# Existence and Uniqueness Theorem for Frenet Frame Supercurves

CRISTEA VALENTIN GABRIEL

**Abstract.** In the first part of this paper, using the Banach Grassmann algebra $B_L$ given by Rogers in her paper [10], a new scalar product and a new definition of the orthogonality are introduced on the (m,n)-dimensional total supereuclidean space ${B_L}^{m+n}$ .Using the $GH^\infty$ functions given by Rogers in [10], the new definitions of the supercurve, of the supersmooth supercurve, of the supersmooth supercurve in general position and of the Frenet frame associated to a supersmooth supercurve in general position are given.In the second part of this paper, using the classical results described in [9] , the new existence and uniqueness theorem for some supercurves which admit Frenet frame is proved.

**Keywords**: the (m,n)-dimensional total supereuclidean space ${B_L}^{m+n}$ , the (m,n)-dimensional supereuclidean space $B_L^{m,n}$ , the $GH^\infty$ functions, supersmooth supercurve, supersmooth supercurve in general position, Frenet frame associated to a supersmooth supercurve, Frenet formulas for the supersmooth supercurve.
**Mathematics Subject Classification (1991)**: 58A50

## 1. Supersmooth supercurve in general position and Frenet frame associated to a supersmooth supercurve in general position

Let us consider only algebras over the reals. For each positive integer L, $B_L$ [10] will denote the Grassmann algebra over the reals with generators $1^{(L)}$ ,$\beta_1^{(L)}$,..., $\beta_L^{(L)}$ and relations

(1) $$1^{(L)}\beta_i^{(L)} = \beta_i^{(L)}1^{(L)} = \beta_i^{(L)} \quad i = 1, ..., L,$$

(2) $$\beta_i^{(L)}\beta_j^{(L)} = -\beta_j^{(L)}\beta_i^{(L)} \quad i, j = 1, ..., L.$$

$B_L$ is a graded algebra [12] and can be written as a direct sum

$$B_L = (B_L)_0 \oplus (B_L)_1$$

where $(B_L)_0$ and $(B_L)_1$ are the even and odd part of $(B_L)$ respectively. Let $M_L$ denote (due to Kostant [7]) the set of finite sequences of positive integers $\mu = (\mu_1, ..., \mu_k)$ with $1 \le \mu_1 < ... < \mu_k \le L$. $M_L$ includes the sequence with no elements, denoted $\phi$. As it follows in [10] for each $\mu$ in $M_L$,

(3) $$\beta_\mu^{(L)} := \beta_{\mu_1}^{(L)} \cdots \beta_{\mu_k}^{(L)},$$

and

(4) $\beta_{\phi}^{(L)} := 1^{(L)}$

a typical element $b$ of $B_L$ may be expressed as

(5) $b = \sum_{\mu \in M_L} b^{\mu} \beta_{\mu}^{(L)},$

where the coefficient $b^{\mu}$ are real numbers. We consider the body map (in DeWitt's terminology [4] )

$$\varepsilon^{(L)} : B_L \to \mathbf{R}$$

given by

(6) $\varepsilon^{(L)}(b) = b^{\phi}.$

With the norm on $B_L$ defined by

(7) $||b|| := \sum_{\mu \in M_L} |b^{\mu}|,$

$B_L$ is a Banach algebra [11] . Considering $L'$ also a positive integer, with $L \geq L'$ , then there is a natural injection $i_{L',L} : B_{L'} \to B_L$ [10], which is the unique algebra homomorphism satisfying

(8) $i_{L',L}(\beta_i^{(L')}) = \beta_i^{(L)} \quad i = 1, ..., L \quad i_{L',L}(1^{(L')}) = 1^{(L)}.$

$B_L$ naturally has a $B_{L'}$ module structure [10] with, given $a \in B_{L'}$ and $b \in B_L$,

(9) $ab := i_{L',L}(a)b.$

We define the $(m, n)-$dimensional total supereuclidean space ${B_L}^{m+n}$ [1] as the space which is the cartesian product of $m + n$ copies of $B_L$ and has the graduation

$${B_L}^{m+n} = ({B_L}^{m+n})_0 \oplus ({B_L}^{m+n})_1$$

A typical element of ${B_L}^{m+n}$ is written $(x^1, ..., x^m, \theta^1, ..., \theta^n)$ or simply $(x, \theta)$, an element of $({B_L}^{m+n})_0$ is called $c-$type or even element and is written in the form $(x'^1, ..., x'^m, \theta'^1, ..., \theta'^n)$ with $x'^1, ..., x'^m \in (B_L)_0$ and $\theta'^1, ..., \theta'^n \in (B_L)_1$ and an element of $({B_L}^{m+n})_1$ is called $a-$type or odd element and is written in the form $(x''^1, ..., x''^m, \theta''^1, ..., \theta''^n)$ with $x''^1, ..., x''^m \in (B_L)_1$ and $\theta''^1, ..., \theta''^n \in (B_L)_0$. An even element has the parity 0 and an odd element has the parity 1.

**Definition 1.** [4] *$V$ is called the supervector space over the space $B_L$ if and only if $V$ is the supervector space given by the $1) - 5)$ conditions from the DeWitt's definition (see DeWitt's book pages 14-15) where instead of $\Lambda_\infty$ we put $B_L$ .*

**Example 1.** *The $(m, n)-$dimensional total supereuclidean space ${B_L}^{m+n}$ with the above graduation*

$${B_L}^{m+n} = ({B_L}^{m+n})_0 \oplus ({B_L}^{m+n})_1$$

*is a supervector space of dimension* $(m,n)$.

**Remark 1.** *Let us consider the* $(m,n)-$*dimensional supereuclidean space* $B_L^{m,n}$ *given by Rogers in* [10]. *We note that the space* $B_L^{m,n}$ *is not a supervector space over* $B_L$ *as in definition 1.*

A useful map is [10]

$$\varepsilon_{m,n}^{(L)} : ({B_L}^{m+n})_0 \to \mathbf{R^m}$$

with

(10) $\quad \varepsilon_{m,n}^{(L)}(x'^1, ..., x'^m, \theta'^1, ..., \theta'^n) := (\varepsilon^{(L)}(x'^1), ..., \varepsilon^{(L)}(x'^m))$

and another useful map is

$$\varepsilon_{m,n}'^{(L)} : ({B_L}^{m+n})_1 \to \mathbf{R^n}$$

with

(11) $\quad \varepsilon_{m,n}'^{(L)}(x''^1, ..., x''^m, \theta''^1, ..., \theta''^n) := (\varepsilon^{(L)}(\theta''^1), ..., \varepsilon^{(L)}(\theta''^n)$.

**Remark 2.** *Let us consider the* $(1,1)-$*dimensional total supereuclidean space* ${B_L}^2$ , $(2,0)-$*dimensional total supereuclidean space* ${B_L}^2$ *and* $(0,2)-$*dimensional total supereuclidean space* ${B_L}^2$ *and the element* $(1,0)$ *which belongs to these three spaces.We note that the element* $(1,0)$ *is c-type for the first two spaces and is a-type for the last space. We may write the supervector* $(1,0)$ *in a standard basis* [4] *in the form:*

$$(1,0) = 1 \cdot (1,0) + 0 \cdot (0,\beta^1)$$

*for the* $(1,1)-$*dimensional total supereuclidean space* ${B_L}^2$ *and* $(0,2)-$*dimensional total supereuclidean space* ${B_L}^2$ *where a standard basis* [4] *in these spaces is* $\{(1,0),(0,\beta^1)\}$ *with* $(1,0)$ $c-$*type supervector and* $(0,\beta^1)$ $a-$*type supervector and the supervector* $(1,0)$ *can be written in a standard basis* [4] *in the form:*

$$(1,0) = 1 \cdot (1,0) + 0 \cdot (0,1)$$

*for the* $(2,0)-$*dimensional total supereuclidean space* ${B_L}^2$, *where a standard basis* [4] *in this space is* $\{(1,0),(0,1)\}$ *with* $(1,0)$ $c-$*type supervector and* $(0,1)$ $a-$*type supervector*

**Definition 2.** [10] *Suppose* $U \subset \mathbf{R}^m$ *is open and* $L'$ *is a positive integer with* $L' \leq L$. *Let* $C^\infty(U, B_{L'})$ *denote the* $B_{L'}$ *module of* $C^\infty$ *functions of* $U$ *into* $B_{L'}$ *;(recall that* $B_{L'}$ *is a Banach algebra, and hence a fotiori a Banach space).Then the map*

$$Z_{L',L} : C^\infty(U, B_{L'}) \to [\varepsilon_{m,0}^{(L)}(U)]^{B_L}$$

*is defined by* [10]

(12) $\quad Z_{L',L}(f)(x^1, ..., x^m) = \sum_{i_1=0 \ldots i_m=0}^{L} \frac{1}{i_1! \ldots i_m!} \cdot$

$$\cdot i_{L',L}(\partial_1^{i_1}...\partial_m^{i_m} f(\varepsilon^{(L)}(x^1),...,\varepsilon^{(L)}(x^m))) \times s(x^1)^{i_1} \cdots s(x^m)^{i_m}$$

*where* [10]

(13) $\quad s(x^i) = x^i - \varepsilon^{(L)}(x^i)1, \quad i = 1, ..., m.$

**Definition 3.** [10] *Suppose* $V$ *is open in* $B_L^{m,n}$ *(with respect to its usual finite-dimensional vector space topology) and* $U = \varepsilon_{m,n}^{(L)}(V)$, *suppose* $L > 2n$ *and* $L' = [\frac{1}{2}L]$, *the last integer not less than* $\frac{1}{2}L$. $GH^\infty(V)$ *denotes the set of functions* $f : V \to B_L$ *for which there exist* [10] $f_\mu \in C^\infty(U, B_{L'})$ *such that*

(14) $\quad f(x,\theta) = \sum_{\mu \in M_n} Z_{L',L}(f_\mu)(x)\theta^\mu$

*where*

(15) $\quad \theta^\mu = \theta^{\mu_1} \cdots \theta^{\mu_k}$

*and*

(16) $\quad \theta^\phi = 1^{(L)}.$

**Definition 4.** [10] *With the notation of Definition 2., let* $f$ *be an element of* $GH^\infty(V)$, *with expansion* (12) . *Then, for* $i = 1, ..., m$

$$G_i f : V \to B_L$$

*is defined by* [10]

(17) $\quad G_i f(x;\theta) = \sum_{\mu \in M_n} Z_{L',L}(\partial_i f_\mu)(x)\theta^\mu.$

*Also, for* $j = 1, ..., n$,

$$G_{j+m} f : V \to B_L$$

*is defined by* [10]

(18) $\quad G_{j+m} f(x;\theta) = \sum_{\mu \in M_n} Z_{L',L}(f_\mu)(x)\theta^{\mu/j} \times (-1)^{|f_\mu(x)|}$,

*where* $|f_\mu(x)|$ *is the Grassmann parity of* $f_\mu(x)$, *and*

$$\theta^{\mu/j} = \theta^{\mu_1} \cdots \theta^{\mu_{i-1}} \theta^{\mu_{i+1}} \cdots \theta^{\mu_k} (-1)^{i-1},$$

*if* $j = \mu_i$ *for some* $i$, $1 \le i \le k$, $\theta^{\mu/j}$ *otherwise.*

For the first time, I have introduced on ${B_L}^{m+n}$ with $n = 2r$ , the scalar product

$$\langle v, w\rangle = x^1y^1 + \cdots + x^my^m + \theta^1{\theta_1}^{r+1} + \cdots + \theta^r{\theta_1}^n - \theta^{r+1}{\theta_1}^1 - \cdots - \theta^n{\theta_1}^r$$

$(\forall)\ v = (x^1, \ldots, x^m, \theta^1, \ldots, \theta^n), w = (y^1, \ldots, y^m, {\theta_1}^1, \ldots, {\theta_1}^n) \in {B_L}^{m+n}$ which has these properties:

a) $\langle v,w\rangle = (-1)^{|v|\cdot|w|}\langle w,v\rangle$ (supersymmetry)
b) $\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle$ $(\forall)$ $u,v,w \in {B_L}^{m+n}$ (linearity)
c) $\langle v,\cdot\rangle = 0$ if and only if $v=0$.

**Proposition 1.** *(Number of scalar products on ${B_L}^{m+n}$ ) On ${B_L}^{m+n}$ with $n=2r$ , we may give $r!$ different scalar products.*

**Proof.** There are $r!$ one-to-one functions $f:\{1,\dots,r\}\to\{r+1,\dots,2r\}$ . For each function $f$ we have the following scalar product between $v=(x^1,\dots,x^m,\theta^1,\dots,\theta^n)$ and $w=(y^1,\dots,y^m,{\theta_1}^1,\dots,{\theta_1}^n)$

$$\langle v,w\rangle_f = \sum_{k=1}^{m} x^k y^k + \sum_{j_1=1}^{r} (\theta^{j_1}{\theta_1}^{f(j_1)} - \theta^{f(j_1)}{\theta_1}^{j_1}).$$

One can easily verify the $a), b)$ and $c)$ relations of a scalar product. Let us prove $a)$ in the case when $v$ and $w$ are odd elements of ${B_L}^{m+n}$ , that is, $v$ and $w$ belong to $({B_L}^{m+n})_1$. Then, we shall have $v=(x''^1,\dots,x''^m,\theta''^1,\dots,\theta''^n)$ and $w=(y''^1,\dots,y''^m,\theta_1''^1,\dots,\theta_1''^n)$ .When we compute the scalar product between $v$ and $w$, we shall get

$$\langle v,w\rangle_f = \sum_{k=1}^{m} x''^k\cdot y''^k + \sum_{j_1=1}^{r}(\theta_1''^j\theta_1''^{f(j_1)} - \theta''^{f(j_1)}\theta_1''^{j_1}) =$$

$$= -\sum_{k=1}^{n} y''^k\cdot x''^k + \sum_{j_1=1}^{r}(\theta_1''^{f(j_1)}\theta''^{j_1} - \theta_1''^{j_1}\theta''^{f(j_1)})$$

Now, the scalar product between $v$ and $w$ becomes

$$\langle v,w\rangle_f = -(\sum_{k=1}^{n} y''^k\cdot x''^k + \sum_{j_1=1}^{r}(\theta_1''^{j_1}\theta''^{f(j_1)} - \theta_1''^{f(j_1)}\theta''^{j_1}))$$

Thus, we get

$$\langle v,w\rangle_f = -\ \langle w,v\rangle_f.$$

This ends the proof.

**Definition 5.** *(Orthogonality on ${B_L}^{m+n}$ ) We say that the supervector $v$ of ${B_L}^{m+n}$ is orthogonal to the supervector $w$ of ${B_L}^{m+n}$ if and only if $\varepsilon^{(L)}(\langle v,w\rangle)=0$ .*

The column supervectors $E_1=(1,\dots,0)$, …, $E_m=(0,\dots,1,0,\dots,0)$, where 1 is written on the $m^{th}$ place, $E_{m+1}=(0,\dots,0,-1,0,\dots,0)$, where -1 is written on the $(m+r+1)^{th}$ place, …, $E_{m+r}=(0,\dots,0,-1)$, where -1 is written on the $(m+n)^{th}$ place, $E_{m+r+1}=(0,\dots,0,1,0,\dots,0)$, where 1 is written on the $(m+1)^{th}$ place, …, $E_{m+n}=(0,\dots,0,1,0,\dots,0)$, where 1 is written on the $(m+r)^{th}$ place, form the standard basis on ${B_L}^{m+n}$ [4] where the first $m$ supervectors are $c-$type and the last $n$ supervectors are $a-$type.

**Definition 6.** *(Supersmooth supercurve)* [10] *Let suppose* $L > 2n$ *and let* ${B_L}^{m+n}$ *be an* $(m,n)-$*dimensional total supereuclidean space, let* $V$ *be an open set in* $B_L^{1,1}$ , *let* $c : V \subset B_L^{1,1} \to {B_L}^{m+n}$ *be a function, and for every* $\theta \in V \cap (B_L)_1$ *let define*

$$c_{\theta,0} : V \cap (B_L)_0 \to ({B_L}^{m+n})_0$$

*given by*

$$c_{\theta,0}(t) = (c(t,\theta))_0$$

*where* $(c(t,\theta))_0$ *is the even part of the supervector* $c(t,\theta)$ *and*

$$c_{\theta,B} : V \cap (B_L)_0 \to \mathbf{R}^m$$

*given by*

$$c_{\theta,B}(t) = \varepsilon_{m,n}^{(L)} \circ c_{\theta,0}(t)$$

*for all* $t \in V \cap (B_L)_0$ . *The function* $c$ *will be said to be supercurve if and only if* $c_{\theta,B}|_{V\cap\mathbf{R}}$ *will be a curve. The function* $c$ *is called supersmooth if and only if*

$c^i \in GH^\infty(V)$ $(\forall)$ $i \in \{1,\ldots,m\}$ and $c^{j+m} \in GH^\infty(V)$ $(\forall)$ $j \in \{1,\ldots,n\}$ where $c^i = x^i \circ c$ $(\forall)$ $i \in \{1,\ldots,m\}$ and $c^{j+m} = \theta^j \circ c$ $(\forall)$ $j \in \{1,\ldots,n\}$ .

**Definition 7.** *(Supercurve in general position) Let suppose* $L > 2n$ *and let* ${B_L}^{m+n}$ *be an* $(m,n)-$*dimensional total supereuclidean space, let* $V$ *be an open set in* $B_L^{1,1}$ *and let* $c : V \subset B_L^{1,1} \to {B_L}^{m+n}$ *be a supersmooth supercurve . We say that the supercurve* $c$ *is in general position if and only if*

$$\{G_1c(t,\theta),\ldots,G_1^{(m-1)}c(t,\theta),\ G_2c(t,\theta),G_1G_2c(t,\theta),\ldots$$
$$\ldots,G_1^{(n-1)}G_2c(t,\theta)\}$$

*are linear independent* $(\forall)$ $(t,\theta) \in V \subset B_L^{1,1}$ *and where by* $G_1c(t,\theta)$ *we understand the supervector*

$$(G_1c^1(t,\theta),\ldots,G_1c^m(t,\theta),G_1c^{m+1}(t,\theta),\ldots,G_1c^{m+n}(t,\theta)) \quad (\forall)\ (t,\theta) \in V \subset B_L^{1,1}$$

*by* $G_2c(t,\theta)$ *we understand the supervector*

$$(G_2c^1(t,\theta),\ldots,G_2c^m(t,\theta),G_2c^{m+1}(t,\theta),\ldots,G_2c^{m+n}(t,\theta)) \quad (\forall)\ (t,\theta) \in V \subset B_L^{1,1},$$

$G_1^{(s)}c(t,\theta) = G_1\ldots G_1c(t,\theta)$, *where* $G_1^{(0)}c(t,\theta) = c(t,\theta)$ *and* $G_1^{(1)}c(t,\theta) = G_1c(t,\theta)$ *and* "$\ldots$" *means that* $G_1$ *is applied by* $s$ *times.*

**Definition 8.** *(Frenet Frame associated to a supersmooth supercurve) Let suppose* $L > 2n$ *and let* ${B_L}^{m+n}$ *be an* $(m,n)-$*dimensional total supererclidean space, let* $V$ *be an open set in* $B_L^{1,1}$ *and let* $c : V \subset B_L^{1,1} \to {B_L}^{m+n}$ *be a supersmooth supercurve. By a Frenet frame associated to a supersmooth supercurve* $c : V \subset B_L^{1,1} \to {B_L}^{m+n}$ *we shall mean a system of* $m+n$ *supervector fields* $\{e_1,\ldots,e_{m+n}\}$ *along to the supersmooth supercurve* $c$ *such that* $(\forall)$ $(t,\theta) \in V \subset B_L^{1,1}$ *we have the following properties:*

(19) $\langle e_k(t,\theta), e_h(t,\theta)\rangle = \delta_{kh}$ $\quad(\forall)$ $k,h \in \{1,\ldots,m\}$

$(19')$ $\langle e_{m+r+j_1}(t,\theta), e_{m+j_2}(t,\theta)\rangle = -\delta_{j_1j_2}$ $\quad (\forall)\ j_1, j_2 \in \{1,\ldots,r\}$

$(19'')$ $\langle e_{m+j_1}(t,\theta), e_{m+r+j_2}(t,\theta)\rangle = \delta_{j_1j_2}$ $\quad (\forall)\ j_1, j_2 \in \{1,\ldots,r\}$

$(19''')$ $\langle e_{m+j_1}(t,\theta), e_{m+j_2}(t,\theta)\rangle = 0,$ $\quad (\forall)\ j_1, j_2 \in \{1,\ldots,r\}$

$(19^{IV})$ $\langle e_{m+r+j_1}(t,\theta), e_{m+r+j_2}(t,\theta)\rangle = 0,$ $\quad (\forall)\ j_1, j_2 \in \{1,\ldots,r\}$

$(19^{V})$ $\langle e_i(t,\theta), e_{m+j}(t,\theta)\rangle = \langle e_{m+j}(t,\theta), e_i(t,\theta)\rangle = 0,$

$(\forall)\ i \in \{1,\ldots,m\}$ *and* $j \in \{1,\ldots,n\}$

(20) $span(G_1c(t,\theta),\ldots,G_1^{(k)}c(t,\theta)) = span(e_1(t,\theta),\ldots,e_k(t,\theta))$ $\quad (\forall)\ k \in \{1,\ldots,m-1\}$

*where* $G_1^{(1)}c(t,\theta) = G_1c(t,\theta)$ *and*

$(20')$ $span(G_2c(t,\theta), G_1G_2c(t,\theta),\ldots,G_1^{(j-1)}G_2c(t,\theta)) =$

$$= span(e_{m+1}(t,\theta),\ldots,e_{m+j}(t,\theta)) \qquad (\forall)\ j \in \{1,\ldots,n\}$$

*where* $G_1^{(0)}G_2c(t,\theta) = G_2c(t,\theta)$ *and* $G_1^{(1)}G_2c(t,\theta) = G_1G_2c(t,\theta)$ *and we shall mean by* $span(e_1(t,\theta),\ldots,e_k(t,\theta))$ *the supervector space* [4], *spanned by* $e_1(t,\theta),\ldots,e_k(t,\theta)$ $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ .The systems of supervectors

(21) $\{G_1c(t,\theta),\ldots,G_1^{(k)}c(t,\theta)\}$

and

$$\{e_1(t,\theta),\ldots,e_k(t,\theta)\}$$

are directed in the same way $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ and $(\forall)\ k \in \{1,\ldots,m-1\}$, that is, $\varepsilon^{(L)}(detM_1) > 0$ where $M_1$ is the matrix when we change from the frame $\{G_1c(t,\theta),\ldots,G_1^{(k)}c(t,\theta)\}$ to the frame $\{e_1(t,\theta),\ldots,e_k(t,\theta)\}$ and the systems of supervectors

$(21')$ $\{G_2c(t,\theta), G_1G_2c(t,\theta),\ldots,G_1^{(j-1)}G_2c(t,\theta)\}$

and

$$\{e_{m+1}(t,\theta),\ldots,e_{m+j}(t,\theta)\}$$

are directed in the same way $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ $(\forall)\ j \in \{1,\ldots,n\}$,that is, $\varepsilon^{(L)}(detM_2) > 0$ where $M_2$ is the matrix when we change from the frame

$$\{G_2c(t,\theta), G_1G_2c(t,\theta),\ldots,G_1^{(j-1)}G_2c(t,\theta)\}$$

to the frame

$$\{e_{m+1}(t,\theta),\ldots,e_{m+j}(t,\theta)\}.$$

The system of supervectors

(22) $\quad \{e_1(t,\theta),\ldots,e_{m+n}(t,\theta)\}$

is positive directed $(\forall)\ (t,\theta)\in V\subset B_L^{1,1}$, that is, $\varepsilon^{(L)}(sdet(e_q^s(t,\theta)))>0$ where by $sdet(e_q^s(t,\theta))$ we understand $det(A-C\cdot B^{-1}\cdot D)\cdot(detB)^{-1}$ [2], [8], [4], [1], [5] where

$$(e_q^s(t,\theta))_{1\le s,q\le m+n}=\begin{pmatrix} A & C\\ D & B\end{pmatrix},$$

where $A,B,C$ and $D$ are $m\times m,\ n\times n,\ m\times n$ and $n\times m$ matrices with elements in $B_L$, respectively.

## 2. Existence and Uniqueness Theorem for Frenet Frame Supercurves

**Theorem 1.** *(Existence and Uniqueness Theorem for Frenet Frame Supercurves) Let suppose $L>2n$ and let ${B_L}^{m+n}$ be an $(m,n)-$dimensional total supereuclidean space, let $V$ be an open set of $B_L^{1,1}$ and let $c:V\subset B_L^{1,1}\to {B_L}^{m+n}$ be a supersmooth supercurve in general position which satisfy the following relation:*

(1) $\quad \varepsilon^{(L)}(\langle G_2c(t,\theta),G_1^{(r)}G_2c(t,\theta)\rangle)>0$

(2) $\quad \varepsilon^{(L)}(\langle G_1^{(j_1)}G_2c(t,\theta)),G_1^{(r+j_1)}G_2c(t,\theta)\rangle)>0$

$(\forall)\ j_1\in\{1,\ldots,r-1\}$, *and* $(\forall)\ (t,\theta)\in V\subset B_L^{1,1}$,

(3) $\quad \varepsilon^{(L)}(\langle G_2c(t,\theta),G_1^{(j)}G_2c(t,\theta)\rangle)=0,$

$(\forall)\ j\in\{0,\ldots,n-1\}$, *with* $j\neq r$ *and* $(\forall)\ (t,\theta)\in V\subset B_L^{1,1}$,

(4) $\quad \varepsilon^{(L)}(\langle G_1^{(j')}G_2c(t,\theta)),G_1^{(j)}G_2c(t,\theta)\rangle)=0$

$(\forall)\ j'\in\{1,\ldots,n-1\}$, $(\forall)\ j\in\{1,\ldots,n-1\}$ *with* $j\neq j'+r$ *and* $j'<j$ *and* $(\forall)\ (t,\theta)\in V\subset B_L^{1,1}$. *Then there exists a unique Frenet frame* $\{e_1,\ldots,e_{m+n}\}$ *associated to the supercurve* $c$ *and we have the following formulas* $(\forall)\ (t,\theta)\in V\subset B_L^{1,1}$:

(5) $\quad G_1e_k(t,\theta)=\sum_{h=1}^m a_{kh}(t,\theta)\cdot e_h(t,\theta) \qquad (\forall)\ k\in\{1,\ldots,m\},$

*where*

(6) $\quad a_{kh}(t,\theta)+a_{hk}(t,\theta)=0 \qquad (\forall)\ k,h\in\{1,\ldots,m\}$

*and*

(7) $\quad a_{kh}(t,\theta)=0 \qquad if \qquad h>k+1, \qquad (\forall)\ k,h\in\{1,\ldots,m\}$

(8) $\qquad G_1 e_{m+j}(t,\theta) = \sum_{l=1}^{n} a_{m+j\ m+l}(t,\theta) \cdot e_{m+l}(t,\theta) \qquad (\forall)\ j \in \{1,\ldots,n\},$

*where*

(9) $\qquad a_{m+j_1\ m+j_2}(t,\theta) + a_{m+r+j_2\ m+r+j_1}(t,\theta) = 0 \qquad (\forall)\ j_1, j_2 \in \{1,\ldots,r\},$

(10) $\qquad a_{m+r+j_1\ m+j_2}(t,\theta) - a_{m+r+j_2\ m+j_1}(t,\theta) = 0 \qquad (\forall)\ j_1, j_2 \in \{1,\ldots,r\},$

(11) $\qquad a_{m+j_1\ m+r+j_2}(t,\theta) - a_{m+j_2\ m+r+j_1}(t,\theta) = 0 \qquad (\forall)\ j_1, j_2 \in \{1,\ldots,r\},$

*and*

(12) $\qquad a_{i\ m+j}(t,\theta) = 0, \quad a_{m+j\ i}(t,\theta) = 0 \quad (\forall)\ i \in \{1,\ldots,m\},$

$(\forall)\ j \in \{1,\ldots,n\},$

(13) $\qquad a_{m+j\ m+l}(t,\theta) = 0 \qquad if \qquad l \neq j+1 \qquad (\forall)\ j, l \in \{1,\ldots,n\},$

*where*

(14) $\qquad a_{m+j_1\ m+j_2}(t,\theta) = \langle G_1 e_{m+j_1}(t,\theta), e_{m+r+j_2}(t,\theta)\rangle,$

(15) $\qquad a_{m+j_1\ m+r+j_2}(t,\theta) = -\langle G_1 e_{m+j_1}(t,\theta), e_{m+j_2}(t,\theta)\rangle,$

(16) $\qquad a_{m+r+j_1\ m+j_2}(t,\theta) = \langle G_1 e_{m+r+j_1}(t,\theta), e_{m+r+j_2}(t,\theta)\rangle,$

(17) $\qquad a_{m+r+j_1\ m+r+j_2}(t,\theta) = -\langle G_1 e_{m+r+j_1}(t,\theta), e_{m+j_2}(t,\theta)\rangle,$

$(\forall)\ j_1, j_2 \in \{1,\ldots,r\},$

(18) $\qquad a_{kh}(t,\theta) = \langle G_1 e_k(t,\theta), e_h(t,\theta)\rangle, \qquad (\forall)\ k, h \in \{1,\ldots,m\},$

(19) $\qquad a_{k\ m+j}(t,\theta) = \langle G_1 e_k(t,\theta), e_{m+j}(t,\theta)\rangle, \qquad (\forall)\ k \in \{1,\ldots,m\}$

*and* $\quad (\forall)\ j \in \{1,\ldots,n\},$

(20) $\qquad a_{m+j\ k}(t,\theta) = \langle G_1 e_{m+j}(t,\theta), e_k(t,\theta)\rangle, \qquad (\forall)\ k \in \{1,\ldots,m\}$

*and* $\quad (\forall)\ j \in \{1,\ldots,n\}.$

**Proof.** We use the proof of the existence and uniqueness theorem for Frenet frame curves from [8] in our proof. From the (1) and (2) relations we have:

$$\varepsilon^{(L)}(\langle G_2 c(t,\theta), G_1^{(r)} G_2 c(t,\theta)\rangle) \neq 0$$

and

$$\varepsilon^{(L)}(\langle G_1^{(j_1)} G_2 c(t,\theta)), G_1^{(r+j_1)} G_2 c(t,\theta)\rangle) \neq 0$$

$\forall\ j_1 \in \{1,\ldots,r-1\}$. Let us consider

$$\lambda_1(t,\theta) = \langle G_2 c(t,\theta), G_1^{(r)} G_2 c(t,\theta)\rangle$$

and

$$\lambda_{j_1}(t,\theta) = \langle G_1^{(j_1-1)} G_2 c(t,\theta), G_1^{(r+j_1-1)} G_2 c(t,\theta)\rangle$$

$(\forall)\ j_1 \in \{2,\ldots,r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$. Because of $\varepsilon^{(L)}(\lambda_{j_1}(t,\theta)) \neq 0$ $(\forall)\ j_1 \in \{1,\ldots,r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ it results that there exist $(\lambda_{j_1}(t,\theta))^{-1}$. Let us consider

$$e_{m+1}(t,\theta) = (\lambda_1(t))^{-1} \cdot G_2 c(t,\theta)$$

and

$$e_{m+j_1}(t,\theta) = (\lambda_{j_1}(t))^{-1} \cdot G_1^{(j_1-1)} G_2 c(t,\theta)$$

$(\forall)\ j_1 \in \{2,\ldots,r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ and let us consider

$$e_{m+r+j_2}(t,\theta) = G_1^{(r+j_2-1)} G_2 c(t,\theta)$$

$(\forall)\ j_2 \in \{1,\ldots,r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.

The supervectors $\{e_{m+1}(t,\theta),\ldots,e_{m+n}(t,\theta)\}$ are linear independent because the supersmooth supercurve $c$ is general position. From the definition of these supervectors, we have:

$$span(G_1 c(t,\theta),\ldots,G_1^{(k)} c(t,\theta)) = span(e_1(t,\theta),\ldots,e_k(t,\theta)) \qquad (\forall)\ k \in \{1,\ldots,m-1\}$$

$$span(G_2 c(t,\theta), G_1 G_2 c(t,\theta),\ldots,G_1^{(j-1)} G_2 c(t,\theta)) =$$
$$= span(e_{m+1}(t,\theta),\ldots,e_{m+j}(t,\theta)) \qquad (\forall)\ j \in \{1,\ldots,n\}$$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.

From the (1) and (2) relation we have $\varepsilon^L((\lambda_{j_1}(t,\theta))) > 0$ $(\forall)\ j_1 \in \{1,\ldots,r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$. The " body " of the determinants of the matrices when we change from the system $\{G_2 c(t,\theta), G_1 G_2 c(t,\theta),\ldots,G_1^{(j-1)} G_2 c(t,\theta)\}$ to the system $\{e_{m+1}(t,\theta),\ldots,e_{m+j}(t,\theta)\}$ $(\forall)\ j \in \{1,\ldots,n\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ is

$$\varepsilon^{(L)}\left(\begin{vmatrix} (\lambda_1(t,\theta))^{-1} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & (\lambda_j(t,\theta))^{-1} & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{vmatrix}\right) =$$

$$= \varepsilon^{(L)}((\lambda_1(t,\theta))^{-1} \cdot\ \ldots\ \cdot (\lambda_j(t,\theta))^{-1}) =$$

$$= \varepsilon^{(L)}((\lambda_1(t,\theta))^{-1}) \cdot \ \ldots \ \cdot \varepsilon^{(L)}((\lambda_j(t,\theta))^{-1}) > 0.$$

We have $(\forall)\ j_1 \in \{1, \ldots, r\}$ and $(\forall)\ t \in I$ that

$$\langle e_{m+j_1}(t), e_{m+r+j_1}(t)\rangle = (\lambda_{j_1}(t,\theta))^{-1} \cdot \langle G_1^{(j_1-1)} G_2 c(t,\theta)), G_1^{(r+j_1-1)} G_2 c(t,\theta)\rangle =$$

$$= (\lambda_{j_1}(t))^{-1} \cdot \lambda_{j_1}(t) = 1.$$

Thus, we get the $(19'')$ formula from the first part of this paper. The $(19')$ formula, from the first part of this paper, results from the supersymmetry property of the scalar product $\langle , \rangle$ and the $(19^{IV})$ and $(19''')$ formulas , from the first part of this paper, result from the (3) and (4) relations.We shall specify if a relation belongs to the first part of this paper and we shall not specify if a relation belongs to the second part of this paper.

Because the supermooth supercurve $c$ is in general position, it results that the supervector $f_1(t,\theta) = G_1 c(t,\theta)$ is nonzero $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$. Let us consider $||v||' = \sqrt{\varepsilon^{(L)}(\langle v, v\rangle)}$ (21) $(\forall)\ v \in {B_L}^{m+n}$ .

We put $e_1(t,\theta) = f_1(t,\theta) \cdot (||f_1(t,\theta)||')^{-1}$ (22) $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.

Let the supervector $f_2(t,\theta)$ be $G_1^{(2)} c(t,\theta) + (\varepsilon^{(L)}(A(t,\theta)) + s(A(t,\theta))) \cdot e_1(t,\theta)$ (23). We shall get the functions $\varepsilon^{(L)}(A(t,\theta))$ and $s(A(t,\theta))$ such that the supervectors $f_2(t,\theta)$ and $e_1(t,\theta)$ to be orthogonal, that is,

(24) $\quad \varepsilon^{(L)}(\langle f_2(t,\theta), e_1(t,\theta)\rangle) = 0$

$(t,\theta) \in V \subset B_L^{1,1}$. From (23) and (24), we have

$$0 = \langle G_1^{(2)} c(t,\theta), e_1(t,\theta)\rangle + (\varepsilon^{(L)}(A(t,\theta)) + s(A(t,\theta))) \cdot \langle e_1(t,\theta), e_1(t,\theta)\rangle$$

from which we obtain

$$0 = \varepsilon^{(L)}(\langle G_1^{(2)} c(t,\theta), e_1(t,\theta)\rangle) + s(\langle G_1^{(2)} c(t,\theta), e_1(t,\theta)\rangle) +$$

$$+\varepsilon^{(L)}(A(t,\theta)) \cdot \varepsilon^{(L)}(\langle e_1(t,\theta), e_1(t,\theta)\rangle) + \varepsilon^{(L)}(A(t,\theta)) \cdot s(\langle e_1(t), e_1(t)\rangle) +$$

$$+s(A(t,\theta)) \cdot s(\langle e_1(t,\theta), e_1(t,\theta)\rangle) + s(A(t,\theta)) \cdot \varepsilon^{(L)}(\langle e_1(t,\theta), e_1(t,\theta)\rangle).$$

Thus we have

$$\varepsilon^{(L)}(\langle G_1^{(2)} c(t,\theta), e_1(t,\theta)\rangle) + \varepsilon^{(L)}(A(t,\theta)) \cdot \varepsilon^{(L)}(\langle e_1(t,\theta), e_1(t,\theta)\rangle) = 0$$

and

$$s(\langle G_1^{(2)} c(t,\theta), e_1(t,\theta)\rangle) + \varepsilon^{(L)}(A_B(t,\theta)) \cdot s(\langle e_1(t,\theta), e_1(t,\theta)\rangle) +$$

$$+s(A(t,\theta)) \cdot s(\langle e_1(t,\theta), e_1(t,\theta)\rangle) + s(A(t,\theta)) \cdot \varepsilon^{(L)}(\langle e_1(t,\theta), e_1(t,\theta)\rangle) = 0.$$

Therefore

(25) $\quad \varepsilon^{(L)}(A(t,\theta)) = -\varepsilon^{(L)}(\langle G_1^{(2)} c(t,\theta), e_1(t,\theta)\rangle)$

and

(26) $s(A(t,\theta)) = (-s(\langle G_1^{(2)} c(t,\theta), e_1(t,\theta)\rangle) - \varepsilon^{(L)}(\langle G_1^{(2)} c(t,\theta), e_1(t,\theta)\rangle)\cdot$

$$\cdot s(\langle e_1(t), e_1(t)\rangle)) \cdot (1 + s(\langle e_1(t), e_1(t)\rangle))^{-1}.$$

From (23), (25) and (26), we get

(27) $$f_2(t,\theta) = G_1^{(2)} c(t,\theta) + (-\varepsilon^{(L)}(\langle G_1^{(2)} c(t,\theta), e_1(t,\theta)\rangle) + s(A(t,\theta))) \cdot e_1(t,\theta).$$

Because the supervectors $G_1 c(t,\theta)$ and $G_1^{(2)} c(t,\theta)$ are linear independent $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ from (27) we have $\varepsilon^{(L)}(\langle f_2(t,\theta), f_2(t,\theta\rangle) \neq 0$ $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$. Therefore $||f_2(t,\theta)||' \neq 0$ $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.

We set $e_2(t,\theta) = f_2(t,\theta) \cdot (||f_2(t,\theta)||')^{-1}$ (28).

We note that $||e_1(t,\theta)||' = ||e_2(t,\theta)||' = 1$ but $\langle e_1(t,\theta), e_1(t,\theta)\rangle = 1 + s(\langle e_1(t,\theta), e_1(t,\theta)\rangle)$ where $s(\langle e_1(t,\theta), e_1(t,\theta)\rangle)$ has not importance and we have the same for $\langle e_2(t,\theta), e_2(t,\theta)\rangle$ . From the formulas (22), (27) and (28) formulas we have:

(29) $$G_1 c(t,\theta) = ||G_1 c(t,\theta)||' \cdot e_1(t,\theta)$$

and

(30) $$G_1^{(2)} c(t,\theta) = \varepsilon^{(L)}(\langle G_1^{(2)} c(t,\theta), e_1(t,\theta)\rangle) - s(A(t,\theta)) \cdot e_1(t,\theta) + ||f_2(t,\theta)||' \cdot e_2(t,\theta).$$

The relations (29) and (30) show us that

$$span(G_1 c(t,\theta), G_1^{(2)} c(t,\theta)) = span(e_1(t,\theta), e_2(t,\theta)).$$

Because of

$$\varepsilon^{(L)}\left(\begin{vmatrix} ||G_1 c(t,\theta)||' & 0 \\ \varepsilon^{(L)}(\langle G_1^{(2)} c(t,\theta), e_1(t,\theta)\rangle) - s(A(t,\theta)) & ||f_2(t,\theta)||' \end{vmatrix}\right) > 0,$$

$(t,\theta) \in V \subset B_L^{1,1}$ it results that the systems of supervectors $\{G_1 c(t,\theta), G_1^{(2)} c(t,\theta)\}$ and $\{e_1(t,\theta), e_2(t,\theta)\}$ are directed in the same way $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.

We assume that we have constructed the unit, orthogonal two by two supervectors $e_1(t,\theta), \ldots, e_{h-1}(t,\theta)$ $(h < m)$ with the properties that:

$$span(G_1 c(t,\theta), G_1^{(2)} c(t,\theta), \ldots, G_1^{(h-1)} c(t,\theta)) = span(e_1(t,\theta), \ldots, e_{h-1}(t,\theta))$$

and the systems of supervectors $\{G_1 c(t,\theta), G_1^{(2)} c(t,\theta), \ldots, G_1^{(h-1)} c(t,\theta)\}$ and $\{e_1(t,\theta), \ldots, e_{h-1}(t,\theta)\}$ are directed in the same way $(\forall)(t,\theta) \in V \subset B_L^{1,1}$. Thus we construct the supervector $f_h(t,\theta)$ :

(31) $f_h(t,\theta) = G_1^{(h)} c(t,\theta) + \sum_{k=1}^{h-1} (\varepsilon^{(L)}(A_k(t,\theta)) + s(A_k(t,\theta))) \cdot e_k(t,\theta), \quad h < m$

where $(t,\theta) \to A_k(t,\theta)$ are supersmooth functions $(\forall)\ k \in \{1,\ldots,h-1\}$ which will be determined by the conditions:

(32) $\langle f_h(t,\theta), e_i(t,\theta)\rangle = 0, \quad h < m, \quad i \in \{1,\ldots,h-1\}.$

By (31) and (32) we get

$$\langle G_1^{(h)}c(t,\theta), e_i(t,\theta)\rangle + \sum_{k=1}^{h-1}(\varepsilon^{(L)}(A_k(t,\theta)) + s(A_k(t,\theta)))\cdot\langle e_k(t,\theta), e_i(t,\theta)\rangle = 0$$

$(\forall)\ i \in \{1,\ldots,h-1\}$ or equivalent to

(33) $\langle G_1^{(h)}c(t,\theta), e_i(t,\theta)\rangle + (\varepsilon^{(L)}(A_i(t,\theta)) + s(A_i(t,\theta)))\cdot(1 + s(\langle e_i(t,\theta), e_i(t,\theta)\rangle)) = 0$

$(\forall)\ i \in \{1,\ldots,h-1\}$ . From (33) we have:

(34) $\varepsilon^{(L)}(\langle G_1^{(h)}c(t,\theta), e_i(t,\theta)\rangle) + \varepsilon^{(L)}(A_i(t,\theta)) = 0$

and

(35) $s(\langle G_1^{(h)}c(t,\theta), e_i(t,\theta)\rangle) + \varepsilon^{(L)}(A_i(t,\theta))\cdot s(\langle e_i(t,\theta), e_i(t,\theta)\rangle) +$

$$+s(A_i(t,\theta))\cdot(1 + s(\langle e_i(t,\theta), e_i(t,\theta)\rangle)) = 0 \quad (\forall)\ h < m,\ i \in \{1,\ldots,h-1\}.$$

Because the supervectors $G_1c(t,\theta), G_1^{(2)}c(t,\theta),\ldots,G_1^{(h)}c(t,\theta)$ $(h < m)$ are linear independent it results that $f_h(t,\theta) \neq 0$.

We put (36) $e_h(t,\theta) = f_h(t,\theta)\cdot(||f_h(t,\theta)||')^{-1}, \quad h < m.$

Thus, we have constructed the unit and orthogonal two by two supervectors $e_1(t,\theta),\ldots,e_{m-1}(t,\theta)$. On the other hand, from (31) and (36) we get the following relations for all $h < m$ :

(37) $G_1^{(h)}c(t,\theta) = (\varepsilon^{(L)}(\langle G_1^{(h)}c(t,\theta), e_1(t,\theta)\rangle) - s(A(t,\theta)))\cdot e_1(t,\theta) + \cdots$

$$\cdots + (\varepsilon^{(L)}(\langle G_1^{(h)}c(t,\theta), e_{h-1}(t)\rangle) - s(A_{h-1}(t,\theta)))\cdot e_{h-1}(t,\theta) + ||f_h(t,\theta)||'\cdot e_h(t,\theta).$$

From (29), (30) and (37) we obtain:

$$span(G_1c(t,\theta), G_1^{(2)}c(t,\theta),\ldots,G_1^{(h)}c(t,\theta)\}) = span(e_1(t,\theta),\ldots,e_h(t,\theta)).$$

Taking account of (29), (30), and (37) we get that the " body " of the determinant of the matrix of the linear transformation when we change from the basis $\{e_1(t,\theta),\ldots,e_h(t,\theta)\}$ to the basis $\{G_1c(t,\theta), G_1^{(2)}c(t,\theta),\ldots,G_1^{(h)}c(t,\theta)\}$ $(h < m)$ $\varepsilon^{(L)}(\Delta(t,\theta))$ is given by:

$$\varepsilon^{(L)}(\Delta(t,\theta)) = ||f_1(t,\theta)||'\cdot\ \cdots\ \cdot||f_h(t,\theta)||' > 0 \quad (\forall)\ (t,\theta) \in V \subset B_L^{1,1}.$$

Therefore $\varepsilon^{(L)}(\Delta(t,\theta)) > 0$ $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ and the systems of supervectors $\{e_1(t,\theta),\ldots,e_h(t,\theta)\}$ and $\{G_1c(t,\theta), G_1^{(2)}c(t,\theta),\ldots,G_1^{(h)}c(t,\theta)\}$ $(h < m)$ are directed in the same way $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.

By our construction, the functions $(t,\theta) \to e_k(t,\theta)$ $(\forall)$ $k \in \{1,\ldots,m-1\}$ and $(t,\theta) \to e_{m+j}(t,\theta)$ $(\forall)$ $j \in \{1,\ldots,n\}$ are supersmooth.
We shall get $e_m(t,\theta)$ from the relations

(38) $\langle e_m(t,\theta), e_k(t,\theta)\rangle = 0$ $(\forall)$ $k \in \{1,\ldots,m-1\}$

and

(38) $\langle e_m(t,\theta), e_{m+j}(t,\theta)\rangle = 0$ $(\forall)$ $j \in \{1,\ldots,n\}$

Thus, we have:

$$e_m^1(t,\theta)\cdot e_1^1(t,\theta) + \cdots + e_m^m(t,\theta)\cdot e_1^m(t,\theta) + e_m^{m+1}(t,\theta)\cdot e_1^{m+r+1}(t,\theta) + \cdots$$

$$\cdots + e_m^{m+r}(t,\theta)\cdot e_1^{m+n}(t,\theta) - e_m^{m+r+1}(t,\theta)\cdot e_1^{m+1}(t,\theta) - \ldots - e_m^{m+n}(t,\theta)\cdot e_1^{m+r}(t,\theta) = 0$$

$\vdots$

(39)

$\vdots$

$$e_m^1(t,\theta)\cdot e_{m+n}^1(t,\theta) + \cdots + e_m^m(t,\theta)\cdot e_{m+n}^m(t,\theta) + e_m^{m+1}(t,\theta)\cdot e_{m+n}^{m+r+1}(t,\theta) + \cdots$$

$$\cdots + e_m^{m+r}(t,\theta)\cdot e_{m+n}^{m+n}(t,\theta) - e_m^{m+r+1}(t,\theta)\cdot e_{m+n}^{m+1}(t,\theta) - \ldots - e_m^{m+n}(t,\theta)\cdot e_{m+n}^{m+r}(t,\theta) = 0$$

where $e_k^1(t,\theta),\ldots,e_k^{m+n}(t,\theta)$ are the components of the supervector $e_k(t,\theta)$ $(\forall)$ $k \in \{1,\ldots,m\}$ and $e_{m+j}^1(t,\theta),\ldots,e_{m+j}^{m+n}(t,\theta)$ are the components of the supervector $e_{m+j}(t,\theta)$ $(\forall)$ $j \in \{1,\ldots,n\}$.

Let us consider (39) as a linear and homogeneous system of $m+n-1$ equations with the $m+n$ unknowns $e_m^1(t,\theta),\ldots,e_m^{m+n}(t,\theta)$. Because the supervectors $e_1(t,\theta),\ldots,e_{m-1}(t,\theta),e_{m+1}(t,\theta),\ldots,e_{m+n}(t,\theta)$ $(\forall)$ $(t,\theta)\in V\subset B_L^{1,1}$ are linear independent, it follows that the rank of the matrix $M(t,\theta)$ :

$$\begin{pmatrix}
e_1^1 & \cdots & e_1^m & e_1^{m+r+1} & \cdots & e_1^{m+n} & -e_1^{m+1} & \cdots & -e_1^{m+r}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
e_{m-1}^1 & \cdots & e_{m-1}^m & e_{m-1}^{m+r+1} & \cdots & e_{m-1}^{m+n} & -e_{m-1}^{m+1} & \cdots & -e_{m-1}^{m+r}\\
e_{m+1}^1 & \cdots & e_{m+1}^m & e_{m+1}^{m+r+1} & \cdots & e_{m+1}^{m+n} & -e_{m+1}^{m+1} & \cdots & -e_{m+1}^{m+r}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\
e_{m+n}^1 & \cdots & e_{m+n}^m & e_{m+n}^{m+r+1} & \cdots & e_{m+n}^{m+n} & -e_{m+n}^{m+1} & \cdots & -e_{m+n}^{m+r}
\end{pmatrix}$$

is $m+n-1$, where $e_q^s$ means $e_q^s(t,\theta)$ with $s\in\{1,\ldots,m+n\}$ and $q\in\{1,\ldots,m-1,m+1,\ldots,m+n\}$. Let $\Delta_q(t,\theta)$ be the minor of order $m+n-1$ obtained by omitting the $q$ colunm from the matrix $M(t,\theta)$ $(\forall)$ $q\in\{1,\ldots,m+n\}$ and $(\forall)$ $(t,\theta)\in V\subset B_L^{1,1}$. Then from (39), we get:

(40) $e_m^q(t,\theta) = (-1)^{q-1}\cdot\nu(t,\theta)\cdot\Delta_q(t,\theta),$ $(\forall)$ $q\in\{1,\ldots,m+n\},$

where $\nu(t,\theta)$ has to fulfil the condition:

(41) $\quad ||e_m(t,\theta)||' = 1.$

Because of $rank(M(t,\theta)) = m+n-1$ we have:

(42) $\quad \varepsilon^{(L)}(\Delta(t,\theta)) = (\varepsilon^{(L)}(\Delta_1(t,\theta)))^2 + \cdots + (\varepsilon^{(L)}(\Delta_m(t,\theta)))^2 > 0.$

From (40) and (41) we get:

(43) $\quad \nu(t,\theta) = \eta \cdot (\sqrt{(\Delta(t,\theta))})^{-1}$

where $\eta = 1$ or $\eta = -1$; this resulting from the condition that the frame $\{e_1(t,\theta),\ldots,e_{m+n}(t,\theta)\}$ to be positiv directed.
By (40) and (43) we get:

(44) $\quad e^q_m(t,\theta) = (-1)^{q-1}\cdot\eta\cdot\Delta_q(t,\theta)\cdot(\sqrt{(\Delta(t,\theta))})^{-1} \quad (\forall)\ q\in\{1,\ldots,m+n\},$

where $\eta$ verify the conditions $|\eta| = 1$ and

$$\varepsilon^{(L)}(sdet(e^s_q(t,\theta))_{1\le s,q\le m+n}) > 0.$$

By (44) and (41) it follows that the functions $(t,\theta)\to e^s_m(t,\theta)$, $1\le s\le m+n$ are supersmooth and the function $(t,\theta)\to e_m(t,\theta)$ is supersmooth.
From our construction results (19) and $(19^V)$ from the first part of this paper.
The uniqueness of the Frenet frame results by our construction.
We fix an index $k\in\{1,\ldots,m\}$. We express $G_1e_k(t,\theta)$ in the frame $\{e_1(t,\theta),\ldots,e_{m+n}(t,\theta)\}$ and we have:

(45) $\quad G_1e_k(t,\theta) = \sum_{h=1}^m a_{kh}(t,\theta)\cdot e_h(t,\theta) + \sum_{j=1}^n a_{kj}(t,\theta)\cdot e_{m+j}(t,\theta).$

Computing the scalar product between the (45) relations and $e_i(t,\theta)$ $(\forall)\ i\in\{1,\ldots,m\}$ we get:

(46) $\quad a_{ki}(t,\theta) = \langle G_1e_k(t,\theta), e_i(t,\theta)\rangle$

because $\langle e_{m+j}(t,\theta), e_i(t,\theta)\rangle = 0 \quad (\forall)\ j\in\{1,\ldots,n\}$ and $(\forall)\ (t,\theta)\in V\subset B^{1,1}_L$.
Thus we proved the (18) relation $(\forall)\ k,i\in\{1,\ldots,m\}$ and $(\forall)\ (t,\theta)\in V\subset B^{1,1}_L$.
Computing the scalar product between the (45) relations and $e_{m+l}(t,\theta)$ $(\forall)\ l\in\{1,\ldots,n\}$ we get:

(47) $\quad a_{k\ m+l}(t,\theta) = \langle G_1e_k(t,\theta), e_{m+l}(t,\theta)\rangle$

because $\langle e_h(t,\theta), e_{m+l}(t,\theta)\rangle = 0 \quad (\forall)\ h\in\{1,\ldots,m\}$ and $(\forall)\ (t,\theta)\in V\subset B^{1,1}_L$. Thus we proved the (19) relation $(\forall)\ h\in\{1,\ldots,m\}$, $l\in\{1,\ldots,n\}$ and $(\forall)\ (t,\theta)\in V\subset B^{1,1}_L$.

Derivating the following relation by $G_1$

$$\langle e_k(t,\theta), e_h(t,\theta)\rangle = \delta_{kh} \qquad (\forall)\ k,h \in \{1,\ldots,m\},\ (\forall)(t,\theta) \in V \subset B_L^{1,1}.$$

we get:

$$\langle G_1 e_k(t,\theta), e_h(t,\theta)\rangle + \langle e_k(t,\theta), G_1 e_h(t)\rangle = 0 \qquad (\forall)\ k,h \in \{1,\ldots,m\},$$

$(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.
By the (46) relation we have:

$$a_{kh}(t,\theta) + a_{hk}(t,\theta) = 0$$

that means we proved the (6) relation $(\forall)\ k,h \in \{1,\ldots,m\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.
Because $\{e_1(t,\theta),\ldots,e_{m+n}(t,\theta)\}$ is a Frenet frame we have:

(48) $G_1^{(k)} c(t,\theta) \in span(e_1(t,\theta),\ldots,e_k(t,\theta)),\quad (\forall)\ k \in \{1,\ldots,m-1\}$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ and

(49) $e_k(t,\theta) \in span(G_1 c(t,\theta),\ldots,G_1^{(k)} c(t,\theta))\quad (\forall)\ k \in \{1,\ldots,m-1\}$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.
By (48) and (49) we get:

(50) $G_1 e_k(t,\theta) \in span(G_1 c(t,\theta),\ldots,G_1^{(k)} c(t,\theta), G_1^{(k+1)} c(t,\theta))$

$(\forall)\ k \in \{1,\ldots,m-1\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.
By (48) and (50) we have:

(51) $G_1 e_k(t) \in span(e_1(t,\theta),\ldots,e_{k+1}(t,\theta))\quad (\forall)\ k \in \{1,\ldots,m-1\}$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.
By (51), we note that in the writing

$$G_1 e_k(t,\theta) = \sum_{h=1}^{m} a_{kh}(t,\theta)\cdot e_h(t,\theta) + \sum_{j=1}^{n} a_{k\ m+j}(t,\theta)\cdot e_{m+j}(t,\theta) \quad (\forall)\ k \in \{1,\ldots,m\}$$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ the coeficients $a_{kh}(t,\theta)$ are zero if $h > k+1$ and $a_{k\ m+j}(t,\theta) = 0\ \ (\forall)\ k \in \{1,\ldots,m\},\ \ j \in \{1,\ldots,n\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.
Thus, we get the (5) formula $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ :

$$G_1 e_k(t,\theta) = \sum_{h=1}^{m} a_{kh}(t,\theta)\cdot e_h(t,\theta) \quad (\forall)\ k \in \{1,\ldots,m\}$$

and the (7) formula $a_{kh}(t,\theta) = 0$ if $h > k+1$, $(\forall)\ k,h \in \{1,\ldots,m\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.

Because of our construction of the frame $\{e_1(t,\theta),\ldots,e_{m+n}(t,\theta)\}$, we have:

(52) $G_1^{(j-1)} G_2 c(t,\theta) \in span(e_{m+j}(t,\theta))$,

and

(53) $e_{m+j}(t,\theta) \in span(G_1^{(j-1)} G_2 c(t,\theta))$

$(\forall)\ j \in \{1,\ldots,n\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.
From now, we have:

(54) $G_1 e_{m+j}(t,\theta) \in span(G_1^{(j)} G_2 c(t,\theta))$

$(\forall)\ j \in \{1,\ldots,n\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.
By the (52) and (54) relations we get

(55) $G_1 e_{m+j}(t,\theta) \in span(e_{m+j+1}(t,\theta))$

$(\forall)\ j \in \{1,\ldots,n\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.
By (55) we note that in writing

$$G_1 e_{m+j}(t,\theta) = \sum_{k=1}^{m} a_{m+j\ k}(t,\theta)\cdot e_k(t,\theta) + \sum_{l=1}^{n} a_{m+j\ m+l}(t,\theta)\cdot e_{m+j}(t,\theta)$$

$(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$, the coeficients $a_{m+j\ m+l}(t,\theta)$ are zero if $l \neq j+1$ and $a_{m+j\ k}(t,\theta) = 0$ $(\forall)\ k \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$ $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.

Thus we proved the (8), (12) and (13) relations.

We fix an index $j_1 \in \{1,\ldots,r\}$. We express $G_1 e_{m+j_1}(t,\theta)$ in the frame $\{e_1(t,\theta),\ldots,e_{m+n}(t,\theta)\}$. We have:

(56) $G_1 e_{m+j_1}(t,\theta) = \sum_{h=1}^{m} a_{m+j_1\ h}(t,\theta)\cdot e_h(t,\theta) + \sum_{j=1}^{n} a_{m+j_1\ m+j}(t)\cdot e_{m+j}(t,\theta)$

$(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.
Computing the scalar product between the (56) relation and $e_k(t,\theta)$ $(\forall)\ k \in \{1,\ldots,m\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ we get:

(57) $a_{m+j_1\ k}(t) = \langle G_1 e_{m+j_1}(t,\theta), e_k(t,\theta)\rangle$ $(\forall)\ k \in \{1,\ldots,m\}$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ because $\langle e_{m+j}(t,\theta), e_k(t,\theta)\rangle = 0$ $(\forall)\ k \in \{1,\ldots,m\}$, $(\forall)\ j \in \{1,\ldots,n\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$. Thus, we proved the (20) relation $(\forall)\ j_1 \in \{1,\ldots,r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.
Computing the scalar product between the (56) relation and $e_{m+j_2}(t,\theta)$ $(\forall)\ j_2 \in \{1,\ldots,r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$. we get:

(58) $a_{m+j_1\ m+r+j_2}(t,\theta) = -\langle G_1 e_{m+j_1}(t,\theta), e_{m+j_2}(t,\theta)\rangle$ $(\forall)\ j_2 \in \{1,\ldots,r\}$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$. Thus we proved the (15) relation $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$. Computing the scalar product between the (56) relation and $e_{m+r+j_2}(t,\theta)$ $(\forall)\ j_2 \in \{1,\dots,r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ we get:

$$(59) \quad a_{m+j_1\ m+j_2}(t,\theta) = \langle G_1 e_{m+j_1}(t,\theta), e_{m+r+j_2}(t,\theta)\rangle \quad (\forall)\ j_2 \in \{1,\dots,r\}$$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$. Thus, we proved the (14) relation $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.

We fix an index $j_1 \in \{1,\dots,r\}$. We express $G_1 e_{m+r+j_1}(t,\theta)$ in the frame $\{e_1(t,\theta),\dots,e_{m+n}(t,\theta)\}$. We have:

$$(60) \quad G_1 e_{m+r+j_1}(t,\theta) = \sum_{h=1}^m a_{m+r+j_1\ h}(t)\cdot e_h(t,\theta) +$$

$$+\sum_{j=1}^n a_{m+r+j_1\ m+j}(t,\theta)\cdot e_{m+j}(t,\theta) \quad (\forall)\ (t,\theta) \in V \subset B_L^{1,1}.$$

Computing the scalar product between the (60) relation and $e_k(t,\theta)$ $(\forall)\ k \in \{1,\dots,m\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ we get:

$$(61) \quad a_{m+r+j_1\ k}(t,\theta) = \langle G_1 e_{m+r+j_1}(t,\theta), e_k(t,\theta)\rangle \quad (\forall)\ k \in \{1,\dots,m\}$$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ because $\langle e_{m+j}(t,\theta), e_k(t,\theta)\rangle = 0$ $(\forall)\ k \in \{1,\dots,m\}$, $(\forall)\ j \in \{1,\dots,n\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$. Thus, by (57) and (61), we proved the (20) relation $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.
Computing the scalar product between the (60) relation and $e_{m+j_2}(t,\theta)$ $(\forall)\ j_2 \in \{1,\dots,r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ we get:

$$(62) \quad a_{m+r+j_1\ m+r+j_2}(t,\theta) = -\langle G_1 e_{m+r+j_1}(t,\theta), e_{m+j_2}(t,\theta)\rangle \quad (\forall)\ j_2 \in \{1,\dots,r\}$$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$. Thus we proved the (17) relation $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$. Computing the scalar product between the (60) relation and $e_{m+r+j_2}(t,\theta)$ $(\forall)\ j_2 \in \{1,\dots,r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ we get:

$$(63) \quad a_{m+r+j_1\ m+j_2}(t,\theta) = \langle G_1 e_{m+r+j_1}(t,\theta), e_{m+r+j_2}(t,\theta)\rangle \quad (\forall)\ j_2 \in \{1,\dots,r\}$$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$. Thus, we proved the (16) relation $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$.

Derivating the following relation by $G_1$

$$\langle e_{m+j_1}(t,\theta), e_{m+r+j_2}(t,\theta)\rangle = \delta_{j_1 j_2} \qquad (\forall)\ j_1, j_2 \in \{1,\dots,r\}$$

we get

$$\langle G_1 e_{m+j_1}(t,\theta), e_{m+r+j_2}(t,\theta)\rangle + \langle e_{m+j_1}(t,\theta), G_1 e_{m+r+j_2}(t,\theta)\rangle = 0$$

$(\forall)\ j_1, j_2 \in \{1,\dots,r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ which is equivalent to

$$\langle G_1 e_{m+j_1}(t,\theta), e_{m+r+j_2}(t,\theta)\rangle - \langle G_1 e_{m+r+j_2}(t,\theta), e_{m+j_1}(t,\theta)\rangle = 0$$

$(\forall)\ j_1, j_2 \in \{1, \ldots, r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ and using the (14) and (17) relations we get:

$$a_{m+j_1\ m+j_2}(t,\theta) + a_{m+r+j_2\ m+r+j_1}(t,\theta) = 0 \quad (\forall)\ j_1, j_2 \in \{1, \ldots, r\}$$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ which is the (9) relation.

Derivating the following relation by $G_1$

$$\langle e_{m+j_1}(t,\theta), e_{m+j_2}(t,\theta)\rangle = 0 \qquad (\forall)\ j_1, j_2 \in \{1, \ldots, r\}$$

we get

$$\langle G_1 e_{m+j_1}(t,\theta), e_{m+j_2}(t,\theta)\rangle + \langle e_{m+j_1}(t,\theta), G_1 e_{m+j_2}(t,\theta)\rangle = 0 \quad (\forall)\ j_1, j_2 \in \{1, \ldots, r\}$$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ which is equivalent to

$$\langle G_1 e_{m+j_1}(t,\theta), e_{m+j_2}(t,\theta)\rangle - \langle G_1 e_{m+j_1}(t,\theta), e_{m+j_1}(t,\theta)\rangle = 0 \quad (\forall)\ j_1, j_2 \in \{1, \ldots, r\}$$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ and using the (15) relation we get:

$$-a_{m+j_1\ m+r+j_2}(t,\theta) + a_{m+j_2\ m+r+j_1}(t,\theta) = 0 \quad (\forall)\ j_1, j_2 \in \{1, \ldots, r\}$$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ which is the (11) relation.

Derivating the following relation by $G_1$

$$\langle e_{m+r+j_1}(t,\theta), e_{m+r+j_2}(t,\theta)\rangle = 0 \qquad (\forall)\ j_1, j_2 \in \{1, \ldots, r\}$$

we get

$$\langle G_1 e_{m+r+j_1}(t,\theta), e_{m+r+j_2}(t,\theta)\rangle + \langle e_{m+r+j_1}(t,\theta), G_1 e_{m+r+j_2}(t,\theta)\rangle = 0$$

$(\forall)\ j_1, j_2 \in \{1, \ldots, r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ which is equivalent to

$$\langle G_1 e_{m+r+j_1}(t,\theta), e_{m+r+j_2}(t,\theta)\rangle - \langle G_1 e_{m+r+j_2}(t,\theta), e_{m+r+j_1}(t,\theta)\rangle = 0$$

$(\forall)\ j_1, j_2 \in \{1, \ldots, r\}$ and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ and using the (16) relation we get:

$$a_{m+r+j_1\ m+j_2}(t,\theta) - a_{m+r+j_2\ m+j_1}(t,\theta) = 0 \quad (\forall)\ j_1, j_2 \in \{1, \ldots, r\}$$

and $(\forall)\ (t,\theta) \in V \subset B_L^{1,1}$ which is the (10) relation. $\diamondsuit$

**Corollary 1.** The (5) and (8) relations extend the Frenet formulas for the curves.

**Remark 1.** By (6), (7), (12), (13), (9), (10) and (11) we get:

$$\mathcal{A} = (a_{sq})_{1\le s,q\le m+n} = \begin{pmatrix} A_1 & A_3 \\ A_4 & A_2 \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} 0 & a_{12} & \cdots & 0 & 0 \\ -a_{12} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & a_{m-1\ m} \\ 0 & 0 & \cdots & -a_{m-1\ m} & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} a_{m+1\ m+1} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{m+r\ m+r} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -a_{m+1\ m+1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & -a_{m+r\ m+r} \end{pmatrix}$$

and

$$A_3 = A_4 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

**Example 1.** Let ${B_L}^{2+2}$ be the $(2,2)-$dimensional total supereuclidean space, $V$ be an open subset of $B_L^{1,1}$ and $c : V \subset B_L^{1,1} \to {B_L}^{2+2}$ be a supersmooth supercurve given by

$$c(t,\theta) = (t^2, \theta \cdot \beta^2, \theta + 2 \cdot \beta^1 \cdot t, \theta \cdot t^2)$$

where $V \cap \mathbf{R}$ be the open set $(0,1)$ .

We note that the supercurve $c$ is supersmooth because the functions

$$c^1(t,\theta) = t^2, \qquad c^2(t,\theta) = \theta \cdot \beta^2, \qquad c^3(t,\theta) = \theta + 2 \cdot \beta^1 \cdot t, \qquad c^4(t,\theta) = \theta \cdot t^2$$

are supersmooth. Let us compute $G_1 c(t,\theta)$, $G_2 c(t,\theta)$, and $G_1 G_2 c(t,\theta)$ :

$$G_1 c(t,\theta) = (2 \cdot t, 0, 2 \cdot \beta^1, 2 \cdot \theta \cdot t)$$

$$G_2 c(t,\theta) = (0, \beta^2, 1, t^2) \qquad and \qquad G_1 G_2 c(t,\theta) = (0,0,0,2 \cdot t).$$

By

$$\varepsilon^{(L)}(\langle G_2 c(t,\theta), G_1 G_2 c(t,\theta)\rangle) = \varepsilon^{(L)}(0 \cdot 0 + \beta^2 \cdot 0 + 1 \cdot 2 \cdot t - t^2 \cdot 0) = 2 \cdot t > 0$$

$(\forall)\ t \in (0,1)$,

$$\varepsilon^{(L)}(\langle G_2 c(t,\theta), G_2 c(t,\theta)\rangle) = \varepsilon^{(L)}(0 \cdot 0 + \beta^2 \cdot \beta^2 + 1 \cdot t^2 - t^2 \cdot 1) = 0$$

$(\forall)\ t \in (0,1)$ and

$$\varepsilon^{(L)}(\langle G_1 G_2 c(t,\theta), G_1 G_2 c(t,\theta)\rangle) = \varepsilon^{(L)}(0 \cdot 0 + 0 \cdot 0 + 0 \cdot 2 \cdot t - 2 \cdot 0 \cdot t) = 0$$

we conclude that the supercurve $c$ fulfils the (1), (2), (3), (4), relations from the Theorem 1.

Computing $G_1c(t,\theta)$, $G_2c(t,\theta)$, and $G_1G_2c(t,\theta)$, we get that the supervectors $\{G_1c(t,\theta), G_2c(t,\theta), G_1G_2c(t,\theta)\}$ are linear independent. Thus we conclude that the supercurve $c$ is in general position.

Let us get the Frenet frame of the supercurve $c$, $\{e_1(t,\theta), e_2(t,\theta), e_3(t,\theta), e_4(t,\theta)\}$ and the matrix $\mathcal{A} = (a_{sq})_{1\le s,q\le 4}$. Let $f_1(t,\theta)$ be $G_1c(t,\theta) = (2\cdot t, 0, 2\cdot\beta^1, 2\cdot\theta\cdot t)$ and

$$\langle G_1c(t,\theta), G_1c(t,\theta)\rangle = 2\cdot t\cdot 2\cdot t + 0\cdot 0 + 2\cdot\beta^1\cdot 2\cdot\theta\cdot t - 2\cdot\theta\cdot 2\cdot\beta^1\cdot t = 4\cdot t^2 + 4\cdot\beta^1\cdot\theta\cdot t - 4\cdot\theta\cdot\beta^1\cdot t$$

We have $\langle G_1c(t,\theta), G_1c(t,\theta)\rangle = 4\cdot t^2 + 8\cdot\beta^1\cdot\theta\cdot t$ and $\varepsilon^{(L)}(\langle G_1c(t,\theta), G_1c(t,\theta)\rangle) = = 4\cdot t^2$. Thus we get $||G_1c(t,\theta)||' = \sqrt{\varepsilon^{(L)}(\langle G_1c(t,\theta), G_1c(t,\theta)\rangle)} = \sqrt{4\cdot t^2} = 2\cdot t$.

By the above theorem 1 we compute the Frenet frame of $c$, $\{e_1(t,\theta), e_2(t,\theta), e_3(t,\theta), e_4(t,\theta)\}$ and we get

$$e_1(t,\theta) = (1, 0, 2\cdot\beta^1\cdot(2\cdot t)^{-1}, \theta),$$

$$e_3(t,\theta) = (0, \beta^2\cdot(2\cdot t)^{-1}, (2\cdot t)^{-1}, 2^{-1}\cdot t)$$

and

$$e_4(t,\theta) = (0,0,0,2\cdot t).$$

Let the matrix $M(t,\theta)$ be

$$\begin{pmatrix} 1 & 0 & \theta & -2\cdot\beta^1\cdot(2\cdot t)^{-1} \\ 0 & \beta^2\cdot(2\cdot t)^{-1} & 2^{-1}\cdot t & -(2\cdot t)^{-1} \\ 0 & 0 & 2\cdot t & 0 \end{pmatrix}$$

and computing $e_2(t,\theta)$ we get

$$e_2(t,\theta) = (-2\beta^2\cdot\beta^1\cdot(2\cdot t)^{-1}, 1, 0, -\beta^2).$$

Now, we may compute $a_{12}(t,\theta)$ and $a_{33}(t,\theta)$ and we have

$$a_{12}(t,\theta) = \langle G_1e_1(t,\theta), e_2(t,\theta)\rangle = \beta^1\cdot\beta^2\cdot t^{-2}$$

where

$$G_1e_1(t,\theta) = (0,0,-\beta^1\cdot t^{-2}, 0)$$

and

$$a_{33}(t,\theta) = \langle G_1e_3(t,\theta), e_4(t,\theta)\rangle = -t^{-1},$$

where

$$G_1e_3(t,\theta) = (0, -\beta^2\cdot 2^{-1}\cdot t^{-2}, -2^{-1}\cdot t^{-2}, 2^{-1}).$$

We conclude that

$$\mathcal{A} = (a_{sq})_{1\le s,q\le 4} =$$

$$= \begin{pmatrix} 0 & \beta^1\cdot\beta^2\cdot t^{-2} & 0 & 0 \\ -\beta^1\cdot\beta^2\cdot t^{-2} & 0 & 0 & 0 \\ 0 & 0 & -t^{-1} & 0 \\ 0 & 0 & 0 & t^{-1} \end{pmatrix}.$$

**Acknowledgement**

The author is very grateful to Max-Planck-Institut für Mathematik for financial support and hospitality.

## REFERENCES

1. BARTOCCI, C., BRUZZO, U., HERNANDEZ-RUIPEREZ, D., The Geometry of Supermanifolds, Mathematics and Its Applications, Volume 71, 1991
2. BEREZIN, F.A., Introduction to Superanalysis, D. Reidel Publishing Company, Kluwer Academic Publishers Group Dordrecht, Boston, Lancaster, Tokyo, Volume 7, 1987
3. BEREZIN, F.A., The method of second quantization, Academic Press, New York, 1966
4. DEWITT, B., Supermanifolds, Cambridge, Univ. Press, Cambridge, London, 1984
5. INOUE, A., MAEDA, Y., Foundations of Calculus on Supereuclidean Space based on a Frcehet-Grassmann Algebra, Kodai Math. J. 14(1991)
6. KOBAYASHI, S., NOMIZU, K., Foundations of Differential Geometry, (I) Interscience, New York-London 1963
7. KOSTANT, B., Graded manifolds, Graded Lie Theory and Prequantization, Lect. Notes in Math. no. 570, Springer-Verlag 1977
8. MANIN, Yu.I., Gauge Field Theory and Complex Geometry, A Series of Comprehensive Studies in Mathematics 289, Springer-Verlag 1988
9. NICOLESCU, L., Curs de geometrie, Tipografia Univ. Bucuresti, 1989
10. ROGERS, A., Graded Manifolds, Supermanifolds and Infinite-Dimensional Grassmann Algebras, Commun. Math. Phys. 105, 375-384(1986)
11. ROGERS, A., A global theory of supermanifolds, J. Math. Phys. 21(6), 1352-1365 June 1980
12. SCHEUNERT, M., The theory of Lie superalgebras, Lect. Notes in Math. no. 716, Springer-Verlag 1979

MAX-PLANCK-INSTITUT FÜR MATHEMATIK, PB 7280, D-53072 BONN, GERMANY AND
C.P. 101 O.P. 4 TARGOVISTE 0200 ROMANIA
E-mail address: cristea@mpim-bonn.mpg.de

——————//——————