}} \otimes \Phi_{g_2,d_2}^{\mu_{d_2}} \\ &- (\phi_{g_2,d_2} \otimes \tilde{\Phi}_{\xi_2^{d_2},m})^{\mu_{d_2}} \otimes \Phi_{g_1,d_1}^{\mu_{d_1}} \\ &+ \Phi_{g_1,d_1}^{\mu_{d_1}} \otimes \Phi_{g_2,d_2}^{\mu_{d_2}}.\end{aligned} \tag{3.6.1}$$

On the other hand, we have

$$\tilde{\Phi}_{\xi_1^{d_1},m} = [\mathbf{Q}[\mu_{d_1}]] - [\mathbf{Q}] \tag{3.6.2}$$

in $K_{MH}(\mathbf{C}, \mu_{d_1} \times \mu_m)$, where μ_m acts on μ_{d_1} through its quotient. Therefore we have

$$(\phi_{g_1,d_1} \otimes \tilde{\Phi}_{\xi^{d_1},m})^{\mu_{d_1}} = \phi_{g_1,d_1,\neq 1} = \Phi_{g_1,d_1,\neq 1}.$$

If $(\tilde{\Phi}_{\xi_1^{d_1}+\xi_2^{d_2},m} \otimes \mathbf{C})(\chi_1,\chi_2) \neq 0$ for a charcters χ_1 and χ_2 of μ_{d_1} and μ_{d_2}, then $\chi_1 \neq 1$ and $\chi_2 \neq 1$. Therefore we have

$$(\phi_{g_1,d_1}\otimes\phi_{g_2,d_2}\otimes\tilde{\Phi}_{\xi_1^{d_1}+\xi_2^{d_2},m})^{\mu_{d_1}\times\mu_{d_2}} = (\Phi_{g_1,d_1}\otimes\Phi_{g_2,d_2}\otimes\tilde{\Phi}_{\xi_1^{d_1}+\xi_2^{d_2},m})^{\mu_{d_1}\times\mu_{d_2}} \tag{3.6.3}$$

By (3.6.1), (3.6.2) and (3.6.3), we have the corollary.

References

[Del] Deligne, P, *Théorie de Hodge II*, Publ. Math. IHES **40** (1973), 5-58.

[Dan] Danilov, V.I., *Newton polyheadra and vanishing cohomology*, Funct. Anal. **13** (1979), 103-115.

[St] Steenbrink, J.H.M., *Mixed Hodge structures on vanishing cohomology*, Real and complex singularities, Oslo 1976, Sijthoff-Noordhoff, Alphen a/d Rijn, 1977, pp. 525-563.

[Dur] Durfee, A.H., *Mixed Hodge structures on punctured neighborhoods*, Duke Math. J. **50 no.4** (1983), 1017-1040.

[N-S] Nemethi,A.-Steenbrink, J.H.M., *Spectral pairs, mixed Hodge modules, and series of plane curve singularities*, New York J. of Math. **1** (1995), 149-177.

[Od] Oda, T., *Convex bodies and algebraic geometry, Ergibnis der Math. u. ihrer Grantzengebiete*, vol. 3, Springer-Verlag.

[D-L] Denef, J.-Loeser, F., *Motivic exponential integrals and a motivic Thom-Sebastiani theorem, preprint* (1998).

[S-T] Sebastiani,M.-Thom,R., *Un Résultat sur la monodromie*, Inv. Math. **13** (1971), 90-96.

[V] Varchenko, A., *Asymptotic Hodge structure in the vanishing cohomology*, Math. USSR Izvestija **18** (1982), 479-512.

[S] Saito, M, *Mixed Hodge modules and applications*, Proceedings of the ICM Kyoto, 1991, pp. 725-734.