On a Topology on Symbol Classes of Type 1,0

Ingo Witt

Max-Planck-Arbeitsgruppe
"Partielle Differentialgleichungen und komplexe Analysis"
Universität Potsdam
Am Neuen Palais 10
14469 Potsdam
Germany

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
53225 Bonn

Germany

MPI 96-120

On a Topology on Symbol Classes of Type 1,0

Ingo Witt*
Max-Planck-Arbeitsgruppe
"Partielle Differentialgleichungen und Komplexe Analysis"
D-14415 Potsdam
Germany

August 24, 1996

Abstract

A topology on symbol classes of type $1,0$ for pseudo-differential operators is introduced. This topology indicated by subscript τ is well-behaved under the formation of tensor products, e.g., $S^m_\tau(\mathbb{R}^n; E) = S^m_\tau(\mathbb{R}^n)\tilde{\otimes}_\epsilon E$ for any complete lcs E. Here symbols taking their coefficients in certain function spaces are regarded as vector-valued ones. The relation just mentioned allows to prove continuity of pseudo-differential operators in different situations by considering the functional-analytic properties of E.

1991 Mathematics Subject Classification: 35S05, 46E40

Key Words: Vector-valued symbols, topologies on symbol classes, continuity of pseudo-differential operators

*This research was supported by the Deutsche Forschungsgemeinschaft.

Contents

1 Introduction **3**

2 The Weak Symbol Topology **6**

2.1 Vector-Valued Symbol Classes 6

2.2 Basic Short Exact Sequences 7

2.3 Oscillatory Integrals as Weakly Continuous Functionals 10

2.4 Properties of $S^m_\tau(\mathbb{R}^n)'$ 12

2.5 Linear-Topological Characterization of $S^m_\tau(\mathbb{R}^n)$ 19

2.6 Tensor Product Representation 22

2.7 Appendix: Proof of Lemma 2.5 26

3 Applications to Pseudo-Differential Operators; Examples **30**

3.1 The Case of Smooth Coefficients 30

3.2 Coefficients from Sobolev Spaces 31

A Facts on Locally Convex Spaces **34**

A.1 Some Notation 34

A.2 Short Exact Sequences 34

A.3 Quasi-Normable Lcs and Schwartz spaces 35

A.4 The ϵ-Product 35

A.5 Precompact Sets and Limited Sets 36

A.6 Miscellaneous Topics 37

B Further Tools **39**

B.1 Banach Operator Ideals 39

B.2 2-Tensor Product 41

1 Introduction

In this paper we introduce a topology on symbol classes of type 1, 0 for pseudo-differential operators which we call the weak symbol topology. This topology seems to be of general importance in questions concerning the continuity of pseudo-differential operators due to its striking linear-topological properties. Here we put the emphasis on vector-valued symbols and tensor product techniques.

Our intention in this contribution is to give rigorous proofs for some relevant properties of the weak symbol topology. In a further section we indicate how this topology can be utilized for verifying continuity of pseudo-differential operators. The notation as weak symbol topology is justified by a notion of Hörmander (cf. [7, Definition 18.4.9]) who called functionals, as it turns out, in the dual of a symbol class equipped with the weak symbol topology weakly continuous. Our initial motivation for constructing the weak symbol topology was to establish a pseudo-differential calculus for non-classical operators with certain non-smooth symbols for which an analytic tool for estimating the remainders was required. This calculus has been announced in [14, Section 4.2]; details will be published elsewhere.

The weak symbol topology is defined as follows: Consider the space $S^m(\mathbb{R}^n)$ of constant coefficient symbols on $\mathbb{R}^n$ of order m. Denote for the moment

$$\tilde{\Sigma}^m(\mathbb{R}^n) = \{\Phi \in S^m(\mathbb{R}^n)' ; \text{ the restriction of } \Phi \text{ to any bounded set in } S^m(\mathbb{R}^n) \text{ is continuous for the } C^\infty\text{-topology}\}.$$

It turns out that $\tilde{\Sigma}^m(\mathbb{R}^n)$ is a complemented subspace of $S^m(\mathbb{R}^n)'$. In particular, $\tilde{\Sigma}^m(\mathbb{R}^n)$ contains all oscillatory integrals acting as linear functionals on constant coefficient symbols. In fact, these functionals form a dense subset in $\tilde{\Sigma}^m(\mathbb{R}^n)$. Thus, loosely speaking, $\tilde{\Sigma}^m(\mathbb{R}^n)$ comprises exactly those analytic expressions that proved to be most important in the theory of pseudo-differential operators. It makes sense to investigate the dual pair $\langle S^m(\mathbb{R}^n), \tilde{\Sigma}^m(\mathbb{R}^n)\rangle$. As a matter of fact, $S^m(\mathbb{R}^n) = (S^m(\mathbb{R}^n), \beta(S^m, \tilde{\Sigma}^m))$, but $S^m(\mathbb{R}^n)$ equipped with its strongest locally convex topology yielding $\tilde{\Sigma}^m(\mathbb{R}^n)$ as dual space has the desired properties.

1.1 Definition. *The weak symbol topology, τ, is defined as the Mackey topology on $S^m(\mathbb{R}^n)$ with respect to the dual pair $\langle S^m(\mathbb{R}^n), \tilde{\Sigma}^m(\mathbb{R}^n)\rangle$, i.e.,*

$$S^m_\tau(\mathbb{R}^n) = (S^m(\mathbb{R}^n), \tau(S^m, \tilde{\Sigma}^m)). \tag{1.1}$$

From now on, we use the notation $S^m_\tau(\mathbb{R}^n)'$ instead of $\tilde{\Sigma}^m(\mathbb{R}^n)$. Further we adopt the convention to consider $m \in \mathbb{R}$, $n \in \mathbb{N}$ as be given and fixed. They will not explicitly appear in the statements of results.

Properties of $S^m_\tau(\mathbb{R}^n)$ are studied by investigating its strong dual $S^m_\tau(\mathbb{R}^n)'$. For the latter we provide a representation as topological quotient of the Banach space direct sum $\sum_{|\alpha|\geq 0} {}^1B_{(m-|\alpha|+n/2)}(\mathbb{R}^n)$, where ${}^1B_{(m-|\alpha|+n/2)}(\mathbb{R}^n)$ is the Fourier image of the Besov space ${}^1H_{(m-|\alpha|+n/2)}(\mathbb{R}^n)$ (cf. Subsection 2.2). By means of this description of $S^m_\tau(\mathbb{R}^n)'$, it is possible to state that $S^m_\tau(\mathbb{R}^n)$ is a Schwartz space. The decisive property for that is to show that $\sigma((S^m_\tau)', S^m)$-compact sets in $S^m_\tau(\mathbb{R}^n)'$ are actually compact. That in turn is furnished by proving that

$$\langle \Phi_k, a_k \rangle \to 0 \text{ as } k \to \infty \tag{1.2}$$

is valid for all $\sigma(S^m,(S^m_\tau)')$-null sequences $\{a_k\}_{k=0}^\infty$ in $S^m(\mathbb{R}^n)$ and all $\sigma((S^m_\tau)',S^m)$-null sequences $\{\Phi_k\}_{k=0}^\infty$ in $S^m_\tau(\mathbb{R}^n)'$. Eventually note that $S^m(\mathbb{R}^n)$, $S^m_\tau(\mathbb{R}^n)$ have the same bounded sets and that on these bounded sets the τ-topology coincides with the C^∞-topology. In particular, convergent sequences in $S^m_\tau(\mathbb{R}^n)$ are identified in this way.

Symbols for pseudo-differential operators with coefficients in certain function spaces shall be considered as vector-valued symbols. Then, for the symbol class $S^m(\mathbb{R}^n;E)$, where E is a complete lcs, the relation

$$S^m_\tau(\mathbb{R}^n;E) = S^m_\tau(\mathbb{R}^n)\tilde{\otimes}_\epsilon E. \tag{1.3}$$

holds (cf. Theorem 2.29). Here subscript τ indicates the weak symbol topology and (1.3) is understood as natural identification of two linear spaces, where the right-hand side defines the topology for the left-hand side. An independent definition of $S^m_\tau(\mathbb{R}^n;E)$ will be given in the text.

Property (1.3) is fundamental for proving continuity of pseudo-differential operators. As we shall see, $S^m_\tau(\mathbb{R}^n)$ is not a nuclear space. Thus it is in general impossible to replace the injective tensor product in (1.3) by the projective tensor product. This causes some complications if one tries to check that certain bilinear forms on $S^m_\tau(\mathbb{R}^n)\times E$ are integral, i.e., belong to the dual of $S^m_\tau(\mathbb{R}^n)\tilde{\otimes}_\epsilon E$. Here the functional-analytic properties of the "coefficient space" E which can be assumed to be good in some sense come into play. For applications in this paper, the situation is kept by the fact that for every $u\in H^{t+m}(\mathbb{R}^n)$ there is an absolutely convex, closed 0-neighbourhood U in $S^m_\tau(\mathbb{R}^n)$ such that the continuous mapping $S^m_\tau(\mathbb{R}^n)\to H^t(\mathbb{R}^n)$, $a\mapsto a(D)u$, factors through the local Banach space $S^m_\tau(\mathbb{R}^n)^\sim_{(U)}$ such that the resulting operator

$$S^m_\tau(\mathbb{R}^n)^\sim_{(U)} \to H^t(\mathbb{R}^n),\;\; a+\ker\|\;\|_{(U)}\mapsto a(D)u, \tag{1.4}$$

where $\|\;\|_{(U)}$ is the continuous semi-norm associated with U, is 2-nuclear (cf. Theorem 2.23).

Now we describe the content in more detail. Section 2 is in the heart of the paper. Here we study several properties of the weak symbol topology. In Subsection 2.1, we define the symbol classes $S^m(\mathbb{R}^n;E)$ and give examples. Then, in Subsection 2.2, we recognize $S^m(\mathbb{R}^n)$ as the strong bidual to $\mathring{S}^m(\mathbb{R}^n)$, where $\mathring{S}^m(\mathbb{R}^n)$ is the closure of $\mathcal{S}(\mathbb{R}^n)$ in $S^m(\mathbb{R}^n)$, and derive the mentioned representation of $S^m_\tau(\mathbb{R}^n)'$ as a quotient of a direct sum of Banach spaces. This is accomplished by noting that symbols in $S^m(\mathbb{R}^n)$ are precisely the Fourier transforms of distributions on $\mathbb{R}^n$ conormal to the origin of order $m+n/4$ leading to certain short exact sequences of Fréchet spaces. In an appendix to Section 2, we prove that the spaces $S^m(\mathbb{R}^n)$, $\mathring{S}^m(\mathbb{R}^n)$ are quasi-normable providing the topological exactness of the dual sequences. These observations lead to the definition of the weak symbol topology as $S^m_\tau(\mathbb{R}^n) = (S^m(\mathbb{R}^n),\tau(S^m(\mathbb{R}^n),\mathring{S}^m(\mathbb{R}^n)'))$. In Subsection 2.3, we study oscillatory integrals as main examples of functionals in $S^m_\tau(\mathbb{R}^n)'$. Further, we characterize functionals in $S^m(\mathbb{R}^n)'$ that belong to $S^m_\tau(\mathbb{R}^n)'$. Next, in Subsection 2.4, we describe weakly compact sets in $\sum_{|\alpha|\geq 0}{}^1B_{(m-|\alpha|+n/2)}(\mathbb{R}^n)$ and prove that every weakly compact set in $S^m_\tau(\mathbb{R}^n)'$ is compact. In addition, we investigate point functionals in $S^m_\tau(\mathbb{R}^n)'$. In Subsection 2.5, the linear-topological characterization of $S^m_\tau(\mathbb{R}^n)$ is completed. Especially, we demonstrate that $S^m_\tau(\mathbb{R}^n)$ is a complete, separable Schwartz space which is not nuclear, specify the weak symbol topology on bounded subsets of $S^m(\mathbb{R}^n)$ and perform some consideration around (1.4). In Subsection 2.6, we turn to the investigation of vector-valued symbols. We define the weak symbol topology on $S^m(\mathbb{R}^n; E)$, obtain the relation (1.3) and show that the bounded sets in $S^m(\mathbb{R}^n; E)$, $S^m_\tau(\mathbb{R}^n; E)$ are the same. Moreover, we identify

$$S^m_\tau(\mathbb{R}^n; E) = \mathcal{L}_\gamma(S^m_\tau(\mathbb{R}^n)'; E) \tag{1.5}$$

as linear topological spaces, where γ stands for the topology of uniform convergence on precompact sets of $S^m_\tau(\mathbb{R}^n)'$.

Then Section 3 is devoted to applications to pseudo-differential operators. We provide two examples, namely pseudo-differential operators with coefficients in $C^\infty_b(\mathbb{R}^n)$ and in $H^s(\mathbb{R}^n)$, where $s > n/2$, and discuss continuity between Sobolev spaces. Of course, these examples were known for a long time, but they mainly serve us to demonstrate the flexibility in verifying continuity of pseudo-differential operators established by the weak symbol topology and the use of tensor product techniques. In Appendix A, for quick reader's reference we present some auxiliary material on locally convex spaces that is used in the paper. In Appendix B, we give further information on Banach operator ideals and tensor products needed in the applications to pseudo-differential operators. This matter is partly organized in a way making it applicable to other circumstances, e.g., when coefficients are chosen in Besov and Bessel-potential spaces. It should be mentioned once again that the proposed method for verifying continuity of pseudo-differential operators was originally invented for dealing with such more difficult situations.

2 The Weak Symbol Topology

We give a thorough description of the weak symbol topology, τ, on symbol classes $S^m(\mathbb{R}^n;E)$. The main results are the characterization of $S^m_\tau(\mathbb{R}^n)$ as a complete, separable Schwartz space having the same bounded sets as $S^m(\mathbb{R}^n)$, the identification of the τ-topology on the bounded sets of $S^m(\mathbb{R}^n)$, further providing a substitute for the non-nuclearity of $S^m_\tau(\mathbb{R}^n)$ in Theorem 2.23, and, for E being a complete lcs, recognizing $S^m_\tau(\mathbb{R}^n;E)$ as the completed tensor product $S^m_\tau(\mathbb{R}^n)\tilde{\otimes}_\epsilon E$. Moreover, it is shown that $S^m(\mathbb{R}^n;E)$, $\mathcal{L}(S^m_\tau(\mathbb{R}^n)',E)$ can be identified as linear spaces.

2.1 Vector-Valued Symbol Classes

We introduce the space $S^m(\mathbb{R}^n;E)$ of symbols on $\mathbb{R}^n$ of order m with values in E.

2.1 Definition. *Let E be a complete lcs. Then $S^m(\mathbb{R}^n;E)$ is the space of all functions $a\in C^\infty(\mathbb{R}^n;E)$ satisfying*

$$\sup_{\xi\in\mathbb{R}} \langle\xi\rangle^{-m+|\alpha|}\left\|\partial_\xi^\alpha a(\xi)\right\| < \infty \tag{2.1}$$

for all multi-indices $\alpha\in\mathbb{N}^n$ and all continuous semi-norms $\|\ \|$ on E.

The left-hand sides in (2.1) give rise to a fundamental system of semi-norms for a locally convex topology on $S^m(\mathbb{R}^n;E)$. In the sequel, $S^m(\mathbb{R}^n;E)$ carries that topology when it comes equipped with a topology not explicitly mentioned. The weak symbol topology on $S^m(\mathbb{R}^n;E)$ to be introduced later shall be indicated by subscript τ.

The space $S^m(\mathbb{R}^n;E)$ is complete. If E is a Fréchet space, then $S^m(\mathbb{R}^n;E)$ is a Fréchet space. We employ the standard notation $S^m(\mathbb{R}^n)=S^m(\mathbb{R};\mathbb{C})$. If E undergoes an interpretation as coefficient space, then $S^m(\mathbb{R}^n)$ is the space of symbols with constant coefficients.

Note that the space $S^{-\infty}(\mathbb{R}^n;E)=\bigcap_{m\in\mathbb{R}} S^m(\mathbb{R}^n;E)$ of symbols of order $-\infty$ is not dense in $S^m(\mathbb{R}^n;E)$. We denote its closure in $S^m(\mathbb{R}^n;E)$ by $\mathring{S}^m(\mathbb{R}^n;E)$. A symbol $a\in S^m(\mathbb{R}^n;E)$ belongs to $\mathring{S}^m(\mathbb{R}^n;E)$ if and only if

$$\langle\xi\rangle^{-m+|\alpha|}\|\partial_\xi^\alpha a(\xi)\| \to 0 \quad \text{as } |\xi|\to\infty \tag{2.2}$$

holds for all multi-indices $\alpha\in\mathbb{N}^n$ and all continuous semi-norms $\|\ \|$ on E.

It is useful to recognize the space $S^{-\infty}(\mathbb{R}^n;E)$ as the space $\mathcal{S}(\mathbb{R}^n;E)$ of Schwartz functions on $\mathbb{R}^n$ with values in E. Notice that

$$S^{-\infty}(\mathbb{R}^n;E)=S^{-\infty}(\mathbb{R}^n)\tilde{\otimes}_\pi E, \tag{2.3}$$

which is easily proved taking into account the nuclearity of $S^{-\infty}(\mathbb{R}^n)$.

2.2 Example. In applications, although by no means necessary, E often stands for the coefficient space. For instance, if $E=C_b^\infty(\mathbb{R}^n)$, then we obtain Hörmander's symbol class $S^m(\mathbb{R}^n\times\mathbb{R}^n)$ with global space variable estimates. If $E=C^\infty(\Omega)$, for $\Omega\subseteq\mathbb{R}^n$ an open set, we get $S^m_{1,0}(\Omega\times\mathbb{R}^n)$ with local space variable estimates. Another interesting example is $E=S^{m'}(\mathbb{R}^n)$; in that case we obtain the class $S^{m,m'}(\mathbb{R}^n\times\mathbb{R}^n)$ of symbols satisfying exit conditions in the space variables at infinity (see, e.g., [11, Definition 1.2.31]). Concerning pseudo-differential operators with non-regular symbols, one is, e.g., interested in $E = H^s(\mathbb{R}^n)$, where the exponent $s > n/2$ is sufficiently large. The resulting symbol class should be denoted by $H^sS^m(\mathbb{R}^n \times \mathbb{R}^n)$.

2.2 Basic Short Exact Sequences

The starting point in introducing the weak symbol topology on $S^m(\mathbb{R}^n)$ is the fact that $S^m(\mathbb{R}^n)$ is the strong bidual of $\mathring{S}^m(\mathbb{R}^n)$. In proving that fact we invoke certain short exact sequences of Fréchet spaces and their dual sequences. These sequences are found by exploiting the observation that symbols in $S^m(\mathbb{R}^n)$ are exactly the Fourier transforms of distributions on $\mathbb{R}^n$ that are conormal to the origin of order $m + n/4$.

Introduce the corresponding set-up. The Fourier transform is

$$Fu(\xi) = \hat{u}(\xi) = \int e^{-ix\xi}u(x)\,dx,$$

where integration is over $\mathbb{R}^n$, such that the inverse Fourier transform becomes $u(x) = (2\pi)^{-n}\int e^{ix\xi}\hat{u}(\xi)\,d\xi$. The space $I^{m+n/4}(\mathbb{R}^n, \{0\})$ of distributions on $\mathbb{R}^n$ conormal to the origin of order $m + n/4$ is introduced as follows: Put $\Lambda_0 = \{\xi \in \mathbb{R}^n;\ |\xi| \leq 1\}$, $\Lambda_j = \{\xi \in \mathbb{R}^n;\ 2^{j-1} \leq |\xi| \leq 2^j\}$ for $j = 1, 2, \ldots$ The Besov space ${}^pH_{(t)}(\mathbb{R}^n)$, t, $p \in \mathbb{R}$, $1 \leq p \leq \infty$, consists of all $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfying $\hat{u} \in L^2_{loc}(\mathbb{R}^n)$,

$$\left\{2^{jt}\,\|\hat{u}\|_{L^2(\Lambda_j)}\right\}_{j=0}^{\infty} \in l^p.$$

Note that ${}^pH_{(t)}(\mathbb{R}^n)$ is Hörmander's notation (see [7, Definition B.1.1]), a more common one is $B^t_{2,p}(\mathbb{R}^n)$. The notation used here will be convenient for us in Subsection 2.7. Then $I^{m+n/4}(\mathbb{R}^n, \{0\})$ is defined as the space of all $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$x^\alpha\partial_x^\beta u \in {}^\infty H_{(-m-n/2)}(\mathbb{R}^n) \tag{2.4}$$

for all α, $\beta \in \mathbb{N}^n$, $|\alpha| \geq |\beta|$. The following well-known result may be found, e.g., in [7, Proposition 18.2.2].

2.3 Proposition. *Let a, $u \in \mathcal{S}'(\mathbb{R}^n)$ be related by $a = Fu$. Then $u \in I^{m+n/4}(\mathbb{R}^n, \{0\})$ if and only if $a \in S^m(\mathbb{R}^n)$.*

Proof: Condition (2.4) is equivalent to

$$x^\alpha u \in {}^\infty H_{(-m+|\alpha|-n/2)}(\mathbb{R}^n)$$

for all $\alpha \in \mathbb{N}^n$. In the Fourier image, the latter read $\partial_\xi^\alpha a(\xi) \in L^2_{loc}(\mathbb{R}^n)$,

$$\sup_{R\geq 1} R^{-2(m-|\alpha|+n/2)}\int_{R\leq|\xi|\leq 2R}|\partial_\xi^\alpha a(\xi)|^2\,d\xi < \infty \tag{2.5}$$

for all $\alpha \in \mathbb{N}^n$. Now, if $a \in S^m(\mathbb{R}^n)$, then (2.5) follows from the symbol estimates. Conversely, if $u \in I^{m+n/4}(\mathbb{R}^n, \{0\})$, then, introducing $a_R(\xi) = a(R\,\xi)/R^m$ for $R \geq 1$, (2.5) implies

$$\|\partial_\xi^\alpha a_R\|_{L^2(\Lambda)} \leq C_\alpha, \quad R \geq 1,$$

where $\Lambda = \Lambda_1$ is as above, for $\alpha \in \mathbb{N}^n$ with certain constants $C_\alpha > 0$. By Sobolev's embedding theorem, that means that $\{a_R;\ R \geq 1\}$ forms a bounded set in $C^\infty(\Lambda)$ which shows that a belongs to $S^m(\mathbb{R}^n)$. □

Remind the functional analysis of the spaces ${}^pH_{(t)}(\mathbb{R}^n)$. ${}^pH_{(t)}(\mathbb{R}^n)$ are Banach spaces. For $1 \leq p < \infty$, $\mathcal{S}(\mathbb{R}^n)$ is dense in ${}^pH_{(t)}(\mathbb{R}^n)$. Denote the closure of $\mathcal{S}(\mathbb{R}^n)$ in ${}^\infty H_{(t)}(\mathbb{R}^n)$ by ${}^\infty \mathring{H}_{(t)}(\mathbb{R}^n)$. Then $u \in {}^\infty H_{(t)}(\mathbb{R}^n)$ belongs to ${}^\infty \mathring{H}_{(t)}(\mathbb{R}^n)$ if and only if $\{2^{jt}\,\|\hat{u}\|_{L^2(\Lambda_j)}\}_{j=0}^\infty \in c_0$. For $1 \leq p < \infty$, the dual space to ${}^pH_{(t)}(\mathbb{R}^n)$ is ${}^{p'}H_{(-t)}(\mathbb{R}^n)$, $p' = p/(p-1)$, with respect to an extension of the L^2-duality. The dual space to ${}^\infty \mathring{H}_{(t)}(\mathbb{R}^n)$ is ${}^1H_{(-t)}(\mathbb{R}^n)$. Notice that ${}^1H_{(-t)}(\mathbb{R}^n)$ can be identified with a complemented subspace of ${}^\infty H_{(t)}(\mathbb{R}^n)'$,

$$ {}^\infty H_{(t)}(\mathbb{R}^n)' = {}^1H_{(-t)}(\mathbb{R}^n) \oplus {}^\infty \mathring{H}_{(t)}(\mathbb{R}^n)^\circ, \tag{2.6} $$

where the polar is taken with respect the dual pair $\langle {}^\infty H_{(t)}(\mathbb{R}^n), {}^\infty H_{(t)}(\mathbb{R}^n)' \rangle$. For future reference, let ${}^pB_{(t)}(\mathbb{R}^n)$ be the space consisting of the Fourier transforms of functions belonging to ${}^pH_{(t)}(\mathbb{R}^n)$. Likewise, ${}^\infty \mathring{B}_{(t)}(\mathbb{R}^n)$ is the closure of $\mathcal{S}(\mathbb{R}^n)$ in ${}^\infty B_{(t)}(\mathbb{R}^n)$.

Further, the closure of $\mathcal{S}(\mathbb{R}^n)$ in $I^{m+n/4}(\mathbb{R}^n, \{0\})$ shall be denoted by $\mathring{I}^{m+n/4}(\mathbb{R}^n, \{0\})$. This closure is characterized by requiring $x^\alpha \partial_x^\beta u \in {}^\infty \mathring{H}_{(-m-n/2)}(\mathbb{R}^n)$ instead of (2.4). Proposition 2.3 continues to hold with $S^m(\mathbb{R}^n)$ and $I^{m+n/4}(\mathbb{R}^n, \{0\})$ replaced by $\mathring{S}^m(\mathbb{R}^n)$ and $\mathring{I}^{m+n/4}(\mathbb{R}^n, \{0\})$, respectively.

As a corollary to Proposition 2.3 we obtain:

2.4 Corollary. *Let $a \in C^\infty(\mathbb{R}^n)$. Then $a \in S^m(\mathbb{R}^n)$ if and only if*

$$ \partial_\xi^\alpha a \in {}^\infty B_{(-m+|\alpha|-n/2)}(\mathbb{R}^n) $$

for all $\alpha \in \mathbb{N}^n$. The same statement is true if $S^m(\mathbb{R}^n)$ and ${}^\infty B_{(-m+|\alpha|-n/2)}(\mathbb{R}^n)$ are replaced by $\mathring{S}^m(\mathbb{R}^n)$ and ${}^\infty \mathring{B}_{(-m+|\alpha|-n/2)}(\mathbb{R}^n)$, respectively.

From Corollary 2.4 we get an embedding

$$ S^m(\mathbb{R}^n) \to \prod_{\alpha \in \mathbb{N}^n} {}^\infty B_{(-m+|\alpha|-n/2)}(\mathbb{R}^n), \quad a \mapsto \{\partial_\xi^\alpha a\}_{\alpha \in \mathbb{N}^n}, \tag{2.7} $$

realizing $S^m(\mathbb{R}^n)$ as a closed subspace of $\prod_{\alpha \in \mathbb{N}^n} {}^\infty B_{(-m+|\alpha|-n/2)}\mathbb{R}^n$. Subsequently, $S^m(\mathbb{R}^n)$ should be identified with its image in $\prod_{\alpha \in \mathbb{N}^n} {}^\infty B_{(-m+|\alpha|-n/2)}(\mathbb{R}^n)$. Under this identification, $\mathring{S}^m(\mathbb{R}^n)$ becomes $S^m(\mathbb{R}^n) \cap \prod {}^\infty \mathring{B}_{(-m+|\alpha|-n/2)}(\mathbb{R}^n)$, i.e., a closed subspace of $\prod {}^\infty \mathring{B}_{(-m+|\alpha|-n/2)}(\mathbb{R}^n)$. Notice that $\prod {}^\infty B_{(-m+|\alpha|-n/2)}(\mathbb{R}^n)$, $\prod {}^\infty \mathring{B}_{(-m+|\alpha|-n/2)}(\mathbb{R}^n)$ are Fréchet spaces.

Employing (2.7) and its analogue for $\mathring{S}^m(\mathbb{R}^n)$, we reach the exact sequences mentioned above. In order to formulate them compactly, introduce the following abbreviations:

$$ \mathring{\Pi}^m(\mathbb{R}^n) = \prod_{\alpha \in \mathbb{N}^n} {}^\infty \mathring{B}_{(-m+|\alpha|-n/2)}(\mathbb{R}^n), \quad \Pi^m(\mathbb{R}^n) = \prod_{\alpha \in \mathbb{N}^n} {}^\infty B_{(-m+|\alpha|-n/2)}(\mathbb{R}^n), $$

$$ \Sigma^m(\mathbb{R}^n) = \sum_{\alpha \in \mathbb{N}^n} {}^1B_{(m-|\alpha|+n/2)}(\mathbb{R}^n), $$

where we have used $\sum$ as substitute for $\bigoplus$, and

$$\Delta^m(\mathbb{R}^n) = \Big\{\{f_\alpha\}_{\alpha\in\mathbb{N}^n} \in \Sigma^m(\mathbb{R}^n) \;\Big|\; \sum_{\alpha\in\mathbb{N}^n} (-1)^{|\alpha|}\,\partial_\xi^\alpha f_\alpha \equiv 0\Big\},$$

where the derivatives are understood in the distributional sense. For further reference introduce

$$S_L^m(\mathbb{R}^n) = \{a \in {}^\infty B_{(-m-n/2)}(\mathbb{R}^n);\ \partial_\xi^\alpha a \in {}^\infty B_{(-m+|\alpha|-n/2)}(\mathbb{R}^n) \text{ for } \alpha \in \mathbb{N}^n,\ |\alpha| \le L\},$$

$$\Sigma_L^m(\mathbb{R}^n) = \sum_{|\alpha|\le L} {}^1 B_{(m-|\alpha|+n/2)}(\mathbb{R}^n),$$

$$\Delta_L^m(\mathbb{R}^n) = \Big\{\{f_\alpha\}_\alpha \in \Sigma_L^m(\mathbb{R}^n) \;\Big|\; \sum_{|\alpha|\le L} (-1)^{|\alpha|}\,\partial_\xi^\alpha f_\alpha \equiv 0\Big\}.$$

Notice the relation

$$\Sigma^m(\mathbb{R}^n) = \operatorname{ind\,lim} \Sigma_L^m(\mathbb{R}^n),$$

where the inductive limit is extended as $L \to \infty$, such that $\Sigma^m(\mathbb{R}^n)$ becomes a strict (LB)-space.

2.5 Lemma. *The spaces $S^m(\mathbb{R}^n)$, $\mathring{S}^m(\mathbb{R}^n)$ are quasi-normable.*

The proof will be given in an appendix to this section, since additional notion of no further use have to be introduced.

2.6 Proposition. *The following sequences are topological exact:*

$$0 \longrightarrow \mathring{S}^m(\mathbb{R}^n) \longrightarrow \mathring{\Pi}^m(\mathbb{R}^n) \longrightarrow \mathring{\Pi}^m(\mathbb{R}^n)/\mathring{S}^m(\mathbb{R}^n) \longrightarrow 0, \tag{2.8}$$

$$0 \longrightarrow \Delta^m(\mathbb{R}^n) \longrightarrow \Sigma^m(\mathbb{R}^n) \longrightarrow \mathring{S}^m(\mathbb{R}^n)' \longrightarrow 0, \tag{2.9}$$

$$0 \longrightarrow S^m(\mathbb{R}^n) \longrightarrow \Pi^m(\mathbb{R}^n) \longrightarrow \Pi^m(\mathbb{R}^n)/S^m(\mathbb{R}^n) \longrightarrow 0, \tag{2.10}$$

$$\begin{aligned} 0 \longrightarrow \Delta^m(\mathbb{R}^n) \oplus (S^m(\mathbb{R}^n)^\circ \cap \mathring{\Pi}^m(\mathbb{R}^n)^\circ) &\longrightarrow \Sigma^m(\mathbb{R}^n) \oplus \mathring{\Pi}^m(\mathbb{R}^n)^\circ \\ \longrightarrow \mathring{S}^m(\mathbb{R}^n)' \oplus \mathring{\Pi}^m(\mathbb{R}^n)^\circ/(S^m(\mathbb{R}^n)^\circ \cap \mathring{\Pi}^m(\mathbb{R}^n)^\circ) &\longrightarrow 0, \end{aligned} \tag{2.11}$$

Thereby, (2.9), (2.10), *and* (2.11) *are the dual sequences to* (2.8), (2.9), *and* (2.10), *respectively. In* (2.11), *polars are taken with respect to* $\langle \Pi^m(\mathbb{R}^n), \Pi^m(\mathbb{R}^n)'\rangle$.

Proof: (2.8), (2.10) are topologically exact as short exact sequences of Fréchet spaces. Because $\Sigma^m(\mathbb{R}^n)$ is the strong dual to $\mathring{\Pi}^m(\mathbb{R}^n)$ and $\Pi^m(\mathbb{R}^n)$ is the strong dual to $\Sigma^m(\mathbb{R}^n)$, (2.9) is the dual sequence to (2.8), as some partial integration shows. Furthermore, we conclude that (2.10) is the dual sequence to (2.9) by the same argument. The topological exactness of (2.9) and of the sequence

$$0 \longrightarrow S^m(\mathbb{R}^n)^\circ \longrightarrow \Pi^m(\mathbb{R}^n)' \longrightarrow \Pi^m(\mathbb{R}^n)'/S^m(\mathbb{R}^n)^\circ \longrightarrow 0 \tag{2.12}$$

are consequences of Proposition A.1 and the observation made in Lemma 2.5. That the latter sequence coincides with (2.11), i.e., (2.9) is a direct summand in (2.12), can easily be verified using (2.6). □

Before we proceed we make out some of the properties of spaces appearing in (2.8)–(2.11): $S^m(\mathbb{R}^n)$, $\mathring{S}^m(\mathbb{R}^n)$ are Fréchet spaces. Thanks to Proposition A.9 we obtain that $\mathring{S}^m(\mathbb{R}^n)'$ is a complete barreled (DF)-space, i.e., a complete (LB)-space. Especially, $\beta((\mathring{S}^m)', \mathring{S}^m)$ and $\beta((\mathring{S}^m)', S^m)$ coincide on $\mathring{S}^m(\mathbb{R}^n)'$. The same is trivially true for $\beta(S^m, (\mathring{S}^m)')$, $\beta(S^m, (S^m)')$ on $S^m(\mathbb{R}^n)$. As a further consequence we get:

2.7 Proposition. *$S^m(\mathbb{R}^n)$ is the strong bidual of $\mathring{S}^m(\mathbb{R}^n)$.*

Proposition 2.7 makes it possible to speak about the dual pair $\langle S^m(\mathbb{R}^n), \mathring{S}^m(\mathbb{R}^n)'\rangle$. This allows to define the weak symbol topology on $S^m(\mathbb{R}^n)$ as follows:

$$S^m_\tau(\mathbb{R}^n) = (S^m(\mathbb{R}^n), \tau(S^m, (\mathring{S}^m)')). \tag{2.13}$$

In Lemma 2.12 we shall see that this definition agrees with Definition 1.1 previously given.

2.8 Lemma. *$S^m_\tau(\mathbb{R}^n)$ is complete.*

Proof: This follows from Lemma A.7. □

2.9 Lemma. *$S^m(\mathbb{R}^n)$, $S^m_\tau(\mathbb{R}^n)$ have the same bounded sets.*

Proof: This follows from Proposition A.8 applied to $E = S^m_\tau(\mathbb{R}^n)'$, $F = \mathbb{C}$. □

In Lemma 2.20 we shall characterize the weak symbol topology on the bounded sets of $S^m(\mathbb{R}^n)$.

2.3 Oscillatory Integrals as Weakly Continuous Functionals

Here we discuss the canonical embedding $\mathring{S}^m(\mathbb{R}^n)' \subset S^m(\mathbb{R}^n)'$ in more detail. Functions $v \in \mathcal{S}(\mathbb{R}^n)$ can be viewed as functionals on $\mathring{S}^m(\mathbb{R}^n)$ via the identification $\mathring{S}^m(\mathbb{R}^n) \ni a \mapsto \int a(\xi)v(\xi)\,d\xi$.

2.10 Lemma. *$\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathring{S}^m(\mathbb{R}^n)'$.*

Proof: By (2.9), elements in $\mathring{S}^m(\mathbb{R}^n)'$ can be written in the form

$$\mathring{S}^m(\mathbb{R}^n) \ni a \mapsto \sum_{|\alpha|\leq L} \int f^\alpha(\xi)\partial^\alpha_\xi a(\xi)\,d\xi \tag{2.14}$$

with $f^\alpha \in {}^1B_{(m-|\alpha|+n/2)}(\mathbb{R}^n)$ for $|\alpha| \le L$ and some $L \in \mathbb{N}$. If $f^\alpha \in \mathcal{S}(\mathbb{R}^n)$ for $|\alpha| \le L$, then the functional in (2.14) can be rewritten as

$$\mathring{S}^m(\mathbb{R}^n) \ni a \mapsto \int \Big(\sum_{|\alpha|\le L} (-1)^{|\alpha|} \partial_\xi^\alpha f^\alpha(\xi) \Big) a(\xi)\, d\xi,$$

where $\sum_{|\alpha|\le L} (-1)^{|\alpha|} \partial_\xi^\alpha f^\alpha(\xi) \in \mathcal{S}(\mathbb{R}^n)$, by partial integration. The proof is concluded remarking that, for $t \in \mathbb{R}$, $\mathcal{S}(\mathbb{R}^n)$ is dense in ${}^1B_{(t)}(\mathbb{R}^n)$. □

The prototype of functionals in $\mathring{S}^m(\mathbb{R}^n)'$ are oscillatory integrals regarded as linear functionals on symbols. Such an oscillatory integral is a formal integral expression of the kind

$$\int e^{i\phi(x,\xi)} a(x,\xi) u(x)\, dx d\xi, \tag{2.15}$$

where $\phi \in C^\infty(\Omega \times (\mathbb{R}^n \setminus 0))$ is a phase function, $a \in S^m_{1,0}(\Omega \times \mathbb{R}^n)$, and $u \in C_0^\infty(\Omega)$, with $\Omega \subset \mathbb{R}^n$ some open set. Being a phase function means that ϕ is real-valued, positively homogeneous of degree 1 in ξ, and $\nabla_{x,\xi}\phi(x,\xi)$ does not vanish on the conical support of a. For $a \in S^m_{1,0}(\Omega \times \mathbb{R}^n)$ with $m < -n$, the integral in (2.15) is absolutely convergent. In general, integrals of the kind (2.15) can be regularized by a partial integration procedure (for details, see [7, §7.8]). The value then assigned to (2.15) is usually denoted by $I_\phi(au)$.

Another possibility for computing (2.15) consists in the following. Choose $\chi \in \mathcal{S}(\mathbb{R}^n)$, $\chi(0) = 1$. Then, for $\epsilon > 0$,

$$\int e^{i\phi(x,\xi)} \chi(\epsilon\xi) a(x,\xi) u(x)\, dx d\xi,$$

is absolutely convergent, and we have

$$I_\phi(au) = \lim_{\epsilon\to 0+} I_\phi(a_\epsilon u), \tag{2.16}$$

where we have set $a_\epsilon(\xi) = \chi(\epsilon\xi)a(\xi)$.

Now coming back to functionals in $\mathring{S}^m(\mathbb{R}^n)'$. Clearly,

$$\mathring{S}^m(\mathbb{R}^n) \ni a \mapsto I_\phi(au), \tag{2.17}$$

where ϕ is a phase function, e.g., $\phi(x,\xi) = x \cdot \xi$, and $u \in C_0^\infty(\mathbb{R}^n)$, defines an element in $\mathring{S}^m(\mathbb{R}^n)'$.

2.11 Proposition. *Let $\Phi \in \mathring{S}^m(\mathbb{R}^n)'$. Then the extension of Φ to a functional on $S^m(\mathbb{R}^n)$ according to* Proposition 2.6 *is given by*

$$\langle \Phi, a \rangle = \lim_{\epsilon\to 0+} \langle \Phi, a_\epsilon \rangle, \tag{2.18}$$

where the limit exists independently of the choice of the function χ with the properties stated above. Moreover, for E being a complete lcs, every $a \in S^m(\mathbb{R}^n; E)$ defines a mapping belonging to $\mathcal{L}(\mathring{S}^m(\mathbb{R}^n)', E)$ via formula (2.18).

Proof: By Lemma 2.10, functionals of the kind (2.17), i.e., oscillatory integrals viewed as linear functionals on constant coefficient symbols, form a dense subset in $\mathring{S}^m(\mathbb{R}^n)'$. (To see this, write $\int a(\xi)v(\xi)\,d\xi$ as $(2\pi)^{-n}\int e^{ix\cdot\xi}a(\xi)\hat{v}(x)\,dxd\xi$, where $v \in \mathcal{S}(\mathbb{R}^n)$, $\hat{v} \in C_0^\infty(\mathbb{R}^n)$.) Then either from the regularizing procedure or the topology given on $\mathring{S}^m(\mathbb{R}^n)'$ it is clear that the approximation of Φ by such oscillatory integrals can be performed uniformly on bounded sets in $\mathring{S}^m(\mathbb{R}^n)$. Therefore, by (2.16), the limit (2.18) exists and gives the desired extension of Φ to a functional on $S^m(\mathbb{R}^n)$, since it is valid for functionals of the kind (2.17).

Furthermore, the same reasoning realizes a symbol $a \in S^m(\mathbb{R}^n; E)$ as a linear operator from $\mathring{S}^m(\mathbb{R}^n)'$ to E that is readily seen to be continuous providing the embedding $S^m(\mathbb{R}^n; E) \subset \mathcal{L}(\mathring{S}^m(\mathbb{R}^n)', E)$. □

Let us point out that Proposition 2.11 allows us to recover the fact that $\mathring{S}^m(\mathbb{R}^n)'$ becomes identified with a complemented subspace in $S^m(\mathbb{R}^n)'$. For that the mapping from $\mathring{S}^m(\mathbb{R}^n)'$ into $S^m(\mathbb{R}^n)'$ defined by (2.18) is seen to be continuous. When identifying $\mathring{S}^m(\mathbb{R}^n)'$ with its image in $S^m(\mathbb{R}^n)'$, a continuous projection in $S^m(\mathbb{R}^n)'$ onto $\mathring{S}^m(\mathbb{R}^n)'$ is given by first projecting $S^m(\mathbb{R}^n)'$ onto $\mathring{S}^m(\mathbb{R}^n)'$ using the dual of the embedding $\mathring{S}^m(\mathbb{R}^n) \hookrightarrow S^m(\mathbb{R}^n)$ and then applying the mapping from $\mathring{S}^m(\mathbb{R}^n)'$ to $S^m(\mathbb{R}^n)'$ just defined.

From now on $\mathring{S}^m(\mathbb{R}^n)'$ shall be regarded as the complemented subspace in $S^m(\mathbb{R}^n)'$ as recognized in Proposition 2.11. As a characterization for elements in $S^m(\mathbb{R}^n)'$ belonging to $\mathring{S}^m(\mathbb{R}^n)'$ we get:

2.12 Lemma. *Let $\Phi \in S^m(\mathbb{R}^n)'$. Then Φ belongs to $\mathring{S}^m(\mathbb{R}^n)'$ if and only if the restriction of Φ to any bounded set in $S^m(\mathbb{R}^n)$ is continuous for the C^∞-topology.*

Proof: Each oscillatory integral defined by (2.17) obeys the property mentioned in the lemma. Thus each element in $\mathring{S}^m(\mathbb{R}^n)'$ does.

To conclude the proof it suffices to notice the obvious fact that any functional in $S^m(\mathbb{R}^n)'$ the restrictions of which to bounded sets in $S^m(\mathbb{R}^n)$ are continuous for the C^∞-topology and which at the same time vanishes on $\mathring{S}^m(\mathbb{R}^n)$ is 0. □

Further, by our consideration above, $S^m_\tau(\mathbb{R}^n)'$, $\mathring{S}^m(\mathbb{R}^n)'$ coincide as topological vector spaces. From now on we will write $S^m_\tau(\mathbb{R}^n)'$ instead of $\mathring{S}^m(\mathbb{R}^n)'$.

2.13 Remark. Functionals in $S^m(\mathbb{R}^n)'$ actually belonging to $S^m_\tau(\mathbb{R}^n)'$ are said to be weakly continuous. This notion was introduced in a more general context by Hörmander (see [7, Definition 18.4.9]), who observed the importance of such functionals in questions related to the continuity of pseudo-differential operators.

2.4 Properties of $S^m_\tau(\mathbb{R}^n)'$

In this subsection we continue to studying the properties of $S^m_\tau(\mathbb{R}^n)'$. So we shall recognize that every weakly compact set in $S^m_\tau(\mathbb{R}^n)'$ is compact. Especially, this implies that the weak symbol topology on $S^m(\mathbb{R}^n)$ is the topology of uniform convergence on the precompact sets of $S^m_\tau(\mathbb{R}^n)'$.

But previously we make the statement that $S^m_\tau(\mathbb{R}^n)'$ is a (LB)-space more precise. For definition of bounded retractivity, see Appendix A.5.

2.14 Lemma. *$S^m_\tau(\mathbb{R}^n)'$ is the inductive limit of the Banach spaces $\Sigma^m_L(\mathbb{R}^n)/\Delta^m_L(\mathbb{R}^n)$,*

$$S^m_\tau(\mathbb{R}^n)' = \operatorname{ind\,lim} \Sigma^m_L(\mathbb{R}^n)/\Delta^m_L(\mathbb{R}^n), \tag{2.19}$$

where the limit is extended as $L \to \infty$. The inductive limit is boundedly retractive.

Proof: Under the natural identification $\Sigma^m_L \subset \Sigma^m$ we have $\Delta^m_L = \Sigma^m_L \cap \Delta^m_{L+1}$ and

$$\Sigma^m_L/\Delta^m_L = \Sigma^m_L/(\Sigma^m_L \cap \Delta^m_{L+1}) \cong (\Sigma^m_L + \Delta^m_{L+1})/\Delta^m_{L+1} \subset \Sigma^m_{L+1}/\Delta^m_{L+1}$$

for all $L \in \mathbb{N}$, with a norm-decreasing embedding. Thus Σ^m_L/Δ^m_L becomes a dense subspace of $\Sigma^m_{L+1}/\Delta^m_{L+1}$. Notice that $\mathcal{S}(\mathbb{R}^n)$ as a subspace of Σ^m_0 is dense in Σ^m_L/Δ^m_L for each $L \in \mathbb{N}$. Obviously, $\Sigma^m(\mathbb{R}^n)/\Delta^m(\mathbb{R}^n) \cong \operatorname{ind\,lim} \Sigma^m_L(\mathbb{R}^n)/\Delta^m_L(\mathbb{R}^n)$. Now (2.19) follows, since $S^m_\tau(\mathbb{R}^n)' \cong \Sigma^m(\mathbb{R}^n)/\Delta^m(\mathbb{R}^n)$ by the topological exactness of the sequence (2.9).

Using Lemma A.3 and Proposition A.11 we conclude first that $S^m_\tau(\mathbb{R}^n)'$ satisfies the strict Mackey convergence condition and then that the regular inductive limit $\operatorname{ind\,lim} \Sigma^m_L/\Delta^m_L$ is boundedly retractive. □

In the sequel we shall sometimes make use of the fact that the spaces appearing in the second short exact sequence in Proposition 2.6, i.e., in (2.9), are separable. This has the usual consequences , e.g., if a lcs E is separable, then equi-continuous sets in E' are separable and metrizable for $\sigma(E', E)$. Notice that the spaces appearing in the third short exact sequence in Proposition 2.6, i.e., in (2.10), are not separable; they are, however, separable for the weak topologies coming from the second short exact sequence, i.e., from (2.9).

In the following result weakly compact sets in $\sum_{|\alpha|\geq 0}{}^1 B_{(m-|\alpha|+n/2)}(\mathbb{R}^n)$ are characterized:

2.15 Lemma. *Let $\mathcal{D} \subset \Sigma^m(\mathbb{R}^n)$ be bounded. Then $\mathcal{D}$ is relatively $\sigma(\Sigma^m, \Pi^m)$-compact if and only if*

$$\sup_{f\in\mathcal{D}} \sum_{j>p} 2^{j(m-|\alpha|+n/2)} \|f^\alpha\|_{L^2(\Lambda_j)} \to 0 \text{ as } p \to \infty \tag{2.20}$$

for all $\alpha \in \mathbb{N}^n$, where f^α is the αth component of f. Furthermore, $\mathcal{D} \subset \Sigma^m(\mathbb{R}^n)$ is relatively compact if and only if, in addition, for all $p \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$, the sets

$$\{f^\alpha|_{\bigcup_{j=0}^p \Lambda_j};\ f \in \mathcal{D}\} \subset L^2(\textstyle\bigcup_{j=0}^p \Lambda_j) \tag{2.21}$$

are relatively compact.

Proof: Since $\mathcal{D} \subset \Sigma^m(\mathbb{R}^n)$ is bounded, $\mathcal{D} \subset \Sigma^m_L(\mathbb{R}^n)$ for some $L \in \mathbb{N}$. Moreover, if $\mathcal{D}$ is relatively weakly compact in $\Sigma^m(\mathbb{R}^n)$, then it is relatively weakly compact in $\Sigma^m_L(\mathbb{R}^n)$, if $\mathcal{D}$ is relatively compact in $\Sigma^m(\mathbb{R}^n)$, then it is relatively compact in $\Sigma^m_L(\mathbb{R}^n)$. Therefore, if suffices to characterize relatively weakly compact sets and relatively compact sets in ${}^1B_{(t)}(\mathbb{R}^n)$, $t \in \mathbb{R}$, by properties (2.20) and (2.20), (2.21), respectively, with $m - |\alpha| + n/2$ replaced by t and f^α replaced by f. Especially, property (2.20) then reads

$$\sup_{f\in\mathcal{D}} \sum_{j>p} 2^{jt}\|f\|_{L^2(\Lambda_j)} \to 0 \text{ as } p \to \infty. \tag{2.22}$$

To start with we prove that every weak Cauchy sequence $\{f_k\}_{k=0}^\infty$ in ${}^1B_{(t)}(\mathbb{R}^n)$ obeys (2.22). Going over to a suitable difference sequence $\{f_k - f_l\}$ for k, $l \to \infty$, if necessary, we may assume that $\{f_k\}_{k=0}^\infty$ is a weak null sequence. Suppose that (2.22) with $\mathcal{D} = \{f_k; k \in \mathbb{N}\}$ is not satisfied. Then we construct a function $v \in {}^\infty B_{(-t)}(\mathbb{R}^n)$ fulfilling $\langle f_k, v\rangle \not\to 0$ as $k \to \infty$ as follows (this is the sliding-hump technique, see, e.g., [9, §22, 4.(4)]): The function v shall be found by defining it successfully on Λ_j for $j = 0, 1, 2, \ldots$ through functions $v_j \in L^2(\Lambda_j)$. By our assumption, we find a $\delta > 0$, an increasing sequence $\{p_l\}_{l=0}^\infty \subset \mathbb{N}$ and a subsequence $\{f_{k_l}\}_{l=0}^\infty \subsetneq \{f_k\}_{k=0}^\infty$ such that

$$\sum_{j>p_l} 2^{jt}\|f_{k_l}\|_{L^2(\Lambda_j)} \geq 4\delta$$

for all $l \in \mathbb{N}$. Possibly choosing a further subsequence we equally assume that

$$\sum_{p_l<j\leq p_{l+1}} 2^{jt}\|f_{k_l}\|_{L^2(\Lambda_j)} \geq 3\delta, \qquad \sum_{j>p_{l+1}} 2^{jt}\|f_{k_l}\|_{L^2(\Lambda_j)} \leq \delta,$$

and, moreover, if the functions v_j for $j = 0, 1, \ldots, p_l$ have already been constructed,

$$\Big|\sum_{0\leq j\leq p_l} (f_{k_l}, v_j)_{L^2(\Lambda_j)}\Big| \leq \delta, \tag{2.23}$$

since $\{f_{k|\bigcup_{j=0}^p \Lambda_j}\}_{k=0}^\infty$ converges weakly to 0 in $L^2(\bigcup_{j=0}^p \Lambda_j)$. Here $(\,,\,)_{L^2(\Lambda_j)}$ is the scalar product in $L^2(\Lambda_j)$.

Now assuming all that, choose $v_j \in L^2(\Lambda_j)$ for $j = p_l + 1, \ldots, p_{l+1}$ to satisfy the conditions $\|v_j\|_{L^2(\Lambda_j)} = 2^{jt}$,

$$(f_{k_l}, v_j)_{L^2(\Lambda_j)} = 2^{jt}\|f_{k_l}\|_{L^2(\Lambda_j)}.$$

Choose $v_j \in L^2(\Lambda_j)$ for $j = 0, \ldots, p_0$ to satisfy (2.23) for $l = 0$. Then setting $v_{|\Lambda_j} = v_j$ for $j \in \mathbb{N}$, v belongs to ${}^\infty B_{(-t)}(\mathbb{R}^n)$, since $\sup_{j>p_0} 2^{-jt}\|v\|_{L^2(\Lambda_j)} = 1$, and

$$\begin{aligned}\int_{\mathbb{R}^n} f_{k_l}(\xi)v(\xi)\,d\xi &\geq \sum_{p_l<j\leq p_{l+1}} (f_{k_l}, v)_{L^2(\Lambda_j)} - \Big|\sum_{0\leq j\leq p_l} (f_{k_l}, v)_{L^2(\Lambda_j)}\Big| - \Big|\sum_{j>p_{l+1}} (f_{k_l}, v)_{L^2(\Lambda_j)}\Big| \\ &\geq 3\delta - \delta - \delta = \delta.\end{aligned}$$

for all $l \in \mathbb{N}$. This contradicts the weak convergence of $\{f_k\}_{k=0}^\infty$ in ${}^1B_{(t)}(\mathbb{R}^n)$ to 0.

To conclude the proof of (2.22) for $\mathcal{D} \subset {}^1B_{(t)}(\mathbb{R}^n)$ being relatively weakly compact use the fact that $\mathcal{D}$ is at the same time weakly metrizable. Thus $\mathcal{D}$ is relatively weakly sequentially compact. Now, if (2.22) were not be satisfied, then $\mathcal{D}$ would contain a weak Cauchy sequence violating property (2.22) which is not possible as we have seen. Hence, $\mathcal{D}$ fulfills (2.22).

Conversely, if $\mathcal{D} \subset {}^1B_{(t)}(\mathbb{R}^n)$ is bounded and satisfies (2.22) we show that $\mathcal{D}$ relatively weakly sequentially compact and, therefore, relatively weakly compact by Eberlein's theorem. Let $\{f_k\}_{k=0}^\infty \subset \mathcal{D}$ be a sequence. By a diagonal procedure, we choose a subsequence $\{f_{k_l}\}_{l=0}^\infty \subset \{f_k\}_{k=0}^\infty$ such that $\{f_{k_l|\Lambda_j}\}_{l=0}^\infty \subset L^2(\Lambda_j)$ is weakly convergent for all $j \in \mathbb{N}$. Introduce $h \in {}^1B_{(t)}(\mathbb{R}^n)$ as been defined on Λ_j as the limit of all these sequences, i.e., $f_{k_l|\Lambda_j} \to h_{|\Lambda_j}$ as $l \to \infty$ weakly in $L^2(\Lambda_j)$. From $\sum_{j>p} 2^{jt}\|f_{k_l} - h\|_{L^2(\Lambda_j)} \leq 2\delta_p$, where $\delta_p = \sup_{f\in\mathcal{D}} \sum_{j>p} 2^{jt}\|f\|_{L^2(\Lambda_j)}$, $p \in \mathbb{N}$, we infer that $f_{k_l} \to h$ as $l \to \infty$ weakly in ${}^1B_{(t)}(\mathbb{R}^n)$, since

$$|\langle f_{k_l} - h, a\rangle| \leq \left|\sum_{j\leq p}(f_{k_l} - h, a)_{L^2(\Lambda_j)}\right| + 2\delta_p \sup_{j>p} 2^{-jt}\|a\|_{L^2(\Lambda)}.$$

for all $l \in \mathbb{N}$ and $a \in {}^\infty B_{(-t)}(\mathbb{R}^n)$. Thus $\mathcal{D}$ is relatively weakly sequentially compact.

Finally, if $\mathcal{D} \subset {}^1B_{(t)}(\mathbb{R}^n)$ is relatively compact, then the analogues of the conditions stated in (2.20), (2.21) are obviously fulfilled. Vice versa, if these conditions are satisfied for a subset $\mathcal{D} \subset {}^1B_{(t)}(\mathbb{R}^n)$, then $\mathcal{D}$ is relatively compact in ${}^1B_{(t)}(\mathbb{R}^n)$ by a construction similar to that one just given.

The lemma is completely proved. □

2.16 Lemma. *Every weak null sequence in $S^m_\tau(\mathbb{R}^n)'$ can be lifted to a weak null sequence in $\Sigma^m(\mathbb{R}^n)$.*

Proof: Let $\{\Phi_k\}_{k=0}^\infty$ be a weak null sequence in $S^m_\tau(\mathbb{R}^n)$. By Lemma 2.5 and Lemma A.2, $\{\Phi_k\}_{k=0}^\infty$ is a weak null sequence in Σ^m_L/Δ^m_L for some $L \in \mathbb{N}$. (Notice that the absolutely convex closed hull of $\{\Phi_k;\ k \in \mathbb{N}\}$ in $S^m_\tau(\mathbb{R}^n)$ is $\{\sum_{k=0}^\infty \lambda_k\Phi_k;\ \sum_{k=0}^\infty |\lambda_k| \leq 1\}$ which is $\sigma(S^m_\tau, S^m)$-compact.) For $k \in \mathbb{N}$, represent Φ_k as

$$\langle\Phi_k, a\rangle = \sum_{|\alpha|\leq L} \int f^\alpha_k(\xi)\partial^\alpha_\xi a(\xi)\, d\xi, \quad a \in S^m(\mathbb{R}^n),$$

where $\{f^\alpha_k\}_{k=0}^\infty \subset {}^1B_{(m-|\alpha|+n/2)}(\mathbb{R}^n)$ may be assumed to be bounded for $\alpha \in \mathbb{N}^n$, $|\alpha| \leq L$. Now using the fact that $\mathcal{S}(\mathbb{R}^n) \subset {}^1B_{(t)}(\mathbb{R}^n)$, $t \in \mathbb{R}$, is dense, we find sequences $\{g^\alpha_k\}_{k=0}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ such that

$$\|f^\alpha_k - g^\alpha_k\|_{{}^1B_{(m-|\alpha|+n/2)}} \to 0 \text{ as } k \to \infty$$

for all $\alpha \in \mathbb{N}^n$, $0 < |\alpha| \leq L$. Set $g^0_k = -\sum_{0<|\alpha|\leq L}(-1)^{|\alpha|}g^\alpha_k \in \mathcal{S}(\mathbb{R}^n)$, $0 = (0, \ldots, 0) \in \mathbb{N}^n$. Then $\{g^\alpha_k\}_{|\alpha|\leq L} \in \Delta^m_L$ such that $\{f^\alpha_k - g^\alpha_k\}_{|\alpha|\leq L} \in \Sigma^m_L$ is in the preimage of Φ_k. Moreover, the sequence $\{f^0_k - g^0_k\}_{k=0}^\infty \subset {}^1B_{(m+n/2)}(\mathbb{R}^n)$ is bounded. For a in the image of $S^m_L(\mathbb{R}^n)$ in $S^m_0(\mathbb{R}^n)$, we have

$$\langle f^0_k - g^0_k, a\rangle \to 0 \text{ as } k \to \infty. \tag{2.24}$$

Then (2.24) holds for all $a \in S^m_0(\mathbb{R}^n)$, i.e., $f^0_k - g^0_k \to 0$ weakly in ${}^1B_{(m+n/2)}(\mathbb{R}^n)$. Thus the assertion follows. □

2.17 Proposition. *Every weakly compact set in $S^m_\tau(\mathbb{R}^n)'$ is compact. Furthermore, every compact set in $S^m_\tau(\mathbb{R}^n)'$ is the canonical image of a compact set in $\Sigma^m(\mathbb{R}^n)$.*

Proof: To start with we show that property (1.2) holds, i.e., for every $\sigma(S^m_\tau, (S^m_\tau)')$-null sequence $\{a_k\}_{k=0}^\infty$ and every $\sigma((S^m_\tau)', S^m_\tau)$-null sequence $\{\Phi_k\}_{k=0}^\infty$ we have

$$\langle \Phi_k, a_k \rangle \to 0 \text{ as } k \to \infty.$$

To see this, for some $L \in \mathbb{N}$, represent the functionals Φ_k in accordance with Lemma 2.16 as

$$\langle \Phi_k, a \rangle = \int \sum_{|\alpha| \le L} f_k^\alpha(\xi)\, \partial_\xi^\alpha a(\xi)\, d\xi, \quad a \in S^m(\mathbb{R}^n),$$

where $\{f_k^\alpha\}_{k=0}^\infty \subset {}^1B_{(m-|\alpha|+n/2)}(\mathbb{R}^n)$ are $\sigma({}^1B_{(m-|\alpha|+n/2)}, {}^\infty B_{(-m+|\alpha|-n/2)})$-null sequences. For $\alpha \in \mathbb{N}^n$, $|\alpha| \le L$, we have

$$\sup_{k \in \mathbb{N}} \sum_{j>p} 2^{j(m-|\alpha|+n/2)} \|f_k^\alpha\|_{L^2(\Lambda_j)} \to 0 \text{ as } p \to \infty$$

thanks to Lemma 2.15.

Fix some $\epsilon > 0$. Since the $\sigma(S^m_\tau, (S^m_\tau)')$-null sequence $\{a_k\}$ is bounded in $S^m_\tau(\mathbb{R}^n)$, it is bounded $S^m(\mathbb{R}^n)$, i.e.,

$$\sup_{j \in \mathbb{N}} 2^{j(-m+|\alpha|-n/2)} \|\partial_\xi^\alpha a_k\|_{L^2(\Lambda_j)} \le C_\alpha$$

for $\alpha \in \mathbb{N}^n$ and some constant $C_\alpha > 0$, and all $k \in \mathbb{N}$. Choose $p \in \mathbb{N}$ so large that

$$\sum_{j>p} 2^{j(m-|\alpha|+n/2)} \|f_k^\alpha\|_{L^2(\Lambda_j)} \le \left(2C_\alpha d_L\right)^{-1} \epsilon$$

for $\alpha \in \mathbb{N}^n$, $|\alpha| \le L$, and all $k \in \mathbb{N}$. Here d_L is the number of $\alpha \in \mathbb{N}^n$ with $|\alpha| \le L$. Then decomposing the expression for $\langle \Phi_k, a_k \rangle$,

$$\langle \Phi_k, a_k \rangle = \sum_{|\alpha| \le L} \sum_{0 \le j \le p} \int_{\Lambda_j} f_k^\alpha(\xi)\, \partial_\xi^\alpha a_k(\xi)\, d\xi + \sum_{|\alpha| \le L} \sum_{j>p} \int_{\Lambda_j} f_k^\alpha(\xi)\, \partial_\xi^\alpha a_k(\xi)\, d\xi,$$

we find for the second summand

$$\left| \sum_{|\alpha| \le L} \sum_{j>p} \int_{\Lambda_j} f_k^\alpha(\xi)\, \partial_\xi^\alpha a_k(\xi)\, d\xi \right| \le \sum_{|\alpha| \le L} \sum_{j>p} 2^{j(m-|\alpha|+n/2)} \|f_k^\alpha\|_{L^2(\Lambda_j)}\, 2^{j(-m+|\alpha|-n/2)} \|\partial_\xi^\alpha a_k\|_{L^2(\Lambda_j)} \le \epsilon/2,$$

while the first summand, $\sum_{|\alpha| \le L} \sum_{0 \le j \le p} \int_{\Lambda_j} f_k^\alpha(\xi)\, \partial_\xi^\alpha a_k(\xi)\, d\xi$, tends to 0 as $k \to \infty$ by Lebegue's convergence theorem. Notice for the latter that $f_k^\alpha(\xi)\, \partial_\xi^\alpha a_k(\xi)$ converges to 0 a.e. in $\xi \in \mathbb{R}^n$ as $k \to \infty$, since $S^m(\mathbb{R}^n) \ni a \mapsto \partial_\xi^\alpha a(\xi)$ defines a weakly continuous functional. Thus, for k large enough, we obtain

$$\left|\langle \Phi_k, a_k \rangle\right| \le \epsilon.$$

To conclude the proof that weakly compact sets in $S^m_\tau(\mathbb{R}^n)'$ are compact we make use of the results of Subsection A.5. Weakly compact sets in $S^m_\tau(\mathbb{R}^n)'$ belong to the class $\mathcal{S}'$ defined with respect to the dual pair $\langle S^m_\tau(\mathbb{R}^n), S^m_\tau(\mathbb{R}^n)' \rangle$, since they are weakly metrizable.

Therefore, by Proposition A.6, they belong to the class $\mathcal{L}'$, i.e., they are limited, with respect to the same dual pair. That means that they are precompact in $S^m_\tau(\mathbb{R}^n)'$, since $S^m_\tau(\mathbb{R}^n)'$ contains a countable weakly total subset. Thus weakly compact sets in $S^m_\tau(\mathbb{R}^n)'$ are compact.

Finally, if $\mathcal{C} \subset S^m_\tau(\mathbb{R}^n)'$ is compact, then $\mathcal{C} \subset \Sigma^m_L/\Delta^m_L$ is compact for some $L \in \mathbb{N}$. Thus $\mathcal{C}$ is the canonical image of some compact set $\mathcal{D} \subset \Sigma^m_L$.

The proof is finished. □

Next we consider functionals used in the proof of Proposition 2.17 in more detail. Let $\xi \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$. Denote the linear functional

$$S^m(\mathbb{R}^n) \ni a \mapsto (-1)^{|\alpha|}\partial^\alpha_\xi a(\xi) \tag{2.25}$$

by $\partial^\alpha \delta_\xi$. In view of Lemma 2.12, $\partial^\alpha \delta_\xi \in S^m_\tau(\mathbb{R}^n)'$.

2.18 Lemma. *Let $\xi \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$. Then the functional $\partial^\alpha\delta_\xi$ belongs to $\Sigma^m_{n+|\alpha|}/\Delta^m_{n+|\alpha|}$. Furthermore, the estimate*

$$\left\|\partial^\alpha\delta_\xi\right\|_{\Sigma^m_{n+|\alpha|}/\Delta^m_{n+|\alpha|}} \le C_{m,\alpha}\langle\xi\rangle^{m-|\alpha|} \tag{2.26}$$

is true with some constant $C_{m,\alpha} > 0$ not depending on $\xi \in \mathbb{R}^n$. For $k \in \mathbb{N}$, the function ψ_α defined by $\psi_\alpha(\xi) = \partial^\alpha\delta_\xi$ belongs to $C^k(\mathbb{R}^n; \Sigma^m_{n+|\alpha|+k}/\Delta^m_{n+|\alpha|+k})$, where

$$\partial^\gamma_\xi \psi_\alpha(\xi) = (-1)^{|\gamma|}\partial^{\alpha+\gamma}\delta_\xi \tag{2.27}$$

for $\gamma \in \mathbb{N}^n$, $|\gamma| \le k$.

Proof: First we prove (2.26). Suppose $m < |\alpha|$. Represent $\partial^\alpha\delta_\xi$ as follows: if $\xi_1 \ge 0, \dots, \xi_n \ge 0$,

$$\begin{aligned}\langle \partial^\alpha\delta_\xi, a\rangle &= (-1)^{n+|\alpha|} \times \\ &\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} H(\eta_1-\xi_1)\dots H(\eta_n-\xi_n)(\partial_{\eta_1}\dots\partial_{\eta_n}\partial^\alpha_\eta a)(\eta)\,d\eta_1\dots d\eta_n,\end{aligned} \tag{2.28}$$

while, for general ξ, for those j for which $\xi_j < 0$ holds we replace the expression $H(\eta_j-\xi_j)$ under the integral sign by $-H(\xi_j-\eta_j)$. Here H is the Heaviside function, i.e., $H(t) = 1$ for $t \ge 0$, $H(t) = 0$ for $t < 0$.

A calculation shows that $H((\mathrm{sgn}\xi_1)\eta_1 - |\xi_1|)\dots H((\mathrm{sgn}\xi_n)\eta_n - |\xi_n|)$ as a function of η belongs to ${}^1B_{(m-|\alpha|-n/2)}(\mathbb{R}^n)$. More precisely, we have

$$\left\|H((\mathrm{sgn}\xi_1)\eta_1 - |\xi_1|)\dots H((\mathrm{sgn}\xi_n)\eta_n - |\xi_n|)\right\|_{{}^1B_{(m-|\alpha|-n/2)}} \le C_{m,\alpha}\langle\xi\rangle^{m-\alpha}$$

with some constant $C_{m,\alpha} > 0$ independent of ξ. Therefore, $\partial^\alpha\delta_\xi \in \Sigma^m_{n+|\alpha|}/\Delta^m_{n+|\alpha|}$ and

$$\left\|\partial^\alpha\delta_\xi\right\|_{\Sigma^m_{n+|\alpha|}/\Delta^m_{n+|\alpha|}} \le C_{m,\alpha}\langle\xi\rangle^{m-|\alpha|}$$

provided that $m < |\alpha|$.

For general $m \in \mathbb{R}$, denote for the moment the functional $\partial^\alpha \delta_\xi$ acting on $S^m(\mathbb{R}^n)$ by $\Phi^m_{\xi,\alpha}$. Then, for arbitrary m, $m' \in \mathbb{R}$, we get by induction on $|\alpha|$ that

$$\Phi^m_{\xi,\alpha} = \langle\xi\rangle^{m'} \sum_{\gamma\leq\alpha} c_\gamma \, \Phi^{m-m'-|\gamma|}_{\xi,\alpha-\gamma} \, d_\gamma \tag{2.29}$$

for suitable $c_\gamma \in \mathbb{Z}$, $d_\gamma \in S^{-m'-|\gamma|}(\mathbb{R}^n)$. Especially, $c_\gamma = 1$, $d_\gamma = \langle\xi\rangle^{-m'}$ for $|\gamma| = 0$. Here d_γ stands for the multiplication operator and $\langle\xi\rangle^{m'}$ is multiplication by a constant. Since multiplication by a function in $S^\mu(\mathbb{R}^n)$ realizes a bounded operator from ${}^1B_{(t)}(\mathbb{R}^n)$ to ${}^1B_{(t-\mu)}(\mathbb{R}^n)$, $t \in \mathbb{R}$, we obtain

$$\begin{aligned} \|\Phi^m_{\xi,\alpha}\|_{\Sigma^m_{n+|\alpha|}/\Delta^m_{n+|\alpha|}} &\leq C\,\langle\xi\rangle^{m'} \sum_{\gamma\leq\alpha} \|\Phi^{m-m'-|\gamma|}_{\xi,\alpha-\gamma}\|_{\Sigma^{m-m'-|\gamma|}_{n+|\alpha|-|\gamma|}/\Delta^{m-m'-|\gamma|}_{n+|\alpha|-|\gamma|}} \\ &\leq C\,\langle\xi\rangle^{m'} \sum_{\gamma\leq\alpha} C_{m-m'-|\gamma|,\alpha-\gamma}\,\langle\xi\rangle^{m-m'-|\alpha|} \end{aligned}$$

with some constant $C > 0$ if $m' > m - |\alpha|$. This implies (2.26).

To verify (2.27) it suffices to treat the case $\gamma = \varepsilon_j$, $j = 1, \dots, n$, where ε_j is the multiindex of length 1 having 1 at the jth place. By symmetry, suppose that $j = 1$. Further suppose that $m < |\alpha| + 3/2$. Replace the representation (2.28) by

$$\begin{aligned} \langle\, \partial^\alpha\delta_\xi, a \,\rangle &= (-1)^{n+|\alpha|+1} \times \\ &\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} (\eta_1 - \xi_1)\, H(\eta_1 - \xi_1) \dots H(\eta_n - \xi_n)(\partial^2_{\eta_1}\partial_{\eta_2} \dots \partial_{\eta_n}\partial^\alpha_\eta a)(\eta)\, d\eta_1 \dots d\eta_n \end{aligned}$$

if $\xi_1 \geq 0, \dots, \xi_n \geq 0$, and the analog expression with $(\eta_1 - \xi_1)H(\eta_1 - \xi_1)$ replaced by $-(\xi_1 - \eta_1)H(\xi_1 - \eta_1)$ if $\xi_1 < 0$ and $H(\eta_j - \xi_j)$ replaced by $H(\xi_j - \eta_j)$ if $\xi_j < 0$, $j = 2, \dots, n$. We have to consider $\partial^\alpha\delta_{\xi+he_1} - \partial^\alpha\delta_\xi + h\partial^{\alpha+\epsilon}\delta_\xi$ as $h \to 0$. In case $\xi_1 \geq 0, \dots, \xi_n \geq 0$ we obtain

$$\begin{aligned} &\langle\, \partial^\alpha\delta_{\xi+he_1} - \partial^\alpha\delta_\xi + h\,\partial^{\alpha+e_1}\delta_\xi, a \,\rangle = \\ &(-1)^{n+|\alpha|+1} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} (\eta_1 - \xi_1 - h)\,\big(H(\eta_1 - \xi_1 - h) - H(\eta_1 - \xi_1)\big) \times \\ &\qquad H(\eta_2 - \xi_2) \dots H(\eta_n - \xi_n)(\partial^2_{\eta_1}\partial_{\eta_2} \dots \partial_{\eta_n}\partial^\alpha_\eta a)(\eta)\, d\eta_1 \dots d\eta_n, \end{aligned}$$

and an analog expression in the general case. For the function under the integral sign in front of $\partial^2_{\eta_1}\partial_{\eta_2} \dots \partial_{\eta_n}\partial^\alpha_\eta a$, i.e, $(\mathrm{sgn}\xi_1)(\eta_1 - \xi_1 - h)\big(H((\mathrm{sgn}\xi_1)(\eta_1 - \xi_1 - h)) - H((\mathrm{sgn}\xi_1)\eta_1 - |\xi_1|)\big) H((\mathrm{sgn}\xi_2)\eta_2 - |\xi_2|) \dots H((\mathrm{sgn}\xi_n)\eta_n - |\xi_n|)$, as a function of $\eta \in \mathbb{R}^n$ we get, for $h \in \mathbb{R}$, the estimate

$$\begin{aligned} &\big\|(\eta_1 - \xi_1 - h)\big(H((\mathrm{sgn}\xi_1)(\eta_1 - \xi_1 - h)) - H((\mathrm{sgn}\xi_1)\eta_1 - |\xi_1|)\big) \times \\ &\quad H((\mathrm{sgn}\xi_2)\eta_2 - |\xi_2|) \dots H((\mathrm{sgn}\xi_n)\eta_n - |\xi_n|)\big\|_{{}^1B_{(m-|\alpha|-1-n/2)}} \leq C_{m,\alpha}\,\langle\xi\rangle^{m-|\alpha|-3/2}\,|h|^{3/2} \end{aligned}$$

for some constant $C_{m,\alpha} > 0$ independent of ξ, h as long as $|h| \leq |\xi|$, since $|\xi_1 - \eta_1| \leq h$ on $\mathrm{supp}\big(H((\mathrm{sgn}\xi_1)(\eta_1 - \xi_1 - h)) - H((\mathrm{sgn}\xi_1)\eta_1 - |\xi_1|)\big)$.

In particular, for $\xi \in \mathbb{R}^n$ fixed, it follows that

$$\big\|\partial^\alpha\delta_{\xi+he_1} - \partial^\alpha\delta_\xi + h\,\partial^{\alpha+e_1}\delta_\xi\big\|_{\Sigma^m_{n+|\alpha|+1}/\Delta^m_{n+|\alpha|+1}} = o(|h|) \text{ as } |h| \to 0.$$

For general $m \in \mathbb{R}$, the same estimate can be obtained by considerations similar to those above using formula (2.29). This yields $\partial_{\xi_j}\psi_\alpha(\xi) = -\partial^{\alpha+e_j}\delta_\xi$ for $j = 1, \dots, n$. □

2.5 Linear-Topological Characterization of $S^m_\tau(\mathbb{R}^n)$

We proceed with providing relevant functional-analytic properties of $S^m_\tau(\mathbb{R}^n)$. From Proposition 2.17, by Lemma A.10, we already know that bounded sets in $S^m_\tau(\mathbb{R}^n)$ are precompact. But even more is true:

2.19 Proposition. *$S^m_\tau(\mathbb{R}^n)$ is a complete, separable Schwartz space.*

Proof: We show first that any convergent sequence in $S^m_\tau(\mathbb{R}^n)'$ is equi-continuously convergent. Let $\{\Phi_k\}_{k=0}^\infty \subset S^m_\tau(\mathbb{R}^n)'$ be a convergent sequence. We may suppose that $\{\Phi_k\}_{k=0}^\infty$ converges to 0. By Lemma 2.14, $\{\Phi_k\}_{k=0}^\infty$ is contained and convergent to 0 in Σ^m_L/Δ^m_L for some $L \in \mathbb{N}$. Further there exists a sequence $\{\rho_k\}_{k=0}^\infty$ of positive reals with $\rho_k \to \infty$ as $k \to \infty$ such that $\{\rho_k\Phi_k\}_{k=0}^\infty$ converges to 0 in Σ^m_L/Δ^m_L and therefore in $S^m_\tau(\mathbb{R}^n)'$ (see [8, 10.1.3, 10.1.4]). Then $U = \{\rho_k\Phi_k;\ k \in \mathbb{N}\}^\circ$ is 0-neighbourhood in $S^m_\tau(\mathbb{R}^n)$, and we have

$$\sup_{a\in U} |\langle\Phi_k, a\rangle| \to 0 \text{ as } k \to \infty,$$

since $|\langle\Phi_k, a\rangle| \le \rho_k^{-1}$ for $k \in \mathbb{N}$ and $a \in U$. Therefore, $\{\Phi_k\}_{k=0}^\infty$ converges uniformly on U.

Now let $\mathcal{C} \subset S^m_\tau(\mathbb{R}^n)'$ be compact. It suffices to show that $\mathcal{C}$ is contained in the closed absolutely convex hull of some equi-continuous null sequence $\{\Phi_k\}_{k=0}^\infty$ in $S^m_\tau(\mathbb{R}^n)'$. $\mathcal{C}$ is contained and compact in the Banach space Σ^m_L/Δ^m_L for some $L \in \mathbb{N}$. Hence, $\mathcal{C} \subset \{\sum_k \lambda_k\Phi_k;\ \sum_k |\lambda_k| \le 1\}$ for certain null sequence $\{\Phi_k\}_{k=0}^\infty$ in Σ^m_L/Δ^m_L. But $\mathcal{C} \subset \{\sum_k \lambda_k\Phi_k;\ \sum_k |\lambda_k| \le 1\}$ is the closed absolutely convex hull of the sequence $\{\Phi_k\}_{k=0}^\infty$ in $S^m_\tau(\mathbb{R}^n)'$, while $\{\Phi_k\}_{k=0}^\infty$ equi-continuously converges to 0. □

Notice further properties of $S^m_\tau(\mathbb{R}^n)$. Since bounded sets in $S^m_\tau(\mathbb{R}^n)$ are relatively compact, $S^m_\tau(\mathbb{R}^n)$ is a semi-Montel space. $S^m_\tau(\mathbb{R}^n)$ is not quasi-barreled, therefore, neither bornological nor barreled, since otherwise $S^m_\tau(\mathbb{R}^n)$ would be a Montel space, i.e., reflexive. The associated bornological space to $S^m_\tau(\mathbb{R}^n)$ is $S^m(\mathbb{R}^n)$.

The fact behind the next lemma is that the weak symbol topology is actually the strongest locally convex topology on $S^m_\tau(\mathbb{R}^n)$ which agrees on the bounded sets with the weak topology $\sigma(S^m_\tau, (S^m_\tau)')$ (see, e.g., [8, 9.3.7]).

2.20 Lemma. *On the bounded sets of $S^m_\tau(\mathbb{R}^n)$, the τ-topology coincides with any of the following topologies: the topology of point-wise convergence, the C^∞-topology, and the topology induced by $S^{m'}(\mathbb{R}^n)$ for $m' > m$.*

Proof: By a classical result (see [6, p. 88]), the latter three topologies are the same on bounded sets of $S^m(\mathbb{R}^n)$. Moreover, each of these topologies is weaker than the τ-topology. Now the result follows, since closed bounded sets in $S^m(\mathbb{R}^n)$ are compact for the τ-topology and a bijective and continuous mapping from a compact space onto a Hausdorff space is a homeomorphism. □

2.21 Remark. Note the strong analogy of the situation under consideration to the sequence spaces c_0, l^1, and l^∞, and their non-reflexivity. Here $\mathring{S}^m(\mathbb{R}^n)$, $\mathring{S}^m(\mathbb{R}^n)'$, and

$S^m(\mathbb{R}^n)$ correspond to c_0, l^1, and l^∞, respectively. For instance, every weakly compact set in l^1 is compact, and $l^\infty_\tau = (l^\infty, \tau(l^\infty, l^1))$ is a universal complete, separable Schwartz space.

Next we discuss nuclearity for $S^m_\tau(\mathbb{R}^n)$. We will see in a moment that $S^m_\tau(\mathbb{R}^n)$ is not a nuclear space. The property by which in the applications we have in mind the non-nuclearity of $S^m_\tau(\mathbb{R}^n)$ is absorbed is given in Theorem 2.23 below.

2.22 Lemma. *For every $t \in \mathbb{R}$, the mapping*

$$S^m_\tau(\mathbb{R}^n) \to \mathcal{L}_\sigma(H^{t+m}(\mathbb{R}^n), H^t(\mathbb{R}^n)), \ a \mapsto a(D) \tag{2.30}$$

is continuous.

Proof: For $u \in H^{t+m}(\mathbb{R}^n)$,

$$a \mapsto \|a(D)u\|_{H^t} = \sup_{\|v\|_{H^{-t}} \le 1} \left| \int a(D)u(x)v(x)\,dx \right|$$

is a continuous semi-norm on $S^m_\tau(\mathbb{R}^n)$, since the set $\{\Phi_v;\ \|v\|_{H^{-t}} \le 1\} \subset S^m_\tau(\mathbb{R}^n)'$, where we have set $\langle \Phi_v, a\rangle = \int a(D)uv\,dx$, is weakly compact for the weak compactness of the unit ball in $H^{-t}(\mathbb{R}^n)$. □

For the notion of local Banach space used in the next theorem, see Appendix A.1.

2.23 Theorem. *For each $u \in H^{t+m}(\mathbb{R}^n)$, there is an absolutely convex, closed 0-neighbourhood U in $S^m_\tau(\mathbb{R}^n)$ such that the mapping $S^m_\tau(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$, $a \mapsto a(D)u$, factors through $S^m_\tau(\mathbb{R}^n)^\sim_{(U)}$ in a way such that the resulting operator*

$$S^m_\tau(\mathbb{R}^n)^\sim_{(U)} \to H^t(\mathbb{R}^n), \ a + \ker\|\ \|_{(U)} \mapsto a(D)u \tag{2.31}$$

is 2-nuclear.

Proof: We can reduce to the case $m = t = 0$. Since then $\{\|\hat{u}\|_{L^2(\Lambda_j)}\} \in l^2$, there is a sequence $\{\gamma_j\} \in c_0$ of positive reals such that

$$\sum_{j=0}^\infty \gamma_j^{-1} \|\hat{u}\|^2_{L^2(\Lambda_j)} < \infty \tag{2.32}$$

(see [8, p. 27]). Fix some $L \in \mathbb{N}$, $L > n/2$, set $\delta_j = \gamma_j^{1/2}$ for $j \in \mathbb{N}$, and define the absolutely convex, compact set $\mathcal{C} \subset S^0_\tau(\mathbb{R}^n)'$ as the union of the absolutely convex, compact set in $S^0_\tau(\mathbb{R}^n)'$ given in the proof of Lemma 2.22 as the set of all functionals $a \mapsto \int a(\xi)\hat{u}(\xi)v(\xi)\,d\xi$ with $\|v\|_{L^2} \le 1$ and the absolutely convex, compact set in $S^0_\tau(\mathbb{R}^n)'$ which is the image in $S^0_\tau(\mathbb{R}^n)'$ of the weakly compact set $\mathcal{D} \subset \Sigma^0_L(\mathbb{R}^n)$ given by requiring

$$\sup_{f \in \mathcal{D}} \sum_{j=p}^\infty 2^{j(-|\alpha|+n/2)} \|f^\alpha\|_{L^2(\Lambda_j)} \le \delta_p$$

for all $p \in \mathbb{N}$ and all $\alpha \in \mathbb{N}^n$, $|\alpha| \leq L$ (see Lemma 2.15). Set further $V = \mathcal{C}^\circ$. Then the mapping $S^0_\tau(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $a \mapsto a(D)u$, factors through the continuous mapping

$$S^0_\tau(\mathbb{R}^n)^{\sim}_{(V)} \to L^2(\mathbb{R}^n), \quad a + \ker \| \,\|_{(V)} \mapsto a(D)u. \tag{2.33}$$

We prove that this mapping is 2-integral. By Proposition B.1, it suffices to show that the mapping $S^0_\tau(\mathbb{R}^n)^{\sim}_{(V)} \to L^2(\mathbb{R}^n)$ induced by $S^0_\tau(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $a \mapsto a\hat{u}$, is order bounded, i.e., there is an $h \in L^2(\mathbb{R}^n)$ such that

$$|a(\xi)\hat{u}(\xi)| \leq h(\xi) \text{ a.e. } \xi \in \mathbb{R}^n \tag{2.34}$$

for all $a \in V$. For that end, we derive an estimate of the modulus of functions in V. Clearly, given $\alpha \in \mathbb{N}^n$, $|\alpha| \leq L$, $j \in \mathbb{N}$, elements $\{f^\beta\} \in \Sigma^0(\mathbb{R}^n)$ with $f^\beta = 0$ if $\beta \neq \alpha$, $f^\alpha{}_{|\Lambda_k} = 0$ if $k \neq j$ and $\|f^\alpha\|_{L^2(\Lambda_j)} \leq 2^{j(|\alpha|-n/2)}\,\delta_j$ belong to $\mathcal{D}$. Therefore, for $a \in V$ we obtain

$$\|\partial^\alpha_\xi a\|_{L^2(\Lambda_j)} \leq 2^{j(-|\alpha|+n/2)}\,\delta_j^{-1}.$$

From that, using a rescaling argument and Sobolev's embedding theorem as in the proof of Proposition 2.3 we arrive at the estimate

$$|a(\xi)| \leq C\,\delta_j^{-1}, \quad \xi \in \Lambda_j,$$

which holds for $a \in V$ and $j \in \mathbb{N}$ with some constant $C > 0$ independent of a, j. Now, by (2.32), in (2.34) we may put

$$h(\xi) = C\,\delta_j^{-1}\,|\hat{u}(\xi)|, \quad \xi \in \Lambda_j.$$

Having shown the 2-integrality of the operator (2.33), we choose an absolutely convex, closed 0-neighbourhood U in $S^0_\tau(\mathbb{R}^n)$ absorbed by V such that the canonical mapping $S^0_\tau(\mathbb{R}^n)^{\sim}_{(U)} \mapsto S^0_\tau(\mathbb{R}^n)^{\sim}_{(V)}$ is compact and get, by Proposition B.2, that the operator

$$S^0_\tau(\mathbb{R}^n)^{\sim}_{(U)} \to L^2(\mathbb{R}^n), \quad a + \ker \| \,\|_{(U)} \mapsto a(D)u$$

is 2-nuclear. □

2.24 Proposition. *Let $u \in H^{t+m}(\mathbb{R}^n)$. Then the mapping $S^m_\tau(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$, $a \mapsto a(D)u$, is nuclear if and only if $u \in {}^1H_{(t+m)}(\mathbb{R}^n)$, where ${}^1H_{(t+m)}(\mathbb{R}^n)$ is the Besov space considered in* Subsection 2.2. *Moreover, an equivalent condition is that there exists an absolutely convex, closed 0-neighbourhood U of $S^m_\tau(\mathbb{R}^n)$ such that the mapping $S^m_\tau(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$, $a \mapsto a(D)u$, factors through $S^m_\tau(\mathbb{R}^n)^{\sim}_{(U)}$ with the resulting operator $S^m_\tau(\mathbb{R}^n)^{\sim}_{(U)} \to H^t(\mathbb{R}^n)$ being 1-summing.*

Proof: Assume again $m = t = 0$. Suppose that there exists an absolutely convex, closed 0-neighbourhood U of $S^m_\tau(\mathbb{R}^n)$ such that the mapping $S^m_\tau(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$, $a \mapsto a(D)u$, factors through $S^m_\tau(\mathbb{R}^n)^{\sim}_{(U)}$ with the resulting operator $S^m_\tau(\mathbb{R}^n)^{\sim}_{(U)} \to H^t(\mathbb{R}^n)$ being 1-summing. Let $\{\chi_j\}_{j=0}^\infty$ be a dyadic decomposition of unity on $\mathbb{R}^n$, i.e., $\chi_j \in C^\infty_0(\mathbb{R}^n)$, $\operatorname{supp}\chi_1 \subset \{\xi \in \mathbb{R}^n;\, 1 \leq \xi \leq 2\}$, $\chi_j(\xi) = \chi_1(2^{-j}\xi)$ for $j \geq 1$, and $\sum_{j=0}^\infty \chi_j(\xi) = 1$. It is easy to see that the sequence $\{\chi_j\}_{j=0}^\infty$ is unconditional summable in $S^m_\tau(\mathbb{R}^n)$, in particular, unconditional weakly summable. Therefore, by our assumption, we must have

$\sum_{j=0}^{\infty} \|\chi(D)u\|_{L^2} < \infty$. By a modification of this example we further see that we get $\sum_{j=0}^{\infty} \|\hat{u}\|_{L^2(\Lambda_j)} < \infty$, i.e., $u \in {}^1H_{(0)}(\mathbb{R}^n)$.

Vice versa, suppose that $u \in {}^1H_{(0)}(\mathbb{R}^n)$. Then there is a sequence $\{\gamma_j\}_{j=0}^{\infty} \in c_0$ such that $\sum_{j=0}^{\infty} \gamma_j^{-1} \|\hat{u}\|_{L^2(\Lambda_j)} < \infty$. Choose a orthonormal basis $\{v_j\}_{j=0}^{\infty}$ in $L^2(\mathbb{R}^n)$ as follows: $\hat{v}_j(\xi) = c_j$ if $\xi \in \Lambda_j$ and $\hat{v}_j(\xi) = 0$ if $\xi \notin \Lambda_j$, where the constant c_j is determined by the condition $\|v_j\|_{L^2} = 1$. In particular, c_j behaves like $2^{-jn/2}$ times some constant as $j \to \infty$. Represent the operator $S_\tau^0(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $a \mapsto a(D)u$, as

$$a \mapsto \sum_{j=0}^{\infty} \gamma_j^{-1} \|\hat{u}\|_{L^2(\Lambda_j)} \langle \Phi_j, a \rangle v_j, \tag{2.35}$$

where the functional $\Phi_j \in S_\tau^0(\mathbb{R}^n)'$ is given by $\langle \Phi_j, a \rangle = \gamma_j \|\hat{u}\|_{L^2(\Lambda_j)}^{-1} c_j \int_{\Lambda_j} a(\xi)\hat{u}(\xi)\, d\xi$ for $a \in S^0(\mathbb{R}^n)$. Now we see that $\langle \Phi_j, a \rangle \to 0$ as $j \to \infty$ for all $a \in S^0(\mathbb{R}^n)$, since

$$|\langle \Phi_j, a \rangle| \leq \gamma_j c_j \|a\|_{L^2(\Lambda_j)}$$

and $\sup_{j \in \mathbb{N}} c_j \|a\|_{L^2(\Lambda_j)} < \infty$. This yields that (2.35) is a nuclear representation. □

Notice that in the proof of Proposition 2.24 we have actually shown that the sequence $\{\chi_j\}_{j=0}^{\infty}$ appearing in a dyadic decomposition of unity on $\mathbb{R}^n$ is unconditional summable, but not absolutely summable in $S_\tau^0(\mathbb{R}^n)$. Especially, we have obtained:

2.25 Corollary. *$S_\tau^m(\mathbb{R}^n)$ is not a nuclear space.*

2.6 Tensor Product Representation

In this subsection the vector-valued symbol classes $S^m(\mathbb{R}^n; E)$ are studied. In particular, it is proved that $S_\tau^m(\mathbb{R}^n; E) = S_\tau^m(\mathbb{R}^n)\hat{\otimes}_\epsilon E$ holds for any complete lcs E.

2.26 Definition. *Let E be a complete lcs. Then the weak symbol topology on $S^m(\mathbb{R}^n; E)$ is given by the semi-norm system*

$$S^m(\mathbb{R}^n; E) \ni a \mapsto \sup_{\Phi \in C} \|\langle \Phi, a \rangle\|, \tag{2.36}$$

where C is any weakly compact set in $S_\tau^m(\mathbb{R}^n)'$ and $\| \ \|$ is any continuous semi-norm on E. $S^m(\mathbb{R}^n; E)$ equipped with this topology is denoted by $S_\tau^m(\mathbb{R}^n; E)$.

From Proposition 2.11 recall that functionals belonging to $S_\tau^m(\mathbb{R}^n)'$ can be applied to functions in $S_\tau^m(\mathbb{R}^n; E)$ yielding elements in E. For $E = \mathbb{C}$, Definition 2.26 agrees with the definition previously given for $S_\tau^m(\mathbb{R}^n)$.

2.27 Proposition. *Let E be a complete lcs. Then the space $S_\tau^m(\mathbb{R}^n; E)$ is complete.*

Proof: Suppose that $\{a_\iota\}$ is a Cauchy net in $S_\tau^m(\mathbb{R}^n; E)$. Since $S_\tau^m(\mathbb{R}^n; E) \hookrightarrow C^\infty(\mathbb{R}^n; E)$, there is an $a \in C^\infty(\mathbb{R}^n; E)$ such that $\{a_\iota\}$ converges to a in $C^\infty(\mathbb{R}^n; E)$. For every $\phi \in E'$,

$\{\langle\phi, a_\iota\rangle\}$ is a Cauchy net in $S^m_\tau(\mathbb{R}^n)$. By Lemma 2.8, $\{\langle\phi, a_\iota\rangle\}$ converges to $\langle\phi, a\rangle$ in $S^m_\tau(\mathbb{R}^n)$, thereby, uniformly on equi-continuous sets in E', i.e.,

$$\sup_{\phi\in H} \|\langle\phi, a_\iota\rangle - \langle\phi, a\rangle\|_{(U)} \to 0 \tag{2.37}$$

for every equi-continuous set $H \subset E'$ and any 0-neighbourhood $U \subset S^m_\tau(\mathbb{R}^n)$. Especially, we have

$$\sup_{\phi\in H} \|\langle\phi, a\rangle\|_{(U)} < \infty. \tag{2.38}$$

Property (2.38) means that, for any equi-continuous set $H \subset E'$, $\{\langle\phi, a\rangle;\ \phi \in H\}$ is bounded in $S^m_\tau(\mathbb{R}^n)$. Now $S^m(\mathbb{R}^n)$, $S^m_\tau(\mathbb{R}^n)$ have the same bounded sets. It follows that (2.38) holds also for 0-neighbourhoods $U \subset S^m(\mathbb{R}^n)$. Consequently, $a \in C^\infty(\mathbb{R}^n; E)$ belongs to $S^m(\mathbb{R}^n; E)$, and $\{a_\iota\}$ converges to a in $S^m_\tau(\mathbb{R}^n; E)$ by (2.37). □

Before stating the next result we need:

2.28 Lemma. *Let E be a complete lcs. Further let $a \in S^m(\mathbb{R}^n; E)$. Then the convergence in (2.18), i.e.,*

$$\langle\Phi, a_\epsilon\rangle \to \langle\Phi, a\rangle \text{ in } E \text{ as } \epsilon \to 0^+, \tag{2.39}$$

holds uniformly for Φ in compact sets in $S^m_\tau(\mathbb{R}^n)'$.

Proof: Regarding a_ϵ for $0 < \epsilon < 1$ as linear mappings in $\mathcal{L}(S^m_\tau(\mathbb{R}^n)', E)$, we see that $\{a_\epsilon;\ 0 < \epsilon < 1\}$ is a bounded set in $\mathcal{L}(S^m_\tau(\mathbb{R}^n)', E)$ for the simple topology. Moreover, $\{a_\epsilon\}$ converges to a as $\epsilon \to 0^+$ in $\mathcal{L}_\sigma(S^m_\tau(\mathbb{R}^n)', E)$ by (2.18), therefore, in $\mathcal{L}_\gamma(S^m_\tau(\mathbb{R}^n)', E)$ by the Banach-Steinhaus theorem. This is the assertion. □

2.29 Theorem. *Let E be a complete lcs. Then*

$$S^m_\tau(\mathbb{R}^n; E) = S^m_\tau(\mathbb{R}^n)\tilde{\otimes}_\epsilon E. \tag{2.40}$$

Proof: We first show that $S^m(\mathbb{R}^n)\otimes E$ in the topology induced by $S^m_\tau(\mathbb{R}^n; E)$ is $S^m_\tau(\mathbb{R}^n)\otimes_\epsilon E$. For that it suffices to notice that for $\mathcal{C}$ being a compact set in $S^m_\tau(\mathbb{R}^n)'$ and U being a 0-neighbourhood of E we have

$$\sup_{\Phi\in\mathcal{C}} \left\|\langle\Phi, \sum_{j=1}^{\kappa} \alpha_j \otimes a_j\rangle\right\|_{(U)} = \sup_{\Phi\otimes\phi\in\mathcal{C}\otimes U^\circ} \left|\sum_{j=1}^{\kappa}\langle\phi, \alpha_j\rangle\langle\Phi, a_j\rangle\right|$$

for all $\alpha_j \in E$, $a_j \in S^m_\tau(\mathbb{R}^n)$ and $\kappa \in \mathbb{N}$. (On the right-hand side there stands the generic semi-norm of $S^m_\tau(\mathbb{R}^n; E)$, on the left-hand side the generic semi-norm of $S^m_\tau(\mathbb{R}^n) \otimes_\epsilon E$.)

It remains to prove that $S^m(\mathbb{R}^n) \otimes E$ is dense in $S^m_\tau(\mathbb{R}^n; E)$. First, $S^{-\infty}(\mathbb{R}^n; E)$ is sequentially dense in $S^m_\tau(\mathbb{R}^n; E)$ by Lemma 2.28. (Choose $\chi \in C_0^\infty(\mathbb{R}^n)$ in (2.39).) Therefore, it is enough to show that $S^{-\infty}(\mathbb{R}^n)\otimes E$ is dense in $S^{-\infty}(\mathbb{R}^n; E)$ when $S^{-\infty}(\mathbb{R}^n; E)$ carries the weak symbol topology induced by $S^m_\tau(\mathbb{R}^n; E)$. But this holds, since by (2.3) $S^{-\infty}(\mathbb{R}^n) \otimes E$ is dense in $S^{-\infty}(\mathbb{R}^n; E)$ in an even stronger topology. □

Using Lemma 2.18 we now derive another representation for $S^m_\tau(\mathbb{R}^n; E)$.

2.30 Proposition. *Let E be a complete lcs. Then for every $H \in \mathcal{L}(S^m_\tau(\mathbb{R}^n)', E)$ there exists an $a \in S^m(\mathbb{R}^n; E)$ such that*

$$H(\Phi) = \langle \Phi, a\rangle \tag{2.41}$$

holds for all $\Phi \in S^m_\tau(\mathbb{R}^n)'$. In this way, $S^m(\mathbb{R}^n; E)$, $\mathcal{L}(S^m_\tau(\mathbb{R}^n)', E)$ are identified as linear spaces. Under this identification,

$$S^m_\tau(\mathbb{R}^n; E) = \mathcal{L}_\gamma(S^m_\tau(\mathbb{R}^n)', E), \tag{2.42}$$

where subscript γ stands for the topology of uniform convergence on all precompact sets in $S^m_\tau(\mathbb{R}^n)'$.

Proof: By Proposition 2.11 we already know that every $S^m(\mathbb{R}^n; E)$ gives rise to an operator in $\mathcal{L}(S^m_\tau(\mathbb{R}^n)', E)$ via formula (2.18). Thus the other direction has to be proved. Let $H \in \mathcal{L}(S^m_\tau(\mathbb{R}^n)', E)$. Define the function $a(\xi)$ on $\mathbb{R}^n$ with values in E by

$$a(\xi) = H(\delta_\xi),$$

where δ_ξ is the functional given in (2.25). Then a belongs to $C^\infty(\mathbb{R}^n; E)$, where

$$\partial^\alpha_\xi a(\xi) = (-1)^{|\alpha|} H(\partial^\alpha \delta_\xi)$$

in view of (2.27), and, further, to $S^m_\tau(\mathbb{R}^n; E)$ in view of (2.26).

We show that a satisfies (2.41). Let $\Phi \in S^m_\tau(\mathbb{R}^n)'$. Represent Φ according to (2.9), i.e.,

$$\langle \Phi, b\rangle = \sum_{|\alpha|\le L} \int f^\alpha(\xi)\, \partial^\alpha_\xi b(\xi)\, d\xi, \quad b \in S^m(\mathbb{R}^n), \tag{2.43}$$

for some $f^\alpha \in {}^1B_{(m-|\alpha|+n/2)}(\mathbb{R}^n)$ and $L \in \mathbb{N}$. Then

$$\langle \Phi, b\rangle = \sum_{|\alpha|\le L} (-1)^{|\alpha|} \int f^\alpha(\xi)\, \langle \partial^\alpha \delta_\xi, b\rangle\, d\xi = \Big\langle \sum_{|\alpha|\le L} (-1)^{|\alpha|} \int f^\alpha(\xi)\, \partial^\alpha \delta_\xi\, d\xi, b \Big\rangle.$$

This is permitted, since $\int f^\alpha(\xi)\, \partial^\alpha \delta_\xi\, d\xi$ exists as Bochner integral in $\Sigma^m_{n+|\alpha|}/\Delta^m_{n+|\alpha|}$ by (2.26). We obtain

$$\Phi = \sum_{|\alpha|\le L} (-1)^{|\alpha|} \int f^\alpha(\xi)\, \partial^\alpha \delta_\xi\, d\xi$$

and

$$H(\Phi) = \sum_{|\alpha|\le L} (-1)^{|\alpha|} \int f^\alpha(\xi)\, H(\partial^\alpha \delta_\xi)\, d\xi = \sum_{|\alpha|\le L} \int f^\alpha(\xi)\, \partial^\alpha_\xi a(\xi)\, d\xi = \langle \Phi, a\rangle.$$

Thereby, the latter equality is another possibility to define the value of $\langle \Phi, a\rangle$ for $\Phi \in S^m_\tau(\mathbb{R}^n)'$ having the representation (2.43).

Thus we have identified $S^m(\mathbb{R}^n;E)$, $\mathcal{L}(S^m_\tau(\mathbb{R}^n)',E)$ as linear spaces. Then (2.42) follows in advance, since, for $\mathcal{C}$ being a compact set in $S^m_\tau(\mathbb{R}^n)'$ and U an absolutely convex, closed 0-neighbourhood of $S^m_\tau(\mathbb{R}^n)$, the semi-norm on $\mathcal{L}_\gamma(S^m_\tau(\mathbb{R}^n)',E)$,

$$H \mapsto \sup_{\Phi\in\mathcal{C}} \|H(\Phi)\|_{(U)}$$

is the same as in (2.36). □

It is seen that under the identification in Proposition 2.30 we also have that

$$S^m(\mathbb{R}^n;E) = \mathcal{L}_\beta(S^m_\tau(\mathbb{R}^n)',E). \tag{2.44}$$

topologically. Further, property (2.42) means that $S^m_\tau(\mathbb{R}^n;E)$ is the ϵ-product of $S^m_\tau(\mathbb{R}^n)$ and E. Thus, by Proposition A.5, as another corollary to Proposition 2.30 we get:

2.31 Corollary. *$S^m_\tau(\mathbb{R}^n)$ has the approximation property.*

Now we take advantage of (2.42), (2.44) to derive the following results:

2.32 Lemma. *Let E be a complete lcs. Then $S^m(\mathbb{R}^n;E)$, $S^m_\tau(\mathbb{R}^n;E)$ have the same bounded sets.*

Proof: Every bounded set in $\mathcal{L}_\gamma(S^m_\tau(\mathbb{R}^n)',E)$ is bounded for the topology of pointwise convergence and, therefore, bounded in $\mathcal{L}_\beta(S^m_\tau(\mathbb{R}^n)',E)$ by Proposition A.8. □

We also obtain a description of the bounded sets in $S^m(\mathbb{R}^n;E)$ as follows: $H\subset S^m(\mathbb{R}^n;E)$ is bounded if and only if the sets $\{\langle\Phi,a\rangle; a\in H, \Phi\in\mathcal{C}\}$ are bounded in E for all compact sets $\mathcal{C}\subset S^m_\tau(\mathbb{R}^n)'$.

Sometimes it is useful to know:

2.33 Lemma. *Let E be a complete lcs. Then the canonical embedding*

$$S^m_\tau(\mathbb{R}^n;E)\subset S^{m'}(\mathbb{R}^n;E) \tag{2.45}$$

is continuous for $m'>m$.

Proof: For $\mathcal{B}\subset S^{m'}_\tau(\mathbb{R}^n)'$ being bounded, $\mathcal{C}$ the image of $\mathcal{B}$ under the natural mapping $S^{m'}_\tau(\mathbb{R}^n)'\to S^m_\tau(\mathbb{R}^n)'$, and U a 0-neighbourhood in E, the 0-neighbourhood $\{a\in S^{m'}(\mathbb{R}^n;E); \langle\Phi,a\rangle\in U \text{ for all } \Phi\in\mathcal{B}\}$ of $S^{m'}(\mathbb{R}^n;E)$ contains the 0-neighbourhood $\{a\in S^m(\mathbb{R}^n;E); \langle\Phi,a\rangle\in U \text{ for all } \Phi\in\mathcal{C}\}$ of $S^m_\tau(\mathbb{R}^n;E)$. The latter is a 0-neighbourhood, since $\mathcal{C}$ is relatively weakly compact in $S^m_\tau(\mathbb{R}^n)'$. □

2.7 Appendix: Proof of Lemma 2.5

We come to the proof of Lemma 2.5. But preliminary we study the spaces $I^{m+n/4}(\mathbb{R}^n, \{0\})$, $\mathring{I}^{m+n/4}(\mathbb{R}^n, \{0\})$.

Recall that $I^{m+n/4}(\mathbb{R}^n, \{0\})$ is the space of all $u \in \mathcal{S}'(\mathbb{R}^n)$ satisfying

$$x^\alpha u \in {}^\infty H_{(-m+|\alpha|-n/2)}(\mathbb{R}^n) \tag{2.46}$$

for all $\alpha \in \mathbb{N}^n$. Similarly for $\mathring{I}^{m+n/4}(\mathbb{R}^n, \{0\})$, ${}^\infty\mathring{H}_{(-m+|\alpha|-n/2)}(\mathbb{R}^n)$.

From (2.46) we infer that

$$[1-\omega]\, I^{m+n/4}(\mathbb{R}^n, \{0\}) = [1-\omega]\, \mathring{I}^{m+n/4}(\mathbb{R}^n, \{0\}) = [1-\omega]\, \mathcal{S}(\mathbb{R}^n). \tag{2.47}$$

Here $[1-\omega]\, I^{m+n/4}(\mathbb{R}^n, \{0\})$ denotes the closure of $\{(1-\omega)\, u;\ u \in I^{m+n/4}(\mathbb{R}^n, \{0\})\}$ in $I^{m+n/4}(\mathbb{R}^n, \{0\})$; the space $[1-\omega]\, \mathring{I}^{m+n/4}(\mathbb{R}^n, \{0\})$, $[1-\omega]\, \mathcal{S}(\mathbb{R}^n)$ is similarly defined. $\omega \in C_0^\infty(\mathbb{R}^n)$ is a cut-off function which equals 1 close to 0.

Thus it remains to describe the behaviour near 0. Introduce the following function spaces: Let Y be a closed compact manifold, $\dim Y = n-1$, and $t,\ p \in \mathbb{R}$, $1 \le p \le \infty$. Then ${}^pH_{(t)}(\mathbb{R} \times Y)$ denotes the space of all $u \in \mathcal{S}'(\mathbb{R};\ {}^pH_{(t)}(Y))$ such that $\hat{u} \in L^2_{loc}(\mathbb{R};\ {}^pH_{(t)}(Y))$,

$$\|u\|_{{}^pH_{(t)}(\mathbb{R}\times Y)} = \left\{ \sum_{j=0}^{\infty} \Big(\int_{\Lambda_j} \|R^t(\xi)\hat{u}(\xi)\|^2_{{}^pH_{(0)}(Y)}\, d\xi \Big)^{p/2} \right\}^{1/p} < \infty \tag{2.48}$$

(modification for $p=\infty$). The annuli Λ_j are defined in Subsection 2.2 (now in the one-dimensional case), $R^t(\lambda) \in L^t_{cl}(Y; \mathbb{R}_\lambda)$ is a parameter-dependent order-reduction, i.e., a parameter-dependent family of elliptic pseudo-differential operators such that $R^t(\lambda) : {}^pH_{(t)}(Y) \to {}^pH_{(0)}(Y)$ realizes an isomorphism for all $\lambda \in \mathbb{R}$. For $p=2$ we also write $H_{(t)}(\mathbb{R}\times Y) = {}^2H_{(t)}(\mathbb{R}\times Y)$.

Starting from (2.48), one can show that, for $t,\ t_0,\ t_1 \in \mathbb{R}$, $1 \le p \le \infty$, $0 < \theta < 1$,

$$\left[H_{(t_0)}(\mathbb{R}\times Y), H_{(t_1)}(\mathbb{R}\times Y)\right]_{\theta,p} = {}^pH_{(t)}(\mathbb{R}\times Y) \tag{2.49}$$

holds provided that $t = (1-\theta)t_0 + \theta t_1$. Here $[\ ,\ ]_{\theta,p}$ denotes the real interpolation functor (see [1]).

Next introduce corresponding Besov spaces over the open cone $\mathbb{R}_+ \times Y$. On $\mathbb{R}_+$, replace the Fourier transform by the Mellin transform, i.e.,

$$Mv(z) = \tilde{v}(z) = \int_0^\infty r^{z-1} v(r)\, dr.$$

The variables r, x in resp. $\mathbb{R}_+$ and $\mathbb{R}$ and the corresponding covariables z, ξ in resp. the Mellin and Fourier image are supposed to be related by

$$x = \log r,\quad z = n/2 - i\xi.$$

Moreover, introduce the mapping Φ that assigns functions v on $\mathbb{R}_+ \times Y$ to functions u on $\mathbb{R}\times Y$ via the formula

$$\Phi u = v \ \text{ iff } \ u(x) = r^{n/2} v(r).$$

Under these hypotheses, it is seen that

$$(Fu)(\xi) = (Mv)(z) \text{ if } \Phi u = v.$$

The space ${}^p\mathcal{H}_{(t),\gamma}(\mathbb{R}_+ \times Y)$ is explained as the image of $e^{\gamma x}\, {}^pH_{(t)}(\mathbb{R} \times Y)$ under Φ, where $e^{\gamma x}$ is multiplication by this function. A norm on ${}^p\mathcal{H}_{(t),\gamma}(\mathbb{R}_+ \times Y)$ is given by

$$\|v\|_{{}^p\mathcal{H}_{(t),\gamma}(\mathbb{R}_+\times Y)} = \left\{ \sum_{j=0}^{\infty} \left(\int_{\substack{\mathrm{Re}\, z = n/2-\gamma \\ \mathrm{Im}\, z \in \Lambda_j}} \|R^t(\mathrm{Im} z)\tilde{v}(z)\|^2_{{}^pH_{(0)}(Y)}\, dz \right)^{p/2} \right\}^{1/p} \tag{2.50}$$

(modification for $p = \infty$). Especially, we have ${}^p\mathcal{H}_{(t),\gamma}(\mathbb{R}_+ \times Y) \doteq r^{\gamma-\delta}\, {}^p\mathcal{H}_{(t),\delta}(\mathbb{R}_+ \times Y)$. Denote $\mathcal{H}_{(t),\gamma}(\mathbb{R}_+ \times Y) = {}^2\mathcal{H}_{(t),\gamma}(\mathbb{R}_+ \times Y)$. From (2.49) it follows that

$$\left[\mathcal{H}_{(t_0),\gamma_0}(\mathbb{R}_+ \times Y), \mathcal{H}_{(t_1),\gamma_1}(\mathbb{R}_+ \times Y)\right]_{\theta,p} = {}^p\mathcal{H}_{(t),\gamma}(\mathbb{R}_+ \times Y)$$

if $t = (1-\theta)t_0 + \theta t_1$, $\gamma = (1-\theta)\gamma_0 + \theta\gamma_1$, $0 < \theta < 1$.
For $Y = S^{n-1}$ being the unit sphere, also write ${}^p\mathcal{H}_{(t),\gamma}(\mathbb{R}^n)$ instead of ${}^p\mathcal{H}_{(t),\gamma}(\mathbb{R}_+ \times S^{n-1})$, thinking of (r, y), where y is the coordinate on S^{n-1}, as introduced as polar coordinates in $\mathbb{R}^n$.

2.34 Remark. For a thorough discussion of Besov spaces on complete Riemannian manifolds with positive injectivity radius and bounded geometry, see [13, Chapter 7]. The approach to the spaces ${}^p\mathcal{H}_{(t),\gamma}(\mathbb{R}_+ \times Y)$ via the integral (2.50) in case $p = 2$ is taken from [11].

For $t \in \mathbb{R}$, $|t| - n/2 \notin \mathbb{N}$, introduce

$$T_{(t)}(\mathbb{R}^n) = \begin{cases} \left\{\omega(x) \sum_{|\alpha|<t-n/2} a_\alpha x^\alpha;\ a_\alpha \in \mathbb{C}\right\}, & t > n/2, \\ \left\{\sum_{|\alpha|<|t|-n/2} a_\alpha\, \delta^{(\alpha)}(x);\ a_\alpha \in \mathbb{C}\right\}, & t < -n/2, \end{cases}$$

where $\delta(x)$ is the Dirac measure. For $|t| < n/2$, set $T_{(t)}(\mathbb{R}^n) = \{0\}$. Notice that $T_{(t)}(\mathbb{R}^n)$ is finite-dimensional.

2.35 Lemma. *Let $t, p \in \mathbb{R}$, $1 \le p \le \infty$. Then, if $|t| - n/2 \notin \mathbb{N}$,*

$$[\omega]\, {}^pH_{(t)}(\mathbb{R}^n) = [\omega]\, {}^p\mathcal{H}_{(t),t}(\mathbb{R}^n) + T_{(t)}(\mathbb{R}^n). \tag{2.51}$$

Proof: For $p = 2$, (2.51) may be found in [2, Theorem AA.7] if $t \ge 0$. (For a proof in a similar situation using pseudo-differential techniques and Mellin expansion, see [11, Theorem 2.1.39].) Then, for $p = 2$ and $t < 0$, (2.51) follows by duality and, for general p, t, by interpolation. □

For the next lemma, denote ${}^\infty\mathcal{H}_{(\infty),\gamma}(\mathbb{R}^n) = \bigcap_{t\in\mathbb{R}} {}^\infty\mathcal{H}_{(t),\gamma}(\mathbb{R}^n)$, ${}^\infty\mathring{\mathcal{H}}_{(\infty),\gamma}(\mathbb{R}^n) = \bigcap_{t\in\mathbb{R}} {}^\infty\mathring{\mathcal{H}}_{(t),\gamma}(\mathbb{R}^n)$.

2.36 Lemma. *Let $m \in \mathbb{R} \setminus \mathbb{Z}$, $|m + n/2| < n/2$. Then*

$$I^{m+n/4}(\mathbb{R}^n, \{0\}) = [\omega]\, {}^{\infty}\mathcal{H}_{(\infty),-m-n/2}(\mathbb{R}^n) + [1-\omega]\, \mathcal{S}(\mathbb{R}^n), \quad (2.52)$$

$$\mathring{I}^{m+n/4}(\mathbb{R}^n, \{0\}) = [\omega]\, {}^{\infty}\mathring{\mathcal{H}}_{(\infty),-m-n/2}(\mathbb{R}^n) + [1-\omega]\, \mathcal{S}(\mathbb{R}^n). \quad (2.53)$$

Proof: We only show (2.52). The proof of (2.53) is similar. Let $s = -m - n/2$. First, $m \notin \mathbb{Z}$ implies $|s+k| - n/2 \notin \mathbb{N}$ for all $k \in \mathbb{N}$. Further, from $|m + n/2| < n/2$ we infer that $T_{(s)}(\mathbb{R}^n) = \{0\}$.

Let $u \in I^{m+n/4}(\mathbb{R}^n, \{0\})$. Using (2.46) and Lemma 2.35 we see that $x^\alpha \omega u = \omega v_\alpha + w_\alpha$ for $\alpha \in \mathbb{N}^n$, where $v_\alpha \in {}^{\infty}\mathcal{H}_{(s+|\alpha|),s+|\alpha|}(\mathbb{R}^n)$ and $w_\alpha \in T_{(s+|\alpha|)}(\mathbb{R}^n)$. Writing $r^k = \sum_{|\alpha|=k} \chi_\alpha x^\alpha$ for certain $\chi_\alpha \in C^\infty(S^{n-1})$, $k \in \mathbb{N}$, we obtain $\omega u = r^{-k}\omega v_k + r^{-k}\sum_{|\alpha|=k}\chi_\alpha w_\alpha$, where $v_k \in {}^{\infty}\mathcal{H}_{(s+k),s+k}(\mathbb{R}^n)$, i.e., $r^{-k} v_k \in {}^{\infty}\mathcal{H}_{(s+k),s}(\mathbb{R}^n)$. Now it is easy to see that $r^{-k}\sum_{|\alpha|=k}\chi_\alpha w_\alpha \notin {}^{\infty}\mathcal{H}_{(s),s}(\mathbb{R}^n)$ unless $w_\alpha = 0$ for all $|\alpha| = k$. Therefore, $\omega u \in {}^{\infty}\mathcal{H}_{(s+k),s}(\mathbb{R}^n)$ for all $k \in \mathbb{N}$, i.e., $\omega u \in {}^{\infty}\mathcal{H}_{(\infty),s}(\mathbb{R}^n)$.

Using (2.47) we get $I^{m+n/4}(\mathbb{R}^n, \{0\}) \subseteq [\omega]\, {}^{\infty}\mathcal{H}_{(\infty),s}(\mathbb{R}^n) + [1-\omega]\, \mathcal{S}(\mathbb{R}^n)$; the other direction is obvious. □

Note that using Lemma 2.35 one can equally prove that, for $m \in \mathbb{R} \setminus \mathbb{Z}$,

$$I^{m+n/4}(\mathbb{R}^n, \{0\}) = [\omega]\, {}^{\infty}\mathcal{H}_{(\infty),-m-n/2}(\mathbb{R}^n) + [1-\omega]\, \mathcal{S}(\mathbb{R}^n) + T_{(-m-n/2)}(\mathbb{R}^n).$$

Proof of Lemma 2.5: It is enough to verify Lemma 2.5 for some $m \in \mathbb{R}$, since $S^m(\mathbb{R}^n) = \langle\xi\rangle^m S^0(\mathbb{R}^n)$, and similarly for $\mathring{S}^m(\mathbb{R}^n)$. In the sequel we shall assume that $m \in \mathbb{R} \setminus \mathbb{Z}$, $|m + n/2| < n/2$.

By Proposition 2.3, it is enough to verify that $I^{m+n/4}(\mathbb{R}^n, \{0\})$, $\mathring{I}^{m+n/4}(\mathbb{R}^n, \{0\})$ are quasi-normable. Then, by Lemma 2.36, it is sufficient to prove that ${}^{\infty}\mathcal{H}_{(\infty),-m-n/2}(\mathbb{R}^n)$, ${}^{\infty}\mathring{\mathcal{H}}_{(\infty),-m-n/2}(\mathbb{R}^n)$ are quasi-normable, since the direct sum of two quasi-normable spaces and the quotient of a quasi-normable space is quasi-normable again (see [5, p. 177]), and $\mathcal{S}(\mathbb{R}^n)$ is a nuclear space.

We show that the spaces ${}^{\infty}H_{(\infty)}(\mathbb{R} \times Y)$, ${}^{\infty}\mathring{H}_{(\infty)}(\mathbb{R} \times Y)$, with Y being a closed compact manifold, are quasi-normable and apply it to $Y = S^{n-1}$. On $\mathbb{R} \times Y$, there exists a family of pseudo-differential operators $\{J_\epsilon;\ 0 < \epsilon \le 1\} \subset L^{-\infty}(\mathbb{R} \times Y)$, with global symbol estimates in $\mathbb{R}$-direction, such that $\{J_\epsilon;\ 0 < \epsilon \le 1\}$ is a bounded set in $L^0(\mathbb{R} \times Y)$, $J_\epsilon u \to u$ in ${}^{p}H_{(t)}(\mathbb{R} \times Y)$ as $\epsilon \to 0^+$ for $u \in {}^{p}H_{(t)}(\mathbb{R} \times Y)$, $t, p \in \mathbb{R}$, $1 \le p \le \infty$, and

$$\|u - J_\epsilon u\|_{{}^{p}H_{(t)}(\mathbb{R}\times Y)} \le C_{t,r}\, \epsilon^r\, \|u\|_{{}^{p}H_{(t+r)}(\mathbb{R}\times Y)}, \quad (2.54)$$

$$\|J_\epsilon u\|_{{}^{p}H_{(t)}(\mathbb{R}\times Y)} \le C_{t,r}\, \epsilon^{-r} \|u\|_{{}^{p}H_{(t-r)}(\mathbb{R}\times Y)} \quad (2.55)$$

for some constant $C_{t,r} > 0$ provided that $r \ge 0$. (For a construction in a similar situation, see [12, Lemma 1.3.A].)

Now let $U = \{u \in {}^{\infty}H_{(\infty)}(\mathbb{R} \times Y);\ \|u\|_{{}^{\infty}H_{(k)}(\mathbb{R}\times Y)} \le 1\}$ be a given 0-neighbourhood in ${}^{\infty}H_{(\infty)}(\mathbb{R}\times Y)$ for some $k \in \mathbb{N}$. Put $V = \{u \in {}^{\infty}H_{(\infty)}(\mathbb{R} \times Y);\ \|u\|_{{}^{\infty}H_{(k+1)}(\mathbb{R}\times Y)} \le 1\}$. Now let $\lambda > 0$. Then, by (2.54), $u - J_\epsilon u \in \lambda U$ for $\epsilon = C_{k,1}^{-1}\lambda$ and $u \in V$, whereas (2.55) implies that, for $u \in V$, $J_\epsilon u$ belongs to a bounded set M in ${}^\infty H_{(\infty)}(\mathbb{R} \times Y)$. Thus $V \subseteq M + \lambda U$ which shows that ${}^\infty H_{(\infty)}(\mathbb{R} \times Y)$ is quasi-normable. The proof for ${}^\infty \mathring{H}_{(\infty)}(\mathbb{R} \times Y)$ is concluded noting that J_ϵ, $0 < \epsilon \leq 1$, maps $\mathcal{S}(\mathbb{R}; C^\infty(Y))$ and, therefore, ${}^\infty \mathring{H}_{(t)}(\mathbb{R} \times Y)$ into itself. □

3 Applications to Pseudo-Differential Operators; Examples

As one application of the weak symbol topology introduced in the previous section we verify Sobolev space continuity of pseudo-differential operators in the following situations: first we consider pseudo-differential operators with symbols in $S^m(\mathbb{R}^n \times \mathbb{R}^n)$, after that pseudo-differential operators the symbols of which have coefficients in $H^s(\mathbb{R}^n)$. Characteristic for the first case is that good order reductions are at our disposal, while in the second case multiplication by a coefficient is controlled in a special manner.

3.1 The Case of Smooth Coefficients

In this subsection we demonstrate the use of tensor product techniques in the verification of continuity of pseudo-differential operators if one is concerned with the "standard situation". This means that we are going to prove continuity of the mapping

$$S^m(\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{L}(H^{t+m}(\mathbb{R}^n), H^t(\mathbb{R}^n)), \;\; a \mapsto a(x, D). \tag{3.1}$$

In the next proposition, $S^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$ stands for $S^m_\tau(\mathbb{R}^n; C^\infty_b(\mathbb{R}^n))$.

3.1 Proposition. *Let $m, t \in \mathbb{R}$. Then the mapping*

$$S^m_\tau(\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{L}_\sigma(H^{t+m}(\mathbb{R}^n), H^t(\mathbb{R}^n)), \;\; a \mapsto a(x, D) \tag{3.2}$$

is continuous.

Proof: The assertion can be reduced to the case $m = t = 0$. We have to show that, for each $u \in L^2(\mathbb{R}^n)$, the mapping

$$S^0_\tau(\mathbb{R}^n \times \mathbb{R}^n) \to L^2(\mathbb{R}^n), \;\; a \mapsto a(x, D)u$$

is continuous. By completeness of $L^2(\mathbb{R}^n)$, it suffices to show that the mapping

$$C^\infty_b(\mathbb{R}^n) \otimes_\epsilon S^0_\tau(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \;\; \alpha \otimes a \mapsto \alpha(x)\, a(D)u \tag{3.3}$$

is continuous, since, obviously, $a(x, D)u \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $a \to 0$ in $S^0_\tau(\mathbb{R}^n \times \mathbb{R}^n)$.

Fix some $u \in L^2(\mathbb{R}^n)$. We show that the bilinear form

$$C^\infty_b(\mathbb{R}^n) \times S^0_\tau(\mathbb{R}^n) \to \mathbb{C}, \;\; (\alpha, a) \mapsto \int \alpha(x)\, a(D)u(x)v(x)\, dx \tag{3.4}$$

belongs to $(C^\infty_b(\mathbb{R}^n) \otimes_\epsilon S^0_\tau(\mathbb{R}^n))'$, i.e., is integral, and runs through an equi-continuous set in $(C^\infty_b(\mathbb{R}^n) \otimes_\epsilon S^0_\tau(\mathbb{R}^n))'$ if v runs through the closed unit ball in $L^2(\mathbb{R}^n)$.

First we show that that, for $v \in L^2(\mathbb{R}^n)$, the bilinear form given in (3.4) is integral. According to Theorem 2.23 there exists an absolutely convex, closed 0-neighbourhood U of $S^0_\tau(\mathbb{R}^n)$ such that the mapping $S^0_\tau(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $a \mapsto a(D)u$, factors through $S^0_\tau(\mathbb{R}^n)^\sim_{(U)}$ in a way such that the arising operator $S^0_\tau(\mathbb{R}^n)^\sim_{(U)} \to L^2(\mathbb{R}^n)$ is 2-nuclear. Let ν_2 denote its 2-nuclear norm. As an easy calculation shows, the multiplication operator

$L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$, $w \mapsto vw$, is absolutely summing with 1-summing norm $\|v\|_{L^2}$. Thus we obtain that the composed operator $S^0_\tau(\mathbb{R}^n)^\sim_{(U)} \to L^1(\mathbb{R}^n)$, $a \mapsto (a(D)u)v$, is nuclear with 1-nuclear norm not exceeding $\nu_2 \|v\|_{L^2}$. In particular, by Proposition B.1, the image of the closed unit ball of $S^0_\tau(\mathbb{R}^n)^\sim_{(U)}$ is order bounded in $L^1(\mathbb{R}^n)$, i.e., there exists a non-negative function $h \in L^1(\mathbb{R}^n)$ such that

$$\left|a(D)u(x)v(x)\right| \le h(x) \quad \text{a.e. } x \in \mathbb{R}^n$$

holds for all $a \in U$. Thereby, h can be chosen to satisfy $\|h\|_{L^1} \le \nu_2 \|v\|_{L^2}$. It follows that the mapping $S^0_\tau(\mathbb{R}^n)^\sim_{(U)} \to L^\infty(\mathbb{R}^n, h(x)dx)$, $a \mapsto h^{-1}(a(D)u)v$, is continuous, where we have set $h^{-1}(x)a(D)u(x)v(x) = 0$ if $h(x) = 0$. Further, the regular Borel measure $h(x)dx$ is finite on $\mathbb{R}^n$. Thus we have found an integral representation,

$$L^\infty(\mathbb{R}^n) \times S^0_\tau(\mathbb{R}^n)^\sim_{(U)} \to \mathbb{C}, \quad (\alpha, a) \mapsto \int \alpha(x)\, h(x)^{-1} a(D)u(x)v(x)\, h(x)dx.$$

At the same time we have seen that, for $\|v\|_{L^2} \le 1$, these bilinear forms belong to a bounded set in $(L^\infty(\mathbb{R}^n) \otimes_\epsilon S^0_\tau(\mathbb{R}^n)^\sim_{(U)})'$. Therefore, these bilinear forms belong to an equi-continuous set in $(C^\infty_b(\mathbb{R}^n) \otimes_\epsilon S^0_\tau(\mathbb{R}^n))'$.

Thus continuity of (3.3) is proved. □

Now continuity of the mapping (3.1) is concluded as follows: Under the continuous mapping $S^m_\tau(\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{L}_\sigma(H^{t+m}(\mathbb{R}^n), H^t(\mathbb{R}^n))$, $a \mapsto a(x, D)$, bounded sets are mapped into bounded sets. Further, $S^m(\mathbb{R}^n \times \mathbb{R}^n)$, $S^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$ and $\mathcal{L}(H^{t+m}(\mathbb{R}^n), H^t(\mathbb{R}^n))$, $\mathcal{L}_\sigma(H^{t+m}(\mathbb{R}^n), H^t(\mathbb{R}^n))$, respectively, have the same bounded sets. Consequently, the mapping (3.1) also maps bounded sets into bounded sets and, therefore, is continuous, since $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ is a bornological space.

3.2 Coefficients from Sobolev Spaces

Next we are concerned with the continuity of the mapping

$$H^sS^m(\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{L}(H^{t+m}(\mathbb{R}^n), H^t(\mathbb{R}^n)), \quad a \mapsto a(x, D) \tag{3.5}$$

for $s, t \in \mathbb{R}$, $s > n/2$, $|t| \le s$. Whereas in the example in Subsection 3.1 the integrality of certain bilinear forms was directly shown, here we take into account linear-topological properties of the coefficients space to "raise" the injective tensor product.

3.2 Lemma. *Let $s, t \in \mathbb{R}$, $s > n/2$, $0 \le t \le s$, and $v \in H^{-t}(\mathbb{R}^n)$. Then the operator*

$$H^s(\mathbb{R}^n) \to H^{-t}(\mathbb{R}^n), \quad \alpha \mapsto \alpha v \tag{3.6}$$

is 2-summing. Moreover, its 2-summing norm does not exceed $C\|v\|_{H^{-t}}$, where the constant $C > 0$ only depends on s, t.

Proof: It suffices to prove that the operator in (3.6) is a Hilbert-Schmidt operator. In fact, given $v \in H^{-t}(\mathbb{R}^n)$, the operator $H^s(\mathbb{R}^n) \to H^{-t}(\mathbb{R}^n)$, $\alpha \mapsto \alpha v$, is Hilbert-Schmidt as the composition of the isometry $H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $\alpha \mapsto \langle\xi\rangle^s\hat{\alpha}(\xi)$, the Hilbert-Schmidt operator

$$L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \ w \mapsto (2\pi)^{-n} \int \frac{\langle\eta\rangle^t}{\langle\xi-\eta\rangle^s\langle\xi\rangle^t} w(\xi-\eta)\langle\eta\rangle^{-t}\hat{v}(\eta)\, d\eta \tag{3.7}$$

and the isometry $L^2(\mathbb{R}^n) \to H^{-t}(\mathbb{R}^n)$, $w \mapsto F^{-1}_{\xi-x}(\langle\xi\rangle^t w)$. In order to see that the operator in (3.7) is Hilbert-Schmidt we show that its kernel belongs to $L^2(\mathbb{R}^n \times \mathbb{R}^n)$.
Up to a constant, the kernel of (3.7) equals $\langle\xi\rangle^{-t}\langle\eta\rangle^{-s}\hat{v}(\xi-\eta)$. Its L^2-norm is finite, since

$$\iint \frac{\langle\eta\rangle^{2t}}{\langle\xi-\eta\rangle^{2s}\langle\xi\rangle^{2t}}\langle\eta\rangle^{-2t}|\hat{v}(\eta)|^2\, d\xi d\eta \le \sup_{\eta\in\mathbb{R}^n}\left(\int \frac{\langle\eta\rangle^{2t}}{\langle\xi-\eta\rangle^{2s}\langle\xi\rangle^{2t}}\, d\xi\right)\int \langle\eta\rangle^{-2t}|\hat{v}(\eta)|^2\, d\eta < \infty$$

for $\sup_{\eta\in\mathbb{R}^n}\int\langle\eta\rangle^{2t}\langle\xi-\eta\rangle^{-2s}\langle\xi\rangle^{-2t}\, d\xi < \infty$ because of $s > n/2$, $0 \le t \le s$. □

As before continuity of (3.5) is implied by the following observation:

3.3 Proposition. *Let $m \in \mathbb{R}$. Further let $s, t \in \mathbb{R}$, $s > n/2$, $|t| \le s$. Then*

$$H^s S^m_\tau(\mathbb{R}^n \times \mathbb{R}^n) \to \mathcal{L}_\sigma(H^{t+m}(\mathbb{R}^n), H^t(\mathbb{R}^n)), \ a \mapsto a(x,D) \tag{3.8}$$

is continuous.

Proof: Again, it suffices to show that, for each $u \in H^{t+m}(\mathbb{R}^n)$, the mapping

$$H^s(\mathbb{R}^n) \otimes_\epsilon S^m_\tau(\mathbb{R}^n) \to H^t(\mathbb{R}^n), \ \alpha\otimes a \mapsto \alpha(x)\, a(D)u$$

is continuous. By Proposition B.4, for that is enough to show that the mapping

$$H^s(\mathbb{R}^n) \otimes_2 S^m_\tau(\mathbb{R}^n) \to H^t(\mathbb{R}^n), \ \alpha\otimes a \mapsto \alpha(x)\, a(D)u \tag{3.9}$$

is continuous.
Fix some $u \in H^{t+m}(\mathbb{R}^n)$. Treat the case $t < 0$ first. Then, by Lemma 2.22 and Lemma 3.2, the mapping

$$S^m_\tau(\mathbb{R}^n) \to \Pi_2(H^s(\mathbb{R}^n), H^t(\mathbb{R}^n)), \ a \mapsto (\,\alpha \mapsto \alpha(x)a(D)u\,),$$

where Π_2 denotes the Banach ideal of 2-summing operators, is continuous, since the 2-summing norm of the operator $H^s(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$, $\alpha \mapsto \alpha(x)a(D)u$, does not exceed $C\,\|a(D)u\|_{H^t}$ and $a \mapsto \|a(D)u\|_{H^t}$ is a continuous semi-norm on $S^m_\tau(\mathbb{R}^n)$. According to Theorem 2.23 there exists an absolutely convex, closed 0-neighbourhood U of $S^0_\tau(\mathbb{R}^n)$ such that the mapping $S^0_\tau(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $a \mapsto a(D)u$, factors through $S^0_\tau(\mathbb{R}^n)^\sim_{(U)}$ such that the arising operator $S^0_\tau(\mathbb{R}^n)^\sim_{(U)} \to L^2(\mathbb{R}^n)$ is 2-nuclear. This operator is then 2-summing which implies the desired estimate:

$$\begin{aligned}\|\alpha(x)a(D)u\|_{H^t} &\le C\,\|a(D)u\|_{H^t}\left\{\int_{B_{H^{-s}}}|(v,\alpha)|^2\, d\nu(v)\right\}^{1/2}\\ &\le C\left\{\int_{U^\circ}|\langle\Phi,a\rangle|^2\, d\mu(\Phi)\right\}^{1/2}\left\{\int_{B_{H^{-s}}}|(v,\alpha)|^2\, d\nu(v)\right\}^{1/2}\end{aligned}$$

for suitable probability measures μ and ν on resp. U° and $B_{H^{-s}}$, where U° is the polar to U in $S^m_\tau(\mathbb{R}^n)'$ and $B_{H^{-s}}$ is the closed unit ball in $H^{-s}(\mathbb{R}^n)$. The assertion follows from Proposition B.3.

In case $t \geq 0$, we argue by duality. Let $v \in H^{-t}(\mathbb{R}^n)$. Then, by Propositon B.3, the bilinear form

$$H^s(\mathbb{R}^n) \times S^m_\tau(\mathbb{R}^n) \to \mathbb{C}, \ (\alpha, a) \mapsto \int a(D)u\, \alpha v\, dx, \tag{3.10}$$

is an element of $(H^s(\mathbb{R}^n) \otimes_2 S^m_\tau(\mathbb{R}^n))'$, since we have the estimate

$$\left| \int a(D)u\, \alpha v\, dx \right| \leq \|a(D)u\|_{H^t} \|\alpha v\|_{H^{-t}},$$

and both operators $S^m_\tau(\mathbb{R}^n)^\sim_{(U)} \to H^t$, $a + \ker\| \ \|_{(U)} \mapsto a(D)u$, for a suitably chosen absolutely convex, closed 0-neighbourhood U of $S^m_\tau(\mathbb{R}^n)$, and $H^s(\mathbb{R}^n) \to H^{-t}(\mathbb{R}^n)$, $\alpha \mapsto \alpha v$, are 2-summing. Furthermore, it is seen that the bilinear form in (3.10) runs through an equi-continuous set in $(H^s(\mathbb{R}^n) \otimes_2 S^m_\tau(\mathbb{R}^n))'$ if v runs through the closed unit ball in $H^{-t}(\mathbb{R}^n)$. This implies continuity of (3.9). □

A Facts on Locally Convex Spaces

In this appendix we provide some general facts concerning locally convex spaces. It is not intented as a self-contained introduction into the theory of locally convex topological vector spaces; for that purpose we refer to standard text books on that topic, e.g., [5], [8], [9]. Instead of we briefly present the necessary prerequisites for understanding the constructions leading to the weak symbol topology and deriving its properties.

A.1 Some Notation

We shall employ standard notion from the theory of locally convex spaces. Locally convex spaces (henceforth abbreviated as lcs) are always assumed to be Hausdorff. If we wish to indicate the topology τ on a lcs E explicitly, we write (E, τ). For a lcs $E, \mathcal{U} = \mathcal{U}_E$ denotes a fixed basis of 0-neighbourhoods consisting of absolutely convex, closed sets. Eventually $\mathcal{B}_E$ is a basis for the bounded sets in E. For $U \in \mathcal{U}_E$, the associated local Banach space is $\tilde{E}_{(U)}$, i.e., the completion of $E_{(U)}$, where $E_{(U)} = E/\ker\| \, \|_{(U)}$ is canonically normed and $\| \, \|_{(U)}$ denotes the continuous semi-norm on E associated with U, i.e., $\|u\|_{(U)} = \inf\{\lambda > 0;\ u \in \lambda U\}$ for $u \in E$. For a disk $B \subset E$, i.e., B is absolutely convex and weakly bounded, $E_B = \bigcup_{n\in\mathbb{N}} nB$ is the natural normed linear span of B. Recall that, for any $U \in \mathcal{U}_E$, we have $\left(E_{(U)}\right)' \cong E'_{U^\circ}$ as Banach spaces. Here E' is the dual to E and $U^\circ \subset E'$ is the polar to U, i.e., $U^\circ = \{\phi \in E';\ |\langle \phi, u\rangle| \leq 1 \text{ for } u \in U\}$. If no otherwise stated, E' is assumed to carry the strong topology. The dual pair between linear spaces E, F is denoted by $\langle E, F\rangle$. Then $\sigma(E,F)$, $\tau(E,F)$, and $\beta(E,F)$ refer to the weak, Mackey, and strong topology on E, respectively, with respect to the dual pair $\langle E, F\rangle$.

For E, F being lcs, $\mathcal{L}(E,F)$ denotes the space of linear continuous mappings from E into F. Then $\mathcal{L}_\sigma(E,F)$, $\mathcal{L}_\gamma(E,F)$, $\mathcal{L}_\beta(E,F)$ is this space equipped with the topology of uniform convergence on all finite subsets of E, on all precompact sets of $\tilde{E}$, and on all bounded sets of E, respectively.

A.2 Short Exact Sequences

By a short exact sequence we mean a sequence of locally convex spaces E, F, G,

$$0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0\,, \tag{A.1}$$

which is algebraically exact and for which all mappings are continuous. (A.1) is called topologically exact, if, in addition, E and G carry the induced and the quotient topology, respectively. A short exact sequence of Fréchet spaces is always topologically exact.

For a short exact sequence (A.1) the dual sequence

$$0 \longrightarrow G' \longrightarrow F' \longrightarrow E' \longrightarrow 0 \tag{A.2}$$

is exact. Recall that in (A.2) the spaces G' and E' may be identified algebraically with $E^\circ = \{\phi \in F' \,|\, \langle \phi, E\rangle = 0\}$ and F'/E°, respectively.

We have the following result (cf. [10, Satz 26.17]):

A.1 Proposition. *Let E be a Fréchet space. Then E is quasi-normable if and only if for any short exact sequence* (A.1) *of Fréchet spaces the dual sequence* (A.2) *is topologically exact.*

A.3 Quasi-Normable Lcs and Schwartz spaces

Recall that a lcs E is called quasi-normable if for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ contained in U such that the topologies induced on U° by resp. E' and E'_{V° are the same. An equivalent characterization is that for every $U \in \mathcal{U}$ we find a $V \in \mathcal{U}$ such that for every $\delta > 0$ there is a $M \in \mathcal{B}_E$ satisfying

$$V \subset M + \delta U. \tag{A.3}$$

We need the following lemmas (cf. [5, Chap. 4, Part 4, Sect. 1, Exer. 2], [5, Chap. 4, Part 4, Sect. 3, Thm. 2]):

A.2 Lemma. *Let E be a quasi-normable lcs. Let $A \subset E'$ be equi-continuous, absolutely convex and $\sigma(E', E'')$-compact. Then there exists a equi-continuous, absolutely convex and weakly closed subset $B \subset E'$ containing A such that A is weakly compact in E'_B.*

A.3 Lemma. *Let E be a quasi-barreled lcs. Then E is quasi-normable if and only if its strong dual E' satisfies the strict Mackey convergence condition.*

A lcs E is called a Schwartz space if for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ contained in U such that U° is compact in E'_{V°. An equivalent characterization is that every mapping in $\mathcal{L}(E, F)$ into an arbitrary Banach space F is compact. A lcs E is Schwartz if and only if it is quasi-normable and its bounded sets are precompact. A sequence $\{\phi_k\}_{k=0}^\infty$ in E' is said to be equi-continuously convergent or $\mathcal{E}$-convergent to 0 if $\{\langle \phi_k, u\rangle\}_{k=0}^\infty$ converges to 0 uniformly in $u \in U$ for some $U \in \mathcal{U}_E$. The topology on a Schwartz space E is the topology of uniform convergence on the $\mathcal{E}$-null sequences in E'. Moreover, for an arbitrary lcs (E, τ), the topology of uniform convergence on the $\mathcal{E}$-null sequences in E' is the strongest Schwartz topology on E which is weaker than τ.

Further details on quasi-normable lcs and Schwartz spaces may be found, e.g., in [5, Chap. 4, Part 4], [8, 10.4, 10.7]).

A.4 The ϵ-Product

We recall some basic facts concerning the injective tensor product, the ϵ-product, and the approximation property. For the approximation property, we follow the presentation in [8, Chapter 18].

Let E, F be lcs. A fundamental system of semi-norms for the topology, ϵ, of the injective tensor product on $E \otimes F$ is given by

$$\|w\|_{(U),(V);\otimes_\epsilon} = \sup_{\phi\otimes\psi\in U^\circ\otimes V^\circ} \left|\sum_{j=1}^{K} \langle \phi, u_j\rangle\langle \psi, v_j\rangle\right|,$$

where $w = \sum_{j=1}^{\kappa} u_j \otimes v_j \in E \otimes F$ and the expression on the right-hand side does not depend on the representation chosen for w. It is the weakest locally convex topology on $E \otimes F$ for which $E' \otimes F'$ is contained in the dual space and, for all $U \in \mathcal{U}_E$, $V \in \mathcal{U}_F$, $U^\circ \otimes V^\circ \subset E' \otimes F'$ is an equi-continuous set. The completion of $E \otimes_\epsilon F$ shall be denoted by $E \tilde{\otimes}_\epsilon F$.

We need a characterization of the dual to $E \otimes_\epsilon F$ (see, e.g., [8, 17.4.1.]):

A.4 Proposition. *Let E, F be lcs. Further let B be a continuous bilinear form on $E \times F$. Then the associated linear form to B on $E \otimes F$ is continuous for the ϵ-topology if and only if there exist a finite measure μ on a locally compact space X and mappings $S \in \mathcal{L}(E; L^\infty(\mu))$, $T \in \mathcal{L}(F; L^\infty(\mu))$ such that*

$$B(u, v) = \int_X Su(x)Tv(x)\, d\mu(x) \tag{A.4}$$

holds for all $u \in E$, $v \in F$.

In Proposition A.4, the finite measure μ is supposed to be a regular Borel measure and we write $L^\infty(\mu)$ instead of $L^\infty(X, \mu)$. Bilinear forms having the property described in the proposition are called integral. It is readily seen that the space of all integral bilinear forms on $E \times F$ is the dual to $E \otimes_\epsilon F$.

The ϵ-product $E\epsilon F$ of E, F is $\mathcal{L}_e(E'_\gamma, F)$, where E'_γ is the dual to E equipped with the topology of uniform convergence on all precompact sets in $\tilde{E}$ and e stands for the topology of uniform convergence on all equi-continuous sets in E'. $E \otimes_\epsilon F$ is a subspace of $E\epsilon F$. Moreover, $E\epsilon F$ is complete if and only if F is complete. In this case, $E \tilde{\otimes}_\epsilon F$ is a closed subspace of $E\epsilon F$. If E, F are both complete, then $E\epsilon F = F\epsilon E$ canonically.

A lcs E is said to have the approximation property if $E' \otimes E$, i.e., the space of all continuous finite-rank operators in E, is dense in $\mathcal{L}_\gamma(E)$. Here $\mathcal{L}_\gamma(E) = \mathcal{L}_\gamma(E, E)$. (Grothendieck's original definition is that $E' \otimes E$ is dense in $\mathcal{L}_{pc}(E)$, where pc stands for the topology of uniform convergence on precompact sets in E. In case E is quasi-complete, both concepts coincide.)

For the next result, see [8, 18.1.8]:

A.5 Proposition. *Let F be a lcs. Then F possesses the approximation property if and only if $E \otimes F$ is dense in $E\epsilon F$ for any lcs E. In this case, we have*

$$E \tilde{\otimes}_\epsilon F = \mathcal{L}_e(E'_\gamma, F) \tag{A.5}$$

if F is complete.

A.5 Precompact Sets and Limited Sets

We provide a frame for discussing property (1.2). It was found in [5, Chap. 5, Part 3, Supp. Exer., Exer. 3–4].

Let E be a lcs. A subset $A \subset E$ is called limited if every weakly convergent sequence in E' converges uniformly on A. An equivalent condition is that A is precompact for the topology of uniform convergence on the subsets of E' for which every sequence contained possesses a subsequence which is weakly Cauchy. Denote the latter class of subsets of E' by $\mathcal{S}'$. Denote further the class of limited subsets of E by $\mathcal{L}$. Since these constructions only depend on the dual pair $\langle E, E' \rangle$, we have adequate notions for the dual pair $\langle E', E \rangle$ and corresponding classes $\mathcal{S}$, $\mathcal{L}'$ of subsets of resp. E, E'. Each $\tau(E, E')$-precompact set in E is limited. Conversely, if E contains a countable weakly total subset, then each limited set is $\tau(E, E')$-precompact. Notice, however, that in l^∞ there are weak Cauchy sequences which are not weakly convergent supplying examples of limited sets in a Banach space that are even not relatively weakly compact.

A.6 Proposition. *Let E be a lcs. Then $\mathcal{S} \subset \mathcal{L}$ if and only if*

$$\langle \phi_k, x_k \rangle \to 0 \quad as\ k \to \infty. \tag{A.6}$$

for every $\sigma(E, E')$-null sequence $\{x_k\}$ in E and every $\sigma(E', E)$-null sequence $\{\phi_k\}$ in E'.

The conditions in Proposition A.6 are also equivalent to $\mathcal{S}' \subset \mathcal{L}'$. A further equivalent condition states that every weakly convergent sequence in E is limited. By [5, Chap. 2, Sect. 18, Exer. 4c], a separable Banach space E satisfies these conditions if and only if every weakly compact set in E is compact.

A.6 Miscellaneous Topics

Here we collect remaining facts also needed in the main body of the paper.

For the first result, see [9, §28, 5.(1)]:

A.7 Lemma. *Let E be a sequentially complete bornological lcs. Then $E'_\tau = (E', \tau(E', E))$ is complete.*

The next result follows from the Banach-Mackey theorem (see [9, §39, 2.(3)]):

A.8 Proposition. *Let E, F be lcs. Suppose that E is sequentially complete. Then every subset of $\mathcal{L}(E, F)$ which is bounded for $\mathcal{L}_\sigma(E, F)$ is bounded for $\mathcal{L}_\beta(E, F)$.*

The following proposition is found in [9, §29, 4.(2), (3)], [10, Corollar 26.18]:

A.9 Proposition. *Let E be a metrizable lcs. Then its strong dual E' is a complete (DF)-space. The associated bornological space to E' is $(E', \beta(E', E''))$. Especially, E' is barreled if and only if E' is bornological.*

Moreover, these conditions are satisfied if E is a quasi-normable Fréchet space.

We also make use of (see [5, Chap. 2, Sec. 18, Thm. 12, Cor. 5]):

A.10 Lemma. *Let E be a lcs. Then the bounded sets in E are precompact if and only if the equi-continuous sets in E' are relatively compact.*

Finally recall the notion of a boundedly retractive inductive limit. Let $\{E_k\}_{k=0}^{\infty}$ be an increasing sequence of linear subspaces of $E = \operatorname{ind}\lim E_k$, where the limit is extended as $k \to \infty$, with continuous embeddings $E_k \subset E_{k+1}$ for all $k \in \mathbb{N}$. Then the inductive limit $\operatorname{ind}\lim E_k$ is said to be boundedly retractive if each bounded set $B \subset E$ is contained and bounded in E_k for some $k \in \mathbb{N}$ and the topologies induced by resp. E and E_k coincide on B. We have the following result:

A.11 Proposition. *Let $E = \operatorname{ind}\lim E_k$ and $\{E_k\}_{k=0}^{\infty}$ as above. Suppose that the inductive limit $\operatorname{ind}\lim E_k$ is regular. Then the inductive limit $\operatorname{ind}\lim E_k$ is boundedly retractive if and only if E satisfies the strict Mackey convergence condition.*

B Further Tools

Statements about locally convex spaces can often be traced back to statements about Banach spaces. In that respect, in connection with the continuity of pseudo-differential operators, Banach operator ideals and tensor norms are of special interest. Here we introduce into these subjects only as far as it is necessary for the intended applications in Section 3.

B.1 Banach Operator Ideals

Throughout this subsection, let E, F be Banach spaces with norms resp. $\| \|_E$ and $\| \|_F$. We use B_E to denote the unit ball in E. For dual spaces E', $B_{E'}$ shall often be considered as $\sigma(E', E)$-compact Hausdorff space. Details on Banach operator ideals may be found, e.g., in [3], [4].

An operator $T \in \mathcal{L}(E, F)$ is called 1-summing (or absolutely summing) if it maps weakly summable sequences into summable sequences. It is called 2-summing if it maps weakly 2-summable sequences into 2-summable sequences. A sequence $\{u_k\}_{k=0}^{\infty} \subset E$ is called weakly summable, summable, weakly 2-summable, and 2-summable, if $\sum_{k=0}^{\infty} |\langle \phi, u_k \rangle| < \infty$ for all $\phi \in E'$, $\sum_{k=0}^{\infty} \|u_k\|_E < \infty$, $\sum_{k=0}^{\infty} |\langle \phi, u_k \rangle|^2 < \infty$ for all $\phi \in E'$, and $\sum_{k=0}^{\infty} \|u_k\|_E^2 < \infty$, respectively. Equivalent conditions for $T \in \mathcal{L}(E, F)$ being 1-summing and 2-summing are

$$\begin{aligned} \sum_{k=0}^{\kappa} \|Tu_k\|_F &\leq C \sup\{\sum_{k=0}^{\kappa} |\langle \phi, u_k \rangle|;\ \phi \in E', \|\phi\|_{E'} \leq 1\}, \\ \Big(\sum_{k=0}^{\kappa} \|Tu_k\|_F^2\Big)^{1/2} &\leq C \sup\{(\sum_{k=0}^{\kappa} |\langle \phi, u_k \rangle|^2)^{1/2};\ \phi \in E', \|\phi\|_{E'} \leq 1\}, \end{aligned}$$

respectively, for all finite sequences $\{u_k\}_{k=0}^{\kappa} \subset E$. The infimums over all constants $C > 0$ therein are the 1-summing norm $\pi_1(T)$ and 2-summing norm $\pi_2(T)$, respectively. Every 1-summing operator is 2-summing. By the Pietsch domination theorem, $T \in \mathcal{L}(E, F)$ is 2-summing if and only if there exists a probability measure μ (as always assumed to be a regular Borel measure) on $B_{E'}$ such that for every $u \in E$

$$\|Tu\|_F \leq C \left\{ \int_{B_{E'}} |\langle \phi, u \rangle|^2 \, d\mu(\phi) \right\}^{1/2} \tag{B.1}$$

holds with some constant $C > 0$. The infimum over all C for all possible μ is the 2-summing norm $\pi_2(T)$ (see [4, Theorem 2.12]). An operator $T \in \mathcal{L}(E, F)$ between Hilbert spaces E, F is 2-summing if and only if it is a Hilbert-Schmidt operator with coincidence of the corresponding norms (see [4, Theorem 4.10]).

An operator $T \in \mathcal{L}(E, F)$ is called 1-integral (or integral) if there are a probability measure μ and operators $R \in \mathcal{L}(E, L^1(\mu))$, $S \in \mathcal{L}(L^\infty(\mu), F'')$ such that we have the factorization

$$k_F \cdot T = S \cdot \imath_1 \cdot R, \tag{B.2}$$

where $k_F : F \to F''$ is the isometric embedding and $\imath_1$ denotes the canonical mapping from $L^1(\mu)$ into $L^\infty(\mu)$ (see also Proposition A.4; $T \in \mathcal{L}(E, F)$ is integral if and only if the bilinear form $E \times F' \to \mathbb{C}$, $(u, \psi) \mapsto \langle \psi, Tu \rangle$, is integral). The 1-integral norm

$\iota_1(T)$ is inf $\|R\|\,\|S\|$, where the infimum is extended over all possible factorizations (B.2). An operator $T \in \mathcal{L}(E,F)$ is called 2-integral if there are a probability measure μ and operators $R \in \mathcal{L}(E, L^2(\mu))$, $S \in \mathcal{L}(L^\infty(\mu), F)$ such that we have

$$T = S \cdot \imath_2 \cdot R, \tag{B.3}$$

where $\imath_2$ is the canonical mapping from $L^2(\mu)$ into $L^\infty(\mu)$. The 2-integral norm $\iota_2(T)$ is inf $\|R\|\,\|S\|$, where this time the infimum is extended over all possible factorizations (B.3). Every 1-integral operator is 2-integral. The 1-integral operators are always 1-summing. The 2-integral operators and 2-summing operator are the same.

Next we state a sufficient condition for an operator $T \in \mathcal{L}(E, L^2(\mu))$ to be 2-integral. For that we make use of the Banach lattice structure of $L^2(\mu)$. Recall that a set $M \subset L^2(\mu)$ is called order bounded if there exists a function $h \in L^2(\mu)$, $h \geq 0$, such that

$$|f(x)| \leq h(x) \quad \mu\text{-a.e.}$$

for all $f \in M$. Then the set $\{|f|;\ f \in M\}$ has a supremum in $L^2(\mu)$. Similar results apply to the space $L^1(\mu)$, but this time the condition to state that an operator $T \in \mathcal{L}(E, L^1(\mu))$ is 1-integral turns out also to be necessary. A proof of the following proposition may be found in [4, Proposition 5.18, Theorem 5.19].

B.1 Proposition. *Let $T \in \mathcal{L}(E, L^2(\mu))$. Then T is 2-integral if $T(B_E)$ is order bounded in $L^2(\mu)$. In that case,*

$$\iota_2(T) \leq \Big\| \sup_{u \in B_E} |Tu| \Big\|_{L^2(\mu)}.$$

If $T \in \mathcal{L}(E, L^1(\mu))$, then T is 1-integral if and only if $T(B_E)$ is order bounded in $L^{(}\mu)$. In that case, $\iota_1(T) = \|\sup_{u \in B_E} |Tu|\|_{L^1(\mu)}$.

An operator $T \in \mathcal{L}(E,F)$ is called 1-nuclear (or nuclear) if there are sequences $\{\phi_k\}_{k=0}^\infty \subset E'$, $\{v_k\}_{k=0}^\infty \subset F$ and $\{\lambda_k\}_{k=0}^\infty \in l^1$ such that $\{\phi_k\}_{k=0}^\infty$ is bounded in E', $\{v_k\}_{k=0}^\infty$ is bounded in F and

$$T = \sum_{k=0}^{\infty} \lambda_k\, \phi_k \otimes v_k, \tag{B.4}$$

where the series converges in $\mathcal{L}(E,F)$. Here, for $\phi \in E'$, $v \in F$, $\phi \otimes v$ denotes the rank-one operator $E \to F$, $u \mapsto \langle \phi, u \rangle v$. The 1-nuclear norm $\nu_1(T)$ is inf $\{\sum_k |\lambda_k| \sup_k \|\phi_k\|_{E'} \cdot \sup_k \|v_k\|_F\}$, where the infimum is extended over all possible representations (B.4). An operator $T \in \mathcal{L}(E,F)$ is called 2-nuclear if there are sequences $\{\phi_k\}_{k=0}^\infty \subset E'$, $\{v_k\}_{k=0}^\infty \subset F$ and $\{\lambda_k\}_{k=0}^\infty \in l^2$ such that $\{\phi_k\}_{k=0}^\infty$ is bounded in E', $\{\langle \psi, v_k \rangle\}_{k=0}^\infty \in l^2$ for all $\psi \in F'$ and, again, (B.4) is satisfied. In this case, the 2-nuclear norm is inf $\{(\sum_k |\lambda_k|^2)^{1/2} \sup_k \|\phi_k\|_{E'} \sup_{\|\psi\|_{F'} \leq 1} (\sum_k |\langle \psi, v_k \rangle|^2)^{1/2}\}$, where the infimum is extended over all representations (B.4). Every 1-nuclear operator is 1-integral, every 2-nuclear operator is 2-integral. Furthermore, every 1-nuclear operator is 2-nuclear. The composition of an 2-nuclear operator S and a 2-summing operator T is 1-nuclear with 1-nuclear norm not exceeding $\pi_2(T)\nu_2(S)$ (see [4, Theorem 5.29]).

We need the following result (see [4, Theorem 5.28]):

B.2 Proposition. *Let $S \in \mathcal{L}(E,F)$, $T \in \mathcal{L}(F,G)$, where G is a further Banach space. Suppose that S is compact and T is 2-integral. Then the operator $T \cdot S$ is 2-nuclear.*

The ideals of 1-summing, 2-summing, 1-integral, 2-integral, 1-nuclear, and 2-nuclear operators are denoted by $\Pi_1(E,F)$, $\Pi_2(E,F)$, $\mathcal{I}_1(E,F)$, $\mathcal{I}_2(E,F)$, $\mathcal{N}_1(E,F)$, and $\mathcal{N}_2(E,F)$, respectively. These spaces equipped with the corresponding norms are Banach spaces. Moreover, they obey the ideal property, e.g., $R \in \mathcal{L}(E_0, E)$, $S \in \Pi_1(E,F)$, $T \in \mathcal{L}(F, F_0)$, where E_0, F_0 are Banach spaces, implies that $T \cdot R \cdot S \in \Pi_1(E_0, F_0)$.

B.2 2-Tensor Product

Here we introduce a certain tensor product used in Subsection 3.2. For further details on tensor products, see [3].

Let E, F be lcs. Then, for $w \in E \otimes F$,

$$\|w\|_{(U),(V);\otimes_2} = \inf \sup_{\phi\otimes\psi\in U^\circ\otimes V^\circ} \left\{\sum_{j=1}^{\kappa} |\langle\phi, u_j\rangle|^2\right\}^{1/2} \left\{\sum_{j=1}^{\kappa} |\langle\psi, v_j\rangle|^2\right\}^{1/2},$$

where the infimum is extended over all finite representations $w = \sum_{j=1}^{\kappa} u_j \otimes v_j$, $u_j \in E$, $v_j \in F$, defines a semi-norm on $E \otimes F$. The space $E \otimes F$ equipped with the semi-norm system $\{\| \|_{(U),(V);\otimes_2};\ U \in \mathcal{U}_E, V \in \mathcal{U}_F\}$ is denoted by $E \otimes_2 F$ and is termed the 2-tensor product of E, F. Its completion is denoted by $E\tilde{\otimes}_2 F$.

In the following statement, in particular, the dual space to $E \otimes_2 F$ is described:

B.3 Proposition. *Let E, F, G be Banach spaces with norms $\| \|_E$, $\| \|_F$, and $\| \|_G$, respectively. Further let $T : E \otimes F \to G$ be linear. Then $T \in \mathcal{L}(E \otimes_2 F, G)$ if and only if there are probability measures μ, ν on resp. $B_{E'}$ and $B_{F'}$ such that for the bilinear form $B : E \times F \to G$ associated with T the following estimate is valid:*

$$\|B(u,v)\|_G \leq C \left\{\int_{B_{E'}} |\langle\phi, u\rangle|^2 \, d\mu(\phi)\right\}^{1/2} \left\{\int_{B_{F'}} |\langle\psi, v\rangle|^2 \, d\nu(\psi)\right\}^{1/2} \tag{B.5}$$

for all $u \in E$, $v \in F$ and some constant $C > 0$. Thereby, the infimum over all possible C in (B.5) is the norm of T in $\mathcal{L}(E \otimes_2 F, G)$.

A proof of this result for $G = \mathbb{C}$ may be found in [3, Theorem 19.2]. It carries over to the case of general G. Notice that, by the Pietsch domination theorem, an equivalent condition for (B.5) is that there exist 2-summing operators $R \in \mathcal{L}(E, E_0)$, $S \in \mathcal{L}(F, F_0)$, where E_0, F_0 are arbitrary Banach spaces, such that

$$\|B(u,v)\|_G \leq \|Ru\|_{E_0} \|Sv\|_{F_0} \tag{B.6}$$

for all $u \in E$, $v \in F$. For the next result, see [3, Corollary 35.4].

B.4 Proposition. *Let E be a lcs. Then E is Hilbertizable, i.e., its topology is given by a system of Hilbert semi-norms, if and only if*

$$E \otimes_\epsilon F = E \otimes_2 F$$

holds topologically for any lcs F.

References

[1] Bergh, J., Löfström, J.: *Interpolation Spaces.* Grundlehren Math. Wiss., Vol. 223, Springer, Berlin, 1976.

[2] Dauge, M.: *Elliptic Boundary Value Problems on Corner Domains.* Lecture Notes in Math., Vol. 1341, Springer, Berlin, 1988.

[3] Defant, A., Floret, K.: *Tensor Norms and Operator Ideals.* Math. Studies, Vol. 176, North-Holland, Amsterdam, 1993.

[4] Diestel, J., Jarchow, H., Tonge, A.: *Absolutely Summing Operators.* Cambridge Stud. Adv. Math., Vol. 43, Cambridge Univ. Press, Cambridge, 1995.

[5] Grothendieck, A.: *Topological Vector Spaces.* Notes Math. Appl., Gordon and Breach, 1972.

[6] Hörmander, L.: *Fourier integral operators.* Acta Math. **127**, 79–183, 1971.

[7] Hörmander, L.: *The Analysis of Linear Partial Differential Operators I, III.* Grundlehren Math. Wiss., Vols. 256, 274, Springer, Berlin, 1983, 1985.

[8] Jarchow, H.: *Locally Convex Spaces.* Math. Leitfäden, Teubner, Stuttgart, 1981.

[9] Köthe, G.: *Topological Vector Spaces I, II.* Grundlehren Math. Wiss., Vols. 159, 237, Springer, Berlin, 1969, 1979.

[10] Meise, R., Vogt, D.: *Einführung in die Funktionalanalysis.* Viehweg Stud. Aufbaukurs Math., Viehweg, Braunschweig, 1992.

[11] Schulze, B.-W.: *Pseudo-Differential Boundary Value Problems, Conical Singularities and Asymptotics.* Math. Topics, Vol. 4, Akademie Verlag, Berlin, 1994.

[12] Taylor, M.: *Pseudodifferential Operators and Nonlinear PDE.* Progr. Math., Vol. 100, Birkhäuser Boston, Boston, 1991.

[13] Triebel, H: *Theory of Function Spaces II.* Monographs Math., Vol. 84, Birkhäuser, Basel, 1992.

[14] Witt, I.: *Non-Linear Hyperbolic Equations in Domains with Conical Points.* Math. Research, Vol. 84, Akademie Verlag, Berlin, 1995.