ON THE GORENSTEIN PROPERTY OF FORM RINGS

M. Hermann, J. Ribbe, P.Schenzel

M. Hermann
J. Ribbe
Mathematisches Institut
der Universität zu Köln
Weyertal 86 - 90
D - 5000 Köln 30

P. Schenzel
Fachbereich Mathematik & Informatik
Martin-Luther-Universität
Halle-Wittenberg
Postfach
D - 4010 Halle - Saale

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

MPI / 92 - 30

ON THE GORENSTEIN PROPERTY OF FORM RINGS

M. Herrmann, J. Ribbe, P. Schenzel

Introduction.

For a Noetherian ring A and an ideal I in A, the blowing up of A along I is given by $\mathrm{Proj}(R(I))$, where $R(I) = \bigoplus_{n\geq 0} I^n t^n$ is the Rees algebra of I. While the local properties of the blowing up and its exceptional fibre $\mathrm{Proj}(G(I))$, where $G(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ is the form ring of I, are the same for I and any power I^r of I, the arithmetic of the specific underlying coordinate rings $R(I)$ and $R(I^r)$ as well as $G(I)$ and $G(I^r)$ are quite different in general.

In this note we relate the Gorenstein property of $G(I^r)$ and $R(I^r)$, $r \geq 1$, to the Gorensteinness of $G(I)$. It is known (s.[2],(3.1)) that - for ideals I of height $ht(I) \geq 2$ - at most one power I^r of I has a Gorenstein Rees ring. One of our main observations in section 2 is that the Gorensteinness of $R(I^r)$ requires - at least for $ht(I) \geq 2$ - the Gorenstein property of the form ring $G(I)$. More precise we prove the following fact (s. theorem (2.3)), which was partly indicated by Ooishi [8] as a question for $\mathfrak{m}$-primary ideals: For any ideal I with $ht(I) \geq 2$ in a local Gorenstein ring $(A,\mathfrak{m})$ the following conditions are equivalent:

(i) $R(I^r)$ is Gorenstein and $R(I)$ is Cohen-Macaulay

(ii) $G(I)$ is Gorenstein with $a(G(I)) = -(r+1)$,

where $a = a(G(I))$ is the a-invariant of $G(I)$, s. [1], (36.13).

By an example we show that this equivalence is not valid if we omit "$R(I)$ is Cohen-Macaulay" in (i).

This theorem (2.3) is a consequence of our key result (s. theorem (2.1)) which shows that for ideals I with $ht(I) \geq 2$ and Cohen-Macaulay form ring $G(I)$ the following statements are equivalent:

(i) $G(I)$ is a Gorenstein ring

(ii) $G(I^r)$ is a Gorenstein ring for all $r \mid a+1$

(iii) There is an integer $r \geq 1$ such that $G(I^r)$ is a Gorenstein ring.

While the assumption on the height of I is essential in our proof of (iii) ⇒ (i) in theorem (2.3), it is not necessary in order to show (i) ⇒ (ii). By an example it follows that theorem (2.1) is not true for an ideal of height zero. It is not clear to the authors whether the implication (iii) ⇒ (i) is also valid under the assumption of height one. In section 3 we present some examples to illustrate this special situation.

Besides of the above-mentioned notation we follow the terminology introduced in [1]. In addition to that we abbreviate by $\mathcal{R}(I) = \bigoplus_{n \in \mathbf{Z}} I^n t^n$ the extended Rees algebra of I, which we use frequently as a technical tool.

We would like to thank N.T. Cuong, C. Ionescu and T. Korb for stimulating discussions during the preparation of this work.

1. The canonical module of certain form rings.

Given a Noetherian ring B we denote by K_B the canonical module of B, provided it exists. Of course K_B is a graded B-module if B is a graded ring. For a local Gorenstein ring A the canonical module of $G(I)$, I an ideal of A, exists since $G(I)$ is the homomorphic image of a Gorenstein ring. The canonical module $K_{G(I)}$ is a finitely generated graded G-module, s.[4], 5.16. So there exists an integer $\min\{n \in \mathbf{Z} : [K_{G(I)}]_n \neq 0\}$ which coincides with the a-invariant of $G(I)$. In the following we give an explicit description of the canonical module of the form rings of the powers of an ideal.

Proposition (1.1). Let I be an ideal in A. Assume $G(I)$ is a Gorenstein ring with a-invariant a. For an arbitrary integer $r \geq 1$ put $a+1 = e+kr$ with $k \in \mathbf{Z}$ and $0 < e \leq r$. Then $G(I^r)$ is a Cohen-Macaulay ring and

$$K_{G(I^r)}(-k+1) \cong \bigoplus_{n\geq -1} I^{nr+e} / I^{(n+1)r+e} \quad .$$

Proof: Let $\mathcal{R}$ denote the extended Rees ring $\mathcal{R}(I)$. Since $\mathcal{R}/(u) \cong G(I)$, $u = t^{-1}$, $\mathcal{R}$ is Gorenstein and the a-invariant of $\mathcal{R}$ is $a(\mathcal{R}) = a+1$, i.e.

$$K_{\mathcal{R}} \cong \mathcal{R}(a+1) \quad .$$

By the basic properties of the r-th Veronesean functor it follows that

$$K_{\mathcal{R}^{(r)}} \cong (K_{\mathcal{R}})^{(r)} \cong (\mathcal{R}(a+1))^{(r)} \cong \bigoplus_{n\in\mathbf{Z}} I^{rn+a+1}t^{rn} \quad .$$

Because of the presentation $a+1 = e+kr$, as given above, by a certain shift it turns out that $K_{\mathcal{R}^{(r)}}$ can be taken as an ideal of $\mathcal{R}^{(r)}$:

$$K_{\mathcal{R}^{(r)}}(-k) \cong \bigoplus_{n\in\mathbf{Z}} I^{rn+e}t^{rn} \cong (I^e, u^r)\mathcal{R}^{(r)} \quad ,$$

where I^e is considered as sitting in degree 0.

Since $\mathcal{R}^{(r)}/(u^r) \cong G(I^r)$ we get for $G := G(I^r)$:

$$K_G(-k+1) \cong (I^e, u^r)\mathcal{R}^{(r)} / u^r(I^e, u^r)\mathcal{R}^{(r)} \quad ,$$

s. [4], 6.3. By an easy calculation we obtain

$$K_G(-k+1) \cong \bigoplus_{n\geq -1} I^{nr+e} / I^{(n+1)r+e}$$

as required.

Corollary (1.2). With the notations and assumptions of (1.1) it follows that

$$\operatorname{type}(G(I^r)) = \begin{cases} 1+\mu(I^e) & \text{for } 0<e<r \\ 1 & \text{for } e=r \end{cases},$$

where $\mu(I^e)$ is the least number of generators of I^e .

Proof: The Cohen-Macaulay type of a Cohen-Macaulay ring - in our case $G(I^r)$ - is given by the minimal number of generators of its canonical module, s.[4],6.11. Put $M = (\mathfrak{m},G_+)$, where G_+ denotes the ideal generated by all forms of positive degree of $G := G(I^r)$. By the homogeneous version of Nakayama's lemma, s.[1],(36.5), the minimal number of generators of K_G is equal to the G/M-vector space dimension of

$$K_G(-k+1) \otimes (G/M) \cong \begin{cases} A/\mathfrak{m} \oplus I^e/\mathfrak{m}I^e & \text{for } 0<e<r \\ A/\mathfrak{m} & \text{for } e=r \end{cases}.$$

Note that the residue field $A/\mathfrak{m}$ is sitting in degree -1 . Counting the dimension proves the claim of (1.2).

As shown in the proof of (1.1) the canonical module $K_{R^{(r)}}(-k)$ can be considered as an ideal of $R^{(r)}$. This observation yields the following corollary.

Corollary (1.3): With the notations and assumptions of (1.1) the graded ring

$$Q_r := \bigoplus_{n\geq 0} I^{rn}/I^{rn+e}$$

is Gorenstein with the a-invariant $a(Q_r) = k$.

Proof: Note that $R^{(r)}$ is Cohen-Macaulay, s.[1],(27.8). Adopting the arguments of [4] in the proof of 6.13 we get from the exact sequence

(1) $0 \longrightarrow K(-k) \longrightarrow B \longrightarrow B/K(-k) \longrightarrow 0$,

where $B := R^{(r)}$ and $K := K_B$, the exact sequence

$$0 \longrightarrow Hom(B,K) \longrightarrow Hom(K(-k),K) \longrightarrow Ext^1(B/K(-k),K_B) \longrightarrow 0 .$$

By definition $Ext^1(B/K(-k),K) \cong K_{B/K(-k)}$, s. [1],(36.14). On the other hand we have $Hom(K,K) \cong B$. Altogether this yields the short exact sequences

(2) $0 \longrightarrow K \longrightarrow B(k) \longrightarrow K_{B/K(-k)} \longrightarrow 0$,

and - by shifting -

(3) $0 \longrightarrow K(-k) \longrightarrow B \longrightarrow K_{B/K(-k)}(-k) \longrightarrow 0$.

Comparing (2) and (3) we obtain

(4) $K_{B/K_B(-k)}(-k) \cong B/K_B(-k)$.

Since $B/K_B(-k) \cong Q_r$, (4) proves $a(Q_r) = k$ and that Q_r is a Gorenstein ring.

2. On the Gorenstein property of Rees and form rings

In this section we shall prove the main results concerning the Gorenstein properties of $G(I^r)$ resp. $R(I^r)$ for an integer $r \geq 1$ and an ideal I .

Theorem (2.1). Let I denote an ideal of a local Gorenstein ring $(A,\mathfrak{m})$. Suppose that $ht\, I \geq 2$ and that $G(I)$ is a Cohen-Macaulay ring with the a-invariant $a = a(G(I))$. Then the following conditions are equivalent:

i) $G(I)$ is a Gorenstein ring.
ii) $G(I^r)$ is a Gorenstein ring for all $r \mid a+1$.
iii) There is an integer $r \geq 1$ such that $G(I^r)$ is a Gorenstein ring.

Proof: The implication i) $\Rightarrow$ ii) is a particular case of (1.2). Moreover, the implication ii) $\Rightarrow$ iii) is obviously true. So let us show iii) $\Rightarrow$ i) for $r \geq 2$ in order to complete the proof. Since $G(I) = \mathcal{R}/u\mathcal{R}$, $\mathcal{R} = \mathcal{R}(I)$, is a Cohen-Macaulay ring it follows that $\mathcal{R}$ is also a Cohen-Macaulay ring. Hence

$$P := \mathcal{R}/u^r\mathcal{R} \cong \bigoplus_{n \geq -r+1} I^n/I^{n+r}$$

is a graded Cohen-Macaulay ring. Put

$$N := (It, u^r)\mathcal{R}/u^r\mathcal{R}$$

the ideal generated by the forms of degree one in P. It is easy to see that

$$P/N \cong \bigoplus_{n=-r+1}^{-1} A/I \quad .$$

Therefore $\operatorname{grade} N = \operatorname{ht} N = \dim A - \dim A/I = \operatorname{ht} I \geq 2$ (note that P is a Cohen-Macaulay ring and $\dim P = \dim A$). Furthermore, there is a natural isomorphism

$$P^{(r)} \cong G(I^r) \quad ,$$

where $P^{(r)}$ denotes the r-th Veronesean subring of P. By the assumption $P^{(r)}$ is a Gorenstein ring and

$$(K_P)^{(r)} \cong K_{P^{(r)}} \cong P^{(r)}(b) = P(rb)^{(r)}$$

for a certain integer b. Now we claim that

$$K_P \cong P(rb) \quad ,$$

i.e., P is a Gorenstein ring. For that choose an element $f \in \left[K_{P^{(r)}}\right]_{-b} \cong \left[K_P\right]_{-rb}$ that generates $K_{P^{(r)}}$ as an $P^{(r)}$-module. Next we show the triviality of its annihilator, i.e.,

$$0 :_P f = \operatorname{Ann}_P f = 0 \quad .$$

To this end recall that

$$(0 :_P f)^{(r)} = 0 :_{P^{(r)}} f = 0$$

because f generates $K_{P^{(r)}}$ as a $P^{(r)}$-module. Therefore there exists a power of N, say N^s, such that

$$N^s(0 :_P f) = 0 \quad .$$

Whence $0 :_P f \subseteq 0 :_P N^s = 0$ because grade $N \geq 2$. Therefore it follows that

$$P = P/\mathrm{Ann}_P f \cong P\, f\,(-rb) \quad .$$

Because $f \in [K_P]_{-rb}$, the graded P-module generated by f is a submodule of K_P. So there is a short exact sequence of graded P-modules

$$0 \longrightarrow P(rb) \longrightarrow K_P \longrightarrow C \longrightarrow 0 \quad ,$$

where C denotes the cokernel of the corresponding embedding. By the choice of f and the Gorenstein property of $P^{(r)}$ the r-th Veronesean functor applied to the short exact sequence yields $C^{(r)} = 0$. That is, there exists an integer s such that $N^s C = 0$. In other words, C is an N-torsion module. But now

$$\mathrm{grade}_N P \geq 2 \quad \text{and} \quad \mathrm{grade}_N K_P \geq 1 \quad .$$

So the above short exact sequence provides that $C = 0$. That is, $K_P \cong P(rb)$ and P is a Gorenstein ring. But then R is a Gorenstein ring and

$$K_R \cong R((b+1)r)$$

as follows because of $P = R/u^r R$. But finally

$$G = R/uR$$

is a Gorenstein ring with $K_G \cong G((b+1)r-1)$. □

The situation that $G(I^r)$ is a Gorenstein ring for all $r \geq 1$ is of a particular interest. As shown in the following it provides the a-invariant $a(G(I))$.

Proposition (2.2). Let I denote an ideal of a local Gorenstein ring $(A,\mathfrak{m})$. Suppose that $ht(I) \geq 2$ and that $G(I)$ is a Cohen-Macaulay ring. Then $G(I^r)$ is a Gorenstein ring for all $r \geq 1$ if and only if $G(I)$ is a Gorenstein ring with $a(G(I)) = -1$.

Proof: It is known that $G(I^r)$ is a Cohen-Macaulay ring provided $G(I)$ is a Cohen-Macaulay ring. So the equivalence turns out by view of 1.2. □

For an ideal I of height at least 2 in a local Gorenstein ring S. Ikeda, s. [7],(3.1), has shown that the Rees ring $R(I)$ is a Gorenstein ring provided $G(I)$ is a Gorenstein ring with $a(G(I)) = -2$. This is the converse to a result shown by C. Huneke, s. [6],1.2 . The implication i) ⇒ ii) of the following theorem extends these results and solves the question of Ooishi mentioned in the introduction (ii) ⇒ i) is due to [2], (3.5)).

Theorem (2.3). Let I be an ideal of a local Gorenstein ring $(A,\mathfrak{m})$. Suppose that $ht\, I \geq 2$. For an integer $r \geq 1$ the following conditions are equivalent:

i) $R(I)$ is a Cohen-Macaulay ring and $R(I^r)$ is a Gorenstein ring.
ii) $G(I)$ is a Gorenstein ring with $a(G(I)) = -(r+1)$.

Proof: First we show i) ⇒ ii) . By [6],1.1, it is known that $G(I)$ is a Cohen-Macaulay ring with $a(G(I)) < 0$. Furthermore, by the main result of [7],(3.1), applied to the Rees ring $R(I^r)$, it turns out that $G(I^r)$ is a Gorenstein ring with $a(G(I^r)) = -2$.

Now (2.1) provides that $G(I)$ is a Gorenstein ring. Put $a(G(I)) = a$. Then by (1.2) and (1.1) it follows that

$$a + 1 = r + (-2)r \quad ,$$

whence $a = -(r+1)$, as required. In order to prove the reverse implication first note that $R(I)$ is a Cohen-Macaulay ring. On the other hand by (1.1) we see that $G(I^r)$ is a Gorenstein ring. Note that $r \mid a(G(I)) + 1$. Furthermore $a(G(I^r)) = -2$ as follows by 1.1 because

$$-r = r + (-2)r \quad .$$

Finally, by [7], (3.1), we see that $R(I^r)$ is a Gorenstein ring.

□

If $r = 1$ in (2.3), then it is just Huneke's result, see [6],1.1 , which we had to use in our proof of (2.3). By view of this particular case one might ask whether the Cohen-Macaulay property of $R(I)$ in i) of (2.3) is superfluous. This is not true as shown in the following example.

Example (2.4). Let k be a field. Put $A = k[[x_1,\dots,x_{11}]]/(x_1^2)$ where $k[[x_1,\dots,x_{11}]]$ denotes the formal power series ring in $x_1,\dots,x_{11}$ over k. Note that A is a hypersurface ring with $\mathfrak{m}^2 = (x_2,\dots,x_{11})\mathfrak{m}$, where $\mathfrak{m}$ denotes the maximal ideal of A . By [10],2.1 , $G(\mathfrak{m})$ is a Gorenstein ring with $a(G(\mathfrak{m})) = -9$. By (2.3) it follows that $R(\mathfrak{m}^8)$ is a Gorenstein ring and $R(\mathfrak{m})$ is a Cohen-Macaulay ring. Now let I denote the ideal of A generated by all monomials of degree 4 in $x_2,\dots,x_{11}$ different from $x_2^2 \cdot x_3^2$. Then $I^2 = \mathfrak{m}^8$ and $R(I^2)$ is a Gorenstein ring. On the other hand there is a short exact sequence

$$0 \longrightarrow R(I) \longrightarrow R(\mathfrak{m}^4) \longrightarrow k \cdot x_2^2 x_3^2 (-1) \longrightarrow 0$$

as easily seen. The Cohen-Macaulayness of $R(\mathfrak{m}^4)$ yields that $R(I)$ is a Buchsbaum ring with $\operatorname{depth} R(I) = 1$ and $\dim R(I) = 11$. For (for $R = R(I)$)

$$H_M^i(R) = 0 \quad , \quad i \neq 1,11 \ , \text{ and } \quad H_M^1(R) \cong k(-1) \quad ,$$

where $M = (\mathfrak{m},R_+)$. Note that even $G(I)$ is not Cohen-Macaulay.

There are several results concerning the Gorenstein property of $R(\mathfrak{m}^r)$, $r \geq 1$, for a local Gorenstein ring $(A,\mathfrak{m})$, see e.g. [2]. It is known that

$r \leq d-1$, $d = \dim A$, provided $R(\mathfrak{m}^r)$ is a Gorenstein ring. The extremal situations $r = d-1$, $d-2$, and $d-3$ are classified in [2],[8]. In addition to these results we consider a lower bound for r . Here $e(A)$ denotes the multiplicity of A .

Proposition (2.5). Let $(A,\mathfrak{m})$ denote a local Gorenstein ring. Assume that $G(\mathfrak{m})$ is a Cohen-Macaulay ring. For an integer $r \leq d-3$, $d = \dim A$, let $R(\mathfrak{m}^r)$ be a Gorenstein ring. Then

$$r \geq \mu(\mathfrak{m}) - e(A) - 1 \quad .$$

Equality holds if and only if either A is a hypersurface or $e(A) = \mu(\mathfrak{m}) - d+2$.

Proof: Without loss of generality we may assume that A has an infinite residue field. By (2.3) $G(\mathfrak{m})$ is a Gorenstein ring with $a(G(\mathfrak{m})) = -(r+1)$. Choose a minimal reduction $\underline{x}$ of $\mathfrak{m}$. Then

$$a(G(\mathfrak{m}) / \underline{x}^*) = d-(r+1) \quad .$$

Let $h_i = \dim_{A/\mathfrak{m}}[G(\mathfrak{m}) / \underline{x}^*]_i$. Then $h_i > 0$ for $0 \leq i \leq d-r-1$ and

$$e(A) = \sum_{i=0}^{d-r-1} h_i \geq 1 + \mu(\mathfrak{m}) - d + d - r - 2 \quad .$$

This shows the inequality. Because $G(\mathfrak{m})/\underline{x}^*$ is a Gorenstein ring $h_i = h_{d-i-r-1}$. That proves the statement about the equality. □

It would be interesting to get a similar result as in (2.5) without the Cohen-Macaulay assumption on $G(\mathfrak{m})$.

3. Ideals of small height.

In the proof of (iii) ⇒ (i) of (2.1) we have decisively used the assumption $ht(I) \geq 2$. For the height zero case this implication is not true as the following example shows.

Example (3.1). Consider the Gorenstein ring $B = k[[t^5, t^6, t^9]]$. Then t^5B is a minimal reduction of the maximal ideal $\mathfrak{n} = (t^5, t^6, t^9) \subset B$ because $\mathfrak{n}^4 = t^5 \cdot \mathfrak{n}^3$. We know by [10],3.6 , that $G(\mathfrak{n}) = \bigoplus_{n\geq 0} \mathfrak{n}^n/\mathfrak{n}^{n-1}$ is Cohen-Macaulay but not Gorenstein. Now let

$$A = B/t^5B \quad \text{and} \quad \mathfrak{m} = \mathfrak{n}/t^5B \quad .$$

A is an Artinian Gorenstein ring and $\mathfrak{m}^4 = 0$. Therefore $G(\mathfrak{m}^4) = A$ is Gorenstein, but

$$G(\mathfrak{m}) = G(\mathfrak{n})/(t^5)^* \quad ,$$

where $(t^5)^*$ is the initial form of t^5 in $G(\mathfrak{n})$, is not Gorenstein, since $(t^5)^*$ is a non-zero-divisor on $G(\mathfrak{n})$.

In the sequel we give examples of height one ideals I in a local Gorenstein ring A for which the above-mentioned implication (iii) ⇒ (i) also holds.

We will say in the following that I is an almost complete intersection ideal if $\mu(I) = ht(I) + 1$ and $\mu(I_P) = ht(P)$ for all $P \in Min(A/I)$.

Proposition (3.2). Let I be an almost complete intersection ideal of height one in a local Gorenstein ring. Assume that

(i) $G(I)$ is not Gorenstein but
(ii) $R(I)$ is Gorenstein.

Then $G(I^r)$ is not Gorenstein for all $r \geq 2$.

Before proving (3.2) we give two examples where the assumptions (i) and (ii) are fulfilled.

Example (3.2). Let $A = k[[X,Y]]$, k a field and X,Y indeterminates, and $I = (X^2, X\cdot Y)$. Note that I is an almost complete intersection of height one. In [3], example 1 it was shown that $R(I)$ is Gorenstein. But $G(I)$ is not Gorenstein s.[11] , example 9.

Example (3.3). Let $A = k[[X,Y,Z]]$ and $I = (X\cdot Y, X\cdot Z)$, s. [3], example 2. Again $R(I)$ is Gorenstein but $G(I)$ is not.

Proof of (3.2): Note that $G(I)$ is Cohen-Macaulay, since $R(I)$ is so. Assume that $G(I^r)$ is Gorenstein for some $r \geq 2$. Then - using the structure-theorem for the canonical module $K_{R(I^r)}$ in [5], (2.5) and also [2], (3.3) - we get

$$K_{R(I^r)} = (1,t)^{-a-2}(-1) \quad ,$$

where $a = a(G(I^r))$, the a-invariant of $G(I^r)$, and $(1,t)^n$ denotes the $R(I^r)$-submodule of the polynomial ring $A[t]$ which is generated by $1,t\ldots,t^n$ in case $n \geq 0$ or $(1,t)^{-1} = I^r R(I^r)$ in case $n = -1$. Moreover by [2], (2.6)

$$a = \left[\frac{a(G(I))}{r}\right]$$

and $a(G(I)) = -\mathrm{ht}(I) = -1$, s. [9]. This means $a = -1$ if $r \geq 2$. Therefore we have

$$(1) \qquad K_{R(I^r)} = (1,t)^{-1}(-1) = I^r R(I^r)(-1) \quad .$$

On the other hand assumption (ii) implies

$$(2) \qquad K_{R(I^r)} \overset{\sim}{=} K_{R(I)}^{(r)} \cong R(I)(-1)^{(r)} \quad .$$

Comparing the degrees in (1) and (2) one gets an A-isomorphism $I^r \cong I^{r-1}$. Since the analytic spread of I is $\ell(I) = 2$ (otherwise I would be a complete intersection), this implies $r = r-1$ by the proof of (3.2) in [2], a contradiction.

Remark (3.4):

a) If I is an almost complete intersection ideal of height one and if A/I is Cohen-Macaulay (so that $G(I)$ is Gorenstein by [11], Corollary 11), then $G(I^r)$ is Gorenstein even for all $r \geq 2$. This follows from (2.2) because $a(G(I)) = -1$.

b) If $(A,\mathfrak{m})$ is a 1-dimensional Gorenstein ring with a Cohen-Macaulay form ring $G(\mathfrak{m})$ and the reduction exponent $\delta(\mathfrak{m}) \leq 3$ we can show that the implication (iii) $\Rightarrow$ (i) in (2.1) holds for the maximal ideal $\mathfrak{m}$.
For instance in the above-mentioned Sally's example (3.1) it turns out that no power of the maximal ideal $\mathfrak{m}$ of the one-dimensional ring $k[[t^5,t^6,t^9]]$ has a Gorenstein form ring. In this case the reduction exponent is $\delta(\mathfrak{m}) = 3$. For us the situation is not clear if $\delta(\mathfrak{m}) \geq 4$.

c) In [10] J. Sally has shown the following result: If $(A,\mathfrak{m})$ is a local Gorenstein ring and $\delta(\mathfrak{m}) = 2$, then $G_A(\mathfrak{m})$ is a Gorenstein ring. Because of the remark given in b) the corresponding result for an $\mathfrak{m}$-primary ideal is not true.

Acknowledgment:

The third named author is grateful to the hospitality of the Max-Planck-Institut für Mathematik (Bonn) during the preparation of this note.

References

[1] M. Herrmann, S. Ikeda, U. Orbanz, Equimultiplicity and Blowing Up, Springer Verlag, Berlin-Heidelberg (1988).

[2] M. Herrmann, J. Ribbe, S. Zarzuela, On the Gorenstein Property of Rees and Form Rings of Powers of Ideals, to appear in Transactions of the American Mathematical Society.

[3] M. Herrmann, J. Ribbe, S. Zarzuela, On Rees and Form Rings of Almost Complete Intersections, to appear in Communications in Algebra.

[4] J. Herzog, E. Kunz, Der kanonische Modul eines Cohen-Macaulay-Rings, Lecture Notes in Mathematics 238, Springer Verlag, Berlin-Heidelberg (1971).

[5] J. Herzog, A. Simis, W. Vasconcelos, On the Canonical Module of the Rees Algebra and the Associated Graded Ring of an Ideal, J. Algebra 105 (1987), 285-302.

[6] C. Huneke, On the Associated Graded Ring of an Ideal, Illinois J. Math. 26 (1982), 121-137.

[7] S. Ikeda, On the Gorensteinness of Rees Algebras over Local Rings, Nagoya Math. J. 102 (1986), 135-154.

[8] A. Ooishi, On the Gorenstein Property of the Associated Graded Ring and the Rees Algebra of an Ideal, Preprint (1990).

[9] J. Ribbe, Dissertation, Universität zu Köln 1991.

[10] J. Sally, Tangent Cones at Gorenstein Singularities, Compositio Mathematica 40 (2) (1980), 167-175.

[11] G. Valla, A Property of almost Complete Intersections, Quart. J. Math. Oxford (2) 33 (1982), 487-492.

M. Herrmann
J. Ribbe
Mathematisches Institut
der Universität zu Köln
Weyertal 86-90

D-5000 Köln 30

P. Schenzel
Fachbereich Mathematik & Informatik
Martin-Luther-Universität Halle-Wittenberg
Postfach

D-4010 Halle - Saale