Kontsevich integral for Kauffman polynomial

Le Tu Quoc Thang
Jun Murakami

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

Germany

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LE TU QUOC THANG AND JUN MURAKAMI

MAX-PLANCK INSTITUT FÜR MATHEMATIK
GOTTFRIED-CLAREN- STRASSE 26
5300 BONN 3, GERMANY

ABSTRACT. We prove that Kontsevich integral, via a weight system coming from Lie algebra so_N, coincides with the Kauffman polynomials. As a corollary we get some relations between mixed Euler numbers.

1. INTRODUCTION

Kontsevich integral is a knot invariant which contains in itself all knot invariants of finite type, or Vassiliev invariants. The value of this integral lies in an algebra $\mathcal{A}_0$, spanned by cord diagrams, subject to relations corresponding to the flatness of the Knizhnik-Zamolodchikov equation, or the so called infinitesimal pure braid relations ([11]).

For a Lie algebra $\mathfrak{g}$ with a bilinear invariant form and a representation $\rho : \mathfrak{g} \to \mathrm{End}(V)$ one can associate a linear mapping $W_{\mathfrak{g},t,\rho}$ from $\mathcal{A}_0$ to $\mathbb{C}[[h]]$, called the weight system of $\mathfrak{g}, t, \rho$. Here t is the invariant element in $\mathfrak{g} \otimes \mathfrak{g}$ corresponding to the bilinear form. Combining with the Kontsevich integral we get a knot invariant with values in $\mathbb{C}[[h]]$. The coefficient of h^n is a Vassiliev invariant of degree n.

From the other hand for a simple Lie algebra $\mathfrak{g}, t, \rho$ there is another knot invariant, constructed from the quantum R-matrix corresponding to $\mathfrak{g}, t, \rho$. Here R-matrix is the image of the universal quantum R-matrix lying in $\mathcal{U}_q\mathfrak{g} \otimes \mathcal{U}_q\mathfrak{g}$ through the representation ρ. The construction is given, for example, in [17, 18, 20]. The invariant is a Laurant polynomial in q, by putting $q = \exp(h)$ we get a formal series in h. By a theorem of Bar-Natan, Birman and Lin the coefficient of h^n is a Vassiliev invariant of degree n, and its n-th derivative is the same as that of the invariant defined in the previous case. Bar-Natan conjectured that these two invariants are the same.

Kontsevich invented his integral by using ideas from Drinfeld's works on quasi-Hopf algebras [6, 7]. From these works it is also clear more or less that the Kontsevich integral, via weight system, should be the same as the invariant coming from quantum groups. But since Drinfeld's work does not treat knot invariant thoroughly, and since Kontsevich integral was invented after this work and now is presented in literature without quasi-Hopf algebra theory, here we present a direct proof that these invariants are the same for the case when $\mathfrak{g}$ is Lie algebra of series B, C, D and ρ is the fundamental representation. The case of series A was treated in our previous work [13]. From the coincidence of the two invariants we can derive some unexpected relations between the so called mixed Euler numbers which recently gains much interest among number theorists. Arnold [2] calls these numbers Zagier zeta function, here we use the terminology suggested by D.Zagier.

In section 2 we recall definition of Kontsevich integral for knots and links, and then give a generalization for framed links. Kontsevich integral for framed links is technically more convenient for our purpose. In section 3 we prove the main theorem about coincidence of the two invariants. In the last section we derive some relations between mixed Euler numbers. In Appendix we give a description of the Drinfeld associator and compute its coefficients.

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2. Kontsevich's integral for framed links

In this section, we extend the definition of Kontsevich integral for framed links. For details on "usual" Kontsevich integral see [12] and [3].

2.1. Algebra and modules of cord diagrams. Let k be a positive integer. A *cord diagram* is k oriented circles with finitely many cords, which will be represented as dashed lines, marked on it, regarded up to orientation and component preserving diffeomorphisms of the circles. Here dashed lines mean that two different cords never intersect each other. We also suppose that the vertices of cords are all different. The circles are called the Wilson loops, we suppose that they are numbered. Denote the collection of all cord diagrams on k circles by $\mathcal{D}^{(k)}$. This collection is naturally graded by the number of cords in such a diagram. Denote the piece of degree d of $\mathcal{D}$ by $\mathcal{G}_d\mathcal{D}^{(k)}$. $\mathcal{G}_d\mathcal{D}^{(k)}$ is simply the collection of all cord diagrams having precisely d cords.

Let the vector space $\mathcal{A}^{(k)}$ be the quotient

$$\mathcal{A}^{(k)} = \operatorname{span}(\mathcal{D}^{(k)})/\operatorname{span}(4-\text{term relations}).$$

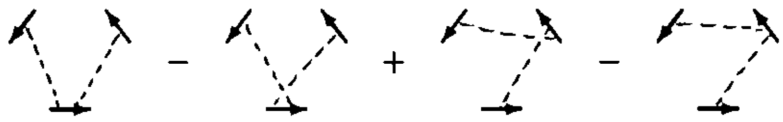
 $= 0$

FIGURE 1. 4-term relation

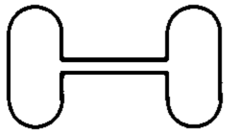


FIGURE 2. Multiplication

The 4-term relation is described in fig.1.

We denote also $\mathcal{A}^{(k)}$ for the completion of $\mathcal{A}^{(k)}$ by the graduation $\mathcal{G}_d\mathcal{A}^{(k)}$. The module $\mathcal{A}^{(1)}$ will also be denoted by $\mathcal{A}$.

Let D_1, D_2 be two cord diagrams, each with a noted Wilson loops. Remove an arc on each noted Wilson loop which does not contains any vertex and then using two lines to combine the two Wilson lines into one single loop (fig.2) we get a cord diagram called the product (or connected sum) of D_1, D_2 along the noted Wilson loops. As in [3] it can be proved that this operation does not depend on the location of the arcs removed.

With this multiplication $\mathcal{A}$ becomes an algebra. Unit in this algebra is the circle without any cord. Using connected sum we can define an action of $\mathcal{A}$ on $\mathcal{A}^{(k)}$ if the number of the Wilson loop to be acted is indicated. And there is an action of $\mathcal{A}^{\otimes k}$ on $\mathcal{A}^{(k)}$.

Suppose X is a compact 1-dimensional oriented piece-wise smooth manifold with or without boundary. The components of X are circles or lines. *A cord digram with support* X is a set of dashed cords with end points lying in the interior of X, regarded up to diffeomorphism which preserves each component and the orientation of X. Connected component of X are called Wilson lines or Wilson loops. Let $\mathcal{A}(X)$ be the space spanned by cord diagrams with support in X subject to the 4-term relation. Let $\mathcal{A}_0(X)$ be the space spanned by cord diagrams with support in X subject to the 4-term relation and every cord diagram containing a part like in figure 3 is equal to zero. If $f : X \to X'$ is a homeomorphism then there is an associate isomorphism between $\mathcal{A}(X)$ and $\mathcal{A}(X')$. If X is a circle then $\mathcal{A}(X)$ is isomorphic to $\mathcal{A}$. We denote by $\mathcal{A}_0$ the factored algebra of $\mathcal{A}$ by the ideal generated by Θ where Θ is the cord diagram in figure 4. Using connected sum

 $= 0$

FIGURE 3. Extra relation

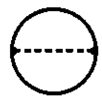


FIGURE 4. The cord diagram Θ

and an evident isomorphism we can define an action of $\mathcal{A}$ on $\mathcal{A}(X)$ if the Wilson line or loop to be acted is indicated. The action is the connected sum with the indicated Wilson line. As in [3] it is proved that this action is well-defined. Similarly $\mathcal{A}_0$ acts on $\mathcal{A}_0(X)$.

2.2. Tangles. We will consider $\mathbb{R}^3$ as the product of $\mathbb{R}$ and $\mathbb{C}$ with a fixed orientation. A point of $\mathbb{R}^3$ has coordinates (t, z), let $z = x + iy$. A plane parallel to $\mathbb{C}$ is called horizontal.

A tangle T is a 1-dimensional compact oriented piece-wise smooth submanifold of $\mathbf{R}^3$ lying between two horizontal planes, called the top plane and the bottom plane of this tangle, such that all the boundary points of T is lying in the top plane or in the bottom plane. There may be some interior point of T lying in the top or bottom planes. This definition is a little more general than, for example, that of [21, 18].

A tangle T is called of type 1 (or braid-like tangle) if :

a)Except for endpoints T has no local maximum or minimum.

b)T contains an even number of connected components, each is called a Wilson line of T. The orientation of half of the components point upwards, the others point downwards.

A tangle is of type 2 if :

a')Except for endpoints T has exactly one extremal point which lies in the top or bottom plane. Of course this point is a maximal (minimal) point if it lies in the top(bottom) plane. The component containing this point is called the distinguished Wilson line of this tangle.

b') Except for the distinguished line there are an even number of components, half of them are directed upwards.

A tangle of type 2 can be treated as a tangle of type 1 if we consider the only extremal point as two end points. Two tangles of type 1 (or type 2) are horizontal equivalent if there is an isotopy transferring the first into the second such that the isotopy preserves every horizontal plane and every point of the top and bottom planes are fixed.

For two tangles T_1, T_2 of both types we can define the product $T_1 \times T_2$ if the bottom plane of T_1 coincides with the top plane of T_2, the end points in this plane are also coincident, and if we combine the two tangle then the orientation on each component is definitely defined. The product is the combined 1-dimensional manifold. The product of two tangles of type 1 is a tangle of type 1, but the product of two tangles of different type or two tangles of type 2 may not be neither of type 1 nor type 2, but it is a tangle.

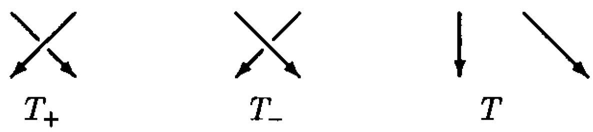


FIGURE 5.

If $T = T_1 \times T_2$ then for $D_1 \in \mathcal{A}(T_1), D_2 \in \mathcal{A}(T_2)$ we can define $D_1 \times D_2 \in \mathcal{A}(T)$ in an obvious way, just combining the two diagrams.

Consider n straight line parallel to $\mathbf{R}$ and going through $(0,i), i = 1,\dots,n$. Let T_1 (rep. T_2, T) be the tangle which is the intersection of these line with the set $0 \le t \le 1$ (res. $1 \le t \le 2, 0 \le t \le 2$). Then $T = T_1 \times T_2$. But there is an obvious homeomorphism $T \approx T_1 \approx T_2$, by shrinking. Let $\mathcal{B}_n = \mathcal{A}(T)$. Then $\mathcal{B}_n$ is an algebra. Unit is the cord diagram without any cord.

2.3. Kontsevich integral. Tangle of type 1: Let T be a tangle of type 1 with $2n$ Wilson lines, numbered by $1,2,...2n$. Suppose the bottom (top) plane is defined by $t = t_{min} (t = t_{max})$. An applicable state (abbreviation AS) of degree m of T is m unordered pairs $(j_1, j_1'), \dots, (j_m, j_m')$, each is a pair of distinct numbers from $\{1, 2, \dots, 2n\}$. For $t \in [t_{min}, t_{max}]$ let $z_i(t), z_i'(t)$ be the projections onto $\mathbf{C}$ of the intersection points of the Wilson lines numbered j_i, j_i' with the horizontal plane going through t, for $i = 1, \dots, m$. Fix $t_{min} < t_1 < t_2 < \dots < t_m < t_{max}$, let D_P be the cord diagram of $\mathcal{A}(T)$ obtained by connecting pairs $z_i(t_i), z_i'(t_i)$ by dashed lines. Let $\#P\downarrow$ be the number of points of the form $z_i(t_i), z_i'(t_i), i = 1, 2 \dots, m$ at which the orientation of T is downwards. Of course D_P and $\#P\downarrow$ do not depend on the choice of $t_1, \dots, t_m$. We define $Z(T) \in \mathcal{A}(T)$ as follow:

$$Z(T) = \sum_{m=0}^{\infty} \frac{1}{(2\pi i)^m} \int_{t_{min} < t_1 < \dots < t_m < t_{max}} \sum_{\text{AS of degree m}} (-1)^{\#P\downarrow} \wedge \frac{dz_i - dz_i'}{z_i - z_i'} D_P$$

In fact this is the holonomy along the corresponding curve in the configuration space with a flat connection (see [3]).

Example 2.1. Consider three tangles T_+, T_-, T in figure 5, the first two have end points $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, in the third the distance between two top end points is l_t, the distance between two bottom end points is l_b. Then

$$Z(T_+) = \exp(\Omega/2)$$

$$Z(T_-) = \exp(-\Omega/2)$$

$$Z(T) = \exp(\Omega \log \frac{l_t}{l_b}) = (\frac{l_t}{l_b})^{\Omega}$$

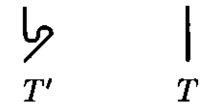


FIGURE 6.

Here in each case the symbol Ω^d for a non-negative integer d is the cord diagram containing d cords (= dashed lines), each is parallel to the plane $\mathbb{C}$ and connects the two Wilson lines.

Tangle of type 2: Now suppose T is a tangle of type 2. In the integral $Z(T)$ the coefficient of some applicable states is infinity. To get rid of these irregularities we take the value of the integral not in $\mathcal{A}$ but in $\mathcal{A}_0$. Formally one can proceed as follows. Let T_ε be the tangle obtained from T by cutting a part near the plane containing extremal point. The cut is carried by a horizontal plane lying between top and bottom planes and having distance ε to the plane containing the extremal point. Of course T_ε is a submanifold of T and hence there is a mapping $\mathcal{A}(T_\varepsilon) \to \mathcal{A}(T)$. Combining with $\mathcal{A}(T) \to \mathcal{A}_0(T)$ we can view $Z(T_\varepsilon)$ as an element of $\mathcal{A}_0(T)$. Then there exists the limit $\lim_{\varepsilon\to 0} Z(T_\varepsilon)$ which belongs to $\mathcal{A}_0(T)$. The reason is if the coefficient of an applicable state of T_ε tends to infinity when $\varepsilon \to 0$ then the cord diagram of this applicable state is zero in $\mathcal{A}_0(T)$ (but not in $\mathcal{A}(T_\varepsilon)$). Denote this integral also by $Z(T)$, it is a well defined element of $\mathcal{A}_0$.

General tangle: Now suppose T is a tangle. Using horizontal planes going through maximal and minimal points of T and some other planes lying between them we can decompose T into the product of several tangles of type 2, $T = T_1 \times T_2 \times \cdots \times T_n$. Put $Z(T) = Z(T_1) \times \cdots \times Z(T_n)$. It can be proved that $Z(T)$ does not depend on the decomposition, if $T = T' \times T''$ then $Z(T) = Z(T') \times Z(T'')$. The most important of $Z(T)$ is the following

Proposition 2.1 ([3, 12]). *a)$Z(T)$ remains unchanged under isotopy which preserves the bottom and the top planes and does not change the number of maximal and minimal points of each Wilson line or loop.*

b)If T' differ from T only in a neighborhood of a ball in which T and T' look like in fig.6 then

$$Z(T') = \gamma . Z(T) \tag{1}$$

where γ is the Kontsevich integral of the tangle U in fig.7, γ belongs to $\mathcal{A}_0$ and the right side of this equality should be understood as the action of γ on the Wilson line containing

FIGURE 7. Diagram U

FIGURE 8. d-th power of ω

this part of the tangle.

Suppose L is an embedding of k circle into $\mathbf{R}^3$ in generic position. The components of L are numbered. For $i = 1, \ldots, k$ let s_i be the number of maximal points of the i-th string. Let

$$\hat{Z}(L) = \gamma^{-s_1} \otimes \cdots \otimes \gamma^{-s_k}.Z(L)$$

here in the right hand side we use the action of $(\mathcal{A}_0)^{\otimes k}$ on $\mathcal{A}_0^{(k)}$.

Theorem 2.2. *$\hat{Z}(L)$ is an isotopy invariant of oriented links.*

Proof. Using proposition 2.1 one easily proves that $\hat{Z}_f(L)$ is invariant under all the moves listed in [21, Theorem 3.2]. Hence $\hat{Z}_f L$ is an ambient isotopy invariant. □

This is an easy generalization of Kontsevich integral for links.

2.4. A generalization for framed links. *A tangle of type* 3 is a tangle of type 2 such that a neighborhood of the only extremal point is lying in the plane (t, x), the extremal point is not a smooth point and in a neighborhood of this point the two parts of the distinguished lines are straight lines forming an angle $\pi/4$, (see fig.8).

Suppose T is a tangle of type 3. Let T_ε for small $\varepsilon \in \mathbb{R}, \varepsilon > 0$ be the tangle obtained from T by cutting a part near the extremal point by a horizontal plane. Here ε is the distance between two intersection points of the distinguished line with the cutting horizontal plane. Then $T = T_\varepsilon \times (T - T_\varepsilon)$ and T_ε is a tangle of type 1. We can define $Z(T_\varepsilon)$ which belongs to $\mathcal{A}(T_\varepsilon)$. Let ω^d stands for the cord diagram in $\mathcal{A}(T - T_\varepsilon)$ which consists of d parallel dashed lines near the maximal (minimal) point and connecting points of the distinguished lines as in fig.8. We regard ω^d as the formal d-th power of ω.

Lemma 2.3 (Lemma about regularization of Kontsevich integral). *If T is a tangle of type 3 containing a minimal point then there exist*

$$Z_f(T) = \lim_{\varepsilon \to 0} Z(T_\varepsilon) \exp(-\frac{\omega}{2\pi i} \log \varepsilon)$$

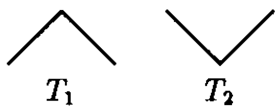


FIGURE 9.

which belongs to $\mathcal{A}(T)$. If T contains a maximal point then there exists

$$Z_f(T) = \lim_{\varepsilon\to 0} \exp(\frac{\omega}{2\pi i}\log\varepsilon) Z(T_\varepsilon)$$

This lemma is proved in [14]. We will write u^ω for $\exp(\omega \log u)$.

Example 2.2. Suppose T_1, T_2 are two tangles in figure 9, the distance between two end points in both tangles is l. Then

$$Z_f(T_1) = l^{\omega/2\pi i}$$

$$Z_f(T_2) = l^{-\omega/2\pi i}$$

While $Z(T_1), Z(T_2)$ are "unit", that is, cord diagram without any cord.

Now suppose L is a framed link. We represent L as a framed link diagram on the plane (t, x) with blackboard framing (see for example [10]). We suppose that all points of L belongs to the plane (t, x) except for a neighborhood of double points. After a deformation at extremal points we can decompose L into several tangles of the type 3, $L = T_1 \times \cdots \times T_n$. Put $Z_f(L) = Z_f(T_1) \times \cdots \times Z_f(T_n)$. It is easy to see that $Z_f(L)$ does not depend on the decomposition. We call it the framed Kontsevich integral of L. Let ϕ be the framed Kontsevich integral of the framed link diagram U in fig.7. Let s_i be defined as in theorem 2.2. Put

$$\hat{Z}_f(L) = \phi^{-s_1} \otimes \cdots \otimes \phi^{-s_k}.Z_f(L)$$

Here in the right hand side we use the action of $\mathcal{A}^{\otimes k}$ on $\mathcal{A}^{(k)}$.

Theorem 2.4. *a) $\hat{Z}_f$ is an invariant of framed oriented links.*

b) If L, L_+, L_- are three framed link represented in the black board framing by diagrams coincident every where except for a disk in which they look like in fig.10 then

$$\hat{Z}_f(L_+) = \exp(\Theta/2)\hat{Z}_f(L)$$

$$\hat{Z}_f(L_-) = \exp(-\Theta/2)\hat{Z}_f(L)$$

here $\exp(\Theta/2)$ *and* $\exp(-\Theta/2)$ *belong to $\mathcal{A}$ and the right hand sides of these equalities should be understood as the action on the Wilson loop concerned.*

This theorem is also proved in [14].

In fact what we get is an invariant of colored framed links.

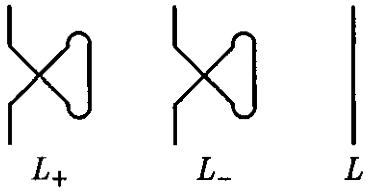


FIGURE 10. Changing frame

Remark. 1) The relation between $\hat{Z}_f(L)$ and $\hat{Z}(L)$ is very simple and is explained in [14].

2)Suppose I is a $\mathbb{C}$-valued invariant of *framed links* of finite type, this means there is $d \in \mathbb{N}$ such that the $(d+1)$-th derivative $V_{d+1}(I)$ of I is zero, then the d-th derivative of I defined a linear mapping from $\mathcal{G}_d(\oplus_k \mathcal{A}^{(k)})$ into $\mathbb{C}$. Conversely every functional on $\mathcal{G}_d(\oplus_k \mathcal{A}^{(k)})$ is the d-th derivative of a framed knot invariant of degree $(d+1)$. This is proved in exactly the same way as in the case of invariant of "unframed knots" using the integral Z_f instead of Z.

3) The invariant $\hat{Z}_f$ contains every framed knot invariant of finite type. This means if $\hat{Z}_f(K_1) = \hat{Z}_f(K_2)$ then $I(K_1) = I(K_2)$ for every framed knot invariant of finite type. Hence the question the system of invariant of finite type is complete is reduced to the question: are there two different framed knots with the same framed Kontsevich integral?

3. Weight systems and Kauffman polynomial

3.1. Semi-simple Lie algebra. A weight systems on $\mathcal{A}^{(k)}$ (resp. on $\mathcal{A}_0^{(k)}$) is a linear mapping from $\mathcal{A}^{(k)}$ (resp. from $\mathcal{A}_0^{(k)}$) to $\mathbb{C}$.

Let $\mathfrak{g}$ be a Lie algebra with a non-degenerate invariant bilinear form. Let t be the corresponding Casimir element in $U\mathfrak{g} \otimes U\mathfrak{g}$. Suppose $\rho_i : \mathfrak{g} \to \mathrm{End}(V_i), i = 1, 2, \ldots$ be a set of representations of $\mathfrak{g}$.We generalize the notion of weight system of [3, 15] as follow. Choose a base e_j^i for each vector space V_i. Let D be a cord diagram such that each Wilson loop or line is enhanced with a number from $\mathbb{N}$, called the color of this Wilson loop (or line).

A connected subset of Wilson loops and lines of D is called an *arc* if it has no vertex (of cords) in the interior of it and its boundaries are vertices or a boundary of the strings. An arc of D is called *internal arc* if both boundary points of it are vertices of cords, and is called *boundary arc* if otherwise. Define a state as a mapping from arcs of D to $\{e_j^i\}$ such that an arc of the Wilson loop or line of color i is e_j^i for some j. For a cord in fig.11 we associate the number ht_{cd}^{ab}, called the weight of this cord in this state, where h is a formal parameter, a, b, c, d are the values of the four arcs under the state-mapping,

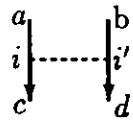


FIGURE 11. State of a cord

i, i' are the colors of the Wilson lines (or lines), and t here is considered as the mapping $t : V_i \otimes V_{i'} \to V_i \otimes V_{i'}$ corresponding to the Casimir element t. Suppose D contains only loops. Let

$$W_{\{\rho_i\}}(D) = \sum_{\text{states}} \prod_{\text{cords}} ht^{ab}_{cd} \tag{2}$$

$W_{\{\rho_i\}}(D)$ is called the weight of D.

Correctness of this mapping follows from the fact that t satisfies the following equation, the graphical representation of which is the 4-term relation:

$$[t_{12}, t_{13}] + [t_{12}, t_{23}] = 0 \tag{3}$$

A colored framed link is a framed link such that each component is enhanced with a number from $\mathbf{N}$ called the color of the component. Combining the above mapping with $\hat{Z}_f$ we get an invariant of colored framed link with valued in $\mathbb{C}[[h]]$.

As a special case for any operator $r : V \otimes V \to V \otimes V$ satisfying (3) we get a weight system W_r.

3.2. The case $\mathfrak{g} = so_N$ and all ρ_i are the fundamental representation. First consider the case when $\rho_i = \rho : \mathfrak{g} \to \mathrm{End}(V)$. We will denote W_ρ simply by W. Suppose D is a cord diagram, maybe with open Wilson lines. Denote the set of all end points of D at which the orientation of D is inwards (outwards) by D_{in} (D_{out}).Then the number of points in D_{in} is equal the number of points in D_{out}. We define $W(D)$ as an operator from $V(D_{out})$ to $V(D_{in})$, where $V(D_{out}) = \otimes_{p \in D_{out}} V(p), V(D_{in}) = \otimes_{p \in D_{in}} V(p)$ and all $V(p)$ are equal to V. Let $e_1, \ldots, e_n$ be a base of V. Consider the state sum (2) as above, only with fixed values of the external arcs. Then by varying the values of the external arcs we get an operator from $V(D_{out})$ to $V(D_{in})$. This operator does not depend on the choice of the base e_i and commute with the action of $\mathfrak{g}$ on $V(D_{out})$ and $V(D_{in})$ (see [3]).

Proposition 3.1. *Suppose ρ is irreducible, D_1, D_2 are two diagrams from $\mathcal{A}^{(k)}$, and $D_1 \# D_2$ is the connected sum along arbitrary components, then*

$$W(D_1 \# D_2) = W(D_1) W(D_2) / \dim V$$

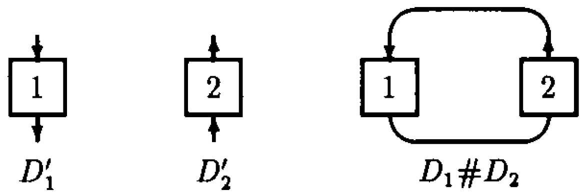


FIGURE 12.

Proof. After removing a small arc from D_1 and D_2 we get cord diagrams D'_1, D'_2, each has two end points (see fig.12) Then $W(D'_1), W(D'_2)$ are operator from V to V. It is easy to see that $W(D_i) = \operatorname{Tr} W(D'_i), i = 1,2$. While $W(D_1\#D_2) = \operatorname{Tr}[W(D'_1)W(D'_2)]$. Since ρ is irreducible and both $W(D'_i), i = 1,2$ commute with actions of ρ we have $W(D'_1) = \text{const.id}, W(D'_2) = \text{const.id}$. It follows that

$$\operatorname{Tr}[W(D'_1)W(D'_2)] = [\operatorname{Tr} W(D'_1)][\operatorname{Tr} W(D'_1)]/\dim V \quad \square$$

Now suppose $\mathfrak{g} = so_N$ and $\rho : so_N \to \operatorname{End}(V) = \operatorname{End}(\mathbb{C}^N)$ is the fundamental representation. After a normalization we have

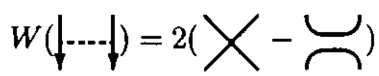
$$t^{ab}_{cd} = 2(\delta^a_d\delta^c_b - \delta^{ab}\delta_{cd})$$

A specific property of this case is the following important:

Lemma 3.2. *We have $t = -t^{t_1} = -t^{t_2}$ where t^{t_1} is the transpose of t on the first space and t^{t_2} is the transpose of t on the second space.*

The proof is trivial and follows from the explicit form of t.

The following graphical representation of t allows us to compute quickly $W(D)$.



$$W(\bigcirc) = N, W(D \sqcup \bigcirc) = NW(D)$$

where D is a cord diagram and $D \sqcup \bigcirc$ is the union of D and a circle which is far away.

Example 3.1. For the tangle T_+, T_- in fig.5 $W(T_+), W(T_-)$ are operators from $V \otimes V$ to $V \otimes V$. Then

$$W(Z_f(T_+)) = P\exp(h\rho(t)/2) \tag{4}$$

$$W(Z_f(T_-)) = P\exp(-h\rho(t)/2) \tag{5}$$

where P is the permutation $P(x \otimes y) = y \otimes x$. This can be proved easily by using the result of example 2.1.

Note that for a cord diagram D in general if we change the orientation of a Wilson line then $W(D)$ changes.

Proposition 3.3. *If we change the orientation of a Wilson line then the operator $\rho(T) = W(Z_f(T))$ remains unchanged.*

Proof. When we change the orientation of a Wilson line, then for an Applicable state, for a cord having one vertex on this Wilson line the term $\#P\downarrow$ is changed by -1, and the associate number of this cord must be replaced by the corresponding number of the matrix obtained from t by transpose on the first or on the second place. From lemma 3.2 we see that the result is unchanged. For a cord having both vertices on the string we have to change t to $t^{t_1 t_2}$ which is equal to t. $\square$

If L is a framed link then it follows that $W(\hat{Z}_f(L))$ does not depends on the orientation of the link.

3.3. Kauffmann polynomial.

Lemma 3.4. *Let* $\eta = N^2/(\frac{\exp[(N-1)h] - \exp[(1-N)h]}{\exp h - \exp(-h)} + 1)$. *Then*

$$[P\exp(h\rho(t)/2) - P\exp(-h\rho(t)/2)]^{ac}_{bd} = (\exp h - exp(-h))[\mathrm{id} - \delta^{ac}\delta_{bd}/\eta]$$

This lemma is proved by explicit calculating matrices $\exp(h\rho(t))$, $\exp(h\rho(t))$. Recall that P is the permutation acting on $V \otimes V$.

The operator $\delta^{ac}\delta_{bd}$ can be represented graphically as .

We have seen that $W(\hat{Z}_f)$ is an isotpy invariant of framed links, but $W(\hat{Z}_f(\bigcirc)) \neq 1$. We will use another normalization. Let $\kappa(L) = N^{-2}W(\phi)W(\hat{Z}_f(L))$, in this case $\kappa(\bigcirc) = 1$. If L is a framed link diagram with s maximal points then from proposition 3.1

$$\kappa(L) = \frac{N^{s-2}W(Z_f(L))}{(W(\phi))^{s-1}} \tag{6}$$

Denote $W(\exp(\Theta/2))/N$ by σ. Then $W(\exp(-\Theta/2)/N = \sigma^{-1}$, by proposition 3.1. We have seen that $\kappa(L)$ is a formal power series on h and is an invariant of framed oriented links.

Proposition 3.5. *If L, L_+, L_- are three framed links such that in some blackboard representation they differ only in a disk in which they look like in fig.10 then $\kappa(L_+) = \sigma\kappa(L), \kappa(L_-) = \sigma^{-1}\kappa(L)$. Hence $\sigma^{-w(L)}\kappa(L)$ where $w(L)$ is the writhe number of the framed link L is an ambient isotopy invariant of oriented links.*

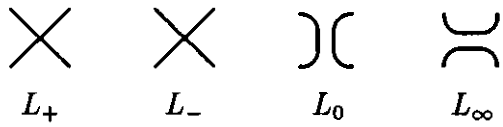


FIGURE 13.

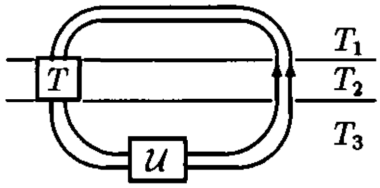


FIGURE 14. Scheme to prove theorem 3.6

This proposition follows from theorem 2.4 and proposition 3.1.

Let $\kappa(L) = \sum_{i=0}^{\infty} \kappa_i h^i$ then from the construction of the integral it follows that each κ_i is an invariant of framed links of degree i and its i-th derivative is computed by the weight system W. From the other hand the coefficients of Kauffmann polynomials are also invariant of finite type with the same derivatives (see [15]). We will prove that these two invariants are the same.

Theorem 3.6. *$\kappa(L)$ does not depend on the orientation of L and if L_+, L_-, L_0, L_∞ are four framed link diagrams coincident outside some disk and looking as in fig.13 in this disk then*

$$\kappa(L_+) - \kappa(L_-) = (\exp(h) - \exp(-h))[\kappa(L_0) - \kappa(L_\infty)] \tag{7}$$

Hence $\kappa(L)$ is the Kauffman polynomial.

Proof. The fact that $\kappa(L)$ does not depend on the orientation follows from proposition 3.3. By regular isotopy we can push the local part containing the difference of the four links far away as in figure 14. In this figure the different parts of the four link are in the box denoted T. The complement parts are the same and is denoted by X. We suppose that the end points of X are $(0,0),(0,1),(1,0),(1,1)$. In figure 14 L is decompose into three tangle, the top is denoted by T_1, the middle by T_2, the bottom by T_3. The middle contains T and two extra lines parallel to the straight line $\mathbb{R}$. We suppose the upper end points of these two lines are $(l,1),(l+1,1)$. We will consider the limit when $l \to \infty$, and write $T_1(l), T_2(l), T_3(l)$. Let $Z(T_2(l)) = A + B(l)$ where $B(l)$ is the part containing all the cord diagrams with at least one "long" cord connecting a Wilson line of the left part of T_2 and a Wilson line of the right part of T_2, $A = Z(T)$ is the remaining. Of course A

does not depend on l. The coefficient of a diagram of $B(l)$ tends to zero when l tends to infinity at least as fast as $\log(1+1/l)$. This follows easily from the formula of the integral.

For all cord diagrams with less than k cords of $Z_f(T_1(l))$ or $Z_f(T_3(l))$, the coefficients tends to infinity when l tends to infinity, but at most as fast as $(\log l)^k$. This also follows easily from the integral formula. Using $\lim_{l\to\infty} \log(1+1/l)(\log l)^k = 0$ we see that

$$Z_f(L) = \lim_{l\to\infty} Z_f(T_1(l)) \times Z_f(T) \times Z_f(T_3(l))$$

Now let T respectively the diagram of fig.13, for the first three cases we use the orientation which points downwards in T. For the last case use arbitrary orientation in T, and then change the orientation of some component of X such that the combining diagram have definite orientation. Using lemmas 3.3, 3.4, equations (4), (5) and the previous equation we get

$$W[Z_f(L_+) - Z_f(L_-)] = (\exp(h) - \exp(-h))W[Z_f(L_0) - Z_f(L_\infty)/\eta] \tag{8}$$

Now consider the case when the part $\mathcal{U}$ in figure 14 is trivial (just two parallel lines), taking into account the number of maxima (see (6)) and using proposition 3.3 we get

$$\left(\frac{\exp(N-1)h - \exp(1-N)h}{\exp(h) - \exp(-h)}\right) W(\phi) = N^2 - W(\phi)^2/\eta$$

or $(W(\phi) + N^2)(W(\phi)/\eta - 1) = 0$. Because $W(\phi)$ depends on h we see that $W(\phi) = \eta$ hence

$$W(\phi) = N^2/(\frac{\exp[(N-1)h] - \exp[(1-N)h]}{\exp h - \exp(-h)} + 1) \tag{9}$$

Now using $W(\phi) = \eta$ in (8) and proposition 3.3 concerning the inversion of one string of a tangle we get (7).

Besides $\kappa(\bigcirc) = 1$. Together with (7) this defines κ uniquely as an invariant of framed link. Hence $\kappa(L)$ is the Kauffmann polynomial. In the notation of Turaev [20, §4.3.4] it is equal to $\tilde{Q}_{m,1}(L)$ with $q = \exp(h)$ □

Remark. An anologous proof yields the following result: For a weight Wif $W(\exp(\rho))$ satisfies the polynomial-equation $f(t) = 0$ then this polynomial annihilates the invariants $W(\hat{Z}_f)$ in the sense of Turaev [20].

4. Some computations and corollaries

We compute explicitly the series $W(\phi)$ and deduce some unexpected relations between the so called mixed Euler numbers.

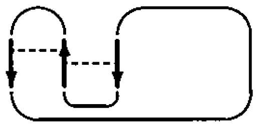


FIGURE 15. Closing

4.1. Elements γ, ϕ. Recalled that $\gamma = Z(U)$ is in $\mathcal{A}_0$ while $\phi = Z_f(U)$ is in $\mathcal{A}$. Let $f : \mathcal{A} \to \mathcal{A}$ be the mapping which takes a cord diagram with k cords into $(-1)^k\Theta^k$. Recall that there is a comultiplication Δ defined on $\mathcal{A}$(see [3, 12] for details). Put

$$\psi(D) = \mathbf{m}[(f \otimes \mathrm{id})\Delta(D)]$$

where D is a cord diagram and $\mathbf{m}$ is the usual multiplication in $\mathcal{A}$. It follows that ψ is an algebra homomorphism, its kernel is the ideal generated by Θ hence it also defines a homomorphism from $\mathcal{A}_0$ to $\mathcal{A}$. The composition $\mathcal{A}_0 \xrightarrow{\psi} \mathcal{A} \to \mathcal{A}_0$ is identity, and $\psi^2 : \mathcal{A} \to \mathcal{A}$ is equal to ψ. The operator ψ was also introduced in [3, 16].

Proposition 4.1. *We have $\phi = \psi(\gamma)$.*

The proof is presented in Appendix.

Consider the algebra $\mathcal{B}_3$ introduced in §I.2. Let Ω_{12} be the cord digram with one cord connecting the first and the second Wilson lines, Ω_{23} be the cord digram with one cord connecting the second and the third Wilson lines. For a multi-index $I = (i_1, \dots, i_k)$ put $Y_I = \Omega_{12}\Omega_{23}^{i_1-1}\Omega_{12}\Omega_{23}^{i_2-1}\dots\Omega_{12}\Omega_{23}^{i_k-1}$. Now converting the orientation of the second Wilson line then closing the three Wilson lines into one Wilson loop as in figure 15 we get a closing mapping cl: $\mathcal{B}_3 \to \mathcal{A}$. Let $D_I = \mathrm{cl}(Y_I)$. We also consider $D(I)$ as an element of $\mathcal{A}_0$, by the natural projection $\mathcal{A} \to \mathcal{A}_0$. Consider the set $\mathfrak{I} = \{I = (i_1, \dots, i_k), i_\nu \in \mathbb{N}, i_k > 1\}$. Let $|I| = \sum_{\nu=1}^k i_\nu$, $k(I)$ be the number of indices in I. For $I = (i_1, \dots, i_k) \in \mathfrak{I}$ set

$$\zeta(i_1, \dots, i_k) = \sum_{m_1 < \dots < m_k \in \mathbb{N}} \frac{1}{m_1^{i_1} \dots m_k^{i_k}}$$

It is called a mixed Euler number. When $k = 1$ it is the value of the zeta function at some natural number.

Proposition 4.2. *The element $\gamma \in \mathcal{A}_0$ is given by*

$$\gamma = 1 + \sum_{I \in \mathfrak{I}} \frac{1}{(2\pi i)^{|I|}}(-1)^{k(I)}\zeta(I)D(I)$$

This proposition is proved in Appendix (see also [13]).

4.2. Relations between mixed Euler number. It is easy to check that $\sum_b[\rho(t)]^{ab}_{bd} = 2(m-1)\delta^a_d$. Let

$$r = \rho(t) - 2(m-1)\,\mathrm{id}$$

and W_r be the weight system corresponding to r. Then $W_r(\Theta) = 0$ and hence W_r is a weight system on $\mathcal{A}_0$.

Proposition 4.3. *For every cord diagram $D \in \mathcal{A}_0$ we have $W_r(D) = W(\psi(D))$.*

This follows immediately from the definition of ψ and r. As a corollary of this and proposition 4.1 we get

Proposition 4.4. *One has $W_r(\gamma) = W(\phi)$.*

We use a normalization of mixed Euler number by putting $\tilde{\zeta}(I) = (-1)^{k(I)}(\pi i)^{-|I|}\zeta(I)$. Now define $g(i_1, \ldots, i_k)$ as follow. Let

$$u = h\begin{pmatrix} N-1 & 1 & -1 \\ 1 & N-1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, v = h\begin{pmatrix} N-1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & N-1 \end{pmatrix}$$

Matrix v is obtained from u by permutation of the second and the third coordinates. Let

$$g(i_1, \ldots, i_k) = (0,1,0)uv^{i_1-1}uv^{i_2-1}\ldots uv^{i_k-1}\begin{pmatrix} 1 \\ 1 \\ N \end{pmatrix}$$

Proposition 4.5. *For $I \in \mathfrak{I}$*

$$W_r(D(I)) = (-2)^{|I|}Ng(I)$$

This is proved by induction on the number of cords, using the graphical representation of t.

Theorem 4.6. *We have*

$$1 + \sum_{I\in\mathfrak{I}} g(I)\tilde{\zeta}(I)h^{|I|} = \frac{N(\exp(h) - \exp(-h))}{\exp(N-1)h - \exp(1-N)h + \exp(h) - \exp(-h)} \tag{10}$$

Proof. Note that if k is odd then $\sum_{|I|=k} g(I)\zeta(I) = 0$ due to the inversion formula for $\zeta(I)$ (see [13] and Appendix). Hence the left side of (10) does not contain terms with odd power of h. Using propositions 4.2, 4.4, 4.5 to compute the left hand side of (9) we get the result. $\square$

Both sides of (10) belong to $\mathbb{C}[N][[h^2]]$. By comparing the coefficients of $N^p h^q$ we get different relations between mixed Euler numbers. It is interesting to notice that these relations are not established by traditional methods of number theory, but by isotopy invariant of links.

For example by comparing the lowest order of N in both sides of (10) when the order of h is fixed we get

$$\zeta(\underbrace{2,2,\dots,2}_{n}) = \pi^{2n}/(2n+1)! \tag{11}$$

This can also be derived from the similar formula gotten from the HOMFLY polynomial case [13]. From (11), by induction one can easily reprove a famous theorem of Euler which expresses $\zeta(2n)$ in terms of Bernoulli numbers: $\zeta(2n) = \dfrac{(2\pi)^{2n}}{2(2n)!}|B_{2n}|$.

By combining all relations among Euler mixed number with $|I| \le 6$, $|I|$ even, coming from the HOMFLY and the Kauffmann polynomials we can compute all these numbers. The case $|I| = 6$ these numbers are computed modulo $\zeta(3)$. For example

$$\zeta(1,2,3) = \frac{-29}{6480}\pi^6 + 3\zeta(3)^2, \zeta(2,4) = \frac{-4}{2835}\pi^6 + \zeta(3)^2$$

For the case when $|I| \ge 8$ these relations are not enough to determine $\zeta(I)$, when $|I|$ is odd we can not get any relation.

Appendix A. Iterated integral, mixed Euler numbers and Drinfeld's associator

A.1. Iterated integral, mixed Euler numbers. We recall here the definition and some properties of iterated integral (see [5, 8]). Suppose $\xi_1, \dots, \xi_k$ are 1-forms on $[a,b]$, that is $\xi_i = f_i(u)\,du, u \in [a,b]$, define

$$\int_a^b \xi_1\xi_2\dots\xi_k = \int_a^b du_1(f_1(u_1)\int_a^{u_1} du_2(f_2(u_2)\int_a^{u_2}\cdots\int_a^{u_{k-1}} du_k f_k(u_k)\dots))$$

Let $\omega_0 = \dfrac{1}{2\pi i}\dfrac{du}{u}, \omega_1 = \dfrac{1}{2\pi i}\dfrac{du}{u-1}$. For example for $0 < a < b < 1$ we have

$$\int_a^b (\omega_0)^k = \frac{1}{(2\pi i)^k k!}[\log(\frac{b}{a})]^k \tag{12}$$

$$\int_a^b (\omega_1)^k = \frac{1}{(2\pi i)^k k!}[\log(\frac{1-b}{1-a})]^k \tag{13}$$

The following properties of iterated integral are well known.

Proposition A.1. *Suppose $\xi_1, \xi_2, \ldots, \xi_{k+l}$ are 1-forms on $[a,b]$ then*

$$\int_a^b \xi_1 \ldots \xi_k \int_a^b \xi_{k+1} \ldots \xi_{k+l} = \sum_\sigma \int_a^b \xi_{\sigma(1)} \ldots \xi_{\sigma(k+l)}$$

where the summation is performed with respect to all permutation $\sigma \in S_{k+l}$ such that $\sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(k), \sigma^{-1}(k+1) < \cdots < \sigma^{-1}(k+l)$.

Note that the sum in the right hand side contains C^k_{k+l} terms.

Proposition A.2. *The iterated integral along the inverse path is given by*

$$\int_b^a \xi_1 \ldots \xi_k = (-1)^k \int_a^b \xi_k \ldots \xi_1$$

Proposition A.3. *For $a < b < c$*

$$\int_a^c \xi_1 \ldots \xi_k = \int_a^b \xi_1 \ldots \xi_k + \sum_{j=1}^{k-1} \int_a^b \xi_1 \ldots \xi_j \int_b^c \xi_{j+1} \ldots \xi_k + \int_b^c \xi_1 \ldots \xi_k$$

Now consider $\int_0^1 \xi_1 \ldots \xi_k$ when ξ_i is either ω_0 or ω_1. If $\xi_1 = \omega_1$ or $\xi_k = \omega_0$ then it is easy to verify that the integral does not converge. Otherwise the integral exists and its value can be computed explicitly as follow. Let $p_1, q_1, p_2, q_2, \ldots, p_n, q_n$ are natural number. Set

$$\tau(p_1, q_1, \ldots, p_n, q_n) = (-1)^{q_1 + \cdots + q_n} \int_0^1 {\omega_0}^{p_1} {\omega_1}^{q_1} \ldots {\omega_0}^{p_n} {\omega_1}^{q_n}$$

Recall that mixed Euler number $\zeta(i_1, \ldots, i_k)$ is defined for natural numbers $i_1, \ldots, i_k$ with $i_k > 1$ by

$$\zeta(i_1, \ldots, i_k) = \sum_{m_1 < \cdots < m_k \in \mathbb{N}} \frac{1}{m_1^{i_1} \ldots m_k^{i_k}}$$

If $i_k = 1$ the right hand side does not converge.

Proposition A.4. *We have*

$$\tau(p_1, q_1, \ldots, p_n, q_n) = \zeta(\underbrace{1, \ldots, 1}_{p_1 - 1}, q_1 + 1, \underbrace{1, \ldots, 1}_{p_2 - 1}, q_2 + 1, \ldots, q_n + 1) \tag{14}$$

Proof. Let for $u \in (0,1)$

$$F(i_1, \ldots, i_k; u) = \sum_{m_1 < \cdots < m_k \in \mathbb{N}} \frac{u^{i_k}}{m_1^{i_1} \ldots m_k^{i_k}}$$

then one verifies at once that

$$\int_0^u \frac{F(i_1, \ldots, i_k; v) dv}{1 - v} = F(i_1, \ldots, i_k, 1; u)$$

$$\int_0^u \frac{F(i_1,\dots,i_k;v)dv}{v} = F(i_1,\dots,i_k+1;u)$$

Using these equalities and induction on $\sum p_j + q_j$ it is easy to prove (14). □

Note that another form equivalent to (14) is

$$\zeta(i_1,\dots,i_k) = \int_0^1 \omega_0\omega_1{}^{i_1-1}\omega_0\omega_1{}^{i_2-1}\dots\omega_0\omega_1{}^{i_k-1}$$

A corollary of the previous Proposition and Proposition A.2 is

Corollary A.5 (Inversion formula for mixed Euler numbers). *We have*

$$\tau(p_1,q_1,\dots,p_n,q_n) = \tau(q_n,p_n,\dots,q_1,p_1) \tag{15}$$

For example $\zeta(1,2)=\zeta(3), \zeta(1,3)=\zeta(4)$. We see that the function τ is more "symmetric" and we will use this function instead of ζ.

A.2. Drinfeld's associator. We can not present here Drinfeld's theory of quasi-Hopf algebras (see [6, 7]). We only mention that the category of representations of a quasi-triangular quasi-Hopf algebra is a quasi-tensorial category, from which one can construct invariants of framed links (see [18, 19, 1]). In a quasitriangular quasi-Hopf algebra A there are two important objects, an element $R \in A\otimes A$, called R-matrix and an element $\Phi \in A\otimes A\otimes A$, called the associator. There are gauge transformations which change R,Φ but do not change the category of representations of A and hence the corresponding link invariant remains unchanged. For a class of quasi-Hopf algebras Drinfeld showed that by gauge transformations one can make R-matrix very simple (of type $R=\exp(t)$), and all the difficulties are placed on Φ. Drinfeld gave an explicit way to construct Φ in this case. We will decribe this Φ, but without any quasi-Hopf algebra, and point out the connection to element ϕ,γ defined in section 4.

Let $M_1 = \mathbb{C}\ll A,B\gg$ be the module of non-commutative formal series on two symbols A,B. With the natural multiplication M_1 is a non-commutative algebra. Consider the equation

$$G'(u) = \frac{1}{2\pi i}\left(\frac{A}{u}+\frac{B}{u-1}\right)G(u) \tag{16}$$

where $G:(0,1)\to M_1$ is a formal series on A,B with coefficients which are analytic functions on u. Then for any $0<a<1$ there is a unique solution to (16) with $G(a)=1$, let denote the value of this solution at $b\in(0,1)$ by $Z_a^b(A,B)$. We can write

$$Z_a^b(A,B) = 1+\sum f_X(a,b)X \tag{17}$$

where the summation is over all monomes X in M_1, $f_X(a,b)$ is an analytic function on a, b. Here a monome in M_1 is the product of a finite number of symbols, each is A or B. By induction one sees that the coefficient $f_X(a,b)$ in (17) is given by

$$f_X(a,b) = \int_a^b X \tag{18}$$

where the right hand side is the iterated integral on $[a,b]$ in which each symbol A in X is replaced by ω_0, each symbol B is replaced by ω_1. Hence if X is a monome which begins with A and ends with B then there exists the limit $\lim_{\varepsilon\to 0} f_X(\varepsilon, 1-\varepsilon)$. Otherwise the limit is ∞.

We will say that a sequence of elements in $M-1$ converges to an element of M_1 if the coefficients of each monome converge to the corresponding coefficient of the limit element.

In order to regularize the limit $\lim_{\varepsilon\to 0} Z_\varepsilon^{1-\varepsilon}$ we can use the following two approaches.

Consider module M_2 which is a submodule of M_1 containing only formal series on monomes beginning with A and ending with B. At the same time M_2 is a factor of M_1: $M_2 = M_1/(BM_1 = 0, M_1A = 0)$. Let $\psi_{12} : M_1 \to M_2$ be the factor map. Then we see that there exists the limit

$$\Gamma = \lim_{\varepsilon\to 0} \psi_{12}(Z_\varepsilon^{1-\varepsilon})$$

which belongs to M_2.

If we write $\Gamma = 1 + \sum \Gamma_X X$ where the summation is over all monomes in M_2, then

$$\Gamma_X = \int_0^1 X \tag{19}$$

This integral is convergent because X begins with A and ends with B. From (14it follows that each coefficient Γ_X is a mixed Euler number.

Another way to regularize $\lim_{\varepsilon\to 0} Z_\varepsilon^{1-\varepsilon}$ is the following. There exists uniquely one solution $G_1(u)$ of (16) with asymptotic $G_1(u) \approx u^{A/2\pi i}$ (for $u \to 0$) where $u^A = \exp(A \log u)$ and $G_1(u) \approx u^{A/2\pi i}$ means that $G_1(u)u^{-A/2\pi i}$ has an analytic continuation into a neighborhood of $u = 0$ and becomes 1 at this point.

Similarly there exists uniquely one solution $G_2(t)$ of (16) with asymptotic $G_2(t) \approx (1-t)^{B/2\pi i}$ $(t \to 1)$. Let

$$\Phi = G_2^{-1} G_1$$

Then Φ does not depend on t and is an element of M_1, it is the Drinfeld's associator and plays important role in the theory of quasi-Hopf algebras and invariants of links. Let us

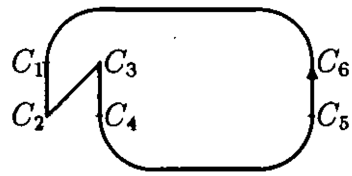


FIGURE 16.

write

$$\Phi(A,B) = 1 + \sum \Phi_X X \tag{20}$$

We will compute Φ_X for each monome X. It is clear from the definition of Φ that

$$\Phi(A,B) = \lim_{\varepsilon \to 0} \varepsilon^{-B/2\pi i} Z_\varepsilon^{1-\varepsilon} \varepsilon^{A/2\pi i} \tag{21}$$

Note that Φ belongs to M_1 while Γ belongs to M_2. We want to find a relation between Φ and Γ.

Recall that $\mathcal{B}_3$ is the algebra of cord diagrams whose support is three lines parallel to $\mathbb{R}$ and lying between two horizontal planes $\{t=0\}, \{t=1\}$.

There is an operator closure $D \in \mathcal{B}_3 \to \mathrm{cl}(D) \in \mathcal{A}$ indicated in fig.15. Recall that $\phi = Z_f(U), \gamma = Z(U)$.

Proposition A.6. *We have*

$$\phi = \mathrm{cl}(\Phi(-\Omega_{12}, -\Omega_{23})) \tag{22}$$

$$\gamma = pr(\mathrm{cl}(\Gamma(-\Omega_{12}, -\Omega_{23}))) \tag{23}$$

Where $pr : \mathcal{A} \to \mathcal{A}_0$ is the natural projection.

Proof. We prove (22), the second identity can be proved in a similar manner, even more easily.

Using horizontal deformation we can deform U into a diagram $U'(l)$ lying in the plane (t,x) like in figure 16. In this figure the points $C_1, C_2, C_3, C_4, C_5, C_6$ have coordinates respectively $(0,1), (0,0), (1,1), (1,0), (l,0), (l,1)$. Using two horizontal planes $\{t=0\}, \{t=1\}$ we cut $U'(l)$ into three tangles: the top is $T_1(l)$, the middle $T_2(l)$ and the bottom $T_3(l)$. Then $Z_f(T_1(l)) = l^{\omega/2\pi i}$, $Z_f(T_3(l)) = (l-1)^{-\omega/2\pi i}$. While $Z_f(T_2(l)) = Z_f(T) + \log(1+1/l)O(1)$ (for $l \to \infty$). Here T is the part from point C_1 to C_4 and $Z_f(T)$ is defined in exactly the same manner as for any tangle of type 3. By definition

$$Z_f(T) = \lim_{\varepsilon \to 0} \varepsilon^{\Omega_{23}/2\pi i} Z(T_\varepsilon^{1-\varepsilon}) \varepsilon^{-\Omega_{12}/2\pi i}$$

where $Z(T_\varepsilon^{1-\varepsilon})$ is the Kontsevich integral of the tangle obtained from T by cutting upper and lower parts by two planes $\{t = 1-\varepsilon\}, \{t = \varepsilon\}$.

Let l tend to infinity, using $\lim_{l\to\infty}(\log l)^k \log(1+1/l) = 0$ we see that $\phi = \mathrm{cl}(Z_f(T))$. From the definition of the integral we see that $Z(T_\varepsilon^{1-\varepsilon}) = Z_\varepsilon^{1-\varepsilon}(-\Omega_{12}, -\Omega_{23})$. Hence

$$\phi = Z_f(U) = \mathrm{cl}(Z_f(T)) = \mathrm{cl}(\Phi(-\Omega_{12}, -\Omega_{23})) \quad \square$$

From (23) and (19) we get proposition 4.2.

Now consider the module M_3 which consists of formal series with coefficients in $\mathbb{C}$ on four symbols A, B, α, β such that α, β commute with every other symbols. Then with the obvious multiplication M_3 is an algebra. Every monome in M_3 can be represented uniquely in the form $\beta^p X \alpha^q$ where X is a monome in M_1. Consider the mapping ψ_{31} : $M_3 \to M_1, \psi_{31}(\beta^p X \alpha^q) = B^p X A^q$. Note that this is a module homomorphism, but not an algebra homomorphism. Note also that $\psi_{31}((B-\beta)Y) = 0$ and $\psi_{31}(Y(A-\alpha)) = 0$ for every element $Y \in M_3$.

Let $\psi_{13} : M_1 \to M_3$ be the map $\psi_{13}(H(A,B)) = H(A-\alpha, B-\beta)$ where $H(A,B)$ is an element of M_1. Denote $\Psi : M_1 \to M_1$ the composition $\psi_{31}\psi_{13}$. This Ψ is a module homomorphism, but not an algebra homomorphism, and if X is a monome in M_1 begins with B or ends with A then $\Psi(X) = 0$, hence Ψ can be regarded also as a homomorphism from M_2 to M_1. If X is a monome in M_1 then $\Psi(X) = X + Y$ where Y is the sum of monomes which begin with B or ends with A. Hence the composition

$$M_2 \hookrightarrow M_1 \xrightarrow{\Psi} M_1 \to M_2$$

is identity. Besides, Ψ^2, as a homomorphism from M_1 to M_1, is coincident with Ψ.

Proposition A.7. *We have*

$$\Psi(\Phi) = \Phi \tag{24}$$

Proof. Note that $G_1(A-\alpha, B-\beta)$ is a solution to equation

$$G' = \frac{1}{2\pi i}\left(\frac{A-\alpha}{u} + \frac{B-\beta}{u-1}\right)G \tag{25}$$

with asymptotic $u^{(A-\alpha)/2\pi i}$ when $u \to 0$. The function $u^{-\alpha/2\pi i}(1-u)^{-\beta/2\pi i}G_1(A,B)$ is also a solution to (25) with the same asymptotic. Hence

$$G_1(A-\alpha, B-\beta) = u^{-\alpha/2\pi i}(1-u)^{-\beta/2\pi i}G_1(A,B)$$

Similarly we get

$$G_2(A-\alpha, B-\beta) = u^{-\alpha/2\pi i}(1-u)^{-\beta/2\pi i}G_2(A,B)$$

Hence $(G_2^{-1}G_1)(A,B) = (G_2^{-1}G_1)(A-\alpha, B-\beta)$. In this identity both sides are elements of M_3. From this we get (24) immediately. $\square$

Theorem A.8. *The following identity holds true*

$$\Psi(\Gamma) = \Phi \tag{26}$$

Proof. Applying Ψ to both sides of (21) we get

$$\lim_{\varepsilon\to 0} \Psi(\varepsilon^{-B/2\pi i} Z_\varepsilon^{1-\varepsilon} \varepsilon^{-A/2\pi i}) = \Psi(\Phi) \tag{27}$$

For every element $Y \in M_1$ one has $\Psi(BY) = \Psi(AY) = 0$ hence the left hand side of (27) is $\lim_{\varepsilon\to 0} \Psi(Z_\varepsilon^{1-\varepsilon})$ which is $\Psi(\Gamma)$. While the right hand side of (27) is Φ by (24. $\square$

As a corollary one can prove proposition 4.1. In fact $\mathrm{cl}(\Psi(\Gamma(-\Omega_{12},\Omega_{23})))$ is just $\psi(\mathrm{cl}(\Gamma(-\Omega_{12},\Omega_{23})))$ by definition of Ψ, and of ψ in §4. Combining with (22),(23) we get proposition 4.1.

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MAX-PLANCK-INSTITUT FÜR MATHEMATIK, GOTTFRIED-CLAREN- STRASSE 26, 5300 BONN 3, GERMANY

E-mail address: letu@mpim-bonn.mpg.de, jun@mpim-bonn.mpg.de