Volumes of arithmetic hyperbolic n-manifolds

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We start with a definition. Let k be a number field and H/k is a semi-simple algebraic group defined over k. Fix a k-embedding of $H(k)$ into $GL(n)$ for a big enough n. Subgroup Λ of $H(k)$ is called *arithmetic* if it is commensurable with the group of k-integral points $GL(n, \mathcal{O}_k)$, that is, the intersection $\Lambda \cap GL(n, \mathcal{O}_k)$ has finite indexes in both subgroups. It can be shown that the notion of arithmeticity does not depend on the choice of a particular k-embedding of H into $GL(n)$.

If G is a semi-simple Lie group then consider all the algebraic groups H/k that admit surjective homomorphism $\phi : H(k \otimes_{\mathbb{Q}} \mathbb{R}) \to G$ with a compact kernel. Subgroups of G which are commensurable with the images of arithmetic subgroups under such homomorphisms are called *arithmetic subgroups of* G, and the field k is called *field of definition* of the corresponding arithmetic subgroups.

For a non-compact semi-simple Lie group G we can associate its symmetric space

$$G/K_G$$

where K_G is a maximal compact subgroup of G, and locally symmetric spaces

$$\Gamma \backslash G/K_G$$

that are defined by the discrete subgroups $\Gamma \subset G$. We call a locally symmetric space arithmetic if the corresponding discrete subgroup Γ is an arithmetic subgroup of G, and we say that the space is defined over k if k is the field of definition of Γ.

If G is an orthogonal group $SO(1, n)^o$ then its symmetric space is the hyperbolic n-space, and the locally symmetric spaces are the hyperbolic orbifolds or manifolds (if Γ is torsion-free). The Haar measure on G induces the measures on the factor spaces. It can be normalized in such a way that the induced measures are the hyperbolic volumes or Euler-Poincare characteristic (for even dimensions) of the orbifolds. We are studying the extremal examples, namely those which have the smallest possible volume.

For $n = 2$ the solution of this problem was known for a long time. The smallest Fuchsian group was obtained by Hurwitz, it is triangle group $(2, 3, 7)$. This group has a lot of interesting properties and was much studied. In particular, Hurwitz used it to prove that the order of the automorphisms group of a compact orientable Riemann surface of genus $g \geq 2$ is bounded from above by $84(g - 1)$. It can be checked that the group $(2, 3, 7)$ is an arithmetic subgroup of $SO(1, 2)^o \cong PSL(2, \mathbb{R})$ defined over $\mathbb{Q}[\cos(2\pi/7]$.

For $n = 3$ the smallest arithmetic orbifold was described by Chinburg and Fridman (Invent. Math., 86 (1986), 507–527). Conjecturally, it is the smallest hyperbolic 3-manifold. For higher dimensions very little was known. Probably the next most interesting case for the applications is the hyperbolic dimension 4.

Our main results are given by the following two theorems.

THEOREM 1. For any $n = 2r \geq 4$ there exists a *unique* compact orientable arithmetic hyperbolic n-orbifold O^n_{min} of the smallest volume. It is defined over the field $k = \mathbb{Q}[\sqrt{5}]$ and has Euler characteristic

$$|\chi(O^n_{min})| = \frac{\lambda(r)}{N(r)4^{r-1}} \prod_{i=1}^{r} |\zeta_k(1-2i)|,$$

where ζ_k is the Dedekind zeta function of k, $N(r) \in \mathbb{Z}$ and $\lambda(r) \in \mathbb{Q}$ are constants such that $1 \leq N(r) \leq 4$, $\lambda(r) = 1$ for even r and $1 \leq N(r) \leq 8$, $\lambda(r) = 2^{-1}(4^r - 1)$ for r odd.

THEOREM 2. If there exists a compact orientable arithmetic hyperbolic 4-manifold M having $\chi(M) \leq 24$, then it satisfies one of the following conditions:

1) M is defined over $\mathbb{Q}[\sqrt{5}]$ and has the form $\Gamma_M \backslash \mathcal{H}^4$ with Γ_M is a torsion-free subgroup of index $7200\chi(M)$ of the group Γ_1 of the smallest arithmetic 4-orbifold;
2) M has Euler characteristic 22, is defined over $\mathbb{Q}[\sqrt{2}]$, and its group is a torsion-free subgroup of index 11520 of Γ_2 which is the smallest arithmetic subgroup of $SO(1,4)^o$ defined over $\mathbb{Q}[\sqrt{2}]$.

In the lecture I discuss the proof of Theorem 1. The argument uses G. Prasad's volume formula for the arithmetic quotients of semi-simple groups, Bruhat-Tits theory, the bound for the index of a principal arithmetic subgroup in its normalizer provided by the Rholfs' exact sequence for the Galois cohomology of k and some other ingredients. The technical part consists of the number theoretical estimates which make essential use of the Odlyzko bounds for discriminants of number fields and some particular information about fields of low degrees. To prove the uniqueness of the smallest orbifold we use the week approximation property for the adjoint groups H over the adéles of k. All the methods are general and can be applied to the other groups, in particular to the symmetries of the odd dimensional hyperbolic spaces, but the technical part will be different.

Theorem 2 addresses another important problem: to describe the smallest compact hyperbolic 4-manifold. The smallest currently known example is due to Davis, it has Euler characteristic 26. Our theorem does not provide any smaller examples explicitly, but it reduces the problem of determining the smallest compact arithmetic hyperbolic 4-manifold to an extensive computation.