ON THE COOPERATION ALGEBRA OF THE CONNECTIVE ADAMS SUMMAND

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Abstract. The aim of this paper is to gain explicit information about the multiplicative structure of $\ell_*\ell$, where ℓ is the connective Adams summand. Our approach differs from Kane's or Lellmann's because our main technical tool is the MU-based Künneth spectral sequence. We prove that the algebra structure on $\ell_*\ell$ is inherited from the multiplication on a Koszul resolution of ℓ_*BP.

Introduction

Our goal in these notes is to shed light on the structure, in particular on the multiplicative structure, of $\ell_*\ell$, where we work at an odd prime p and ℓ is the Adams summand of the p-localization of the connective K-theory spectrum ku. This was investigated by Kane [5] and Lellmann [9] using Brown-Gitler spectra. Our approach is different and exploits the fact that MU is a commutative $\mathbb{S}$-algebra in the sense of Elmendorf, Kriz, Mandell and May [4] and ℓ is a MU-ring spectrum (in fact it is even an MU-algebra). As a calculational tool, we make use of a Künneth spectral sequence (2.2) converging to $\ell_*\ell$ where we work with a concrete Koszul resolution. Our approach bears some similarities to old work of Landweber [8], who worked without the benefit of the modern development of structured ring spectra. The multiplicative structure on the Koszul resolution gives us control over the convergence of the spectral sequence and the multiplicative structure of $\ell_*\ell$. In particular, it sheds light on the torsion.

The outline of the paper is as follows. We recall some basic facts about complex cobordism, MU, in Section 1 and describe the Künneth spectral sequence in Section 2. Some background on the Bockstein spectral sequence is given in Section 3. The multiplicative structure on the E^2-term of this spectral sequence is made precise in section 4 where we introduce the Koszul resolution we will use later in terms of its multiplicative generators. We study the torsion part in $\ell_*\ell$ and the torsion-free part separately. The investigation of ordinary and L-homology of ℓ in Section 5 leads to the identification of the p-torsion in $\ell_*\ell$ with the u-torsion where $\ell_* = \mathbb{Z}_{(p)}[u]$ with u being in degree $2p-2$. In Section 6 we show how to exploit the cofibre sequence

$$\ell \xrightarrow{p} \ell \longrightarrow \ell/p$$

to analyse the Künneth spectral sequence and relate the simpler spectral sequence for ℓ/p to that for ℓ. To that end we prove an auxiliary result on connecting homomorphisms in the Künneth spectral sequence, which is analogous to the well-known geometric boundary theorem (see for instance [14, chapter 2, §3]). We summarize our calculation of $\ell_*\ell$ at the end of that section.

In Section 7 we use classical tools from the Adams spectral sequence in order to study torsion phenomena in $\ell_*\ell$. We use the fact that the p and u-torsion is all simple to show that the Künneth spectral sequence for $\ell_*\ell$ collapses at the E^2-term and that there are no extension issues. We can describe the torsion in $\ell_*\ell$ in terms of familiar elements which are certain coaction-primitives in the $\mathbb{F}_p$-homology of ℓ.

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We summarize our results on the multiplicative structure on $\ell_*\ell$ at the end of Section 8 where we establish congruence relations in the zero line of the Künneth spectral sequence and describe the map from the torsion-free part of $\ell_*\ell$ to $\ell_*\ell\otimes\mathbb{Q}$. Taking this together with the explicit formulæ of the multiplication in the torsion part in $\ell_*\ell$ gives a rather comprehensive, though not complete, description of the multiplicative structure of $\ell_*\ell$.

1. Recollections on MU and ℓ

Throughout, we will assume all spectra are localized at p for some odd prime p.

Let ku denote connective complex K-theory and let ℓ be the Adams summand, also known as $BP\langle 1\rangle$, so that

$$ku_{(p)}\sim\bigvee_{0\leqslant i\leqslant p-2}\Sigma^{2i}\ell.$$

We have $\ell_*=\pi_*\ell=\mathbb{Z}_{(p)}[u]$ with $u\in\ell_{2(p-1)}$. We will denote the Adams summand of $KU_{(p)}$ by L; then $L_*=\ell_*[u^{-1}]$.

Let us recall some standard facts for which convenient sources are [1, 15]. Since ℓ is complex oriented,

$$\ell_*MU=\ell_*[m'_n : n\geqslant 1],$$

where $m'_n\in\ell_{2n}MU$ agrees with the m^ℓ_n of Adams [1]. By the Hattori-Stong theorem, the Hurewicz homomorphism $MU_*\longrightarrow\ell_*MU$ is a split monomorphism, so we will view MU_* as a subring of ℓ_*MU. Now

$$MU_*=\mathbb{Z}_{(p)}[x_n : n\geqslant 1],$$

where $x_n\in MU_{2n}$ and using Milnor's criterion for polynomial generators of MU_* we can arrange that

$$x_n\equiv\begin{cases} pm'_{p^k-1} \text{ mod decomposables} & \text{if } n=p^k-1 \text{ for some } k,\\ m'_n \quad\text{mod decomposables} & \text{otherwise.}\end{cases}$$

In fact, we can take $x_{p^k-1}=v_k$ to be the Hazewinkel generator which lies in $BP_*\subseteq MU_*$. The following formula recursively determines the Hurewicz image of v_k in $H_*MU=\mathbb{Z}_{(p)}[m_k : k\geqslant 1]$:

$$v_k=pm_{p^k-1}-\sum_{1\leqslant j\leqslant k-1}m_{p^j-1}v_{k-j}^{p^j}. \tag{1.1}$$

In H_*BP with $\ell_k=m_{p^k-1}$, this corresponds to the familiar formula

$$v_k=p\ell_k-\sum_{1\leqslant j\leqslant k-1}\ell_j v_{k-j}^{p^j}. \tag{1.2}$$

We note that

$$\ell_*MU/(x_n : n\neq p^k-1 \text{ for any } k)=\ell_*[t_k : k\geqslant 1]=\ell_*BP, \tag{1.3}$$

where $t_k\in\ell_{2p^k-2}BP$ is the image of the standard polynomial generator $t_k\in BP_*BP$ of [1].

Now recall that the natural complex orientation of ℓ factors as

$$\sigma\colon MU\longrightarrow BP\longrightarrow\ell$$

and we can choose the generators x_n so that

$$\sigma_*(x_n)=\begin{cases} u & \text{if } n=p-1,\\ 0 & \text{otherwise.}\end{cases}$$

In particular, the kernel of the map $BP_*\longrightarrow\ell_*$ is the ideal generated by the Hazewinkel generators $v_2,v_3,\ldots$.

We can also find useful expressions for the v_n. Using standard formulæ for the right unit $\eta_R\colon BP_*\longrightarrow BP_*BP$ which can be found in [15], we have for $n\geqslant 2$,

$$v_n=pt_n+ut_{n-1}^p-u^{p^{n-1}}t_{n-1}+ps'_n+us''_n, \tag{1.4}$$

where $s'_n,s''_n\in\mathbb{Z}_{(p)}[u,t_1,\ldots,t_{n-1}]$. We also have $v_1=pt_1+u$.

We now make some useful deductions. To ease notation we write v_n for the image $\mathrm{e}(v_n) \in E_*BP$ of v_n under the Hurewicz homorphism $\mathrm{e}\colon BP_* \longrightarrow E_*BP$.

Proposition 1.1. *In the ring $\mathbb{Q} \otimes \ell_*BP$, the sequence $v_2, v_3, \ldots, v_n, \ldots$ is regular and*

$$\mathbb{Q} \otimes \ell_*BP/(v_n : n \geqslant 2) = \mathbb{Q} \otimes \ell_*[t_1] = \mathbb{Q} \otimes \ell_*[v_1].$$

Proof. For each $n \geqslant 1$, pt_n is a polynomial generator for $\mathbb{Q} \otimes \ell_*BP = \mathbb{Q} \otimes \ell_*[t_i : i \geqslant 1]$ over $\mathbb{Q} \otimes \ell_*$. □

Proposition 1.2. *In the ring L_*BP, the sequence $v_2, v_3, \ldots, v_n, \ldots$ is regular and*

$$L_*BP/(v_n : n \geqslant 2) = L_*[t_k : k \geqslant 1]/(t_n^p - u^{p^n-1}t_n + pu^{-1}s'_{n+1} + s''_{n+1} + pu^{-1}t_{n+1} : n \geqslant 1).$$

In the ring $L_*BP/(p)$, the sequence $v_2, v_3, \ldots, v_n, \ldots$ is regular and

$$L_*BP/(p, v_n : n \geqslant 2) = L_*/(p)[t_k : k \geqslant 1]/(t_n^p - u^{p^n-1}t_n + s''_{n+1} : n \geqslant 1).$$

Proof. The case of L_*BP follows from Proposition 2.2, but here is a direct proof.

By induction, for each $m \geqslant 2$ there is a monomorphism of L_*-algebras

$$L_*BP/(v_n : m \geqslant n \geqslant 2) \longrightarrow \mathbb{Q} \otimes L_*[t_1, t_k : k \geqslant m+1]$$

under which the image of v_{m+1} is clearly not a zero-divisor. This shows that the v_n form a regular sequence in L_*BP. □

2. A Künneth spectral sequence for $\ell_*\ell$

We will describe a calculation of $\ell_*\ell = \pi_*(\ell \wedge \ell)$ that makes use of the Künneth spectral sequence of [4] for MU-modules. This is different from the approach taken by Kane [5], and we feel it offers some insight into the form of answer, especially with regard to multiplicative structure.

For any ring spectrum E and MU-ring spectrum F, there is a homologically graded spectral sequence

$$\text{(2.1)} \qquad \mathrm{E}^2_{s,t} = \mathrm{Tor}^{MU_*}_{s,t}(\pi_*(E \wedge MU), \pi_*F) \Longrightarrow \pi_*((E \wedge MU) \wedge_{MU} F) \cong \pi_*(E \wedge F) = E_*F.$$

Note that this spectral sequence is actually multiplicative [3]. Taking $E = F = \ell$ we obtain a multiplicative spectral sequence

$$\text{(2.2)} \qquad \mathrm{E}^2_{s,t} = \mathrm{Tor}^{MU_*}_{s,t}(\pi_*(\ell \wedge MU), \pi_*\ell) \Longrightarrow \ell_*\ell.$$

Now consider the MU_*-module ℓ_*. We can assume that the complex orientation gives rise to a ring isomorphism

$$MU_*/(x_n : n \neq p-1) \xrightarrow{\cong} \ell_*.$$

There is a Koszul resolution of ℓ_* as a module over MU_*,

$$\Lambda_{MU_*}(e_r : 0 < r \neq p-1) \longrightarrow \ell_* \to 0,$$

where $\Lambda_{MU_*}(e_r : 0 < r \neq p-1)$ is the exterior algebra generated by elements e_r of bidegree $(1, 2r)$ whose differential d is the derivation which satisfies $d(e_r) = x_r$.

For arbitrary E and $F = \ell$, the E^2-term of the spectral sequence (2.1) is the homology of the complex

$$E_*MU \otimes_{MU_*} \Lambda_{MU_*}(e_r : 0 < r \neq p-1) \cong \Lambda_{E_*MU}(e_r : 0 < r \neq p-1)$$

with differential $\mathrm{id}\otimes d$ which corresponds to the differential d taking values in the latter complex. From (1.3) we find that the homology of this complex is

$$\text{(2.3)} \qquad \mathrm{H}_*(\Lambda_{E_*MU}(e_r : 0 < r \neq p-1), d) = \mathrm{H}_*(\Lambda_{E_*BP}(\varepsilon_r : r \geqslant 2), d),$$

where ε_r has bidegree $(1, 2p^r - 2)$ and $d(\varepsilon_r) = v_r$.

Proposition 2.1. *Suppose that the E-theory Hurewicz images of the v_k in E_*BP with $k \geqslant 2$ form a regular sequence. Then the complex*

$$\Lambda_{E_*BP}(\varepsilon_r : r \geqslant 2) \longrightarrow E_*BP/(v_r : r \geqslant 2) \to 0$$

is acyclic and

$$\mathrm{Tor}^{MU_*}_{s,*}(E_*MU, \ell_*) = \begin{cases} E_*BP/(v_r : r \geqslant 2) & \text{if } s = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Therefore the Künneth spectral sequence of (2.1) *degenerates to give an isomorphism*

$$E_*BP/(v_r : r \geqslant 2) \xrightarrow{\cong} E_*\ell.$$

The regularity condition of this result occurs for the cases $E = \ell\mathbb{Q}$, L/p by Propositions 1.2 and 1.1. We do not have a proof that it holds for the case $E = L$, however the following provides a substitute.

Proposition 2.2. *Suppose that E is a p-local Landweber exact spectrum. Then the complex*

$$\Lambda_{E_*BP}(\varepsilon_r : r \geqslant 2) \longrightarrow E_*BP/(v_r : r \geqslant 2) \to 0$$

is acyclic and the conclusion of Proposition 2.1 *is valid.*

Proof. The hypothesis means that the functor $E_* \otimes_{MU_*} (\ \ )$ is exact on the category of left MU_*MU-comodules. Then

$$\begin{aligned} E_*MU \otimes_{MU_*} \Lambda_{MU_*}(e_r : r \geqslant 2) &\cong E_* \otimes_{MU_*} MU_*MU \otimes_{MU_*} \Lambda_{MU_*}(e_r : r \geqslant 2) \\ &\cong E_* \otimes_{MU_*} \Lambda_{MU_*MU}(e_r : r \geqslant 2) \\ &\cong E_* \otimes_{MU_*} \Lambda_{MU_*BP}(\varepsilon_r : r \geqslant 2). \end{aligned}$$

But now it is easy to see that the sequence $\eta_R(v_2), \eta_R(v_3), \ldots$ is regular in MU_*BP since

$$\eta_R(v_k) = v_k \bmod (t_i : i \geqslant 1)$$

and $v_2, v_3, \ldots$ is a regular sequence in MU_*. Therefore

$$\Lambda_{MU_*BP}(\varepsilon_r : r \geqslant 2) \longrightarrow MU_*BP/(\eta_R(v_r) : r \geqslant 2) \to 0$$

is exact and so is

$$E_* \otimes_{MU_*} \Lambda_{MU_*BP}(\varepsilon_r : r \geqslant 2) \longrightarrow E_* \otimes_{MU_*} MU_*BP/(\eta_R(v_r) : r \geqslant 2) \to 0,$$

since in each homological degree, $\Lambda_{MU_*BP}(\varepsilon_r : r \geqslant 2)$ is a free MU_*BP-module and therefore an MU_*MU-comodule. From this we obtain the result. □

Of course, this result applies when $E = L$. Later we will also consider some cases where these regularity conditions do not hold.

3. Bockstein spectral sequences

We follow [16, p158] in this account. Let R be a graded commutative ring and suppose that we have an exact couple of graded R-modules

$$\begin{tikzcd} A^0_* \arrow[rr, "x\cdot"] & & A^0_* \arrow[dl, "j^0"] \\ & B^0_* \arrow[ul, "\delta^0"] & \end{tikzcd}$$

where δ^0 is a map of degree $|x|-1$ and $x\cdot$ is multiplication by $x \in R$. Then there are inductively defined exact couples

$$\begin{tikzcd} A^r_* \arrow[rr, "x\cdot"] & & A^r_* \arrow[dl, "j^r"] \\ & B^r_* \arrow[ul, "\delta^r"] & \end{tikzcd}$$

and an associated spectral sequence (B^r, d^r) with $B^{r+1}_* = \mathrm{H}(B^r_*, d^r)$. For each $r \geqslant 1$, there are exact sequences

$$0 \to A^0_n/(xA^0_{n-|x|} + {}_{x^r}A^0_n) \xrightarrow{\bar{j^r}} B^r_n \xrightarrow{\delta^r} {}_xA^0_{n+|x|+r-1} \cap x^r A^0_{n+|x|+r-r|x|-1} \to 0, \tag{3.1}$$

where

$${}_{x^r}A^0_n = \ker(x^r\colon A^0_n \longrightarrow A^0_{n+r|x|}), \quad {}_{x^\infty}A^0_n = \bigcup_{r \geqslant 1} {}_{x^r}A^0_n.$$

In particular, if $B^1_n = B^\infty_n = 0$ for some n, we obtain the following:

$${}_{x^\infty}A_n = {}_xA_n, \tag{3.2}$$

$$\ker \delta^0 = \ker d^0 = \operatorname{im} j^0. \tag{3.3}$$

4. Generalized Koszul complexes and Bockstein spectral sequences

Let R be a commutative ring and $x \in R$ a non-zero divisor which is also not a unit. Let $w_1, w_2, w_3, \ldots$ be a (possibly finite) regular sequence in R which reduces to a regular sequence in $R/(x)$.

The Koszul complex $(\Lambda_R(e_r : r \geqslant 1), d)$ whose differential is the R-derivation determined by $d(e_r) = w_r$ provides a resolution

$$\Lambda_R(e_r : r \geqslant 1) \longrightarrow R/(w_r : r \geqslant 1) \to 0$$

of $R/(w_r : r \geqslant 1)$ by R-modules.

Now consider the sequence $xw_1, xw_2, xw_3, \ldots$ which is not regular in R since for $s > r$,

$$w_r(xw_s) = w_s(xw_r).$$

The Koszul complex $(\Lambda_R(e'_r : r \geqslant 1), d')$ with differential satisfying $d'(e'_r) = xw_r$ is no longer exact but does augment onto $R/(xw_r : r \geqslant 1)$. Notice that there is a monomorphism of R-dga's

$$j\colon \Lambda_R(e'_r : r \geqslant 1) \longrightarrow \Lambda_R(e_r : r \geqslant 1); \quad j(e'_r) = xe_r,$$

and this covers the reduction map $R/(xw_r : r \geqslant 1) \longrightarrow R/(w_r : r \geqslant 1)$. Using this, we will view $\Lambda_R(e'_r : r \geqslant 1)$ as a subcomplex of $\Lambda_R(e_r : r \geqslant 1)$. We want to determine the homology of $(\Lambda_R(e'_r : r \geqslant 1), d')$.

Suppose that $z \in \Lambda_R(e'_r : r \geqslant 1)_n$ with $n > 0$ and $d'(z) = 0$. Then working in $\Lambda_R(e_r : r \geqslant 1)$ we have $d(j(z)) = 0$, so by exactness of the latter complex, there is an element

$$y = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_{n+1}} y_{i_1,i_2,\ldots,i_{n+1}} e_{i_1} e_{i_2} \cdots e_{i_{n+1}} \in \Lambda_R(e_r : r \geqslant 1)_{n+1}$$

for which $d(y) = j(z)$. But

$$d(y) = \sum_{\substack{1 \leqslant i_1 < i_2 < \cdots < i_{n+1} \\ 1 \leqslant k \leqslant n+1}} (-1)^k w_{i_k} y_{i_1,i_2,\ldots,i_{n+1}} e_{i_1} e_{i_2} \cdots \widehat{e_{i_k}} \cdots e_{i_{n+1}}.$$

Since we have

$$j(z) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_n} x^n z_{i_1,i_2,\ldots,i_n} e_{i_1} e_{i_2} \cdots e_{i_n},$$

using the regularity assumption we find that each $y_{i_1,i_2,\ldots,i_{n+1}}$ has the form

$$y_{i_1,i_2,\ldots,i_{n+1}} = x^n y'_{i_1,i_2,\ldots,i_{n+1}}$$

for some $y'_{i_1,i_2,\dots,i_{n+1}} \in R$ and therefore

$$z = \sum_{\substack{1\leqslant i_1<i_2<\cdots<i_{n+1}\\ 1\leqslant k\leqslant n+1}} (-1)^k w_{i_k} y'_{i_1,i_2,\dots,i_{n+1}} e'_{i_1} e'_{i_2} \cdots \widehat{e'}_{i_k} \cdots e'_{i_{n+1}}.$$

Notice that

$$xz = d'\left(\sum_{1\leqslant i_1<i_2<\cdots<i_{n+1}} y'_{i_1,i_2,\dots,i_{n+1}} e'_{i_1} e'_{i_2} \cdots e'_{i_{n+1}}\right),$$

hence x annihilates the n-th homology of $\Lambda_R(e'_r : r \geqslant 1)$, hence it is an $R/(x)$-module spanned by the elements

$$\Delta_x(i_1, i_2, \dots, i_{n+1}) = \sum_{1\leqslant k\leqslant n+1} (-1)^k w_{i_k} e'_{i_1} e'_{i_2} \cdots \widehat{e'}_{i_k} \cdots e'_{i_{n+1}} \tag{4.1}$$

for collections of distinct integers $i_1, i_2, \dots, i_{n+1} \geqslant 1$. Clearly for a permutation $\sigma \in \mathrm{S}_{n+1}$,

$$\Delta_x(i_{\sigma(1)}, i_{\sigma(2)}, \dots, i_{\sigma(n+1)}) = \operatorname{sign}\sigma\, \Delta_x(i_1, i_2, \dots, i_{n+1}).$$

Thus we will often restrict attention to indexing sequences satisfying

$$1 \leqslant i_1 < i_2 < \cdots < i_{n+1}.$$

These elements satisfy some further additive and multiplicative relations.

Proposition 4.1. *Let $r, s \geqslant 2$ and suppose that $i_1, i_2, \dots, i_r \geqslant 1$ and $j_1, j_2, \dots, j_s \geqslant 1$ are sequences of distinct integers. Let*

$$t = \#\{i_1, i_2, \dots, i_r\} \cup \{j_1, j_2, \dots, j_s\}$$

and write

$$\{k_1, k_2, \dots, k_t\} = \{i_1, i_2, \dots, i_r\} \cup \{j_1, j_2, \dots, j_s\}$$

with $1 \leqslant k_1 < k_2 < \cdots < k_t$. Then the following identities are satisfied in each of $\Lambda_R(e'_r : r \geqslant 1)$ and $\mathrm{H}_*(\Lambda_R(e'_r : r \geqslant 1), d')$.

$$\Delta_x(i_1, i_2, \dots, i_r)\Delta_x(j_1, j_2, \dots, j_s) = \begin{cases} 0 & \text{if } t \leqslant r+s-2, \\ (-1)^a w_{k_m} \Delta_x(k_1, k_2, \dots, k_t) & \text{if } t = r+s-1 \ \&\ k_m = i_a = j_b, \\ \displaystyle\sum_{j=1}^{r} (-1)^{j+s+1} w_{i_j} \Delta_x(i_1, i_2, \dots, \widehat{i_j}, \dots i_r, j_1, j_2, \dots, j_s) & \text{if } t = r+s, \end{cases} \tag{4.2a}$$

$$\sum_{j=1}^{r} (-1)^j w_{i_j} \Delta_x(i_1, i_2, \dots, \widehat{i_j}, \dots i_r) = 0. \tag{4.2b}$$

Theorem 4.2. *The homology of $(\Lambda_R(e'_r : r \geqslant 1), d')$ is given by*

$$\mathrm{H}_n(\Lambda_R(e'_r : r \geqslant 1), d') = \begin{cases} R/(xw_r : r \geqslant 1) & \text{if } n = 0, \\ R/(x)\{\Delta_x(i_1, i_2, \dots, i_{n+1}) : 1 \leqslant i_1 < i_2 < \cdots < i_{n+1}\} & \text{if } n > 0 \end{cases}$$

where in the second case, the $R/(x)$-module is generated by the $\Delta_x(i_1, i_2, \dots, i_{n+1})$ indicated, subject to relations given in (4.2b).

Proof. Consider the long exact sequence obtained by taking homology of the exact sequence

$$0 \to R \otimes_R \Lambda_R(e'_r : r \geqslant 1) \to R \otimes_R \Lambda_R(e'_r : r \geqslant 1) \to R/(x) \otimes_R \Lambda_R(e'_r : r \geqslant 1) \to 0.$$

The associated exact couple has

$$A^0_* = \mathrm{H}_*(\Lambda_R(e'_r : r \geqslant 1), d'),$$
$$B^0_* = \mathrm{H}_*(\Lambda_{R/(x)}(e'_r : r \geqslant 1), d') = \Lambda_{R/(x)}(e'_r : r \geqslant 1).$$

Making use of the formula $d^0 e'_r = w_r$ we find that

$$B^1_* = R/(x, w_1, w_2, \ldots),$$

and therefore the x-torsion in A^0_* is all simple. $\square$

Notice that the quotient R-module $R/(xw_r : r \geqslant 1)$ has x-torsion, as does the higher homology, at least if the sequence of w_r's has at least two terms.

5. Ordinary and L-homology of ℓ

We can compute $H_*\ell$ using the spectral sequence $(\mathrm{E}^r_{*,*}(H), d^r)$ obtained from (2.1) by taking $E = H = H\mathbb{Z}_{(p)}$ and $F = \ell$. This can be compared with the spectral sequence $(\mathrm{E}^r_{*,*}(H\mathbb{Q}), d^r)$ for $H\mathbb{Q}_*\ell$ making use of the morphism of spectral sequences

$$\mathrm{E}^r_{*,*}(H) \longrightarrow \mathrm{E}^r_{*,*}(H\mathbb{Q})$$

induced by the natural map $H \longrightarrow H\mathbb{Q}$. We will also consider the spectral sequence $(\mathrm{E}^r_{*,*}(\bar{H}), d^r)$ associated with $\bar{H} = H\mathbb{F}_p$.

By (1.2), the sequence $v_2, v_3, \ldots, v_n, \ldots$ in the polynomial ring $H\mathbb{Q}_*BP = \mathbb{Q}[\ell_i : i \geqslant 1]$ is regular. So by Proposition 2.1 we have

$$\mathrm{E}^2_{s,*}(H\mathbb{Q}) = \begin{cases} \mathbb{Q}[\ell_i : i \geqslant 1]/(v_k : k \geqslant 2) & \text{if } s = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Hence this spectral sequence collapses at E^2 and we have

$$H\mathbb{Q}_*\ell = \mathbb{Q}[\ell_1] = \mathbb{Q}[v_1],$$

where $v_1 = p\ell_1$. The image of ℓ_n in $H\mathbb{Q}_*\ell$ can be recursively computed with the aid of the following formula derived from (1.2):

$$\ell_n = \frac{v_1^{p^{n-1}} \ell_{n-1}}{p}. \tag{5.2}$$

So we have

$$\ell_n = \frac{v_1^{(p^n-1)/(p-1)}}{p^n} = p^{p^{n-1}+p^{n-2}+\cdots+p+1-n}\, \ell_1^{(p^n-1)/(p-1)}. \tag{5.3}$$

Notice that for a monomial in the ℓ_j's in $H\mathbb{Q}_{2m(p-1)}\ell$, we have

$$\ell_1^{r_1} \cdots \ell_n^{r_n} = \frac{v_1^m}{p^{r_1+2r_2+\cdots+nr_n}},$$

for which

$$r_1 + 2r_2 + \cdots + nr_n \leqslant r_1 + r_2\frac{p^2-1}{p-1} + \cdots + r_n\frac{p^n-1}{p-1} = m.$$

This calculation shows that the images of the monomials in the ℓ_j's in $H\mathbb{Q}_{2m(p-1)}\ell$ are contained in the cyclic $\mathbb{Z}_{(p)}$-module generated by $\ell_1^m = v_1^m/p^m$. Turning to the spectral sequence $\mathrm{E}^r_{*,*}(H)$, we see that

$$\mathrm{E}^2_{0,*}(H) = H_*BP/(v_j : j \geqslant 2)$$

and the natural map

$$H_{2m(p-1)}BP/(v_j : j \geqslant 2) \longrightarrow H\mathbb{Q}_{2m(p-1)}BP/(v_j : j \geqslant 2)$$

has image equal $\mathbb{Z}_{(p)}\ell_1^m$. In [1], the analogous result for ku was obtained using the Adams spectral sequence.

Proposition 5.1. *For $m \geqslant 0$,*

$$\mathrm{im}[H_{2m(p-1)}\ell \longrightarrow H\mathbb{Q}_{2m(p-1)}\ell] = \mathbb{Z}_{(p)}\ell_1^m = \mathbb{Z}_{(p)}\frac{v_1^m}{p^m}.$$

Hence,

$$\mathrm{im}[H_*\ell \longrightarrow H\mathbb{Q}_*\ell] = \mathbb{Z}_{(p)}[\ell_1] = \mathbb{Z}_{(p)}[v_1/p].$$

The spectral sequence $(\mathrm{E}^r_{*,*}(\bar{H}), d^r)$ is easy to determine. As $v_k = 0$ in $\bar{H}_*BP$, we find that

$$\mathrm{E}^\infty_{*,*}(\bar{H}) = \mathrm{E}^2_{*,*}(\bar{H}) = \Lambda_{\bar{H}_*BP}(\varepsilon_r : r \geqslant 2).$$

Thus we recover the well-known result that

$$\bar{H}_*\ell = \mathbb{F}_p[t_k : k \geqslant 1] \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}(\varepsilon_r : r \geqslant 2),$$

where t_k has degree $2p^k - 2$ and ε_r has degree $2p^r - 1$.

From Propositions 1.2 and 2.1 we have

$$\mathrm{Tor}^{MU_*}_{*,*}(L_*MU, \ell_*) = L_*BP/(v_r : r \geqslant 2),$$
$$\mathrm{Tor}^{MU_*}_{*,*}(\bar{L}_*MU, \ell_*) = \bar{L}_*BP/(v_r : r \geqslant 2),$$

where $\bar{L} = L/p$ denotes the spectrum L smashed with the mod p Moore spectrum. As a consequence, the Künneth spectral sequences for $L_*\ell$ and $\bar{L}_*\ell$ degenerate to give

$$L_*BP/(v_r : r \geqslant 2) \cong L_*\ell, \quad \bar{L}_*BP/(v_r : r \geqslant 2) \cong \bar{L}_*\ell.$$

Since L_*MU is a free $\mathbb{Z}_{(p)}$-module, multiplication by p gives an exact sequence of right MU_*-modules

$$0 \to L_*MU \xrightarrow{p} L_*MU \longrightarrow \bar{L}_*MU \to 0$$

which in turn induces a long exact sequence on the homological functor $\mathrm{Tor}^{MU_*}_*(\quad, \ell_*)$ which collapses to the short exact sequence

$$0 \to \mathrm{Tor}^{MU_*}_{0,*}(L_*MU, \ell_*) \xrightarrow{p} \mathrm{Tor}^{MU_*}_{0,*}(L_*MU, \ell_*) \longrightarrow \mathrm{Tor}^{MU_*}_{0,*}(\bar{L}_*MU, \ell_*) \to 0.$$

From this we see that there is a short exact sequence

$$0 \to L_*\ell \xrightarrow{p} L_*\ell \longrightarrow \bar{L}_*\ell \to 0.$$

On tensoring with $\mathbb{Q}$ we easily see that $\mathbb{Q} \otimes \ell_*\ell \longrightarrow \mathbb{Q} \otimes L_*\ell$ is a monomorphism. Hence we have

Proposition 5.2. *$L_*\ell$ has no p-torsion and the natural map $\ell_*\ell \longrightarrow L_*\ell$ induces an exact sequence*

$$0 \to {}_{p^\infty}(\ell_*\ell) \longrightarrow \ell_*\ell \longrightarrow L_*\ell.$$

Corollary 5.3. *We have*

$${}_{p^\infty}(\ell_*\ell) = {}_{u^\infty}(\ell_*\ell).$$

Proof. Since $\ell_* \longrightarrow L_* = \ell_*[u^{-1}]$ is a localization, we have $L_*\ell = \ell_*\ell[u^{-1}]$ and

$$\ker(\ell_*\ell \longrightarrow L_*\ell) = {}_{u^\infty}(\ell_*\ell),$$

hence ${}_{u^\infty}(\ell_*\ell) = {}_{p^\infty}(\ell_*\ell)$. □

In order to gain control over the p-torsion in $\mathrm{Tor}^{MU_*}_{*,*}(\ell_* MU, \ell_*)$, we will exploit the cofibre sequence

$$\ell \xrightarrow{p} \ell \xrightarrow{\varrho} \bar{\ell} \xrightarrow{\delta} \Sigma\ell, \tag{6.1}$$

where $\bar{\ell} = \ell/p$. To this end we will relate the geometric connecting morphisms of cofibre sequences to morphisms of Künneth spectral sequences. The method of proof we use in this part is analogous to that of the geometric boundary theorem in [14, II.3].

Let W be a cofibrant R-module which we fix from now on. Then for any R-module Z there is a Künneth spectral sequence with

$$\mathrm{E}^2_{s,t}(Z) = \mathrm{Tor}^{R_*}_{s,t}(Z_*, W_*) \Longrightarrow \pi_*(Z \wedge_R W).$$

Lemma 6.1. *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

be a cofibre sequence of R-modules with $X \simeq \bigvee_{i=1}^m \Sigma^{n_i} R$ and $\pi_* f$ surjective. Then there is a map of Künneth spectral sequences

$$\mathrm{E}^r_{s,t}(Y) \xrightarrow{\psi^r} \mathrm{E}^r_{s-1,t}(\Sigma^{-1} Z) \quad (r \geqslant 2),$$

such that ψ^2 is the connecting homomorphism

$$\mathrm{Tor}^{R_*}_{s,t}(Y_*, W_*) \xrightarrow{\cong} \mathrm{Tor}^{R_*}_{s-1,t}((\Sigma^{-1}Z)_*, W_*).$$

Proof. Since $\pi_* f$ is surjective, there is a short exact sequence

$$0 \to (\Sigma^{-1}Z)_* \longrightarrow \bigoplus_{i=1}^m \Sigma^{n_i} R_* \longrightarrow Y_* \to 0.$$

This induces a long exact sequence of Tor-groups, in which every third term is trivial, because $\bigoplus_{i=1}^m \Sigma^{n_i} R_*$ is R_*-free. Therefore we have an isomorphism

$$\mathrm{Tor}^{R_*}_{s,t}(Y_*, W_*) \xrightarrow{\cong} \mathrm{Tor}^{R_*}_{s-1,t}((\Sigma^{-1}Z)_*, W_*).$$

On the level of projective resolutions, we can splice a resolution $P_{\bullet,*}$ for Y_* together with a resolution $Q_{\bullet,*}$ of $(\Sigma^{-1}Z)_*$ to obtain a trivial split resolution for $\bigoplus_{i=1}^m \Sigma^{n_i} R_*$. Thus we obtain a map between exact couples and so obtain the desired map of spectral sequences. □

Theorem 6.2. *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

be a cofibre sequence with $\pi_* f$ surjective. Then there is an induced map of Künneth spectral sequences

$$\mathrm{E}^r_{s,t}(Y) \xrightarrow{\varphi^r} \mathrm{E}^r_{s-1,t}(\Sigma^{-1} Z) \quad (r \geqslant 2)$$

such that φ^2 is the connecting homomorphism

$$\mathrm{Tor}^{R_*}_{s,t}(Y_*, W_*) \xrightarrow{\cong} \mathrm{Tor}^{R_*}_{s-1,t}((\Sigma^{-1}Z)_*, W_*).$$

Proof. Choose a map $f' \colon \bigvee_{i=1}^m \Sigma^{n_i} R \longrightarrow Y$ with $\pi_* f'$ surjective and consider the cofibre sequence

$$\bigvee_{i=1}^m \Sigma^{n_i} R \xrightarrow{f'} Y \xrightarrow{j} \mathrm{cone}(f').$$

By Lemma 6.1 there is a map of Künneth spectral sequences

$$\mathrm{E}^r_{s,t}(Y) \xrightarrow{\psi^r} \mathrm{E}^r_{s-1,t}(\Sigma^{-1}\,\mathrm{cone}(f')).$$

As $\pi_* f$ is surjective, the composition $g \circ f'$ is trivial and there is a factorization $g = \xi \circ j$.

$$\begin{tikzcd} & \mathrm{cone}(f') \arrow[dr, dotted, "\xi"] & & \\ X \arrow[r, "f"] & Y \arrow[u, "j"] \arrow[r, "g"] & Z \arrow[r, "h"] & \Sigma X \\ & \bigvee_{i=1}^{m} \Sigma^{n_i} R \arrow[u, "f'"] & & \end{tikzcd}$$

Now we may define φ^r to be $(\Sigma^{-1}\xi)_* \circ \psi^r$. $\square$

For the connective Adams summand ℓ, we will consider the cofibre sequence

$$\ell \xrightarrow{\varrho} \bar{\ell} \xrightarrow{\delta} \Sigma\ell \xrightarrow{\Sigma p} \Sigma\ell$$

obtained from (6.1). The reduction map ϱ is surjective in homotopy and therefore we can apply Theorem 6.2 to obtain a map of Künneth spectral sequences

$$\mathrm{E}^r_{s,t}(\bar{\ell} \wedge MU) \xrightarrow{\varphi^r} \mathrm{E}^r_{s-1,t}(\ell \wedge MU) \quad (r \geqslant 2).$$

In particular, this yields a connecting homomorphism

$$\varphi^2\colon \mathrm{Tor}^{MU_*}_{s,t}(\bar{\ell}_* MU, \ell_*) \longrightarrow \mathrm{Tor}^{MU_*}_{s-1,t}(\ell_* MU, \ell_*).$$

Theorem 6.3. *Each p-torsion element of $\mathrm{Tor}^{MU_*}_{*,*}(\ell_* MU, \ell_*)$ is the image of an element of $\mathrm{Tor}^{MU_*}_{*+1,*}(\bar{\ell}_* MU, \ell_*)$ under the connecting homomorphism and is an infinite cycle.*

Proof. Making use of the long exact sequence on Tor-groups associated to the short exact sequence

$$0 \to \ell_* \xrightarrow{p} \ell_* \xrightarrow{\varrho_*} \bar{\ell}_* \to 0,$$

the claim about the p-torsion in $\mathrm{Tor}^{MU_*}_{*,*}(\ell_* MU, \ell_*)$ follows.

We will prove that the elements $\Delta_u(i_1, \ldots, i_n)$ are infinite cycles in the Künneth spectral sequence for $\bar{\ell}_*\ell$. Our proof is by induction on $n \geqslant 2$. For $n = 2$ the elements $\Delta_u(i_1, i_2)$ are infinite cycles for degree reasons. Now suppose that for all $n \leqslant k$, the $\Delta_u(i_1, \ldots, i_n)$ are infinite cycles. From Proposition 4.1 we know that

$$w_{i_2}\Delta_u(i_1, \ldots, i_{k+1}) = \Delta_u(i_1, i_2)\Delta_u(i_2, \ldots, i_{k+1}).$$

By assumption, the two factors are infinite cycles and therefore their product is an infinite cycle as well. The scalar factor w_{i_2} acts as a regular element on the $R/(u)$-module generated by the Δ_u elements, so we can conclude that $\Delta_u(i_1, \ldots, i_{k+1})$ has to be an infinite cycle as well.

So the Künneth spectral sequence for $\bar{\ell}_*\ell$ collapses at the E^2-page and as the connecting homomorphism induces a map of spectral sequences, every p-torsion class in $\mathrm{Tor}^{MU_*}_{*,*}(\ell_* MU, \ell_*)$ has to be an infinite cycle. $\square$

Theorem 6.3 gives an explicit description of the p-torsion classes in $\mathrm{Tor}^{MU_*}_{*,*}(\ell_* MU, \ell_*)$ as the image of the elements $\Delta_u(i_1, \ldots, i_n)$ in $\mathrm{Tor}^{MU_*}_{*,*}(\bar{\ell}_* MU, \ell_*)$ under the boundary homorphism.

Corollary 6.4. *Since the $\Delta_u(i_1, \ldots, i_n)$ generate the Künneth spectral sequence for $\bar{\ell}_*\ell$ additively, this spectral sequence collapses at the E^2-page. For the same reasons, the Künneth spectral sequence for $\ell_*\ell$ collapses as well.*

Remark 6.5. To summarize, the calculation of the rational homology of ℓ in (5.1) tells us that the torsion-free part of $\ell_*\ell$ has to have its origin in the zero-line of the Künneth spectral sequence. The torsion part is imported from the Künneth spectral sequence for $\bar{\ell}_*\ell$ via the geometric boundary result. The Künneth spectral sequence for $\ell_*\ell$ collapses at the E^2-page. Furthermore, Corollary 7.4 implies that there are no extension problems.

7. Detecting homotopy in the Adams spectral sequence

In this section we recall some results about the classical Adams spectral sequence for $\ell_*\ell$. We make heavy use of standard facts about Hopf algebras and the Steenrod algebra [13, 12]. In the following we generically write I for identity morphisms, φ for products and actions, ψ for coproducts and coactions, η for units and ε for counits and we use $\bar{x}$ for the antipode on an element x. Undecorated tensor products are taken over the ground field.

We write $\bar{H}_*(\ )$ for $H_*(\ ;\mathbb{F}_p)$ and $\mathcal{A}_*$ for the dual Steenrod algebra,

$$\mathcal{A}_* = \mathbb{F}_p[\zeta_n : n \geqslant 1] \otimes \Lambda(\bar{\tau}_n : n \geqslant 0),$$

where the coaction is given by

$$\psi(\zeta_n) = \sum_{i=0}^{n} \zeta_i \otimes \zeta_{n-i}^{p^i}, \quad \psi(\bar{\tau}_n) = 1 \otimes \bar{\tau}_n + \sum_{i=0}^{n} \bar{\tau}_i \otimes \zeta_{n-i}^{p^i}.$$

The sub-comodule algebra

$$\mathcal{B}_* = \mathbb{F}_p[\zeta_n : n \geqslant 1] \otimes \Lambda(\bar{\tau}_n : n \geqslant 2)$$

gives rise to a quotient Hopf algebra

$$\mathcal{E}_* = \mathcal{A}_* /\!/ \mathcal{B}_* = \Lambda(\alpha, \beta),$$

where α, β are the residue classes of $\bar{\tau}_0, \bar{\tau}_1$ respectively. Then

$$\mathcal{B}_* = \mathcal{A}_* \square_{\mathcal{E}_*} \mathbb{F}_p.$$

Now the natural map $\ell \longrightarrow \bar{H}$ induces an isomorphism

$$\bar{H}_*(\ell) \xrightarrow{\cong} \mathcal{B}_* \subseteq \mathcal{A}_*$$

and there are isomorphisms of $\mathcal{A}_*$-comodule algebras

$$\bar{H}_*(\ell \wedge \ell) \xrightarrow{\cong} \bar{H}_*(\ell) \otimes \bar{H}_*(\ell) \xrightarrow{\cong} \mathcal{B}_* \otimes \mathcal{B}_* \xrightarrow{\cong} \mathcal{A}_* \square_{\mathcal{E}_*} \mathcal{B}_*. \tag{7.1}$$

The E_2-term of the Adams spectral sequence converging to $\pi_*(\ell \wedge \ell) = \ell_*\ell$ has the form

$$\mathrm{E}^2_{s,t} = \mathrm{Cotor}^{\mathcal{A}_*}_{s,t}(\mathbb{F}_p, \bar{H}_*(\ell \wedge \ell)) \cong \mathrm{Cotor}^{\mathcal{A}_*}_{s,t}(\mathbb{F}_p, \mathcal{A}_* \square_{\mathcal{E}_*} \mathcal{B}_*)$$

and so by making use of a standard change of rings result, we have

$$\mathrm{E}^2_{s,t} \cong \mathrm{Cotor}^{\mathcal{E}_*}_{s,t}(\mathbb{F}_p, \mathcal{B}_*). \tag{7.2}$$

Note that by results of [5], the torsion in $\ell_*\ell$ is detected by the edge homomorphism (which is essentially the Hurewicz homomorphism) into the 0-line

$$\mathrm{E}^2_{0,*} \cong \mathrm{Cotor}^{\mathcal{E}_*}_{0,*}(\mathbb{F}_p, \mathcal{B}_*) = \mathbb{F}_p \square_{\mathcal{E}_*} \mathcal{B}_*.$$

The map involved here is obtained by composing the following $\mathcal{A}_*$-comodule algebra homomorphisms and suitably restricting the codomain:

$$\begin{aligned}\pi_*(\ell \wedge \ell) \longrightarrow \bar{H}_*(\ell) \otimes \bar{H}_*(\ell) \xrightarrow{\cong} \mathcal{B}_* \otimes \mathcal{B}_* \xrightarrow{I \otimes \psi} \mathcal{B}_* \otimes (\mathcal{A}_* \square_{\mathcal{E}_*} \mathcal{B}_*) \\ \xrightarrow{\varphi \otimes I} \mathcal{A}_* \square_{\mathcal{E}_*} \mathcal{B}_* \longrightarrow \mathcal{E}_* \square_{\mathcal{E}_*} \mathcal{B}_* \xrightarrow{\cong} \mathcal{B}_*.\end{aligned}$$

The final isomorphism is the composition

$$\mathcal{E}_* \square_{\mathcal{E}_*} \mathcal{B}_* \xrightarrow{\mathrm{incl}} \mathcal{E}_* \otimes \mathcal{B}_* \xrightarrow{\varepsilon \otimes I} \mathbb{F}_p \otimes \mathcal{B}_* \xrightarrow{\cong} \mathcal{B}_*.$$

A careful check of what the composition does on primitives shows that it can be expressed as

$$\pi_*(\ell \wedge \ell) \longrightarrow \bar{H}_*(\ell \wedge \ell) \xrightarrow{(\nu \wedge \mathrm{id})_*} \bar{H}_*(\ell), \tag{7.3}$$

where $\nu \colon \bar{H} \wedge \ell \longrightarrow \bar{H}$ is the natural pairing. In particular, this implies that the image of the Hurewicz map for $\ell \wedge \ell$ maps monomorphically into $\bar{H}_*(\ell)$.

It will be useful to know how to compute the inverse of the map

$$\mathbb{F}_p \square_{\mathcal{A}_*} (\mathcal{B}_* \otimes \mathcal{B}_*) \longrightarrow \mathbb{F}_p \square_{\mathcal{E}_*} \mathcal{B}_*.$$

This is just

$$\mathbb{F}_p\square_{\mathcal{E}_*}\mathcal{B}_* \xrightarrow{\text{incl}} \mathbb{F}_p\otimes\mathcal{B}_* \xrightarrow{I\otimes\psi} \mathbb{F}_p\otimes(\mathcal{A}_*\otimes\mathcal{B}_*),$$

whose image is in fact contained in $\mathbb{F}_p\square_{\mathcal{A}_*}(\mathcal{B}_*\otimes\mathcal{B}_*)$.

Given these results, we can use them to detect elements of $\ell_*\ell$ in $\mathcal{B}_*$, in particular we can detect the torsion this way. To do this, we need to understand $\mathcal{B}_*$ as an $\mathcal{E}_*$-comodule, in particular the non-trivial $\mathcal{E}_*$-parallelograms of the form

(7.4)
$$\begin{tikzcd}
x'' \arrow[d, "\alpha"'] & x \arrow[l, "-\beta"'] \arrow[d, "\alpha"] \\
x''' & x' \arrow[l, "\beta"]
\end{tikzcd}$$

in which the $\mathcal{E}_*$-coaction satisfies

(7.5) $\psi(x) = 1\otimes x+\alpha\otimes x'-\beta\otimes x''+\beta\alpha\otimes x''', \quad \psi(x') = 1\otimes x'+\beta\otimes x''', \quad \psi(x'') = 1\otimes x''+\alpha\otimes x'''.$

Then x''' is an element of $\mathbb{F}_p\square_{\mathcal{E}_*}\mathcal{B}_*$ which corresponds to an $H\mathbb{F}_p$ wedge summand in $\ell\wedge\ell$ and a correponding torsion element. Of course, these elements can be expressed in terms of the homology action of Q_0 and Q_1, *i.e.*,

$$x' = Q_0x, \quad x'' = -Q_1x, \quad x''' = Q_1Q_0x.$$

Now by Margolis [10, chapter 18 theorem 5] dualized to a homology version for $\mathcal{E}_*$-comodules tells us that $\mathcal{B}_*$ uniquely decomposes into a coproduct of comodules isomorphic (up to grading) to $\mathcal{E}_*$, together with a comodule containing no free summand and isomorphic to a coproduct of lightning flash comodules. The latter summand does not concern us for now since all the torsion in $\ell_*\ell$ comes from the $H\mathbb{F}_p$ wedge summands as above corresponding to the free summand. In fact, Adams and Priddy [2, proof of proposition 3.12] determine the stable type of the lightning flash comodules, in particular, the stable class of the $\mathcal{E}_*$-comodule $\mathcal{B}_*$ is shown to be

$$\bigotimes_{r\geqslant 0}(1+L_r+L_r^2+\cdots+L_r^{p-1}), \tag{7.6}$$

where

$$L_r = \Sigma^{a(r)}J^{b(r)}, \quad a(r)+b(r) = 2(p-1)p^r, \quad b(r) = p^{r-1}+\cdots+p+1.$$

Here $J = \mathcal{E}_*/\mathbb{F}_p$ is the coaugmentation coideal of $\mathcal{E}_*$, represented by the following diagram

$$\begin{tikzcd}
 & \bullet \arrow[dl, "\beta"'] \arrow[d, "\alpha"] \\
\bullet & \bullet
\end{tikzcd}$$

and Σ is the trivial comodule $\mathbb{F}_p$ assigned degree 1. Furthermore, all products are tensor products over $\mathbb{F}_p$ taken in the stable comodule category of $\mathcal{E}_*$.

Now the most obvious candidates for the tops of $\mathcal{E}_*$-parallelograms are the elements

$$\bar\tau_{i_1}\bar\tau_{i_2}\cdots\bar\tau_{i_{n+1}} \qquad (1<i_1<i_2<\cdots<i_{n+1}, \quad n\geqslant 1).$$

These can be multiplied by monomials in the ζ_j to obtain others.

Theorem 7.1. *Consider the $\mathbb{F}_p$-vector subspace $\mathcal{V}\subseteq\mathbb{F}_p\square_{\mathcal{E}_*}\mathcal{B}_*$ spanned by $\mathbb{F}_p[\zeta_i : i\geqslant 1]$-scalar multiples of the elements* 1 *and*

$$Q_1Q_0(\bar\tau_{i_1}\bar\tau_{i_2}\cdots\bar\tau_{i_{n+1}}) \qquad (1<i_1<i_2<\cdots<i_{n+1}, \quad n\geqslant 1). \tag{7.7}$$

Then $\mathcal{V}$ consists of all the elements in $\mathbb{F}_p\square_{\mathcal{E}_*}\mathcal{B}_*$ which are the images of torsion elements under the composition of the Hurewicz homomorphism $\pi_*(\ell\wedge\ell)\longrightarrow\bar H_*(\ell\wedge\ell)$ and the identification of the homology $\bar H_*(\ell\wedge\ell)$ with $\mathbb{F}_p\square_{\mathcal{E}_*}\mathcal{B}_*$.

Proof. Clearly $\mathbb{F}_p[\zeta_i : i \geqslant 1] \subseteq \mathbb{F}_p\square_{\mathcal{E}_*}\mathcal{B}_*$. Now we know that the Künneth spectral sequence for $\ell_*\ell$ collapses and there are no additive extension problems. We need to understand the mod p Hurewicz images of elements represented by the elements arising from the $\Delta_u(i_1, \ldots, i_{s+2})$ in $\mathrm{Tor}^{MU_*}_{s+1,*}(\bar{\ell}_*MU, \ell_*)$, since these will give an additive basis for the p-torsion in $\ell_*\ell$.

$$
\begin{tikzcd}
\mathrm{Tor}^{MU_*}_{s+1,*}(\bar{\ell}_*MU, \ell_*) \arrow[dr, bend left, "\delta"] & & & \\
& \mathrm{Tor}^{MU_*}_{s,*}(\ell_*MU, \ell_*) \arrow[r, dash, dotted] \arrow[d] & \pi_*(\ell \wedge \ell) \arrow[d] \arrow[dr, bend left] & \\
& \mathrm{Tor}^{MU_*}_{s,*}((\bar{H} \wedge \ell)_*MU, \ell_*) \arrow[r, dash, dotted] \arrow[d] & \pi_*(\bar{H} \wedge \ell \wedge \ell) \arrow[d] & \mathbb{F}_p\square_{\mathcal{A}_*}\bar{H}_*(\ell \wedge \ell) \arrow[l] \arrow[d] \\
& \mathrm{Tor}^{MU_*}_{s,*}(\bar{H}_*MU, \ell_*) \arrow[r, dash, dotted] & \bar{H}_*(\ell) & \mathbb{F}_p\square_{\mathcal{E}_*}\bar{H}_*(\ell) \arrow[l]
\end{tikzcd}
$$

The Künneth spectral sequence (2.1) for $E_*\ell$ is natural for maps of ring spectra $E \longrightarrow F$. Therefore the map (7.3) corresponds in the spectral sequence to the composition of the two vertical maps in the left column in the diagram above. As the Hurewicz homomorphism has its image in the primitives of $\bar{H}_*(\ell \wedge \ell)$, it follows that the elements $\Delta_u(i_1, \ldots, i_{s+2})$ up to a unit map to

$$Q_0Q_1(\bar{\tau}_{i_1} \cdots \bar{\tau}_{i_{s+2}}) = \sum_{1<t<r\leqslant s+2} (-1)^{r+t}(\zeta_{i_r}\zeta^p_{i_t-1} - \zeta_{i_t}\zeta^p_{i_r-1})\,\bar{\tau}_{i_1}\bar{\tau}_{i_2} \cdots \widehat{\bar{\tau}}_{i_t} \cdots \widehat{\bar{\tau}}_{i_r} \cdots \bar{\tau}_{i_{s+2}}. \qquad \square$$

Remark 7.2. Since the torsion in $\pi_*(\ell \wedge \ell)$ maps injectively into $\mathbb{F}_p\square_{\mathcal{A}_*}(\mathcal{B}_* \otimes \mathcal{B}_*)$ which in turn is identified with $\mathbb{F}_p\square_{\mathcal{E}_*}\mathcal{B}_*$, Theorem 7.1 shows that the elements $Q_1Q_0(\bar{\tau}_{i_1}\bar{\tau}_{i_2} \cdots \bar{\tau}_{i_n})$ with $n \geqslant 3$ correspond to nilpotent elements; only elements of the form $Q_1Q_0(\bar{\tau}_r\bar{\tau}_s)$ are not nilpotent.

From Corollary 5.3 we know that the p-torsion and u-torsion in $\ell_*\ell$ agree. We recall a fact from [5, proposition 9.1].

Proposition 7.3. *All torsion in $\ell_*\ell$ is simple,* i.e., *for every torsion-class $x \in \ell_*\ell$ we have $px = 0$ which is equivalent to $ux = 0$.*

Corollary 7.4. *The Künneth spectral sequence for $\ell_*\ell$ collapses at the* E^2*-page and there are no non-trivial extensions.*

Example 7.5. For every prime p, the first torsion class in $\mathrm{Tor}^{BP_*}_{*,*}(\ell_*BP, \ell_*)$ occurs in degree $2(p^3 + p^2 - p - 1)$ and this class survives to $\ell_*\ell$. The lowest degree element appearing as the bottom of a parallelogram is

$$Q_1Q_0(\bar{\tau}_2\bar{\tau}_3) = \zeta_2^{p+1} - \zeta_1^p\zeta_3.$$

The coaction map ψ sends this element to the Hurewicz image of the corresponding torsion element of $\ell_*\ell$ in $\bar{H}_*(\ell \wedge \ell)$.

8. Multiplicative structure of $\ell_*\ell$

In this section we establish congruence relations in the zero line of the Künneth spectral sequence. These are derived in BP_*BP and mapped under the natural map. In fact they are first produced in $\mathbb{Q} \otimes BP_*BP$ then interpreted in the subring BP_*BP.

We describe the map from the torsion-free part of $\ell_*\ell$ to $\ell_*\ell \otimes \mathbb{Q}$ and summarize our results about the multiplicative structure of $\ell_*\ell$ at the end of this section.

It will be useful to have the following generalization of a well-known result (which corresponds to the case where $t = 1$).

Lemma 8.1. *Let R be a commutative ring, p a prime and $t \in R$. If $x, y, z \in R$ satisfy $z \equiv px + ty \bmod (pt)$, then for all $k \geqslant 0$,*

$$z^{p^k} \equiv p^{p^k} x^{p^k} + t^{p^k} y^{p^k} \bmod (p^{k+1}t).$$

Proof. We prove this by induction on k, the case $k = 0$ being known. Suppose it is true for some $k \geqslant 0$. Choose a $w \in R$ for which

$$z^{p^k} = p^{p^k} x^{p^k} + t^{p^k} y^{p^k} + p^{k+1}tw.$$

Then working $\bmod(p^{k+2}t)$ we have

$$\begin{aligned}
z^{p^{k+1}} &= (p^{p^k} x^{p^k} + t^{p^k} y^{p^k})^p + p^{k+2}t^p w^p + \sum_{1 \leqslant i \leqslant p-1} \binom{p}{i} (p^{p^k} x^{p^k} + t^{p^k} y^{p^k})^{p-i} p^{k+1+i} t^i w^i \\
&\equiv (p^{p^k} x^{p^k} + t^{p^k} y^{p^k})^p \\
&\equiv p^{p^{k+1}} x^{p^{k+1}} + t^{p^{k+1}} y^{p^{k+1}} + \sum_{1 \leqslant i \leqslant p-1} \binom{p}{i} p^{ip^k} x^{ip^k} t^{(p-i)p^k} y^{(p-i)p^k} \\
&\equiv p^{p^{k+1}} x^{p^{k+1}} + t^{p^{k+1}} y^{p^{k+1}}.
\end{aligned}$$

Hence the result holds for $k + 1$. $\square$

We will work with the Hazewinkel generators v_n of (1.2). The following standard formula for the right unit $\eta_R \colon \mathbb{Q} \otimes BP_* \longrightarrow \mathbb{Q} \otimes BP_*BP$ can be found in [15, p24]:

$$\eta_R(\ell_n) = \sum_{0 \leqslant j \leqslant n} \ell_j t_{n-j}^{p^j}. \tag{8.1}$$

On combining this with (1.2) we obtain

$$\eta_R(v_n) = \sum_{0 \leqslant i \leqslant n} p\ell_i t_{n-i}^{p^i} - \sum_{\substack{1 \leqslant i \leqslant n-1 \\ 0 \leqslant j \leqslant i}} \ell_j t_{i-j}^{p^j} \eta_R(v_{n-i})^{p^i}$$

and hence

$$\eta_R(v_n) = \sum_{0 \leqslant i \leqslant n} p\ell_i t_{n-i}^{p^i} - \sum_{0 \leqslant i \leqslant n-1} \ell_i t_{n-1-i}^{p^i} \eta_R(v_1)^{p^{n-1}} - \sum_{\substack{1 \leqslant i \leqslant n-2 \\ 0 \leqslant j \leqslant i}} \ell_j t_{i-j}^{p^j} \eta_R(v_{n-i})^{p^i}. \tag{8.2}$$

Remark 8.2. The left hand side of equation (8.2) lies in $BP_*BP \subseteq \mathbb{Q} \otimes BP_*BP$, therefore so does the right hand side. However, because of the presence of denominators in the terms involving the ℓ_r, care needs to be exercised when using this equation. For example, since $cp_r = p^r \ell_r \in BP_*$ we can certainly deduce that in BP_*BP modulo the ideal $(\eta_R(v_2), \ldots, \eta_R(v_{n-1})) \triangleleft BP_*BP$,

$$\begin{aligned}
&p^{n-1}\eta_R(v_n) \equiv \\
&\qquad \sum_{0 \leqslant i \leqslant n} p^{n-i} cp_i t_{n-i}^{p^i} - \sum_{0 \leqslant i \leqslant n-1} p^{n-1-i} cp_i t_{n-1-i}^{p^i} \eta_R(v_1)^{p^{n-1}} \bmod (\eta_R(v_2), \ldots, \eta_R(v_{n-1})).
\end{aligned}$$

We will see later that similar phenomena in ℓ_*BP give rise to congruences in $\ell_*\ell$.

We will now derive some formulæ in ℓ_*BP. The natural map of ring spectra $BP \longrightarrow \ell$ is determined on homotopy by

$$v_r \longmapsto \begin{cases} u & \text{if } r = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8.3}$$

Recalling (5.3), we see that in $\operatorname{im}[H_*\ell \longrightarrow H\mathbb{Q}_*\ell]$, the logarithm series for the factor of ℓ is

$$\log^\ell T = \sum_{n \geqslant 0} \ell_n T^{p^n} = \sum_{n \geqslant 0} \frac{u^{p^{n-1}+\cdots+p+1}}{p^n} T^{p^n}.$$

We can project (8.2) into ℓ_*BP, with η_R being replaced by the ℓ-theory Hurewicz homomorphism $\underline{\ell}\colon BP_* \longrightarrow \ell_*BP$. This yields

$$\begin{aligned}\underline{\ell}(v_n) &= pt_n - t_{n-1}\underline{\ell}(v_1)^{p^{n-1}}\\ &\quad + \sum_{1\leqslant i\leqslant n} \frac{u^{p^{i-1}+\cdots+p+1}t_{n-i}^{p^i}}{p^{i-1}} - \sum_{1\leqslant i\leqslant n-1} \frac{u^{p^{i-1}+\cdots+p+1}t_{n-1-i}^{p^i}\underline{\ell}(v_1)^{p^{n-1}}}{p^i}\\ &\quad - \sum_{1\leqslant i\leqslant n-2} t_i\underline{\ell}(v_{n-i})^{p^i} - \sum_{\substack{1\leqslant i\leqslant n-2\\ 1\leqslant j\leqslant i}} \frac{u^{p^{j-1}+\cdots+p+1}t_{i-j}^{p^j}\underline{\ell}(v_{n-i})^{p^i}}{p^j}.\end{aligned}$$

and the equivalent formula

$$\begin{aligned}(8.4)\quad \underline{\ell}(v_n) &= pt_n + (ut_{n-1}^p - \underline{\ell}(v_1)^{p^{n-1}}t_{n-1})\\ &\quad + \sum_{1\leqslant i\leqslant n-1} \frac{u^{p^{i-1}+\cdots+p+1}(u^{p^i}t_{n-1-i}^{p^{i+1}} - \underline{\ell}(v_1)^{p^{n-1}}t_{n-1-i}^{p^i})}{p^i}\\ &\quad - \sum_{1\leqslant i\leqslant n-2} t_i\underline{\ell}(v_{n-i})^{p^i} - \sum_{\substack{1\leqslant i\leqslant n-2\\ 1\leqslant j\leqslant i}} \frac{u^{p^{j-1}+\cdots+p+1}t_{i-j}^{p^j}\underline{\ell}(v_{n-i})^{p^i}}{p^j}.\end{aligned}$$

Thus we have

$$\begin{aligned}\underline{\ell}(v_2) &= pt_2 + (ut_1^p - \underline{\ell}(v_1)^pt_1) + \frac{u(u^p - \underline{\ell}(v_1)^p)}{p}\\ &= pt_2 + (1-p^{p-1})ut_1^p - \underline{\ell}(v_1)^pt_1 - \sum_{1\leqslant i\leqslant p-1}\binom{p}{i}p^{i-1}u^{p+1-i}t_1^i.\end{aligned}$$

By the Hattori-Stong theorem, the element $\underline{\ell}(v_n) \in \ell_*BP$ is not divisible by p, but notice that on multiplying by p^{n-2} we have

$$\begin{aligned}p^{n-2}\underline{\ell}(v_n) &= p^{n-1}t_n + p^{n-2}(ut_{n-1}^p - \underline{\ell}(v_1)^{p^{n-1}}t_{n-1})\\ &\quad + \sum_{1\leqslant i\leqslant n-1} \frac{u^{p^{i-1}+\cdots+p+1}(u^{p^i}t_{n-1-i}^{p^{i+1}} - \underline{\ell}(v_1)^{p^{n-1}}t_{n-1-i}^{p^i})}{p^{i-n+2}}\\ &\quad - \sum_{1\leqslant i\leqslant n-2} p^{n-2}t_i\underline{\ell}(v_{n-i})^{p^i} - \sum_{\substack{1\leqslant i\leqslant n-2\\ 1\leqslant j\leqslant i}} p^{n-2-j}u^{p^{j-1}+\cdots+p+1}t_{i-j}^{p^j}\underline{\ell}(v_{n-i})^{p^i}.\end{aligned}$$

and so

$$\begin{aligned}&p^{n-1}t_n + p^{n-2}(ut_{n-1}^p - \underline{\ell}(v_1)^{p^{n-1}}t_{n-1})\\ &\quad + \sum_{1\leqslant i\leqslant n-1} \frac{u^{p^{i-1}+\cdots+p+1}(u^{p^i}t_{n-1-i}^{p^{i+1}} - \underline{\ell}(v_1)^{p^{n-1}}t_{n-1-i}^{p^i})}{p^{i-n+2}} \equiv 0 \mod (\underline{\ell}(v_2),\ldots,\underline{\ell}(v_n)).\end{aligned}$$

Using the identity $\underline{\ell}(v_1) = u + pt_1$ and the resulting congruences (see Lemma 8.1),

$$\underline{\ell}(v_1)^{p^m} \equiv u^{p^m} \bmod (p^{m+1}) \qquad (m \geqslant 1),$$

we deduce that when $n \geqslant 2$,

$$(8.5) \quad \underline{\ell}(v_n) \equiv (pt_n - p^{p^{n-1}} t_1^{p^{n-1}} t_{n-1}) + (ut_{n-1}^p - u^{p^{n-1}} t_{n-1}) + \sum_{1 \leqslant i \leqslant n-2} \frac{u^{p^{i-1}+\cdots+p+1}(u^{p^i} t_{n-1-i}^{p^{i+1}} - u^{p^{n-1}} t_{n-1-i}^{p^i})}{p^i} - \sum_{1 \leqslant i \leqslant n-2} t_i \underline{\ell}(v_{n-i})^{p^i} - \sum_{\substack{1 \leqslant i \leqslant n-2 \\ 1 \leqslant j \leqslant i}} \frac{u^{p^{j-1}+\cdots+p+1} t_{i-j}^{p^j} \underline{\ell}(v_{n-i})^{p^i}}{p^j} \mod (pu).$$

Thus when $n = 2$ we have

$$\begin{aligned}
\underline{\ell}(v_2) &\equiv (pt_2 - p^p t_1^p t_1) + (ut_1^p - u^p t_1) \mod (pu) \\
&\equiv ut_1^p - u^p t_1 \mod (p).
\end{aligned}$$

When working in the image of the rationalization map $H_*(\ell \wedge \ell) \longrightarrow H\mathbb{Q}_*(\ell \wedge \ell)$, we will denote by u and v the images of $u \in \ell_{2p-2}$ under the left and right units for $\ell \wedge \ell$.

Now reinterpreting (8.2) in $H\mathbb{Q}_*(\ell \wedge \ell)$, for each $n \geqslant 2$ we have $\eta_R(v_n) \longmapsto 0$ and so

$$pt_n + ut_{n-1}^p + \sum_{1 \leqslant h \leqslant n-1} \frac{u^{(p^h+p^{h-1}+\cdots+p+1)} t_{n-h-1}^{p^{h+1}}}{p^h} = t_{n-1} v^{p^{n-1}} + \sum_{1 \leqslant k \leqslant n-1} \frac{u^{(p^{k-1}+p^{k-2}+\cdots+p+1)} t_{n-1-k}^{p^k} v^{p^{n-1}}}{p^k}.$$

On rearranging this, we obtain

$$(8.6) \qquad pt_n = v^{p^{n-1}} t_{n-1} - ut_{n-1}^p + \sum_{1 \leqslant k \leqslant n-1} \frac{u^{(p^{k-1}+p^{k-2}+\cdots+p+1)}(v^{p^{n-1}} t_{n-1-k}^{p^k} - u^{p^k} t_{n-k-1}^{p^{k+1}})}{p^k}.$$

For small values of $n = 1$ we have

$$\begin{aligned}
pt_1 &= v - u, \\
pt_2 &= v^p t_1 - ut_1^p + \frac{u(v^p - u^p)}{p}, \\
pt_3 &= v^{p^2} t_2 - ut_2^p + \frac{u(v^{p^2} t_1^p - u^p t_1^{p^2})}{p} + \frac{u^{p+1}(v^{p^2} - u^{p^2})}{p^2}, \\
pt_4 &= v^{p^3} t_3 - ut_3^p + \frac{u(v^{p^3} t_2^p - u^p t_2^{p^2})}{p} + \frac{u^{p+1}(v^{p^3} t_1^{p^2} - u^{p^2} t_1^{p^3})}{p^2} + \frac{u^{p^2+p+1}(v^{p^3} - u^{p^3})}{p^3}.
\end{aligned}$$

We want to draw some general conclusions about these expressions.

Lemma 8.3. *In $\ell_*\ell$, for $n \geqslant 1$, we have the congruences*

$$(8.7) \qquad pt_n \equiv v^{p^{n-1}} t_{n-1} - ut_{n-1}^p \mod (pu),$$

$$(8.8) \qquad pt_n - p^{p^{n-1}} t_1^{p^{n-1}} \equiv u^{p^{n-1}} t_{n-1} - ut_{n-1}^p \mod (pu).$$

Proof. We will prove this by induction on n, the case $n = 1$ being noted above. So suppose that

$$pt_k \equiv v^{p^{k-1}} t_{k-1} - ut_{k-1}^p \mod (pu).$$

whenever $1 \leqslant k < n$ for some n. Then for every such k we have

$$v^{p^{k-1}} t_{k-1} \equiv ut_{k-1}^p \mod (p).$$

By Lemma 8.1, for every $m \geqslant 1$,

$$(v^{p^{k-1}} t_{k-1})^{p^m} \equiv (ut_{k-1}^p)^{p^m} \mod (p^{m+1}),$$

i.e.,

$$v^{p^{m+k-1}} t_{k-1}^{p^m} \equiv u^{p^m} t_{k-1}^{p^{m+1}} \bmod (p^{m+1}).$$

Now when $1 \leqslant k \leqslant n-1$,

$$v^{p^{n-1}} t_{n-1-k}^{p^k} - u^{p^k} t_{n-k-1}^{p^{k+1}} \equiv 0 \bmod (p^{k+1}),$$

hence in the formula for pt_n in (8.6), the summand

$$u^{(p^{k-1}+p^{k-2}+\cdots+p+1)} \frac{(v^{p^{n-1}} t_{n-1-k}^{p^k} - u^{p^k} t_{n-k-1}^{p^{k+1}})}{p^k}$$

must be divisible by pu. Therefore we have the congruence

$$pt_n \equiv v^{p^{n-1}} t_{n-1} - ut_{n-1}^p \bmod (pu).$$

Using the expansion

$$v^{p^{n-1}} = u^{p^{n-1}} + \sum_{1 \leqslant j \leqslant p^{n-1}} \binom{p^{n-1}}{j} u^{p^{n-1}-j} p^j t_1^j$$

we obtain

$$pt_n - p^{p^{n-1}} t_1^{p^{n-1}} \equiv u^{p^{n-1}} t_{n-1} - ut_{n-1}^p \bmod (pu). \qquad \square$$

Summary. Kane [5, (19:6:1)], using Adams' criterion [1, III,17.6], worked out what the image of the torsion-free part of $\ell_*\ell$ is when we pass to $\ell_*\ell \otimes \mathbb{Q}$. The generators for the image of $\ell_*\ell$ in $\ell_*\ell \otimes \mathbb{Q}$ are

$$t_{n,i} = \frac{u^i v(v-(p-1)u) \cdot \ldots \cdot (v-(n-1)(p-1)u)}{p^i}, \quad 0 \leqslant i \leqslant \nu_p(n!).$$

Obviously, the relation $ut_{n,i} = pt_{n,i+1}$ holds and it is clear how to multiply elements like that.

To summarize our results on the multiplicative structure of $\ell_*\ell$, we have the following:

- When we start with two non-torsion elements in $\ell_*\ell$, we can take their image in $\ell_*\ell \otimes \mathbb{Q}$, take their product there and interpret the result as a non-torsion element in $\ell_*\ell$.
- Any two elements coming from the zero-line of the Künneth spectral sequence multiply according to the congruence relations we specified in (8.3) up to (8.3). These element might be torsion or non-torsion, but there is no non-torsion in higher filtrations.
- Torsion elements in non-zero filtration have their origin in the generators Δ_u and for these we spelled out the multiplication in (4.2a).
- As the Δ_u-expression allow coefficients from ℓ_*BP, the multiplication of non-torsion elements in the zero-line with torsion elements in higher filtration is determined as well.

We agree that the recursive nature of the congruences for $\ell_*BP \otimes_{BP_*} \ell_*$ might hamper the calculation, but our approach leads to more information about the multiplication in $\ell_*\ell$ than the known sources (compare *e.g.*, [5, page with wrong adams formula]).

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