Fréchet Algebra Techniques for Boundary Value Problems on Noncompact Manifolds: Fredholm Criteria and Functional Calculus via Spectral Invariance

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Abstract: A Boutet de Monvel type calculus is developed for boundary value problems on (possibly) noncompact manifolds. It is based on a class of weighted symbols and Sobolev spaces. If the underlying manifold is compact, one recovers the standard calculus. The following is proven:

(1) The algebra $\mathcal{G}$ of Green operators of order and type zero is a spectrally invariant Fréchet subalgebra of $\mathcal{L}(H)$, H a suitable Hilbert space, i.e.

$$\mathcal{G} \cap \mathcal{L}(H)^{-1} = \mathcal{G}^{-1}.$$

(2) Focusing on the elements of order and type zero is no restriction since there are order reducing operators within the calculus.

(3) There is a necessary and sufficient criterion for the Fredholm property of boundary value problems, based on the invertibility of symbols modulo lower order symbols, and

(4) There is a holomorphic functional calculus for the elements of $\mathcal{G}$ in several complex variables.

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Contents

Introduction

After earlier work by Vishik and Eskin [63], Boutet de Monvel's calculus [2], established in 1971, showed a new way of treating boundary value problems by pseudodifferential methods and in the framework of operator algebras. In particular, the parametrix construction within the calculus gave necessary and sufficient conditions for the Fredholm property of boundary value problems on smooth compact manifolds generalizing the classical conditions of Lopatinski and Shapiro.
Functional calculus for boundary value problems on compact manifolds is a central topic of G. Grubb's 1986 monograph [18].
The present paper deals with both Fredholm criteria and functional calculus in the context of manifolds that may be noncompact, also with noncompact boundaries – in a situation, where the classical methods fail. It offers a new approach and a solution based on operator algebra techniques.
The class of manifolds I am considering is described in an axiomatic way. It includes Euclidean space, compact manifolds and all manifolds with finitely many cylindrical ends, in particular the manifolds 'Euclidean at Infinity' of Choquet-Bruhat and Christodoulou [5], those considered by Lockhart and McOwen [30], Rabinovich [35], Parenti [34], and considerably more. For more details see [13], and [49].
Instead of a direct analysis of particular boundary problems, I am developing a version of Boutet de Monvel's calculus adapted to the noncompact situation. There, I am focusing on the algebra $\mathcal{G}$ of elements of order and type zero. It is a Fréchet-*-subalgebra of $\mathcal{L}(H)$, where H is a Hilbert space these operators are naturally acting on. Moreover, I show that $\mathcal{G}$ is *spectrally invariant*: $\mathcal{G} \cap \mathcal{L}(H)^{-1} = \mathcal{G}^{-1}$.
The importance of spectral invariance in Fréchet algebras was observed by Gramsch [14]. He introduced the notion of Ψ^*-algebras: By definition, a *Ψ^*-subalgebra of $\mathcal{L}(H)$*, H a Hilbert space, is a spectrally invariant, symmetric, continuously embedded Fréchet subalgebra with the same unit.
Establishing the Ψ^*-property is the crucial step towards a variety of interesting results.
Here, I will first show a Fredholm criterion for boundary problems on noncompact manifolds; it is new even for differential problems on the half-space $\mathbf{R}^n_+$. Based on spectral invariance and general Fredholm theory, the proof is much simpler than earlier concepts that have been used e.g. in the case of classical pseudodifferential operators, based on variants of Gohberg's lemma.
In connection with results of Waelbroeck, the spectral invariance also yields a holomorphic functional calculus for the elements of $\mathcal{G}$ in several complex variables.
Let us specialize for a moment to the case where the underlying manifold is compact. The above algebra $\mathcal{G}$ then coincides with the standard Boutet de Monvel algebra constructed from pseudodifferential symbols in the Hörmander class $S^0_{1,0}$. Spectral invariance and the Ψ^*-property have been open questions also in this case.
Grubb had suggested a different method to obtain functional calculus in the compact situation [18]. She constructed a parameter-dependent version of Boutet de Monvel's calculus. The precise analysis of the resolvent then allowed various applications. As far as functional calculus is concerned, however, the present result not only extends those in [18] to several complex variables; it gives a stronger version even in one variable and

without the need to first establish a parameter-dependent calculus. Specializing further to the algebra of classical (pseudohomogeneous) elements, one recovers G. Grubb's theorem for one variable and gets an extension to several. For this case, however, B.-W. Schulze had proven spectral invariance in 1989, [52].
The Ψ^*-property gives access to results in perturbation theory, on non-abelian cohomology and Oka principle (in case the algebra is additionally submultiplicative), on analytic Fréchet submanifolds [14], [15], or for the division problem for operator-valued distributions [16]. Connes and Bost have shown that the K-theory of a Ψ^*-algebra coincides with that of its C^*-closure.
Spectral invariance for pseudodifferential operators was first proven by Beals in 1977, [1], cf. [62]. Since then it has shown to hold in many interesting cases ([9], [41], [43], [44], [45]), although it fails in slightly different situations, [67], [12].
In many algebras of pseudodifferential operators, there is a close connection between the facts that the algebra is spectrally invariant and that the Fredholm property can be characterized by ellipticity in a suitable sense. This has been observed and systematically exploited in [44] and [45]. Already in 1989, Schulze has shown how to deduce spectral invariance in an abstract setting, provided that only elliptic operators are Fredholm [52].
Questions related to spectral invariance naturally come up in analysis. While it is already important for the parametrix construction in the theory of boundary value problems that the inverses of the boundary symbol operators belong to the calculus (Boutet de Monvel [2], Grubb [18], Section 3.2, Rempel-Schulze [37], Section 3.1), the study of spectral invariance becomes indispensable in connection with the analysis of operators on manifolds with singularities, cf. Schulze [51], [53], [54]. The characterization of the Fredholm property and the parametrix construction require the invertibility of operator-valued symbols on various levels within the calculus, hence spectral invariance of algebras of operators on the lower-dimensional skeletons of the manifold.
Compared to the Banach algebra techniques established by H.O. Cordes and his associates, cf. [8], [10], [11], the present approach has the advantage that it yields existence and regularity results at the same time: An elliptic operator is a Fredholm operator, and whenever an operator in the calculus is Fredholm, there is a Fredholm inverse which is a parametrix in the calculus. Since all these operators respect the whole scale of Sobolev spaces, this allows conclusions on the regularity of solutions in the spirit of Weyl's lemma: As soon as the Fredholm property is established between one fixed pair of Sobolev spaces, the application of the parametrix will give a hold on the regularity of the solution given the regularity of the right hand side.

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Essential parts of these results were presented at the conference "25 Years of Microlocal Analysis", Irsee, in July 1990. Except for theorem 4.1 this paper contains a part of the results of the author's Habilitationsschrift, Mainz, Fall 1991. A concise version was published in [46].
Meanwhile, G. Grubb has proven spectral invariance for an algebra of boundary value problems on a different class of manifolds [20]; in that case, however, it seems impossible to deduce Fredholm criteria. Rabinovich [36] has obtained Fredholm criteria for boundary value problems on certain noncompact manifolds in terms of 'limit operators'.

1 The SG-Calculus for Pseudodifferential Operators. SG-Manifolds.

In order to overcome the basic difficulties stemming from the non-compactness of the underlying manifold, we are going to use symbol classes and Sobolev spaces with a very controlled behavior near infinity. On $\mathbf{R}^n$, the following concept is due to Shubin [58], Parenti [33], and Cordes [7].

1.1 Definition. For $m = (m_1, m_2) \in \mathbf{R}^2$, $SG^m = SG^m(\mathbf{R}^n)$ is the space of all smooth functions p on $\mathbf{R}^n \times \mathbf{R}^n$ such that for all multi-indices α, β

$$D_\xi^\alpha D_x^\beta p(x,\xi) \le C_{\alpha,\beta} \langle\xi\rangle^{m_1-|\alpha|} \langle x\rangle^{m_2-|\beta|}, \tag{1}$$

with $\langle x\rangle = (1+|x|^2)^{1/2}$. We will call m the *order* of the *symbol* p. The intersection $SG^{-\infty} = \bigcap SG^m$ is the space of *regularizing symbols.*
One may also introduce SG 'double' symbols: For $m = (m_1, m_2, m_3)$ we say that $p \in SG^m$, if it is a smooth function on $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n$ with

$$D_\xi^\alpha D_x^\beta D_y^\gamma p(x,y,\xi) = \mathrm{O}(\langle\xi\rangle^{m_1-|\alpha|} \langle x\rangle^{m_2-|\beta|} \langle y\rangle^{m_3-|\gamma|}) \tag{2}$$

for every choice of multi-indices α, β, γ.

In general, all symbols will take values in matrices. Occacionally, I will also consider the case that E and F are Hilbert spaces, $p(x,y,\xi) \in \mathcal{L}(E,F)$ and $\|D_\xi^\alpha D_x^\beta D_y^\gamma p\|_{\mathcal{L}(E,F)}$ satisfies the estimates (2). For the sake of clarity this will always be indicated.
As usual the pseudodifferential operator $\mathrm{Op}\, p$ or $p(x,D)$ associated with the symbol p is defined by

$$[p(x,D)f](x) = [\mathrm{Op}\, p]f(x) = (2\pi)^{-n} \int\int e^{i(x-y)\xi} p(x,y,\xi) f(y)\, dy\, d\xi;$$

this reduces to

$$(2\pi)^{-n/2} \int e^{ix\xi} p(x,\xi) \hat{f}(\xi)\, d\xi,$$

if p is independent of y.
Here, f is a rapidly decreasing function; $\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int e^{-ix\xi} f(x)\, dx$ is its Fourier transform, and p is called the symbol of $\mathrm{Op}\, p$.
Like in the standard theory, the SG double symbols play a minor role, since for $p \in SG^{(m_1,m_2,m_3)}$, there is a $q \in SG^{(m_1,m_2+m_3)}$ such that $\mathrm{Op}\, p = \mathrm{Op}\, q$.

1.2 Theorem. (Shubin, Parenti, Cordes) *The SG-classes are closed under compositions and adjoints: If p, q are SG-symbols of orders m and m', respectively, then $\mathrm{Op}\, p \circ \mathrm{Op}\, q = \mathrm{Op}\, r$ for a symbol r of order $m + m'$, and $(\mathrm{Op}\, p)^* = \mathrm{Op}\, s$, where s also has order m.*
The pseudodifferential operators with regularizing symbols are precisely the integral operators with rapidly decreasing kernel functions.

The SG-pseudodifferential operators naturally act on weighted Sobolev spaces.

1.3 Definition. For $s = (s_1, s_2) \in \mathbf{R}^2$ let

$$H^s = H^s(\mathbf{R}^n) = \{u \in \mathcal{S}'(\mathbf{R}^n) : \langle x\rangle^{s_2} (1-\Delta)^{s_1/2} u \in L^2(\mathbf{R}^n)\}.$$

If E is a Hilbert space, then $H^s(\mathbf{R}^n, E)$ denotes the vector-valued analog.

A symbol p of order m yields a bounded linear operator $\operatorname{Op} p : H^s \to H^{s-m}$ for all s.

1.4 Definition. A symbol $p \in SG^m$ is called *elliptic*, if, for large $|x| + |\xi|$, $p(x,\xi)$ is invertible and $p(x,\xi)^{-1} = O(\langle\xi\rangle^{-m_1} \langle x\rangle^{-m_2})$.

Ellipticity in this sense allows the construction of a parametrix modulo regularizing operators. Given an elliptic $p \in SG^m$, there is a $q \in SG^{-m}$ such that $\operatorname{Op} p \circ \operatorname{Op} q - I$ and $\operatorname{Op} q \circ \operatorname{Op} q - I$ are regularizing.
It is obvious that we will only be able to transfer these symbols to manifolds if the manifold has a special structure near infinity.

1.5 Definition. (Schrohe [40], Erkip & Schrohe [13]) Let Ω be an n-dimensional manifold without boundary. Call Ω *SG − compatible* if conditions (SG1) - (SG3) hold.

- (SG1) There are finitely many coordinate charts that cover Ω, say $\Omega = \cup_{j=1}^J \Omega_j$.
- (SG2) This cover has a good shrinking.
- (SG3) All the changes of coordinates χ satisfy $\partial^\alpha \chi(x) = O(\langle x\rangle^{1-\alpha})$.

Let X be an n-dimensional submanifold of Ω with boundary $\partial X = Y$, where Y is an $n-1$-dimensional submanifold without boundary. Assume additionally that

- (SG4) The coordinate charts $\kappa_j : \Omega_j \longrightarrow \mathbf{R}^n$ map $X \cap \Omega_j$ to $\mathbf{R}^n_+, Y \cap \Omega_j$ to $\partial\mathbf{R}^n_+$, and $\Omega_j \cap (\Omega\backslash\bar{X})$ to $\mathbf{R}^n_-$,
- (SG5) There is a fixed Riemannian metric g on Ω whose tensor g_{ij} satisfies (in local coordinates) the estimates $\partial^\alpha g_{ij}(x) = O(\langle x\rangle^{-\alpha}), g^{-1}(x) = O(1)$.

We then call the tuple (Ω, X, Y, g) an *SG-manifold with boundary.* The existence of a good shrinking in (SG2) means that Ω also is the union of sets $\Omega'_j \subseteq \Omega_j$, and there is an $\epsilon > 0$ such that $B(x, \epsilon\langle x\rangle) \subseteq \kappa_j(\Omega_j)$ for every $x \in \Omega'_j$.
This is a typical condition for SG-manifolds. More generally, for open subsets U, U' of $\mathbf{R}^n$ we shall say that U is a *conic neighorhood* of U', if there is an $\epsilon > 0$ such that $B(x, \epsilon\langle x\rangle) \subseteq U$ for every $x \in U'$.
$\mathcal{S}(\Omega)$ and $\mathcal{S}(X)$ denote the spaces of all smooth functions on Ω and X, respectively, that satisfy the estimates for rapidly decreasing functions in all local coordinates. $H^s(\Omega)$ is the space of all distributions on Ω that belong to $H^s(\mathbf{R}^n)$ in local coordinates; $H^s(X) = \{u|_X : u \in H^s(\Omega)\}, H^s_0(X) = \{u \in H^s(\Omega) : \operatorname{supp} u \subseteq \bar{X}\}$. All notions are justified by (SG3).

1.6 Remark. A metric with (SG5) always exists: Simply patch together the metrics induced from the Euclidean metric via the coordinate charts, using a partition of unity of the type in Theorem 1.7(a). Here, I just want to fix such a metric in order to fix normal coordinates near the boundary.
Any choice of such a metric, however, will make the manifold Ω asymptotically flat: the Christoffel symbols satisfy $D^\alpha\Gamma(x) = O(\langle x\rangle^{-1-\alpha})$. It does not, however, imply finiteness of the Betti numbers. Examples for $SG-$manifolds include the infinite-holed torus. More examples for $SG-$manifolds with boundary were given in [13].

1.7 Theorem. (Schrohe [40]) (a) *Given an SG-compatible manifold Ω and a finite cover $\{\Omega_1, \ldots, \Omega_J\}$ as in* (SG1), *there always is a partition of unity $\{\phi_1, \ldots, \phi_J\}$ and a set of cut-off functions $\{\psi_1, \ldots, \psi_J\}$ such that*

(i) $\operatorname{supp}\phi_j$, $\operatorname{supp}\psi_j \subseteq \Omega_j$,

(ii) $\phi_j\psi_j = \phi_j$,

(iii) $D^\alpha\phi_j(x) = O(\langle x\rangle^{-|\alpha|})$, $\quad D^\alpha\psi_j(x) = O(\langle x\rangle^{-|\alpha|})$.

(b) *The symbol classes $SG^m(\mathbf{R})^n$ are invariant under changes of coordinates that satisfy conditions* (SG2) *and* (SG3).
Hence SG-pseudodifferential operators may be defined on Ω, using a partition of unity $\{\phi_j\}$ and cut-off functions $\{\psi_j\}$ as in (a) *and asking that the nonlocal terms be integral operators with rapidly decreasing kernels and that the local terms be defined by an SG-symbol. More precisely: For $A : \mathcal{S}(\Omega) \to \mathcal{S}(\Omega)$ write*

$$A = \sum_{j=1}^{J} \phi_j A\psi_j + \sum_{j=1}^{J} \phi_j A(1 - \psi_j).$$

The operators $\phi_j A\psi_j$ induce operators A_j on $\mathbf{R}^n$ by $A_j f(x) = [\phi_j A\psi_j(f \circ \kappa_j)](\kappa_j^{-1}(x))$. We shall say that A belongs to $SG^m(\Omega)$, if each A_j has a symbol in $SG^m(\mathbf{R}^n)$, and each of the operators $\phi_j A(1 - \psi_j)$ is an integral operator with a kernel density in $\mathcal{S}(\Omega \times \Omega)$.

1.8 Theorem. (Erkip & Schrohe [13]). *If (Ω, X, Y, g) is SG-compatible, then one can switch to normal coordinates near the boundary within the calculus: One can introduce additional coordinate neighborhoods $\{\Omega_j\}$ covering Y such that for suitable $\delta > 0$*

$$\Omega_j = \{(y,t) : y \in \Omega_j \cap Y,\ |t| < \delta\,\langle y\rangle\};$$

the changes of coordinates are of the form $(y,t) \mapsto (\bar{\chi}(y), t)$ with a function $\bar{\chi}$ satisfying (SG3) *on $\mathbf{R}^{n-1}$.*
The normal derivative (which is defined in a neighborhood of the boundary as the generator of the flow induced by the geodesics starting at the boundary with unit inward normal speed) then is an operator with a symbol in $SG^{(1,0)}$.

2 The Algebra of Green Operators on SG-manifolds

2.1 (Standard Notation). Let (Ω, X, Y, g) be SG-compatible with boundary.
(a) By r^+ denote restriction of functions or distributions on Ω to X; e^+ denotes extension (by zero) from X to Ω, provided it makes sense.
(b) Given a pseudodifferential operator P on Ω, define P_+ by $P_+ = \mathrm{r}^+ P \mathrm{e}^+$.
(c) The weighted Sobolev spaces on X are defined by restriction: $H^s(X) = \mathrm{r}^+ H^s(\Omega)$.
(d) As in [37], let $H = H' \oplus H^+ \oplus H_0^-$, where H' is the space of all polynomials on $\mathbf{R}$, and

$$H^+ = \{(\mathrm{e}^+ f)^\wedge : f \in \mathcal{S}(\mathbf{R}_+)\}, H_0^- = \{(\mathrm{e}^- f)^\wedge : f \in \mathcal{S}(\mathbf{R}_-)\}$$

with e^- denoting extension by zero from $\mathbf{R}_-$ to $\mathbf{R}$.
Let $H^- = H_0^- \cup H'$, $H_d = \{f \in H^- : f(\xi) = \mathrm{O}(\langle\xi\rangle^{d-1})\}, d \in \mathbf{N}_0, H_d^- = H_d \cap H^-$.
We endow H^+ and H_0^- with the Fréchet topology induced from $\mathcal{S}(\mathbf{R}_+)$ and $\mathcal{S}(\mathbf{R}_-)$, resp.; H_d' carries the trivial finite dimensional vector space topology. This makes H_+, H_d^- and H_d nuclear Fréchet spaces.
The operators Π^+, Π_0^-, Π^-, and Π_0 are defined as the canonical projections mapping H to H^+, H_0^-, H^-, and H_0, respectively. We have $\Pi^+ = \mathcal{F}\mathrm{r}^+\mathcal{F}^{-1}; \Pi_0^- = \mathcal{F}\mathrm{r}^-\mathcal{F}^{-1}$.
(e) For a sufficiently smooth function f on $\mathbf{R}_+$ let

$$\gamma_j f = \lim_{t\to 0^+} \partial_t^j f(t).$$

(f) Define $\Pi' : H \to \mathbf{C}$ by $\Pi' = (2\pi)^{-\frac{1}{2}}\gamma_0\mathcal{F}^{-1}$.

For the definition of the symbol classes let us first assume that $(\Omega, X, Y, g) = (\mathbf{R}^n, \mathbf{R}^n_+, \mathbf{R}^{n-1}$, Euclidean metric). Write $\mathbf{R}^n_+ = \{(x', x_n) : x_n > 0\}$. The presentation in definitions 2.2 and 2.4 follows the classical route. A new approach using operator-valued symbols on spaces with group actions is contained in Schrohe & Schulze [50]; cf. also [54] for earlier work in this direction and [21], [47], [48] for additional characterizations of the transmission property and the singular Green operators.

2.2 Definition. A symbol $p \in SG^m$ has the *transmission property*, if for every $k \in \mathbf{N}_0$,

$$\partial^k_{x_n} p(x', x_n, \xi', \langle\xi'\rangle\,\xi_n)|_{x_n=0} \in SG^{m-(0,k)}_{x',\xi'} \hat{\otimes}_\pi H_{d,\xi_n}. \tag{1}$$

Here, $d = \max\{\mathrm{entier}(m_1) + 1, 0\}$; the indices x', ξ', ξ_n refer to the arguments of the functions.
For a 'double' symbol $p(x, y, \xi)$ we ask that for all $k, l \in \mathbf{N}_0$

$$\partial^k_{x_n}\partial^l_{y_n} p(x', x_n, y', y_n, \xi', \langle\xi'\rangle\,\xi_n)|_{x_n=y_n=0} \in SG^{(m_1, m_2-k, m_3-l)} \hat{\otimes}_\pi H_{d,\xi_n}. \tag{2}$$

Write $p \in \mathcal{A}^m$. Together with the Fréchet topology on SG^m, (1) or (2) yield a Fréchet topology for $\mathcal{A}^m$.

2.3 Lemma. *It is no restriction to ask that the symbol p be defined on $\mathbf{R}^n \times \mathbf{R}^n$ and not only on $\mathbf{R}^n_+ \times \mathbf{R}^n$: Let $p = p(x,\xi) \in C^\infty(\mathbf{R}^n_+ \times \mathbf{R}^n)$ be a function satisfying the estimates*

$$|D^\alpha_\xi D^\beta_x p(x,\xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m_1 - |\alpha|} \langle x \rangle^{m_2 - |\beta|}$$

for all $(x,\xi) \in \mathbf{R}^n_+ \times \mathbf{R}^n$. Then there is an extension $\tilde{p}$ of p to $\mathbf{R}^n \times \mathbf{R}^n$ satisfying

$$|D^\alpha_\xi D^\beta_x \tilde{p}(x,\xi)| \leq D_{\alpha\beta} \langle \xi \rangle^{m_1 - |\alpha|} \langle x \rangle^{m_2 - |\beta|}$$

for all $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}^n$. The constants $D_{\alpha\beta}$ differ from the $C_{\alpha\beta}$ only by universal factors independent of p.

Proof. Use Seeley's extension procedure, cf. [55]. There is a sequence $\{a_k\}$ such that (i) $\sum_0^\infty |a_k| 2^{kn} < \infty$ for $n = 0, 1, \ldots$, (ii) $\sum_0^\infty a_k(-2^k)^n = 1$ for $n = 0, 1, \ldots$.
Now choose $\theta \in C^\infty(\mathbf{R})$ such that $\theta = 1$ on $(-\infty, 1], \theta = 0$ on $[2, \infty)$. On $\{x_n < 0\}$ let

$$\tilde{p}(x,\xi) = \sum_{k=0}^\infty a_k \theta(-\frac{2^k x_n}{\langle x' \rangle}) p(x', -2^k x_n, \xi).$$

For each x_n the sum is finite, and $\tilde{p}(x,\xi) = 0$ whenever $x_n < -2\langle x' \rangle$.

2.4 Definition. (a) A function $g \in C^\infty(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \mathbf{R} \times \mathbf{R})$ is called a *singular Green symbol of order m and type d*, written $g \in \mathcal{B}^{m,d}$, provided

$$g(x', \xi', \langle \xi' \rangle \nu, \langle \xi' \rangle \eta) \in SG^m_{x',\xi'} \hat{\otimes}_\pi H^+_\nu \hat{\otimes}_\pi H^-_{d,\eta}. \tag{1}$$

(b) $t \in C^\infty(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \mathbf{R})$ is a *trace symbol of order m and type d*, written $t \in \mathcal{T}^{m,d}$, if

$$t(x', \xi', \langle \xi' \rangle \nu) \in SG^m_{x',\xi'} \hat{\otimes}_\pi H^-_{d,\nu}. \tag{2}$$

(c) $k \in C^\infty(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \mathbf{R})$ is a *potential symbol of order m* , written $k \in \mathcal{K}^m$, if

$$k(x', \xi', \langle \xi' \rangle \nu) \in SG^m_{x',\xi'} \hat{\otimes}_\pi H^+_\nu. \tag{3}$$

Relations (1), (2), and (3) define Fréchet topologies on $\mathcal{B}^{m,d}, \mathcal{T}^{m,d}$, and $\mathcal{K}^m$.
In all cases, one could also use SG double symbols, i.e. have $g(x', y', \xi', \langle \xi' \rangle \nu, \langle \xi' \rangle \eta) \in SG^m \hat{\otimes}_\pi H^+_\nu \hat{\otimes}_\pi H^-_{d,\eta}$, $t(x', y', \xi', \langle \xi' \rangle \nu) \in SG^m \hat{\otimes}_\pi H^-_{d,\nu}$, and $k(x', y', \xi', \langle \xi' \rangle \nu) \in SG^m \hat{\otimes}_\pi H^+_\nu$.

2.5 Definition. The various symbols induce boundary symbol operators (acting in the normal direction only) in the standard way: Let $f \in \mathcal{S}(\mathbf{R}_+), c \in \mathbf{C}$. For fixed x', ξ' define

$$\begin{aligned}
p_+(x, \xi', D_n) f(x_n) &= \mathrm{r}^+ p(x', x_n, \xi', D_n) \mathrm{e}^+ f; && (1)\\
g(x', \xi', D_n) f(x_n) &= (2\pi)^{-\frac{1}{2}} \int e^{ix_n\xi_n} \Pi'_{\eta_n} g(x', \xi', \xi_n, \eta_n) (\mathrm{e}^+ f)\hat{}(\eta_n) d\xi_n && (2)\\
t(x', \xi', D_n) f &= \Pi'_{\xi_n} \{ t(x', \xi) (\mathrm{e}^+ f)\hat{}(\xi_n) \}; && (3)\\
[k(x', \xi', D_n) c](x_n) &= (2\pi)^{-\frac{1}{2}} \int e^{ix_n\xi_n} k(x', \xi) d\xi_n \cdot c && (4)\\
& && (5)
\end{aligned}$$

The full operators are defined from the boundary symbol operators by pseudodifferential action in the (x', ξ')-variables, denoted here by Op$'$:

pseudodifferential operators:	$\mathrm{Op}_+ p = (\mathrm{Op}\, p)_+ = \mathrm{r}^+ \mathrm{Op}\, p\, \mathrm{e}^+ = \mathrm{Op}'\, p_+(x, \xi', D_n)$,
singular Green operators:	$\mathrm{Op}_G g = \mathrm{Op}'\, g(x', \xi', D_n)$,
trace operators:	$\mathrm{Op}_T t = \mathrm{Op}'\, t(x', \xi', D_n)$,
potential operators:	$\mathrm{Op}_K k = \mathrm{Op}' k(x', \xi', D_n)$.

Similarly for double symbols. The resulting spaces of operators are the same: this is a consequence of the corresponding result for $SG-$pseudodifferential opertors, cf. the remark after 1.1.

2.6 Lemma. *Let $m \in \mathbf{R}^2$. Instead of the symbols one can also use the notion of symbol kernels, cf.* [18]. *We have the following equivalent characterizations*
(a) *$g \in \mathcal{B}^{m,0}$ iff for all $f \in \mathcal{S}(\mathbf{R}_+)$*

$$[g(x', \xi', D_n)f](x_n) = \int_0^\infty \tilde{g}(x', \xi', x_n, y_n) f(y_n) dy_n$$

with a function $\tilde{g} \in C^\infty(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \mathbf{R}_+ \times \mathbf{R}_+)$ satisfying the estimates

$$\| x_n^k y_n^r D_{x_n}^{k'} D_{y_n}^{r'} D_{\xi'}^\alpha D_{x'}^\beta \tilde{g}(x', \xi', x_n, y_n) \|_{L^2(\mathbf{R}_+ \times \mathbf{R}_+)} = O(\langle x' \rangle^{m_2 - |\beta|} \langle \xi' \rangle^{m_1 + 1 - |\alpha| - k + k' - r + r'}) \quad (1)$$

for all multi-indices α, β and all $k, k', r, r' \in \mathbf{N}_0$.
(b) *$t \in \mathcal{T}^{m,0}$, iff for all $f \in \mathcal{S}(\mathbf{R}_+)$*

$$t(x', \xi', D_n)f = \int_0^\infty \tilde{t}(x', \xi', y_n) f(y_n) dy_n$$

with a function $\tilde{t} \in C^\infty(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \mathbf{R}_+)$ satisfying the estimates

$$\| y_n^r D_{y_n}^{r'} D_{\xi'}^\alpha D_{x'}^\beta \tilde{t}(x', \xi', y_n) \|_{L^2(\mathbf{R}_+)} = O(\langle x' \rangle^{m_2 - |\beta|} \langle \xi' \rangle^{m_1 + \frac{1}{2} - |\alpha| - r + r'}) \quad (2)$$

for all multi-indices α, β and all $r, r' \in \mathbf{N}$.
(c) *$k \in \mathcal{K}^m$ iff for all $c \in \mathbf{C}$*

$$k(x', \xi', D_n)c = \int_0^\infty \tilde{k}(x', \xi', x_n) dx_n\, c$$

with a function $\tilde{k} \in C^\infty(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \mathbf{R}_+)$ satisfying the estimates (2).

All these estimates are immediate from the definitions in 2.4 in connection with the fact that $p_{kk'}(f) = \| x_n^k D_{x_n}^{k'} f \|_{L^2}$ is a defining semi-norm system for the topology of $\mathcal{S}(\mathbf{R}_+)$.

2.7 Theorem. *Let $m \in \mathbf{R}^2, p \in \mathcal{A}^m$. Then*

$$\mathrm{Op}_+ p : \mathcal{S}(X) \to \mathcal{S}(X)$$

is continuous. In particular, for all $(x', \xi') \in \mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$,

$$p_+(x', \xi', D_n) : \mathcal{S}(\mathbf{R}_+) \longrightarrow \mathcal{S}(\mathbf{R}_+)$$

is continuous. The symbol topology is stronger than the operator topology.

Proof: *Step* 1. Choose $\phi, \psi \in C_0^\infty(\mathbf{R})$, equal to 1 in a neighborhood of zero, $\phi\psi = \psi$. Write

$$P_+ = P_+(1-\psi(\frac{x_n}{\langle x'\rangle})) + \phi(\frac{x_n}{\langle x'\rangle})P_+\psi(\frac{x_n}{\langle x'\rangle}) + (1-\phi(\frac{x_n}{\langle x'\rangle}))P_+\psi(\frac{x_n}{\langle x'\rangle}). \tag{1}$$

Clearly, the first and the third term on the right hand side have the desired mapping property. Let us analyze the second.

Step 2. We start with a Taylor expansion for p :

$$p(x,\xi) = \sum_{j=0}^{M-1} \frac{x_n^j}{j!}\partial_{x_n}^j p(x',0,\xi) + x_n^M p_M(x,\xi).$$

Since $p \in \mathcal{A}^m$ we have $\langle x'\rangle^j \partial_{x_n}^j p(x',0,\xi) = d_j(x',\xi) + e_j(x',\xi)$ with a differential symbol d_j with coefficients in $SG^{m-(k,0)}(Y)$, and

$$e_j(x',\xi',\langle\xi'\rangle\xi_n) \in SG^m(Y)\hat{\otimes}_\pi H_0. \tag{2}$$

Step 3. Consider first $\mathrm{Op}_+\left[\phi(x_n/\langle x'\rangle)\,(x_n/\,\langle x'\rangle)^j\, d_j(x',\xi)\right]$. Since $\phi(t)t^j$ has compact support, this is a differential SG operator and clearly maps $\mathcal{S}(X)$ to $\mathcal{S}(X)$.

Step 4. In order to treat the operators involving e_j, (2) allows us to confine ourselves to the case

$$e_j(x',\xi) = c(x',\xi')h(\frac{\xi_n}{\langle\xi'\rangle})$$

with $c(x',\xi') \in SG^m(Y)$ and $h \in H_0$. Since $\mathcal{S}(X) = \mathcal{S}(\mathbf{R}^{n-1})\hat{\otimes}_\pi\mathcal{S}(\mathbf{R}_+)$, we even may restrict the attention to the operator $\mathrm{Op}_+h(\frac{\xi_n}{\langle\xi'\rangle})$ and show that, given $v \in \mathcal{S}(X)$, for all α, β, k

$$\left| \mathrm{r}^+ D_{x_n}^k D_{x'}^\alpha (x')^\beta \iint e^{ix'\xi' + ix_n\langle\xi'\rangle\nu}\hat{h}(\nu)v(\xi',\langle\xi'\rangle\nu)\,\langle\xi'\rangle\; d\nu\, d\xi' \right| \tag{3}$$

$$= \left| \iint e^{ix'\xi' + ix_n\langle\xi'\rangle\nu}\Pi_\nu^+\left\{[(\langle\xi'\rangle\nu]^k \xi'^\alpha h(\nu) D_{\xi'}^\beta[\hat{v}(\xi',\langle\xi'\rangle\nu)\,\langle\xi'\rangle]\right\}\, d\nu\, d\xi' \right|$$

is finite. We do not have to worry about powers of x_n, since we will eventually multiply with $\phi(x_n/\langle x'\rangle)$.
In order to estimate (3), first note that $D_{\xi'}^\beta\hat{v}(\xi',\langle\xi'\rangle\nu)$ is a linear combination of terms of the form

$$(D_{\xi'}^\gamma D_\nu^\mu \hat{v})(\xi',\langle\xi'\rangle\nu)\nu^\mu e_{\beta\gamma\mu}(\xi')$$

with $|\gamma| + \mu \le |\beta|, e_{\beta\gamma\mu} \in SG^0\left(\mathbf{R}^{n-1}\right)$, and universal coefficients.
Let us show that all H^+-semi-norms for $\Pi^+\{\ldots\}$ in (3) can be estimated by an arbitrarily negative power of $\langle\xi'\rangle$. Since H is an algebra with continuous multiplication, and $\Pi^+ : H \to H^+$ is continuous, it suffices to show the corresponding estimate for the H-semi-norms of $(D_{\xi'}^\gamma D_\nu^\mu\hat{v})(\xi',\langle\xi'\rangle\nu)$. Now, an $H-$semi-norm for $(D_{\xi'}^\gamma D_\nu^\mu\hat{v})(\xi',\langle\xi'\rangle\nu)$ differs from the corresponding semi-norm for $(D_{\xi'}^\gamma D_\nu^\mu\hat{v})(\xi',\nu)$ by a power of $\langle\xi'\rangle$, the exponent depending only on the semi-norm. Since $D_{\xi'}^\alpha D_\nu^\mu\hat{v}$ decays arbitrarily fast with respect to ξ', so do all its H-semi-norms.
Therefore the inner integral in (3) furnishes $\left(\mathcal{F}_{\nu\to x_n}^{-1}\Pi^+\{\ldots\}\right)(x_n\langle\xi'\rangle)$, which is $\mathrm{O}(\langle\xi'\rangle^{-N})$ for arbitrary N. Altogether expression (3) is finite; the bound depends only on the semi-norms for h in H_0 and for v in $\mathcal{S}(X)$.

Step 5. It remains to analyze

$$\mathrm{Op}_+[\phi(\frac{x_n}{\langle x'\rangle})x_n^M p_M(x,\xi)] = \phi(\frac{x_n}{\langle x'\rangle})\sum_{j=0}^{M}(-1)^j\binom{M}{j}\mathrm{Op}_+\left(D_{\xi_n}^j p_M(x,\xi)\right)x_n^{M-j}. \qquad (4)$$

Now x_n^k maps $e^+\mathcal{S}(X)$ to $H^{(k,0)}(\mathbf{R}^n)$ continuously, and $\phi(\frac{x_n}{\langle x'\rangle})D_{\xi_n}^j p_M \in SG^{(m_1-j,m_2-M)}(\mathbf{R}^n)$. Therefore, the operator in (4) yields a continuous map from $\mathcal{S}(X)$ to $H^{(M,M)-m}(\mathbf{R}^n_+)$. Since M could be chosen arbitrarily, the proof is complete.

2.8 Theorem. *Let $m \in \mathbf{R}^2, d \in \mathbf{N}_0, k \in \mathcal{K}^m, t \in \mathcal{T}^{m,d}, g \in \mathcal{B}^{m,d}$. Then the following mappings are continuous.*
(a) $\mathrm{Op}_G g : \mathcal{S}(X) \to \mathcal{S}(X)$; $\quad g(x',\xi',D_n) : \mathcal{S}(\mathbf{R}_+) \to \mathcal{S}(\mathbf{R}_+)$, $x',\xi' \in \mathbf{R}^{n-1}$ *fixed,*
(b) $\mathrm{Op}_K k : \mathcal{S}(Y) \to \mathcal{S}(X)$; $\quad k(x',\xi',D_n) : \mathbf{C} \to \mathcal{S}(\mathbf{R}_+)$, $x',\xi' \in \mathbf{R}^{n-1}$ *fixed,*
(c) $\mathrm{Op}_T t : \mathcal{S}(X) \to \mathcal{S}(Y)$; $\quad t(x',\xi',D_n) : \mathcal{S}(\mathbf{R}_+) \to \mathbf{C}$, $x',\xi' \in \mathbf{R}^{n-1}$ *fixed.*
In all cases, the symbol topology is stronger than the operator topology.

Proof. Let us first show (b). For $f \in \mathcal{S}(Y)$,

$$(\mathrm{Op}_K k)\, f(x) = (2\pi)^{\frac{n-1}{2}}\int e^{ix'\xi'}\tilde{k}(x',\xi',x_n)\hat{f}(\xi')d\xi',$$

where $\tilde{k}$ is the symbol kernel for k. The semi-norm system $\{p_{\alpha\beta rr'} : \alpha,\beta \in \mathbf{N}_0^{n-1}, r,r' \in \mathbf{N}_0\}$ given by

$$p_{\alpha\beta rr'}(f) = \sup_{x'}\|x_n^r D_{x_n}^{r'}(x')^\alpha D_{x'}^\beta f\|_{L^2(\mathbf{R}_+)}$$

is defining for the topology of $\mathcal{S}(X)$. Using integration by parts, estimating $p_{\alpha\beta rr'}(\mathrm{Op}_K kf)$ reduces to applying the estimates in 2.6 for $\tilde{k}$ and the $\mathcal{S}(Y)$-estimates for $\hat{f}$.
(c) First decompose $t = t_0 + t_d$, where $t_0 \in \mathcal{T}^{m,0}$ and $t_d(x',\xi',\nu) = \sum_{j=0}^{d-1} a_j(x',\xi')\gamma^j$ with $a_j \in SG^{m-(j,0)}(\mathbf{R}^{n-1})$. The existence of this decomposition is a consequence of 2.4(2) and the definition of H_d^-. $\mathrm{Op}_T t_d : \mathcal{S}(X) \to \mathcal{S}(Y)$ certainly is continuous. Using the symbol kernel $\tilde{t}_0$ write

$$[\mathrm{Op}_T t_0]\, f(x') = (2\pi)^{\frac{n-1}{2}}\int\int_0^\infty \tilde{t}_0(x',\xi',y_n)(\mathcal{F}_{y'\to\xi'}v)(\xi',y_n)dy_n d\xi'.$$

With Cauchy-Schwarz' estimate for the interior integral we are again reduced to applying the symbol kernel estimates for $\tilde{t}_0$ and the $\mathcal{S}(X)$-estimates for v.
The proof of (a) now is a combination of the methods in (b) and (c): First write $g = g_0 + g_d$ with $g_0 \in \mathcal{B}^{m,0}$ and $g_d(x',\xi',\nu,\eta) = \sum k_j(x',\xi',\nu)\eta^j, k_j \in \mathcal{K}^{m-(j,0)}$ Then $\mathrm{Op}_G g_d = \mathrm{Op}_K k_j \circ (-i)^j\gamma_j : \mathcal{S}(X) \to \mathcal{S}(X)$ is clearly continuous in view of (b), while the analysis of

$$[\mathrm{Op}_G g_0]\, f(x) = \int\int_0^\infty e^{ix'\xi'}\tilde{g}_0(x',\xi',x_n,y_n)(\mathcal{F}_{y'\to\xi'}f)(\xi',y_n)dy_n d\xi'$$

with the symbol kernel $\tilde{g}_0$ of g_0 is similar as before.

2.9 Definition. A *Green operator of order m and type d* is a matrix A of operators

$$A = \begin{bmatrix} \mathrm{Op}_+ p + \mathrm{Op}_G g & \mathrm{Op}_K k \\ \mathrm{Op}_T t & \mathrm{Op}\, s \end{bmatrix} : \begin{matrix} \mathcal{S}(\mathbf{R}^n_+)^{n_1} \\ \oplus \\ \mathcal{S}(\mathbf{R}^{n-1})^{n_3} \end{matrix} \longrightarrow \begin{matrix} \mathcal{S}(\mathbf{R}^n_+)^{n_2} \\ \oplus \\ \mathcal{S}(\mathbf{R}^{n-1})^{n_4} \end{matrix} \qquad (1)$$

where $p \in \mathcal{A}^m, g \in \mathcal{B}^{m-(1,0),d}, k \in \mathcal{K}^{m-(1,0)}, t \in \mathcal{T}^{m,d}, s \in SG^m(\mathbf{R}^{n-1})$. Write $A \in \mathcal{G}^{m,d}$. The *boundary symbol operator associated with* A is the operator

$$a(x',\xi',D_n) = \begin{bmatrix} p_+(x,\xi',D_n) + g(x',\xi',D_n) & k(x',\xi',D_n) \\ t(x',\xi',D_n) & s(x',\xi') \end{bmatrix} : \begin{array}{c} \mathcal{S}(\mathbf{R}_+)^{n_1} \\ \oplus \\ \mathbf{C}^{n_3} \end{array} \longrightarrow \begin{array}{c} \mathcal{S}(\mathbf{R}_+)^{n_2} \\ \oplus \\ \mathbf{C}^{n_4} \end{array}.$$

For fixed $(x',\xi'), a(x',\xi',D_n)$ still is an x_n-dependent operator. The entries are assumed to be matrix-valued with the obvious sizes, i.e. p and g are $n_2 \times n_1$-matrices, k is $n_2 \times n_3$, t is $n_4 \times n_1$, and s is $n_4 \times n_3$. In order to save notation, call this an $(n_2,n_4) \times (n_1,n_3)$-matrix.

2.10 Theorem. (a) *The regularizing Green operators of type zero (those in $\cap_m \mathcal{G}^{m,0}$) are precisely the integral operators with rapidly decreasing kernel functions over the respective spaces: A regularizing singular Green operator of type zero has a kernel in $\mathcal{S}(X \times X)$, a regularizing trace operator of type zero has a kernel in $\mathcal{S}(Y \times X)$, a regularizing potential operator a kernel in $\mathcal{S}(X \times Y)$, and finally a regularizing pseudodifferential operator on the boundary has a kernel in $\mathcal{S}(Y \times Y)$; as before $X = \mathbf{R}^n_+, Y = \mathbf{R}^{n-1}$.*
In particular: If G is a regularizing singular Green operator of type zero, then there is a regularizing pseudodifferential operator P such that $G = P_+$.
(b) *A regularizing singular Green operator G of type $d \in \mathbf{N}$ has the form*

$$G = \sum_{j=0}^{d-1} K_j\gamma_j + G_0,$$

where the K_j are regularizing potential operators, G_0 is a regularizing singular Green operator of type zero, and γ_j is the trace operator in 2.1.
A regularizing trace operator T of type d can be written

$$T = \sum_{j=0}^{d-1} S_j\gamma_j + T_0$$

with regularizing pseudodifferential operators S_j on Y and a regularizing trace operator T_0 of type zero.
(c) *Let $\phi \in C_0^\infty(\mathbf{R}), \phi \equiv 1$ near zero, and let G, T, K be singular Green, potential and trace operators of order m and type d. By Φ denote for the moment the operator of multiplication by the function $1 - \phi(x_n/\langle x'\rangle)$. Then ΦK is a regularizing potential operator, $G\Phi$ and $T\Phi$ are regularizing singular Green and trace operators, respectively, of type zero, and ΦG is a regularizing singular Green operator of type d.*

Proof. (a) This is a consequence of the symbol kernel estimates in 2.6 in connection with the fact that the regularizing SG pseudodifferential operators on $\mathbf{R}^{n-1}$ are precisely the integral operators with kernel functions in $\mathcal{S}(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1})$, cf. Theorem 1.2.
(b) For the symbols g and t of G and T, resp., write $g = g_d + g_0$, $t = t_d + t_0$, where g_0, t_0 are regularizing of type zero, and $g_d(x',\xi',\nu,\eta) = \sum_{j=0}^{d-1} k_j(x',\xi',\nu)\eta^j$, $t_d(x',\xi',\nu) = \sum_{j=0}^{d-1} a_j(x',\xi')\nu^j$, with $k_j \in \mathcal{K}^{-\infty}, a_j \in SG^{-\infty}(\mathbf{R}^{n-1})$; this is possible by Definition 2.4. Now (a) yields the assertion.
(c) Write $1 - \phi(x_n/\langle x'\rangle) = \left[[1 - \phi(x_n/\langle x'\rangle)]\,(x_n/\langle x'\rangle)^{-N}\right](x_n/\langle x'\rangle)^N, N \in \mathbf{N}$. If K has the symbol $k \in \mathcal{K}^m$, then $(x_n/\langle x'\rangle)^N K$ has the symbol $\langle x'\rangle^{-N} D^N_{\xi_n} k \in \mathcal{K}^{m-(N,N)}$. Since

$(1-\phi)(t)t^{-N} \in C_b^\infty(\mathbf{R})$, (a) shows the assertion. The argument for trace and singular Green operators is similar. Note that if the multiplication is from the right and N is sufficiently large, then the evaluations $\gamma_j(x_n^N f)$ yield zero, hence only the type zero parts of the symbols will contribute.

2.11 Theorem. *Let $A \in \mathcal{G}^{m,d}, A' \in \mathcal{G}^{m',d'}$ with matrix sizes so that the composition makes sense. Then $AA' \in \mathcal{G}^{m'',d''}$, where $m'' = m + m'$, and $d'' = \max\{m' + d, d'\}$. If $m \le 0$ and $d = 0$, then the adjoint A^* of A also belongs to $\mathcal{G}^{m,0}$.*
For the various compositions of operators, the classical asymptotic expansion formulas (cf. sections 2.6, 2.7 in [18]) hold with respect to the SG-calculus.
In particular, $\mathcal{G}^{0,0}$ is a *-algebra, if the matrix size is $(n_1, n_2) \times (n_1, n_2)$.

Proof. Clearly, the analysis has to take into account the behavior of the x–derivatives and therefore is somewhat more subtle than in the standard case. The modifications that have to be made, however, are essentially the same for all 13 compositions involved. For this reason I will give the details for one of these compositions and refer to the detailed analysis in [44] for the others.
Claim. Let $p \in \mathcal{A}^m, k \in \mathcal{K}^{m'-(1,0)}$. Then for fixed x', ξ', the operator (with respect to the x_n–variable)

$$k'(x', \xi', D_n) = p_+(x, \xi', D_n) \circ_n k(x', \xi', D_n)$$

is an operator with a potential symbol $k'(x', \xi)$ and

$$k'(x', \xi) \sim \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \Pi^+_{\xi_n}(D^j_{\xi_n}\{\partial^j_{x_n} p(x', 0, \xi) k(x', \xi)\})$$

Here $\circ_n$ denotes composition with respect to the action in the x_n–variable.
Proof of the claim. Write

$$p(x, \xi) = \sum_{j=0}^{M-1} \frac{x_n{}^j}{j!} \partial^j_{x_n} p(x', 0, \xi) + x_n^M p_M(x, \xi)$$

with $p_M(x, \xi) = \text{const.} \int_0^1 \partial^M_{x_n} p(x', \tau x_n, \xi)(1-\tau)^{M-1} d\tau$, so that for large M

$$D^\alpha_\xi D^\beta_x p_M(x, \xi) = O\left(\langle \xi \rangle^{m_1 - |\alpha|} \langle x' \rangle^{m_2 - |\beta| - M}\right). \tag{1}$$

For $v \in \mathcal{S}(Y)$,

$$\begin{aligned} & p_+(x, \xi', D_n) \circ_n k(x', \xi', D_n) v(x) \\ = \; & \mathrm{r}^+ \int e^{ix_n\xi_n} \sum_{j<M} \frac{x_n{}^j}{j!} \partial^j_{x_n} p(x', 0, \xi) k(x', \xi) + x_n{}^M p_M(x, \xi') k(x', \xi)) \cdot v d\xi_n \\ = \; & \sum_{j<M} \frac{1}{j!} \mathcal{F}^{-1}_{\xi_n \to x_n} \Pi_+ \{(-D_{\xi_n})^j \{\partial^j_{x_n} p(x', 0, \xi) k(x', \xi)\}\} \cdot v + \mathrm{r}^+ \int e^{ix_n\xi_n} a_M(x, \xi) d\xi_n \cdot v \end{aligned}$$

with $a_M(x, \xi) = (-D_{\xi_n})^M \{p_M(x, \xi) k(x', \xi)\}$. We have used that $r^+ \mathcal{F}^{-1} = \mathcal{F}^{-1} \Pi^+$. The terms under the summation give the right start of the asymptotic expansion in the claim. In order to justify the expansion we will show that the operator $\mathrm{Op}_K a_M : v \mapsto \int e^{ix'\xi'} \int e^{ix_n\xi_n} a_M(x, \xi) d\xi_n \hat{v}(\xi') d\xi'$ (a slight modification of the standard definition to x_n-dependent symbols) can be written as an integral operator with an integral kernel that

will satisfy any *fixed* number of the estimates required for a function in $\mathcal{S}(X \times Y)$, provided M is sufficiently large.
Choose a function $\varphi \in C_0^\infty(\mathbf{R}), \varphi \equiv 1$ near zero. Write

$$p_M(x,\xi) = \varphi(\frac{x_n}{\langle x'\rangle})p_M(x,\xi) + (1-\varphi(\frac{x_n}{\langle x'\rangle}))p_M(x,\xi).$$

Now let

$$b_M(x,\xi) = \varphi(\frac{x_n}{\langle x'\rangle})(-D_{\xi_n})^M\left(p_M(x,\xi)k(x',\xi)\right). \tag{2}$$

The H_+-estimates for k imply that

$$\|D_{\xi'}^\alpha D_{x'}^\beta \xi_n^j D_{\xi_n}^r k(x',\xi',\xi_n)\|_{L^2_{\xi_n}} = \mathrm{O}\left(\langle x'\rangle^{m_2'-|\beta|}\langle\xi'\rangle^{m_1'-|\alpha|+\frac{1}{2}+j-r}\right)$$

for $j \le r$; the estimate $\|f\|_{\sup} \le \|f\|_{L^2}\|\partial_t f\|_{L^2}$, valid for functions in $H^1(\mathbf{R})$, implies that

$$|D_{\xi'}^\alpha D_{x'}^\beta \xi_n^j D_{\xi_n}^r k(x',\xi',\xi_n)| = \mathrm{O}\left(\langle x'\rangle^{m_2'-|\beta|}\langle\xi'\rangle^{m_1'-|\alpha|+j-r}\right). \tag{3}$$

Applying Leibniz' rule to (2), and noting that $\varphi(x_n/\langle x'\rangle)D_{\xi_n}^r p_M \in SG^{m-(r,M)}$, we conclude from (3) that any fixed $\mathcal{S}$-semi-norm for b_M will be bounded provided M is large. Therefore $\mathrm{Op}_K b_M$ is an integral operator whose kernel functions will satisfy any fixed $\mathcal{S}(X \times Y)$ semi-norm for large M. In order to analyze $c_M = a_M - b_M$, we first observe that for arbitrary fixed $L \ge 0$

$$D_x^\alpha\{(1-\varphi(\frac{x_n}{\langle x'\rangle}))(\frac{x_n}{\langle x'\rangle})^{-L}\} \le C_\alpha,$$

independent of $x = (x', x_n)$. Then

$$\begin{aligned} & r^+\int e^{ix_n\xi_n}c_M(x,\xi)d\xi_n \\ = \ & \langle x'\rangle^{-L}\{(1-\varphi)(\frac{x_n}{\langle x'\rangle})(\frac{x_n}{\langle x'\rangle})^{-L}\}\langle x_n\rangle^{-2K} \\ & \cdot(2\pi)^{-\frac{1}{2}}\int e^{ix_n\xi_n}(1-\Delta_{\xi_n})^K(-D_{\xi_n})^{L+M}\{p_M(x,\xi)k(x',\xi)\}d\xi_n. \end{aligned}$$

The last integral converges and is bounded, independent of x, whenever M is large. The same argument applies to derivatives. The corresponding kernel function for $\mathrm{Op}_K c_M$ will then also satisfy any fixed estimate for a rapidly decreasing function on $X \times Y$. This concludes the proof of the claim.

2.12 Theorem. *Let $A \in \mathcal{G}^{m,d}$ be an $(n_2,n_4)\times(n_1,n_3)$-matrix. Then*

$$A = \begin{bmatrix} P_+ + G & K \\ T & S \end{bmatrix} : \begin{matrix} H^s(\mathbf{R}^n_+)^{n_1} \\ \oplus \\ H^{s-(\frac{1}{2},0)}(\mathbf{R}^{n-1})^{n_3} \end{matrix} \longrightarrow \begin{matrix} H^{s-m}(\mathbf{R}^n_+)^{n_2} \\ \oplus \\ H^{s-m-(\frac{1}{2},0)}(\mathbf{R}^{n-1})^{n_4} \end{matrix}.$$

is bounded, provided $s_1 > d - \frac{1}{2}$. If $d = 0$, then we may extend the result to the case $s_1 < -\frac{1}{2}$ by using $H_0^s(\mathbf{R}^n_+)$ on the left hand side and replacing $H^{s-m}(\mathbf{R}^n_+)$ on the right hand side by $H_0^{s-m}(\mathbf{R}^{n-1})$ whenever $s_1 - m_1 < -\frac{1}{2}$.
The topologies on the symbol spaces are stronger than the corresponding topologies of bounded operators.

For the proof of Theorem 2.12 we need the following lemma; it is essentially well-known, cf. [19], [37].

2.13 Lemma. *Let $\chi \in \mathcal{S}(\mathbf{R})$ be a function with* $\operatorname{supp} \mathcal{F}^{-1}\chi \subset \mathbf{R}_-$. *For $k \in \mathbf{Z}$, $\mu \in \mathbf{R}$, and $a \gg 0$ define*

$$\lambda^k(\xi,\mu) = \left(\chi(\frac{\xi_n}{a\,\langle\xi',\mu\rangle})\,\langle\xi',\mu\rangle - i\xi_n\right)^k .$$

Then, for sufficiently large a,

$$\lambda^k \in \mathcal{A}^{(k,0)}(\mathbf{R}^n_\xi \times \mathbf{R}_\mu),$$

and $\Lambda^k(\mu) := \operatorname{Op}_+\lambda^k(\mu) : H^s(\mathbf{R}^n_+) \to H^{s-(k,0)}(\mathbf{R}^n_+)$ is an isomorphism for all $s \in \mathbf{R}^2$ with $s_1 > -\frac{1}{2}$ and large $|\mu|$. In general a and μ will not be mentioned.
Due to its particular form, $\Lambda^k(\mu)$ is independent of the choice of the extension $H^s(\mathbf{R}^n_+) \to H^s(\mathbf{R}^n)$. By defining $\Lambda^k(\mu)_+$ with the help an arbitrary extension, we obtain an operator which (i) *agrees with the preceding one for $s_1 > -\frac{1}{2}$, and* (ii) *furnishes an isomorphism for all $s \in \mathbf{R}^2$.*

Proof of Theorem 2.12. Let us consider the various entries of A separately, starting with the potential operator K.

Step 1.

$$K : H^{s-(\frac{1}{2},0)}(\mathbf{R}^{n-1}) \to H^{s-m}(\mathbf{R}^n_+) \tag{1}$$

is bounded: By interpolation we may assume $s_1 - m_1 \in \mathbf{Z}$. K in (1) is bounded, if and only if

$$K' := \langle x\rangle^{s_2-m_2}\,\Lambda^{s_1-m_1} K\,\langle D'\rangle^{-s_1+1}\,\langle x'\rangle^{-s_2} : H^{(\frac{1}{2},0)}(\mathbf{R}^{n-1}) \to H^0(\mathbf{R}^n_+)$$

is bounded. K' is a potential operator; it has a symbol $k'(x',\xi) \in \mathcal{K}^0$ which we may convert into a symbol $\tilde{k}(y',\xi)$. Applying 2.4(3),

$$\tilde{k}(y',\xi) = \sum_{j=0}^{\infty} \lambda_j d_j(y',\xi') h_j(\frac{\xi_n}{\langle\xi'\rangle})$$

with $\{\lambda_j\} \in \ell^1$, $\{d_j\}$ a null sequence in SG^0, and $\{h_j\}$ a null sequence in H^+. Hence

$$\operatorname{Op}_K\tilde{k} = \sum \lambda_j \operatorname{Op}_K h_j \operatorname{Op}' d_j.$$

Now the assertion follows from the facts that $\operatorname{Op}' d_j : H^{(\frac{1}{2},0)}(\mathbf{R}^{n-1}) \to H^{(\frac{1}{2},0)}(\mathbf{R}^{n-1})$ is bounded and that $\operatorname{Op}_K h_j(\frac{\xi_n}{\langle\xi'\rangle}) : H^{(\frac{1}{2},0)}(\mathbf{R}^{n-1}) \to H^0(\mathbf{R}^n_+)$ is bounded. The latter is a consequence of the identity

$$\|\operatorname{Op}_K h_j(\frac{\xi_n}{\langle\xi'\rangle})v\|_{L^2(\mathbf{R}^n_+)} = \|h_j(\frac{\xi_n}{\langle\xi'\rangle})\hat{v}\|_{L^2(\mathbf{R}^n)} = \|h_j\|_{L^2(\mathbf{R})}\|v\|_{H^{\frac{1}{2}}(\mathbf{R}^{n-1})}.$$

Step 2. If T is a trace operator with a symbol in $\mathcal{T}^{m,d}$ and $s_1 > d - \frac{1}{2}$, then

$$T : H^s(\mathbf{R}^n_+) \to H^{s-m-(\frac{1}{2},0)}(\mathbf{R}^{n-1}) \tag{2}$$

is bounded: We decompose the symbol t of $T: t = t_0 + t_d$, where $t_0 \in \mathcal{T}^{m,0}$ and $\operatorname{Op} t_d = \sum_{j=0}^{d-1} P_j\gamma_j$ with $P_j \in SG^{m-(j,0)}(\mathbf{R}^{n-1})$. With an argument like that before, it is easily seen that $\operatorname{Op}_{\mathcal{T}} t_0$ has the mapping property (2) even without the restriction $s_1 > d - \frac{1}{2}$. For $\operatorname{Op}_{\mathcal{T}} t_d$ we obtain the assertion from the standard trace theorem and the properties of the SG–operators.

Step 3. Let G be a singular Green operator with a symbol g in $\mathcal{B}^{m-(1,0),d}$. We have to show the continuity of

$$G : H^s(\mathbf{R}^n_+) \to H^{s-m}(\mathbf{R}^n_+) \tag{3}$$

for $s_1 > d - \frac{1}{2}$. According to 2.4(1) we may write

$$g(x', \xi', \nu, \tau) = \sum_{j=0}^{\infty} \lambda_j k_j(x', \xi', \nu) t_j(\xi', \tau)$$

with $\{\lambda_j\} \in l^1, k_j$ a null sequence in $\mathcal{K}^{m-(1,0)}$, and t_j a null sequence in $\mathcal{T}^{0,d}$. Letting $T_j = \operatorname{Op}_{\mathcal{T}} t_j, K_j = \operatorname{Op}_{\mathcal{K}} k_j$, $G = \sum_{j=0}^{\infty} \lambda_j K_j T_j$. The assertion follows from the continuity of $T_j : H^s(\mathbf{R}^n_+) \to H^{s-(\frac{1}{2},0)}(\mathbf{R}^{n-1})$ and $K_j : H^{s-(\frac{1}{2},0)}(\mathbf{R}^{n-1}) \to H^{s-m}(\mathbf{R}^n_+)$.

Step 4. Suppose P is a pseudodifferential operator in $\operatorname{Op} \mathcal{A}^m$. Let us show the boundedness of

$$P_+ : H^s(\mathbf{R}^n_+) \to H^{s-m}(\mathbf{R}^n_+). \tag{4}$$

Extension by zero e^+ is defined for $s_1 > -\frac{1}{2}$; it is continuous for $-\frac{1}{2} < s_1 < \frac{1}{2}$. Restriction $H^s(\mathbf{R}^n) \to H^s(\mathbf{R}^n_+)$ is bounded for all s. So there is nothing to show for $-\frac{1}{2} < s_1 < \frac{1}{2}$. Using interpolation we may assume that $s_1 \in \mathbf{N}$. Clearly, P_+ has property (4) if and only if

$$P'_+ = P_+ \Lambda^{-s_1} \langle x \rangle^{-s_2} : H^0(\mathbf{R}^n_+) \to H^{s-m}(\mathbf{R}^n_+)$$

is bounded. From the calculus we know that $P_+ \Lambda^{-s_1} \langle x \rangle^{-s_2} = R + G$ with R a pseudodifferential operator of order $m - s$ and G a singular Green operator with a symbol in $\mathcal{B}^{m-s-(1,0),0}$. In view of step 3 this yields the assertion.

For the case of $s_1 < -\frac{1}{2}$ and the H^s_0 Sobolev spaces one works with another reduction of the order, namely that given by $\operatorname{Op}_+ \overline{\lambda}^k$ which is an isomorphism $H^s_0(\mathbf{R}^n_+) \to H^{s-k}_0(\mathbf{R}^n_+)$.

We can now start to define Green operators on an arbitrary SG-manifold (Ω, X, Y, g).

2.14 Definition.

(a) Call a vector bundle over Ω an *SG-(vector) bundle*, if it is trivial over the coordinate charts Ω_j and the transition matrices (a_{ij}) satisfy the estimates

$$\partial_x^\alpha a_{ij}(x) = \mathrm{O}(\langle x \rangle^{-|\alpha|}), \ \ (a_{ij}(x))^{-1} = \mathrm{O}(1).$$

SG-bundles over X are simply the restrictions of SG-bundles over Ω.

(b) It is straightforward to introduce rapidly decreasing sections into a bundle E and weighted Sobolev spaces of distribution sections. The notation is $\mathcal{S}(\Omega, E), \mathcal{S}(X, E)$, and $H^s(\Omega, E), H^s(X, E), s \in \mathbf{R}^2$, respectively. Identify $H^0(\cdot)$ and $L^2(\cdot)$.

(c) If ϕ is a C^∞-function on Ω, then simply write ϕA for $\begin{bmatrix} \phi & 0 \\ 0 & \phi|_Y \end{bmatrix} A$.

2.15 Coordinate Charts. By 1.8 there is a conic neighborhood of the boundary Y with normal coordinates. Given any coordinate neighborhood, we may assume that either it does not intersect a conic neighborhood of Y or else it is one of those with normal coordinates.
In order to fix the notation, suppose that the above neighborhood of Y is the set Y_1 defined as the union of the sets $\{(y,t) : y \in \Omega_j \cap Y, |t| < \langle y\rangle\}, j = 1,\ldots,J$. Call these coordinate neighborhoods boundary charts. Furthermore suppose that the charts for the interior ("interior charts") do not intersect the sets $\{(y,t) : y \in \Omega_j \cap Y, |t| < \frac{1}{2}\langle y\rangle\}$, where $\langle y\rangle$ is to be understood as a suitable globalization of the corresponding local notion. For convenience we will assume that also the homeomorphic images of the charts in Euclidean space have these properties. Choose the enumeration so that the boundary charts are $\Omega_1,\ldots,\Omega_{j_0}$ for some $1 \leq j_0 \leq J$.
Now choose a partition of unity $\{\phi_1\ldots\phi_J\}$ and cut-off functions $\{\psi_1\ldots\psi_J\}$ with the properties (i) – (iii) in 1.7.

2.16 Definition. Let E_1, E_2, and F_1, F_2 be SG-bundles over X and Y, respectively. A *Green operator of order m and type d* is a matrix of operators

$$A = \begin{bmatrix} P_+ + G & K \\ T & S \end{bmatrix} : \begin{matrix} \mathcal{S}(X,E_1) \\ \oplus \\ \mathcal{S}(Y,F_1) \end{matrix} \longrightarrow \begin{matrix} \mathcal{S}(X,E_2) \\ \oplus \\ \mathcal{S}(Y,F_2) \end{matrix}$$

with the following properties

(i) P is a pseudodifferential operator of order m on Ω, $P_+ = \mathrm{r}^+Pe^+$,

(ii) S is a pseudodifferential operator of order m on Y,

(iii) writing $G = \sum_{j=1}^{j_0} \phi_j G\psi_j + \sum_{j=1}^{j_0} \phi_j G(1 - \psi_j) + (1 - \sum_{j=1}^{j_0} \phi_j)G =: \sum_{j=1}^{j_0} G_j + \sum_{j=1}^{j_0} R_j + R'$

- each G_j is – in local coordinates – a singular Green operator of order $m-(1,0)$ and type d,
- each R_j is an integral operator with a kernel density in $\mathcal{S}(X \times X)$, and
- R' can be written $R' = \sum_{l=0}^{d-1} K_l\gamma_l + R''$; K_l and R' are integral operators with rapidly decreasing kernel densities on $X \times Y$ and $X \times X$, respectively.

(iv) Writing $K = \sum_{j=1}^{j_0} \phi_j K\,\psi_j + \sum_{j=1}^{j_0} \phi_j K(1 - \psi_j) + (1 - \sum_{j=1}^{j_0} \phi_j)K =: \sum_{j=1}^{j_0} K_j + \sum_{j=1}^{j_0} L_j + L'$

- each K_j is – in local coordinates – a potential operator of order $m-(1,0)$ and type zero,
- L_j and L' are integral operators with kernel densities in $\mathcal{S}(X \times Y)$

(v) Writing $T = \sum_{j=1}^{j_0} \phi_j T\,\psi_j + \sum_{j=1}^{j_0} \phi_j T(1 - \psi_j) + (1 - \sum_{j=1}^{j_0} \phi_j)T =: \sum_{j=1}^{j_0} T_j + \sum_{j=1}^{j_0} Q_j + Q'$

- each T_j is – in local coordinates – a trace operator of order m and type d,
- each of the operators Q_j, Q' can be written $\sum_{l=0}^{d-1} S_l\gamma_l$ with $S_l \in SG^{-\infty}(Y)$.

2.17 Theorem. *The above definitions are invariant under changes of coordinates satisfying* (SG3) *and* (SG4). *The* $\mathbf{R}^n_+$ *results of* 2.11, 2.12 *extend to general SG-manifolds and bundles with one restriction: The boundary symbol operators are only defined in the above neighborhood of* Y *with normal coordinates.*

2.18 Definition. Let $A \in \mathcal{G}^{m,d}, d \le \max\{m_1, 0\}$.
(a) A *parametrix* to A is an operator $B \in \mathcal{G}^{-m,d'}$, $d' \le \max\{-m_1, 0\}$, such that $AB - I$ and $BA - I$ are both regularizing.
(b) Call A *elliptic of order* m, if the following holds:

(i) The pseudodifferential operator P is elliptic on X, i.e. in all local coordinates

$$p(x,\xi)^{-1} = O(\langle\xi\rangle^{-m_1}\langle x\rangle^{-m_2}) \tag{1}$$

for large $|x| + |\xi|, x \in X$.

(ii) Near the boundary, the boundary symbol operator of A is locally invertible by boundary symbols of order $-m$ and type $d' \le \max\{-m_1, 0\}$. This means the following. In addition to the functions ϕ_j, ψ_j of 2.15 choose functions θ_j supported in Ω_j, satisfying $D_x^\alpha\theta_j(x) = O(\langle x\rangle^{-\alpha})$ and $\theta_j\psi_j = \psi_j$. Denote the boundary symbol of $\psi_j A\theta_j$ by a_j. Then ask that there are boundary symbol operators b_j with

$$a_j b_j \phi_j - \phi_j = g_{1j} \quad \text{and} \quad \phi_j b_j a_j - \phi_j = g_{2j} \tag{2}$$

are regularizing boundary symbols. Here the composition is with respect to the action in the normal direction. In view of the composition rules, the types of the g_{1j} are $\le \max\{-m_1, 0\}$; those of the g_{2j} are $\le \max\{m_1, 0\}$.

It would have been equivalent to ask that the orders of the operators g_{1j} and g_{2j} be $(-\epsilon, -\epsilon)$ for some $\epsilon > 0$.

2.19 Theorem. *Let* $A \in \mathcal{G}^{m,d}, d \le \max\{m_1, 0\}$.
(a) *There is a parametrix* B *to* A *if and only if* A *is elliptic.*
(b) *For* $s_1 > d - \frac{1}{2}$, *ellipticity of* A *implies the Fredholm property of*

$$A : H^s(X, E_1) \oplus H^{s-(\frac{1}{2},0)}(Y, F_1) \longrightarrow H^{s-m}(X, E_2) \oplus H^{s-(\frac{1}{2},0)}(Y, F_2). \tag{1}$$

Proof. (a) A parametrix construction just like in the standard case – cf. [37], [18] – shows that ellipticity is sufficient for the existence of a parametrix. Vice versa, suppose we have found a parametrix $B \in \mathcal{G}^{-m,d'}, d' = \max\{-m_1, 0\}$. Then $\psi_j A\theta_j B\phi_j - \phi_j$ and $\phi_j B\psi_j A\theta_j$ are regularizing for the functions ϕ_j, ψ_j, and θ_j of 2.18. Let a_j and b_j denote boundary symbols for $\psi_j A\theta_j$ and $\theta_j B\psi_j$, respectively. Then the composition rules imply that

$$a_j b_j \phi_j - \phi_j \quad \text{and} \quad \phi_j b_j a_j - \phi_j$$

are of order $(-1,-1)$.
It is not so straightforward to check the estimates for the pseudodifferential part. Write p, q for (arbitrary) symbols for the pseudodifferential part of A and B. Using the partition of unity and cut-off functions, we may assume that $X = \mathbf{R}^n_+$. The pseudodifferential parts of $AB - I$ and $BA - I$ can be taken to be zero. On the other hand, this pseudodifferential part equals $pq - 1$ and $qp - 1$, respectively, modulo SG symbols of order $(-1,-1)$. Applying Theorem 3.19, below, we see that for $(x,\xi) \in X \times \mathbf{R}^n$ with $|x| + |\xi|$ large, $|p(x,\xi)q(x,\xi) - 1| < \frac{1}{2}$ and $|q(x,\xi)p(x,\xi) - 1| < \frac{1}{2}$. This shows that $p(x,\xi)$ is invertible with bounded inverse.
(b) This is immediate from the fact that regularizing operators map into $\mathcal{S}(X, E_j) \oplus \mathcal{S}(Y, F_j), j = 1, 2$, which is compact in $H^s(X, E_1) \oplus H^{s-(\frac{1}{2},0)}(Y, F_2)$ and $H^{s-m}(X, E_2) \oplus H^{s-m-(\frac{1}{2},0)}(Y,_2 F)$, respectively.

2.20 Remark. In the half-space case, the boundary symbol operators are globally defined. It turns out that then the existence of a parametrix B to A is equivalent to the existence of a boundary symbol operator b that inverts the boundary symbol operator a of A modulo regularizing singular Green boundary symbol operators g_1, g_2, i.e.

$$a(x', \xi', D_n) \circ b(x', \xi', D_n) - 1 = g_1(x', \xi', D_n)$$

and

$$b(x', \xi', D_n) \circ a(x', \xi', D_n) - 1 = g_2(x', \xi', D_n)$$

are regularizing boundary symbol operators.

3 Spectral Invariance

Denote by $\mathcal{G} = \mathcal{G}^{0,0}$ the algebra of Green operators of order and type zero. By 2.12 and 2.17, $\mathcal{G} \hookrightarrow \mathcal{L}(H)$ for a natural Hilbert space H. It is the aim of this section to show

3.1 Theorem. *$\mathcal{G}$ is a unital, symmetric continuously embedded Fréchet subalgebra of $\mathcal{L}(H)$ with the spectral invariance property:*

$$\mathcal{G} \cap \mathcal{L}(H)^{-1} = \mathcal{G}^{-1}.$$

In other words, $\mathcal{G}$ is a Ψ^*–subalgebra of $\mathcal{L}(H)$ in the sense of Gramsch [14].

Again, we shall consider first the case of the half-space $X = \mathbf{R}^n_+$ with boundary $\partial X = Y = \mathbf{R}^{n-1}$; the manifold case will be an easy consequence.
Suppose that the operators in $\mathcal{G}$ are of matrix size $(n_1, n_2) \times (n_1, n_2)$. They define bounded maps on the Hilbert space

$$H = L^2(X)^{n_1} \oplus H^{-(\frac{1}{2},0)}(Y)^{n_2}.$$

Clearly, the actual choice of n_1 and n_2 does not matter, so assume that $n_1 = n_2 = 1$. It is our first goal to give $\mathcal{G}$ a Fréchet topology. We need some preparations.

3.2 Lemma. *$\mathcal{G}$ is symmetric with respect to the inner product in H.*

Proof. For $P \in \operatorname{Op}\mathcal{A}^0$, we have $(P_+)^* = (P^*)_+$, and the $L^2(\mathbf{R}^n)$-adjoint P^* belongs to $\operatorname{Op}\mathcal{A}^0$.
If G is a singular Green operator with symbol kernel $\tilde{g}(x', y', \xi', x_n, y_n)$, then G^* has the symbol kernel $\bar{\tilde{g}}(y', x', \xi', y_n, x_n)$. By 2.6, $G^* \in \operatorname{Op}_G \mathcal{B}^{(-1,0),0}$.
If T is a trace operator in $\operatorname{Op}_T \mathcal{T}^{0,0}$ with the symbol kernel $\tilde{t}(x', y', \xi, y_n)$, then the operator L with the symbol kernel $\tilde{l}(x', y', \xi', x_n) = \bar{\tilde{t}}(y', x', \xi', x_n)$ belongs to $\operatorname{Op}_K \mathcal{K}^0$ and satisfies

$$(Tf, g)_{L^2(\mathbf{R}^{n-1})} = (f, Lg)_{L^2(\mathbf{R}^n_+)}.$$

Letting $L' = L\,\langle D' \rangle^{-1} \in \operatorname{Op}_K \mathcal{K}^{(-1,0)}$, we obtain the adjoint with respect to the duality in H.
Similarly for potential operators: If $K \in \operatorname{Op}_K \mathcal{K}^{(-1,0)}$ has the symbol kernel $\tilde{k}(x', y', \xi', x_n)$, then its adjoint R with respect to the corresponding L^2 inner products is given by the symbol kernel $\tilde{r}(x', y', \xi', y_n) = \bar{\tilde{k}}(y', x', \xi', y_n)$, so its adjoint with respect to the inner product in H is $\langle D' \rangle R \in \operatorname{Op}_T \mathcal{T}^{0,0}$.
Finally the $H^{(-\frac{1}{2},0)}$–adjoint of S is $S^* = \langle D' \rangle S' \langle D' \rangle^{-1}$ with the L^2–adjoint S', and $S^* \in \operatorname{Op} SG^0(\mathbf{R}^{n-1})$.

3.3 Proposition. (a) *For $a \in \mathcal{A}^0$, the mapping $a \mapsto \operatorname{Op} a \in \mathcal{L}(L^2(\mathbf{R}^n))$ is injective. Let σ be a right inverse. Then*

$$\# : \mathcal{A}^0 \times \mathcal{A}^0 \to \mathcal{A}^0, \qquad p\#q = \sigma(\operatorname{Op} p \operatorname{Op} q)$$

defines a continuous multiplication.
(b) *For $g \in \mathcal{B}^{(-1,0),0}$, the mapping $g \mapsto \mathrm{Op}_G g \in \mathcal{L}(L^2(\mathbf{R}^n_+))$ is injective. Let σ_G be a right inverse. Then the following mappings are continuous:*

$$\begin{array}{lccll} m_1: & \mathcal{A}^0 \times \mathcal{B}^{(-1,0),0} & \to & \mathcal{B}^{(-1,0),0}; & m_1(p,g) = \sigma_G(\mathrm{Op}^+ p\, \mathrm{Op}_G g) \\ m_2: & \mathcal{B}^{(-1,0),0} \times \mathcal{A}^0 & \to & \mathcal{B}^{(-1,0),0}; & m_2(g,p) = \sigma_G(\mathrm{Op}_G g\, \mathrm{Op}^+ p) \\ m_3: & \mathcal{B}^{(-1,0),0} \times \mathcal{B}^{(-1,0),0} & \to & \mathcal{B}^{(-1,0),0}; & m_3(g,h) = \sigma_G(\mathrm{Op}_G g\, \mathrm{Op}_G h) \\ m_4: & \mathcal{A}^0 \times \mathcal{A}^0 & \to & \mathcal{B}^{(-1,0),0}; & m_4(p,q) = \sigma_G(L(\mathrm{Op}\, p, \mathrm{Op}\, q)) \end{array}$$

Proof. (a) The symbol of a pseudodifferential operator is uniquely determined, cf. [27], chapter 2. The continuity of the multiplication follows from the fact that $\mathcal{A}^0$ is a Fréchet space and separate continuity.
(b) $\mathrm{Op}_G g = \mathrm{Op}' g(x', \xi', D_n)$ may be considered as a pseudodifferential operator with an operator-valued symbol, and $\mathrm{Op}_G g = 0$ iff $g(x', \xi', D_n) = 0$ iff $g = 0$. For details see Remark 3.11.
Since $\mathcal{A}^0$ and $\mathcal{B}^{(-1,0),0}$ are Fréchet spaces, the continuity of m_1 through m_4 is implied by separate continuity. This in turn follows from the fact that the symbol topology is stronger than the operator topology.

3.4 Corollary. By defining

$$(p,g)(q,h) = (p\#q, m_1(p,h) + m_2(g,q) + m_3(g,h) + m_4(p,q))$$

we can make $\mathcal{A}^0 \times \mathcal{B}^{(-1,0),0}$ a Fréchet algebra with a continuous associative multiplication.

The following lemma is simple.

3.5 Lemma. *Let $\mathcal{A}$ be a Fréchet algebra with a closed two-sided ideal $\mathcal{J}$. Then $\mathcal{A}/\mathcal{J}$ also is a Fréchet algebra.*

We reach the first goal:

3.6 Proposition. *$\mathcal{G}$ is a symmetric unital Fréchet subalgebra of $\mathcal{L}(H)$ with a stronger topology.*

Proof. $\mathcal{G}$ is symmetric by 3.2. In order to see that it is a Fréchet algebra let us confine ourselves to the case where $n_2 = 0$. Then $\mathcal{G}$ is canonically isomorphic to $\mathcal{A}^0 \times \mathcal{B}^{(-1,0),0}/\mathcal{J}$, where $\mathcal{A}^0 \times \mathcal{B}^{(-1,0),0}$ carries the multiplication introduced in 3.4, and

$$\mathcal{J} = \{(p,q) : \mathrm{Op}_+ p + \mathrm{Op}_G g = 0\}.$$

Since $\mathcal{J}$ is a closed two-sided ideal we obtain the assertion from 3.4 and 3.5.

The rather delicate problem now is to establish the spectral invariance. First note the corresponding result for SG-manifolds without boundary.

3.7 Theorem. (Schrohe [41]) *Let Ω be $SG-$compatible without boundary. The algebra of pseudodifferential operators with symbols in $SG^0(\Omega)$ is a Ψ^*-subalgebra of $\mathcal{L}(H^s(\Omega))$ for every $s \in \mathbf{R}$.*

The lemma, below shows that it suffices to consider a neighborhood of the identity:

3.8 Lemma. *cf.* [14], 5.7. *Let $(C, \|\cdot\|)$ be a unital C^*-algebra, and let $\mathcal{A}$ be a symmetric subalgebra with the same unit e. If there is an $\epsilon > 0$ such that*

$$(e-x)^{-1} \in \mathcal{A} \text{ for all } x \in \mathcal{A} \text{ with } \|x\| < \epsilon, \tag{1}$$

then $\mathcal{A}$ is spectrally invariant in C.

Proof. $\mathcal{A}$ is dense in its C^*-closure $\mathcal{B} := C^*(\mathcal{A})$. Suppose $a \in \mathcal{A}$ is invertible with inverse b. Then $b \in \mathcal{B}$. Choose a sequence $\{b_j\}$ in $\mathcal{A}$ with $b_j \to b$. Then $c_j = e - ab_j$ tends to zero in $\mathcal{B}$. Thus $(e-c_j)^{-1} \in \mathcal{A}$ for sufficiently large j by (1), and $a^{-1} \in \mathcal{A}$.

3.9 Lemma. *It is sufficient to show that there is an $\epsilon > 0$ such that $I - (P_+ + G)$ is invertible within the calculus for all $P_+ + G$ in a neighborhood of zero in $\mathcal{L}(L^2(X))$.*

Proof. Write a matrix $A \in \mathcal{G}$ as $A = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$. Assuming that g_{11} is invertible,

$$A = \begin{bmatrix} I & 0 \\ g_{21}g_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} g_{11} & 0 \\ 0 & g_{22} - g_{21}g_{11}^{-1}g_{12} \end{bmatrix} \begin{bmatrix} I & g_{11}^{-1}g_{12} \\ 0 & I \end{bmatrix}.$$

For g_{11} and g_{22} close to the identity I and g_{12}, g_{21} close to zero in the corresponding operator norms, Theorem 3.7 shows that A is invertible within the calculus whenever this is true for g_{11}.

3.10 The Problem. We have therefore reduced the proof of 3.1 to showing the spectral invariance of the algebra

$$C = \{\mathrm{Op}_+ p + \mathrm{Op}_G g : p \in \mathcal{A}^0, g \in \mathcal{B}^{(-1,0),0}\}$$

in $\mathcal{L}(L^2(X))$; in view of Lemma 3.8 it is even sufficient to show that $(I + \mathrm{Op}_+ p + \mathrm{Op}_G g)^{-1} = I + \mathrm{Op}_+ \tilde{p} + \mathrm{Op}_G \tilde{g}$ for suitable $\tilde{p} \in \mathcal{A}^0, \tilde{g} \in \mathcal{B}^{(-1,0),0}$ whenever $\|\mathrm{Op}_+ p + \mathrm{Op}_G g\|_{\mathcal{L}(L^2(X))}$ is small.
We can even simplify somewhat more. Suppose that both $P = \mathrm{Op}_+ p$ and $G = \mathrm{Op}_G g$ are small in $\mathcal{L}(L^2(X))$. The identity

$$I + P + G = (I+P)(I + (I+P)^{-1}G)$$

then shows that we may consider the invertibility of the singular Green and the pseudodifferential part separately. It will turn out that this assumption is justified; this is why I shall start with an analysis of the singular Green operators.

3.11 Remark. (a) The estimates in 2.6(1) imply that for $g \in \mathcal{B}^{m,0}$

$$(x', \xi') \mapsto g(x', \xi', D_n) \in SG^{m+(1,0)}(\mathbf{R}^{n-1}, \mathcal{L}(L^2(\mathbf{R}_+))).$$

In fact,

$$\begin{aligned}&\left\|D_{\xi'}^{\alpha}D_{x'}^{\beta}g(x',\xi',D_n)\right\|_{\mathcal{L}(L^2(\mathbf{R}_+))} \leq \left\|D_{\xi'}^{\alpha}D_{x'}^{\beta}g(x',\xi',D_n)\right\|_{HS(L^2(\mathbf{R}_+))}\\ = \ &\left\|D_{\xi'}^{\alpha}D_{x'}^{\beta}\tilde{g}\,(x',\xi',x_n,y_n)\right\|_{L^2(\mathbf{R}_+\times\mathbf{R}_+)} \leq c_{\alpha\beta}\,\langle\xi'\rangle^{m_1+1-|\alpha|}\,\langle x'\rangle^{m_2-|\beta|}.\end{aligned}$$

Here, HS indicates the Hilbert-Schmidt norm, and $\tilde{g}$ denotes the symbol kernel of g.
(b) As an operator-valued symbol, $g(x',\xi',D_n)$ is uniquely determined: If $g\in\mathcal{B}^{-(1,0),0}$ and $\operatorname{Op}_G g=0$ then $g(x',\xi',D_n)=0$ for all x',ξ'. This follows from the well-known corresponding result for the scalar-valued case, cf. Kumano-go [27], Chapter 2, Proposition 1.2: Simply choose $f_1,\ f_2\in\mathcal{S}(\mathbf{R}^{n-1})$, $v_1,\ v_2\in E=L^2(\mathbf{R}_+)$; then $f_jv_j\in\mathcal{S}(\mathbf{R}^{n-1},E)$, $j=1,2$, and $0=\langle\operatorname{Op}_G g\,(f_1v_1)\,,f_2v_2\rangle_{L^2(\mathbf{R}^{n-1},E)}=\langle\operatorname{Op}'\langle g(x',\xi',D_n)v_1,v_2\rangle_E\,f_1,f_2\rangle_{L^2(\mathbf{R}^{n-1})}$.
(c) Consequently, the symbol of a singular Green operator is uniquely determined. If $g(x',\xi',D_n)=0$ for all x',ξ', then the symbol kernel $\tilde{g}\,(x',\xi',x_n,y_n)$ vanishes everywhere, and so does $g\,(x',\xi',\xi_n,\eta_n)=\mathcal{F}_{x_n\to\xi_n}\bar{\mathcal{F}}_{y_n\to\eta_n}\tilde{g}\,(x',\xi',x_n,y_n)$.

An important observation is the following

3.12 Theorem. *Let $a\in SG^0(\mathbf{R}^n,\mathcal{L}(E))$, E a Hilbert space, and let $A=\operatorname{Op}a$: $L^2(\mathbf{R}^n,E)\to L^2(\mathbf{R}^n,E)$. Given $\varepsilon>0$ there is a compact set $K=K(\varepsilon)\subset\mathbf{R}^n\times\mathbf{R}^n$ such that*

$$\|a(x,\xi)\|_{\mathcal{L}(E)}\ \leq\ (1+\varepsilon)\,\|A\|_{\mathcal{L}(L^2(\mathbf{R}^n,E))}$$

for all $(x,\xi)\notin K$.

Proof. The proof uses techniques developed by Hörmander for a similar question in the scalar-valued case. He showed the following [26], Theorem 3.3: If $p\in S^0_{\rho,\delta}(\mathbf{R}^n\times\mathbf{R}^n)$, $\rho>\delta$, is a pseudodifferential symbol with support in a set $C\times\mathbf{R}^n$ with $C\subseteq\mathbf{R}^n$ bounded, then

$$\limsup_{\xi\to\infty}\ \sup_x\ |p(x,\xi)|\ \leq\ \|\operatorname{Op}p\|_{\mathcal{L}(L^2(\mathbf{R}^n))}.$$

Here, I will show that

$$\limsup_{\xi\to\infty}\ \sup_x\ \|a(x,\xi)\|_{\mathcal{L}(E)}\ \leq\ \|A\|, \tag{1}$$

and

$$\limsup_{x\to\infty}\ \sup_\xi\ \|a(x,\xi)\|_{\mathcal{L}(E)}\ \leq\ \|A\|. \tag{2}$$

Together (1) and (2) imply the assertion. In order to see (1), let $\alpha=\limsup_{\xi\to\infty}\sup_x\|a(x,\xi)\|_{\mathcal{L}(E)}$. Choose a sequence $\{(x^\nu,\xi^\nu):\nu=1,2,\ldots\}$ with $\xi^\nu\to\infty$ and $\|a\,(x^\nu,\xi^\nu)\|_{\mathcal{L}(E)}\to\alpha$, a sequence $\{e_\nu\}\subseteq E$ with $\|e_\nu\|_E=1$ and

$$\|a\,(x^\nu,\xi^\nu)\,e_\nu\|_E\to\alpha,$$

and pick a function $u\in C_0^\infty(\mathbf{R}^n)$ with $\|u\|_{L^2}=1$. In this set-up we may essentially copy Hörmander's proof. The idea is to consider the sequence $\{u_\nu:\nu=1,2,..\}\subseteq\mathcal{S}(\mathbf{R}^n,E)$ defined by

$$u_\nu(x)=|\xi^\nu|^{\frac{n}{4}}\,u\left((x-x^\nu)\,|\xi^\nu|^{\frac{1}{2}}\right)e^{ix\xi^\nu}e_\nu$$

and to show that

$$\liminf_{\nu\to\infty}\|Au_\nu\|_{L^2(\mathbf{R}^n,E)}\ \geq\ \alpha.$$

In view of the fact that $\|u_\nu\|_{L^2(\mathbf{R}^n,E)} = 1$, this proves (1).

The proof of (2) is similar after the following change. Define the operator B on $L^2(\mathbf{R}^n, E)$ by

$$Bu(x) = (2\pi)^{-\frac{n}{2}} \int e^{ix\xi} a(x,\xi) u(\xi)\, d\xi,$$

i.e. $B = A \circ \mathcal{F}^{-1}$. Since the Fourier transform is an isometric isomorphism, $\|B\| = \|A\|$. Let $\beta = \limsup_{x\to\infty} \sup_\xi \|a(x,\xi)\|_{\mathcal{L}(E)}$. Now pick a function $u \in C_0^\infty(\mathbf{R}^n)$ with $\|u\|_{L^2(\mathbf{R}^n)} = 1$; then choose a sequence $\{(x^\nu, \xi^\nu) : \nu = 1,2,...\}$ in $\mathbf{R}^n \times \mathbf{R}^n$ and a sequence $\{e_\nu\}$ in E such that $\|a(x^\nu,\xi^\nu) e_\nu\|_E \to \beta$ while $\|e_\nu\|_E = 1$. Define the functions u_ν by

$$u_\nu(\xi) \;=\; e^{-ix^\nu(\xi-\xi^\nu)} \left|x^\nu\right|^{\frac{n}{4}} u\left((\xi - \xi^\nu)\left|x^\nu\right|^{\frac{1}{2}}\right) e_\nu$$

and copy the proof of part (a) with the roles of x^ν and ξ^ν reversed. This leads to the inequality

$$\beta \le \liminf \|Bu_\nu\|_{L^2(\mathbf{R}^n,E)},$$

so that $\|A\| = \|B\| \ge \beta$.

The next result we will need concerns invertibility at the boundary symbol level.

3.13 Theorem. *Let $g \in \mathcal{B}^{(-1,0),0}$, and suppose that for all $(x',\xi') \in \mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$*

$$\|g(x',\xi',D_n)\|_{\mathcal{L}(L^2(\mathbf{R}_+))} \le 0.1.$$

Then the operator

$$1 - g(x',\xi',D_n) \,:\, L^2(\mathbf{R}_+) \;\to\; L^2(\mathbf{R}_+)$$

is invertible for all x', ξ', and there is a symbol $k \in \mathcal{B}^{(-1,0),0}$ with

$$(1 - g(x',\xi',D_n))^{-1} = 1 - k(x',\xi',D_n).$$

Here I am writing 1 for the identity boundary symbol operator. Clearly, the constant 0.1 is not optimal. Although the objective is different, the proof uses techniques of [18], Proposition 3.2.1 and Theorem 3.2.3.

Proof. *Step* 1. The symbol g may be written

$$g(x',\xi',\xi_n,\eta_n) \;=\; \sum_{k,m=0}^{\infty} c_{km}(x',\xi')\, \hat\varphi_k(\xi_n, \langle\xi'\rangle)\, \bar{\hat\varphi}_m(\eta_n, \langle\xi'\rangle)$$

with a sequence $\{c_{km}\}$ which is rapidly decreasing in $SG^0(\mathbf{R}^{n-1})$ with respect to $k+m$ ("Laguerre function expansion"). This is an immediate generalization of the corresponding fact for the standard singular Green symbols, where the sequence $\{c_{km}\}$ is rapidly decreasing in $S^0_{1,0}$, cf. [18], section 2.2. The modified Laguerre functions $\varphi_k(\cdot,\sigma)$ and their Fourier transforms $\hat\varphi(t,\sigma) = \left(\frac{\sigma}{\pi}\right)^{\frac{1}{2}} \frac{(\sigma - it)^k}{(\sigma+it)^{k+1}}$, $k \in \mathbf{Z}$, form complete orthonormal systems: $\{\varphi_k : k = 0,1,...\}$ for $L^2(\mathbf{R}_+)$, $\{\varphi_k : k = -1,-2,...\}$ for $L^2(\mathbf{R}_-)$. Consequently, there is an $M \in \mathbf{N}$ such that

$$\left(\sum_{k+m>M} |c_{km}(x',\xi')|^2\right)^{\frac{1}{2}} < 0.1.$$

Step 2. Let $g_1 = \sum_{k+m>M} c_{km}\hat\varphi_k\bar{\hat\varphi}_m$. Since the $\hat\varphi_k$ are orthonormal, $\|g_1(x',\xi',D_n)\|_{\mathcal{L}(L^2(\mathbf{R}_+))} \le \|g_1(x',\xi',\xi_n,\eta_n)\|_{L^2(\mathbf{R}^2)} < 0.1$, and $1-g_1(x',\xi',D_n)$ is invertible on $L^2(\mathbf{R}_+)$ for all x',ξ'.

Step 3. There is a symbol $h \in \mathcal{B}^{(-1,0),0}$ with

$$(1-g_1(x',\xi',D_n))^{-1} = 1 + h(x',\xi',D_n):$$

Clearly, $h = \sum_{k=1}^{\infty} g_1(x',\xi',D_n)^k$. Let us check that the series converges not only in $\mathcal{L}(L^2(\mathbf{R}_+))$ but also in the space of boundary symbol operators. Given the integral kernel $\tilde g_1(x',\xi',\cdot,\cdot)$ of $g_1(x',\xi',D_n)$, the integral kernel for $g_1(x',\xi',D_n)^k$ is

$$f_k(x',\xi',x_n,y_n) = \int_{\mathbf{R}_+}\cdots\int_{\mathbf{R}_+} g_1(x_n,w_1)\,\tilde g_1(w_1,w_2)\cdot\ldots\cdot\tilde g_1(w_{k-1},y_n)\,dw_1\ldots dw_{k-1},$$

where I have omitted the arguments x',ξ' under the integrals.
In order to prove the assertion, it suffices to show that for all fixed $\alpha,\beta,m,m'r,r'$

$$\left\|x_n^m D_{x_n}^{m'} y_n^r D_{y_n}^{r'} D_{\xi'}^{\alpha} D_{x'}^{\beta} f_k(x',\xi',x_n,y_n)\right\|_{L^2(\mathbf{R}_+\times\mathbf{R}_+)} \le d_k\,\langle\xi'\rangle^{-|\alpha|-m+m'-r+r'}\,\langle x'\rangle^{-|\beta|}.$$

with a sequence $d_k = d_k(\alpha,\beta,m,m',r,r') \in \ell^1$.
This however, is an immediate consequence of the following facts:

(i) For arbitrary functions $h_j \in L^2(\mathbf{R}_+\times\mathbf{R}_+)$

$$\begin{aligned}&\left\|\int\cdots\int h_1(x,w_1)\,h_2(w_1,w_2)\cdot\ldots\cdot h_r(w_{r-1},y)\,dw_1\ldots dw_{r-1}\right\|_{L^2_{xy}}\\ \le\;& \|h_1\|_{L^2_{xw_1}}\cdot\ldots\cdot\|h_r\|_{L^2_{w_{r-1},y}}.\end{aligned}$$

(ii) For a derivation δ on a Banach algebra, $\delta(a_1\ldots a_r) = \sum_j a_1\ldots\delta(a_j)\ldots a_r$; in particular, $\|\delta(a^r)\| \le \|a\|^{r-1}\,\|\delta(a)\|$.

Step 4. Let $g_2 = g - g_1$. Then $(1-g)(1-g_1)^{-1} = 1 - g_2(1-g_1)^{-1} =: 1-g'$, omitting the arguments (x',ξ',D_n).
The image of $g'(x',\xi',D_n)$ is contained in that of $g_2(x',\xi',D_n)$. This in turn is a subspace of the span of $\varphi_0(\cdot,\langle\xi'\rangle),\ldots,\varphi_M(\cdot,\langle\xi'\rangle)$, since $g_2(x',\xi',D_n)$ has the integral kernel $\sum_{k+m\le M} c_{km}(x',\xi')\,\varphi_k(x_n,\langle\xi'\rangle)\,\varphi_m(y_n,\langle\xi'\rangle)$.
Now write, similarly as before,

$$\begin{aligned} g'(x',\xi',\xi_n,\eta_n) &= \sum_{k,m=0}^{\infty} d_{km}(x',\xi')\hat\varphi_k(\xi_n,\langle\xi'\rangle)\,\bar{\hat\varphi}_m(\eta_n,\langle\xi'\rangle)\\ = \sum_{k+m\le L}\ldots + \sum_{k+m>L}\ldots &=: g_2'(x',\xi',\xi_n,\eta_n) + g_1'(x',\xi',\xi_n,\eta_n),\end{aligned}$$

where L is chosen such that $\left(\sum_{k+m>L}|d_{km}(x',\xi')|^2\right)^{\frac12} < 0.1$.
Notice that $d_{km} = 0$ for $k > M$ in view of the above consideration on the range. Moreover,

$$(1-g_1')^{-1}(1-g') = 1 - (1-g_1')^{-1}g_2' =: 1-g''$$

with arguments (x', ξ', D_n) omitted and $g'' \in \mathcal{B}^{(-1,0),0}$ by the argument in Step 3. We have

$$\begin{aligned}\operatorname{range} g''(x',\xi',D_n) &\subseteq \operatorname{range}\left(\sum_{k=0}^{\infty} g_1^k(x',\xi',D_n)\, g'(x',\xi',D_n)\right)\\ &\subseteq \overline{\operatorname{range} g_1'(x',\xi',D_n)} + \operatorname{range} g'(x',\xi',D_n)\end{aligned}$$

contained in the span of $\varphi_0(\cdot,\langle\xi'\rangle),\ldots,\varphi_M(\cdot,\langle\xi'\rangle)$. On the other hand, $g_2'(x',\xi',D_n)$ vanishes on the span of $\{\varphi_j(\cdot,\langle\xi'\rangle : j > L\}$, hence so does $g''(x',\xi',D_n)$.

Step 5. Let $M' = \max(L, M)$, and let U be the linear space spanned by $\{\varphi_0(\cdot,\langle\xi'\rangle),\ldots,\varphi_{M'}(\cdot,\langle\xi'\rangle)\}$. We may represent the operator $g''(x',\xi',D_n)$ as a matrix with respect to this basis

$$g''(x',\xi',D_n) \sim ((e_{km}(x',\xi')))$$

with entries $e_{km} \in SG^0(\mathbf{R}^{n-1})$, $0 \le k, m \le M'$. The identity

$$1 - g'' = (1 - g_1')^{-1}(1 - g)(1 - g_1)^{-1} \tag{1}$$

implies that $1-g''(x',\xi',D_n)$ is invertible on $L^2(\mathbf{R}_+)$ and that $\|g''(x',\xi',D_n)\|_{\mathcal{L}(L^2(\mathbf{R}_+))} < \frac{1}{2}$ for all x',ξ'. Since $g''(x',\xi',D_n)$ vanishes on $U^\perp$, $(1 - g''(x',\xi',D_n))^{-1}$ will also map U to itself; it can be given as the operator associated with the inverse matrix to

$$((\delta_{km} - e_{km}(x',\xi'))) \tag{2}$$

on U and the identity on $U^\perp$. Moreover, the $\mathcal{L}(L^2(\mathbf{R}_+))$ norm of this inverse is bounded on U, independent of x',ξ'. Hence all entries in the inverse matrix to (2) will be bounded functions, and so will be its determinant. By Cramer's rule the inverse matrix has entries in SG^0. Hence $(1-g''(x',\xi',D_n))^{-1}$ has the form $1-g_3(x',\xi',D_n)$ with some $g_3 \in \mathcal{B}^{(-1,0),0}$. Together with identity (1) this yields the assertion.

Theorem 3.13 is a crucial step towards the proof of the following theorem.

3.14 Theorem. *Let $G = \mathrm{Op}_G g$ for some $g \in \mathcal{B}^{(-1,0),0}$, and suppose that*

$$\|G\|_{\mathcal{L}(L^2(X))} < 0.1.$$

Then the operator $I - G$ is invertible on $L^2(X)$, and there is a symbol $h \in \mathcal{B}^{(-1,0),0}$ such that $(I - G)^{-1} = I - \mathrm{Op}_G h$.

The constant 0.1 is not optimal; once this theorem is proven, it follows from Lemma 3.8 that the same is true for $\|G\|_{\mathcal{L}(L^2(X))} < 1$.

As a preparation for the proof of Theorem 3.14 we need the following observations.

3.15 Lemma. *The following assertions are equivalent*

(a) $b \in \mathcal{B}^{-\infty,0}$.

(b) $\mathrm{Op}_G b$ *is an integral operator with a kernel in* $\mathcal{S}(X \times X)$.

(c) $\mathrm{Op}_G b$: $\mathcal{S}'(X) \to \mathcal{S}(X)$ *is continuous.*

Lemma 3.15 follows essentially from Theorem 2.10.

3.16 Lemma. *The set* $\{\lambda I + S : \lambda \in \mathbf{C},\, S \in \mathrm{Op}_G\mathcal{B}^{-\infty,0}\}$ *is a* Ψ^**−subalgebra of* $\mathcal{L}(L^2(X))$.

Proof. It follows from Theorem 2.11 that these operators form a unital symmetric algebra in $\mathcal{L}(L^2(X))$. The representation $\lambda I + S$ is unique, and the symbol of S is uniquely determined by Remark 3.11(c). So we can give the above set the topology induced from the symbol space which is stronger than the operator topology and Fréchet. Finally, the spectral invariance in $\mathcal{L}(L^2(X))$ follows from the identity

$$(I - S)^{-1} = I + S + S(I - S)^{-1}S$$

in connection with 3.15.

The following lemma is a consequence of the asymptotic expansion formulae for the compositions.

3.17 Lemma. *Let* $g_1, g_2 \in \mathcal{B}^{(-1,0),0}$, *and suppose that*

$$(1 - g_2(x', \xi', D_n))\,(1 - g_1(x', \xi', D_n)) = 1.$$

Then

$$(I - \mathrm{Op}_G g_2)\,(I - \mathrm{Op}_G g_1) = I - \mathrm{Op}_G g_3$$

for some $g_3 \in \mathcal{B}^{(-2,-1),0}$.

3.18 Remark. Recall the following facts from the operator theory. For a proof see Taylor, [60], Theorems 5.41-G, 5.5-E, 5.8-A. If $A \in \mathcal{L}(E)$, E a Banach space, is of the form $A = I + S$, S compact, then there is an $r \in \mathbf{N}_0$ such that

$$\begin{aligned} \mathcal{N}(A^r) &= \mathcal{N}\left(A^{r+1}\right), \mathcal{R}(A^r) = \mathcal{R}\left(A^{r+1}\right), & (1)\\ E &= \mathcal{N}(A^r) \oplus \mathcal{R}(A^r), & (2)\\ A &: \mathcal{R}(A^r) \to \mathcal{R}(A^r) \text{ bijective.} & (3) \end{aligned}$$

The spectrum of A is discrete with only accumulation point $\lambda = 1$. Let Γ be a small circle about $\lambda = -1$ and

$$P = \frac{1}{2\pi i}\int_\Gamma (\lambda I - S)^{-1} d\lambda; \tag{4}$$

then

$$\mathcal{N}(A^r) = \mathcal{R}(P), \mathcal{R}(A^r) = \mathcal{N}(P). \tag{5}$$

The range $\mathcal{R}(P)$ is finite-dimensional, since S is compact.

Let us now start with the proof of Theorem 3.14.

Step 1. By Theorem 3.12 in connection with Remark 3.11 there is a compact set K in $\mathbf{R}^{n-1} \times \mathbf{R}^{n-1}$ such that

$$\|g(x', \xi', D_n)\|_{\mathcal{L}(L^2(\mathbf{R}_+))} < 0.1$$

for $(x',\xi') \notin K$. Choose a function $\varphi = \varphi(x',\xi')$ in $C_0^\infty(\mathbf{R}^{n-1} \times \mathbf{R}^{n-1})$, equal to 1 on K and with $0 \le \varphi \le 1$. Let $g_0(x',\xi',\xi_n,\eta_n) = \varphi(x',\xi)g(x',\xi',\xi_n,\eta_n)$, $g_1(x',\xi',\xi_n,\eta_n) = (1-\varphi(x',\xi'))\,g(x',\xi',\xi_n,\eta_n)$. Then $g_1 \in \mathcal{B}^{(-1,0),0}$, $g_0 \in \mathcal{B}^{-\infty,0}$. Moreover, for all (x',ξ'),

$$\|g_1(x',\xi',D_n)\|_{\mathcal{L}(L^2(\mathbf{R}_+))} \le |1-\varphi(x',\xi')|\,\|g(x',\xi',D_n)\|_{\mathcal{L}(L^2(\mathbf{R}_+))} \le 0.1.$$

Step 2. By Theorem 3.13 there is a symbol $g_2 \in \mathcal{B}^{(-1,0),0}$ with

$$(1-g_2(x',\xi',D_n))\,(1-g_1(x',\xi',D_n)) = 1$$

so that

$$(I - \mathrm{Op}_G g_2)\,(I - \mathrm{Op}_G g_1) = I + \mathrm{Op}_G g_3$$

with a symbol $g_3 \in \mathcal{B}^{(-2,-1),0}$, using Lemma 3.17. The usual parametrix construction yields a symbol $h \in \mathcal{B}^{(-2,-1),0}$ with

$$(I - \mathrm{Op}_G h)\,(I - \mathrm{Op}_G g_2)\,(I - \mathrm{Op}_G g_1) - I \in \mathrm{Op}_G \mathcal{B}^{-\infty,0}.$$

Since $g - g_1 \in \mathcal{B}^{-\infty,0}$, the relation remains true if we replace g_1 by g, i.e. there is a $k \in \mathcal{B}^{(-1,0),0}$ with

$$(I - \mathrm{Op}_G k)\,(I - G) = I + S$$

for an $S \in \mathrm{Op}_G\mathcal{B}^{-\infty,0}$.

Step 3. Write $B = I - \mathrm{Op}_G k$, $C = I - G$. Then apply the construction in Remark 3.18 to $A = BC$. In view of the integral representation 3.18(4) in connection with 3.16, $P = \lambda I + S'$ for some $S' \in \mathrm{Op}_G\mathcal{B}^{-\infty,0}$. Since the range of P is finite-dimensional, $\lambda = 0$; in particular $\mathcal{R}(P) \subseteq \mathcal{S}(X)$.

Step 4. Let us show that there is a relative inverse F to CP in $\mathrm{Op}_G\mathcal{B}^{-\infty,0}$ i.e., there is an operator $F \in \mathrm{Op}_G\mathcal{B}^{-\infty,0}$ with $FCPF = F$ and $CPFCP = CP$:
Choose an orthonormal basis $\{e_1,...,e_k\}$ of $\mathcal{R}(P)$. Define $f_j = CP\,e_j = Ce_j$, $j = 1,...,k$. Since $C = I - G$: $\mathcal{S}(X) \to \mathcal{S}(X)$ is continuous, $f_j \in \mathcal{S}(X)$. Moreover, they are linearly independent, for C is bijective. So we can define $F : L^2(X) \to L^2(X)$ by

$$\begin{aligned} F(f_j) &= e_j \quad \text{on span}\,\{f_1,...,f_k\} \\ F &= 0 \quad \text{on span}\,\{f_1,...,f_k\}^\perp. \end{aligned}$$

It is easy to check that F has the desired properties.

Step 5. Let $F' = PF$. Then F' also is a relative inverse to CP. Moreover, for the number r of 3.18

$$\left((BC)^{r-1}B + F'\right)C = I + T, \tag{1}$$

where $T \in \mathrm{Op}_G\mathcal{B}^{-\infty,0}$ and $I + T$ is invertible on $L^2(X)$:
Clearly, the fact that $BC \in I + \mathrm{Op}_G\mathcal{B}^{-\infty,0}$ and $F' \in \mathrm{Op}\,\mathcal{B}^{-\infty,0}$ implies that T also belongs to this class. Therefore $I + T$ is a Fredholm operator of index zero. In order to check its invertibility, assume that there is an $h \in L^2(X)$ with $(I+T)h = 0$. Decomposing $h = h_n + h_r$ with $h_n \in \mathcal{N}(P)$, $h_r \in \mathcal{R}(P)$, one concludes from 3.18(3), (5) that $h = 0$.

Step 6. Conclusion: Since $I - T$ is invertible, identity (1) and Lemma 3.16 give the assertion.

The next theorem is similar to Theorem 3.12. The difference is that we control the operator norm only on the half-space $X \times \mathbf{R}^n$.

3.19 Theorem. *Let* $P = \operatorname{Op} p \in \operatorname{Op} \mathcal{A}^0$, $G \in \operatorname{Op}_G \mathcal{B}^{(-1,0),0}$ *and* $\|P_+ + G\|_{\mathcal{L}(L^2(X))} = 1$. *Then*

$$\limsup_{x \in X, (x,\xi) \to \infty} |p(x,\xi)| \leq 1. \tag{1}$$

The proof is rather lengthy, and I only give a sketch. The idea is the following. Let $d = \limsup_{(x,\xi)\to\infty} |p(x,\xi)|$ on $X \times \mathbf{R}^n$. We consider two possible cases.

(i) There is a sequence (x^ν, ξ^ν) with $\xi^\nu \to \infty$ and $|p(x^\nu, \xi^\nu)| \to d$; without loss of generality we may assume that $p(x^\nu, \xi^\nu) \to \alpha$ with $|\alpha| = d$.

(ii) There is a sequence (x^ν, ξ^ν) with ξ^ν bounded, $x^\nu \to \infty$ and $p(x^\nu, \xi^\nu) \to \alpha$ with $|\alpha| = d$. Without loss of generality we may assume $\xi^\nu \to \xi^0 \in \mathbf{R}^n$.

In view of (1) we may assume that $d > 0$. More is true. Let $\varepsilon > 0$ be arbitrary. Settling for a sequence (x^ν, ξ^ν) with $p(x^\nu, \xi^\nu) \to \beta$ and $|\beta| \geq d - \varepsilon$, we may assume that $x_n^\nu \geq c_\varepsilon \langle x^\nu \rangle$ for some $c_\varepsilon > 0$. This follows from the fact, that

$$\partial_{x_n} p(x,\xi) = O\left(\langle x \rangle^{-1}\right).$$

In case (i), the assertion follows from a more elaborate version of the proof of 3.12: Choose a function $0 \neq u \in C_0^\infty(X)$, consider the sequence $u^\nu = |\xi^\nu|^{\frac{n}{4}} u((x - x^\nu)|\xi^\nu|^{\frac{1}{2}}) e^{ix\xi^\nu}$ and show that $|\alpha| \leq \lim \|P_+ u^\nu\|_{L^2(X)}$. For large ν, $u^\nu = 0$ on $\left\{x : x_n \leq \frac{1}{2} c_\varepsilon \langle x \rangle\right\}$. So we can choose a function $\varphi \in C_0^\infty(\mathbf{R})$, $\varphi \equiv 1$ near zero with

$$G u^\nu = G(1 - \varphi(\frac{x_n}{\langle x' \rangle})) u^\nu.$$

Now u^ν converges weakly to zero, and $G(1-\varphi)(\frac{x_n}{\langle x' \rangle}) \in \operatorname{Op}_G \mathcal{B}^{-\infty,0}$ is compact, so $Gu^\nu \to 0$. This allows us to conclude, similarly as before, that $d - \varepsilon \leq \|P_+ + G\|_{\mathcal{L}(L^2(X))}$.

In case (ii) one may apply an idea of Grushin [23], proof of Theorem 3.3: Choose a function $\psi \in C_0^\infty(\mathbf{R}^p)$, $0 \leq \psi \leq 1$, with $\psi(\xi^0) = 1$. By Arzela-Ascoli, there is a subsequence $\{x^{\nu_k}\}$ of $\{x^\nu\}$ and a $\varphi \in C_0^\infty(\mathbf{R}^n)$ such that $p(x^{\nu_k}, \xi)\psi(\xi) - \varphi(\xi) \to 0$ in all derivatives. Clearly, $\varphi(\xi^0) = d$. Moreover, there is a $u \in C_0^\infty(X)$ with $\|u\|_{L^2(X)} = 1$ and $\|\varphi(D)u\|_{L^2(\mathbf{R}^n)} \geq d - \varepsilon$ for arbitrary $\varepsilon > 0$. The idea then is to show that for $u^\nu(x) = u(x - x^\nu)$ we have $\|P_+ r^+ \psi(D) u^\nu\|_{L^2(X)} \geq d - 2\varepsilon$ for all $\nu \geq \nu_0$. Since u^ν weakly tends to zero, and since, for large ν, the support of u^ν is contained in $\{x_n \geq c\langle x \rangle\}$, $c > 0$, we obtain the assertion just as in (i).

The next theorem shows that the asymptotic behavior of the symbol determines how good an operator can be approximated by regularizing operators.

3.20 Theorem. *Let* $P_+ \in \operatorname{Op}_+ SG^0\left(\mathbf{R}^n_+\right)$ *be a pseudodifferential operator with a symbol* p *defined on* $X \times \mathbf{R}^n$, *and assume that*

$$\limsup_{(x,\xi)\to\infty, x \in X} |p(x,\xi)| \leq 1.$$

Then there is

- *a universal constant* K, *independent of* p,

- *an extension* $\tilde{p} \in SG^0(\mathbf{R}^n)$ *of* p *to* $\mathbf{R}^n \times \mathbf{R}^n$, *and*

- *a regularizing operator* R *on* $\mathbf{R}^n$ *such that*

$$\|P - R\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \le K. \tag{1}$$

In (1), P is the operator with the extended symbol. Obviously, $\tilde{p}$ has the transmission property whenever p does.

The proof of 3.20 will be prepared by the following lemmata.

3.21 Lemma. (a) *There is an* $N \in \mathbf{N}$ *and a constant* C_{CV} *such that for all* $q \in S^0_{0,0}(\mathbf{R}^n \times \mathbf{R}^n)$

$$\|\mathrm{Op}\, q\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \le C_{CV} \max_{|\alpha|,|\beta| \le N} \left\| q^{(\alpha)}_{(\beta)}(x,\xi) \right\|_{sup}.$$

Remember the notation $q^{(\alpha)}_{(\beta)} = D^\alpha_\xi D^\beta_x q$.
(b) *Let* $q \in SG^0(\mathbf{R}^n)$ *with* $|q(x,\xi)| \le 1$ *for all* $|x| + |\xi| \ge M$. *Then there is a function* $\varphi \in C^\infty_0(\mathbf{R}^n)$, $0 \le \varphi \le 1$, *and a combinatorial constant* C_{com} *such that*

$$\|(1-\varphi)(x)\,\mathrm{Op}\; q(1-\varphi)(D)\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \le C_{com} C_{CV} \tag{1}$$

Proof. (a) is Calderon and Vaillancourt's theorem.
(b) Choose a function φ with $\varphi \equiv 1$ on a set $\{|x| \le M'\}$ satisfying $|D^\alpha_x \varphi(x)| \le 1$ for all $|\alpha| \le N$; N is the number in (a). Applying (a) to the symbol $r(x,\xi) = (1-\varphi)(x)\, q(x,\xi)\,(1-\varphi)(\xi)$, we can estimate the norm on the left hand side of (1) by finitely many derivatives of r.
Now $|q(x,\xi)| \le 1$ for $|x| + |\xi| \ge M$, and for $(\alpha,\beta) \ne (0,0)$, the SG estimates imply that $\left|q^{(\alpha)}_{(\beta)}(x,\xi)\right| \le 1$ for large $|x|$ and $|\xi|$. For large M', we thus have $\left|r^{(\alpha)}_{(\beta)}(x,\xi)\right| \le C_{com}$ for a combinatorial constant arising from the binomial coefficients in Leibniz' rule for derivatives of r.

A direct calculation in oscillatory integrals shows the lemma, below.

3.22 Lemma. *For* $p \in SG^0(\mathbf{R}^n)$,

$$\mathrm{Op}\, p = \mathcal{F}^{-1}\mathrm{Op}\, q\, \mathcal{F} + \mathcal{F}^{-1}\mathrm{Op}\, r\, \mathcal{F}$$

with $q(x,\xi) = p(-\xi, x) \in SG^0(\mathbf{R}^n)$, *and suitable* $r \in SG^{(-1,-1)}(\mathbf{R}^n)$.

Finally, the following is a result of Hörmander [25], proof of Theorem 3.3.

3.23 Lemma. *Let* $q \in S^0_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$, *and suppose that* $q(x,\xi) = 0$ *for all* x *outside a compact set. Then for every* $\varepsilon > 0$ *there is a symbol* $r(x,\xi) \in C^\infty_0(\mathbf{R}^n \times \mathbf{R}^n)$ *with*

$$\|\mathrm{Op}\, q - \mathrm{Op}\, r\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \le \limsup_{\xi} \sup_{x} |q(x,\xi)| + \varepsilon.$$

Now the proof of Theorem 3.20 is easy. There is an extension $\tilde{p}$ of p satisfying

$$\limsup_{(x,\xi)\to\infty} |\tilde{p}(x,\xi)| < C_s,$$

with a universal constant C_s, cf. 2.3. Let $\tilde{P} = \mathrm{Op}\,\tilde{p}$. By Lemma 3.21(b) there is a compactly supported φ with

$$\|(1-\varphi)(x)\,\tilde{P}(1-\varphi)(D)\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \le C_{com} C_{CV} C_s. \qquad (1)$$

Now write

$$\begin{aligned} \tilde{P} &= \varphi(x)\,\tilde{P}\varphi(D) + \varphi(x)\tilde{P}(1-\varphi)(D) + \\ &\quad (1-\varphi)(x)\,\tilde{P}\varphi(D) + (1-\varphi)(x)\,\tilde{P}(1-\varphi)(D) \\ &= P_1 + P_2 + P_3 + P_4 \end{aligned}$$

with the obvious notation. Then the assertion follows from (i) - (iv), below.

(i) P_1 is regularizing.

(ii) P_2 has the symbol $\varphi(x)\tilde{p}(x,\xi)(1-\varphi)(\xi)$. By Lemma 3.23, there is a compactly supported r_2 with

$$\|P_2 - \mathrm{Op}\,r_2\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \le C_s.$$

(iii) P_3 has the symbol $(1-\varphi)(x)\tilde{p}(x,\xi)\varphi(\xi)$. By 3.22, $\mathcal{F}P_3\mathcal{F}^{-1} = \mathrm{Op}\,q + \mathrm{Op}\,r$ with

$$q(x,\xi) = (1-\varphi)(-\xi)\tilde{p}(-\xi,x)\varphi(x),$$

and $r \in SG^{(-1,-1)}(\mathbf{R}^n)$. Applying Lemma 3.23 to q, we obtain a compactly supported symbol r_{31} with

$$\|\mathrm{Op}\,q - \mathrm{Op}\,r_{31}\|_{\mathcal{L}(L^2(\mathbf{R}^n))} < C_s.$$

By Calderon and Vaillancourt's theorem 3.21(a) we find a compactly supported symbol r_{32} with

$$\|\mathrm{Op}\,r - \mathrm{Op}\,r_{32}\|_{\mathcal{L}(L^2(\mathbf{R}^n))} < 1.$$

Let $r_3 = r_{31} + r_{32}$, $R_3 = \mathcal{F}^{-1}\mathrm{Op}\,r_3\mathcal{F}$. Then R_3 is regularizing, and $\|P_3 - R_3\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \le C_s + 1$, since $\mathcal{F}$ is an isometry.

(iv) From (1) we know that $\|P_4\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \le C_{com} C_{CV} C_s$.

We get the following corollary

3.24 Corollary. There is a universal constant $C > 0$ such that for all operators $P_+ + G \in \mathcal{C}$ with $\|P_+ + G\|_{\mathcal{L}(L^2(\mathbf{R}^n))} < 1$ we can find $P' \in \mathrm{Op}\,\mathcal{A}^0$, $G' \in \mathrm{Op}_G\mathcal{B}^{(-1,0),0}$ with $P_+ + G = P'_+ + G'$, $\|P'\|_{\mathcal{L}((\mathbf{R}^n))} \le C$, and $\|G'\|_{\mathcal{L}((\mathbf{R}^n))} \le C$.

Proof. This is immediate from 3.19, 3.20 and the fact that if R is a regularizing pseudodifferential operator, then R_+ also is a regularizing singular Green operator, cf. 2.10.

As a further step towards the proof of the spectral invariance of the algebra $\mathcal{C}$ we have the following lemma.

3.25 Proposition. *Suppose $P \in \operatorname{Op}\mathcal{A}^0$ is a pseudodifferential operator with $\|P\|_{\mathcal{L}(L^2(\mathbf{R}^n))} <$ 0.01. Then $I - P_+$ is invertible in $\mathcal{L}(L^2(X))$, and*

$$(I - P_+)^{-1} = I - \left(P'_+ + G'\right),$$

where $P' \in \operatorname{Op}\mathcal{A}^0$, $G' \in \operatorname{Op}_G\mathcal{B}^{(-1,0),0}$.

Proof. $I - P$ is invertible on $L^2(\mathbf{R}^n)$. By Theorem 3.7, there is a $Q \in \operatorname{Op} SG^0$ with $(I - Q) = (I - P)^{-1}$. In particular, $I - Q$ also is a parametrix to $I - P$, so Q has the transmission property, and $Q \in \operatorname{Op}\mathcal{A}^0$. Now

$$(I - Q)_+(I - P_+) = I - L,$$

where $L = L(I - Q, I - P)$ is the singular Green left-over term. Clearly,

$$\begin{aligned} \|L\|_{\mathcal{L}(L^2(X))} &\le \|Q\|_{\mathcal{L}(L^2(\mathbf{R}^n))} + \|P\|_{\mathcal{L}(L^2(\mathbf{R}^n))} + \|P\|_{\mathcal{L}(L^2(\mathbf{R}^n))}\|Q\|_{\mathcal{L}(L^2(\mathbf{R}^n))} \\ &< 0.1. \end{aligned}$$

By Theorem 3.14, $(I - L)^{-1} = (I - H)$ for some $H \in \operatorname{Op}_G\mathcal{B}^{(-1,0),0}$. This gives the desired result, since $(I - L)^{-1}(I - Q)_+(I - P_+) = I$.

3.26 Conclusion. Suppose $P_+ + G \in \mathcal{C}$, and $\|P_+ + G\|$ is small in $\mathcal{L}(L^2(X))$. Find small representatives P', G' according to 3.24. Then

$$I - (P_+ + G) = \left(I - P'_+\right)\left(I - \left(I - P'_+\right)^{-1} G'\right). \tag{1}$$

By 3.25, $I + P'_+$ has an inverse in $\mathcal{C}$. Consequently, $\left(I - P'_+\right)^{-1} G'$ is a singular Green operator of small norm. By Theorem 3.14, also the second factor in (1) is invertible in $\mathcal{C}$. Hence $\mathcal{C}$ is spectrally invariant.

The Manifold Case: The spectral invariance result also holds for general SG manifolds with boundary:

3.27 Theorem. *Let (Ω, X, Y, g) be SG-compatible, E, F fixed SG-bundles over Ω, Y, respectively. Then the algebra $\mathcal{G}$ of Green operators of order and type zero is a Ψ^*-subalgebra of $\mathcal{L}(H)$, where $H = H^0(X, E) \oplus H^{(-\frac{1}{2},0)}(Y, F)$.*

Again we shall break up the proof into a series of smaller results.

3.28 Remark. With the definitions in 2.16 and Theorem 2.17 $\mathcal{G}$ is a unital symmetric algebra in $\mathcal{L}(H)$. We can introduce a Fréchet topology for $\mathcal{G}$ starting from the half-space situation. Again, this topology is stronger than the operator topology of $\mathcal{L}(H)$.
We therefore only have to prove the spectral invariance in $\mathcal{L}(H)$. By Lemma 3.8 we may confine ourselves to showing that for some $\epsilon > 0$

$$(I - A)^{-1} \in \mathcal{G},$$

provided $\|A\|_{\mathcal{L}(H)} < \epsilon$.

The lemma, below, is a simple consequence of the characterization of the regularizing Green operators of type zero via rapidly decreasing kernel sections.

3.29 Lemma. *A is a regularizing Green operator in $\mathcal{G}$ if and only if A has a continuous extension $A : \mathcal{S}'(X,E) \oplus \mathcal{S}'(Y,F) \longrightarrow \mathcal{S}(X,E) \oplus \mathcal{S}(Y,F)$. If A is regularizing and $I - A$ is invertible in $\mathcal{L}(H)$, then $(I-A)^{-1} = I + A + A(I-A)^{-1}A = I - A'$ for a regularizing $A' \in \mathcal{G}$.*

3.30 Lemma. *Let $A \in \mathcal{G}$, and let ϕ, ψ be SG^0-functions supported in a single coordinate neighborhood. Then $\phi A \psi$ induces a Green operator $(\phi A\psi)_*$ on the half-space by*

$$(\phi A \psi)_* f = [(\phi A \psi)(f \circ \kappa)] \circ \kappa^{-1}, \tag{1}$$

where κ is the coordinate homeomorphism from the manifold to Euclidean space. Suppose that for the corresponding operator norm on Euclidean space we have $\|(\phi A\psi)_*\| < 1$. Then $I - \phi A \psi$ is invertible, and $(I - \phi A\psi)^{-1} \in \mathcal{G}$.

Proof. Write $(\phi A\psi)_* = \phi_* A_* \psi_*$ with a Green operator A_* on Euclidean space. Since $I + \phi_* A_* \psi_*$ is invertible; its inverse by 3.26 is a Green operator of order and type zero on $\mathbf{R}^n_+$. The identity

$$(I - (\phi A\psi)_*)^{-1} = I + \phi_* A \psi_* + \phi_* A_* \psi_* (I - (\phi A \psi)_*)^{-1} \phi_* A_* \psi_*$$

implies that the inverse is of the form $I + \phi_* B_* \psi_*$ with a Green operator B_* of order and type zero.
Let $\phi B \psi$ be the corresponding operator in $\mathcal{L}(H)$ induced via the coordinate maps. It is easily checked that $I + \phi B\psi$ is the inverse to $I + \phi A \psi$.

This allows us to obtain the spectral invariance also for the manifold case.

3.31 Conclusion in the Manifold Case. Choose an SG-partition of unity $\{\phi_1, \ldots, \phi_J\}$ and cut-off functions $\{\psi_1, \ldots, \psi_J\}$ subordinate to the cover $\{\Omega_1, \ldots, \Omega_J\}$ as in 1.7(a). Then write for sufficiently small A:

$$I - A = I - \phi_1 A - \phi_2 A - \ldots - \phi_J A = (I - \phi_2 A (I - \phi_1 A)^{-1} - \ldots)(I - \phi_1 A),$$

We have $I - \phi_1 A = (I - \phi_1 A \psi_1)(I - (I - \phi_1 A \psi_1)^{-1} \phi_1 A (I - \psi_1))$. Both factors are invertible within the calculus by 3.29 and 3.30 provided $\|A\|_{\mathcal{L}(H)}$ is small; note that $\phi_1 A(I - \psi_1)$ is regularizing. Induction over J gives the assertion.

4 Applications: Fredholm Criteria, Functional Calculus, and the Case of Compact Manifolds

Throughout this section let E, F be $SG-$bundles over X and Y, respectively, and let $\mathcal{G}$ be the algebra of Green operators of order and type zero on the Hilbert space $H = H^0(X,E) \oplus H^{(-\frac{1}{2},0)}(Y,F)$. The first theorem concerns the existence of reductions of the order in the calculus.

4.1 Theorem. *Let E be an SG-vector bundle over X. For every $m \in \mathbf{Z}\times\mathbf{R}$ there is a family $\{R_-^m(\mu) : \mu \in \mathbf{R}\}$ of pseudodifferential operators such that for all $s \in \mathbf{R}\times\mathbf{R}, s_1 > -\frac{1}{2}$,*

$$R_-^m(\mu)_+ : H^s(X,E) \longrightarrow H^{s-m}(X,E)$$

is invertible for large μ.

The proof uses ideas of Shubin [59], Rempel and Schulze [37], [38] and Grubb [19]. Again, it is split up into several steps.

Step 1. Preparation. Without loss of generality we can assume that the bundle E is trivial one-dimensional and that $m_2 = 0, m_1 = m \in \mathbf{Z}$. The former is due to the locality of the construction, while the latter is a consequence of the fact that multiplication with powers of $\langle x\rangle$ – or rather a global version of it – has the desired properties. As we shall see, we may even confine ourselfves to the construction for $m = 1$. With this we shall start. So in the following, $m = 1$.
Pick coordinates as in 2.15. Fix a function χ as in 2.13 and choose a function $\tilde{\tau} \in C_0^\infty(\mathbf{R}), 0 \geq \tilde{\tau} \geq 1$, equal to 1 on $[-\frac{1}{4},\frac{1}{4}]$ and vanishing outside $[-\frac{1}{3},\frac{1}{3}]$. Let $\tau(x) = \tilde{\tau}(x_n/\langle x'\rangle)$.
Step 2. We now choose symbols in the local coordinate charts on Euclidean space. For a boundary chart Ω_j pick the symbol

$$a_j(x,\xi,\mu) = \left[\chi(\frac{\xi_n}{a\langle\xi',\mu\rangle})\langle\xi',\mu\rangle - i\xi_n\right]^{\tau(x)} \langle\xi,\mu\rangle^{1-\tau(x)}; \tag{1}$$

for an interior chart take

$$a_j(x,\xi,\mu) = \langle\xi,\mu\rangle. \tag{2}$$

Like in 2.13, $a \gg 0$ is a parameter to be chosen later on. We then use a partition of unity$\{\phi_j\}$ and cut-off functions $\{\psi_j\}$ as in 1.7 and define the operator $R_-(\mu)$ on $C_0^\infty(\Omega)$ by

$$R_-(\mu) = \sum_{j=1}^{J}[\psi_j \mathrm{Op}\, a_j\phi_j]_*,$$

where the asterisk indicates that the operators are given via the transport by the coordinate functions, cf. 3.30(1).
Step 3. The above symbols are parameter-dependent in a strong sense. They satisfy the relations

$$D_\xi^\alpha D_x^\beta D_\mu^\gamma a_j(x,\xi,\mu) = \mathrm{O}(\langle\xi,\mu\rangle^{1-|\alpha|-|\gamma|}\langle x\rangle^{-\beta}), \tag{3}$$

so μ plays the role of an additional covariable. Consequently, also the local symbols r_j of $R_-(\mu)$ – where we have to take into account the effects of the symbols induced from other coordinate neighborhoods via the changes of coordinates – will satisfy these relations. Moreover, by making the parameter a large, we may achieve that $r_j(x,\xi,\mu)$ is invertible, and

$$r_j(x,\xi,\mu)^{-1} = O(\langle\xi,\mu\rangle^{-1}). \tag{4}$$

Step 4. The estimates (4) and (5) allow us to construct a parameter-dependent parametrix following Seeley [56] – for details cf. e.g. [42], section 2.3. For sufficiently large μ, $R_-(\mu) : H^s(\Omega) \to H^{s-(1,0)}(\Omega)$ will be invertible, independent of s. Applying 3.7, its inverse $R_-^{-1}(\mu)$ is a pseudodifferential operator with a symbol in $SG^{(-1,0)}(\Omega)$, even in $\mathcal{A}^{(-1,0)}$, since $R_-^{-1}(\mu)$ also is a parametrix to $R_-(\mu)$. Moreover, also for $R_-^{-1}(\mu)$ the parameter μ plays the role of an additional covariable.

Step 5. The symbols in (2) belong to H^- as functions of ξ_n. Therefore the local symbols of $R_-(\mu)$ are H^- functions of ξ_n, up to perturbations that are regularizing in x,ξ, and μ. Also the symbol of $R_-^{-1}(\mu)$ will be given by local symbols that are H^- functions of ξ_n, up to regularizing perturbations. This follows from the parametrix construction in connection with the fact that also $r_j^{-1}(x,\xi,\mu)$ is an H^- function modulo regularizing perturbations. In view of the calculus, this implies that the singular Green symbol of $L(R_-^{\pm 1}(\mu), R^k(\mu))$ is regularizing with μ in the role of an additional covariable. In particular,

$$R_-(\mu)_+ R^{-1}(\mu)_+ = I + O(\langle\mu\rangle^{-\infty}) = R_-^{-1}(\mu)_+ R_-(\mu)_+.$$

Thus $R_-^{\pm 1}(\mu)_+ : H^s(X) \to H^{s\pm(1,0)}(X)$ is invertible for large μ, provided $s_1 > -\frac{1}{2}$. Moreover, $[R_-(\mu)_+]^2 \equiv [R_-^2(\mu)]_+$ modulo regularizing operators. By iteration we get the assertion.

The first application is the theorem, below.

4.2 Theorem. *Let $A \in \mathcal{G}$. Then $A : H \longrightarrow H$ is a Fredholm operator iff it is elliptic.*

By 2.19, ellipticity implies the Fredholm property, so we only have to prove the converse. We shall need the following two lemmata.

4.3 Lemma. *Suppose $B \in \mathcal{G}$, and $B : H \to H$ has finite-dimensional range. Then B is a regularizing Green operator of type zero.*

Proof of 4.3. Let $\mathcal{S} = \mathcal{S}(X,E) \oplus \mathcal{S}(Y,F)$. $B : \mathcal{S} \to \mathcal{S}$ is continuous, and $\mathcal{S}$ is dense in H, so $B\mathcal{S}$ is dense in the finite-dimensional space BH. Hence both are equal, $\operatorname{im} B \subseteq \mathcal{S}$, and we may write $Bf = \sum (f,u_j) f_j$ for suitable $u_j \in H$, and orthonormal $f_j \in \mathcal{S}$. Considering B^* we conclude that B is an integral operator with a kernel density in $\mathcal{S} \otimes_{alg} \mathcal{S}$.

4.4 Lemma. (*cf.* [14], *Bemerkung* 5.7) *Suppose $A \in \mathcal{G}$, and $A : H \to H$ is Fredholm. Then there are $R_1, R_2 \in \mathcal{G}$ such that*

$$F_1 = R_1 A - I \qquad \text{and} \qquad F_2 = AR_2 - I \tag{1}$$

are operators of finite rank in $\mathcal{G}$.

Proof of 4.4. Since $A : H \to H$ is a Fredholm operator on a Hilbert space, the projections P_1 onto its kernel and P_2 onto the kernel of A^* can both be given in terms of resolvent integrals. The Ψ^*-property implies that both are elements in $\mathcal{G}$. Then $R_1 = (P_1 + A^*A)^{-1}A^*$ and $R_2 = A^*(P_2 + AA^*)^{-1}$ are the desired operators.

Now the proof of Theorem 4.2 is simple: Let R_1, R_2, F_1, F_1 be the operators as in Lemma 4.4. First consider the pseudodifferential parts of A, R_1, and R_2. Denote them by P, P_1, and P_2, respectively. By 4.3 and 4.4(1)

$$(PP_1)_+ - I = G_1 \quad \text{and} \quad (P_2P)_+ - I = G_2$$

are singular Green operators. Applying 3.19 to $0 = (PP_1)_+ - I - G_1$ and $0 = (P_2P)_+ - I - G_2$, any pseudodifferential symbol of $PP_1 - I$ and $P_2P - I$ is small for large $|x| + |\xi|, x \in X$. Hence the symbol $p(x, \xi)$ of P is invertible for large $|x| + |\xi|$, and $p(x, \xi)^{-1}$ is bounded. The argument for the boundary symbol operators is even easier. Choose cut-off functions $\phi, \theta, \psi \in SG^0(\Omega)$ supported in the usual collar neighborhood of Y, and suppose that

$$\phi\psi = \phi; \quad \theta\psi = \psi.$$

Denote by a, r_1, r_2, f_1, f_2 the boundary symbols for A, R_1, R_2, F_1, F_2, respectively, defined in this neighborhood. By 4.4 in connection with the calculus,

$$\begin{aligned} \phi R_1 \psi A \theta - \phi I &= \phi F_1 \theta + S_1, \\ \psi A \theta R_2 \phi - \phi I &= \psi F_2 \phi + S_2, \end{aligned}$$

where S_1, S_2 are regularizing of type zero. Let $\equiv$ denote equivalence of boundary symbol operators of order and type zero modulo symbols of order $\leq (-1, -1)$ and type zero. Then

$$\begin{aligned} 0 &\equiv \phi\, r_1\, \psi\, a\, \theta - \phi I, \text{and} \\ 0 &\equiv \psi\, a\, \theta\, r_2\, \phi - \phi I, \end{aligned}$$

so that $\psi a \theta$ has a left and right inverse modulo lower order terms.

4.5 Theorem. *Let A be a Green operator of order $m \in \mathbf{Z} \times \mathbf{R}$ and type $d \leq \max\{m_1, 0\}$, and let E and F be SG-bundles over X and Y, respectively. Suppose*

$$A : \begin{matrix} H^s(X, E) \\ \oplus \\ H^{s-(\frac{1}{2},0)}(X, F) \end{matrix} \to \begin{matrix} H^{s-m}(X, E) \\ \oplus \\ H^{s-m-(\frac{1}{2},0)}(X, F) \end{matrix} \tag{1}$$

is invertible for some $s \in \mathbf{N}_0 \times \mathbf{R}, s_1 \geq m_1$.
Then the inverse A^{-1} is again a Green operator. Its order is $-m$, its type a priori *is* $\max\{s_1 - m_1, 0\}$.

Proof. Use the pseudodifferential reduction of the order constructed in Theorem 4.1 to find pseudodifferential isomorphisms

$$R_1^{-s} : L^2(X, E) \oplus H^{-(\frac{1}{2},0)}(X, F) \to H^s(X, E) \oplus H^{s-(\frac{1}{2},0)}(X, F)$$

and

$$R_2^{s-m} : H^{s-m}(X,E) \oplus H^{s-m-(\frac{1}{2},0)}(X,F) \to L^2(X,E) \oplus H^{-(\frac{1}{2},0)}(X,F)$$

of orders $-s$ and $s-m$, respectively. Consider $B = R_2^{s-m} A R_1^{-s}$. B is invertible on $L^2(X,E) \oplus H^{-(\frac{1}{2},0)}(X,F)$. Moreover, it is an invertible Green operator of order and type zero. By Theorem 3.27 also its inverse, C, is of order and type zero. Hence $A^{-1} = R_1^{-s} C R_2^{s-m}$ is of order $-m$ and type $\max\{s_1 - m_1, 0\}$.

4.6 Corollary. Let A and s be as in 4.5. Then

$$A : \begin{array}{c} H^{s'}(X,E) \\ \oplus \\ H^{s'-(\frac{1}{2},0)}(X,F) \end{array} \to \begin{array}{c} H^{s'-m}(X,E) \\ \oplus \\ H^{s'-m-(\frac{1}{2},0)}(X,F) \end{array} \qquad (1)$$

is invertible for all $s' \in \mathbf{R}^2$ with $s'_1 \geq s_1$.
In particular: If A is of order and type zero and invertible on $H^0(X,E) \oplus H^{-(\frac{1}{2},0)}(Y,F)$ then A is invertible on $H^\sigma(X,E) \oplus H^{\sigma-(\frac{1}{2},0)}(Y,F)$ for all $\sigma \in \mathbf{R}^2, \sigma_1 > -\frac{1}{2}$.

Proof. The above inverse yields a bounded operator from $H^{s'-m}(X,E) \oplus H^{s'-m-(\frac{1}{2},0)}(Y,F)$ to $H^{s'}(X,E) \oplus H^{s'-(\frac{1}{2},0)}(Y,F)$ which inverts A on this space.

4.7 Corollary. Let A be of order and type zero. Suppose that for some $\sigma \in \mathbf{N}_0 \times \mathbf{Z}, A$ is invertible on $H^\sigma(X,E) \oplus H^{\sigma-(\frac{1}{2},0)}(Y,F)$, and that A is additionally elliptic. Then A is invertible on $H^s(X,E) \oplus H^{s-(\frac{1}{2},0)}(Y,F)$ for all $s \in \mathbf{R}^2, s_1 > -\frac{1}{2}$.

By 4.6 it is sufficient to show invertibility on $H = H^0(X,E) \oplus H^{-(\frac{1}{2},0)}(Y,F)$. In view of the ellipticity, A is a Fredholm operator on H. Applying the parametrix, we see that the kernel of A is a subspace of $\mathcal{S} = \mathcal{S}(X,E) \oplus \mathcal{S}(Y,F)$. Since it is trivial on $H^s(X,E) \oplus H^{s-(\frac{1}{2},0)}(Y,F)$, it is also trivial on H.
On the other hand suppose that there is an $0 \neq f \in H$, which is orthogonal to $\mathcal{R}(A)$. Then f belongs to the kernel of the H-adjoint A^* of A, which also is elliptic. Hence $f \in \mathcal{S}$, so $0 \neq f$ belongs to $H^s(X,E) \oplus H^{s-(\frac{1}{2},0)}(Y,F)$ but not to the range of A which is impossible.

4.8 Theorem. *Let A be a Green operator of order $m \in \mathbf{Z} \times \mathbf{R}$ and type $d \leq \max\{m_1, 0\}$, and let E and F be SG-bundles over X and Y, respectively. Suppose*

$$A : \begin{array}{c} H^{s}(X,E) \\ \oplus \\ H^{s-(\frac{1}{2},0)}(X,F) \end{array} \to \begin{array}{c} H^{s-m}(X,E) \\ \oplus \\ H^{s-m-(\frac{1}{2},0)}(X,F) \end{array} \qquad (1)$$

is invertible for $s \in \mathbf{N}_0 \times \mathbf{R}, s_1 = \max\{m_1, 0\}$.
Then there is a parametrix B of order $-m$ and type $\max\{-m_1, 0\}$ to A. In particular, A is elliptic.

The proof is the same as that of 4.5, except that now we use 4.2 instead of the spectral invariance.

4.9 Theorem. *There is a holomorphic functional calculus for the elements in $\mathcal{G}$ in several complex variables.*

Proof. Given k commuting elements $A_1, \ldots, A_k \in \mathcal{G}$, choose $\mathcal{A}$ as a maximal commutative subalgebra of $\mathcal{G}$ containing $I, A_1, \ldots, A_k$. $\mathcal{G}$ is a Ψ^*-subalgebra of $\mathcal{L}(H)$, thus has an open group of invertible elements. As a closed subalgebra of $\mathcal{G}$, A is Fréchet with an open group of invertible elements. Hence inversion is continuous, cf. Waelbroeck [66]. Again by [66], VI, Proposition 4, there is a functional calculus for the holomorphic functions on the joint spectrum of $A_1, \ldots, A_k$ in $\mathcal{A}$ with values in $\mathcal{A} \subseteq \mathcal{G}$.

4.10 Remark. Using methods of Gramsch [14], [28], it can be shown that also the holomorphic functional calculus of J.L. Taylor [61] is applicable.

4.11 Corollary. (a) On a compact manifold with boundary, the algebra $\mathcal{G}$ coincides with the algebra of elements of order and type zero in Boutet de Monvel's algebra for symbols based on the Hörmander class $S^0_{1,0}$. The notion of SG-vector bundles reduces to usual bundles; $H^s(X,E), H^s(Y,F), s \in \mathbf{R}^2$, coincide with $H^{s_1}(X,E), H^{s_1}(Y,F)$.
In this situation, Theorem 4.9 not only gives an extension of G. Grubb's theorem on functional calculus [18], Theorem 3.4.4 to several complex variables. Even in the case of one variable only, one obtains a stronger result, since in Theorem 4.9 we have no restriction on the choice of the paths, cf. [18], p. 356f.
(b) The *classical* elements in Boutet de Monvel's algebra on a compact manifolds are those where all symbols have asymptotic expansions into homogeneous terms in the respective classes ('polyhomogeneous'). Denote the corresponding algebra of classical elements of order and type zero by $\mathcal{G}^{cl}$. It is then a consequence of 3.27 and the calculus that $\mathcal{G}^{cl}$ is a Ψ^*-subalgebra of $\mathcal{L}(H)$, too, for details see the proof, below.
This also gives a functional calculus for the elements of $\mathcal{G}^{cl}$. In the case of one variable, one recovers G. Grubb's result [18], Theorem 3.4.4. For $\mathcal{G}^{cl}$, spectral invariance was shown earlier by B.-W. Schulze [52] using different methods.

Proof. Clearly, $\mathcal{G}^{cl}$ is unital and symmetric. Following an idea of Guillemin [24], Schulze [52] showed how $\mathcal{G}^{cl}$ can be endowed with a Fréchet topology. Now suppose that $A \in \mathcal{G}^{cl}$ is invertible. Since $\mathcal{G}^{cl} \subseteq \mathcal{G}$ there is an inverse B to A in $\mathcal{G}$. In particular, A is elliptic in the sense of 2.18. On a compact manifold, this notion coincides with the standard notion of ellipticity for classical operators (see e.g. Rempel and Schulze [37], Section 3.1.1, Definition 1), for both notions are equivalent to the Fredholm property, cf. Theorem 4.2 here and [37], 3.1.1.1, Theorems 2' and 7. This implies the existence of a parametrix in $\mathcal{G}^{cl}$, cf. [37], Section 3.1.1.1, Theorem 2. On the other hand, this operator also is a parametrix in $\mathcal{G}$, and two parametrices differ only by an operators which is regularizing of type zero. Those, however, belong to $\mathcal{G}^{cl}$. Therefore $B \in \mathcal{G}^{cl}$, and $\mathcal{G}^{cl}$ is spectrally invariant.

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