Polynomial invariants for 4–manifolds of type (1,n) and a calculation for $S^2 \times S^2$

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Introduction

In this article we lay out a framework for some formulations of differential invariants for certain 4–manifolds and then calculate an example for the product manifold $S^2 \times S^2$. To be more precise, let X be a smooth compact simply–connected oriented 4–manifold with $b_2^+(X) = 1$. For integers $k > 0$, denote $\mathscr{C}_X^k$ the set of connected components of the positive cone Ω_X in $H^2(X;\mathbb{R})$ after dividing by the system of walls $\bigcup_{1 \le \ell \le k} W_\ell$, where

$$W_\ell = \bigcup \{ \langle e \rangle^\perp \subset H^2(X;\mathbb{R}) \mid e \cdot e = -\ell \ ; \ e \in H^2(X;\mathbb{Z}) \} \ .$$

Write simply $\mathscr{C}_X$ for $\mathscr{C}_X^1$. In [D3] Donaldson introduces a differential invariant

$$\Gamma_X : \mathscr{C}_X \longrightarrow H^2(X;\mathbb{Z})$$

using Yang–Mills moduli spaces associated to an SU(2)–bundle $P \longrightarrow X$ with

$c_2(P) = 1$. Here we work with a larger second Chern class $c_2(P) = k \geq 2$ and define assignments

$$\Gamma_X^k : \mathscr{C}_X^k \longrightarrow \mathrm{Sym}^{4k-3}(H^2(X;\mathbb{Z}))$$

in the same spirit. This will be explained in § 1. We then apply this framework to the manifold $S^2 \times S^2$ and determine $\Gamma^2_{S^2\times S^2}$ in § 2. In order to state the result, observe first

$$H^2(S^2 \times S^2, \mathbb{R}) \simeq \{a_1h_1 + a_2h_2 \mid a_1,a_2 \in \mathbb{R}\}$$

is spanned by two (integral) generators h_1,h_2 over $\mathbb{R}$ while $\mathscr{C}^2_{S^2\times S^2}$ is a set consisting of regions

$$C_+ = \{a_1 > a_2 > 0\}\ ,\ \ C_- = \{a_2 > a_1 > 0\}$$

together with $-C_+, -C_-$ as elements.

<u>Theorem</u> The assignment

$$\Gamma^2_{S^2\times S^2} : \mathscr{C}^2_{S^2\times S^2} \longrightarrow \mathrm{Sym}^5(H^2(S^2 \times S^2;\mathbb{Z}))$$

is given by

$$\Gamma^2_{S^2\times S^2}(C_+) = h_1^5 - 5(h_1^4\, h_2) + 10(h_1^3\, h_2^2) \quad \text{and}$$

$$\Gamma^2_{S^2\times S^2}(C_-) = h_2^5 - 5(h_1\, h_2^4) + 10(h_1^2\, h_2^3)$$

where $(h_1^{\ell_1} h_2^{\ell_2})$ denotes the symmetrization of $h_1^{\ell_1} h_2^{\ell_2}$ in $(H^2(S^2 \times S^2;\mathbb{Z}))^{\otimes 5}$ for positive integers ℓ_1, ℓ_2.

This will be proved by arguments in algebraic geometry. I should emphasis the polynomials $\Gamma^2_{S^2 \times S^2}(C_+)$, $\Gamma^2_{S^2 \times S^2}(C_-)$ as described above are <u>not</u> Donaldson polynomials and quite on the contrary they reflect the construction of such polynomials depends upon the metrics as $b_2^+(S^2 \times S^2) = 1$. Moreover, in contrast to a result of [FMM], these two polynomials are not polynomials on the intersection form and the canonical class of a quadric surface Q realizing $S^2 \times S^2$.

In [K] one can find similar discussion for differential invariants concerning SO(3)–bundles. Other useful information relevant to our work can also be found in [FM1], [FM2].

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§ 1. The definition of Γ_X^k

In this section we explain the construction of the polynomial $\Gamma_X^k(C)$ in $\mathrm{Sym}^{4k-3}(H^2(X;\mathbb{Z}))$ associated to a chamber $C \in \mathscr{C}_X^k$ in general. To begin with, let P be an SU(2)–bundle over X with $c_2(P) = k$ and $\mathscr{A}$ be the space of connections on P . The gauge group $\mathscr{G} = \mathrm{Aut}\, P$ acts on $\mathscr{A}$ preserving anti–self–dual (ASD) connections and we denote by

$$M_k(m) = \{A \in \mathscr{A} \mid *_m F(A) = -F(A)\}/\ \mathscr{G}$$

the moduli space of ASD connections on P relative to a Riemannian metric m on X . In general $M_k(m)$ is a smooth oriented manifold of (real) dimension 8k–6 assuming $b_2^+(X) = 1$. Associated to any given metric m on X , there is an L_2–normalized self–dual harmonic 2–form ω_m which is unique up to a sign. A choice of ω_m determines a standard orientation of $M_k(m)$ and we write $M_k(\omega_m)$ for $M_k(m)$ with such an assigned orientation understood. In this convention $M_k(-\omega_m)$ has the opposite orientation compared with $M_k(\omega_m)$.

Given any smooth oriented real surface $\Sigma \subset X$ we can define a line bundle over $\mathscr{A}$ by assigning to each connection $A \in \mathscr{A}$ the complex line

$$\mathscr{L}_\Sigma(A) = \Lambda^{\max}(\ker \not{\partial}_{A|_\Sigma})^* \otimes \Lambda^{\max}(\mathrm{coker}\, \not{\partial}_{A|_\Sigma})$$

where $\not{\partial}_{A|_\Sigma}$ denotes the Dirac operator coupled with $A|_\Sigma$. If the metric m on X is sufficiently general, such assignments factor through the gauge group action and descend to the manifold $M_k(m)$ defining a line bundle $\mathscr{L}_\Sigma \longrightarrow M_k(m)$ provided the surfaces Σ are suitably chosen. In this situation we consider the zero sets $V_\Sigma \cap M_k(m)$ for certain transversal sections of $\mathscr{L}_\Sigma$ and by working with $4k-3$ such surfaces as a whole we obtain transversal intersections $V_{\Sigma_1} \cap \ldots \cap V_{\Sigma_{4k-3}} \cap M_k(m)$ consisting of points as elements. Apart from the (special) case $k = 1$, one can arrange such intersections to be <u>compact</u> assuming $b_2^+(X) = 1$ (cf. [D5] lemma (3.1)). Indeed, this is the case should one work with sufficiently general metrics m on X so that all moduli spaces $M_1(m),\ldots,M_{k-1}(m)$ in addition to $M_k(m)$ are smooth manifolds of formal dimension containing no U(1)–reduction. For this purpose one is to assume $[\omega_m]$ lies in some chamber $C \in \mathscr{C}_X^k$ in order that it does not meet the system of walls $\bigcup_{1 \le \ell \le k} W_\ell$. In such cases, we can define a symmetric multi–linear map

$$q_{k,X}(\omega_m) : \underbrace{H_2(X;\mathbb{Z}) \times \ldots \times H_2(X;\mathbb{Z})}_{4k-3 \text{ times}} \longrightarrow \mathbb{Z}$$

using assignments

$$([\Sigma_1],\ldots,[\Sigma_{4k-3}]) \longmapsto \text{the algebraic sum of a transversal intersection}$$
$$V_{\Sigma_1} \cap \ldots \cap V_{\Sigma_{4k-3}} \cap M_k(\omega_m) .$$

Should we write $\mu([\Sigma])$ for $c_1(\mathscr{L}_\Sigma)$, these intersection numbers are given by the natural pairings

$$<\mu([\Sigma_1]) \cup \ldots \cup \mu([\Sigma_{4k-3}]),\ [M_k(\omega_m)]>

and if we consider

$$q_{k,X}(\omega_m) = \langle \mu^{4k-3}, [M_k(\omega_m)] \rangle$$

an element in $\mathrm{Sym}^{4k-3}(H^2(X;\mathbb{Z}))$ this construction gives an assignment

$$\begin{aligned} \Gamma_X^k : \mathscr{C}_X^k &\longrightarrow \mathrm{Sym}^{4k-3}(H^2(X;\mathbb{Z})) \\ C &\longmapsto q_{k,X}(\omega_m) \end{aligned}$$

assuming $[\omega_m] \in C$. In this way we obtain the definition of the assignment Γ_X^k .

This discussion provides a way of defining differential invariants for X but for a complete definition one is to describe the (universal) difference between $\Gamma_X^k(C_1), \Gamma_X^k(C_2)$ for two different chambers $C_1, C_2 \in \mathscr{C}_X^k$. Despite not knowing of such comparison formulas in general, we should point out it is still possible to derive as a special case the difference

$$\Gamma^2_{S^2\times S^2}(C_+) - \Gamma^2_{S^2\times S^2}(C_-) = (h_1-h_2)^5$$

directly from Yang–Mills theory. We shall discuss this elsewhere.

§ 2. The calculation of $\Gamma^2_{S^2\times S^2}$

In this section we determine the invariant $\Gamma^2_{S^2\times S^2}$ for the standard $S^2\times S^2$. For simplicity we write

$$q_+ = \Gamma^2_{S^2\times S^2}(C_+) \quad \text{and} \quad q_- = \Gamma^2_{S^2\times S^2}(C_-) \ .$$

To find q_+ and q_- we are to use some arguments in algebraic geometry. It is a well–known fact that $S^2\times S^2$ can be realized as a complex quadric surface $Q \simeq \mathbb{P}_1\times\mathbb{P}_1$ in the complex projective 3–space $\mathbb{P}_3$ and all the ample line bundles H_{r_1,r_2} on Q are of the form $\mathcal{O}(r_1,r_2) = pr_1^*\mathcal{O}_{\mathbb{P}_1}(r_1)\otimes pr_2^*\,\mathcal{O}_{\mathbb{P}_1}(r_2)$ where r_1,r_2 are strictly positive integers and pr_i denotes the projection map from $Q\simeq\mathbb{P}_1\times\mathbb{P}_1$ to the i–th factor for $i = 1,2$. For each ample line bundle H_{r_1,r_2}, let M_{r_1,r_2} be the moduli space of H_{r_1,r_2}–stable 2–bundles E over Q with $\Lambda^2E\simeq\mathcal{O}_Q$ and $c_2(E) = 2$. (In this setting, E is H_{r_1,r_2}–stable if $\mathcal{L}\cdot H_{r_1,r_2} < 0$ for every holomorphic line bundle $\mathcal{L}\longrightarrow Q$ admitting a non–zero bundle map $\mathcal{L}\longrightarrow E$.) The moduli spaces M_{r_1,r_2} are smooth and if $r_1\neq r_2$ they are naturally identified with Yang–Mills moduli spaces $M_2(m)$ for compatible Kähler metrics m on Q by a theorem of Uhlenbeck and Yau. It follows from the general theory to determine q_+ and q_- it suffices to pick two moduli spaces M_{r_1,r_2}, one for $r_1 > r_2$ and one for $r_1 < r_2$. (The case $r_1 = r_2$ is special.) As we shall see however, the moduli spaces themselves are in fact divided into three kinds, according to the comparison between r_1 and r_2. This can be summarized as follows.

(2.1) Proposition. Associated to a quadric surface there are three spaces M_Δ, M_+, M_- such that

$$M_{r_1,r_2} = \begin{cases} M_+ & \text{if } r_1 > r_2 \\ M_\Delta & \text{if } r_1 = r_2 \\ M_- & \text{if } r_1 < r_2 \end{cases} .$$

In addition, we have

$$M_+ = M_\Delta \coprod \mathbb{P}_2^+ , \quad M_- = M_\Delta \coprod \mathbb{P}_2^-$$

where $\mathbb{P}_2^+$, $\mathbb{P}_2^-$ are two (distinct) copies of the complex projective plane parametrizing respectively non–trivial extensions of the following exact sequences:

$$0 \longrightarrow \mathcal{O}(1,-1) \longrightarrow E \longrightarrow \mathcal{O}(-1,1) \longrightarrow 0 \quad \text{and}$$
$$0 \longrightarrow \mathcal{O}(-1,1) \longrightarrow E \longrightarrow \mathcal{O}(1,-1) \longrightarrow 0 .$$

This proposition is a special case of the discussions in [M] and therefore we omit the proof here. Now let $L_i = pr_i^{-1}(\cdot) \simeq \mathbb{P}_1$ be the fibres of the projection map pr_i on $Q \simeq \mathbb{P}_1 \times \mathbb{P}_1$ for $i = 1,2$. To obtain q_+, q_- it suffices to establish the following table of evaluations for μ^5 .

(2.2) Table

Number of		$<\mu^5,[M_+]>$	$<\mu^5,[M_-]>$
L_1–lines	L_2–lines		
0	5	1	0
1	4	–1	0
2	3	1	0
3	2	0	1
4	1	0	–1
5	0	0	1

We shall only check the column for $<\mu^5,[M_+]>$ as the evaluation for $<\mu^5,[M_-]>$ is similar. Note that $q_+ = <\mu^5,[M_+]>$ in $\mathrm{Sym}^5(H^2(X;\mathbb{Z}))$.

Our calculation for q_+ hinges on the fact that a line L_i on the quadric Q is a copy of $\mathbb{P}_1$ and so one can adapt an argument in [D1] to show that the zero sets $V_{L_i} \cap M_+$ used to define q_+ can be taken to have the following concrete form:

$$\{[E] \in M_+ \mid E|_{L_i} \text{ is not trivial}\} . \tag{2.3}$$

(It is well–known that holomorphic 2–bundles on a projective line $\mathbb{P}_1 \simeq L_i$ always split and so we have

$$E|_{L_i} \simeq \mathcal{O}_{L_i}(a) \oplus \mathcal{O}_{L_i}(-a)$$

for some integer $a \geq 0$. The condition $E|_{L_i}$ is not trivial in (2.3) means $a \neq 0$ in the splitting of $E|_{L_i}$. In this case we say L_i is a jumping line of the bundle E.) Suggested by this, it is natural to investigate the splitting behaviour of an element $[E] \in M_+$ when restricted to a line L on the quadric Q. We first observe its splitting type is rather confined.

(2.4) Lemma. For a stable 2–bundle E over Q with $c_1(E) = 0$, $c_2(E) = 2$, we have either

$$E|_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L \qquad \text{(trivial)} \quad \text{or}$$
$$E|_L \simeq \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1) \qquad \text{(jumping)}$$

for all lines L on Q.

Proof. The argument is a direct consequence of the Riemann–Roch formular. Associated to each L_1–line there is an exact sequence

$$0 \longrightarrow E(-1,0) \longrightarrow E \longrightarrow E|_{L_1} \longrightarrow 0 \ . \tag{2.5}$$

The stability of E gives $h^0(E) = 0$ and therefore the corresponding long exact sequence of (2.5) reads

$$0 \longrightarrow H^0(E|_{L_1}) \longrightarrow H^1(E(-1,0)) \longrightarrow H^1(E) \longrightarrow \dots \ .$$

One checks readily by the Riemann–Roch formula

$$\chi(E(r_1,r_2)) = 2(r_1+1)(r_2+1)-2$$

that $h^1(E) = 0$ and $h^1(E(-1,0)) = 2$. It follows then

$$h^0(E|_{L_1}) = h^0(\mathcal{O}_{L_1}(a) \oplus \mathcal{O}_{L_1}(-a)) = 2$$

which can possibly happen only when $a = 0,1$. The argument for L_2–lines is similar and this proves the lemma.

Now we come to count the number of jumping lines a stable bundle $E \longrightarrow Q$ can possibly have. We denote for instance $H_{r_1 \geq r_2}$ the ample line bundle H_{r_1,r_2} on Q if $r_1 \geq r_2 > 0$.

(2.6) Lemma. An $H_{r_1 \geq r_2}$–stable bundle E can have at most two jumping lines in the line system $L_1 = pr_1^{-1}(\cdot)$. Similarly, an $H_{r_1 \leq r_2}$–stable bundle E can have at most two jumping lines in the line system $L_2 = pr_2^{-1}(\cdot)$.

As the moduli space M_0 is contained in M_+ and M_- by proposition (2.1), the following corollary is immediate.

(2.7) Corollary. A bundle $E \longrightarrow Q$ can have at most two jumping lines in each line system of Q if $[E] \in M_0$.

To prove lemma (2.6), we show first for $[E] \in M_+$ the splitting type $E|_{L_1}$ is generically trivial. Suppose not, one finds by lemma (2.4)

$$E|_{L_1} \simeq \mathcal{O}_{L_1}(1) \oplus \mathcal{O}_{L_1}(-1)$$

holds uniformly for all L_1 and consequently that

$$E(0,-1)|_{L_1} \simeq \mathcal{O}_{L_1} \oplus \mathcal{O}_{L_1}(-2)$$

on all such lines. Thus $(pr_1)_* E(0,-1)$ defines a line bundle, say, $\mathcal{O}_{\mathbb{P}_1}(\ell)$ over the base curve $\mathbb{P}_1$. It follows then

$$pr_1^*((pr_1)_* E(0,-1)) \simeq \mathcal{O}(\ell,0)$$

defines a line subbundle of $E(0,-1)$ fitting into an exact sequence

$$0 \longrightarrow \mathcal{O}(\ell,0) \xrightarrow{\text{ev}} E(0,-1) \longrightarrow \mathcal{O}(-\ell,-2) \longrightarrow 0$$

via the natural evaluation map ev. As $c_2(E(0,-1)) = 2$, one finds

$$\mathcal{O}(\ell,0) \cdot \mathcal{O}(-\ell,-2) = -2\ell = 2$$

which gives that $\ell = -1$. We conclude therefore E comes from an extension

(2.8) $$0 \longrightarrow \mathcal{O}(-1,1) \longrightarrow E \longrightarrow \mathcal{O}(1,-1) \longrightarrow 0 \ .$$

This however contradicts the $H_{r_1 \geq r_2}$–stability of E since

$$H_{r_1 \geq r_2} \cdot \mathcal{O}(-1,1) = r_1 - r_2 \geq 0 \ .$$

Thus for those $[E] \in M_+$ the restrictions $E|_{L_1}$ is generically trivial.

To determine the number of L_1–jumping lines an $H_{r_1 \geq r_2}$–stable bundle can possibly have, it is easiest to consider E as a family of holomorphic bundles over a projective line $\mathbb{P}_1 \simeq L_2$. In this interpretation, the number of L_1–jumping lines for E is exactly the number of elements in the zero set $V_{L_1} \cap L_2$ described in (2.3). As $V_{L_1} \cap L_2$ represents the zero set of a (non–trivial) section of $\mathcal{L}_{L_1} \longrightarrow L_2$ with

$$c_1(\mathcal{L}_{L_1}) = c_2(E)/[L_1] = 2h_1$$

(cf. [D2]), we conclude $V_{L_1} \cap L_2$ contains at most two points and therefore $E|_{L_1}$ is non–trivial for at most two L_1–lines. The argument for $H_{r_1 \leq r_2}$–stable bundle E is similar and this proves the lemma.

One infers easily from this lemma that in table (2.2)

$$q_+(L_1^5) = q_+(L_1^4\, L_2) = q_+(L_1^3\, L_2^2) = 0 \tag{2.9}$$

by using this kind of zero sets $V_{L_1(z_i)}$ on M_+ associated to three (distinct) lines $L_1(z_i)$, $i = 1,2,3$. Indeed, in these situations the number of L_1–lines we are working with is no less than three and so (2.9) follows if one can show

$$V_{L_1(z_1)} \cap V_{L_1(z_2)} \cap V_{L_1(z_3)} \cap M_+ = \phi \ .$$

This is however a trivial consequence of lemma (2.6) as no $H_{r_1 \geq r_2}$–stable bundle E can "jump" on three distinct L_1–lines.

To find remaining evaluations for q_+ we apply the same argument to three (distinct) L_2–lines. This time we get a non–empty (set) intersection

$$\text{(2.10)} \qquad \left\{ \bigcup_{i=1}^{3} V_{L_2(w_i)} \right\} \cap M_+$$

$$= \left\{ \bigcup_{i=1}^{3} V_{L_2(w_i)} \cap M_0 \right\} \coprod \left\{ \bigcap_{i=1}^{3} V_{L_2(w_i)} \cap \mathbb{P}_2^+ \right\} \quad \text{(by proposition (2.1))}$$

$$= \bigcap_{i=1}^{3} V_{L_2(w_i)} \cap \mathbb{P}_2^+ \quad \text{(by lemma (2.6))}$$

$$= \mathbb{P}_2^+ \ .$$

We shall briefly explain in a moment the intersection (2.10) is transversal in general. Assuming this, we can proceed to determine q_+ in the remaining cases by studying the universal bundle $\mathcal{E}_+$ over the product space $\mathbb{P}_2^+ \times Q$:

$$\text{(2.11)} \qquad 0 \longrightarrow P^*_{\mathbb{P}_2^+} \mathcal{O}_{\mathbb{P}_2^+}(1) \otimes P_Q^* \mathcal{O}(1,-1) \longrightarrow \mathcal{E}_+ \longrightarrow P_Q^* \mathcal{O}(-1,1) \longrightarrow 0$$

where $P_{\mathbb{P}_2^+}$ and P_Q are the obvious projection maps (cf. [R] lemma 2.3).

To show $q_+(L_1^2 L_2^3) = 1$, we consider the intersection between $\mathbb{P}_2^+$ and two (more) zero sets $V_{L_1(z_i)}$, $i = 1,2$. These zero sets on $\mathbb{P}_2^+$ represent the determinant bundle $\mathscr{L}_{L_1}$ with

$$c_1(\mathscr{L}_{L_1}) = c_2(\mathscr{E}_+)/[L_1] = h_+ \ ,$$

where h_+ denotes the standard generator of $H^2(\mathbb{P}_2^+;\mathbb{Z})$. It follows up to a sign $q_+(L_1^2 L_2^3)$ is given by

$$< c_2(\mathscr{E}_+)/_{[L_1(z_1)]} \cup c_2(\mathscr{E}_+)/_{[L_1(z_2)]}, [\mathbb{P}_2^+] > = h_+ \cdot h_+ = 1 \ .$$

As the algebraic sum associated to an intersection of five zero sets on M_+ defined by holomorphic sections must be non–negative, we conclude $\mathbb{P}_2^+$ has its usual complex orientation and therefore $q_+(L_1^2 L_2^3) = 1$, as stated in (2.2). Similarly, using

$$c_1(\mathscr{L}_{L_2}) = c_2(\mathscr{E}_+)/_{[L_2]} = -h_+$$

and one derives

$$q_+(L_1 L_2^4) = h_+ \cdot (-h_+) = -1 \ ,$$
$$q_+(L_2^5) = (-h_+) \cdot (-h_+) = 1$$

as wished.

To see the intersection $\left\{\bigcap_{i=1}^{3} V_{L_2(w_i)}\right\} \cap M_+ \simeq \mathbb{P}_2^+$ is transversal in general, we observe first the restriction map

$$r_{L_2} : H^1(\mathrm{End}\ E) \longrightarrow H^1((\mathrm{End}\ E)|_{L_2}) \simeq H^1(\mathcal{O}_{L_2}(-2)) \simeq \mathbb{C}$$

is surjective and fits into the following commutative diagram.

(2.12) <u>Diagram</u>.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(E(1,-1)) & \longrightarrow & H^1(\mathrm{End}\ E) & \longrightarrow & H^1(E(-1,1)) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle r_{L_2}} & & \downarrow & & \\
 & & 0 & \longrightarrow & H^1((\mathrm{End}\ E)|_{L_2}) & \longrightarrow & H^1(E(-1,1)|_{L_2}) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Thus, for an element $[E]$ in the zero set $V_{L_2} \cap M_+$ we are working with, one finds the tangent space $T_{[E]}(V_{L_2} \cap M_+)$ identifies with $\mathrm{Ker}\ r_{L_2}$ in diagram (2.12). Now by tracing diagrams we can show three such tangent spaces meet transversely in $H^1(\mathrm{End}\ E)$ for three general L_2–lines and this will prove the transversality for the intersection (2.10). We leave the detail of this argument to the reader.

Now we wish to explain why q_+ , q_- are not polynomials of the intersection form

$$q_Q = h_1h_2 + h_2h_1$$

and the canonical class

$$k_Q = -2h_1 - 2h_2$$

on a quadric surface $Q \simeq S^2 \times S^2$. Supposing on the contrary q_+ , say, admits such an expression, the coefficient a_0 of k_Q^5 would then be detected by the evaluation $q_+(L_1^5)$ or $q_+(L_2^5)$ as the intersection form is zero in either case. A contradiction is immediate since we have $a_0 \neq 0$ by $q_+(L_2^5) = 1$ while $q_+(L_1^5) = 0$ gives $a_0 = 0$.

Obviously the failure of q_+, q_- admitting such expressions lies in the fact that the construction of these polynomials depends upon the choice of metrics on Q . However we can get around this dependence just by averaging, or taking the sum of q_+ and q_- . Thus, as Q is a complete intersection, we can apply [FMM] theorem 5 to conclude $q_+ + q_-$ is a polynomial on q_Q and k_Q . Indeed, one can find by a direct calculation

$$q_+ + q_- = -\tfrac{1}{32}\, k_Q^5 + \tfrac{5}{8}\,(k_Q^3\, q_Q) - \tfrac{15}{4}\,(k_Q\, q_Q^2)$$

where the brackets ( ) denote symmetrizations of k_Q and q_Q .

References

[D1] Donaldson, S.K.
"4–manifolds with infinite intersection form"
Lecture notes in mathematics (1111), Springer–Verlag, Berlin (1984)

[D2] Donaldson, S.K.
"Connection, cohomology and the intersection forms of 4–manifolds"
J. Diff. Geom. 24 (1986) 275–341

[D3] Donaldson, S.K.
"Irrationality and the h–cobordism conjecture"
J. Diff. Geom. 26 (1987) 141–168.

[D4] Donaldson, S.K.
"The orientation of Yang–Mills moduli spaces and 4–manifold topology"
J. Diff. Geom. 26 (1987) 397–428

[D5] Donaldson, S.K.
"Polynomial invariants for smooth four–manifolds"
Preprint

[FM1] Friedman, R., Morgan, J.W.
"On the diffeomorphism types of certain algebraic surfaces I"
J. Diff. Geom. 27 (1988) 297–369

[FM2] Friedman, R., Morgan, J.W.
"On the diffeomorphism types of certain algebraic surfaces II"
J. Diff. Geom. 27 (1988) 371–398.

[FMM] Friedman, R. Moishezon, B., Morgan, J.W.
"On the C^{∞} invariance of the canonical classes of certain algebraic surfaces"
Bull. Amer. Math. Soc. (N.S.) 17 (1987) 283–286

[K] Kotschick, D.
Oxford D. Phil. Thesis (1989)

[M] Mong, K.C.
"Moduli spaces of stable 2–bundles and polarizations"
Max–Planck–Institut preprint (1989)

[R] Ramann, S.
"The moduli spaces of vector bundles over an algebraic curve"
Math. Ann. 200, 69–84 (1973)