Holomorphic maps between circular domains

by

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of Bonn and of the Max-Planck-Institut für Mathematik. We would like to thank them both for their support and hospitality.

1. Let $U = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disk in $\mathbb{C}$. If M is a complex (Banach) manifold, Denote by $H(U,M)$ the set of all holomorphic maps from U to M. The <u>Kobayashi metric</u> of M is defined by

$$(1.1) \qquad F_M(p,v) = \inf\{|u| \mid u \in T_0(U) \text{ and } df(0,u) = v \text{ for some } f \in H(U,M) \text{ with } f(0)=p\}$$

for all $p \in M$ and $v \in T_p(M)$. Here we identify $T_0(U)$ with $\mathbb{C}$ and $|\ |$ denotes the euclidean norm. The <u>indicatrix</u> of M at p with respect to the Kobayashi metric is defined by

$$(1.2) \qquad I_p(M) = \{v \in T_p(M) \mid F_M(p,v) < 1\}.$$

Let V be a complex Banach space. We say that a domain $D \subset V$ is <u>starlike</u> if, for all $t \in [0,1]$, we have

$$tD = \{tZ \mid Z \in D\} \subset D.$$

The <u>Minkowski functional</u> of a starlike domain $D \subset V$ is a function $m_D = m : V \to \mathbb{R}_+$ defined by

$$(1.3) \qquad m(Z) = \begin{cases} 0 & \text{if } Z = 0 \\ \inf\{1/t \mid t > 0 \text{ and } tZ \notin D\} & \text{if } Z \neq 0. \end{cases}$$

Then, for all $Z \in V - \{0\}$, we have $(m(Z))^{-1}Z \in \partial D$ and $m(Z) = 1$ if and only if $Z \in \partial D$. Also $D = \{Z \in V \mid m(Z) < 1\}$. We say that $G \subset V$ is a <u>complete circular domain</u> if G is a connected, bounded, open set with the property that if $Z \in G$ and $\lambda \in \overline{U}$, then $\lambda Z \in G$. Such a domain G is clearlv starlike and $0 \in G$. Moreover the associated Minkowski functional m_G has the following homogeneity property:

(1.4) $\quad m_G(\lambda Z) = |\lambda| m_G(Z)$ for all $\lambda \in \mathbb{C}$ and $Z \in V$.

Throughout the paper we shall identify the tangent space $T_0(G)$ of G at the origin with V. Then, in particular, the indicatrix $I_0(G)$ will be a subset of V. Also we shall denote the restriction $F_G(0,\cdot)$ of the Kobayashi metric to $T_0(G)$ by F_G^0. Then F_G^0, under the above identification, is defined on V. The following result is due to Barth [B]:

THEOREM 1.1 Let $G \quad V$ be a complete circular domain. Then $G \subset I_0(G)$. Moreover, if G is pseudoconvex, then $G = I_0(G)$ and, in fact, $m_G = F_G^0$.

2. The Kobayashi metric is a biholomorphic invariant and plays an important role in the classification of complex manifolds. On the other hand one can classify complete circular domains by means of their Minkowski functionals (cfr. [PW]). Barth's theorem implies that for complete circular domains the two approaches coincide even in the infinite dimensional case. Precisely we have the following result.

THEOREM 2.1 Let G, G' be two pseudoconvex, complete circular domains in a complex Banach space V. The following statements are equivalent:

(i) G is biholomorphic to G'.

(ii) There exists a linear isomorphism $A : V \to V$ such that $m_G = m_{G'} \circ A$.

(iii) There exists a linear isomorphism $A : V \to V$ such that $F_G^o = F_{G'}^o \circ A$.

PROOF. Clearly (ii)<==>(iii) because of Theorem 1.1. Assume that (ii) holds. Then, if $Z \in G$, we have $m_{G'}(A(Z)) = m_G(Z) < 1$ and hence $A(G) \subset G'$. Similarly one shows that $G' \subset A(G)$ so that $A_{|G} : G \to G'$ is a biholomorphic map and (i) holds. Finally if (i) is true, because of a theorem of Braun, Kaup and Upmeier [BKU], then there exists a linear isomorphism $A : V \to V$ such that $A(G) = G'$ and thus (ii) follows.

q.e.d.

A simple and interesting consequence of Theorem 2.1 is the following (cfr. [PW] for a weaker statement):

COROLLARY 2.2 Let $G \subset \mathbb{C}^n$ be a pseudoconvex, complete circular domain with C^2 boundary and let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$. The following statements are equivalent:

(i) G is biholomorphic to $\mathbb{B}$.

(ii) There exists $A \in GL(n,\mathbb{C})$ such that $m_G(Z) = F_G^o(Z) = \|A(Z)\|$.

(iii) $m_G = F_G^o : \mathbb{C}^n \to \mathbb{R}_+$ is of class C^2 at the origin.

PROOF. Clearly (i)==>(ii), (i) ==>(iii), (ii)==>(iii). Also since $m_{\mathbb{B}} = \| \|$, Theorem 2.1 implies that (ii)==>(i). We shall show that (iii)==>(ii). Define $M = (m_G)^2$. If (iii) holds, then M is of class C^2 on $\mathbb{C}^n$. Moreover because of (1.4), we have for all $Z \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$

$$(2.1) \qquad M(\lambda Z) = |\lambda|^2 M(Z).$$

Let $Z \in \mathbb{C}^n - \{0\}$. Differentiating (2.1) for λ, we get

$$(2.2) \qquad 0 < M(Z) = \frac{\partial^2 M(\lambda Z)}{\partial\lambda\partial\bar{\lambda}} = \sum_{\mu,\nu=1}^{n} \frac{\partial^2 M(\lambda Z)}{\partial z^\mu \partial \bar{z}^\nu} z^\mu \bar{z}^\nu.$$

Taking limit in (2.2) as $\lambda \to 0$, we can conclude that M is a positive definite hermitian form and hence (ii) follows.

q.e.d.

We shall now prove a Schwarz's lemma for holomorphic maps between circular domains. This result is partly contained in [S] in the finite dimensional version. However our proof based on Barth's theorem is much simpler and holds in the infinite dimensional setting (cfr. also [FV] Theorem III.2.3 and [R] Theorem 8.1.2).

THEOREM 2.3 Let G, G' be complete circular domains in a complex Banach space V. Let m, m' be the respective Minkowski functionals and assume that G' is pseudoconvex. If $\phi : G \to G'$ is a holomorphic map with $\phi(0) = 0$, then

(i) $m'(\phi(Z)) \leq m(Z)$ for all $Z \in G$ and if $m'(\phi(Z)) = m(Z)$ for some $Z \in G$, then, for all $\lambda \in \mathbb{C}$ with $|\lambda| < (m(Z))^{-1}$, we have $m'(\phi(\lambda Z)) = m(\lambda Z) = |\lambda| m(Z)$.

(ii) If $A = d\phi(0)$ is the differential of ϕ at 0, then $A(G) \subset G'$.

PROOF. First of all we observe that if $f \in H(U, G')$ and $f(0) = 0$, then $m(f(z)) \leq |z|$ for all $z \in U$. In fact, if we define $g_r : \bar{U} \to V$ by $g_r(z) = z^{-1} f(rz)$ for $r \in (0,1)$, then g_r is holomorphic on U, continuous on $\bar{U}$ and $g_r(\partial U) \subset G'$. But then, since

G' is pseudoconvex, by the Kontinuitätssatz, $g_r(U) \subset G'$ for all $r \in (0,1)$. Taking limit as $r \to 1^-$, this implies $m'(f(z)) \leq |z|$. Let $Z \in G$. Then $Z = tc$ where $t = m(Z)$ and $c = t^{-1}Z \in \partial G$. Define $\phi_Z \in H(U,G)$ by $\phi_Z(z) = zc$. Then, as observed above, $m'(\phi_Z(z)) \leq |z|$ for all $z \in U$. In particular we get

$$m'(\phi(Z)) = m'(\phi_Z(t)) \leq t = m(Z).$$

Since G' is pseudoconvex then m' is a plurisubharmonic function (cfr. [B], Theorem 1). Thus the function h defined by $h(\lambda) = m'(\lambda^{-1}\phi(\lambda Z))$ on the disk $U' = \{\lambda \in \mathbb{C} \mid |\lambda| < m(Z)^{-1}\}$ is subharmonic for every $Z \in G$. Since $\sup_{U'} h = m(Z)$, if $m'(\phi(Z)) = m(Z)$, then h must be constant and hence $m'(\phi(\lambda Z)) = m(\lambda Z) = |\lambda| m(Z)$ and the proof of (i) is complete.

Part (ii) follows immediately from Barth's theorem because under the hypothesis we have

$$A(G) \subset A(I_O(G)) \subset I_O(G')) = G'.$$

q.e.d.

3. From now on we shall give applications of the above results to the study of holomorphic maps between circular domains. First we shall deal with isometries of the Kobayashi metric. Let M, N be complex manifolds and $\phi : M \to N$ be a holomorphic map. We say that ϕ is an isomety of the Kobayashi metric (<u>K-isometry</u> for short) at $p \in M$ if for all $v \in T_p(M)$ we have $F_M(p,v) = F_N(\phi(p), d\phi(p,v))$.

<u>THEOREM 3.1</u> Let G, G' be pseudoconvex, complete circular domains in $\mathbb{C}^n$ and $\phi : G \to G'$ be a holomorphic map with $\phi(O) = O$. If ϕ is a K-isometry at O, then ϕ is a linear biholomorphic map of G into G'.

PROOF. Let $A = d\phi(0) : \mathbb{C}^n \to \mathbb{C}^n$ be the differential of ϕ at 0. By hypothesis $F^0_{G'} = F^0_G \circ A$. Thus A is non singular and, by Theorem 2.1, G is biholomorphic to G' and, in fact, $G' = A(G)$. But then $\psi = \phi \circ A^{-1} : G \to G$ is a holomorphic map such that $\psi(0) = 0$ and $d\psi(0) = \mathrm{Id}$. By Cartan's uniqueness theorem we conclude that $\psi = \mathrm{Id}$ and hence $\phi = A = d\phi(0)$.

q.e.d.

COROLLARY 3.2 Let G, G' be pseudoconvex, complete circular domains in $\mathbb{C}^n$ and $\phi : G \to G'$ be a holomorphic map.

(i) If G is homogeneous and $Z \in \phi^{-1}(0)$ exists such that ϕ is a K-isometry at Z, then ϕ is a biholomorphic map.

(ii) If G' is homogeneous and ϕ is a K-isometry at $0 \in G$, then ϕ is a biholomorphic map.

In particular if both G, G' are homogeneous, then ϕ is biholomorphic if and only if ϕ is a K-isometry at some point $Z \in G$.

PROOF. The claim is a straightforward consequence of Theorem 3.1 since automorphisms are K-isometries at every point.

q.e.d.

4. In this section we restrict our considerations to more special domains. First we need some terminology. Let V be a complex Banach space and $K \subset V$. A point $p \in K$ is called a complex extreme point of K if $q = 0$ is the only vector in V such that $\{p + \lambda q \mid \lambda \in U\} \subset K$. Let D be a convex domain in V. We say that D is E-convex if every point $p \in \partial D$ is a complex extreme point of $\bar{D}$. We also recall the definition of the Kobayashi pseudodistance δ_M for a connected complex (Banach) manifold M.

Let $p, q \in M$. A <u>holomorphic chain</u> for p, q is a pair $\{(f_j)_{j=1,\dots,N}, (z_j)_{j=0,\dots,N}\}$ such that $f_j \in H(U,M)$, $z_j \in U$ and $p = f_1(z_0)$, $f_1(z_1) = f_2(z_1)$, $\dots$, $f_j(z_j) = f_{j+1}(z_j), \dots$, $f_N(z_N) = q$. Then, if ρ denotes the hyperbolic distance on U,

$$(4.1) \qquad \delta_M(p,q) = \inf \{ \sum_j \rho(z_j, z_{j+1}) \mid \{(f_j),(z_j)\} \text{ is a holomorphic chain for } p,q\}.$$

In fact, if D is a convex domain in a complex Banach space V, then, because of a theorem of Lempert [L] which carries over to the infinite dimensional case with no change, δ_D is a distance and, if $p,q \in D$, we have

$$(4.2) \qquad \delta_D(p,q) = \inf \{\rho(z,w) \mid z,w \in U \text{ and } f(z)=p,\ f(w)=q \text{ for some } f \in H(U,D)\}.$$

A <u>complex geodesic</u> (cfr. [V]) for the Kobayashi distance d_M is a map $f \in H(U,M)$ such that $\delta_M(f(z),f(w)) = \rho(z,w)$ for all $z,w \in U$. It is known ([V]) that if G is a E-convex, complete circular domain in a complex Banach space V, then the complex geodesics of δ_G are the linear maps $f_c \in H(U,G)$ defined by $f_c(z) = zc$ where c is any point in ∂G. The following observation (cfr. also [P] Theorem 3.1') will be useful later.

<u>PROPOSITION 4.1</u> Let G be a E-convex, complete circular domain in a complex Banach space V. Let m be its Minkowski functional and F^O be the Kobayashi metric of G at the origin. Then for all $Z \in G$ we have

$$(4.3) \qquad \delta_G(O,Z) = \frac{1}{2}\log\frac{1+m(Z)}{1-m(Z)} = \frac{1}{2}\log\frac{1+F^O(Z)}{1-F^O(Z)}.$$

<u>PROOF.</u> If $Z \in G$ and $c = (m(Z))^{-1}Z \in \partial G$, then the map

$f_c \in H(u,G)$ defined as above is a complex geodesic and hence the first equality holds. The second one is a consequence of Barth's theorem.

q.e.d.

<u>THEOREM 4.2</u> Let G, G' be E-convex, complete circular domains in $\mathbb{C}^n$ and let m, m' be the respective Minkowski functionals. If $\phi : G \to G'$ is a holomorphic map with $\phi(0) = 0$, then the following statements are equivalent:

(i) ϕ is a linear biholomorphic map.

(ii) $\delta_G(0,Z) = \delta_{G'}(0,\phi(Z))$ for all $Z \in G$.

(iii) For some $r \in (0,1)$, if $m(Z) = r$, then $m'(\phi(Z)) = r$.

<u>PROOF.</u> Since biholomorphic maps preserve the Kobayashi distance, (i) $\Longrightarrow$(ii). Also (ii) $\Longrightarrow$(iii) because of (4.3). Again because of (4.3) and of (i) of Theorem 2.3, we have that (iii) $\Longrightarrow$(ii). Only (ii) $\Longrightarrow$(i) remains to be shown. Given any $c \in \partial G$, the map $f_c \in H(U,G)$ defined by $f_c(z) = zc$ is a complex geodesic of G. If (ii) holds, then the map $q = \phi \circ f_c \in H(U,G')$ is a complex geodesic of G' and $G(0) = 0$. Thus, as remarked above, q must be linear and hence

$$\phi(zc) = g(z) = dq(0,z) = d\phi(0,zc).$$

Since this holds for all $c \in \partial G$, ϕ is linear. Also since δ_G is a distance, if $\phi(Z) = 0$, then $0 = \delta_{G'}(0,\phi(Z)) = \delta_G(0,Z)$ and hence $Z = 0$ i.e. ϕ is injective and therefore surjective. Because of (4.3) then $F^0_{G'} = F^0_G \circ \phi = F^0_G \circ d\phi(0)$ i.e. ϕ is K-isometric at 0 and the claim follows from Theorem 3.1.

q.e.d.

As with Corollary 3.2, since biholomorphic maps are Kobayashi distance preserving, one shows at once the following consequence of the above theorem.

COROLLARY 4.3 Let G, G' be E-convex, complete circular domains in $\mathbb{C}^n$ and let $\phi : G \to G'$ be a holomorphic map.

(i) If G is homogeneous and there exists $Z \in G$ such that $\phi(Z) = 0$ and $\delta_G(Z,W) = \delta_{G'}(0,\phi(W))$ for all $W \in G$, then ϕ is biholomorphic.

(ii) If G' is homogeneous and $\delta_G(0,W) = \delta_{G'}(\phi(0),\phi(W))$ for all $W \in G$, then ϕ is biholomorphic.

In particular, if both G and G' are homogeneous, ϕ is biholomorphic if and only if there exists $Z \in G$ such that $\delta_G(Z,W) = \delta_{G'}(\phi(Z),\phi(W))$ for all $W \in G$.

5. We shall now study the case of proper holomorphic maps between circular domains. Recently Bell [B] characterized among them the polynomial maps using Bergman kernel tecniques. Here we use the previous results to characterize the biholomorphic maps. The following fact about Monge-Ampère equations will be useful.

PROPOSITION 5.1 Let G be a pseudoconvex, complete circular domain in $\mathbb{C}^n$ with C^2 boundary. Let $m : \mathbb{C}^n \to \mathbb{R}_+$ be its Minkowski functional. Then $u_c = c \log m$ is the unique solution to the problem

$$\left.\begin{array}{ll} (5.1) & w \text{ is plurisubharmonic on } G \text{ and } w \in C^0(\bar{G} - \{0\}) \\ (5.2) & \det\left(\frac{\partial^2 w}{\partial z^\mu \partial \bar{z}^\nu}\right) = 0 \\ (5.3) & w(Z) - c \log\|Z\| = O(1) \quad \text{as} \quad Z \to 0 \\ (5.4) & w(Z) = 0 \quad \text{if} \quad Z \in \partial G \end{array}\right\} (P_c)$$

for all $c > 0$.

PROOF. It is known that u_c satisfies (5.1), (5.2) (cfr. for example [BT2]) and (5.4). If $R_0 = \min_{\|Z\|=1} m(Z)$ and $R_1 = \max_{\|Z\|=1} m(Z)$, then $R_0\|Z\| \leq m(Z) \leq R_1\|Z\|$ and therefore also (5.3) holds for u_c. Let v be another solution to the the problem (P_c). Then, given any $\eta > 0$, there exists δ so that, if $\mathbb{B}(\delta)$ is a ball of radius δ around the origin and $G(\delta) = G - \bar{\mathbb{B}}(\delta)$, then

$$(5.5) \qquad (1 - \eta)v \leq u_c \leq (1 + \eta)v \quad \text{on} \quad \partial G(\delta).$$

Thus, by the Bedford-Taylor minimum principle for the Monge-Ampère operator (cfr. [BT1]), (5.5) holds on $G(\delta)$. Then, letting $\eta \to 0$, it follows that $u_c = v$.

q.e.d.

COROLLARY 5.2 Let G, G' be two pseudoconvex, complete circular domain in $\mathbb{C}^n$ with C^2 boundaries and let m, m' be the respective Minkowski functionals. If $\phi : G \to G'$ is a proper holomorphic map such that

(i) $\phi(0) = 0$,

(ii) there exists an integer $N > 0$ and constants $C, K > 0$ such that $C\|Z\|^N \leq \|\phi(Z)\| \leq K\|Z\|^N$ for all $Z \in G$,

(iii) ϕ extends to a homeomorphism $\hat{\phi} : \bar{G} \quad \bar{G'}$,

then $m' \circ \phi = m^N$.

PROOF. Under the hypothesis $v = \log\|z\|$ solves the problem (P_N) and hence Proposition 5.1 implies the claim.

q.e.d.

The question on when a proper holomorphic map $\phi : D \to D'$ between two pseudoconvex domains extends to a homeomorphism $\hat{\phi} : \bar{D} \to \overline{D'}$ has not yet been completely settled. Here we restrict ourselves to the use of the theorem of Henkin and Vormoor (cfr. [DL] for a detailed proof) which states that if D, D' have C^2 boundaries, D has a plurisubharmonic defining function, D' is strictly pseudoconvex, then the proper map $\phi : D \to D'$ extends to a homeomorphism $\hat{\phi} : \bar{D} \to \overline{D'}$.

THEOREM 5.3 Let G, G' be complete circular domains in $\mathbb{C}^n$. Assume that they have C^2 boundaries, that G is pseudoconvex and G' is strictly pseudoconvex. Let $\phi : G \to G'$ be a proper holomorphic map regular at O and such that $\phi(O) = O$. Then ϕ is a linear biholomorphic map and, in particular, G is strictly pseudoconvex.

PROOF. Let m, m' be the Minkowski functionals of G, G' respectively. Then m is a plurisubharmonic defining function for G. Because of the theorem of Henkin and Vormoor and since ϕ is regular at O, the assumption of Corollary 5.2 are verified in our situation with $N = 1$. Thus we can conclude $m' \circ \phi = m$. Let $c \in \mathbb{C}^n$ so that $F^O_G(c) = m(c) = 1$ so that $c \in \partial I_O(G) = \partial G$. Define $\phi_c : U \to G'$ by $\phi_c(z) = \phi(zc)$. Then $m'(\phi_c(z)) = m(zc) = |z|$. Also, since ϕ_c is holomorphic, we have the power series development $\phi_c(z) = \sum_{k \geq 1} A_k z^k$ for $z \in U$. But then $m'(A_1 + \sum_{k \geq 2} A_k z^k) = 1$ so that taking limit as $z \to O$, we get

$$(5.6) \qquad F^{O}_{G'}(d\phi(O,c)) = m'\left(\frac{d}{dz}\phi_c(z)\Big|_{z=O}\right) = m'(A_1) = 1 = F^{O}_{G}(c).$$

Since (5.6) holds for all $c \in \mathbb{C}^n$ with $F^{O}_{G}(c) = 1$, Then ϕ is a K-isometry at O and thus, by Theorem 3.1, the claim follows.

q.e.d.

A direct consquence of Theorem 5.3 is the following result originally proved by Alexander [A].

COROLLARY 5.4 Let $\mathbb{B}$ be the unit ball in $\mathbb{C}^n$ and $\phi : \mathbb{B} \to \mathbb{B}$ be a proper map. Then ϕ is an automorphism of $\mathbb{B}$.

PROOF. Let $Z \in \mathbb{B}$ be a regular point of ϕ, and let $W = \phi(Z)$. Let $\alpha, \beta \in \mathrm{Aut}(\mathbb{B})$ be such that $\alpha(O) = Z$ and $\beta(W) = O$. Then the claim follows applying Theorem 5.3 to the map $\psi = \beta \circ \phi \circ \alpha$.

q.e.d.

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