ON SOME SYSTEMS OF DIFFERENCE EQUATIONS. Part 12.

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§2.0. Foreword.

All calculations below are made by hands.

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§12.1. Return from ν to (τ, μ) in the case $l = 2$.

According to the Theorem 1 (see Foreword in [12]),

(1) $$A_0^{\sim}(z;\nu) = S_0^{\sim} + z\sum_{i=0}^{1} \nu^{-i} V_0^{\sim *}(i)$$

with

(2) $$S_2^{\sim} = \begin{pmatrix} 1 & -8 & 32 & -88 & 192 & -360 & 608 & -952 \\ 0 & 1 & -8 & 32 & -88 & 192 & -360 & 608 \\ 0 & 0 & 1 & -8 & 32 & -88 & 192 & -360 \\ 0 & 0 & 0 & 1 & -8 & 32 & -88 & 192 \\ 0 & 0 & 0 & 0 & 1 & -8 & 32 & -88 \\ 0 & 0 & 0 & 0 & 0 & 1 & -8 & 32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

(3)

$$V_2^{\sim*}(0) = 8\begin{pmatrix} 176 & -249 & -364 & 545 & 280 & -431 & -76 & 119 \\ -119 & 176 & 227 & -364 & -169 & 280 & 45 & -76 \\ 76 & -119 & -128 & 227 & 92 & -169 & -24 & 45 \\ -45 & 76 & 61 & -128 & -43 & 92 & 11 & -24 \\ 24 & -45 & -20 & 61 & 16 & -43 & -4 & 11 \\ -11 & 24 & -1 & -20 & -5 & 16 & 1 & -4 \\ 4 & -11 & 8 & -1 & 4 & -5 & 0 & 1 \\ -1 & 4 & -7 & 8 & -7 & 4 & -1 & 0 \end{pmatrix},$$

(4)

$$V_2^{\sim*}(1) = 8\begin{pmatrix} 455 & -1020 & -113 & 1552 & -603 & -628 & 357 & 0 \\ -300 & 682 & 44 & -996 & 404 & 394 & -228 & 0 \\ 185 & -428 & -3 & 592 & -253 & -228 & 135 & 0 \\ -104 & 246 & -16 & -316 & 144 & 118 & -72 & 0 \\ 51 & -124 & 19 & 144 & -71 & -52 & 33 & 0 \\ -20 & 50 & -12 & -52 & 28 & 18 & -12 & 0 \\ 5 & -12 & 1 & 16 & -9 & -4 & 3 & 0 \\ 0 & -2 & 8 & -12 & 8 & -2 & 0 & 0 \end{pmatrix},$$

(5)

$$V_2^{\sim*}(2) = 8\begin{pmatrix} 400 & -1243 & 972 & 542 & -1028 & 357 & 0 & 0 \\ -259 & 808 & -642 & -332 & 653 & -228 & 0 & 0 \\ 156 & -489 & 396 & 186 & -384 & 135 & 0 & 0 \\ -85 & 268 & -222 & -92 & 203 & -72 & 0 & 0 \\ 40 & -127 & 108 & 38 & -92 & 33 & 0 & 0 \\ -15 & 48 & -42 & -12 & 33 & -12 & 0 & 0 \\ 4 & -13 & 12 & 2 & -8 & 3 & 0 & 0 \\ -1 & 4 & -6 & 4 & -1 & 0 & 0 & 0 \end{pmatrix},$$

(6)
$$V_2^{\sim*}(3) = 8\begin{pmatrix} 119 & -476 & 714 & -476 & 119 & 0 & 0 & 0 \\ -76 & 304 & -456 & 304 & -76 & 0 & 0 & 0 \\ 45 & -180 & 270 & -180 & 45 & 0 & 0 & 0 \\ -24 & 96 & -144 & 96 & -24 & 0 & 0 & 0 \\ 11 & -44 & 66 & -44 & 11 & 0 & 0 & 0 \\ -4 & 16 & -24 & 16 & -4 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In view of (3) in Part 4,

(7)
$$A_2^*(z;\nu) = \nu^7 T_{8,\nu)} A_0^{\sim}(z;\nu)(T_{8,\nu)})^{-1}.$$

We denote the element, which stands in the matrix $A_2^*(z;\nu)$ on the intersection of i-th row and k-th column, by $a^*_{2,i,k}(z;\nu)$, where $\{i,\, k\} \subset \{1, ..., 8\}$.

In view of (2) – (7),

(8) $$a^*_{2,1,1}(z;\nu) = \nu^7 + 8z(176\nu^7 + 455\nu^6 + 400\nu^5 + 119\nu^4) =$$

$$1409\nu^7 + 3640\nu^6 + 3200\nu^5 + 952\nu^4 +$$
$$(z-1)(1408\nu^7 + 3640\nu^6 + 3200\nu^5 + 952\nu^4),$$

(9) $$a^*_{2,1,2}(z;\nu) = -8\nu^6 + 8z(-249\nu^6 - 1020\nu^5 - 1243\nu^4 - 476\nu^3) =$$

$$-2000\nu^6 - 8160\nu^5 - 9944\nu^4 - 3808\nu^3 +$$
$$(z-1)(-1992\nu^6 - 8160\nu^5 - 9944\nu^4 - 3808\nu^3),$$

(10) $$a^*_{2,1,3}(z;\nu) = 32\nu^5 +$$

$$8z(-364\nu^5 - 113\nu^4 + 972\nu^3 + 714\nu^2) =$$
$$-2880\nu^5 - 904\nu^4 + 7776\nu^3 + 5712\nu^2 +$$
$$(z-1)(-2912\nu^5 - 904\nu^4 + 7776\nu^3 + 5712\nu^2),$$

(11) $$a^*_{2,1,4}(z;\nu) = -88\nu^4 +$$

$$8z(545\nu^4 + 1552\nu^3 + 542\nu^2 - 476\nu) =$$
$$4272\nu^4 + 12416\nu^3 + 4336\nu^2 - 3808\nu +$$
$$(z-1)(4360\nu^4 + 12416\nu^3 + 4336\nu^2 - 3808\nu),$$

(12) $$a^*_{2,1,5}(z;\nu) = 192\nu^3 + 8z(280\nu^3 - 603\nu^2 - 1028\nu + 119) =$$

$$2432\nu^3 - 4824\nu^2 - 8224\nu + 952 +$$
$$(z-1)(2240\nu^3 - 4824\nu^2 - 8224\nu + 952),$$

(13) $$a^*_{2,1,6}(z;\nu) = -360\nu^2 + 8z(-431\nu^2 - 628\nu + 357) =$$

$$-3808\nu^2 - 5024\nu + 2856 +$$
$$(z-1)(-3448\nu^2 - 5024\nu + 2856),$$

(14) $$a^*_{2,1,7}(z;\nu) = 608\nu + 8z(-76\nu + 357) =$$

$$2856 + (z-1)(-608\nu + 2856),$$

(15) $$a^*_{2,1,8}(z;\nu) = -952\nu + 8z(119) = (z-1)952,$$

(16) $$a^*_{2,2,1}(z;\nu) = -952\nu^8 - 2400\nu^7 - 2072\nu^6 - 608\nu^5 +$$

$$(z-1)(-952\nu^8 - 2400\nu^7 - 2072\nu^6 - 608\nu^5),$$

(17) $$a^*_{2,2,2}(z;\nu) = \nu^7 + 8z(176\nu^7 + 682\nu^6 + 808\nu^5 + 304\nu^4) =$$

$$1409\nu^7+5456\nu^6+6464\nu^5+2432\nu^4+$$
$$(z-1)(1408\nu^7+5456\nu^6+6464\nu^5+2432\nu^4),$$

(18) $$a^*_{2,2,3}(z;\nu)=-8\nu^6+8z(227\nu^6+44\nu^5-642\nu^4-456\nu^3)=$$
$$1808\nu^6+352\nu^5-5136\nu^4-3648\nu^3+$$
$$(z-1)(1816\nu^6+352\nu^5-5136\nu^4-3648\nu^3),$$

(19) $$a^*_{2,2,4}(z;\nu)=32\nu^5+$$
$$8z(-364\nu^5-996\nu^4-332\nu^3+304\nu^2)=$$
$$-2880\nu^5-7968\nu^4-2656\nu^3+2432\nu^2+$$
$$(z-1)(-2912\nu^5-7968\nu^4-2656\nu^3+2432\nu^2),$$

(20) $$a^*_{2,2,5}(z;\nu)=-88\nu^4+8z(-169\nu^4+404\nu^3+653\nu^2-76\nu)=$$
$$-1440\nu^4+3232\nu^3+5224\nu^2-608\nu+$$
$$(z-1)(-1352\nu^4+3232\nu^3+5224\nu^2-608\nu),$$

(21) $$a^*_{2,2,6}(z;\nu)=192\nu^3+8z(280\nu^3+394\nu^2-228\nu)=$$
$$2432\nu^3+3152\nu^2-1824\nu+$$
$$(z-1)(2240\nu^3+3152\nu^2-1824\nu),$$

(22) $$a^*_{2,2,7}(z;\nu)=-360\nu^2+8z(45\nu^2-228\nu)=$$
$$-1824\nu+(z-1)(360\nu^2-1824\nu),$$

(23) $$a^*_{2,2,8}(z;\nu)=608\nu+8z(-76\nu)=-(z-1)608\nu,$$

(24) $$a^*_{2,3,1}(z;\nu)=608\nu^9+1480\nu^8+1248\nu^7+360\nu^6+$$
$$(z-1)(608\nu^9+1480\nu^8+1248\nu^7+360\nu^6),$$

(25) $$a^*_{2,3,2}(z;\nu)=-952\nu^8-3424\nu^7-3912\nu^6-1440\nu^5+$$
$$(z-1)(-952\nu^8-3424\nu^7-3912\nu^6-1440\nu^5),$$

(26) $$a^*_{2,3,3}(z;\nu)=\nu^7+8z(-128\nu^7-3\nu^6+396\nu^5+270\nu^4)=$$
$$-1023\nu^7-24\nu^6+3168\nu^5+2160\nu^4+$$
$$(z-1)(-1024\nu^7-24\nu^6+3168\nu^5+2160\nu^4),$$

(27) $$a^*_{2,3,4}(z;\nu)=-8\nu^6+8z(227\nu^6+592\nu^5+186\nu^4-180\nu^3)=$$

$$1808\nu^6 + 4736\nu^5 + 1488\nu^4 - 1440\nu^3 +$$
$$(z-1)(1816\nu^6 + 4736\nu^5 + 1488\nu^4 - 1440\nu^3),$$

(28) $$a^*_{2,3,5}(z;\nu) = 32\nu^5 + 8z(92\nu^5 - 253\nu^4 - 384\nu^3 + 45\nu^2) =$$

$$768\nu^5 - 2024\nu^4 - 3072\nu^3 + 360\nu^2 +$$
$$(z-1)(736\nu^5 - 2024\nu^4 - 3072\nu^3 + 360\nu^2),$$

(29) $$a^*_{2,3,6}(z;\nu) = -88\nu^4 + 8z(-169\nu^4 - 228\nu^3 + 135\nu^2) =$$

$$-1440\nu^4 - 1824\nu^3 + 1080\nu^2),$$
$$(z-1)(-1352\nu^4 - 1824\nu^3 + 1080\nu^2),$$

(30) $$a^*_{2,3,7}(z;\nu) = 192\nu^3 + 8z(-24\nu^3 + 135\nu^2) =$$

$$1080\nu^2 + (z-1)(-192\nu^3 + 1080\nu^2),$$

(31) $$a^*_{2,3,8}(z;\nu) = -360\nu^2 + 8z(45\nu^2) = (z-1)360\nu^2,$$

(32) $$a^*_{2,4,1}(z;\nu) = -360\nu^{10} - 832\nu^9 - 680\nu^8 - 192\nu^7 +$$

$$(z-1)(-360\nu^{10} - 832\nu^9 - 680\nu^8 - 192\nu^7),$$

(33) $$a^*_{2,4,2}(z;\nu) = 608\nu^9 + 1968\nu^8 + 2144\nu^7 + 768\nu^6 +$$

$$(z-1)(608\nu^9 + 1968\nu^8 + 2144\nu^7 + 768\nu^6),$$

(34) $$a^*_{2,4,3}(z;\nu) = 488\nu^8 - 128\nu^7 - 1776\nu^6 - 1152\nu^5 +$$

$$(z-1)(488\nu^8 - 128\nu^7 - 1776\nu^6 - 1152\nu^5),$$

(35) $$a^*_{2,4,4}(z;\nu) = -1023\nu^7 - 2528\nu^6 - 736\nu^5 + 768\nu^4 +$$

$$(z-1)(-1024\nu^7 - 2528\nu^6 - 736\nu^5 + 768\nu^4),$$

(36) $$a^*_{2,4,5}(z;\nu) = -352\nu^6 + 1152\nu^5 + 1624\nu^4 - 192\nu^3 +$$

$$(z-1)(-344\nu^6 + 1152\nu^5 + 1624\nu^4 - 192\nu^3),$$

(37) $$a^*_{2,4,6}(z;\nu) = 768\nu^5 + 944\nu^4 - 576\nu^3 +$$

$$(z-1)(736\nu^5 + 944\nu^4 - 576\nu^3),$$

(38) $$a^*_{2,4,7}(z;\nu) = -576\nu^3 +$$

$$(z-1)(88\nu^4 - 576\nu^3),$$

(39) $$a^*_{2,4,8}(z;\nu) = (z-1)(-192\nu^3),$$

(40) $$a^*_{2,5,1}(z;\nu) = 192\nu^1 1 + 408\nu^{10} + 320\nu^9 + 88\nu^8 +$$
$$(z-1)(192\nu^{11} + 408\nu^{10} + 320\nu^9 + 88\nu^8),$$

(41) $$a^*_{2,5,2}(z;\nu) = -360\nu^{10} - 992\nu^9 - 1016\nu^8 - 352\nu^7 +$$
$$(z-1)(-360\nu^{10} - 992\nu^9 - 1016\nu^8 - 352\nu^7),$$

(42) $$a^*_{2,5,3}(z;\nu) = -160\nu^9 + 152\nu^8 + 864\nu^7 + 528\nu^6 +$$
$$(z-1)(-160\nu^9 + 152\nu^8 + 864\nu^7 + 528\nu^6),$$

(43) $$a^*_{2,5,4}(z;\nu) = 488\nu^8 + 1152\nu^7 + 304\nu^6 - 352\nu^5 +$$
$$(z-1)(488\nu^8 + 1152\nu^7 + 304\nu^6 - 352\nu^5),$$

(44) $$a^*_{2,5,5}(z;\nu) = 129\nu^7 - 568\nu^6 - 736\nu^5 + 88\nu^4 +$$
$$(z-1)(128\nu^7 - 568\nu^6 - 736\nu^5 + 88\nu^4),$$

(45) $$a^*_{2,5,6}(z;\nu) = -352\nu^6 - 416\nu^5 + 264\nu^4 +$$
$$(z-1)(-344\nu^6 - 416\nu^5 + 264\nu^4),$$

(46) $$a^*_{2,5,7}(z;\nu) = 264\nu^4 +$$
$$(z-1)(-32\nu^5 + 264\nu^4),$$

(47) $$a^*_{2,5,8}(z;\nu) = 88(z-1)\nu^4,$$

(48) $$a^*_{2,6,1}(z;\nu) = -88\nu^{12} - 160\nu^{11} - 120\nu^{10} - 32\nu^9 +$$
$$(z-1)(-88\nu^{12} - 160\nu^{11} - 120\nu^{10} - 32\nu^9),$$

(49) $$a^*_{2,6,2}(z;\nu) = 192\nu^{11} + 400\nu^{10} + 384\nu^9 + 128\nu^8 +$$
$$(z-1)(192\nu^{11} + 400\nu^{10} + 384\nu^9 + 128\nu^8),$$

(50) $$a^*_{2,6,3}(z;\nu) = -8\nu^{10} - 96\nu^9 - 336\nu^8 - 192\nu^7 +$$
$$(z-1)(-8\nu^{10} - 96\nu^9 - 336\nu^8 - 192\nu^7),$$

(51) $$a^*_{2,6,4}(z;\nu) = -160\nu^9 - 416\nu^8 - 96\nu^7 + 128\nu^6 +$$
$$(z-1)(-160\nu^9 - 416\nu^8 - 96\nu^7 + 128\nu^6),$$

$$a^*_{2,6,5}(z;\nu) = -40\nu^8 + 224\nu^7 + 264\nu^6 - 32\nu^5 + (z-1)(-40\nu^8 + 224\nu^7 + 264\nu^6 - 32\nu^5), \tag{52}$$

$$a^*_{2,6,6}(z;\nu) = 129\nu^7 + 144\nu^6 - 96\nu^5 + (z-1)(128\nu^7 + 144\nu^6 - 96\nu^5), \tag{53}$$

$$a^*_{2,6,7}(z;\nu) = -96\nu^5 + (z-1)(8\nu^6 - 96\nu^5), \tag{54}$$

$$a^*_{2,6,8}(z;\nu) = (z-1)(-32\nu^5), \tag{55}$$

$$a^*_{2,7,1}(z;\nu) = 32\nu^{13} + 40\nu^{12} + 32\nu^{11} + 8\nu^{10} + (z-1)(32\nu^{13} + 40\nu^{12} + 32\nu^{11} + 8\nu^{10}), \tag{56}$$

$$a^*_{2,7,2}(z;\nu) = -88\nu^{12} - 96\nu^{11} - 104\nu^{10} - 32\nu^9 + (z-1)(-88\nu^{12} - 96\nu^{11} - 104\nu^{10} - 32\nu^9), \tag{57}$$

$$a^*_{2,7,3}(z;\nu) = 64\nu^{11} + 8\nu^{10} + 96\nu^9 + 48\nu^8 + (z-1)(64\nu^{11} + 8\nu^{10} + 96\nu^9 + 48\nu^8), \tag{58}$$

$$a^*_{2,7,4}(z;\nu) = -8\nu^{10} + 128\nu^9 + 16\nu^8 - 32\nu^7 + (z-1)(-8\nu^{10} + 128\nu^9 + 16\nu^8 - 32\nu^7), \tag{59}$$

$$a^*_{2,7,5}(z;\nu) = 32\nu^9 - 72\nu^8 - 64\nu^7 + 8\nu^6 + (z-1)(32\nu^9 - 72\nu^8 - 64\nu^7 + 8\nu^6), \tag{60}$$

$$a^*_{2,7,6}(z;\nu) = -40\nu^8 - 32\nu^7 + 24\nu^6 + (z-1)(-40\nu^8 - 32\nu^7 + 24\nu^6), \tag{61}$$

$$a^*_{2,7,7}(z;\nu) = \nu^7 + 24\nu^6 + (z-1)(24\nu^6), \tag{62}$$

$$a^*_{2,7,8}(z;\nu) = (z-1)(8\nu^6), \tag{63}$$

$$a^*_{2,8,1}(z;\nu) = -8\nu^{14} - 8\nu^{12} + (z-1)(-8\nu^{14} - 8\nu^{12}), \tag{64}$$

$$a^*_{2,8,2}(z;\nu) = 32\nu^{13} - 16\nu^{12} + 32\nu^{11}) + (z-1)(32\nu^{13} - 16\nu^{12} + 32\nu^{11})), \tag{65}$$

(66) $$a^*_{2,8,3}(z;\nu) = -56\nu^{12} + 64\nu^{11} - 48\nu^{10} + (z-1)(-56\nu^{12} + 64\nu^{11} - 48\nu^{10}),$$

(67) $$a^*_{2,8,4}(z;\nu) = 64\nu^{11} - 96\nu^{10} + 32\nu^9 + (z-1)(64\nu^{11} - 96\nu^{10} + 32\nu^9),$$

(68) $$a^*_{2,8,5}(z;\nu) = -56\nu^{10} + 64\nu^9 - 8\nu^8 + (z-1)(-56\nu^{10} + 64\nu^9 - 8\nu^8),$$

(69) $$a^*_{2,8,6}(z;\nu) = 32\nu^9 - 16\nu^8 + (z-1)(32\nu^9 - 16\nu^8),$$

(70) $$a^*_{2,8,7}(z;\nu) = -8\nu^8 + (z-1)(-8\nu^8),$$

(71) $$a^*_{2,8,8}(z;\nu) = \nu^7,$$

Clearly,

$$\nu^n = P_n(\mu) + \tau Q_n(\mu),$$
$$P_{n+1}(\mu) = (-1/2)P_n(\mu) + (\mu + 1/4)Q_n(\mu),$$
$$Q_{n+1}(\mu) = P_n(\mu) - (1/2)Q_n(\mu).$$

where

$$\mu = \mu(\nu) = \nu(\nu+1), \tau = \tau(\nu) = \nu + 1/2,$$
$$n \in [0, \infty) \cap \mathbb{Z}, \{P_n(\mu), Q_n(\mu)\} \subset \mathbb{Q}[\mu],$$

(72) $$P_0(\mu) = 1,$$

(73) $$Q_0(\mu) = 0,$$

(74) $$P_1(\mu) = -1/2,$$

(75) $$Q_1(\mu) = 1,$$

(76) $$P_2(\mu) = \mu + 1/2,$$

(77) $$Q_2(\mu) = -1,$$

(78) $$P_3(\mu) = -3\mu/2 - 1/2$$

(79) $$Q_3(\mu) = \mu + 1,$$

$$P_4(\mu) = \mu^2 + 2\mu + 1/2, \tag{80}$$

$$Q_4(\mu) = -2\mu - 1, \tag{81}$$

$$P_5(\mu) = -5\mu^2/2 - 5\mu/2 - 1/2, \tag{82}$$

$$Q_5(\mu) = \mu^2 + 3\mu + 1, \tag{83}$$

$$P_6(\mu) = \mu^3 + 9\mu^2/2 + 3\mu + 1/2, \tag{84}$$

$$Q_6(\mu) = -3\mu^2 - 4\mu - 1, \tag{85}$$

$$P_7(\mu) = -7\mu^3/2 - 7\mu^2 - 7\mu/2 - 1/2, \tag{86}$$

$$Q_7(\mu) = \mu^3 + 6\mu^2 + 5\mu + 1, \tag{87}$$

$$P_8(\mu) = \mu^4 + 8\mu^3 + 10\mu^2 + 4\mu + 1/2, \tag{88}$$

$$Q_8(\mu) = -4\mu^3 - 10\mu^2 - 6\mu - 1, \tag{89}$$

$$P_9(\mu) = -9\mu^4/2 - 15\mu^3 - 27\mu^2/2 - 9\mu/2 - 1/2, \tag{90}$$

$$Q_9(\mu) = \mu^4 + 10\mu^3 + 15\mu^2 + 7\mu + 1, \tag{91}$$

$$P_{10}(\mu) = \mu^5 + 25\mu^4/2 + 25\mu^3 + 35\mu^2/2 + 5\mu + 1/2, \tag{92}$$

$$Q_{10}(\mu) = -5\mu^4 - 20\mu^3 - 21\mu^2 - 8\mu - 1, \tag{93}$$

$$P_{11}(\mu) = -11\mu^5/2 - 55\mu^4/2 - 77\mu^3/2 - \\ 22\mu^2 - 11\mu/2 - 1/2, \tag{94}$$

$$Q_{11}(\mu) = \mu^5 + 15\mu^4 + 35\mu^3 + 28\mu^2 + 9\mu + 1, \tag{95}$$

$$P_{12}(\mu) = \mu^6 + 18\mu^5 + 105\mu^4/2 + \\ 56\mu^3 + 27\mu^2 + 6\mu + 1/2, \tag{96}$$

$$Q_{12}(\mu) = -6\mu^5 - 35\mu^4 - 56\mu^3 - 36\mu^2 - 10\mu - 1, \tag{97}$$

$$P_{13}(\mu) = -13\mu^6/2 - 91\mu^5/2 - 91\mu^4 - \tag{98}$$

$$78\mu^3 - 65\mu^2/2 - 13\mu/2 - 1/2,$$

(99) $$Q_{13}(\mu) = \mu^6 + 21\mu^5 + 70\mu^4 + 84\mu^3 + 45\mu^2 + 11\mu + 1,$$

(100) $$P_{14}(\mu) = \mu^7 + 49\mu^6/2 + 98\mu^5 +$$
$$147\mu^4 + 105\mu^3 + 77\mu^2/2 + 7\mu + 1/2,$$

(101) $$Q_{14}(\mu) = -7\mu^6 - 56\mu^5 - 126\mu^4 -$$
$$120\mu^3 - 55\mu^2 - 12\mu - 1.$$

In view of (7) – (101), we can represent the element

$$a^*_{2,i,k}(z;\nu), \text{ where } \{i,\ k\} \subset \{1,\ ...,\ 8\}$$

in the form

(102) $$a^*_{2,i,k}(z;\nu) = a^{\vee\vee}_{2,i,k}(z;\nu) + \tau a^{\wedge\wedge}_{2,i,k}(z;\nu),$$

where

(103) $$a^{\vee\vee}_{2,i,k}(z;\nu) = \frac{1}{2}r^{\vee\vee}_{2,i,k}(\mu) + (z-1)\frac{1}{2}v^{\vee\vee}_{2,i,k}(\mu),$$

(104) $$a^{\wedge\wedge}_{2,i,k}(z;\nu) = r^{\wedge\wedge}_{2,i,k}(\mu) + (z-1)v^{\wedge\wedge}_{2,i,k}(\mu),$$

$$r^{\vee\vee}_{2,i,k}(\mu),\ v^{\vee\vee}_{2,i,k}(\mu),\ r^{\wedge\wedge}_{2,i,k}(\mu),\ v^{\wedge\wedge}_{2,i,k}(\mu)$$

belong to $\mathbb{Z}[\mu]$. We have

$$r^{\vee\vee}_{2,1,1}(\mu) = -17 - 215\mu - 1062\mu^2 - 2583\mu^3,$$

$$v^{\vee\vee}_{2,1,1}(\mu) = -16 - 208\mu - 1048\mu^2 - 2576\mu^3,$$

$$r^{\wedge\wedge}_{2,1,1}(\mu) = 17 + 181\mu + 734\mu^2 + 1409\mu^3,$$

$$v^{\wedge\wedge}_{2,1,1}(\mu) = 16 + 176\mu + 728\mu^2 + 1408\mu^3,$$

$$r^{\vee\vee}_{2,1,2}(\mu) = 24 + 448\mu + 2912\mu^2 - 4000\mu^3,$$

$$v^{\vee\vee}_{2,1,2}(\mu) = 32 + 496\mu + 2984\mu^2 - 3984\mu^3,$$

$$r^{\wedge\wedge}_{2,1,2}(\mu) = -24 - 400\mu - 2160\mu^2,$$

$$v^{\wedge\wedge}_{2,1,2}(\mu) = -32 - 432\mu - 2184\mu^2,$$

$$r^{\vee\vee}_{2,1,3}(\mu) = -88 - 1120\mu + 12592\mu^2,$$

$$v^{\vee\vee}_{2,1,3}(\mu) = -56 - 960\mu + 12752\mu^2,$$

$$r^{\wedge\wedge}_{2,1,3}(\mu) = 88 + 944\mu - 2880\mu^2,$$

$$v^{\wedge\wedge}_{2,1,3}(\mu) = 56 + 848\mu - 2912\mu^2,$$

$$r^{\vee\vee}_{2,1,4}(\mu) = -11488\mu + 8544\mu^2,$$

$$v^{\vee\vee}_{2,1,4}(\mu) = 88 - 11136\mu + 8720\mu^2,$$

$$r^{\wedge\wedge}_{2,1,4}(\mu) = 3872\mu,$$
$$v^{\wedge\wedge}_{2,1,4}(\mu) = -88 + 3696\mu,$$
$$r^{\vee\vee}_{2,1,5}(\mu) = 2872 - 16944\mu,$$
$$v^{\vee\vee}_{2,1,5}(\mu) = 3064 - 16368\mu,$$
$$r^{\wedge\wedge}_{2,1,5}(\mu) = -968 + 2432\mu,$$
$$v^{\wedge\wedge}_{2,1,5}(\mu) = -1160 + 2240\mu,$$
$$r^{\vee\vee}_{2,1,6}(\mu) = 6928 - 7616\mu,$$
$$v^{\vee\vee}_{2,1,6}(\mu) = 7288 - 6896\mu,$$
$$r^{\wedge\wedge}_{2,1,6}(\mu) = -1216,$$
$$v^{\wedge\wedge}_{2,1,6}(\mu) = -1576,$$
$$r^{\vee\vee}_{2,1,7}(\mu) = 5712,$$
$$v^{\vee\vee}_{2,1,7}(\mu) = 6320,$$
$$r^{\wedge\wedge}_{2,1,7}(\mu) = 0,$$
$$v^{\wedge\wedge}_{2,1,7}(\mu) = -608,$$
$$r^{\vee\vee}_{2,1,8}(\mu) = 0,$$
$$v^{\vee\vee}_{2,1,8}(\mu) = 1904,$$
$$r^{\wedge\wedge}_{2,1,8}(\mu) = 0,$$
$$v^{\wedge\wedge}_{2,1,8}(\mu) = 0,$$
$$v^{\vee\vee}_{2,2,1}(\mu) = -1 - 208\mu - 1048\mu^2 - 2576\mu^3 - 1904\mu^4,$$
$$r^{\wedge\wedge}_{2,2,1}(\mu) = 16 + 176\mu + 728\mu^2 + 1408\mu^3,$$
$$v^{\wedge\wedge}_{2,2,1}(\mu) = 16 + 176\mu + 728\mu^2 + 1408\mu^3,$$
$$r^{\vee\vee}_{2,2,2}(\mu) = 15 + 281\mu + 1922\mu^2 + 1049\mu^3,$$
$$v^{\vee\vee}_{2,2,2}(\mu) = 16 + 288\mu + 1936\mu^2 + 1056\mu^3,$$
$$r^{\wedge\wedge}_{2,2,2}(\mu) = -15 - 251\mu - 1450\mu^2 + 1409\mu^3,$$
$$v^{\wedge\wedge}_{2,2,2}(\mu) = -16 - 256\mu - 1456\mu^2 + 1408\mu^3,$$
$$r^{\vee\vee}_{2,2,3}(\mu) = -32 - 512\mu + 4240\mu^2 + 3616\mu^3,$$
$$v^{\vee\vee}_{2,2,3}(\mu) = -24 - 464\mu + 4312\mu^2 + 3632\mu^3,$$
$$r^{\wedge\wedge}_{2,2,3}(\mu) = 32 + 448\mu - 5072\mu^2,$$
$$v^{\wedge\wedge}_{2,2,3}(\mu) = 24 + 416\mu - 5096\mu^2,$$
$$r^{\vee\vee}_{2,2,4}(\mu) = -4640\mu - 1536\mu^2,$$
$$v^{\vee\vee}_{2,2,4}(\mu) = 32 - 4480\mu - 1376\mu^2,$$
$$r^{\wedge\wedge}_{2,2,4}(\mu) = 4640\mu - 2880\mu^2,$$
$$v^{\wedge\wedge}_{2,2,4}(\mu) = -32 + 4544\mu - 2912\mu^2,$$
$$r^{\vee\vee}_{2,2,5}(\mu) = 1160 - 5008\mu - 2880\mu^2,$$

$$v^{\vee\vee}_{2,2,5}(\mu) = 1248 - 4656\mu - 2704\mu^2,$$
$$r^{\wedge\wedge}_{2,2,5}(\mu) = -1160 + 6112\mu,$$
$$v^{\wedge\wedge}_{2,2,5}(\mu) = -1248 + 5936\mu,$$
$$r^{\vee\vee}_{2,2,6}(\mu) = 2544 - 992\mu,$$
$$v^{\vee\vee}_{2,2,6}(\mu) = 2736 - 416\mu,$$
$$r^{\wedge\wedge}_{2,2,6}(\mu) = -2544 + 2432\mu,$$
$$v^{\wedge\wedge}_{2,2,6}(\mu) = -2736 + 2240\mu,$$
$$r^{\vee\vee}_{2,2,7}(\mu) = 1824,$$
$$v^{\vee\vee}_{2,2,7}(\mu) = 2184 + 720\mu,$$
$$r^{\wedge\wedge}_{2,2,7}(\mu) = -1824,$$
$$v^{\wedge\wedge}_{2,2,7}(\mu) = -2184,$$
$$r^{\vee\vee}_{2,2,8}(\mu) = 0,$$
$$v^{\vee\vee}_{2,2,8}(\mu) = 608,$$
$$r^{\wedge\wedge}_{2,2,8}(\mu) = 0,$$
$$v^{\wedge\wedge}_{2,2,8}(\mu) = -608,$$
$$r^{\vee\vee}_{2,3,1}(\mu) = -16 - 208\mu - 1048\mu^2 - 2576\mu^3 - 2512\mu^4,$$
$$v^{\vee\vee}_{2,3,1}(\mu) = -16 - 208\mu - 1048\mu^2 - 2576\mu^3 - 2512\mu^4,$$
$$r^{\wedge\wedge}_{2,3,1}(\mu) = 16 + 176\mu + 728\mu^2 + 1408\mu^3 + 608\mu^4,$$
$$v^{\wedge\wedge}_{2,3,1}(\mu) = 16 + 176\mu + 728\mu^2 + 1408\mu^3 + 608\mu^4,$$
$$r^{\vee\vee}_{2,3,2}(\mu) = 80\mu + 888\mu^2 + 912\mu^3 - 1904\mu^4,$$
$$v^{\vee\vee}_{2,3,2}(\mu) = 80\mu + 888\mu^2 + 912\mu^3 - 1904\mu^4,$$
$$r^{\wedge\wedge}_{2,3,2}(\mu) = -80\mu - 728\mu^2 + 384\mu^3,$$
$$v^{\wedge\wedge}_{2,3,2}(\mu) = -80\mu - 728\mu^2 + 384\mu^3,$$
$$r^{\vee\vee}_{2,3,3}(\mu) = -1953 - 7959\mu - 1302\mu^2 - 7113\mu^3,$$
$$v^{\vee\vee}_{2,3,3}(\mu) = -1952\mu - 7952\mu^2 - 1288\mu^3 + 7120\mu^4,$$
$$r^{\wedge\wedge}_{2,3,3}(\mu) = 9 + 165\mu - 2898\mu^2 - 1023\mu^3,$$
$$v^{\wedge\wedge}_{2,3,3}(\mu) = 8 + 160\mu - 2904\mu^2 - 1024\mu^3,$$
$$r^{\vee\vee}_{2,3,4}(\mu) = -2688 - 7504\mu - 1072\mu^2 + 3616\mu^3,$$
$$v^{\vee\vee}_{2,3,4}(\mu) = -2680 - 7456\mu - 1000\mu^2 + 3632\mu^3,$$
$$r^{\wedge\wedge}_{2,3,4}(\mu) = 2688 + 2128\mu - 688\mu^2,$$
$$v^{\wedge\wedge}_{2,3,4}(\mu) = 2680 + 2096\mu - 712\mu^2,$$
$$r^{\vee\vee}_{2,3,5}(\mu) = 640 - 2000\mu - 7888\mu^2,$$
$$v^{\vee\vee}_{2,3,5}(\mu) = 672 - 1840\mu - 7728\mu^2,$$
$$r^{\wedge\wedge}_{2,3,5}(\mu) = -640 + 3280\mu - 768\mu^2,$$

$$v^{\wedge\wedge}_{2,3,5}(\mu) = -672 + 3184\mu + 736\mu^2,$$
$$r^{\vee\vee}_{2,3,6}(\mu) = 1464 + 1872\mu - 2880\mu^2,$$
$$v^{\vee\vee}_{2,3,6}(\mu) = 1552 + 2224\mu - 2704\mu^2,$$
$$r^{\wedge\wedge}_{2,3,6}(\mu) = -1464 + 1056\mu,$$
$$v^{\wedge\wedge}_{2,3,6}(\mu) = -1552 + 880\mu,$$
$$r^{\vee\vee}_{2,3,7}(\mu) = 1080 + 2160\mu,$$
$$v^{\vee\vee}_{2,3,7}(\mu) = 1272 + 2736\mu,$$
$$r^{\wedge\wedge}_{2,3,7}(\mu) = -1080,$$
$$v^{\wedge\wedge}_{2,3,7}(\mu) = -1272 - 192\mu,$$
$$r^{\vee\vee}_{2,3,8}(\mu) = 0,$$
$$v^{\vee\vee}_{2,3,8}(\mu) = 360 + 720\mu,$$
$$r^{\wedge\wedge}_{2,3,8}(\mu) = 0,$$
$$v^{\wedge\wedge}_{2,3,8}(\mu) = -360,$$
$$r^{\vee\vee}_{2,4,1}(\mu) = -16 - 208\mu - 1048\mu^2 - 2576\mu^3 - 2872\mu^4 - 720\mu^5,$$
$$v^{\vee\vee}_{2,4,1}(\mu) = -16 - 208\mu - 1048\mu^2 - 2576\mu^3 - 2872\mu^4 - 720\mu^5,$$
$$r^{\wedge\wedge}_{2,4,1}(\mu) = 16 + 176\mu + 728\mu^2 + 1408\mu^3 + 968\mu^4,$$
$$v^{\wedge\wedge}_{2,4,1}(\mu) = 16 + 176\mu + 728\mu^2 + 1408\mu^3 + 968\mu^4,$$
$$r^{\vee\vee}_{2,4,2}(\mu) = -16 - 128\mu - 160\mu^2 - 224\mu^3 - 1536\mu^4,$$
$$v^{\vee\vee}_{2,4,2}(\mu) = -16 - 128\mu - 160\mu^2 - 224\mu^3 - 1536\mu^4,$$
$$r^{\wedge\wedge}_{2,4,2}(\mu) = 16 + 96\mu + 352\mu^3 + 608\mu^4,$$
$$v^{\wedge\wedge}_{2,4,2}(\mu) = 16 + 96\mu + 352\mu^3 + 608\mu^4,$$
$$r^{\vee\vee}_{2,4,3}(\mu) = -8 - 96\mu + 1328\mu^2 + 5152\mu^3 + 976\mu^4,$$
$$v^{\vee\vee}_{2,4,3}(\mu) = -8 - 96\mu + 1328\mu^2 + 5152\mu^3 + 976\mu^4,$$
$$r^{\wedge\wedge}_{2,4,3}(\mu) = 8 + 80\mu - 1472\mu^2 - 2080\mu^3,$$
$$r^{\wedge\wedge}_{2,4,3}(\mu) = 8 + 80\mu - 1472\mu^2 - 2080\mu^3,$$
$$r^{\vee\vee}_{2,4,4}(\mu) = -1 - 1255\mu - 3214\mu^2 + 2105\mu^3,$$
$$v^{\vee\vee}_{2,4,4}(\mu) = -1248\mu - 3200\mu^2 + 2112\mu^3,$$
$$r^{\wedge\wedge}_{2,4,4}(\mu) = 1 + 1253\mu + 710\mu^2 - 1023\mu^3,$$
$$r^{\wedge\wedge}_{2,4,4}(\mu) = 1248\mu + 704\mu^2 - 1024\mu^3,$$
$$r^{\vee\vee}_{2,4,5}(\mu) = 312 - 800\mu - 5680\mu^2 - 704\mu^3,$$
$$v^{\vee\vee}_{2,4,5}(\mu) = 320 - 752\mu - 5608\mu^2 - 688\mu^3,$$
$$r^{\wedge\wedge}_{2,4,5}(\mu) = -312 + 1424\mu + 2208\mu^2,$$
$$v^{\wedge\wedge}_{2,4,5}(\mu) = -320 + 1392\mu + 2184\mu^2,$$

$$r^{\vee\vee}_{2,4,6}(\mu) = 752 + 1664\mu - 1952\mu^2,$$

$$r^{\vee\vee}_{2,4,6}(\mu) = 784 + 1824\mu - 1792\mu^2,$$

$$r^{\wedge\wedge}_{2,4,6}(\mu) = -752 - 160\mu + 768\mu^2,$$

$$v^{\wedge\wedge}_{2,4,6}(\mu) = -784 - 256\mu + 736\mu^2,$$

$$r^{\vee\vee}_{2,4,7}(\mu) = 576 + 1728\mu,$$

$$v^{\vee\vee}_{2,4,7}(\mu) = 664 + 2080\mu + 176\mu^2,$$

$$r^{\wedge\wedge}_{2,4,7}(\mu) = -576 - 576\mu,$$

$$v^{\wedge\wedge}_{2,4,7}(\mu) = -664 - 752\mu,$$

$$r^{\vee\vee}_{2,4,8}(\mu) = 0,$$

$$v^{\vee\vee}_{2,4,8}(\mu) = 192 + 576\mu,$$

$$r^{\wedge\wedge}_{2,4,8}(\mu) = 0,$$

$$v^{\wedge\wedge}_{2,4,8}(\mu) = -192 - 192\mu,$$

$$r^{\vee\vee}_{2,5,1}(\mu) = -16 - 208\mu - 1048\mu^2 - 2576\mu^3 - 3064\mu^4 - 1296\mu^5,$$

$$v^{\vee\vee}_{2,5,1}(\mu) = -16 - 208\mu - 1048\mu^2 - 2576\mu^3 - 3064\mu^4 - 1296\mu^5,$$

$$r^{\wedge\wedge}_{2,5,1}(\mu) = 16 + 176\mu + 728\mu^2 + 1408\mu^3 + 1160\mu^4 + 192\mu^5,$$

$$v^{\wedge\wedge}_{2,5,1}(\mu) = 16 + 176\mu + 728\mu^2 + 1408\mu^3 + 1160\mu^4 + 192\mu^5,$$

$$r^{\vee\vee}_{2,5,2}(\mu) = -32 - 336\mu - 1208\mu^2 - 2032\mu^3 - 2104\mu^4 - 720\mu^5,$$

$$v^{\vee\vee}_{2,5,2}(\mu) = -32 - 336\mu - 1208\mu^2 - 2032\mu^3 - 2104\mu^4 - 720\mu^5,$$

$$r^{\wedge\wedge}_{2,5,2}(\mu) = 32 + 272\mu + 728\mu^2 + 992\mu^3 + 808\mu^4,$$

$$v^{\wedge\wedge}_{2,5,2}(\mu) = 32 + 272\mu + 728\mu^2 + 992\mu^3 + 808\mu^4,$$

$$r^{\vee\vee}_{2,5,3}(\mu) = -24 - 224\mu + 16\mu^2 + 2240\mu^3 + 1744\mu^4,$$

$$v^{\vee\vee}_{2,5,3}(\mu) = -24 - 224\mu + 16\mu^2 + 2240\mu^3 + 1744\mu^4,$$

$$r^{\wedge\wedge}_{2,5,3}(\mu) = 24 + 176\mu - 320\mu^2 - 1344\mu^3 - 160\mu^4,$$

$$v^{\wedge\wedge}_{2,5,3}(\mu) = 24 + 176\mu - 320\mu^2 - 1344\mu^3 - 160\mu^4,$$

$$v^{\vee\vee}_{2,5,4}(\mu) = -8 - 576\mu - 1872\mu^2 + 352\mu^3 + 976\mu^4,$$

$$v^{\vee\vee}_{2,5,4}(\mu) = -8 - 576\mu - 1872\mu^2 + 352\mu^3 + 976\mu^4,$$

$$r^{\wedge\wedge}_{2,5,4}(\mu) = 8 + 560\mu + 768\mu^2 - 800\mu^3,$$

$$v^{\wedge\wedge}_{2,5,4}(\mu) = 8 + 560\mu + 768\mu^2 - 800\mu^3,$$

$$r^{\vee\vee}_{2,5,5}(\mu) = 127 - 279\mu - 3062\mu^2 - 2039\mu^3,$$

$$v^{\vee\vee}_{2,5,5}(\mu) = 128 - 272\mu - 3048\mu^2 - 2032\mu^3,$$

$$r^{\wedge\wedge}_{2,5,5}(\mu) = -127 + 533\mu + 1742\mu^2 + 129\mu^3,$$

$$v^{\wedge\wedge}_{2,5,5}(\mu) = -128 + 528\mu + 1736\mu^2 + 128\mu^3,$$

$$r^{\vee\vee}_{2,5,6}(\mu) = 328 + 1024\mu - 560\mu^2 - 704\mu^3,$$

$$v^{\vee\vee}_{2,5,6}(\mu) = 336 + 1072\mu - 488\mu^2 - 688\mu^3,$$
$$r^{\wedge\wedge}_{2,5,6}(\mu) = -328 - 368\mu + 640\mu^2,$$
$$v^{\wedge\wedge}_{2,5,6}(\mu) = -336 - 400\mu + 616\mu^2,$$
$$r^{\vee\vee}_{2,5,7}(\mu) = 264 + 1056\mu + 528\mu^2,$$
$$v^{\vee\vee}_{2,5,7}(\mu) = 296 + 1216\mu + 688\mu^2,$$
$$r^{\wedge\wedge}_{2,5,7}(\mu) = -264 - 528\mu,$$
$$v^{\wedge\wedge}_{2,5,7}(\mu) = -296 - 624\mu - 32\mu^2,$$
$$v^{\vee\vee}_{2,5,8}(\mu) = 0,$$
$$v^{\vee\vee}_{2,5,8}(\mu) = 88 + 352\mu + 176\mu^2,$$
$$r^{\wedge\wedge}_{2,5,8}(\mu) = 0,$$
$$v^{\wedge\wedge}_{2,5,8}(\mu) = -88 - 176\mu,$$
$$r^{\vee\vee}_{2,6,1}(\mu) = -16 - 208\mu - 1048\mu^2 - 2576\mu^3 - 3152\mu^4 - 1648\mu^5 - 176\mu^6,$$
$$v^{\vee\vee}_{2,6,1}(\mu) = -16 - 208\mu - 1048\mu^2 - 2576\mu^3 - 3152\mu^4 - 1648\mu^5 - 176\mu^6,$$
$$r^{\wedge\wedge}_{2,6,1}(\mu) = 16 + 176\mu + 728\mu^2 + 1408\mu^3 + 1248\mu^4 + 368\mu^5,$$
$$v^{\wedge\wedge}_{2,6,1}(\mu) = 16 + 176\mu + 728\mu^2 + 1408\mu^3 + 1248\mu^4 + 368\mu^5,$$
$$r^{\vee\vee}_{2,6,2}(\mu) = -48 - 544\mu - 2256\mu^2 - 4256\mu^3 - 3760\mu^4 - 1312\mu^5,$$
$$v^{\vee\vee}_{2,6,2}(\mu) = -48 - 544\mu - 2256\mu^2 - 4256\mu^3 - 3760\mu^4 - 1312\mu^5,$$
$$r^{\wedge\wedge}_{2,6,2}(\mu) = 48 + 448\mu + 1456\mu^2 + 2048\mu^3 + 1264\mu^4 + 192\mu^5,$$
$$v^{\wedge\wedge}_{2,6,2}(\mu) = 48 + 448\mu + 1456\mu^2 + 2048\mu^3 + 1264\mu^4 + 192\mu^5,$$
$$r^{\vee\vee}_{2,6,3}(\mu) = -56 - 560\mu - 1720\mu^2 - 1552\mu^3 - 8\mu^4 - 16\mu^5,$$
$$v^{\vee\vee}_{2,6,3}(\mu) = -56 - 560\mu - 1720\mu^2 - 1552\mu^3 - 8\mu^4 - 16\mu^5,$$
$$r^{\wedge\wedge}_{2,6,3}(\mu) = 56 + 448\mu + 936\mu^2 + 352\mu^3 - 56\mu^4,$$
$$v^{\wedge\wedge}_{2,6,3}(\mu) = 56 + 448\mu + 936\mu^2 + 352\mu^3 - 56\mu^4,$$
$$r^{\vee\vee}_{2,6,4}(\mu) = -32 - 448\mu - 1504\mu^2 - 928\mu^3 + 608\mu^4,$$
$$v^{\vee\vee}_{2,6,4}(\mu) = -32 - 448\mu - 1504\mu^2 - 928\mu^3 + 608\mu^4,$$
$$r^{\wedge\wedge}_{2,6,4}(\mu) = 32 + 384\mu + 800\mu^2 - 32\mu^3 - 160\mu^4,$$
$$v^{\wedge\wedge}_{2,6,4}(\mu) = 32 + 384\mu + 800\mu^2 - 32\mu^3 - 160\mu^4,$$
$$r^{\vee\vee}_{2,6,5}(\mu) = 32 - 144\mu - 1400\mu^2 - 1680\mu^3 - 80\mu^4,$$
$$v^{\vee\vee}_{2,6,5}(\mu) = 32 - 144\mu - 1400\mu^2 - 1680\mu^3 - 80\mu^4,$$
$$r^{\wedge\wedge}_{2,6,5}(\mu) = -32 + 208\mu + 920\mu^2 + 384\mu^3,$$
$$v^{\wedge\wedge}_{2,6,5}(\mu) = -32 + 208\mu + 920\mu^2 + 384\mu^3,$$
$$r^{\vee\vee}_{2,6,6}(\mu) = 111 + 441\mu - 30\mu^2 - 615\mu^3,$$
$$r^{\vee\vee}_{2,6,6}(\mu) = 112 + 448\mu - 16\mu^2 - 608\mu^3,$$

$$r^{\wedge\wedge}_{2,6,6}(\mu) = -111 - 219\mu + 246\mu^2 + 129\mu^3,$$
$$v^{\wedge\wedge}_{2,6,6}(\mu) = -112 - 224\mu + 240\mu^2 + 128\mu^3,$$
$$r^{\vee\vee}_{2,6,7}(\mu) = 96 + 480\mu + 480\mu^2,$$
$$v^{\vee\vee}_{2,6,7}(\mu) = 104 + 528\mu + 552\mu^2 + 16\mu^3,$$
$$r^{\wedge\wedge}_{2,6,7}(\mu) = -96 - 288\mu - 96\mu^2,$$
$$v^{\wedge\wedge}_{2,6,7}(\mu) = -104 - 320\mu - 120\mu^2,$$
$$r^{\vee\vee}_{2,6,8}(\mu) = 0,$$
$$v^{\vee\vee}_{2,6,8}(\mu) = 32 + 160\mu + 160\mu^2,$$
$$r^{\wedge\wedge}_{2,6,8}(\mu) = 0,$$
$$v^{\wedge\wedge}_{2,6,8}(\mu) = -32 - 96\mu - 32\mu^2,$$

$$r^{\vee\vee}_{2,7,1}(\mu) = -16 - 208\mu - 1048\mu^2 - 2576\mu^3 - 3184\mu^4 - 1808\mu^5 - 336\mu^6,$$
$$v^{\vee\vee}_{2,7,1}(\mu) = -16 - 208\mu - 1048\mu^2 - 2576\mu^3 - 3184\mu^4 - 1808\mu^5 - 336\mu^6,$$
$$r^{\wedge\wedge}_{2,7,1}(\mu) = 16 + 176\mu + 728\mu^2 + 1408\mu^3 + 1280\mu^4 + 464\mu^5 + 32\mu^6,$$
$$v^{\wedge\wedge}_{2,7,1}(\mu) = 16 + 176\mu + 728\mu^2 + 1408\mu^3 + 1280\mu^4 + 464\mu^5 + 32\mu^6,$$
$$r^{\vee\vee}_{2,7,2}(\mu) = -64 - 752\mu - 3304\mu^2 - 6704\mu^3 - 6272\mu^4 - 2320\mu^5 - 176\mu^6,$$
$$v^{\vee\vee}_{2,7,2}(\mu) = -64 - 752\mu - 3304\mu^2 - 6704\mu^3 - 6272\mu^4 - 2320\mu^5 - 176\mu^6,$$
$$r^{\wedge\wedge}_{2,7,2}(\mu) = 64 + 624\mu + 2184\mu^2 + 3328\mu^3 + 2128\mu^4 + 432\mu^5,$$
$$v^{\wedge\wedge}_{2,7,2}(\mu) = 64 + 624\mu + 2184\mu^2 + 3328\mu^3 + 2128\mu^4 + 432\mu^5,$$
$$r^{\vee\vee}_{2,7,3}(\mu) = -104 - 1104\mu - 4168\mu^2 - 6640\mu^3 - 4088\mu^4 - 688\mu^5,$$
$$v^{\vee\vee}_{2,7,3}(\mu) = -104 - 1104\mu - 4168\mu^2 - 6640\mu^3 - 4088\mu^4 - 688\mu^5,$$
$$r^{\wedge\wedge}_{2,7,3}(\mu) = 104 + 896\mu + 2584\mu^2 + 2848\mu^3 + 1016\mu^4 + 64\mu^5,$$
$$v^{\wedge\wedge}_{2,7,3}(\mu) = 104 + 896\mu + 2584\mu^2 + 2848\mu^3 + 1016\mu^4 + 64\mu^5,$$
$$r^{\vee\vee}_{2,7,4}(\mu) = -88 - 880\mu - 2968\mu^2 - 3760\mu^3 - 1320\mu^4 - 16\mu^5,$$
$$v^{\vee\vee}_{2,7,4}(\mu) = -88 - 880\mu - 2968\mu^2 - 3760\mu^3 - 1320\mu^4 - 16\mu^5,$$
$$r^{\wedge\wedge}_{2,7,4}(\mu) = 88 + 704\mu + 1736\mu^2 + 1344\mu^3 + 168\mu^4,$$
$$v^{\wedge\wedge}_{2,7,4}(\mu) = 88 + 704\mu + 1736\mu^2 + 1344\mu^3 + 168\mu^4,$$
$$r^{\vee\vee}_{2,7,5}(\mu) = -32 - 368\mu - 1336\mu^2 - 1648\mu^3 - 432\mu^4,$$
$$v^{\vee\vee}_{2,7,5}(\mu) = -32 - 368\mu - 1336\mu^2 - 1648\mu^3 - 432\mu^4,$$
$$r^{\wedge\wedge}_{2,7,5}(\mu) = 32 + 304\mu + 792\mu^2 + 544\mu^3 + 32\mu^4,$$
$$v^{\wedge\wedge}_{2,7,5}(\mu) = 32 + 304\mu + 792\mu^2 + 544\mu^3 + 32\mu^4,$$
$$r^{\vee\vee}_{2,7,6}(\mu) = 16 + 48\mu - 136\mu^2 - 368\mu^3 - 80\mu^4,$$
$$v^{\vee\vee}_{2,7,6}(\mu) = 16 + 48\mu - 136\mu^2 - 368\mu^3 - 80\mu^4,$$

$$r^{\wedge\wedge}_{2,7,6}(\mu) = -16 - 16\mu + 136\mu^2 + 128\mu^3,$$

$$r^{\wedge\wedge}_{2,7,6}(\mu) = -16 - 16\mu + 136\mu^2 + 128\mu^3,$$

$$r^{\vee\vee}_{2,7,7}(\mu) = 23 + 137\mu + 202\mu^2 + 41\mu^3,$$

$$v^{\vee\vee}_{2,7,7}(\mu) = 24 + 144\mu + 216\mu^2 + 48\mu^3,$$

$$r^{\wedge\wedge}_{2,7,7}(\mu) = -23 - 91\mu - 66\mu^2 + \mu^3,$$

$$v^{\wedge\wedge}_{2,7,7}(\mu) = -24 - 96\mu - 72\mu^2,$$

$$r^{\vee\vee}_{2,7,8}(\mu) = 0,$$

$$v^{\vee\vee}_{2,7,8}(\mu) = 8 + 48\mu + 72\mu^2 + 16\mu^3,$$

$$r^{\wedge\wedge}_{2,7,8}(\mu) = 0,$$

$$v^{\wedge\wedge}_{2,7,8}(\mu) = -8 - 32\mu - 24\mu^2,$$

$$r^{\vee\vee}_{2,8,1}(\mu) = -16 - 208\mu - 1048\mu^2 - 2576\mu^3 - 3192\mu^4 - 1856\mu^5 - 408\mu^6 - 16\mu^7, ,$$

$$v^{\vee\vee}_{2,8,1}(\mu) = -16 - 208\mu - 1048\mu^2 - 2576\mu^3 - 3192\mu^4 - 1856\mu^5 - 408\mu^6 - 16\mu^7,$$

$$r^{\wedge\wedge}_{2,8,1}(\mu) = 16 + 176\mu + 728\mu^2 + 1408\mu^3 + 1288\mu^4 + 496\mu^5 + 56\mu^6,$$

$$v^{\wedge\wedge}_{2,8,1}(\mu) = 16 + 176\mu + 728\mu^2 + 1408\mu^3 + 1288\mu^4 + 496\mu^5 + 56\mu^6,$$

$$r^{\vee\vee}_{2,8,2}(\mu) = -80 - 960\mu - 4352\mu^2 - 9248\mu^3 - 9264\mu^4 - 3840\mu^5 - 448\mu^6,$$

$$v^{\vee\vee}_{2,8,2}(\mu) = -80 - 960\mu - 4352\mu^2 - 9248\mu^3 - 9264\mu^4 - 3840\mu^5 - 448\mu^6,$$

$$r^{\wedge\wedge}_{2,8,2}(\mu) = 80 + 800\mu + 2912\mu^2 + 4704\mu^3 + 3280\mu^4 + 800\mu^5 + 32\mu^6,$$

$$v^{\wedge\wedge}_{2,8,2}(\mu) = 80 + 800\mu + 2912\mu^2 + 4704\mu^3 + 3280\mu^4 + 800\mu^5 + 32\mu^6,$$

$$r^{\vee\vee}_{2,8,3}(\mu) = -168 - 1856\mu - 7520\mu^2 - 13600\mu^3 - 10600\mu^4 - 2816\mu^5 - 112\mu^6 m,$$

$$r^{\vee\vee}_{2,8,3}(\mu) = -168 - 1856\mu - 7520\mu^2 - 13600\mu^3 - 10600\mu^4 - 2816\mu^5 - 112\mu^6 m,$$

$$r^{\wedge\wedge}_{2,8,3}(\mu) = 168 + 1520\mu + 4816\mu^2 + 6336\mu^3 + 3160\mu^4 + 400\mu^5,$$

$$v^{\wedge\wedge}_{2,8,3}(\mu) = 168 + 1520\mu + 4816\mu^2 + 6336\mu^3 + 3160\mu^4 + 400\mu^5,$$

$$r^{\vee\vee}_{2,8,4}(\mu) = -192 - 1952\mu - 7040\mu^2 - 10688\mu^3 - 6208\mu^4 - 896\mu^5,$$

$$v^{\vee\vee}_{2,8,4}(\mu) = -192 - 1952\mu - 7040\mu^2 - 10688\mu^3 - 6208\mu^4 - 896\mu^5,$$

$$r^{\wedge\wedge}_{2,8,4}(\mu) = 192 + 1568\mu + 4288\mu^2 + 4480\mu^3 + 1472\mu^4 + 64\mu^5,$$

$$v^{\wedge\wedge}_{2,8,4}(\mu) = 192 + 1568\mu + 4288\mu^2 + 4480\mu^3 + 1472\mu^4 + 64\mu^5,$$

$$r^{\vee\vee}_{2,8,5}(\mu) = v^{\vee\vee}_{2,8,5}(\mu) = -128 - 1200\mu - 3848\mu^2 - 4848\mu^3 - 1992\mu^4 - 112\mu^5,$$

$$r^{\wedge\wedge}_{2,8,5}(\mu) = v^{\wedge\wedge}_{2,8,5}(\mu) = 128 + 944\mu + 2216\mu^2 + 1792\mu^3 + 344\mu^4,$$

$$r^{\vee\vee}_{2,8,6}(\mu) = v^{\vee\vee}_{2,8,6}(\mu) = -48 - 416\mu - 1184\mu^2 - 1216\mu^3 - 320\mu^4,$$

$$r_{2,8,6}^{\wedge\wedge}(\mu)=v_{2,8,6}^{\wedge\wedge}(\mu)=48+320\mu+640\mu^2+384\mu^3+32\mu^4,$$
$$r_{2,8,7}^{\vee\vee}(\mu)=v_{2,8,7}^{\vee\vee}(\mu)=-8-64\mu-160\mu^2-128\mu^3-16\mu^4,$$
$$r_{2,8,7}^{\wedge\wedge}(\mu)=v_{2,8,7}^{\wedge\wedge}(\mu)=8+48\mu+80\mu^2+32\mu^3,$$

$$r_{2,8,8}^{\vee\vee}(\mu)=-1-7\mu-14\mu^2-7\mu^3,$$
$$v_{2,8,8}^{\vee\vee}(\mu)=0,$$
$$r_{2,8,8}^{\wedge\wedge}(\mu)=1+5\mu+6\mu^2+\mu^3,$$
$$v_{2,8,8}^{\wedge\wedge}(\mu)=0.$$

Therefore

$$A_2^*(z;\nu)=(1/2)U_2^{\vee\vee}(z;\mu)+\tau U_2^{\wedge\wedge}(z;\mu), \tag{105}$$

where

$$U_2^{\vee\vee}(z;\mu)=\left(\sum_{k=0}^{7}\mu^k R_2^{\vee\vee}(k)\right)+ (z-1)\left(\sum_{k=0}^{7}\mu^k V_2^{\vee\vee}(k)\right), \tag{106}$$

$$U_2^{\wedge\wedge}(z;\mu)=\left(\sum_{k=0}^{6}\mu^k R_2^{\wedge\wedge}(k)\right)+ (z-1)\left(\sum_{k=0}^{6}\mu^k V_2^{\wedge\wedge}(k)\right), \tag{107}$$

$$R_2^{\vee\vee}(0)=\begin{pmatrix}
-17 & 24 & -88 & 0 & 2872 & 6928 & 5712 & 0\\
-16 & 15 & -32 & 0 & 1160 & 2544 & 1824 & 0\\
-16 & 0 & -9 & 0 & 640 & 1464 & 1080 & 0\\
-16 & -16 & -8 & -1 & 312 & 752 & 576 & 0\\
-16 & -32 & -24 & -8 & 127 & 328 & 264 & \\
-16 & -48 & -56 & -32 & 32 & 111 & 96 & 0\\
-16 & -64 & -104 & -88 & -32 & 16 & 23 & 0\\
-16 & -80 & -168 & -192 & -128 & -48 & -8 & -1
\end{pmatrix},$$

$$R_2^{\vee\vee}(1)=\begin{pmatrix}
-215 & 448 & -1120 & -11488 & -16944 & -7616 & 0 & 0\\
-208 & 281 & -512 & -4640 & -5008 & -992 & 0 & 0\\
-208 & 80 & -183 & -2560 & -2000 & 1872 & 2160 & 0\\
-208 & -128 & -96 & -1255 & -800 & 1664 & 1728 & 0\\
-208 & -336 & -224 & -576 & -279 & 1024 & 1056 & 0\\
-208 & -544 & -560 & -448 & -144 & 441 & 480 & 0\\
-208 & -752 & -1104 & -880 & -368 & 48 & 137 & 0\\
-208 & -960 & -1856 & -1952 & -1200 & -416 & -64 & -7
\end{pmatrix},$$

$$R_2^{\vee\vee}(2)=$$

$$\begin{pmatrix}
-1062 & 2912 & 12592 & 8544 & 0 & 0 & 0 & 0\\
-1048 & 1922 & 4240 & -1536 & -2808 & 0 & 0 & 0\\
-1048 & 888 & 2586 & -4432 & -7888 & -2880 & 0 & 0\\
-1048 & -160 & 1328 & 3214 & -5680 & 1952 & 0 & 0\\
-1048 & 1208 & 16 & -1872 & -3062 & -560 & 528 & 0\\
-1048 & -2256 & -1720 & -1504 & -1400 & -30 & 480 & 0\\
-1048 & -3304 & -4168 & -2968 & -1336 & -136 & 202 & 0\\
-1048 & -4352 & -7520 & 7040 & -3848 & -1184 & -160 & -14
\end{pmatrix},$$

$$R_2^{\vee\vee}(3) =$$

$$\begin{pmatrix}
-2583 & -4000 & 0 & 0 & 0 & 0 & 0 & 0\\
-2576 & 1049 & 3616 & 0 & 0 & 0 & 0 & 0\\
-2576 & 912 & 7113 & 3616 & 0 & 0 & 0 & 0\\
-2576 & -224 & 5152 & 2105 & -704 & 0 & 0 & 0\\
-2576 & -2032 & 2240 & 352 & -2039 & -704 & 0 & 0\\
-2576 & -4256 & -1552 & -928 & -1680 & -615 & 0 & 0\\
-2576 & -6704 & -6640 & -3760 & -1648 & -368 & 41 & 0\\
-2576 & -9248 & -13600 & 10688 & -4848 & -1216 & -128 & -7
\end{pmatrix},$$

$$R_2^{\vee\vee}(4) = V_2^{\vee\vee}(4) =$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1904 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-2512 & -1904 & 0 & 0 & 00 & 0 & 0 & 0\\
-2872 & -1536 & 976 & 0 & 0 & 0 & 0 & 0\\
-3064 & -2104 & 1744 & 976 & 0 & 0 & 0 & 0\\
-3152 & -3760 & -8 & 608 & -80 & 0 & 0 & 0\\
-3184 & -6272 & -4088 & -1320 & -432 & -80 & 0 & 0\\
-3192 & -9264 & -10600 & -6208 & -1992 & -320 & -16 & 0
\end{pmatrix},$$

$$R_2^{\vee\vee}(5) = V_2^{\vee\vee}(5) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-720 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1296 & -720 & 0 & 0 & 0 & 0 & 0 & 0\\
-1648 & -1312 & -16 & 0 & 0 & 0 & 0 & 0\\
-1808 & -2320 & -688 & -16 & 0 & 0 & 0 & 0\\
-1856 & -3840 & -2816 & -896 & -112 & 0 & 0 & 0
\end{pmatrix},$$

$$R_2^{\vee\vee}(6) = V_2^{\vee\vee}(6) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-176 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-336 & -176 & 0 & 0 & 0 & 0 & 0 & 0\\
-408 & -448 & -112 & 0 & 0 & 0 & 0 &
\end{pmatrix},$$

$$R_2^{\vee\vee}(7)=V_2^{\vee\vee}(7)=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\-16&0&0&0&0&0&0&0\end{pmatrix},$$

$$V_2^{\vee\vee}(0)=\begin{pmatrix}-16&32&-56&88&3064&7288&6320&1904\\-16&16&-24&32&1248&2736&2184&608\\-16&0&-8&8&672&1552&1272&360\\-16&-16&-8&0&320&784&664&192\\-16&-32&-24&-8&128&336&296&88\\-16&-48&-56&-32&32&112&104&32\\-16&-64&-104&-88&-32&16&24&8\\-16&-80&-168&-192&-128&-48&-8&0\end{pmatrix},$$

$$V_2^{\vee\vee}(1)=\begin{pmatrix}-208&496&-960&-11136&-16368&-6896&0&0\\-208&288&-464&-4480&-4656&-416&720&0\\-208&80&-176&-2512&-1840&2224&2736&720\\-208&-128&-96&-1248&-752&1824&2080&576\\-208&-336&-224&-576&-272&1072&1216&352\\-208&-544&-560&-448&-144&448&528&160\\-208&-752&-1104&-880&-368&48&144&48\\-208&-960&-1856&-1952&-1200&-416&-64&0\end{pmatrix},$$

$$V_2^{\vee\vee}(2)=\begin{pmatrix}-1048&2984&12752&8720&0&0&0&0\\-1048&1936&4312&-1376&-2704&0&0&0\\-1048&888&2600&-4360&-7728&-2704&0&0\\-1048&-160&1328&-3200&-5608&-1792&176&0\\-1048&-1208&16&-1872&-3048&-488&688&176\\-1048&-2256&-1720&-1504&-1400&-16&552&160\\-1048&-3304&-4168&-2968&-1336&-136&216&72\\-1048&-4352&-7520&-7040&-3848&-1184&-160&0\end{pmatrix},$$

$$V_2^{\vee\vee}(3)=$$

$$\begin{pmatrix}-2576&-3984&0&0&0&0&0&0\\-2576&1056&3632&0&0&0&0&0\\-2576&912&7120&3632&0&0&0&0\\-2576&-224&5152&2112&-688&0&0&0\\-2576&-2032&2240&352&-2032&-688&0&0\\-2576&-4256&-1552&-928&-1680&-608&16&0\\-2576&-6704&-6640&-3760&-1648&-368&48&16\\-2576&-9248&-13600&-10688&-4848&-1216&-128&0\end{pmatrix},$$

$$R_2^{\wedge\wedge}(0)=\begin{pmatrix}17&-24&88&0&-968&-1216&0&0\\16&-15&32&0&-1160&-2544&-1824&0\\16&0&9&0&-640&-1464&-1080&0\\16&16&8&1&-312&-752&-576&0\\16&32&24&8&-127&-328&-264&0\\16&48&56&32&-32&-111&-96&0\\16&64&104&88&32&-16&-23&0\\16&80&168&192&128&48&8&1\end{pmatrix},$$

$$R_2^{\wedge\wedge}(1)=\begin{pmatrix}181&-400&944&3872&2432&0&0&0\\176&-251&448&4640&6112&2432&0&0\\176&-80&165&2560&3280&1056&0&0\\176&96&80&1253&1424&-160&-576&0\\176&272&176&560&533&-368&-528&0\\176&448&448&384&208&-219&-288&0\\176&624&896&704&304&-16&-91&0\\176&800&1520&1568&944&320&48&5\end{pmatrix},$$

$$R_2^{\wedge\wedge}(2)=\begin{pmatrix}734&-2160&-2880&0&0&0&0&0\\728&-1450&-5072&-2880&0&0&0&0\\728&-728&-2898&-688&768&0&0&0\\728&0&-1472&710&2208&768&0&0\\728&728&-320&768&1742&640&0&0\\728&1456&936&800&920&246&-96&0\\728&2184&2584&1736&792&136&-66&0\\728&2912&4816&4288&2216&640&80&6\end{pmatrix},$$

$$R_2^{\wedge\wedge}(3)=\begin{pmatrix}1409&0&0&0&0&0&0&0\\1408&1409&0&0&0&0&0&0\\1408&384&-1023&0&0&0&0&0\\1408&352&-2080&-1023&0&0&0&0\\+1408&992&-1344&-800&129&0&0&0\\1408&2048&352&-32&384&129&0&0\\1408&3328&2848&1344&544&128&1&0\\1408&4704&6336&4480&1792&384&32&1\end{pmatrix},$$

$$R_2^{\wedge\wedge}(4)=V_2^{\wedge\wedge}(4)=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\608&0&0&0&0&0&0&0\\968&608&0&0&0&0&0&0\\1160&808&-160&0&0&0&0&0\\1248&1264&-56&-160&0&0&0&0\\1280&2128&1016&168&32&0&0&0\\1288&3280&3160&1472&344&32&0&0\end{pmatrix},$$

$$R_2^{\wedge\wedge}(5)=V_2^{\wedge\wedge}(5)=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\192&0&0&0&0&0&0&0\\368&192&0&0&0&0&0&0\\464&432&64&0&0&0&0&0\\496&800&400&64&0&0&0&0\end{pmatrix},$$

$$R_2^{\wedge\wedge}(6)=V_2^{\wedge\wedge}(6)=\begin{pmatrix}0&0&0&0&0&0&0&0\\+0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\32&0&0&0&0&0&0&0\\56&32&0&0&0&0&0&0\end{pmatrix},$$

$$V_2^{\wedge\wedge}(0)=\begin{pmatrix}16&-32&56&-88&-1160&-1576&-608&0\\16&-16&24&-32&-1248&-2736&-2184&-608\\16&0&8&-8&-672&-1552&-1272&-360\\16&16&8&0&-320&-784&-664&-192\\16&32&24&8&-128&-336&-296&-88\\16&48&56&32&-32&-112&-104&-32\\16&64&104&88&32&-16&-24&-8\\16&80&168&192&128&48&8&0\end{pmatrix},$$

$$V_2^{\wedge\wedge}(1)=\begin{pmatrix}176&-432&848&3696&2240&0&0&0\\176&-256&416&4544&5936&2240&0&0\\176&-80&160&2528&3184&880&-192&0\\176&96&80&1248&1392&-256&-752&-192\\176&272&176&560&528&-400&-624&-176\\176&448&448&384&208&-224&-320&-96\\176&624&896&704&304&-16&-96&-32\\176&800&1520&1568&944&320&48&0\end{pmatrix},$$

$$V_2^{\wedge\wedge}(2)=\begin{pmatrix}+728&-2184&-2912&0&0&0&0&0\\728&-1456&-5096&-2912&0&0&0&0\\728&-728&-2904&712&736&0&0&0\\728&0&-1472&704&2184&736&0&0\\728&728&-320&768&1736&616&-32&0\\728&1456&936&800&920&240&-120&-32\\728&2184&2584&1736&792&136&-72&-24\\728&2912&4816&4288&2216&640&80&0\end{pmatrix},$$

$$V_2^{\wedge\wedge}(3) = \begin{pmatrix} 1408 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1408 & 1408 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1408 & 384 & -1024 & 0 & 0 & 0 & 0 & 0 \\ 1408 & 352 & -2080 & -1024 & 0 & 0 & 0 & 0 \\ 1408 & 992 & -1344 & -800 & 128 & 0 & 0 & 0 \\ 1408 & 2048 & 352 & -32 & 384 & 128 & 0 & 0 \\ 1408 & 3328 & 2848 & 1344 & 544 & 128 & 0 & 0 \\ 1408 & 4704 & 6336 & 4480 & 1792 & 384 & 32 & 0 \end{pmatrix}.$$

Moreover,

$$a^{\vee}_{2,i,j}(z;\nu) = a^{\vee\vee}_{2,i,j}(z;\nu),\ a^{\wedge}_{2,i,j}(z;\nu) = a^{\wedge\wedge}_{2,i,j}(\nu),$$

for $i = 1,\ ...,\ 8$ and $j = 1,\ ...,\ 8.$,

$$R_2^{\vee\vee}(k) = R_k^{\vee},\ V_2^{\vee\vee}(k) = V_2^{\vee}(k),$$

for $k = 0,\ ...,\ 7$,

$$R_k^{\wedge\wedge} = R_k^{\wedge},\ V_k^{\wedge\wedge} = V_k^{\wedge},$$

for $k = 0,\ ...,\ 6$, where

$$a^{\vee}_{2,i,j}(z;\nu), a^{\wedge}_{2,i,j}(z;\nu), \text{ for } \{\imath, j\} \subset \{1,\ ...,\ 8\},$$

$$R_2^{\vee}(k),\ V_2^{\vee}(k), \text{ for } k = 0,\ ...,\ 7,$$

and

$$R_2^{\wedge}(k),\ V_2^{\wedge}(k), \text{ for } k = 0,\ ...,\ 2$$

are pointed in the [6], in section 3.2 of [7] and in section 6.2 of [10]. Let

$$R_l^{*\vee}(\mu) = \sum_{k=0}^{3+2l} \mu^k R_2^{\vee}(k),$$

$$V_l^{*\vee}(\mu) = \sum_{k=0}^{3+2l} \mu^k V_2^{\vee}(k),$$

$$R_l^{*\wedge}(\mu) = \sum_{k=0}^{2+2l} \mu^k R_2^{\wedge}(k),$$

$$V_l^{*\wedge}(\mu) = \sum_{k=0}^{2+2l} \mu^k V_2^{\wedge}(k),$$

where $l = 0,\ 1,\ 2$. According to (32) and (33) of the [6],

(108) $$\nu^7 X^{\wedge}_{2,k}(z;\nu-1) = A_2^*(z;\nu) X_{2,k}(z;\nu)^{\wedge},$$

(109) $$(-\nu)^7 X^{\wedge}_{2,k}(z;\nu) = A_2^*(z;-\nu) X_{2,k}(z;\nu-1)^{\wedge},$$

where $k = 1,\ 2,\ 3,\ 5,\ 7 |z| > 1, \nu \in \mathbb{Z}$,

(110) $$A_2^*(z;\nu) = (1/2) U_2^{\vee}(z;\mu) + \tau U_2^{\wedge}(z;\mu),$$

where

(111) $$\mu = \nu(\nu+1),\ \tau = \nu + \frac{1}{2},$$

(112) $$U_l^{\vee}(z,\mu) = \sum_{k=0}^{3+2l} \mu^k (R_l^{\vee}(k) + (z-1)V_l^{\vee}(k)) =$$
$$R_l^{*\vee}(\mu) + (z-1)V_l^{*\vee}(\mu),$$

(113) $$U_l^{\wedge}(z,\mu) = \sum_{k=0}^{2+2l} \mu^k (R_l^{\wedge}(k)) + (z-1)V_l^{\wedge}(k)) =$$
$$R_l^{*\wedge}(\mu) + (z-1)V_l^{*\wedge}(\mu)$$

for $l = 0,\ 1,\ 2$. So, (105) coincides with (110).

§12.2. Properties of the considered systems as abstract systems. The case $l = 2,\ z = 1$.

As in §2 of [6], we let

(114) $$\mu = \mu(\nu) = \nu(\nu+1),\ \tau = \tau(\nu) = \nu + \frac{1}{2}.$$

Then, clearly,

(115) $$\mu(-\nu) = \mu(\nu-1) = \mu - 2\tau + 1,\ \tau(\nu-1) =$$
$$\tau - 1,\ \tau^2 = \mu + \frac{1}{4},$$

(116) $$\mu^k = \sum_{\kappa=0}^{k} \binom{k}{\kappa} \nu^{k+\kappa},$$

where $k \in \mathbb{N}_0$.

Further we have

(117) $$\mu(-\nu)^k = \mu(\nu-1)^k = P_k^*(\mu) + Q_k^*(\mu)\tau,$$

where
$$\{P_k^*(\mu),\ Q_k^*(\mu)\} \subset \mathbb{Q}[\mu],\ k \in \mathbb{N}_0.$$

Moreover,

(118) $$P_0^*(\mu) = 1,\ Q_0^*(\mu) = 0,\ P_1^*(\mu) =$$
$$\mu + 1,\ Q_1^*(\mu) = -2\ P_2^*(\mu) =$$
$$\mu^2 + 6\mu + 2,\ Q_2(\mu) = -4\mu - 4,\ P_3^*(\mu) =$$
$$\mu^3 + 15\mu^2 + 18\mu + 4,\ Q_3^*(\mu) = -6\mu^2 - 20\mu - 8,\ P_4(\mu) =$$
$$\mu^4 + 28\mu^3 + 76\mu^2 + 48\mu + 8,\ Q_4^*(\mu) =$$

$$-8\mu^3 - 56\mu^2 - 64\mu - 16,\ P_5(\mu)^* =$$
$$\mu^5 + 45\mu^4 + 220\mu^3 + 280\mu^2 + 120\mu + 16,\ Q_5^*(\mu) =$$
$$-10\mu^4 - 120\mu^3 - 272\mu^2 - 176\mu - 32,\ P_6^*(\mu) =$$
$$\mu^6 + 66\mu^5 + 510\mu^4 + 1104\mu^3 + 888\mu^2 + 288\mu + 32,\ Q_6^*(\mu) =$$
$$-12\mu^5 - 220\mu^4 - 832\mu^3 - 1008\mu^2 - 448\mu - 64,\ P_7^*(\mu) = \mu^7 +$$
$$91\mu^6 + 1022\mu^5 + 3388\mu^4 + 4424\mu^3 + 2576\mu^2 + 672\mu + 64,\ Q_7^*(\mu) =$$
$$-14\mu^6 - 364\mu^5 - 2072\mu^4 - 4048\mu^3 - 3232\mu^2 - 1088\mu - 128.$$

We considered in [6] – [12] the system, which has in the case $l = 2$ the form

(119) $$\nu^7 X(\nu - 1) = A_2^*(z;\nu)X(\nu)$$

with

(120) $$X(\nu) = X_{2,k}(z;\nu),$$

where $k \in \{1, 2, 3, 5, 7\}$ and $|z| > 1$. We consider now this system as abstract system. Let

(121) $$r_{2,1}(\nu) = \mu^4,\ r_{2,2}(\nu) = -4\mu^3,\ r_{2,3}(\nu) =$$
$$-4\mu^3 + 6\mu^2,\ r_{2,4}(\nu) = 12\mu^2 - 4\mu,\ r_{2,5}(\nu) =$$
$$6\mu^2 - 12\mu + 1,\ r_{2,6}(\nu) = -12\mu + 4,\ r_{2,7}(\nu) =$$
$$-4\mu + 6,\ r_{2,8}(\nu) = 4.$$

In view of (121),

(122) $$r_{2,k}(\nu - 1) = r_{2,k}^{\vee\vee *}(\nu) + r_{2,k}^{\vee\wedge *}(\nu)\tau,$$

where $k = 1, 2, 3, 4, 5, 6, 7, 8,$

(123) $$r_{2,1}^{\vee\vee *}(\nu) = P_4^*(\mu) =$$
$$\mu^4 + 28\mu^3 + 76\mu^2 + 48\mu + 8,\ r_{2,1}^{\vee\wedge *}(\nu) =$$
$$Q_4^*(\mu) = -8\mu^3 - 56\mu^2 - 64\mu - 16,\ r_{2,2}^{\vee\vee *}(\nu) = -4P_3^*(\mu) =$$
$$-4\mu^3 - 60\mu^2 - 72\mu - 16,\ r_{2,2}^{\vee\wedge *}(\nu) = -4Q_3^*(\mu) =$$
$$24\mu^2 + 80\mu + 32,\ r_{2,3}^{\vee\vee *}(\nu) = -4P_3^*(\mu) + 6P_2^*(\mu) =$$
$$-4\mu^3 - 54\mu^2 - 36\mu - 4,\ r_{2,3}(\nu)^{\vee\wedge *} = -4Q_3^*(\mu) + 6Q_2^*(\mu) =$$
$$24\mu^2 + 56\mu + 8,\ r_{2,4}(\nu)^{\vee\vee *} = 12P_2^*(\mu) - 4P_1^*(\mu) =$$
$$12\mu^2 + 68\mu + 20,\ r_{2,4}(\nu)^{\vee\wedge *} = 12Q_2^*(\mu) - 4Q_1^*(\mu) =$$
$$-48\mu - 40,\ r_{2,5}(\nu)^{\vee\vee *} = 6P_2^*(\mu) - 12P_1^*(\mu) + 1 =$$
$$6\mu^2 + 24\mu + 1,\ r_{2,5}(\nu)^{\vee\wedge *} = 6Q_2^*(\mu) - 12Q_1 =$$
$$-24\mu,\ r_{2,6}(\nu)^{\vee\vee *} = -12P_1^*(\mu) + 4 =$$
$$-12\mu - 8,\ r_{2,6}(\nu)^{\vee\wedge *} =$$

$$-12Q_1^*(\mu) = 24,\ r_{2,7}(\nu)^{\vee\vee*} = -4P_1^*(\mu) + 6 =$$
$$-4\mu + 2,\ r_{2,7}(\nu)^{\vee\wedge*} =$$
$$-4Q_1^*(\mu) = 8,\ r_{2,8}(\nu)^{\vee\vee*} = 4,\ r_{2,8}(\nu)^{\vee\wedge*} = 0.$$

Then

(124) $$r_2^{hh\vee*}(\nu) := \sum_{k=1}^{8} r_{2,k}(\nu)^{\vee\vee*} =$$
$$\mu^4 + 20\mu^3 - 20\mu^2 + 16\mu + 7,$$

(125) $$r_2^{hh\wedge*}(\nu) := \sum_{k=1}^{8} r_{2,k}(\nu)^{\vee\wedge*} =$$
$$-8\mu^3 - 8\mu^2 + 16.$$

Let $R_2(\nu)$, $R_2^{\vee\vee*}(\nu)$, $R_2^{\vee\wedge*}(\nu)$ are rows, consisting of 8 elements, k-th of which is equal respectively to

$$r_{2,k}(\nu),\ r_{2,k}(\nu)^{\vee\vee*},\ r_{2,k}(\nu)^{\vee\wedge*},$$

where $k = 1, 2, 3, 4, 5, 6, 7, 8$. Let

$$R_2^{\wedge*}(\nu) :=$$
$$R_2(\nu - 1)A_2^*(1;\nu) - \nu^7 R_2(\nu).$$

Let

$$R_2^{\wedge\vee*}(\nu) := R_2^{\vee\vee*}(\nu)A_2^{\vee}(1;\nu)+$$
$$(\mu + 1/4)R_2^{\vee\wedge*}(\nu)A_2^{\wedge}(1;\nu) - P_7(\mu)R_2(\nu),$$
$$R_2^{\wedge\wedge*}(1,\nu) := R_2^{\vee\vee*}(0,\nu)A_2^{\wedge}(1;\nu)+$$
$$R_2^{\vee\wedge}(0,\nu)A_{\alpha,0}^{\vee}(1;\nu) - Q_7(\mu)R_2(1,\nu).$$

Since $A_2^{\vee}(z;\nu) = (1/2)U_2^{\vee}(z;\nu)$, $A_2^{\wedge}(z;\nu) = U_2^{\wedge}(z;\nu)$ (see end of §22 in [6], it foloows from (112), (113), that

(126) $$A_2^{\vee}(1;\nu) = \frac{1}{2}R_2^{*\vee}(\mu),\ A_2^{\wedge}(1;\nu) = R_2^{*\wedge}(\mu).$$

Let $r_{2,k}^{\wedge\vee*}(\nu)$ and $r_{2,k}^{\wedge\wedge*}(\nu)$ denote $k-$th element of the rows respetively $R_2^{\wedge\vee*}(\nu)$ and $R_2^{\wedge\vee*}(\nu)$.

Let, as before, we denote the elements, which stand in the matrices $A_2^{\vee}(z;\nu)$ and $A_2^{\wedge}(z;\nu)$ on the intersetions of their i-th row and k-th column by respectively $a_{2,i,k}^{\vee}(1;\nu)$ and $a_{2,i,k}^{\wedge}(1;\nu)$, where $i, k = 1, 2, 3, 4, 5, 6, 7, 8$. Clearly,

(127) $$R_2(\nu - 1) = R_2^{\vee\vee*}(\nu) + R^{\vee\wedge*}(\nu)_2\tau.$$

In view of (86), (87)

(128) $$r_{2,k}^{\wedge\vee*}(\nu) =$$

$$\frac{1}{2}(7\mu^3+14\mu^2+7\mu+1)r_{2,k}(\nu)+$$

$$\sum_{i=1}^{8}(r^{\vee\vee*}_{2,i}(\nu)a^{\vee}_{2,i,k}(1;\nu)+r^{\vee\wedge*}_{2,i}(\nu)a^{\wedge}_{2,i,k}(1;\nu)(\mu+1/4),$$

(129) $$r^{\wedge\wedge}_{2,k}(\nu)=$$

$$-(\mu^3+6\mu^2+5\mu+1)r_{2,k}(\nu)+$$

$$\sum_{i=1}^{8}(r^{\vee\vee*}_{2,i}(\nu)a^{\wedge}_{2i,k}(1;\nu)+r^{\vee\wedge*}_{2,k}(\nu)a^{\vee}_{2,i,k}(1;\nu))$$

for $k=1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8.$

Lemma 12.2.1. . *The equality*

(130) $$r^{\wedge\vee}_{2,1}(\nu)=0$$

holds.

Proof. In view of (123), (124), (125), let

(131) $$r^{h\vee\vee*}_{2,1}(\nu):=r^{hh\vee*}_{2}(\nu)-r^{vee\vee*}_{2,1}(\nu)=$$

$$-8\mu^3-96\mu^2-32\mu-1,$$

(132) $$r^{h\vee\wedge*}_{2,1}(\nu):=r^{hh\wedge*}_{2}(\nu)-r^{\vee\wedge*}_{2,1}(\nu)=$$

$$48\mu^2+64\mu+32,$$

Let

(133) $$a^{\vee*}_{2,1}(\nu):=(-2576\mu^3-1048\mu^2-208\mu-16)/2,$$

(134) $$a^{\wedge*}_{2,1}(\nu):=(1408\mu^3+728\mu^2+176\mu+16)/4=$$

$$352\mu^3+182\mu^2+44\mu+4,$$

(135) $$a^{\vee\vee*}_{2,i,1}(\nu)=\mu^{-4}(a^{\vee}_{2,i,1}(\nu)-a^{\vee*}_{2,1}(\nu)),$$

(136) $$a^{\wedge\wedge*}_{2,i,1}(\nu)=\mu^{-4}(a^{\wedge}_{2,i,1}(\nu)-4a^{\wedge*}_{2,1}(\nu))/4,$$

where $i=1,\ ...,\ 8$. Then

(137) $$a^{\vee}_{2,i,1}(\nu)=a^{\vee*}_{2,1}(\nu)+\mu^4a^{\vee\vee*}_{2,i,1}(\nu),$$

(138) $$a^{\wedge}_{2,i,1}(\nu)=4a^{\wedge*}_{2,1}(\nu)+4\mu^4a^{\wedge\wedge*}_{2,i,1}(\nu),$$

where $i=2,\ ...,\ 8$, It follows from results of §1, that

(139) $$a^{\vee\vee*}_{2,2,1}(\nu)=-952,\ a^{\vee\vee*}_{2,3,1}(\nu)=-1256,\ a^{\vee\vee*}_{2,4,1}(\nu)=$$

$$-360\mu - 1436,\ a^{\vee\vee *}_{2,5,1}(\nu) =$$
$$-648\mu - 1532,\ a^{\vee\vee *}_{2,6,1}(\nu) = -88\mu^2 - 824\mu -$$
$$1576,\ a^{\vee\vee *}_{2,7,1}(\nu) = -168\mu^2 - 94\mu -$$
$$1592,\ a^{\vee\vee *}_{2,8,1}(\nu) = -8\mu^3 - 204\mu^2 - 928\mu -$$
$$1596,\ a^{\wedge\wedge *}_{2,2,1}(\nu) = 0,\ a^{\wedge\wedge *}_{2,3,1}(\nu) =$$
$$152,\ a^{\wedge\wedge *}_{2,4,1}(\nu) = 242,\ a^{\wedge\wedge *}_{2,5,1}(\nu) =$$
$$48\mu + 290,\ a^{\wedge\wedge *}_{2,6,1}(\nu) = 92\mu +$$
$$316,\ a^{\wedge\wedge *}_{2,7,1}(\nu) = 8\mu^2 + 116\mu +$$
$$1280,\ a^{\wedge\wedge *}_{2,8,1}(\nu) = 14\mu^2 + 124\mu + 322.$$

Let $k \in \mathbb{N}$ and $\mathfrak{M}_k$ denotes the main ideal $\mu^k\mathbb{Q}[\mu]$ in $\mathbb{Q}[\mu]$ with generator μ^k. It follows from (126), (133) – (139), and (131) – (132), that

(140)
$$r^{\wedge\vee}_{2,1}(\nu) = r^{h\vee\vee *}_{2,1}(\nu) a^{\vee *}_{2,1}(\nu) +$$
$$r^{a\wedge\vee}_{2,1}(\nu) +$$
$$r^{b\wedge\vee}_{2,1}(\nu) +$$
$$r_2(\nu)^{h\vee\wedge *} a^{\wedge *}_{2,1}(4\mu + 1) +$$
$$r^{c\wedge\vee}_{2,1}(\nu)(4\mu + 1) +$$
$$r^{d\wedge\vee}_{2,1}(\nu)(4\mu + 1),$$

where (see (86), (87), (123), (103), (104), (139)

(141)
$$r^{a\wedge\vee}_{2,1}(\nu) :=$$
$$-P_7(\nu) r_{2,1}(\nu) + r^{\vee\vee *}_{2,1}(\nu) a^{\vee}_{2,1,1}(1,\nu) =$$
$$\frac{1}{2}(7\mu^3 + 14\mu^2 + 7\mu + 1)\mu^4 +$$
$$(\mu^4 + 28\mu^3 + 76\mu^2 + 48\mu + 8)(-2583\mu^3 - 1062\mu^2 - 215\mu - 17)/2 =$$
$$-\mu^4(1288\mu^3 + 524\mu^2 + 104\mu + 8) +$$
$$2(7\mu^3 + 19\mu^2 + 12\mu + 2)(-2583\mu^3 - 1062\mu^2 - 215\mu - 17) \equiv$$
$$2(7\mu^3 + 19\mu^2 + 12\mu + 2)(-2583\mu^3 - 1062\mu^2 - 215\mu - 17) \equiv$$
$$-2(22114\mu^3 + 5027\mu^2 + 634\mu + 34 \ (\mod \mathfrak{m}_4),$$

(142)
$$r^{b\wedge\vee}_{2,1}(\nu) :=$$
$$\sum_{k=2}^{8} r^{\vee\vee *}_{2,k}(\nu) a^{\vee\vee *}_{2,k,1}(\nu)\mu^4 =$$
$$(-4\mu^3 - 60\mu^2 - 72\mu - 16)(-952)\mu^4 +$$
$$(-4\mu^3 - 54\mu^2 - 36\mu - 4)(-1256)\mu^4 +$$
$$(12\mu^2 + 68\mu + 20)(-360\mu - 1436)\mu^4 +$$

$$(6\mu^2+24\mu+1)(-648\mu-1532)\mu^4+$$
$$(-12\mu-8)(-88\mu^2-824\mu-1576)\mu^4+$$
$$(-4\mu+2)(-168\mu^2-904\mu-1592)\mu^4+$$
$$4(-8\mu^3-204\mu^2-928\mu-1596)\mu^4,$$

(143) $$r^{c\wedge\vee}_{2,1}(\nu) := r^{\vee\wedge *}_{2,1}(\nu)a^{\wedge}_{2,1,1}(1,\nu)/4 =$$

$$(-8\mu^3-56\mu^2-64\mu-16)(1409\mu^3+734\mu^2+181\mu+17)/4 =$$
$$-2((\mu^3+7\mu^2+8\mu+2)(1409\mu^3+734\mu^2+181\mu+17) \equiv$$
$$-2((17+1267+5872+2818)\mu^3+(119+1448+1468)\mu^2+(136+362)\mu+34) \equiv$$
$$-2(9974\mu^3+3035\mu^2+498\mu+34) \pmod{\mathfrak{m}_4}.$$

(144) $$r^{d\wedge\vee}_{2,1}(\nu) :=$$

$$\sum_{k=2}^{8} r^{\vee\wedge *}_{2,k}(\nu)a^{\wedge\wedge *}_{2,k,1}(\nu)\mu^4 =$$
$$(24\mu^2+56\mu+8)(152)\mu^4+$$
$$(-48\mu-40)(242)\mu^4+$$
$$(-24\mu)(48\mu+290)\mu^4+$$
$$24(92\mu+312)\mu^4+$$
$$8(8\mu^2+116\mu+320)\mu^4.$$

Coefficient at μ^n in the polynomial $r^{\wedge\vee}_{2,1}(\nu)$ is equal to 0, if $n > 7$. In view of (131) and (133)

(145) $$r^{h\vee\vee *}_{2,1}(\nu)a^{\vee *}_{2,1}(\nu) =$$

$$(-8\mu^3-96\mu^2-32\mu-1)(-1288\mu^3-524\mu^2-104\mu-8) =$$
$$4(-8\mu^3-96\mu^2-32\mu-1)(-322\mu^3-131\mu^2-26\mu-2) \equiv$$
$$2(14052\mu^3+2310\mu^2+180\mu+4) \pmod{\mathfrak{m}_4}.$$

In view of (145) and (141),

(146) $$r^{h\vee\vee *}_{2}(\nu)a^{\vee *}_{2,1}(\nu) + r^{a\wedge\vee}_{2,1}(\nu) \equiv$$

$$2(14052\mu^3+2310\mu^2+180\mu+4)-$$
$$2(22114\mu^3+5027\mu^2+634\mu+34 \equiv$$
$$-2(8062\mu^3+2717\mu^2+454\mu+30) \pmod{\mathfrak{m}_4}.$$

In view of (132) and (134),

(147) $$r_2(\nu)^{h\vee\wedge *}a^{\wedge *}_{2,1}(\nu) =$$

$$(352\mu^3+182\mu^2+44\mu+4)(48\mu^2+64\mu+32) =$$
$$32(3\mu^2+4\mu+2)(176\mu^3+91\mu^2+22\mu+2)) \equiv$$

$$32((66+364+352)\mu^3+(6+88+182)\mu^2+52\mu+4)\equiv$$
$$64(391\mu^3+138\mu^2+26\mu+2)\equiv$$
$$2(12512\mu^3+4416\mu^2+832\mu+64)\ (\mod \mathfrak{m}_4).$$

In view of (147) and (143),

(148) $$r_2(\nu)^{h\vee\wedge*}a_{2,1}^{\wedge*}(\nu)+r_{2,1}^{c\wedge\vee}(\nu)\equiv$$
$$2(12512\mu^3+4416\mu^2+832\mu+64)-$$
$$2(9974\mu^3+3035\mu^2+498\mu+34)\equiv$$
$$(2538\mu^3+1381\mu^2+334\mu+30)\ (\mod \mathfrak{m}_4).$$

Therefore

(149) $$r_2(\nu)^{h\vee\wedge*}a_{2,1}^{\wedge*}(\nu)(4\mu+1)+$$
$$(4\mu+1)r_{2,1}^{c\wedge\vee}(\nu)\equiv$$
$$2(2538\mu^3+1381\mu^2+334\mu+30)(4\mu+1)\equiv$$
$$2(8062\mu^3+2717\mu^2+454\mu+30)\ (\mod \mathfrak{m}_4).$$

In view of (140), (146) and (142),

(150) $$r_{2,1}^{\wedge\vee}(\nu)\in\mathfrak{m}_4,$$

Let $\lambda=1/\mu$. Let $k\in\mathbb{N}$ and $\mathfrak{L}_k$ denotes the main ideal $\lambda^k\mathbb{Q}[\lambda]$ in $\mathbb{Q}[\lambda]$ with generator λ^k. We will finish the proof of equality (130), if we show that

(151) $$\lambda^7 r_{2,1}^{\wedge\vee}(\nu)\in\mathfrak{l}_4.$$

In view of (140)),

(152) $$\lambda^7 r_{2,1}^{\wedge\vee}(\nu)=(\lambda^4 r_2^{h\vee\vee*}(\nu))(\lambda^3(a_{2,1}^{\vee*}(\nu))+$$
$$\lambda^7 r_{2,1}^{a\wedge\vee}(\nu)+$$
$$\lambda^7 r_{2,1}^{b\wedge\vee}(\nu)+$$
$$(\lambda^4 r_2(\nu)^{h\vee\wedge*})(\lambda^2 a_{2,1}^{\wedge*})(4+\lambda)+$$
$$\lambda^6 r_{2,1}^{c\wedge\vee}(\nu)(4+\lambda)+$$
$$\lambda^6 r_{2,1}^{d\wedge\vee}(\nu)(4+\lambda).$$

In view of (145)),

(153) $$\lambda^4 r_2(\nu)^{h\vee\vee*}\lambda^3 a_{2,1}^{\vee*}(\nu)=$$
$$(-8\lambda-96\lambda^2-32\lambda^3-\lambda^4)(-1288\mu^3-524\mu^2-104\mu-8)=$$
$$4(8\lambda+96\lambda^2+32\lambda^3+\lambda^4)(322+131\lambda+26\lambda^2+2\lambda^3)\equiv$$
$$2((416+25152+20608)\lambda^3+(2096+61824)\lambda^2+5152\lambda)\equiv$$
$$2(46176\lambda^3+63920\lambda^2+5152\lambda)\ (\mod \mathfrak{l}_4).$$

In view of (141)),

(154) $$\lambda^7 r^{a\wedge\vee}_{2,1}(\nu) =$$

$$-(1288 + 524\lambda + 104\lambda^2 + 8\lambda^3)+$$
$$2(7\lambda + 19\lambda^2 + 12\lambda^3 + 2\lambda^4)(-2583 - 1062\lambda - 215\lambda^2 - 17\lambda^3) \equiv$$
$$-2((1505 + 20178 + 30996 + 4)\lambda^3 + (7434 + 49077 + 52)\lambda^2+$$
$$(18081 + 262)\lambda + 644) =$$
$$-2((52683\lambda^3 + 56563)\lambda^2 + 18343\lambda + 644).$$

In view of (142)),

(155) $$\lambda^7 r^{b\wedge\vee}_{2,1}(\nu) =$$

$$952(4 + 60\lambda + 72\lambda^2 + 16\lambda^3)+$$
$$1256(4 + 54\lambda + 36\lambda^2 + 4\lambda^3)-$$
$$(12 + 68\lambda + 20\lambda^2)(360 + 1436\lambda)-$$
$$(6 + 24\lambda + \lambda^2)(648 + 1532\lambda)+$$
$$(12 + 8\lambda)(88 + 824\lambda + 1576\lambda^2)-$$
$$(2\lambda - 4)(168 + 904\lambda + 1592\lambda^2)-$$
$$4(8 + 204\lambda + 928\lambda^2 + 1596\lambda^3) =$$
$$(15232 + 5024 - 28720 - 1532 + 12608 - 3184 - 6384)\lambda^3+$$
$$(68544 + 45216 - 104848 - 37416 + 25504 + 4560 - 3712)\lambda^2+$$
$$(57120 + 67824 - 41712 - 24744 + 10592 + 3280 - 816)\lambda+$$
$$3808 + 5024 - 4320 - 3888 + 1056 + 672 - 32 =$$
$$-6956\lambda^3 - 2152\lambda^2 + 71544\lambda + 2320.$$

In view of (143)),

(156) $$\lambda^6 r^{c\wedge\vee}_{2,1}(\nu) =$$

$$-2(1 + 7\lambda + 8\lambda^2 + 2\lambda^3)(1409 + 734\lambda + 181\lambda^2 + 17\lambda^3) \equiv$$
$$-2((2818 + 5872 + 1267 + 17)\lambda^3 + (181 + 5138 + 11272)\lambda^2+$$
$$(9863 + 734)\lambda + 1409) \equiv$$
$$-2(9974\lambda^3 + 16591\lambda^2 + 10597\lambda + 1409) \pmod{\mathfrak{l}}.$$

In view of (144),

(157) $$\lambda^6 r^{d\wedge\vee}_{2,1}(\nu) =$$

$$152(24 + 56\lambda + 8\lambda^2)+$$
$$242(-48\lambda - 40\lambda^2)+$$
$$(-24)(48 + 290\lambda)+$$
$$24(92\lambda + 312\lambda^2)+$$

$$8(8 + 116\lambda + 320\lambda^2) =$$

$$(1216 - 9680 + 7488 + 2560)\lambda^2 + (8512 - 11616 - 6960 + 2208 + 928)\lambda +$$

$$3648 - 1152 + 64 = 1584\lambda^2 - 6928\lambda + 2560).$$

In view of (153), (155) and (154),

(158) $$\lambda^7 r_2(\nu)^{h\vee\vee*} a_{2,1}^{\vee*}(\nu) + \lambda^7 r_{2,1}^{a\wedge\vee}(\nu) + \lambda^7 r_{2,1}^{b\wedge\vee}(\nu) \equiv$$

$$-2(-46176\lambda^3 - 63920\lambda^2 - 5152\lambda) -$$

$$2(52683\lambda^3 + 56563\lambda^2 + 18343\lambda + 644) -$$

$$2(3478\lambda^3 + 1076\lambda^2 - 35772\lambda - 1160) \equiv$$

$$-2(9985\lambda^3 - 6281\lambda^2 - 22581\lambda - 516) \pmod{\mathfrak{l}_4}.$$

In view of (147)),

(159) $$\lambda^6 r_2(\nu)^{h\vee\wedge*} a_{2,1}^{\wedge*}(\nu) =$$

$$(48\lambda + 64\lambda^2 + 32\lambda^3)(352 + 182\lambda + 44\lambda^2 + 4\lambda^3) =$$

$$32(3\lambda + 4\lambda^2 + 2\lambda^3)(176 + 91\lambda + 22\lambda^2 + 2\lambda^3) \equiv$$

$$32((352 + 364 + 66)\lambda^3 + (273 + 704)\lambda^2 + 528\lambda) \equiv$$

$$32(782\lambda^3 + 977\lambda^2 + 528\lambda \equiv)$$

$$2(12512\lambda^3 + 15632\lambda^2 + 8448\lambda) \pmod{\mathfrak{l}_4}.$$

In view of (159), (156) and (157),

(160) $$\lambda^6 r_{2,1}^{h\vee\wedge*}(\nu) a_{2,1}^{\wedge*}(\nu) + \lambda^6 r_{2,1}^{c\wedge\vee}(\nu) +$$

$$\lambda^6 r_{2,1}^{d\wedge\vee}(\nu) \equiv$$

$$2(12512\lambda^3 + 15632\lambda^2 + 8448\lambda) -$$

$$2(9974\lambda^3 + 16591\lambda^2 + 10597\lambda + 1409) +$$

$$2(792\lambda^2 - 3464\lambda + 1280) =$$

Therefore

(161) $$\lambda^6 r_{2,1}^{h\vee\wedge*}(\nu) a_{2,1}^{\wedge*}(\nu)(\lambda + 4) + \lambda^6 r_{2,1}^{c\wedge\vee}(\nu)(\lambda + 4) +$$

$$\lambda^6 r_{2,1}^{d\wedge\vee}(\nu)(\lambda + 4) \equiv$$

$$2(2538\lambda^3 - 167\lambda^2 - 5613\lambda - 129)(\lambda + 4) \equiv$$

$$2(9985\lambda^3 - 6281\lambda^2 - 22581\lambda - 516) \pmod{\mathfrak{l}_4}.$$

If we compare (158) and (161), then we see that (151) holds. ■

Lemma 12.2.2. . *The equality*

(162) $$r_{2,1}^{\wedge\wedge}(\nu) = 0$$

holds.

Proof. It follows from (126), (133) – (139), and (131) – (132), that

$$(163)\qquad r^{\wedge\wedge}_{2,1}(\nu) = r_2^{h\vee\vee*}(\nu)4a^{\wedge*}_{2,1}(\nu) +$$

$$r^{a\wedge\wedge}_{2,1}(\nu)+$$

$$r^{b\wedge\vee}_{2,1}(\nu)+$$

$$r_2(\nu)^{h\vee\wedge*}a^{\vee*}_{2,1}+$$

$$r^{c\wedge\wedge}_{2,1}(\nu)+$$

$$r^{d\wedge\wedge}_{2,1}(\nu),$$

where

$$(164)\qquad r^{a\wedge\wedge}_{2,1}(\nu) :=$$

$$-Q_7(\nu)r_{2,1}(\nu) + r^{\vee\vee*}_{2,1}(\nu)a^{\wedge}_{2,1,1}(1,\nu) =$$

$$-(\mu^3 + 6\mu^2 + 5\mu + 1)\mu^4+$$

$$(\mu^4 + 28\mu^3 + 76\mu^2 + 48\mu + 8)(1409\mu^3 + 734\mu^2 + 181\mu + 17) =$$

$$\mu^4(1408\mu^3 + 728\mu^2 + 176\mu + 16)+$$

$$(28\mu^3 + 76\mu^2 + 48\mu + 8)(1409\mu^3 + 734\mu^2 + 181\mu + 17) \equiv$$

$$(476+13756+35232+11272)\mu^3+(1292+8688+5872)\mu^2+(856+1448)\mu+136 \equiv$$

$$60736\mu^3 + 15852\mu^2 + 2304\mu + 136 \pmod{\mathfrak{m}_4},$$

$$(165)\qquad r^{b\wedge\vee}_{2,1}(\nu) :=$$

$$\sum_{k=2}^{8} r^{\vee\vee*}_{2,k}(\nu)4a^{\wedge\wedge*}_{2,k,1}(\nu)\mu^4 =$$

$$(-4\mu^3 - 54\mu^2 - 36\mu - 4)(608)\mu^4+$$

$$(12\mu^2 + 68\mu + 20)(968)\mu^4+$$

$$(6\mu^2 + 24\mu + 1)(192\mu + 1160)\mu^4+$$

$$(-12\mu - 8)(368\mu + 1248)\mu^4+$$

$$(-4\mu + 2)(32\mu^2 + 464\mu + 1280)\mu^4+$$

$$4(56\mu^2 + 496\mu + 1288)\mu^4,$$

$$(166)\qquad r^{c\wedge\wedge}_{2,1}(\nu) := r^{\vee\wedge*}_{2,1}(\nu)a^{\vee}_{2,1,1}(1,\nu)$$

$$(-8\mu^3 - 56\mu^2 - 64\mu - 16)(-2583\mu^3 - 1062\mu^2 - 215\mu - 17)/2 =$$

$$4(\mu^3 + 7\mu^2 + 8\mu + 2)(2583\mu^3 + 1062\mu^2 + 215\mu + 17) \equiv$$

$$4((17+1505+8496+5166)\mu^3+(119+1720+2124)\mu^2+(136+430)\mu+34) \equiv$$

$$4(15184\mu^3 + 3963\mu^2 + 566\mu + 34) \pmod{\mathfrak{m}_4}.$$

(167)
$$r_{2,1}^{d\wedge\vee}(\nu) :=$$
$$\sum_{k=2}^{8} r_{2,k}^{\vee\wedge *}(\nu) a_{2,k,1}^{\vee\vee *}(\nu)\mu^4 =$$
$$(24\mu^2 + 80\mu + 32)(-952)\mu^4 +$$
$$(24\mu^2 + 56\mu + 8)(-1256)\mu^4 +$$
$$(-48\mu - 40)(-360\mu - 1436)\mu^4 +$$
$$(-24\mu)(-648\mu - 1532)\mu^4 +$$
$$24(-88\mu^2 - 824\mu - 1576)\mu^4 +$$
$$8(-168\mu^2 - 904\mu - 1592)\mu^4,$$

Coefficient at μ^n in the polynomial $r_{2,1}^{\wedge\wedge}(\nu)$ is equal to 0, if $n > 7$. In view of (131) and (134)

(168)
$$r_2(\nu)^{h\vee\vee *} 4a_{2,1}^{\wedge *}(\nu) =$$
$$(-8\mu^3 - 96\mu^2 - 32\mu - 1)(1408\mu^3 + 728\mu^2 + 176\mu + 16) =$$
$$-8(8\mu^3 + 96\mu^2 + 32\mu + 1)(176\mu^3 + 91\mu^2 + 22\mu + 2) \equiv$$
$$-8((16 + 2112 + 2912 + 176)\mu^3 + (192 + 704 + 91)\mu^2 + 86\mu + 2) \equiv$$
$$-8(5216\mu^3 + 987\mu^2 + 86\mu + 2) \pmod{\mathfrak{m}_4}.$$

In view of (164), (165) and (168),

(169)
$$r_{2,1}^{a\wedge\wedge}(\nu) + r_{2,1}^{b\wedge\vee}(\nu) +$$
$$r_2^{h\vee\vee *}(\nu) 4a_{2,1}^{\wedge *}(\nu) \equiv$$
$$-4(10432\mu^3 + 1974\mu^2 + 172\mu + 4) +$$
$$4(15184\mu^3 + 3963\mu^2 + 576\mu + 34) \equiv$$
$$4(4752\mu^3 + 1989\mu^2 + 404\mu + 30) \pmod{\mathfrak{m}_4}.$$

In view of (132) and (133),

(170)
$$r_2(\nu)^{h\vee\wedge *} a_{2,1}^{\vee *}(\nu) =$$
$$(48\mu^2 + 64\mu + 32)(-1288\mu^3 - 524\mu^2 - 104\mu - 8) =$$
$$-64(3\mu^2 + 4\mu + 2)(322\mu^3 + 131\mu^2 + 26\mu + 2) \equiv$$
$$-64((78 + 524 + 644)\mu^3 + (6 + 104 + 262)\mu^2 + (8 + 52)\mu + 4) \equiv$$
$$-64(1246\mu^3 + 372\mu^2 + 60\mu + 4) \equiv$$
$$-4(19936\mu^3 + 5952\mu^2 + 960\mu + 64) \pmod{\mathfrak{m}_4}.$$

In view of (169), (166), (167) and (170),

(171)
$$r_{2,1}^{\wedge\wedge}(\nu) = r_{2,1}^{a\wedge\wedge}(\nu) +$$
$$r_{2,1}^{b\wedge\wedge}(\nu) + r_2(\nu)^{h\vee\vee *} 4a_{2,1}^{\wedge *}(\nu) +$$

$$r_{2,1}^{c\wedge\wedge}(\nu) \equiv +r_{2,1}^{d\wedge\wedge}(\nu)+$$
$$r_2^{h\vee\wedge*}(\nu)a_{2,1}^{\vee*}(\nu) \equiv$$
$$4(4752\mu^3 + 1989\mu^2 + 404\mu + 30)+$$
$$-4(19936\mu^3 + 5952\mu^2 + 960\mu + 64)+$$
$$4(15184\mu^3 + 3163\mu^2 + 566\mu + 34) \in \mathfrak{m}_4).$$

We will finish the proof of equality (162), if we show that

$$(172) \qquad \lambda^7 r_{2,1}^{\wedge\wedge}(\nu) \in \mathfrak{l}_4.$$

In view of (163)),

$$(173) \qquad \lambda^7 r_{2,1}^{\wedge\wedge}(\nu) = \lambda^7 r_2^{h\vee\vee*}(\nu)4a_{2,1}^{\wedge*}(\nu) +$$
$$\lambda^7 r_{2,1}^{a\wedge\wedge}(\nu)+$$
$$\lambda^7 r_{2,1}^{b\wedge\vee}(\nu)+$$
$$\lambda^4 r_2(\nu)^{h\vee\wedge*}\lambda^3 a_{2,1}^{\vee*}+$$
$$\lambda^7 r_{2,1}^{c\wedge\wedge}(\nu)+$$
$$\lambda^7 r_{2,1}^{d\wedge\wedge}(\nu).$$

In view of (164)),

$$(174) \qquad \lambda^7 r_{2,1}^{a\wedge\wedge}(\nu) =$$
$$\lambda^7\mu^4(1408\mu^3 + 728\mu^2 + 176\mu + 16)+$$
$$\lambda^7(28\mu^3 + 76\mu^2 + 48\mu + 8)(1409\mu^3 + 734\mu^2 + 181\mu + 17) \equiv$$
$$4(4\lambda^3 + 44\lambda^2 + 182\lambda + 352)+$$
$$4\lambda(2\lambda^3 + 12\lambda^2 + 19\lambda + 7)(17\lambda^3 + 181\lambda^2 + 734\lambda + 1409) \equiv$$
$$4((4 + 16908 + 13946 + 1267)\lambda^3 + (44 + 26771 + 5138)\lambda^2+$$
$$4((182 + 9863)\lambda + 352) \equiv$$
$$4(32125\lambda^3 + 31953\lambda^2 + 10045\lambda + 352) \pmod{\mathfrak{l}}.$$

In view of (165)),

$$(175) \qquad \lambda^7 r_{2,1}^{b\wedge\vee}(\nu) =$$
$$\lambda^7(-4\mu^3 - 54\mu^2 - 36\mu - 4)(608)\mu^4+$$
$$\lambda^7(12\mu^2 + 68\mu + 20)(968)\mu^4+$$
$$\lambda^7(6\mu^2 + 24\mu + 1)(192\mu + 1160)\mu^4+$$
$$\lambda^7(-12\mu - 8)(368\mu + 1248)\mu^4+$$
$$(-4\mu + 2)(32\mu^2 + 464\mu + 1280)\mu^4+$$
$$4\lambda^7(56\mu^2 + 496\mu + 1288)\mu^4 \equiv$$
$$-152(2 + 27\lambda + 18\lambda^2 + 2\lambda^3)8 + 121(12\lambda + 68\lambda^2 + 20\lambda^3)8+$$

$$(6+24\lambda+\lambda^2)(24+145\lambda)8+$$
$$(-\lambda)(12+8\lambda)(46+156\lambda)8+$$
$$(-4+2\lambda)(4+58\lambda+160\lambda^2)8+$$
$$(28\lambda+248\lambda^2+644\lambda^3)8\equiv$$
$$8(-304+2420+145-1248+320+644)\lambda^3+$$
$$8(-2736+8228+3504-2240-524+248)\lambda^2)+$$
$$+8(-4104+1452+1446-552-224+28)\lambda+8(-304+144-16)\equiv$$
$$8(1977\lambda^3+6480\lambda^2-1954\lambda-176)\,(\mod \mathfrak{l}_4).$$

In view of (168)),

(176) $$\lambda^7 r_2(\nu)^{h\vee\vee*}4a_{2,1}^{\wedge*}(\nu)=$$

$$\lambda^7(-8\mu^3-96\mu^2-32\mu-1)(1408\mu^3+728\mu^2+176\mu+16)=$$
$$-8\lambda(\lambda^3+32\lambda^2+96\lambda+8)(2\lambda^3+22\lambda^2+91\lambda+176)\equiv$$
$$-8\lambda((5632+8736+176)\lambda^2+(16896+728)\lambda+1408)\equiv$$
$$-8\lambda(14544\lambda^2+17624\lambda+1408)\,(\mod \mathfrak{l})_4.$$

In view of (174),(175) and (176),

(177) $$\lambda^7 r_{2,1}^{a\wedge\wedge}(\nu)+\lambda^7 r_{2,1}^{b\wedge\wedge}(\nu)+$$

$$\lambda^7 r_2(\nu)^{h\vee\vee*}4a_{2,1}^{\wedge*}(\nu)\equiv$$
$$4(32125\lambda^3+31953\lambda^2+10045\lambda+352)+$$
$$8(1977\lambda^3+6480\lambda^2-1954\lambda-176)-$$
$$8(14544\lambda^3+17624\lambda^2+1408\lambda)\equiv$$
$$4(6991\lambda^3+9665\lambda^2+3321\lambda\,(\mod \mathfrak{l}_4).$$

In view of (166)),

(178) $$\lambda^7 r_{2,1}^{c\wedge\wedge}(\nu)=$$

$$4\lambda 7(\mu^3+7\mu^2+8\mu+2)(2583\mu^3+1062\mu^2+215\mu+17)\equiv$$
$$4\lambda(1+7\lambda+8\lambda^2+2\lambda^3)(2583+1062\lambda+215\lambda^2+17\lambda^3)\equiv$$
$$4\lambda(2583+19143\lambda+28313\lambda^2)\,(\mod \mathfrak{l}_4).$$

In view of (167),

(179) $$\lambda^7 r_{2,1}^{d\wedge\vee}(\nu)=$$

$$\lambda^7(24\mu^2+80\mu+32)(-952)\mu^4+$$
$$\lambda^7(24\mu^2+56\mu+8)(-1256)\mu^4+$$
$$\lambda^7(-48\mu-40)(-360\mu-1436)\mu^4+$$
$$\lambda^7(-24\mu)(-648\mu-1532)\mu^4+$$

$$\lambda^7 24(-88\mu^2 - 824\mu - 1576)\mu^4 +$$
$$\lambda^7 8(-168\mu^2 - 904\mu - 1592)\mu^4 =$$
$$(-119(6 + 20\lambda + 8\lambda^2) - 157(6 + 14\lambda + 2\lambda^2))32\lambda +$$
$$((6 + 5\lambda)(90 + 359\lambda) + 3(162 + 383\lambda))32\lambda +$$
$$(-3(22 + 206\lambda + 394\lambda^2) - 2(21 + 113\lambda + 199\lambda^2))32\lambda =$$
$$32(-952 - 314 + 1795 - 1182 - 398)\lambda^3 +$$
$$32(-2380 - 2198 + 2604 + 1149 - 618 - 226)\lambda^2 +$$
$$32\lambda(-714 - 942 + 540 + 486 - 66 - 42) = 32(-1051\lambda^3 - 1669\lambda^2 - 738\lambda).$$

In view of (170),

$$\lambda^7 r_{2,1}^{h\vee\wedge *}(\nu) a_{2,1}^{\vee *}(\nu) = \tag{180}$$
$$\lambda^7(48\mu^2 + 64\mu + 32)(-1288\mu^3 - 524\mu^2 - 104\mu - 8) =$$
$$-64\lambda^2(3 + 4\lambda + 2\lambda^2)(322 + 131\lambda + 26\lambda^2 + 2\lambda^3) \equiv$$
$$-64(966\lambda^2 + 1681\lambda^3) \equiv -2(30912\lambda^2 + 53792\lambda^3) (\mod \mathfrak{l}).$$

In view of (178)), (179) and (170),

$$\lambda^7 r_{2,1}^{c\wedge\wedge}(\nu) + \lambda^7 r_{2,1}^{d\wedge\vee}(\nu) + \tag{181}$$
$$\lambda^7 r_{2,1}^{h\vee\wedge *}(\nu) a_{2,1}^{\vee *}(\nu) \equiv$$
$$4(28313\lambda^3 + 19143\lambda^2 + 2583\lambda) +$$
$$4(-8408\lambda^3 - 13352\lambda^2 - 5904\lambda) +$$
$$4(-26896\lambda^3 - 15456\lambda^2) \equiv$$
$$4(-6991\lambda^3 - 9665^2 - 3321\lambda (\mod \mathfrak{l}_4).$$

If we compare (177) and (181), then we see that (172) holds. ■

Lemma 12.2.3. . *The equality*

$$r_{2,2}^{\wedge\vee}(\nu) = 0 \tag{182}$$

holds.

Proof. In view of (86), (121) – (123) and results of previous section,

$$r_{2,2}^{\wedge\vee}(\nu) = -2(7\mu^3 + 14\mu^2 + 7\mu + 1)\mu^3 + \tag{183}$$
$$4(\mu^4 + 28\mu^3 + 76\mu^2 + 48\mu + 8)(-500\mu^3 + 364\mu^2 + 56\mu + 3) +$$
$$16(\mu^3 + 7\mu^2 + 8\mu + 2)(270\mu^2 + 50\mu + 3)(4\mu + 1) +$$
$$2(-\mu^3 - 15\mu^2 - 18\mu - 4)(1049\mu^3 + 1922\mu^2 + 281\mu + 15) +$$

!

$$2(3\mu^2 + 10\mu + 4)(1409\mu^3 - 1450\mu^2 - 251\mu - 15)(4\mu + 1) +$$
$$8(2\mu^3 + 27\mu^2 + 18\mu + 2)(238\mu^4 - 114\mu^3 - 111\mu^2 - 10\mu) +$$
$$16(3\mu^2 + 7\mu + 1)(48\mu^3 - 91\mu^2 - 10\mu)(4\mu + 1)$$

$$32(3\mu^2+17\mu+5)(-96\mu^4-14\mu^3-10\mu^2-8\mu-1)+$$
$$32(-6\mu-5)(38\mu^4+22\mu^3+6\mu+1)(4\mu+1)+$$
$$4(6\mu^2+24\mu+1)(-90\mu^5-263\mu^4-254\mu^3-151\mu^2-42\mu-4)+$$
$$16(-3\mu)(101\mu^4+124\mu^3+91\mu^2+34\mu+4)(4\mu+1)+$$
$$32(3\mu+2)(82\mu^5+235\mu^4+266\mu^3+141\mu^2+34\mu+3)+$$

!

$$32(3)(12\mu^5+79\mu^4+128\mu^3+91\mu^2+28\mu+3)(4\mu+1)+$$
$$8(2\mu-1)(22\mu^6+290\mu^5+784\mu^4+838\mu^3+413\mu^2+94\mu+8)+$$
$$16(54\mu^5+266\mu^4+416\mu^3+273\mu^2+78\mu+8)(4\mu+1)+$$
$$32(-28\mu^6-240\mu^5-579\mu^4-578\mu^3-272\mu^2-60\mu-5).$$

Coefficient at μ^7 in the polynomial $r^{\wedge\vee}_{2,2}(\nu)$ is equal to

(184) $$-2000+3808-2160+352=0.$$

Coefficient at μ^6 in the polynomial $r^{\wedge\vee}_{2,2}(\nu)$ is equal to

(185)
$$-2\times 7+16\times 91+(-64)\times 875+128\times 135+(-2)\times 1049+8\times 4227+$$
$$16\times 27\times 119+(-32)\times 57+1024\times 9-1024\times 9+$$
$$(-512)\times 57+(-64)\times 135+(-8)\times 789+64\times(-303)+$$
$$64\times 123+512\times 9+32\times 145+(-16)\times 11+$$
$$128\times 27-128\times 7=$$
$$2\times(-7-1049)+8\times(4227-789)+16\times(91-11+3213)+$$
$$+32(-57+145)+64(-875-135-303+123)+128(135+27-7)+512(-57+9)=$$
$$16\times(3293+1719)+64\times(-1190+44-33)+128\times 155+512\times(-48)=$$
$$64(1253-1190+11)+128\times 155+512\times 48)=128(37+155)+512(-48)=0.$$

Coefficient at μ^1 in the polynomial $r^{\wedge\vee}_{2,2}(\nu)$ is equal to

(186)
$$32(56+18)+16(24+100+24)-4(562+135)-4(75+502+120)+$$
$$(-32)\times 5+(-32)\times 5+32(-17-40)-32(6+30+20)-8(48+21)-32\times 6+$$
$$32(68+9)+128(3+7)3+16(8-47)+32(39+16)+$$
$$+(-128)\times 15=(-8)\times 69+(-8)\times 69+$$
$$16\times(-39)+32(-10-57-56-6+77+55)+64\times 37+64\times(37)+128\times 15)=$$
$$16(-383-39)+32\times 3+128\times 52=32(-211+3)+128\times 52=0.$$

Clearly,

(187) $$r^{\wedge\vee}_{2,2}(\nu)\big|_{\mu=0}=96+96-120-120-160-160-16+$$

$$192 + 288 - 64 + 128 - 160 = 0.$$

So $r^{\wedge\vee}_{2,2}(\nu) = \mu^2 t_1(\mu)$, where coefficients of $t_1(\mu)$ at μ^k are equal to 0, if $k > 3$. To prove that $t_1(\mu) = 0$ it is sufficient to prove that $t_1(\mu) \in (\mu^4 - 1)\mathbb{Q}[\mu]$. So we most prove that

$$r^{\wedge\vee}_{2,2}(\nu)\big|_{m=\pm 1} = r^{\wedge\vee}_{2,2}(\nu)\big|_{m=i} = 0.$$

We have

$$r^{\wedge\vee}_{2,2}(\nu)\big|_{\mu=-1} = 2 + 4 \times 9 \times 811 + 0 + 0 +$$
$$2(-3)(-2623)(-3)+$$
$$8(9)(251)+$$
$$16(-3)(-129)(-3)$$
$$32(-9)(-95)+$$
$$32(1)(11)(-3)+$$
$$4(-17)(-32)+$$
$$16(3)(38)(-3)+$$
$$32(-1)(-3)+$$
$$32(3)(5)(-3)+$$
$$8(-3)(5)+$$
$$16(-1)(-3)+$$
$$32(-6).$$

Since

$$\omega_1 := 2(1 + 17 \times 64) + 2 \times 9 \times (-2623) = 4 \times 9 \times (-1251),$$

$$\omega_2 := \omega_1 + 4 \times 9 \times 811 = -32 \times 9 \times 55,$$
$$\omega_3 := -32 \times 3 \times 11 + 32 \times 3 - 32 \times 3 \times 2 = -32 \times 9 \times 4,$$
$$\omega_4 := \omega_2 + \omega_3 + 32 \times 9(85 - 5 - 19) = 64 \times 9,$$
$$\omega_5 := 24(-5 + 2) = -8 \times 9, \omega_6 :=$$
$$\omega_5 + 8 \times 9(251 - 258) = -64 \times 9, \omega_4 + \omega_6 = 0,$$

it follows that

(188) $$r^{\wedge\vee}_{2,2}(\nu)\big|_{\mu=-1} = 0.$$

In view of (183)

$$r^{\wedge\vee}_{2,2}(\nu)\big|_{\mu=1} = -2 \times 29+$$
$$4 \times 161 \times (-77) + 16 \times 18 \times 323 \times 5+$$
$$4(-19 \times 3267) - 2 \times 17 \times 307 \times 5+$$
$$8 \times 49 \times 3 + 16 \times 11 \times (-53) \times 5+$$
$$32 \times 25 \times (-129) + 32 \times (-11) \times 67 \times 5+$$

$$4 \times 31 \times (-804) + 16 \times (-3) \times 354 \times 5+$$
$$32 \times 5 \times 761 + 32 \times 341 \times 5+$$
$$8 \times 2449 + 16 \times 1095 \times 5 + 32(-1762)$$

Since

$$\omega_7 := -2 \times 29 - 2 \times 17 \times 307 \times 5 = -8 \times 6531,$$
$$\omega_8 := -4(161 \times 77 + 19 \times 3267) = -8 \times 37235,$$
$$\omega_9 := \omega_7 + \omega_8 + 8(49 \times 3 + 2449) = -16 \times 5 \times 4117,$$
$$\omega_1 0 := \omega_9 + 16 \times 11 \times (-53) \times 5 + 16 \times 1095 \times 5 =$$
$$16 \times 5(-583 + 1095 - 4117) = -16 \times 5 \times 3605,$$
$$\omega_{11} := \omega_{10} + 4 \times 31 \times (-804) = -16(6231 + 18025) = -32 \times 12128,$$
$$\omega_{12} := 16 \times 18 \times 323 \times 5 + 32 \times 25 \times (-129)+$$
$$32 \times 5 \times (-11)67 + 16 \times (-3) \times 354 \times 5 + 32 \times 5 \times 761 + 32 \times 3 \times 341 \times 5 =$$
$$32 \times 5(9 \times 323 - 645 - 737 - 531 + 761 + 1023) = 32 \times 13890,$$
$$\omega_{13} := \omega_1 2 - 32 \times 1762 = 32 \times 12128, \omega_{11} + \omega_1 3 = 0,$$

it follows that

(189) $$r^{\wedge\vee}_{2,2}(\nu)\Big|_{\mu=1} = 0.$$

In view of (183)

$$r^{\wedge\vee}_{2,2}(\nu)\Big|_{\mu=i} = -2 \times 13i+$$
$$4(-67 + 20i)(-361 + 556i) + 16(-5 + 7i)(-267 + 50i)(1 + 4i)+$$
$$2(11 - 17i)(-1907 - 768i) + 2(1 + 10i)(1435 - 1660i)(1 + 4i)+$$
$$8(-25 + 16i)(349 + 104i) + 16(-2 + 7i)(91 - 58i)(1 + 4i)+$$
$$32(2 + 17i)(-87 + 6i) + 32(-5 - 6i)(39 - 16i)(1 + 4i)+$$
$$4(-5 + 24i)(-116 + 122i) + 16(-3i)(14 - 90i)(1 + 4i)+$$
$$32(2 + 3i)(97 - 150i) + 32(3(-9 - 108i))(1 + 4i)+$$
$$8(-1 + 2i)(357 - 454i) + 16(1 - 284i)(1 + 4i)+$$
$$32(-284 + 278i).$$

Since

$$z_1 := -2\times 13i+2(11-17i)(-1907-768i)+2(1+10i)(1435-1660i)(1+4i) =$$
$$4(-33379+54394i),\ z_2 := z1+4(-67+20i)(-361+556i) = 8(-10156+4961i,$$
$$z_3 := z_2 + 8(-25 + 16i)(349 + 104i) + 8(1 - 2i)(357 - 454i)+$$
$$8(-5 + 24i)(-58 + 61i) = 32(-5292 + 1854i),$$
$$z4 := 16(-5 + 7i)(-267 + 50i) + 16(-2 + 7i)(91 - 58i) + 16(1 - 284i) =$$
$$32(605 - 825i),\ z_5 := z_4(1 + 4i) = 32(3905 + 1595i),$$
$$z6 := z5 + z3 = 32(-1387 + 3449i),\ z7 := 32((2 + 17i)(-87 + 6i)+$$

$$(2+3i)(97-150i)+((-5-6i)(39-16i)-3i(7-45i)+3(-9-108i))(1-4i)+$$
$$32(-284+278i)=32(1387-3449i),\ z_6+z_7=0,$$

it follows that

$$r_{2,2}^{\wedge\vee}(\nu)\big|_{\mu=-1}=0. \tag{190}$$

■ **Lemma 12.2.4.** . *The equality*

$$r_{2,2}^{\wedge\vee}(\nu)=0 \tag{191}$$

holds.

Proof. In view of (87), (121) – (123) and results of previous section,

$$r_{2,2}^{\wedge\wedge}(\nu)=4\mu^3(\mu^3+6\mu^2+5\mu+1)+ \tag{192}$$

$$8(\mu^4+28\mu^3+76\mu^2+48\mu+8)(-270\mu^2-50\mu-3)+$$
$$32(-\mu^3-7\mu^2-8\mu-2)(-500\mu^3+364\mu^2+56\mu+3)+$$
$$4(-\mu^3-15\mu^2-18\mu-4)(1409\mu^3-1450\mu^2-251\mu-15)+$$
$$4(3\mu^2+10\mu+4)(1049\mu^3+1922\mu^2+281\mu+15)+$$
$$16(-2\mu^3-27\mu^2-18\mu-2)(48\mu^3-91\mu^2-10\mu)+$$
$$32(3\mu^2+7\mu+1)(-238\mu^4+114\mu^3+111\mu^2+10\mu)+$$
$$64(3\mu^2+17\mu+5)(38\mu^4+22\mu^3+6\mu+1)+$$
$$64(6\mu+5)(96\mu^4+14\mu^3+10\mu^2+8\mu+1)+$$
$$8(6\mu^2+24\mu+1)(101\mu^4+124\mu^3+91\mu^2+34\mu+4)+$$
$$32(-3\mu)(-90\mu^5-263\mu^4-254\mu^3-151\mu^2-42\mu-4)+$$
$$64(-3\mu-2)(12\mu^5+79\mu^4+128\mu^3+91\mu^2+28\mu+3)+$$
$$64(3(-82\mu^5-235\mu^4-266\mu^3-141\mu^2-34\mu-3))+$$
$$16(-2\mu+1)(54\mu^5+266\mu^4+416\mu^3+273\mu^2+78\mu+8)+$$
$$32(-22\mu^6-290\mu^5-784\mu^4-838\mu^3-413\mu^2-94\mu-8)+$$
$$64(2\mu^6+50\mu^5+205\mu^4+294\mu^3+182\mu^2+50\mu+5).$$

4,3 Since

$$\omega_{14}:=4(1-1409)=-512\times 11,\ \omega_{15}:=16(303-135)=128\times 21,$$
$$\omega_{16}:=64(-357+135-27-11)=-256\times 65,\ \omega_{17}:=\omega_1 5+$$
$$128(125+57+1)=128(21+125+57+1)=512\times 51,\ \omega_{18}:=$$
$$omega_{16}-256\times 9=-512\times 37, \omega_{19}:=\omega_{18}+\omega_{17}+$$
$$\omega_{14}-512\times 3=512(-37+51-11-3)=0,$$

it follows from (192) that coefficient at μ^6 in the polynomial $r_{2,2}^{\wedge\wedge}(\nu)$ is equal to 0. Clearly,

$$r_{2,2}^{\wedge\wedge}(\nu)\big|_{\mu=0}= \tag{193}$$

$$8 \times 8 \times (-3) + 32 \times (-2) \times 3+$$
$$4 \times (-4)(-15) + 4 \times 4 \times 15)+$$
$$64 \times 5 + 64 \times 5$$
$$8 \times 4 + 64 \times (-2) \times 3+$$
$$64 \times 3 \times (-3) + 16 \times 8)+$$
$$32 \times (-8)64 \times 5.$$

Since

$$\omega_{20} := 4 \times (-4) \times (-15) + 4 \times 4 \times 15 = 32 \times 15,$$
$$\omega_{21} := \omega_{20} + 32 = 512,\ \omega_{22} := 64(-3 - 3 + 5 + 5 - 9 + 5) = 0,$$
$$\omega_{22} := 128(-3 + 1) = -256,\ \omega_{23} := \omega_{22} - 256 = -512,$$
$$\omega_{21} + \omega_{23} = 0,$$

it follows from (193) that

$$r^{\wedge\wedge}_{2,2}(\nu)\Big|_{\mu=0} = 0.$$

Further we have

(194) $$r^{\wedge\wedge}_{2,2}(\nu)\Big|_{\mu=-1} = -4 +$$

$$8 \times 9(-223) + 0 + 0+$$
$$4 \times (-3) \times 607+$$
$$16 \times (-9) \times (-129)+$$
$$32 \times (-3) \times (-251)+$$
$$64 \times (-9) \times 11+$$
$$64 \times (-1) \times 85+$$
$$8 \times (-17) \times 38+$$
$$32 \times 3 \times (-32)+$$
$$64 \times 5+$$
$$64 \times 3 \times 3+$$
$$16 \times 3 \times (-1)+$$
$$32 \times (-5) + 0.$$

Since

$$\omega_{24} := 4(-1 + (-3) \times 607) = -8 \times 911,\ \omega_{25} := \omega_2 4+$$
$$8 \times 9(-223) = 8(-911 - 2007) = -16 \times 1459,\ \omega_{26} := \omega_{25}+$$
$$16((-9)\times(-129)+(-17)\times 19-3) = 16(-1459+1161-323-3) = -256\times 39,$$
$$\omega_{27} := 32((-3) \times (-251) - 5) = 128 \times 187,\ \omega_{28} :=$$
$$64((-9) \times 11 - 85 + 5 + 9 = -128 \times 85,\ \omega_{29} := \omega_2 8 + \omega_2 7 =$$
$$256 \times 51,\ \omega_{30} := \omega_{29} + \omega_{26} = 256(51 - 39) = 1024 \times 3,$$

it follows from (194) that

$$r^{\wedge\wedge}_{2,2}(\nu)\big|_{\mu=-1} = 0.$$

In view of (192),

(195) $$r^{\wedge\wedge}_{2,2}(\nu)\big|_{\mu=1} = 4 \times 13 +$$

$$8 \times 161 \times (-323)+$$
$$32 \times (-18)(-77)+$$
$$4 \times (-38) \times (-307)+$$
$$4 \times 17 \times (3267)+$$
$$16 \times (-49) \times (-53)+$$
$$32 \times 11 \times (-3)+$$
$$64 \times 25 \times 67+$$
$$64 \times 11 \times 129+$$
$$8 \times 31 \times 354+$$
$$32 \times (-3) \times (-804)+$$
$$64 \times (-5) \times (341)+$$
$$64 \times 3 \times (-761)+$$
$$16 \times (-1) \times 1095+$$
$$32 \times (-2449)+$$
$$64 \times 788.$$

Since

$$\omega_{31} := 4(13 + 17 \times (3267)) = 1024 \times 217,\ \omega_{32} :=$$
$$8(161 \times (-323) + (-19) \times (-307)) = 8(-52003 + 5833) = -16 \times 23085,$$
$$\omega_{33} := \omega_{32} + 16((-49) \times (-53) + 31 \times 177 - 1095) =$$
$$16(-23085 + 2597 + 5487 - 1095) = 16(-24180 + 8084) =$$
$$64(-6045 + 2021) = -512 \times 503,$$
$$\omega_{34} := 32(11 \times (-3) - 2449) = -64 \times 1241,\ \omega_3 5 :=$$
$$\omega_{34} + 64((-9)(-77) + 25 \times 67 + 11 \times 129 + (-5) \times (341) + 3 \times (-761) =$$
$$64(-1241 + 693 + 1675 + 1419 - 1705 - 2283) = -128 \times 721,\ \omega_{36} := \omega_3 5+$$
$$128 \times 603 = -256 \times 59,\ \omega_{37} := \omega_{36} + 256 \times 197 = 512 \times 69,$$
$$\omega_{38} := \omega_{37} + \omega_{33} = -1024 \times 217, \omega_{38} + \omega_3 1 = 0,$$

it follows from (195) that

$$r^{\wedge\wedge}_{2,2}(\nu)\big|_{\mu=1} = 0.$$

Finally, according to (192),

(196) $$r^{\wedge\wedge}_{2,2}(\nu)\big|_{\mu=i} = -4i \times (-5 + 4i) +$$

$$8(-67+20i)(267-50i)+$$
$$32(5-7i)(-361+556i)+$$
$$4(11-17i)(1435-1660i)+$$
$$4(1+10i)(-1907-768i)+$$
$$16\times(25-16i)(91-58i)+$$
$$32(-2+7i)(-349-104i)+$$
$$64(2+17i)(39-16i)+$$
$$64(5+6i)(87-6i)+$$
$$8(-5+24i)(14-90i)+$$
$$32\times(-3i)\times(-116+122i)+$$
$$64(-2-3i)(-9-88i)+$$
$$64\times 3\times(-97+150i)+$$
$$16(1-2i)(1-284i)+$$
$$32\times(-357+454i)+$$
$$64(26-194i).$$

Since

$$z_8 := 4(4+5i+(11-17i)(1435-1660i)+(1+10i)(-1907-768i) =$$
$$8(-3333-31249i),$$
$$z_9 := z_8+8((-67+20i)(267-50i)+(-5+24i)(14-90i)) = 64(-2266-2721i),$$
$$z_{10} := 16((25-16i)(91-58i)+(1-2i)(1-284i)) = 64(195-798i),$$
$$z_{11} := 32((5-7i)(-361+556i)+(-2+7i)(-349-104i)+$$
$$(-357+454i)) = 64(1578+1763i),$$
$$z_{12} := z_9+z_{10}+z_{11} = 64(-493-1756i),$$
$$z_{13} := 64((2+17i)(39-16i)+(5+6i)(87-6i)+183+174i) = 64(1004+1297i),$$
$$z_{14} := 64((-2-3i)(-9-88i)-291+450i+26-194i) = 64(-511+459i,$$
$$z_{12}+z_{13}+z_{14} = 0,$$

it follows from (196) that

$$r^{\wedge\wedge}_{2,2}(\nu)\Big|_{\mu=i} = 0.\blacksquare$$

§12.3. The corrections to the previous parts of this work.

The frase
Then, in view of (150) – (151), clearly,

$$R_{\alpha,0}(1,\nu) = R^{\vee}_{\alpha,0}(1,\nu)+\tau R^{\wedge}_{\alpha,0}(1,\nu).$$

before equality (159) must be removed in [14].

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