A realization of the intersection form in Yang–Mills theory

by

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Introduction

Let S_3 be a (generic) homotopy K3 surface having a multiple fibre F_3 of multiplicity three. In [M1] we have determined for S_3 the moduli space $M_2(\omega)$ of ω–stable 2–bundles $\mathcal{E} \longrightarrow S_3$ with $(c_1(\mathcal{E}), c_2(\mathcal{E})) = (0,2)$ relative to some Kähler form ω of a Kähler metric m_0 on S_3. The purpose of this note is to explain for generic (but not necessarily Kähler) metrics m_0' on S_3 close to m_0 it is possible to make use of the explicit description of $M_2(\omega)$ and apply the method introduced by Donaldson in [D3] of defining polynomial invariants to construct certain symmetric bilinear maps

$$q_{2,S_3}(m_0') : K_{S_3}^{\perp} \times K_{S_3}^{\perp} \longrightarrow \mathbb{Z} \ ,$$

defined on the sublattice $K_{S_3}^{\perp} \subset H_2(S_3;\mathbb{Z})$ annihilated by the canonical class K_{S_3} of S_3. Furthermore we show $q_{2,S_3}(m_0')$ links to the intersection form Q_{S_3} of S_3 on $H_2(S_3;\mathbb{Z})$ in the following way.

Theorem $q_{2,S_3}(m_0') = 2Q_{S_3}$ as symmetric bilinear maps on $K_{S_3}^{\perp} \times K_{S_3}^{\perp}$.

One should note that $q_{2,S_3}(m_0')$ is <u>not</u> a polynomial invariant of S_3 in the sense of [D3]. Nevertheless it shares the property being a polynomial on Q_{S_3} and K_{S_3} ([FMM]).

A natural compactification of $M_2(\omega)$, by its explicit algebro–geometric description obtained in [M1], is to add to it a copy of the symmetric product S^2F_3 of the multiple fibre F_3. Noticing $M_1(\omega)$ is empty, one would incline to think then the Yang–Mills compactification $\overline{M_2(\omega)}$ of $M_2(\omega)$ in this situation could be as nice as that

$$\overline{M_2(\omega)} = M_2(\omega) \cup (S^2F_3 \times \{[\theta]\}) \quad ,$$

where $[\theta]$ denotes the gauge equivalence class of the trivial connection θ on S_3. A main point of the present paper is to explain this is indeed the case. To justify the compatibility of these two compactifications of $M_2(\omega)$, we are to describe a neighbourhood system for the lower stratum

$$\overline{M_2(\omega)} \backslash M_2(\omega) \subset S^2(S_3) \times \{(\theta)\}$$

in $\overline{M_2(\omega)}$, exploiting particularly the structure near the diagonal part $\Delta_{S_3} \times \{[\theta]\}$. This is the most technical part of establishing the theorem granted results obtained in [M1].

The reason we do not work directly with the Kähler metric m_0 on S_3 is that such a metric fails to be generic and the moduli space $M_2(\omega)$, despite being smooth, is of (real) dimension higher than the virtual one by two. This is accountable, as explained in [M1], by the appearance of the "cokernel bundle" $\zeta \longrightarrow M_2(\omega)$ arising from the assignment $\mathcal{E} \longrightarrow H^2(s\ell(\mathcal{E})) \simeq \mathbb{C}$. Here we discuss in such situations how one could sometimes get around this kind of difficulty by working with nearby generic metrics m_0'. This is the point that limits the domain of $\tilde{q}_{2,S_3}(m_0')$ to the sublattice $K_{S_3}^{\perp} \times K_{S_3}^{\perp}$ of $H_2(S_3;\mathbb{Z}) \times H_2(S_3;\mathbb{Z})$ as we shall see.

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§ 1. The definition of $q_{2,S_3}(m_0')$

In Riemannian geometry the homotopy K3 surface S_3 is a smooth compact simply–connected oriented 4–manifold with $b_2^+(S_3) = 3$. For such a surface there defines in [D3] a polynomial invariant $q_{k,S_3} \in \mathrm{Sym}^d(H^2(S_3;\mathbb{Z}))$ of degree $d = 4k-6$ for each integer $k > 3$. To construct such polynomials q_{k,S_3}, it requires amongst other things a suitable choice of generic metrics m on S_3 so that the associated Yang–Mills moduli spaces $M_k(m)$, consisting of equivalence classes of anti–self–dual (ASD) connections on an SU(2)–bundle $P \longrightarrow S_3$ with $c_2(P) = k$, is a smooth manifold of virtual dimension 2d. Here we consider this construction on the surface S_3 for a smaller value $k=2$. It turns out that by working with some special metric m_0' on S_3 we are still able to define certain polynomial $q_{2,S_3}(m_0')$ using essentially the same method of [D3] as we are going to explain.

Recall that the surface $S_3 \xrightarrow{\ \psi\ } \mathbb{P}_1$ is elliptic with canonical bundle $K_{S_3} \simeq [F_3]^{\otimes 2}$, the square of the line bundle $[F_3]$ associated to the multiple fibre F_3. Given any lattice point α in

$$K_{S_3}^{\perp} = \{\alpha \in H_2(S_3;\mathbb{Z}) : \alpha \cdot K_{S_3} = 0\} ,$$

one can always find a smooth oriented real surface $\Sigma \subset S_3$ representing α with the property that the intersection $\Sigma \cap F_3$ is empty. For such a surface Σ, we can define as in [D2] a (complex) line bundle $\mathscr{L}_\Sigma \longrightarrow M_2(m)$ by the assignment

$$A \longmapsto \Lambda^{\max}(\mathrm{Ker}\, \partial\!\!\!/_{A|_\Sigma})^* \otimes \Lambda^{\max}(\mathrm{coker}\, \partial\!\!\!/_{A|_\Sigma})$$

sending a connection A on P to the determinant line associated to the Dirac operator $\not{\partial}_{A|_{\Sigma}}$ coupled with the restricted connection $A|_{\Sigma}$. Provided Σ is suitably chosen, we can find for the bundle $\mathscr{L}_{\Sigma} \longrightarrow M_2(m)$ transversal sections with zero sets $V_{\Sigma} \cap M_2(m)$ containing elements $[A]$ which are non–trivial on Σ. As $M_2(m)$ has virtual dimension four, one can consider then appropriate intersection numbers $|V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m)|$ on $M_2(m)$.

(1.1) Lemma Transversal intersections $V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m_0')$ are compact, provided the surfaces Σ_1, Σ_2 are disjoint from the multiple fibre F_3 and m_0' is a generic metric sufficiently close to m_0.

We shall show this lemma in coming sections. Assuming this for the moment, we obtain an assignment

$$(\Sigma_1,\Sigma_2) \longmapsto |V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m_0')|$$

and hence a symmetric bilinear map

$$q_{2,S_3}(m_0') : K_{S_3}^{\perp} \times K_{S_3}^{\perp} \longrightarrow \mathbb{Z}$$

for a generic metric m_0' close to m_0, as wished.

Despite the framework just described does not apply to the (non–generic) Kähler metric m_0 on S_3, it will be important for us to consider transversal intersections

$V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m_0)$ in order to determine the polynomial $q_{2,S_3}(m_0')$. Note that it still makes good sense to talk of such intersections since the moduli space $M_2(m_0)$ is, after all, a smooth manifold. This nice property of $M_2(m_0)$ follows from a theorem of Uhlenbeck and Yau allowing one to identify $M_2(m_0)$ with the moduli space $M_2(\omega)$ of stable 2–bundles determined in [M1]. Besides, there is no U(1)–reduction in $M_2(m_0)$ to worry about for the following reason.

By choice $\omega = \omega_{S_3} + N\phi^* \omega_{\mathbb{P}_1}$, the sum of an arbitrary Kähler form ω_{S_3} on S_3 and a multiple of the pullback Fubini–Study form $\omega_{\mathbb{P}_1}$ of $\mathbb{P}_1$. Here we take N to be an integer larger than

$$\deg_{\omega_{S_3}} K_{S_3} = \int_{S_3} c_1(K_{S_3}) \wedge \omega_{S_3} .$$

Then, as a result of the following proposition, one finds $M_2(m_0)$ in fact contains no U(1)–reduction at all.

(1.2) <u>Proposition</u> There is no holomorphic line bundle $\mathscr{L} \longrightarrow S_3$ satisfying $\omega \cdot \mathscr{L} = 0$ and $\mathscr{L} \cdot \mathscr{L} \in \{-1,-2,-3\}$.

<u>Proof</u> Suppose on the contrary there is such a bundle $\mathscr{L}$ over S_3. The Riemann–Roch formula gives then

$$h^0(\mathscr{L} \otimes [F_3]) - h^1(\mathscr{L} \otimes [F_3]) + h^0(\mathscr{L}^{-1} \otimes [F_3]) = \tfrac{1}{2}\, \mathscr{L} \cdot \mathscr{L} + 2 .$$

Assuming $\mathscr{L} \cdot \mathscr{L} = -1,-2,-3$, we have either

$$h^0(\mathscr{L}\otimes[F_3]) \geq 1 \text{ or } h^0(\mathscr{L}^{-1}\otimes[F_3]) \geq 1 \ .$$

Consider first the case when $\mathscr{L} \cdot [F_3] = 0$. If $h^0(\mathscr{L}\otimes[F_3]) \geq 1$, we have that the bundle $\mathscr{L}\otimes[F_3]$ is represented by an effective divisor D on S_3 satsifying $[D] \cdot [F_3] = 0$. It follows D is equivalent to a combination of fibres on S_3 and one infers then $\mathscr{L} \cdot \mathscr{L} = 0$, a contradiction to the assumption that $\mathscr{L} \cdot \mathscr{L} \neq 0$. One argues similarly for the case $h^0(\mathscr{L}^{-1}\otimes[F_3]) \geq 1$. Suppose now that $\mathscr{L} \cdot [F_3] \neq 0$. If $h^0(\mathscr{L}\otimes[F_3]) \geq 1$, then we have $(\mathscr{L}\otimes[F_3]) \cdot [F_3] \geq 0$ and the assumption implies $\mathscr{L} \cdot [F_3] \geq 1$. As $\omega \cdot \mathscr{L} = 0$ for such a bundle $\mathscr{L}$, one finds on the one hand

$$\deg_\omega(\mathscr{L}\otimes[F_3]) = \deg_\omega[F_3] = \deg_{\omega_{S_3}}[F_3]$$

while on the other

$$\deg_\omega(\mathscr{L}\otimes[F_3]) = \deg_{\omega_{S_3}}(\mathscr{L}\otimes[F_3]) + N(\mathscr{L} \cdot F) \ .$$

It follows then

$$\deg_{\omega_{S_3}}[F_3] > N(\mathscr{L}\cdot F) > N > \deg_{\omega_{S_3}} K_{S_3} \ .$$

This is a contradiction as $K_{S_3} \simeq [F_3]^{\otimes 2}$. The treatment for the case $h^0(\mathscr{L}^{-1}\otimes[F_3]) \geq 1$ is similar and this proves the proposition.

Note that transversal intersections $V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m_0)$ are smooth (real)

2–dimensional manifolds rather than a finite number of points. They are moreover compact as the argument of proving lemma (1.1) will show. We shall however move on to determine $q_{2,S_3}(m_0')$ first in the next section. The proof of lemma (1.1) is quite technical and will be postponed to § 3 – § 4.

§ 2. The determination of $q_{2,S_3}(m_0')$

The calculation of $q_{2,S_3}(m_0')$ involves more generally the consideration of certain problem that we wish to discuss first. Let X be a smooth compact simply–connected oriented 4–manifold with $b_2^+(X)$ odd. Suppose that for some fixed non–generic metric m on X the moduli space $M_k(m)$ is a smooth compact manifold of dimension $2d+r$, higher than the virtual one $2d$ by r. Assume moreover the second cohomology group $H_A^2(\text{ad } P)$ in the Atiyah–Hitchen–Singer deformation complex is of dimension r for all $[A] \in M_k(m)$ so that the assignment $A \longrightarrow H_A^2(\text{ad } P)$ defines a cokernel bundle $\zeta \longrightarrow M_k(m)$. Then a question one would like to pose is that whether it is possible to recover (up to isotopy) nearby moduli spaces $M_k(m')$ for those generic metrics m' on X sufficiently close to m. The answer to this problem would not be affirmative in general. However, with the additional assumption that $M_k(m)$ and $M_k(m')$ are all compact, it is indeed possible to recover $M_k(m')$ in the following way. Our approach follows [FU].

Let $\mathcal{A}$ be the affine space of connections on P and $\mathcal{R} = C^s(GL(TX))$ the Banach space of C^s–automorphism of the tangent bundle of X for some integer $s >> 0$. Writing $P_+ = \frac{1}{2}(1 + *_m)$, we define a map

$$\begin{aligned} F_{+,\cdot} : \mathcal{A} \times \mathcal{R} &\longrightarrow \Omega_+^2(\text{ad } P) \\ (A,\phi) &\longmapsto P_+((\phi^{-1})^* F(A)) \end{aligned}$$

with the property that for all fixed $\phi \in \mathcal{R}$ the zero set $\{F_{+,\cdot}(A,\phi) = 0\} \subset \mathcal{A}$ consists of all ASD connections on P relative to the pullback metric $\phi^* m$ on X. (We assume here $\Omega^1(\text{ad } P)$ and $\Omega_+^2(\text{ad } P)$ are modelled on certain Hilbert spaces but notations for which are omitted for simplicity.) At the point $(A,\text{id.}) \in \mathcal{A} \times \mathcal{R}$ the partial derivative of

$F_{+,\cdot}$ in the $\mathcal{R}$–factor is the map

$$T_{id.}(R) \longrightarrow \Omega^2_+(\text{ad } P)$$
$$\gamma \longmapsto P_+(\gamma^* F(A))$$

and we always assume $|\gamma| = 1$. For those $[A] \in M_k(m)$ we can define an orthogonal projection

$$\pi_A : \Omega^2_+(\text{ad } P) \longrightarrow H^2_A(\text{ad } P)$$

and thus obtain a section ϵ_γ of the cokernel bundle $\zeta \longrightarrow M_k(m)$ induced by the assignment $A \longrightarrow \pi_A P_+(\gamma^* F(A))$. Let $\phi_t^{-1} = \exp t\gamma \in \mathcal{R}$.

(2.1) Proposition If ϵ_γ vanishes transversally on $M_k(m)$, then the zero set $\{\epsilon_\gamma=0\}$ is diffeomorphic to a nearby $M_k(\phi_t^* m)$ provided the moduli spaces $M_k(\phi_t^* m)$ are compact over a small path of metrics $\phi_t^* m$ on X.

This proposition does not apply directly to the non–compact moduli space $M_2(m_0)$ for S_3 we have been considering. Nevertheless, assuming the compactness of $V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m_0')$ for nearby metrics m_0', one will find an easy modification of the proof for this proposition shows intersection numbers of $V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m_0')$ are in fact that of $V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2^\sigma(m_0)$ on a "cut–down" moduli space $M_2^\sigma(m_0)$, the zero set of a transversal section σ of $\zeta \longrightarrow M_2(m_0)$. We shall determine the polynomial $\tilde{q}_{2,S_3}(m_0')$ by means of computing intersection numbers $|V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2^\sigma(m_0)|$ in respect to the natural orientation of $M_2^\sigma(m_0)$. Note that the compactness assumption on $M_k(\phi_t^* m)$ in the proposition could not be relaxed in view of the pathetic possibility that

$M_k(\phi_t^* m)$ could have "ends" not isotopic to $M_2^{\sigma}(m_0)$. We first need the following lemma to exclude this complexity.

Let $N \longrightarrow M_k(m)$ be the normal bundle of $M_k(m) \hookrightarrow \mathscr{B}_X^*$, the space of equivalence classes of irreducible connection on P . Suppose $U_\epsilon \subset \mathscr{B}_X^*$ is a small tubular neighbourhood of $M_k(m)$ diffeomorphic to the ϵ–ball bundle associated to N .

(2.2) <u>Lemma</u> Suppose for some $t_0 > 0$ moduli spaces $M_k(\phi_t^* m)$ are compact for all $t \in [0,t_0]$. Then given any tubular neighbourhood U_ϵ of $M_k(m)$ there is a small constant $t_\epsilon > 0$ such that

$$M_k(\phi_t^* m) \subset U_\epsilon \quad \text{for all} \quad t \in [0,t_\epsilon]$$

<u>Proof</u> For $0 < c \leq t_0$ we let

$$Y_c = \cup \, \{M_k(\phi_t^* m) \times \{t\} \mid t \in [0,c]\}$$

and by assumption the space Y_{t_0} is compact. If S_ϵ denotes the ϵ–sphere bundle associated to $N \longrightarrow M_k(m)$, then the intersection $Y_{t_0} \cup \{S_\epsilon \times [0,t_0]\}$ is compact and so is its projection image $K \subset [0,t_0]$. As $M_k(m)$ is properly contained in U_ϵ , one finds $0 \notin K$ and hence that the minimum t_K of K is strictly positive. It follows then the space $Y_{\frac{1}{2} t_K}$ can be written as a disjoint union of two compact pieces

$$W_1 = Y_{\frac{1}{2} t_K} \cap \{U_\epsilon \times [0,\tfrac{1}{2} t_K]\} \quad \text{and}$$
$$W_2 = Y_{\frac{1}{2} t_K} \backslash \{U_\epsilon \times [0,\tfrac{1}{2} t_K]\}$$

since $Y_{\frac{1}{2}t_K} \cap \{S_\epsilon \times [0,\frac{1}{2}t_K]\}$ is empty. Now project W_2 to $[0,\frac{1}{2}t_K]$ and obtain a compact subset with minimum $2t_\epsilon$, say. One checks $t_\epsilon > 0$ and that $Y_{t_\epsilon} \subset U_\epsilon \times [0,t_\epsilon]$. The lemma follows.

Assume from now on $t \in [0,t_\epsilon]$ and write ζ_A for $H^2_A(\mathrm{ad}\,P)$ for simplicity. Under the present assumption the map

$$d_A^+ : \{N_A \subset \mathrm{Ker}\, d_A^* \text{ in } \Omega^1(\mathrm{ad}\,P)\} \longrightarrow \zeta_A^\perp$$

is an isomorphism. By the implicit function theorem, we can solve $n_t(A) \in N_A$ for sufficiently small t so that

$$P_+((\phi_t^{-1})^* F(A + n_t(A))) \in \zeta_A$$

and thereby obtain a manifold

$$Z_t = \{A + n_t(A) \mid [A] \in M_k(m)\} / \mathcal{G}$$

in U_ϵ, where $\mathcal{G}$ denotes the gauge transformation group of P. Clearly then we have

$$M_k(\phi_t^* m) = \{P_+((\phi_t^{-1})^* F(A + n_t(A))) = 0\} / \mathcal{G} \subset Z_t$$

which is diffeomorphic to the zero set

$$\mathcal{Z}_t = \{[A] \in M_k(m) \mid P_+((\phi_t^{-1})^* F(A + n_t(A))) = 0\}$$

in the obvious way. To finish the proof of the proposition we show $\mathcal{Z}_t$ is diffeomorphic to $\{\epsilon_\gamma = 0\}$ if $t \neq 0$ is small. On the compact space $M_k(m)$, one finds $\mathcal{Z}_t$ is the zero set of

$$\begin{aligned} &P_+((\phi_t^{-1})^* F(A + n_t(A))) \\ &= \pi_A P_+\{F(A + n_t(A)) + t\,\gamma^* F(A + n_t(A)) + 0(t^2)\} \\ &= \pi_A P_+\{F(A) + t\gamma^* F(A) + 0(t^2)\} \\ &= t\{\epsilon_\gamma(A) + 0(t)\} \end{aligned}$$

by $\pi_A d_A^+ \equiv 0$ and $|n_t| = 0(t)$. As ϵ_γ vanishes transversely by assumption, we conclude $\mathcal{Z}_t$ and $\{\epsilon_\gamma=0\}$ are in fact isotopic for sufficiently small $t \neq 0$. This proves proposition (2.1).

To compute intersection numbers $|V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2^\sigma(m_0)|$ we exploit the fact that on the bundle $\zeta^{\otimes 2} \longrightarrow M_2(m_0)$ there is a section with transversal zero set $\Delta_{\hat{Y}/\mathbb{Z}_2}$ topologically a copy of $\hat{S}_3 \backslash F_3 \subset M_2(m_0)$, where $\hat{S}_3$ denotes the blow–up of S_3 at all the node points on singular fibres of $S_3 \xrightarrow{\psi} \mathbb{P}_1$ (c.f. [M1]). We shall show $|V_{\Sigma_1} \cap V_{\Sigma_2} \cap \Delta_{\hat{Y}/\mathbb{Z}_2}| = 4\Sigma_1 \cdot \Sigma_2$ so that

$$\begin{aligned} q_{2,S_3}(m_0')([\Sigma_1],[\Sigma_2]) &= \tfrac{1}{2}|V_{\Sigma_1} \cap V_{\Sigma_2} \cap \Delta_{\hat{Y}/\mathbb{Z}_2}| \\ &= 2\Sigma_1 \cdot \Sigma_2 \ . \end{aligned}$$

It will follow then $q_{2,S_3}(m_0') = 2Q_{S_3}$ as the theorem asserts. This calculation requires the knowledge of $H_2(\hat{S}_3 \backslash F_3;\mathbb{Z})$ that we wish to discuss now.

It is well–known $H_2(\hat{S}_3\backslash F_3;\mathbb{Z})$ is generated by $H_2(S_3\backslash F_3;\mathbb{Z})$ together with homology classes $\{e_t\}_{t=1}^{24}$ carried by the exceptional divisors on $\hat{S}_3\backslash F_3$. It suffices for us to determine $H_2(S_3\backslash F_3;\mathbb{Z})$. If $\{h_r\}_{r=1}^{21}$ is an integral basis of $K_{S_3}^{\perp}$, we can choose for each h_r a smooth surface Σ_r representing h_r with $\Sigma_r \cap F_3$ empty. Such surfaces Σ_r carry homology classes in $H_2(S_3\backslash F_3;\mathbb{Z})$ which will also be denoted by h_r for simplicity. (Without loss we assume each Σ_r does not contain any node on singular fibres on S_3 .) Using the fact $\pi_1(S_3\backslash F_3) = 0$ established in [K], one finds by a Mayer–Vietories argument $H_2(S_3\backslash F_3;\mathbb{Z})$ is in fact generated by two elements β_1, β_2 in addition to the lifting of $K_{S_3}^{\perp}$ spanned by $\{h_r\}$. These two homology classes β_1, β_2 can be described more easily in $S_3\backslash F_3 \simeq S_0\backslash F_{a_0}$, the complement of a smooth fibre F_{a_0} on an elliptic K3 surface S_0 . To see this, let F be a smooth fibre of S_0 close to F_{a_0} and ℓ_1, ℓ_2 be the two loops on F generating $H_1(F;\mathbb{Z})$. If α is a linking circle of F_{a_0} in S_0 , then β_1, β_2 are simply the homology classes defined respectively by $\alpha \times \ell_1$, $\alpha \times \ell_2$. We have thus showed $H_2(\hat{S}_3\backslash F_3;\mathbb{Z})$ is freely generated by h_r, β_s, e_t . Denote by h_r^*, β_s^*, e_t^* the dual classes of h_r, β_s, e_t in $H^2(\hat{S}_3\backslash F_3;\mathbb{Z})$. A fact that will be useful in our discussion is that the support of β_s can be chosen arbitrarily close to the multiple fibre F_3 by shrinking the linking circle α . Thus we may assume β_r is disjoint from the surfaces Σ_r if so wished.

Now we are ready to compute the algebraic sum of $V_{\Sigma_1} \cap V_{\Sigma_2} \cap \Delta_{\mathfrak{Y}/\mathbb{Z}_2}$ using algebraic geometry. The basic tool of our calculation is the exact sequence

$$(2.3) \qquad 0 \longrightarrow 0 \longrightarrow \not{\mathcal{E}} \otimes pr_1^*\zeta \otimes pr_2^*[F_3] \longrightarrow pr_1^*\zeta \otimes pr_2^*[F_3]^{\otimes 2} \otimes \mathcal{J} \longrightarrow 0$$

for the universal bundle $\not{\mathcal{E}} \longrightarrow M_2(m_0) \times S_3$ when restricted to $\Delta_{\mathfrak{Y}/\mathbb{Z}_2} \times S_3$ (c.f.

[M1]). To begin with, we notice first it is possible to choose zero set V_{Σ_i} of sections on the bundle $\mathscr{L}_{\Sigma_i}$ so that $V_{\Sigma_i} \cap \Delta_{\hat{Y}/\mathbb{Z}_2}$ is compact for $i = 1,2$. To see this, let U be a small tubular neighbourhood of F_3 in $\hat{S}_3$ not meeting Σ_1, Σ_2. We wish to show the bundle $\mathscr{L}_{\Sigma_i}$ are trivial over $U\backslash F_3 \subset \hat{S}_3\backslash F_3 \simeq \Delta_{\hat{Y}/\mathbb{Z}_2}$. If we write

$$c_1(\mathscr{L}_{\Sigma_i}) = \sum_r a_r^i h_r^* + \sum_s b_s^i \beta_s^* + \sum_t c_t^i e_t^* ,$$

then it is enough to check that $b_1^i = b_2^i = 0$ for $i = 1,2$. Using the fact $c_1(\mathscr{L}_{\Sigma_i}) = c_2(\not{\mathscr{E}})/[\Sigma_i]$ (c.f. [D2]) one finds

$$\begin{aligned} b_s^i &= \langle c_2(\not{\mathscr{E}})/[\Sigma_i], \beta_s\rangle \\ &= \langle c_2(\not{\mathscr{E}} \otimes pr_1^* \zeta \otimes pr_2^*[F_3])/[\Sigma_i], \beta_s\rangle \quad \text{if } F_3 \cdot \Sigma_i = 0 . \end{aligned}$$

Deform β_s to be inside $U\backslash F_3$ if necessary we may assume Σ_i and β_s have empty intersection. It follows then $b_s^i = 0$ as the section of $\not{\mathscr{E}} \otimes pr_1^* \zeta \otimes pr_2^*[F_3]$ inducing the exact sequence (2.3) does not vanish on $\Sigma_i \times \beta_s$.

Now, by the fact that $H_2(\hat{S}_3\backslash F_3;\mathbb{Z})$ is free of torsion, we can evaluate the algebraic sum using differential forms as follows. Write $\Phi_N(\Sigma)$ for the Thom class of the normal bundle of a smooth oriented real surface Σ in $\hat{S}_3\backslash F_3$. In the case when Σ is compact, one can assume $\Phi_N(\Sigma)$ in $H^2(\hat{S}_3\backslash F_3;\mathbb{R})$ has compact support (c.f. [BT]). We shall check in a moment

$$\Phi_N(V_{\Sigma_i}) = 2\,\Phi_N(\Sigma_i)$$

when $V_{\Sigma_i} \cap \hat{S}_3 \backslash F_3$ is compact. Granted this, the algebraic sum of $V_{\Sigma_1} \cap V_{\Sigma_2} \cap \Delta_{\tilde{Y}/\mathbb{Z}_2}$ is given by

$$\int_{\hat{S}_3 \backslash F_3} \Phi_N(V_{\Sigma_1}) \wedge \Phi_N(V_{\Sigma_2}) = 4 \int_{\hat{S}_3 \backslash F_3} \Phi_N(\Sigma_1) \wedge \Phi_N(\Sigma_2)$$
$$= 4\, \Sigma_1 \cdot \Sigma_2$$

as what we wish to establish.

It is easy to see that $\Phi_N(\Sigma_i) = \sum_r a_r^i h_r^*$ with $a_r^i = h_r \cdot \Sigma_i$. On the other hand, one notices that $\Phi_N(V_{\Sigma_i})$ is in essence the Poincaré dual of $V_{\Sigma_i} \cap \hat{S}_3 \backslash F_3$ in $\hat{S}_3 \backslash F_3$ which can otherwise be realized as $c_1(\mathscr{L}_{\Sigma_i})$ in this setting (c.f. [BT] p. 67, p. 134). As before, one finds $c_1(\mathscr{L}_\Sigma) \in H^2(\hat{S}_3 \backslash F_3)$ has only h_r^*–components with coefficients

$$\langle c_1(\mathscr{L}_{\Sigma_i}), h_r \rangle = \langle c_2(\not{\mathscr{S}} \otimes pr_1^* \zeta \otimes pr_2^* [F_3]) / [\Sigma_i], h_r \rangle$$
$$= 2\, h_r \cdot \Sigma_i$$

as the section of $\not{\mathscr{S}} \otimes pr_1^* \zeta \otimes pr_2^* [F_3]$ inducing the exact sequence (2.3) vanishes to order two at transversal intersection points of Σ_i and Σ_r . It follows $\Phi_N(V_{\Sigma_i}) = 2\Phi_N(\Sigma_i)$ as wished.

§ 3. The compactness of $V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m_0)$

The rest of this paper will be showing for real surfaces Σ_1, Σ_2 of S_3 disjoint from the multiple fibre F_3, transversal intersections $V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m_0)$ are compact for the Kähler metric m_0 on S_3. The main point we need is that the surfaces Σ_1, Σ_2 are submanifolds of $S_3\backslash F_3$ on where the $\mathbb{R}^3$–bundle $\Lambda^2_{+,m_0} \longrightarrow S_3\backslash F_3$ is naturally trivialized by the Kähler form ω together with $\mathrm{Re}\, \psi$ and $\mathrm{Im}\, \psi$, the real part and imaginary part of a non–zero section $\psi \in H^0(K_{S_3})$. This argument can easily be extended to those generic metric m_0' close to m_0. Indeed for such metrics similar trivializations of the bundle $\Lambda^2_{+,m_0'}$ can be found over $S_3\backslash U_{F_3}$, the complement of a small neighbourhood U_{F_3} of F_3 not meeting Σ_1, Σ_2. Thus one may draw the conclusion $V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m_0')$ are actually compact for such metrics and this ensures in particular the map $\tilde{q}_{2,S_3}(m_0')$ constructed in § 1 is indeed well–defined. For simplicity, we write in what follows $\omega_1, \omega_2, \omega_3$ for the self–dual harmonic forms ω, $\mathrm{Re}\, \psi$, $\mathrm{Im}\, \psi$ on S_3 respectively. Also we assume $\omega_1 \wedge \omega_2 \wedge \omega_3$ orients Λ^2_+ over $S_3\backslash F_3$.

To show $V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m_0)$ is compact, we observe first $M_1(m_0)$ is empty so that

$$\overline{M_2(m_0)}\backslash M_2(m_0) \subset S^2(S_3) \times \{[\theta]\} \ .$$

Thus a sequence $\{[A_i]\}$ of elements in $V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m_0)$ can possibly approach the lower stratum $\overline{M_2(m_0)}\backslash M_2(m_0)$ of $\overline{M_2(m_0)}$ only if there is some subsequence $\{[A_{i'}]\} \subset \{[A_i]\}$ such that either

(a) there are two (distinct) points x_1, x_2 on S_3 away from which $A_{i'} \longrightarrow \theta$ in C^∞, or

(b) there is a point $x_0 \in \Sigma_1 \cap \Sigma_2$ away from which $A_{i'} \longrightarrow \theta$ in C^∞.

Note that in case (a) both surfaces Σ_1, Σ_2 have to contain at least one of the two points x_1, x_2 should $[A_{i'}]$ be in $V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m_0)$ for large i'. We are to show for cases (a) and (b) the self–dual curvature $F_{+,m_0}(A_{i'})$ for $i' >> 0$ must not be zero. This however contradicts the anti–self–duality of $A_{i'}$ and enables us to conclude intersections $V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m_0)$ are compact. The impossibility of case (a) was first pointed out to the author by Donaldson using an "orientation argument". Here we make this idea precise and extend it to cover the less–understood case (b).

We begin with case (a) and consider $\{[A_{i'}]\}$ approaches some $(x_1,x_2;[\theta]) \in (S^2(S_3)\backslash\Delta_{S_3}) \times \{[\theta]\}$ away from the diagonal part $\Delta_{S_3} \times \{[\theta]\}$ for $i' >> 0$. As explained in [D2], a neighbourhood for such an $[A_{i'}]$ can be described in the following way. Let $\widetilde{su(2)}$ be the trivial bundle over S_3 with fibre $su(2)$, the Lie algebra of $SU(2)$. Denote by $\tilde{e}_s$, $s=1,2,3$, the constant sections of $\widetilde{su(2)}$ associated to an orthonormal oriented basis $e_1,\ e_2,\ e_3$ of $su(2)$. Let $\tilde{\mathbb{R}}^3$ be the vector bundle spanned by $\{\tilde{e}_s\}_{s=1}^3$ over $\mathbb{R}$. Define for $x_1,\ x_2$ a 16–dimensional manifold

$$N_{x_1,x_2} = \prod_{i=1}^{2} \left\{ (B(x_i,r) \subset S_3) \times (0,\epsilon) \times SO((\Lambda^2_+)_{y_i \in B(x_i,r)}, su(2)) \right\} ,$$

where $r,\ \epsilon$ are some small constants, and then fix a rule of assigning an element $n = \prod(y_i,\lambda_i,R_i) \in N_{x_1,x_2}$ a cutoff function β_n on S_3 supported away from $O(\sqrt{\lambda_i})$–neighbourhoods of y_i, $i = 1,2$. Finally let $\mathscr{H}^2_+(\simeq \mathbb{R}^3)$ denote the harmonic space of self–dual 2–forms on S_3 relative to the metric m_0. Then the "alternating method" developed in [D2] shows in this situation for $i' >> 0$ a neighbourhood of $[A_{i'}] \in M_2(m_0)$ is modelled on a quotient $\phi^{-1}_{x_1,x_2}(O)/SO(3)$, where ϕ_{x_1,x_2} is an SO(3)–equivariant map

$$\phi_{x_1,x_2} : N_{x_1,x_2} \longrightarrow \mathcal{H}^2_+ \otimes \beta_. \cdot \mathbb{R}^3$$

solving

$$F_+(\tau(n)) = \phi_{x_1,x_2}(n)$$

for some assignment τ sending an element $n \in N_{x_1,x_2}$ to some connection $\tau(n) = A^\infty(n)$ on $P \longrightarrow S_3$.

There is an approximation of the map ϕ_{x_1,x_2} in terms of self–dual harmonic forms on S_3. More precisely for any element $n = \prod(y_i,\lambda_i,R_i) \in N_{x_1,x_2}$, we identify $\phi_{x_1,x_2}(n) \in \mathcal{H}^2_+ \otimes \beta_. \cdot \mathbb{R}^3$ via L^2–projection with the vector

$$q(n) = \sum_{r,s=1}^{3} q_{rs}(n) \cdot \omega_r \otimes \tilde{e}_s \in \mathcal{H}^2_+ \otimes \mathbb{R}^3 .$$

For such elements we have

$$q_{rs}(n) = 8\pi^2 \sum_{i=1}^{2} \lambda_i^2 \langle R_i \omega_r(y_i), e_s \rangle_{su(2)} + O(\overline{\lambda}^3) , \tag{3.1}$$

where $\bar{\lambda} = \max\{\lambda_1,\lambda_2\}$. To see $F_+(A_{i'}) \neq 0$, it is enough to show $q \neq 0$ if $\bar{\lambda} << 1$ and for this suppose we are to consider two different cases separately as follows.

Assume first both x_1, x_2 lie on the surface Σ_1, Σ_2 disjoint from $F_3 = \{\omega_2 = \omega_3 = 0\}$. Rewrite (3.1) into the form

$$q_{rs}(n) = 8\pi^2\bar{\lambda}^2\langle \sum_{i=1}^{2} \frac{\lambda_i^2}{\bar{\lambda}^2} R_i\omega_r(y_i),e_s\rangle_{su(2)} + O(\bar{\lambda}^3)$$

and one sees $q \neq 0$ for small $\bar{\lambda} \neq 0$ if the norm of the vector

$$\underline{v} = \Big[\sum_{i=1}^{2} \frac{\lambda_i^2}{\bar{\lambda}^2} R_i\omega_1(y_i), \sum_{i=1}^{2} \frac{\lambda_i^2}{\bar{\lambda}^2} R_i\omega_2(y_i), \sum_{i=1}^{2} \frac{\lambda_i^2}{\bar{\lambda}^2} R_i\omega_2(y_i)\Big]$$

in $(su(2))^{\oplus 3}$ is definitely bounded away from zero. As $\underline{v}$ is invariant under the transformation $(\lambda_1,\lambda_2) \longrightarrow (t\lambda_1,t\lambda_2)$ for $t \neq 0$, it suffices to show $\underline{v} \neq \underline{0}$ on a compact piece $\{\bar{\lambda} = \bar{\lambda}_0\}$ for some constant $\bar{\lambda}_0 \neq 0$. In the case when, say, $\lambda_1 = 0$ so that $\lambda_2 = \bar{\lambda}_0$ is non–zero, one finds readily $\underline{v}$ is not the zero vector in $(su(2))^{\oplus 3}$. If neither λ_1 nor λ_2 is zero, we prove $\underline{v} \neq \underline{0}$ by an orientation argument as follows. Suppose on the contrary that the vector $\underline{v}$ is the origin of $(su(2))^{\oplus 3}$. Then we get two sets of <u>oriented</u> basis, namely,

$$\{\lambda_1^2 R_1\omega_r(y_1)\}_{r=1}^{3} \text{ and } \{\lambda_2^2 R_2\omega_r(y_2)\}_{r=1}^{3} ,$$

of $su(2)$ which are however related by an <u>orientation reversing</u> transformation

$-1 \in GL(3,\mathbb{R})$. This is clearly absurd and one concludes therefore $q \neq 0$ if $\bar{\lambda} << 1$ in this case.

Now we assume, say, $x_1 \in \Sigma_1 \cap \Sigma_2$ in which case the point x_2 need no longer be on Σ_1 or Σ_2 . The previous argument applies in this situation except possibly for the case when y_2 is a point of F_3 since then $\underline{y}$ takes the form

$$\left\{\sum_{i=1}^{2} \lambda_i^2 R_i \omega_1(y_1),\ \lambda_1^2 R_1 \omega_2(y_1),\ \lambda_1^2 R_1 \omega_3(y_1)\right\} \in (su(2))^{\oplus 3}$$

and the orientation argument breaks down if $\lambda_1 = 0$. In this case however we have $\bar{\lambda}_0 = \lambda_2$ and for small λ_1 , say, $0 \leq \lambda_1 \leq \frac{1}{2} \lambda_2$, the vector

$$\lambda_1^2 R_1 \omega_1(x_1) + \lambda_2^2 R_2 \omega_1(x_2) \in su(2)$$

is non–zero as ω_1 is the Kähler form associated to m_0 . One argues as before $\underline{y} \neq 0$ on $\{\bar{\lambda} = \bar{\lambda}_0 \neq 0\}$ and this completes the proof that intersections $V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m_0)$ stay away from $(x_1,x_2;[\theta]) \in (S^2(S_3)\backslash\Delta_{S_3}) \times \{[\theta]\}$ in case (a).

Now we consider case (b) and suppose $\{[A_{i'}]\}$ approaches some $(x_0,x_0;[\theta])$ in the diagonal part $\Delta_{S_3} \times \{[\theta]\} \subset \overline{M_2(m_0)}$ as $i' \longrightarrow \infty$. Thus for a large i' the connection $A_{i'}$ is close to the trivial connection θ on complements of small geodesic balls about the point $x_0 \in S_3\backslash F_3$. In this case we can define for $A_{i'}$ a "measure of concentration"

(3.2) $$\mu_{i'} = \min_{\rho} \left\{ \int_{B(x,\rho)} |F(A_{i'})|^2 = 16\pi^2 - \eta ;\ x \in S_3 \right\}$$

having the property that $\mu_{i'} \longrightarrow 0$ as $i' \longrightarrow \infty$. Here $\eta > 0$ is a small constant to be specified later. Pick an $x_{i'} \in S_3$ satisfying

$$\int_{B(x_{i'},\mu_{i'})} |F(A_{i'})|^2 = 16\pi^2 - \eta \ ;$$

the choice of which will not be important in our discussion. Now for a fixed small $\rho_0 > 0$ we can dilate neighbourhoods $B(x_{i'},\rho_0)$ of S_3 by a factor $1/\mu_{i'}$ and obtain thereby a sequence of connection $\{\hat{A}_{i'}\}$ defined on large balls of $T_{x_{i'}}S_3 \simeq \mathbb{R}^4$. By passing to subsequences and after gauge transformations, we may assume $\{\hat{A}_{i'}\}$ converges as $i' \longrightarrow \infty$ on compact subsets of $\mathbb{R}^4$ away from a finite point set $\hat{L}$ to an ASD connection $\hat{A}_\infty$, which extends to the whole of $S^4 = \mathbb{R}^4 \cup \{\infty\}$. There are three possibilities:

(3.3)
- (i) $\hat{L}$ is empty,
- (ii) $\hat{L}$ consists of a single point $z_0 \in B^4$, and
- (iii) $\hat{L}$ consists of two distinct points $z_1, z_2 \in B^4$, at least one of which lies on ∂B^4.

We shall show in the next section that $F_+(A_{i'}) \not\equiv 0$ for $i' >> 0$ in these three cases using rather technical arguments. The compactness of $V_{\Sigma_1} \cap V_{\Sigma_2} \cap M_2(m_0)$ will then follow. Note that the method we are going to discuss in fact can be used to describe a neighbourhood system for the diagonal part $S^2(S_3) \times \{[\theta]\}$ in $\overline{M_2(m_0)}$ but we need not go into full details of this to exclude all possibilities in (3.3).

Remark We conclude the whereabouts of the ponts z_0, z_1, z_2 in cases (ii) and (iii) by the fact that

$$\int_{B^4} |F(\hat{A}_{i'})|^2 = 16\pi^2 - \eta \ ,$$

a direct consequence of the way we define $\hat{A}_{i'}$. Such conclusions might sound unfamiliar at first sight but has in fact been considered in [D1]. Indeed, assuming $c_2(P) = 1$ in the above discussion, we can deduce $\{\hat{A}_{i'}\}$ converges to some connection on $\mathbb{R}^4$ for $i' >> 0$ since the only other possibility is that $\{\hat{A}_{i'}\}$ would converge to the trivial connection on $\mathbb{R}^4$ away from a point in $S^3 = \partial B^4$ which however would contradict the choices of center $x(A_{i'})$ for the connections $A_{i'}$. It is the point that we perform such kind of dilation for $A_{i'}$ right at the center $x(A_{i'})$ rather than the point x_0 to secure the convergence of $\hat{A}_{i'}$ in [D1]. In our present situation however, we are to consider some more possibilities as described in cases (ii) and (iii) in addition to the convergence case (i).

§ 4. An elaboration of the alternating method

We show in this section all the three possibilities in (3.3) cannot occur and our main tool is a variation of the alternating method developed in [D2]. Consider first in case (i) that the point set $\hat{L}$ is empty. In this situation one finds $[\hat{A}_\infty] \in M_2(S^4)$. Should we take η in (3.2) to be the small universal constant in the appendix of [D1] required to obtain the decay estimates for ASD conections on $\mathbb{R}^4$, the connections $A_{i'}$ can be put into a standard form over a conformal connected sum $S_3 \underset{\mu_{i'}}{\#} S^4_{x_{i'}}$ relative to the trivial connection θ on S_3 and $\hat{A}_\infty$ on $S^4_{x_{i'}} \simeq T_{x_{i'}};X \cup \{\infty\}$ (c.f. [D2]). Conversely it is possible to the apply the alternating construction to obtain all such ASD connections, parametrized by a quotient $\phi^{-1}_{x_0,\mu_0,\hat{A}_\infty}(O)/SO(3)$ for some SO(3)–equivariant map

$$\phi_{x_0,\mu_0,\hat{A}_\infty} : \{B(x_0,r) \subset S_3\backslash F_3\} \times (0,\mu_0) \times \mho_{\hat{A}_\infty} \times SO((\mathrm{ad}\, P_{\hat{A}_\infty+a})_{x \in B(x_0,r)}, su(2)\}$$

$$\longrightarrow \mathscr{H}^2_+ \otimes \beta.\cdot \mathbb{R}^3 .$$

Here r, μ_0 are small constants with μ_0 depends upon $\hat{A}_\infty$ while $\mho_{\hat{A}_\infty}$ is a slice in a small neighbourhood $U_{\hat{A}_\infty}$ of $[\hat{A}_\infty]$ in $M_2(\mathbb{R}^4)$ transversal to the conformal group action $\mathrm{conf}(\mathbb{R}^4)$. Furthermore, $\hat{A}_\infty+a$ is a connection on P with $[\hat{A}_\infty+a] \in \mho_{\hat{A}_\infty}$. To show $F_+(A_{i'}) \neq 0$ for large i' in this situation, we identify as before the image of $\phi_{x_0,\mu_0,\hat{A}_\infty}$ at a point $n = (x,\mu,\hat{A}_\infty+a)$ with some $q(n) \in \mathscr{H}^2_+ \oplus \mathbb{R}^3$ having components

$$(4.1) \qquad q_{rs}(n) = 8\pi^2\mu^2 \langle F_+(\sigma^*(\hat{A}_\infty+A)_x, \omega_r(x) \otimes e_s\rangle_{su(2)} + O(\mu^3) ,$$

where σ denotes the antipodal map on S^4. Note that $\sigma^*(\hat{A}_\infty+a)$ becomes self–dual as σ is orientation reversing on S^4. This time we wish to check $\langle F_+(\sigma^*(\hat{A}_\infty+a)_x, \omega_r(x)\otimes e_s\rangle_{su(2)}$ is not all zero for $r,s \in \{1,2,3\}$ but it is a consequence of the following lemma as $\{\omega_r(x)\}_{r=1}^3$ constitutes a frame for $(\Lambda_+^2)_x$ once $x \in B(x_0,r)$ is in the complement of $F_3 = \{\omega_2 = \omega_3 = 0\}$.

(4.2) Lemma The curvature field $F_+(\sigma^*A)$ is nowhere vanishing for any $[A] \in M_2(S^4)$.

The proof of this lemma, using the renowned ADHM construction, is rather diverging from the present discussion and so will be postponed to the appendix. Granted this result for the moment, one sees readily in (4.1) that $F_+(A_{i'}) \neq 0$ for $i' >> 0$ and thus the possibility of case (i) can be excluded.

Consider now in case (ii) $\{A_{i'}\}$ converges to some connection $\hat{A}_\infty$ on $\mathbb{R}^4$ away from a point $z_0 \in B^4$ when $i' \longrightarrow \infty$. Note that $[\hat{A}_\infty] \in M_1(S^4)$ in this case. We defined for each $A_{i'}$ with $i' >> 0$ a center $z_{i'} \in \mathbb{R}^4$ and a radius $\lambda_{i'}$ as in [D1]. By dilating small neighbourhoods of $z_{i'} \in \mathbb{R}^4$ in a usual way we can represent $A_{i'}$ is a standard form on the conformal model $S_3 \underset{\mu_{i'}}{\#} S^4_{x_{i'}} \underset{\lambda_{i'}}{\#} S^4_{z_{i'}}$ of S_3 relative to the ASD connections $\theta, \hat{A}_\infty, I$ on $S_3, S^4_{x_{i'}}, S^4_{z_{i'}}$ respectively. Here the two S^4–factors are joining in a <u>row</u> and I denotes the standard ASD connection on $S^4_{z_{i'}}$. (We have thus dilated a neighbourhood of $x_{i'} \in S_3$ <u>twice</u>.) This time to capture $A_{i'}$ for large i' we need a small variation of the alternating construction. Denote by $S_3 \coprod S^4_z$ the disjoint union of S_3 and S^4_z. We then work with the manifolds S^4_x and $S_3 \coprod S^4_z$ in the construction. As a first step of the iteration in the alternating construction, one defines a connection on $X \underset{\mu}{\#} S^4_x \underset{\lambda}{\#} S^4_z$ by cutting off $\theta, \hat{A}_\infty, I$ just as in [D2] when the alternating construction starts. The procedure of shifting error terms, a core feature of the construction, is applied separately to S^4_x and $S_3 \coprod S^4_z$ so that μ, λ can be treated as independent parameters. On may then find small constants μ_0, λ_0 depending on the (fixed) connections θ, $\hat{A}_\infty$, I so that the iteration proceeds indefinitely when $\mu < \mu_0$ and $\lambda < \lambda_0$. In this way we obtain SO(3)–equivariant maps

$$\phi_{x,\mu,z,\lambda,\hat{A}_\infty} : \{B(x,r) \times [0,\epsilon) \times SO((\operatorname{ad} P_{\hat{A}_\infty})_x, su(2))\}$$
$$\times \{B(z,r) \times [0,\epsilon) \times SO((\operatorname{ad} P_{\hat{A}_\infty})_z, (\Lambda^2_+)_z\} \longrightarrow \mathscr{H}^2_+ \otimes \beta.\cdot\tilde{\mathbb{R}}^3$$

with the property that some $\{\phi_{x,\mu,z,\lambda,\hat{A}_\infty} = 0\}/SO(3)$ contains $[A_{i'}]$ if i' is large. This time the projection image $q = \sum_{r,s} q_{rs}\omega_r \otimes \tilde{e}_s \in \mathscr{H}^2_+ \otimes \tilde{\mathbb{R}}^3$ of $\phi_{x,\mu,z,\lambda,\hat{A}_\infty}$ has components

$$q_{rs} = 8\pi^2\mu^2 \langle F_+(\sigma^*\hat{A}_{\infty,\lambda})_x, \ \ \omega_r(x)\otimes e_s\rangle_{su(2)} + O(\mu^3)$$

for some ASD connection $\hat{A}_{\infty,\lambda}$ on $\mathbb{R}^4$ approaching $\hat{A}_\infty$ as $\lambda \longrightarrow 0$. Since $x \notin F_3$, we observe that

$$\{\langle F_+(\sigma^*\hat{A}_{\infty,\lambda})_x, \omega_r(x)\otimes e_s\rangle_{su(2)}\}^3_{r,s=1} \in \mathbb{R}^9$$

is bounded away from the origin as in the limiting case $\lambda = 0$ the curvature field $F_+(\sigma^*\hat{A}_\infty)$ is also non–vanishing. One argues then $F_+(A_{i'}) \neq 0$ for $i' >> 0$ and this excludes the possibility of case (ii).

Now we come to the final case that $\hat{A}_{i'} \longrightarrow \theta$ on $\mathbb{R}^4$ as $i' \longrightarrow \infty$ away from two points z_1, z_2 on B^4 with one of which lies on $S^3 = \partial B^4$. In principle $A_{i'}$ can still be captured eventually and put into certain standard form on some suitable conformal model of S_3. However, we are not able to deduce $F_+(A_{i'}) \neq 0$ for large i' by the previous method for the following reason. The self–dual harmonic forms on $S_3 \underset{\mu}{\#} S^4_x$ have pointwise norm scaled down by a factor of $O(\mu^2)$ in the leading term approximating the ASD equation in this situation. As $\mu \longrightarrow 0$, so that the leading term goes down to zero, the previous argument breaks down and it fails to give $F_+(A_{i'}) \neq 0$ for $i' >> 0$ this time. To get around this difficulty, we observe that for $A_{i'}$ with i' large it is possible to define two (distinct) centers $x_1(A_{i'})$, $x_2(A_{i'})$ on S_3 and respectively two radii $\lambda_1(A_{i'})$, $\lambda_2(A_{i'})$ as in [D1]. Furthermore one finds for any given integer $N >> 0$ the centers $x_1(A_{i'})$, $x_2(A_{i'})$ stay at least $O(N\overline{\lambda}(A_{i'}))$ apart for large i', where $\overline{\lambda}(A_{i'})$ is the larger of the two radii $\lambda_1(A_{i'})$, $\lambda_2(A_{i'})$. Given such nice properties of $A_{i'}$ we may now argue $F_+(A_{i'}) \neq 0$ on S_3 for $i' >> 0$ as in § 3 by a more delicate calculation as follows. Note that there is a brief discussion concerning the existence of such connections on certain definite 4–manifold in [M2].

We begin with a technical lemma for $A_{i'}$ with $i' >> 0$. For such a connection the geodesic balls $B_{N\lambda_1(A_{i'})}(x_1(A_{i'}))$ and $B_{N\lambda_2(A_{i'})}(x_2(A_{i'}))$ can be assumed disjoint and we write $\tilde{S}_3 = S_3 \setminus \bigcup_{\alpha=1}^{2} B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))$. Let

$$(\tilde{S}_3)^\sim = S_3 \setminus \bigcup_{\alpha=1}^{2} B_{\frac{1}{2}N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'})) \ .$$

Then the restricted connection $A_{i'}\big|_{(\tilde{S}_3)^\sim}$ can be extended smoothly to some $(\tilde{A}_{i'})^\sim$ defined on the whole of S_3 in such a way that

$$F((\tilde{A}_{i'})^{\sim}) = 0 \quad \text{on} \quad \bigcup_{\alpha=1}^{2} B_{\frac{1}{4}N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'})) \ .$$

Furthermore, by taking N sufficiently large we obtain a uniform estimate

$$|F(\tilde{A}_{i'})^{\sim}| \leq \text{const.}\,|F(A_{i'})|$$

on the annuli $\bigcup_{\alpha=1}^{2} B_{\frac{1}{2}N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))\backslash B_{\frac{1}{4}N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))$ for all large i' ([D1] Lemma 20). Taking into account of the fact that

$$|F(A_{i'})| \leq \text{const.}\,\frac{1}{N^4\lambda_\alpha^2(A_{i'})} \quad \text{on} \quad B_{\frac{1}{2}N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))\backslash B_{\frac{1}{4}N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))$$

we conclude $\|F(\tilde{A}_{i'})^{\sim}\|_{L^2}$ can be arranged arbitrarily small and therefore $(\tilde{A}_{i'})^{\sim}$ is a connection on the trivial SU(2)–bundle over S_3.

(4.3) <u>Lemma</u> For sufficiently large i' we can find for $(\tilde{A}_{i'})^{\sim}$ a global gauge on which $(\tilde{A}_{i'})^{\sim} = d + (\tilde{a}_{i'})^{\sim}$, where $(\tilde{a}_{i'})^{\sim}$ is a smooth connection matrix satisfying

$$\|(\tilde{a}_{i'})^{\sim}\|_{C^k} \leq \text{const.}\ \|F((\tilde{A}_{i'})^{\sim})\|_{L^2}$$

on $\tilde{S}_3$ for all integer $k \geq 0$.

The proof of this lemma is much in the spirit of [FU] § 8 and [U1] § 3 and so we shall be brief. Recall that the centers $x_1(A_{i'})$, $x_2(A_{i'})$ for $A_{i'}$ are close to some $x_0 \in S_3$ and so we may find some fixed small geodesic ball $B(x_0,r_0)$ on S_3 so that $x_1(A_{i'})$, $x_2(A_{i'})$ are well inside $B(x_0,r_0)$ for $i' >> 0$. Now fix a suitable covering

$\{\mathcal{U}_\alpha\}$ of S_3 with the following properties.

(i) Each $\mathcal{U}_\alpha \simeq B^4$ are small geodesic balls on S_3.

(ii) $\mathcal{U}_1 = B(x_0, r_0)$.

(iii) $\cup \{\frac{1}{2}\mathcal{U}_\alpha\}$ covers S_3.

(iv) $\cup \{\frac{1}{2}\mathcal{U}_\alpha : \alpha \neq 1\}$ is disjoint from $B(x_0, \frac{1}{2}r_0)$.

(v) $B_{\frac{1}{2}N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'})) \subset B(x_0, \frac{1}{2}r_0)$ for $\alpha = 1,2$.

Note that the choice of such a covering is not crucial to this argument. Now we can find for each $\mathcal{U}_\alpha$ a (Coulomb) gauge on which $(\tilde{A}_{i'})^\sim = d + (\tilde{a}_{\alpha,i'})^\sim$ with

$$\|(\tilde{a}_{\alpha,i'})^\sim\|_{C^k} \leq \text{const.}\ \|F((\tilde{A}_{i'})^\sim)\|_{L^2}\ .$$

Moreover, since $\|d^*(F(d + (\tilde{a}_{\alpha,i'})^\sim)\|_{L^2}$ is finite, we can deduce the regularity

$$\|(\tilde{a}_{\alpha,i'})^\sim\|_{C^k} \leq \text{const.}\ \|F((\tilde{A}_{i'})^\sim)\|_{L^2}$$

on $\frac{1}{2}\mathcal{U}_\alpha \cap \tilde{S}_3$ for $(\tilde{a}_{\alpha,i'})^\sim$ just as in [FU] Proposition 8.3 as on $\tilde{S}_3$ the connection $(\tilde{A}_{i'})^\sim$ is ASD. Then we can construct a global gauge on S_3 for $(\tilde{A}_{i'})^\sim$ following [U1] § 3 so that $(\tilde{A}_{i'})^\sim = d + (\tilde{a}_{i'})^\sim$ on this gauge satisfying

$$\|(\tilde{a}_{i'})^\sim\|_{C^k} \leq \text{const.}\ \|F((\tilde{A}_{i'})^\sim)\|_{L^2}$$

on $\tilde{S}_3$. The lemma follows.

Now we define new connections

$$\tilde{A}_{i'} = d + \beta_{i'}(\tilde{a}_{i'})^{\sim}$$

on S_3, where each $\beta_{i'}$ is a smooth cutoff function supported away from $\bigcup_{\alpha=1}^{2} B_{\frac{1}{2}N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))$ and takes the constant value 1 over $\tilde{S}_3$. Note that $\tilde{A}_{i'}|\tilde{S}_3$ is gauge equivalent to $A_{i'}|\tilde{S}_3$. Moreover, we have

$$\begin{aligned} |F(\tilde{A}_{i'})| &\le \text{const.}\,\{|F(A_{i'})| + |d\beta_{i'}|\} \\ &\le \text{const.}\left\{\frac{1}{N^4\lambda_\alpha^2(A_{i'})} + \frac{1}{N\lambda_\alpha(A_{i'})}\right\} \\ &\le \text{const.}\,\frac{1}{N^4\lambda_\alpha^2(A_{i'})} \end{aligned}$$

on $B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))\backslash B_{\frac{1}{2}N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))$. As the connection matrix $\tilde{a}_{i'} = \beta_{i'}\cdot(\tilde{a}_{i'})^{\sim}$ is small in $C^0(S_3)$, we can put $\tilde{A}_{i'}$ into a Coulomb gauge so that $\tilde{A}_{i'} = d + a_{i'}$ on S_3 with $|a_{i'}|$ uniformly small (c.f. [U2]). Now we deduce better estimates

$$\|a_{i'}\|_{L^2}^2 \le \text{const.}\,\frac{\bar{\lambda}^2(A_{i'})}{N^2} \tag{4.4}$$

for such connection matrices $a_{i'}$ by using

$$\|a_{i'}\|_{L^2} \le \text{const.}\ \|a_{i'}\|_{L_1^{4/3}}$$
$$\le \text{const.}\ \{\|F_+(\tilde{A}_{i'})\|_{L^{4/3}} + \|(a_{i'} \wedge a_{i'})_+\|_{L^{4/3}}$$
$$\le \text{const.} \left\{ \frac{\lambda(A_{i'})}{N} + \|a_{i'}\|_{L^2} \cdot \|a_{i'}\|_{L^4} \right\}$$

and the fact that $\|a_{i'}\|_{L^4}$ is uniformly small if $i' >> 0$. As we shall see however, (4.4) cannot possibly hold for $i' >> 0$ if $A_{i'}$ is to be ASD. This contradiction will then rule out this possibility of case (iii) described in (3.3) and we will be home.

To see (4.4) cannot hold for i' large we observe first $F_+(d+a_{i'}) = 0$ on $\tilde{S}_3$ and hence that

$$\int_{\tilde{S}_3} \operatorname{Tr}(d^+ a_{i'} \wedge \omega_{rs}) = -\int_{\tilde{S}_3} \operatorname{Tr}((a_{i'} \wedge a_{i'})_+ \wedge \omega_{rs}) \tag{4.5}$$

for each harmonic elements $\omega_{rs} = \omega_r \otimes \tilde{e}_s$ of $H^2_\theta \hookrightarrow \Omega^2_+(\widetilde{su(2)})$ where $r,s = 1,2,3$. The left hand side of (4.5) induces a vector $u \in \mathbb{R}^9$ with components

$$\left\{ \int_{\tilde{S}_3} \operatorname{Tr}(d^+ a_{i'} \wedge \omega_{rs}) \right\}_{r,s=1}^{3}$$

and for our purpose it suffices to show that the norm $\|u\|_{\mathbb{R}^9}$ of u satisfies

$$\frac{1}{\lambda^2(A_{i'})} \|u\|_{\mathbb{R}^9} \ge \text{const.}\ (\log N) \tag{4.6}$$

so that (4.5) cannot hold for $N >> 0$ in view of (4.4). To establish (4.6) we shall show for certain transition function $\rho_{\alpha,i'}$ that

(4.7)
$$\int_{\tilde{S}_3} \mathrm{Tr}(d^+ a_{i'} \wedge \omega_{rs}) =$$
$$\sum_{\alpha=1}^{2} \int_{B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))\setminus B_{\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))} \mathrm{Tr}(F(I_{\lambda_\alpha(A_{i'})})\wedge\rho^{-1}_{\alpha,i'}\,\omega_{rs}\rho_{\alpha,i'})$$
$$+ O(\bar{\lambda}^2(A_{i'})) ,$$

where $I_{\lambda_\alpha(A_{i'})}$ is a rescaled standard ASD connection of radius $\lambda_\alpha(A_{i'})$ for $\alpha = 1,2$ while $B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))$ has orientation opposite to the usual one so that ω_{rs} becomes ASD. Assuming $B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))$ is a flat 4–ball for simplicity we can deduce then by a straightforward calculation that

(4.8)
$$\left|\int_{B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))\setminus B_{\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))} \mathrm{Tr}(F(I_{\lambda_\alpha(A_{i'})})\wedge\rho^{-1}_{\alpha,i'}\,\omega_{rs}\rho_{\alpha,i'})\right|$$
$$\geq \mathrm{const.}\ \lambda^2_\alpha(A_{i'}) \cdot \log N + O(\lambda^2_\alpha(A_{i'}))$$

and (4.6) will follow should we apply the orientation argument as in § 3 to the leading terms

$$\left\{\sum_{\alpha=1}^{2}\int_{B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))\setminus B_{\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))} \mathrm{Tr}(F(I_{\lambda_\alpha(A_{i'})})\wedge\rho_{\alpha,i'}^{-1}\omega_{rs}\rho_{\alpha,i'})\right\}_{r,s=1}^{3}$$

of (4.7) and take into account of (4.8) in addition.

To see (4.7) holds we observe on the annulus $B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))\setminus B_{\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))$ it is possible to construct for $A_{i'}$ a "transversal gauge" as discussed in [FU] § 9. Indeed, on the 3–sphere $S^3_{N\lambda_\alpha(A_{i'})} = \partial B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))$ we find a gauge for $A_{i'}$ on which the connection matrix $b^\alpha_{i'}$ satisfies

$$\begin{aligned}|b^\alpha_{i'}|_{C^0(S^3_{N\lambda_\alpha(A_{i'})})} &\le \text{const. } N\lambda_\alpha(A_{i'})\cdot|F(A_{i'})|_{C^0(S^3_{N\lambda_\alpha(A_{i'})})}\\ &\le \text{const. } \frac{1}{N^3\lambda_\alpha(A_{i'})}\end{aligned}$$

This gauge extends to one on the whole of $B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))\setminus B_{\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))$ in such a way that the (extended) connection matrix $b^\alpha_{i'}$ for $A_{i'}$ satisfies

$$\text{(4.9)}\qquad |b^\alpha_{i'}(y)| \le \text{const. }\left\{\frac{1}{N^3\lambda_\alpha(A_{i'})}+\int_{d(y,x_\alpha(A_{i'}))}^{N\lambda_\alpha(A_{i'})}\frac{\lambda^2_\alpha(A_{i'})}{r^4}\,dr\right\}$$

$$\le \text{const. }\left\{\frac{1}{N^3\lambda_\alpha(A_{i'})}+\frac{\lambda^2_\alpha(A_{i'})}{(d(y,x_\alpha(A_{i'})))^3}\right\}.$$

As estimate we shall need in a moment is that on $S^3_{\lambda_\alpha(A_{i'})} = \partial B_{\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))$

$$\left| \int_{S^3_{\lambda_\alpha(A_{i'})}} \mathrm{Tr}(b^\alpha_{i'}(y) \wedge \omega_{rs}) \right| \le \text{const.}\ \lambda^2_\alpha(A_{i'}) \tag{4.10}$$

which follows easily from (4.9) as one sees.

The connection matrices $a_{i'}$ and $b^\alpha_{i'}$ (for the trivial connection) on the annulus $B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'})) \setminus B_{\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))$ are related by

$$a_{i'} = -d\rho_{\alpha,i'} \cdot \rho^{-1}_{\alpha,i'} + \rho_{\alpha,i'} b^\alpha_{i'} \rho^{-1}_{\alpha,i'}$$

for some transition function $\rho_{\alpha,i'}$ satisfying

$$|d\rho_{\alpha,i'}| \le |a_{i'}| + |b^\alpha_{i'}| \le 2|b^\alpha_{i'}| \tag{4.11}$$

and so we have

$$\int_{\tilde{S}_3} \mathrm{Tr}(d^+ a_{i'} \wedge \omega_{rs}) = \int_{\partial\tilde{S}_3} \mathrm{Tr}(a_{i'} \wedge \omega_{rs}) \tag{4.12}$$

$$= \int_{\partial\tilde{S}_3} \mathrm{Tr}((\rho_{\alpha,i'} b^\alpha_{i'} \rho^{-1}_{\alpha,i'}) \wedge \omega_{rs}) + O(\lambda^2_\alpha(A_{i'}))$$

$$= \int_{\partial\tilde{S}_3} \mathrm{Tr}(b^\alpha_{i'} \wedge (\rho^{-1}_{\alpha,i'} \omega_{rs} \rho_{\alpha,i'})) + O(\lambda^2_\alpha(A_{i'}))$$

using the estimate

$$\left| \int_{\partial \tilde{S}_3} - d\rho_{\alpha,i'} \cdot \rho^{-1}_{\alpha,i'} \wedge \omega_{rs} \right| \le \text{const.} \frac{1}{N^3 \lambda_\alpha(A_{i'})} \text{vol}(\partial \tilde{S}_3) = 0(\lambda^2_\alpha(A_{i'})) \ .$$

Now we apply the Stokes' theorem on the annulus $B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))\backslash B_{\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))$ to get

(4.13)
$$- \int_{\partial \tilde{S}_3} \text{Tr}(b^\alpha_{i'} \wedge (\rho^{-1}_{\alpha,i'} \omega_{rs} \rho_{\alpha,i'}))$$

$$= \int_{B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))\backslash B_{\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))} \text{Tr}((d^- b^\alpha_{i'}) \wedge (\rho^{-1}_{\alpha,i'} \omega_{rs} \rho_{\alpha,i'}))$$

$$+ \int_{S^3_{\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))} \text{Tr}(b^\alpha_{i'} \wedge (\rho^{-1}_{\alpha,i'} \omega_{rs} \rho_{\alpha,i'})) + O(\lambda^2_\alpha(A_{i'}))$$

by (4.9), (4.11) and the fact that $d_\theta \omega_{rs} = 0$. By (4.9) we can estimate the boundary integral

$$\left| \int_{S^3_{\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))} \text{Tr}(b^\alpha_{i'} \wedge (\rho^{-1}_{\alpha,i'} \omega_{rs} \rho_{\alpha,i'})) \right| \le \text{const.}\ \lambda^2_\alpha(A_{i'})$$

and one finds then (4.12) and (4.13) combine to give

$$\int_{\partial \tilde{S}_3} \mathrm{Tr}(d^+ a_{i'} \wedge \omega_{rs})$$

$$= \int_{B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))\backslash B_{\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))} \mathrm{Tr}((d^- b^\alpha_{i'}) \wedge (\rho^{-1}_{\alpha,i'}\, \omega_{rs} \rho_{\alpha,i'}))$$

$$+ O(\lambda^2_\alpha(A_{i'}))$$

$$= \int_{B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))\backslash B_{\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))} \mathrm{Tr}(F_-(I_{\lambda_\alpha(A_{i'})}) \wedge (\rho^{-1}_{\alpha,i'}\, \omega_{rs} \rho_{\alpha,i'}))$$

$$- \int_{B_{N\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))\backslash B_{\lambda_\alpha(A_{i'})}(x_\alpha(A_{i'}))} \mathrm{Tr}((b^\alpha_{i'} \wedge b^\alpha_{i'})_- \wedge (\rho^{-1}_{\alpha,i'}\, \omega_{rs} \rho_{\alpha,i'}))$$

$$+ O(\lambda^2_\alpha(A_{i'}))$$

which is in essence (4.7) should one apply the estimate of $|b^\alpha_{i'}(y)|$ in (4.9) to this situation. This finishes the proof.

Appendix

Using the ADHM construction, we wish to show here that the curvature field $F = F(A)$ of an $SU(2) \simeq Sp(1)$ connection A on a quaternionic line bundle $E \longrightarrow S^4$ with $c_2(E) = -2$ is nowhere vanishing if A is self–dual. All such self–dual connections can be realized in the following way (c.f. [A]). Regard S^4 as the quaternion projective line $\mathbb{P}_1(\mathbb{H})$ with scalar multiplication on the right. Let C, D be two constant quaternionic 3×2 matrices and define

$$v(x,y) = Cx + Dy$$

for $(x,y) \in \mathbb{H}^{\oplus 2}$. Provided $v(x,y)$ has maximal rank for all $(x,y) \neq (0,0)$, the column vectors of $v(x,y)$ span a quaternionic plane in $\mathbb{H}^{\oplus 3}$ and hence its orthogonal complement $E_{(x,y)}$, as (x,y) varies, defines a quaternionic line bundle $E \longrightarrow S^4 \simeq \mathbb{P}_1(\mathbb{H})$ with $c_2(E) = -2$. We assume $v(x,y)$ always has maximal rank in what follows. By taking orthogonal projections $P_{(x,y)} : \mathbb{H}^{\oplus 3} \longrightarrow E_{(x,y)}$ for all $(x,y) \neq (0,0)$, we obtain an $Sp(1)$–connection on E in a standard way. The associated curvature field can be given in affine coordinates $(x,y) = (x,1)$ by

$$F = PC\, dx\, \rho^2 d\bar{x}\, C^* P$$

where $\rho^2 = v^* v$. In this setting $F_- = 0$ precisely when ρ^2 is real. Now if we pick an orthogonal gauge u of the bundle E, i.e.

$$u^* v = 0 \text{ and } u^* u = 1 \ ,$$

then the curvature F can be expressed in this gauge u by

$$F = u^* C\, dx\, \rho^{-2}\, d\bar{x}\, C^* u \ .$$

We are now to show that F is non–vanishing given ρ^2 is real.

The first thing we notice is that if $\rho^2 = v^* v$ is a real matrix, then it must be symmetric and positive over $\mathbb{R}$. In particular, $\rho^2 = M^T \cdot M$ for some real matrix M and hence we may write $\rho^{-2} = M^{-1} \cdot (M^{-1})^T$. Thus the curvature field can be written into the following special form:

$$\begin{aligned} F &= u^* C\, dx\, M^{-1} \cdot (M^{-1})^T d\bar{x}\, C^* u \\ &= (u^* C\, M^{-1}) dx \wedge d\bar{x}((M^{-1})^T C^* u \\ &= (u^* C\, M^{-1}) dx \wedge d\bar{x}(u^* C\, M^{-1})^* \\ &= (w_1, w_2) dx \wedge d\bar{x} \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 \end{bmatrix} \end{aligned}$$

where $(w_1, w_2) = u^* C\, M^{-1} \in \mathbb{H}^{\oplus 2}$. Note that $(w_1, w_2) \neq (0,0)$ since $u^* C$ is non–vanishing. To see this, we assume on the contrary that $u^* C$ vanishes at some $x_0 = (x_0, 1)$, i.e. $u^*(x_0) \cdot C = 0$. Then we would have

$$\begin{aligned} 0 &= u^* v \\ &= u^* \cdot (Cx + D) \\ &= u^*(x_0) \cdot D \quad \text{at} \quad x = x_0 \end{aligned}$$

which implies in particular that $u^*(x_0) \cdot v(x,y) = 0$ for all $(x,y) \neq (0,0)$. It would follow then the bundle E is spanned trivially by the vector $u(x_0) \in \mathbb{H}^{\oplus 3}$, a contradiction to the assumption that $c_2(E) = -2$.

Now, as required in the argument, we write curvature field F explicitly as follows:

$$\begin{aligned} F = -2\{&(dx^1 \wedge dx^2 + dx^3 \wedge dx^4)(w_1\, i\, \overline{w}_2 + w_2\, i\, \overline{w}_2) \\ &+ (dx^1 \wedge dx^3 + dx^4 \wedge dx^3)(w_1\, j\, \overline{w}_1 + w_2\, j\, \overline{w}_2) \\ &+ (dx^1 \wedge dx^4 + dx^2 \wedge dx^3)(w_1\, k\, w_1 + w_3\, k\, \overline{w}_3)\} \ . \end{aligned}$$

To finish the proof, suppose on the contrary that F vanishes at some x_0, i.e.

$$\begin{cases} w_1(x_0)\ i\ \overline{w_1(x_0)} + w_2(x_0)\ i\ \overline{w_2(x_0)} = 0 \\ w_1(x_0)\ j\ \overline{w_1(x_0)} + w_2(x_0)\ j\ \overline{w_2(x_0)} = 0 \\ w_1(x_0)\ k\ \overline{w_1(x_0)} + w_2(x_0)\ k\ \overline{w_2(x_0)} = 0 \ . \end{cases}$$

Clearly then we have $|w_1(x_0)|^2 = |w_2(x_0)|^2$ which is moreover non–zero since $(w_1, w_2) \neq (0,0)$. Using the fact $w_i(x_0)^{-1} = \overline{w_i(x_0)} / |w_i(x_0)|^2$ one obtains

$$\text{(a.1)} \qquad \begin{cases} w_1(x_0)\ i\ w_1(x_0)^{-1} = -w_2(x_0)\ i\ w_2(x_0)^{-1} \\ w_1(x_0)\ j\ w_1(x_0)^{-1} = -w_2(x_0)\ j\ w_2(x_0)^{-1} \\ w_1(x_0)\ k\ w_1(x_0)^{-1} = -w_2(x_0)\ k\ w_2(x_0)^{-1} \end{cases}$$

and in where we may assume $w_i(x_0) \in Sp(1)$ for $i = 1,2$ via normalizations $w_i(x_0) \longrightarrow w_i(x_0)/_{|w_i(x_0)|}$. By the fact that the adjoint representation of $Sp(1)$, which sends $q \in Sp(1)$ to

$$\pi(q) : \mathrm{Im}\,\mathbb{H} \longrightarrow \mathrm{Im}\,\mathbb{H}\ ; \quad v \longmapsto q\, v\, q^{-1} \ ,$$

is a homomorphism

$$\pi : Sp(1) \longrightarrow SO(3)$$

with images in SO(3) , we conclude each of the following sets of vectors

$$\{w_1(x_0)\, i\, w_1(x_0)^{-1},\ w_1(x_0)\, j\, w_1(x_0)^{-1},\ w_1(x_0)\, k\, w_1(x_0)^{-1}\}\ ,$$
$$\{w_2(x_0)\, i\, w_2(x_0)^{-1},\ w_2(x_0)\, j\, w_2(x_0)^{-1},\ w_2(x_0)\, k\, w_2(x_0)^{-1}\}$$

forms an oriented orthogonal basis for $\mathrm{Im}\,\mathbb{H}$. This however gives a contradiction to (a.1) by the orientation argument and thus lemma (4.1) follows.

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