THE ETA INVARIANT OF PIN MANIFOLDS WITH CYCLIC FUNDAMENTAL GROUPS

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ABSTRACT. Let $\ell = 2^\nu > 1$. Let M be an orientable manifold of odd dimension m with $\pi_1(M) = \mathbf{Z}_\ell$ whose universal cover $\tilde M$ is spin. We define a fixed point free action of $\mathbf{Z}_{2\ell}$ on the product $\tilde M \times \tilde M$ and let $N := \tilde M \times \tilde M / \mathbf{Z}_{2\ell}$; $N(M)$ is non-orientable and admits a natural pin$^-$ structure. We express the eta invariant of $N(M)$ in terms of the eta invariant of M and show the map $M \to N(M)$ extends to a map of suitably chosen equivariant connective K-theory groups. Let X be a non-orientable manifold with $\pi_1(X) = \mathbf{Z}_{2\ell}$ of even dimension $m \geq 6$ whose universal cover is spin. We show that if X admits a metric of positive scalar curvature, then the moduli space of all metrics of positive scalar curvature on X has an infinite number of arc components. If $m \equiv 2 \bmod 4$ and if $w_2(X) = 0$, we show X admits a metric of positive scalar curvature if and only if the $\hat A$ genus of the universal cover $\tilde X$ vanishes; this establishes the Gromov-Lawson conjecture in this special case.

§1 INTRODUCTION

1.1 Notational conventions. We work in the category of smooth manifolds and smooth vector bundles in this paper. All manifolds are assumed to be closed and connected unless otherwise noted. Let $\ell = 2^\nu > 1$ be a non-trivial power of 2 and let $g_\ell := e^{2\pi\sqrt{-1}/\ell}$ be the canonical generator of $\mathbb{Z}_\ell := \{\lambda \in \mathbb{C} : \lambda^\ell = 1\}$. Let $\rho_s(\lambda) = \lambda^s$ define linear representations of $\mathbb{Z}_\ell$ for s in the dual group $\mathbb{Z}_\ell^* = \mathbb{Z}/\ell\mathbb{Z}$. Let π be a finite group. A π structure on a manifold M of dimension m is a map f from M to the classifying space $B\pi$ of π. Let $\mathcal{Z}(B\pi)$ be the classifying principal π bundle over $B\pi$ and let $\mathcal{Z}(M) = f^*\mathcal{Z}(B\pi)$ be the associated principal π bundle over M. If $\pi_1(M) = \pi$, we give M the natural π structure; $\mathcal{Z}(M) = \tilde M$ is the universal cover of M.

If m is odd and if M admits a spin^c structure, let P_M be the Dirac operator on M; P_M is the tangential operator of the spin^c complex. If m is even and if M admits a pin^c structure, let P_M be the Dirac operator on M; P_M is the tangential operator of the pin^c complex. The operator P_M is a self-adjoint elliptic operator. If M has a π structure f and if ρ is a representation of π, let $\eta(M, \rho)$ be the eta invariant of the Dirac operator P_M with coefficients in the associated flat bundle.

We say that a manifold M is a *spherical space form* if M admits a Riemannian metric of constant sectional curvature +1. We say that a finite group π is a *spherical space form group* if π is the fundamental group of a spherical space form or

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equivalently if there exists a fixed point free representation of π to the orthogonal group $O(m+1)$ for some m.

1.2 Previous results for orientable manifolds. In [14, 18] we showed the eta invariant completely detects the reduced complex, real, and quaternion K-theory groups $\tilde{K}(M)$, $\tilde{KO}(M)$, and $\tilde{KSp}(M)$ if M is a spherical space form. In [16], we showed the eta invariant and the equivariant characteristic numbers completely detect the equivariant spin^c bordism groups $MSpin_*^c(B\mathbb{Z}_\ell)$. Later joint work with Bahri, Bendersky, and Davis [3] then gave the additive structure of these groups. In related work [17], we studied the equivariant unitary bordism groups $MU_*(B\mathbb{Z}_\ell)$.

Let $m \geq 5$ be odd. In joint work with B. Botvinnik [7, 8], we used the eta invariant to construct exotic metrics of positive scalar curvature on a wide class of manifolds with finite fundamental groups whose universal cover is spin.

The Gromov-Lawson conjecture as generalized by Rosenberg asserts that a manifold of dimension $m \geq 5$ whose universal cover is spin admits a metric of positive scalar curvature if and only if a generalized index of the Dirac operator vanishes. In joint work with Botvinnik and Stolz [10], we used the eta invariant to prove this conjecture for a spin manifold M whose fundamental group is a spherical space form group. In joint work with Botvinnik [9], we extended these results to the case in which M is an orientable manifold with cyclic fundamental group and whose universal cover admits a spin structure.

The proof of all of these results used formulas of Donnelly [12] which compute the eta invariant of the tangential operators of the classical elliptic complexes in terms of Dedekind sums if M is a spherical space form. We also used generalizations of these formulas to manifolds M which are lens space bundles over S^2.

1.3 Previous results for non orientable manifolds. In [15], we developed the basic theory of the Dirac operator for a pin^c manifold. In joint work with Bahri [4], we used these results to determine the bordism groups $MPin_*^c$. Oledzki [26, 27] has used the eta invariant to detect exotic 4 dimensional projective spaces. In these results for even dimensional manifolds, the only fundamental group which enters is the cyclic group $\mathbb{Z}_2$ of order 2.

1.4 Outline of the paper. This paper is devoted to computing the eta invariant for a wider class of pin$^-$ manifolds than those with fundamental group $\mathbb{Z}_2$ and to giving some applications of this computation. In §2, we discuss the Dirac operator for odd dimensional spin and spin^c manifolds and for even dimensional pin$^\pm$ and pin^c manifolds. Let π be a finite group. If m is even, we will show that the map which sends M to $\eta(M,\rho) \in \mathbb{R}/\mathbb{Z}$ extends to a homomorphism from the equivariant pin$^-$ bordism groups $MPin_m^-(B\pi)$ to $\mathbb{R}/\mathbb{Z}$. If $m \equiv 2$ (8) and if ρ is real or if $m \equiv 6$ (8) and if ρ is quaternion, then this invariant takes values in $\mathbb{R}/2\mathbb{Z}$.

Let f be a $\mathbb{Z}_\ell$ structure on an orientable manifold M of odd dimension m; we assume the associated principal bundle $\mathcal{Z} = \mathcal{Z}(M)$ is spin. Let $\mathcal{D}_M$ denote the action of $\mathbb{Z}_\ell$ on $\mathcal{Z}$. Define a fixed point free action $\mathcal{D}_N$ of $\mathbb{Z}_{2\ell}$ on $\mathcal{Z} \times \mathcal{Z}$ by

$$\mathcal{D}_N(g_{2\ell}) : (z_1, z_2) \mapsto (\mathcal{D}_M(g_\ell)z_2, z_1); \quad \text{let} \quad N(M) := (\mathcal{Z} \times \mathcal{Z})/\mathbb{Z}_{2\ell}. \tag{1.5}$$

In Lemma 3.4 we will show that $N(M)$ inherits a natural pin$^-$ structure. In Theorem 3.7, we will express the eta invariant of $N(M)$ in terms of the eta invariant of

M; this is the main analytical result of this paper. There is an analogous result for $N(M)$ if M is an even dimensional non-orientable manifold due to Barrera-Yanez [5]. We can also take twisted products. Let M be an even dimensional pin$^-$ manifold with a $\mathbb{Z}_\ell$ structure. Let U be an even dimensional spin manifold with a spinor $\mathbb{Z}_\ell$ action. In Theorem 3.11 we express the eta invariant of $U \times_{\mathbb{Z}_\ell} \mathcal{Z}(M)$ in terms of the eta invariant of M and the equivariant index of the spin complex on U.

Let M be either a lens space or a lens space bundle bundle over S^2. In §4, we use the results of §3 to compute the eta invariant of $N(M)$. This gives a whole new class of pin$^-$ manifolds with cyclic fundamental group $\mathbb{Z}_{2\ell}$ for which we can compute the eta invariant combinatorially. We estimate the range of the eta invariant and establish some technical results we will use in later sections.

Let X be a manifold of dimension $m \geq 5$ with $\pi_1(X) = \pi$ whose universal cover is spin and which admits a metric of positive scalar curvature. If m is odd, we assume X is orientable and if m is even, we assume X is not orientable. Let $\mathcal{R}(X)$ be the space of all metrics of positive scalar curvature on X and let $\mathcal{M}(X)$ be the moduli space $\mathcal{R}(X)/\text{Diff}(X)$. In Theorems 5.6, 5.7, 5.10, and 5.12 we show under certain conditions that $\mathcal{M}(X)$ has an infinite number of arc components and that X admits infinitely many metrics of positive scalar curvature which are not concordant. Theorems 5.6 and 5.7 follow from results of [7, 8]; Theorem 5.6 deals with the case m odd and π cyclic and Theorem 5.7 deals with the case of general π if X itself is spin. Theorems 5.10 and 5.12 are new; they use the analytic results of §3. Theorem 5.10 deals with the case m even and π cyclic and Theorem 5.12 deals with the case of general π if X admits a pin$^-$ structure and if $m \equiv 2 \bmod 4$. We also refer to related work by Kreck and Stolz [22] if $m \equiv 3 \bmod 4$, if $H^1(X; \mathbb{Z}_2) = 0$, if the Pontrjagin classes of X vanish, and if X is spin.

In the generalized Gromov-Lawson-Rosenberg conjecture, it is stated that a manifold M of dimension $m \geq 5$ whose universal cover is spin admits a metric of positive scalar curvature if and only if a certain generalized equivariant index of the Dirac operator vanishes. Stolz [34] has established this conjecture if M is simply connected. The fundamental group π of M plays a crucial role. This conjecture has been established if π is a spherical space form group and M is spin by Botvinnik, Gilkey and Stolz [10], if π is cyclic and if M admits a flat spin^c structure by Botvinnik and Gilkey [9] and by Kwasik and Schultz [24], if $\pi = \mathbb{Z}_p \oplus \mathbb{Z}_p$ for p an odd prime by Schultz [32], and for a short list of infinite fundamental groups, including free groups, free abelian groups, and fundamental groups of orientable surfaces by Rosenberg and Stolz [30]. In §6, we prove the Gromov-Lawson conjecture for non-orientable pin$^-$ manifolds of dimension $m = 4k + 2$ with fundamental group $\mathbb{Z}_{2\ell}$. As a byproduct of our investigation, we show the eta invariant completely detects certain twisted connective K-theory groups and we show the map $M \to N(M)$ extends to a map in connective K-theory, see Theorem 6.6 for details.

It is a pleasant task to thank E. Barrera-Yanez, B. Botvinnik, and S. Stolz for helpful conversations regarding these matters.

§2 The Dirac operator for spin and pin manifolds

2.1 Clifford algebras. Let $\text{Clif}^{\pm}(m)$ denote the real Clifford algebra on $\mathbb{R}^m$; this is the universal unital algebra generated by $\mathbb{R}^m$ subject to the Clifford commutation relations $v * w + w * v = \pm(v, w) \cdot 1$. Let $\text{Clif}^c(m) := \text{Clif}^-(m) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification. Let $\text{pin}^\pm(m) \subset \text{Clif}^\pm(m)$ be the multiplicative subgroup generated by the unit sphere of $\mathbb{R}^m$; this is the set of all elements x which can be written as a finite product $x = v_1 * ... * v_k$ of elements v_i of length 1 in $\mathbb{R}^m$. We complexify to define

$$\text{pin}^c(m) = \text{pin}^-(m) \times_{\mathbb{Z}_2} S^1 \subset \text{Clif}^c(m)$$

where we identify $(g, \lambda) \simeq (-g, -\lambda)$.

2.2 Definition. If $v \in V$ and $|v| = 1$, then $v^{-1} = \pm v$ so $(v_1 * ... * v_k)^{-1}$ is given by $(\pm 1)^k v_k * ... * v_1$. Let

(1) $\det(x, \lambda) = \lambda^2 : \text{pin}^c(m) \to S^1$.

(2) $\chi(v_1 * ... * v_k) := (-1)^k : \text{pin}^\pm(m) \to \mathbb{Z}_2$.
$\chi(v_1 * ... * v_k, \lambda) := (-1)^k : \text{pin}^c(m) \to \mathbb{Z}_2$.

(3) $\Xi(x) : w \mapsto \chi(x) x * w * x^{-1} : \text{pin}^\pm(m) \to O(m)$.
$\Xi(x, \lambda) : w \mapsto \chi(x, \lambda) x * w * x^{-1} : \text{pin}^c(m) \to O(m)$.

(4) $\text{spin}(m) = \text{spin}^-(m) := \ker(\chi) \cap \text{pin}^+(m)$ and $\text{spin}^c(m) := \ker(\chi) \cap \text{pin}^c(m)$.

If v is a unit vector in $\mathbb{R}^m$, $\Xi(v)$ is reflection in the hyperplane perpendicular to v. Thus Ξ defines a surjective group homomorphism from $\text{pin}^\pm(m)$ to $O(m)$ and from $\text{spin}(m)$ to $SO(m)$. Let $m \geq 3$. Since $\text{spin}(m)$ is connected, since $\pi_1(SO(m)) = \mathbb{Z}_2$, and since $\ker(\Xi)$ is $\{\pm 1\} \in \text{spin}(m)$, $\text{spin}(m)$ is *the* universal cover of $SO(m)$. A similar argument shows that $\text{pin}^\pm(m)$ is *a* universal cover of $O(m)$. Since $O(m)$ is not connected, the universal cover is not uniquely defined as a group; $\text{pin}^\pm(m)$ are the two possible universal covering groups of $O(m)$.

Let $\epsilon = -$, $\epsilon = +$, or $\epsilon = c$. We say that a manifold M admits a *pin$^\epsilon$ structure* if we can lift the transition functions of the tangent bundle TM from the orthogonal group $O(m)$ to $\text{pin}^\epsilon(m)$. We say that an orientable manifold M admits a *spin structure* or a *spin^c structure* if we can lift the transition functions of the tangent bundle TM from the special orthogonal group $SO(m)$ to $\text{spin}(m)$ or to $\text{spin}^c(m)$. If M admits a (s)pin^c structure s_M, the determinant line bundle $\det(s_M)$ is the associated complex line bundle over M. Let w_i be the Stiefel-Whitney classes of TM. The following is well known; see for example Giambalvo [13].

2.3 Lemma.

(1) *M admits a spin structure* $\iff$ $w_1 = 0$ *and* $w_2 = 0$.

(2) *M admits a spin^c structure* $\iff$ $w_1 = 0$ *and* w_2 *lifts to* $H^2(M; \mathbb{Z})$.

(3) *M admits a pin$^-$ structure* $\iff$ $w_1^2 + w_2 = 0$.

(4) *M admits a pin$^+$ structure* $\iff$ $w_2 = 0$.

(5) *M admits a pin^c structure* $\iff$ w_2 *lifts to* $H^2(M; \mathbb{Z})$.

2.4 Example. Let L be the classifying real line bundle over $\mathbb{RP}^k := S^k/\mathbb{Z}_2$ and let $\nu L = L \oplus ... \oplus L$. Let $\omega_\nu := e_1 * ... * e_\nu$. Then $\Xi(\omega_\nu) = -I_\nu$ is a lift of the transition functions of νL from $O(\nu)$ to $\text{pin}^\epsilon(m)$. Then νL admits a $\text{pin}^\pm$ structure if and only if $\omega_\nu^2 = 1$ or equivalently if $(\pm 1)^\nu (-1)^{\nu(\nu-1)/2} = 1$; by replacing ω_ν by

$\sqrt{-1}\omega_\nu$ if necessary we see νL always admits a pin^c structure. These structures reduce to spin^c structures if and only if ν is even. Since $T(\mathbb{RP}^k)\oplus 1=(k+1)L$,

(1) $(4k+1)L$ and $\mathbb{RP}^{4k}$ admit pin$^+$ structures; $w_2=0$.

(2) $(4k+2)L$ and $\mathbb{RP}^{4k+1}$ admit spin^c structures; $w_1=0$, and w_2 lifts.

(3) $(4k+3)L$ and $\mathbb{RP}^{4k+2}$ admit pin$^-$ strutures; $w_1^2+w_2=0$.

(4) $(4k+4)L$ and $\mathbb{RP}^{4k+3}$ admit spin structures; $w_1=0$ and $w_2=0$.

2.5 Operators of Dirac type. We say a second order partial differential operator D on the space of smooth sections $C^\infty(V)$ of a vector bundle V is of *Laplace type* if locally D has the form $D=-g^{ij}I_V\partial_i\partial_j+A^k\partial_k+B$ or equivalently if the leading symbol of D is given by the metric tensor. We say a first order partial differential operator P is of *Dirac type* if P^2 is of Laplace type. The leading symbol p of P gives V a $\mathrm{Clif}^-(M)$ module structure i.e. p is a linear map from the cotangent bundle to the bundle of endomorphisms of V so that $p(\xi)^2=-|\xi|^2I_V$. Conversely, given a $\mathrm{Clif}^-(TM)$ module structure p on V and a connection ∇ on V, then $P:=p\circ\nabla$ is an operator of Dirac type on V. Fix a fiber metric on V so p is skew-adjoint. The connection ∇ is said to be compatible with the Clifford module structure if ∇ is Riemannian and if $\nabla p=0$. Such.connections always exist and the associated operator P is self-adjoint; see Branson-Gilkey [11].

Let $M_\nu(\mathbb{F})$ be the $\nu\times\nu$ matrix algebra over the division algebra $\mathbb{F}=\mathbb{R}$, $\mathbb{F}=\mathbb{C}$, or $\mathbb{F}=\mathbb{H}$; we use the notation $M_-(\mathbb{F})$ when we wish to simplify notation by omitting the parameter ν. We recall the structure of some of the Clifford algebras; see [1] for further details:

2.6 Lemma.

(1) *The isomorphism $\mathrm{Clif}^c(2k)=M_{2^k}(\mathbb{C})$ defines an irreducible $\mathrm{Clif}^c(2k)$ module Δ_{2k}. Every $\mathrm{Clif}^c(2k)$ module is isomorphic the direct sum of copies of this module.*

(2) *The isomorphism $\mathrm{Clif}^c(2k+1)=M_{2^k}(\mathbb{C})\oplus M_{2^k}(\mathbb{C})$ defines two irreducible $\mathrm{Clif}^c(2k+1)$ modules Δ_{2k+1} and $\tilde\Delta_{2k+1}$; $\tilde\Delta_{2k+1}(\xi)=-\Delta_{2k+1}(\xi)$. Every $\mathrm{Clif}^c(2k+1)$ module is isomorphic to the direct sum of copies of these two modules.*

(3) $\mathrm{Clif}^+(8k)=M_-(\mathbb{R})$ *and* $\mathrm{Clif}^+(8k+1)=\mathrm{Clif}^+(8k)\oplus\mathrm{Clif}^+(8k)$.

(4) $\mathrm{Clif}^-(8k+2)=M_-(\mathbb{H})$ *and* $\mathrm{Clif}^-(8k+3)=\mathrm{Clif}^-(8k+2)\oplus\mathrm{Clif}^-(8k+2)$.

(5) $\mathrm{Clif}^+(8k+4)=M_-(\mathbb{H})$ *and* $\mathrm{Clif}^+(8k+5)=\mathrm{Clif}^+(8k+4)\oplus\mathrm{Clif}^+(8k+4)$.

(6) $\mathrm{Clif}^-(8k+6)=M_-(\mathbb{R})$ *and* $\mathrm{Clif}^-(8k+7)=\mathrm{Clif}^-(8k+6)\oplus\mathrm{Clif}^-(8k+6)$.

Let σ be a representation of $\mathrm{Clif}^c(m)$ on a finite dimensional vector space Σ. If $x\in spin^c(m)$, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R}^m\otimes\Sigma & \xrightarrow{\sigma} & \Sigma \\ \downarrow\Xi(x)\otimes\sigma(x) & & \downarrow\sigma(x)\,. \\ \mathbb{R}^m\otimes\Sigma & \xrightarrow{\sigma} & \Sigma \end{array} \tag{2.7}$$

If M admits a spin^c structure, we use diagram (2.7) to define an elliptic operator $P_\sigma:=\sigma\circ\nabla$ of Dirac type on the associated bundle $\Sigma(M)$. If Δ is the fundamental representation, P is the Dirac operator on the spin bundle $\Delta(M)$. If m is odd, P is the tangential operator of the spin^c complex. If m is even, the decomposition $P = P^+ \oplus P^-$ gives the spin^c complex.

If, on the other hand, M is not orientable, the situation is a bit more complicated. Let $\xi \in \mathbb{R}^m$. Because

$$\sigma(\Xi(x)\xi)\sigma(x) = \chi(x)\sigma(x)\sigma(\xi)\sigma(x)^{-1}\sigma(x) = \chi(x)\sigma(x)\sigma(\xi),$$

diagram (2.7) does not commute if $\chi(x) = -1$. To remedy this difficulty, we introduce χ on the right hand side. Let χ act on the one dimensional vector space L_χ. The following diagram **does** commute:

$$\begin{array}{cccc} & \mathbb{R}^m \otimes \Sigma & \xrightarrow{\sigma} & \Sigma \otimes L_\chi \\ (2.8) & \downarrow \Xi(x) \otimes \sigma(x) & \circ & \downarrow \sigma(x) \otimes \chi(x) \\ & \mathbb{R}^m \otimes \Sigma & \xrightarrow{\sigma} & \Sigma \otimes L_\chi \end{array}$$

Let M be a manifold which admits a pin$^\epsilon$ structure. Then $L_\chi(M)$ is the orientation line bundle. We use diagram(2.8) to define an elliptic symbol and, once a compatible connection ∇ is chosen, an elliptic complex

$$(2.9)\qquad \begin{array}{l} \sigma : T^*M \otimes \Sigma(M) \to \Sigma(M) \otimes L_\chi(M), \text{ and} \\ D_\sigma := \sigma \circ \nabla : C^\infty(\Sigma(M)) \to C^\infty(\Sigma(M) \otimes L_\chi(M)). \end{array}$$

If m is even, there is another way to solve the difficulty involved in diagram (2.7). Let $\{e_i\}$ be the standard orthonormal basis for $\mathbb{R}^m$. Let $\omega_m := \delta(m,\epsilon)e_1 * \ldots * e_m$ where $\delta(m,\epsilon)$ is chosen so that $\omega_m^2 = -1$ if $\epsilon = -, c$ and $\omega_m^2 = +1$ if $\epsilon = +$. More precisely

$$\delta(m,\epsilon) = \begin{cases} 1 & \text{if } m \equiv 2\ (4) \text{ and } \epsilon = - \text{ or } \epsilon = c, \\ 1 & \text{if } m \equiv 0\ (4) \text{ and } \epsilon = +, \\ i & \text{if } m \equiv 0\ (4) \text{ and } \epsilon = - \text{ or } \epsilon = c, \\ i & \text{if } m \equiv 2\ (4) \text{ and } \epsilon = +. \end{cases}$$

Since m is even, ω_m anti-commutes with $\xi \in \mathbb{R}^m \subset Clif^\epsilon(m)$. We extend σ to a representation of $\text{Clif}^\epsilon(m+1)$ on Σ by defining $\sigma(e_{m+1}) = \sigma(\omega_m)$. We also define a new representation $\tilde\sigma$ of $\text{Clif}^-(m)$ on Σ by $\tilde\sigma(\xi) := \sigma(\omega_m)\sigma(\xi)$; even if $\epsilon = +$, $\tilde\sigma$ is a representation of Clif^-. If $x \in \text{pin}^\epsilon(m)$, then $x\omega_m = \chi(x)\omega_m x$ and the following diagram commutes:

$$\begin{array}{cccc} & \mathbb{R}^m \otimes \Sigma & \xrightarrow{\tilde\sigma} & \Sigma \\ (2.10) & \downarrow \Xi(x) \otimes \sigma(x) & \circ & \downarrow \sigma(x)\,. \\ & \mathbb{R}^m \otimes \Sigma & \xrightarrow{\tilde\sigma} & \Sigma \end{array}$$

If M admits a pin$^\epsilon(m)$ structure, then diagram (2.10) gives rise to a self-adjoint operator of Dirac type

$$(2.11)\qquad P_{\tilde\sigma} = \tilde\sigma \circ \nabla : C^\infty(\Sigma(M)) \to C^\infty(\Sigma(M)).$$

We emphasize that the representation σ defines $\Sigma(M)$ and the representation $\tilde\sigma$ defines P. If Σ is complex, σ and $\tilde\sigma$ are abstractly isomorphic, but act differently upon the representation space Σ. If $\epsilon = -$ or if $\epsilon = c$, the roles of σ and $\tilde\sigma$ are symmetric; if we wish to use σ to define an operator P, we can use $\tilde\sigma$ to define the bundle. We take the fundamental representation to define the Dirac operator on M; this is the tangential operator of the pin^c complex.

2.12 Remark. If m is even, let D_σ be the operator defined in equation (2.9) and let $P_{\tilde{\sigma}}$ be the operator defined in equation (2.11). Then $\sigma(\omega_m)$ defines an isomorphism between Σ and $\Sigma \otimes L_\chi$ which extends to an isomorphism between $\Sigma(M)$ and $\Sigma(M) \otimes L_\chi(M)$ and modulo a possible sign convention, $P_{\tilde{\sigma}} = \sigma(\omega_m) D_\sigma$. Furthermore, if we extend σ to a representation of $\text{Clif}^\epsilon(m+1)$, then $P_{\tilde{\sigma}}$ is the tangential operator associated to the resulting elliptic complex on $M \times [0, \infty)$.

2.13 The eta invariant. If P is a self-adjoint operator of Dirac type, let

$$\eta(z, P) := \text{Tr}_{L^2}(P \cdot (P^2)^{-(z+1)/2})$$

be the eta invariant of Atiyah-Patodi-Singer [2]. The function $\eta(z, P)$ has a meromorphic extension to $\mathbb{C}$ with isolated simple poles on the real axis. The origin is a regular value and we define

$$\eta(P) := \tfrac{1}{2}\{\eta(z, P) + \dim\ker(P)\}|_{z=0}$$

as a measure of the spectral asymmetry of P. A representation ρ of a finite group π defines a flat vector bundle $V(\rho)$ over the classifying space $B\pi$. Let f give M a π structure. The pull back bundle $f^*(V(\rho))$ is a flat bundle over M with holonomy ρf_*. If m is even, we assume M has a pin^c structure; if m is odd, we assume M has a spin^c structure. Let $\eta(M, \rho)$ be the eta invariant of the Dirac operator on M with coefficients in $f^*V(\rho)$. If m is even and if M is orientable, then P_ρ is conjugate to $-P_\rho$ and therefore $\eta(M, \rho) = \dim\ker(P_\rho)/2$. This vanishes if M admits a metric of positive scalar curvature. We refer to [19] for further details.

2.14 Equivariant pin bordism. The equivariant bordism groups $MSpin^\epsilon_m(B\pi)$ and $MSpin^\epsilon_m(B\pi)$ consist of triples (M, f, s) where f is a π structure on a manifold M of dimension m and where s is a (s)pin^ϵ structure on M; we assume M is closed but not necessarily connected. We impose the equivalence relation $(M, f, s) \sim 0$ if there exists a compact manifold N (which need not be connected) with boundary M so the structures (f, s) extend over N. The group structure is defined by disjoint union. There are twisted bordism groups we will discuss in §5. Let $R(\pi)$ be the group representation ring of a finite group π and let $R_0(\pi)$ be the augmentation ideal of virtual representations of virtual dimension 0. The map $\rho \to \eta(M, \rho)$ is additive in ρ and extends to $R(\pi)$.

2.15 Lemma.

(1) *If m is odd and if $\rho \in R_0(\pi)$, the map $(M, f, s) \mapsto \eta(M, \rho)$ extends to a homomorphism from $MSpin^\epsilon_m(B\pi)$ to $\mathbb{R}/\mathbb{Z}$ which takes values in $\mathbb{R}/2\mathbb{Z}$ for $MSpin_m(B\pi)$ in the following cases:*

  a) *If $m \equiv 3$ mod 8 and ρ is real.*

  b) *If $m \equiv 7$ mod 8 and ρ is quaternion.*

(2) *If m is even and if $\rho \in R(\pi)$, then the map $(M, f, s) \mapsto \eta(M, \rho)$ extends to a homomorphism from $MPin^\epsilon_m(B\pi)$ to $\mathbb{R}/\mathbb{Z}$ which takes values in $\mathbb{R}/2\mathbb{Z}$ in the following cases:*

  a) *If $m \equiv 0$ mod 8, if $\epsilon = +$, and if ρ is quaternion.*

b) *If $m \equiv 2$ mod 8, if $\epsilon = -$, and if ρ is real.*
c) *If $m \equiv 4$ mod 8, if $\epsilon = +$, and if ρ is real.*
c) *If $m \equiv 6$ mod 8, if $\epsilon = -$, and if ρ is quaternion.*

Proof. We refer to [9] for the proof of assertion (1). Suppose that $M = \partial N$ where the pin$^\epsilon$ and π structures extend from M to N. Choose a metric on N which is product near the boundary. We use Remark 2.12 to see that modulo a possible sign convention, P is the tangential operator of an elliptic complex D defined over N. The Atiyah-Patodi Singer index theorem [2-I, Theorem 3.10] then yields

$$\operatorname{Index}(D(\rho)) = \int_N \mathcal{P} \pm \eta(M, \rho)$$

where $\mathcal{P}$ is the constant term in the asymptotic expansion of the heat equation; $\mathcal{P}$ vanishes as $\dim(N)$ is odd. This proves that $\eta(M,\rho)$ is a bordism invariant with values in $\mathbb{R}/\mathbb{Z}$. To complete the proof, we use Lemma 2.6 to squeeze out an extra factor of 2. In the cases discussed, $\delta(\epsilon, m) = 1$ so we do not have to complexify. If $m \equiv 2$ mod 8, $\Delta(N)$ and D admit a quaternion structure; we assumed ρ is real. If $m \equiv 6$ mod 8, $\Delta(N)$ and D are real; we assumed ρ is quaternion. Thus $\Delta(N) \otimes f_N^*(V(\rho))$ and D_ρ admit a quaternion structure so the (complex) dimension of the kernel and cokernel is even. □

§3 Twisted cyclic actions

3.1 spin^c structures with flat determinant line bundles. Let M be an odd dimensional orientable manifold with fundamental group $\pi_1(M) = \mathbb{Z}_\ell$ whose universal cover $\tilde{M}$ is spin. Let $\mathcal{P}_M(g_\ell)$ be a lift to the principal spin bundle on $\tilde{M}$ of the orientation preserving isometry $\mathcal{D}_M(g_\ell)$ of $\tilde{M}$. Then $\mathcal{P}_M(g_\ell)^\ell = 1 \iff M$ admits a spin structure. If $\mathcal{P}_M(g_\ell)^\ell = -1$, we let $\mathcal{P}_M^c(g_\ell) := e^{\pi\sqrt{-1}/\ell}\mathcal{P}_M(g_\ell)$ on the principal spin^c bundle. Then $\mathcal{P}_M^c(g_\ell)^\ell = 1$ so M admits a spin^c structure with flat associated line bundle given by ρ_1.

Conversely, suppose f is a $\mathbb{Z}_\ell$ structure on a manifold M and suppose that M admits a spin^c structure with associated flat determinant line bundle given by ρ_b. Then $\mathcal{Z}(M)$ admits a spin^c structure with trivial determinant line bundle and hence a spin structure; g_ℓ preserves this spin structure and the action of $\mathbb{Z}_\ell$ on the principal spin^c bundle of $\mathcal{Z}$ is given by $\pm\mathcal{P}_M^c(g_\ell) := e^{b\pi\sqrt{-1}/\ell}\mathcal{P}_M(g_\ell)$.

3.2 Definition. We define the external tensor product $\boxtimes$ as follows. Let

$$P_1 \boxtimes P_2 := P_1 \otimes 1 \otimes a_1 + 1 \otimes P_2 \otimes a_2 = \begin{pmatrix} P_1 \otimes 1 & 1 \otimes P_2 \\ 1 \otimes P_2 & -P_1 \otimes 1 \end{pmatrix} \text{ for}$$

(3.3) $$a_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ a_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ a_3 := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} / \sqrt{2};$$

$$a_1^2 = a_2^2 = a_3^2 = 1,\ a_1 a_2 + a_2 a_1 = 0,\ a_1 a_3 = a_3 a_2,\ a_2 a_3 = a_3 a_1.$$

3.4 Lemma. *Let $\mathcal{Z}$ be the principal $\mathbb{Z}_\ell$ bundle defined by a $\mathbb{Z}_\ell$ structure f on an oriented odd dimensional manifold M. Assume M admits a spin^c structure with associated flat determinant line bundle given by $f^*\rho_b$.*

(1) *The twisted product $N(M) := (\mathcal{Z} \times \mathcal{Z})/\mathbb{Z}_{2\ell}$ is a non-orientable manifold with natural $\mathbb{Z}_{2\ell}$ and pin$^-$ structures.*

(2) *We may identify* $\Delta(\mathcal{Z}\times\mathcal{Z})=\Delta(\mathcal{Z})\otimes\Delta(\mathcal{Z})\otimes\mathbb{C}^2$. *Under this identification,* $P(\mathcal{Z}\times\mathcal{Z})=P(\mathcal{Z})\boxtimes P(\mathcal{Z})$.

(3) *Let* $C(g_\ell)$ *and* $C(g_{2\ell})$ *give the action of* g_ℓ *and* $g_{2\ell}$ *on* $\Delta(\mathcal{Z})$ *and* $\Delta(\mathcal{Z}\times\mathcal{Z})$ *respectively. Let* $\epsilon=1$ *if* $m\equiv 3$ *mod* 4 *and* $\epsilon=\sqrt{-1}$ *if* $m\equiv 1$ *mod* 4. *Then* $C(g_{2\ell})(v_1\otimes v_2\otimes z)=\epsilon C(g_\ell)v_2\otimes v_1\otimes a_3 z$.

Proof. Let $\mathcal{D}_M$ give the natural action of $\mathbb{Z}_\ell$ on $\mathcal{Z}$. Let

$$\mathcal{D}_N(g_{2\ell}):(z_1,z_2)\mapsto(\mathcal{D}_M(g_\ell)z_2,z_1).$$

Then $\mathcal{D}_N(g_{2\ell})^2:(z_1,z_2)\mapsto(\mathcal{D}_M(g_\ell)z_1,\mathcal{D}_M(g_\ell)z_2)$ so $\mathcal{D}_M(g_{2\ell})^{2\ell}=1$. We have $\mathcal{D}_N$ defines a fixed point free isometric action of $\mathbb{Z}_{2\ell}$ on $\mathcal{Z}\times\mathcal{Z}$. Clifford multiplication defines an embedding $\operatorname{spin}(m)\times\operatorname{spin}(m)\subset\operatorname{spin}(2m)$. Since the spin^c structure on M lifts to a spin structure on $\mathcal{Z}$, $\mathcal{Z}\times\mathcal{Z}$ admits a natural spin structure. Since the dimension m of M is odd, the flip $(z_1,z_2)\to(z_2,z_1)$ reverses the orientation of $\mathcal{Z}\times\mathcal{Z}$. Since $\mathcal{D}_M(g_\ell)$ preserves the orientation of $\mathcal{Z}$, $\mathcal{D}_N(g_{2\ell})$ reverses the orientation of $\mathcal{Z}\times\mathcal{Z}$ so N is not orientable. Let $\mathcal{P}_N(g_{2\ell})$ be a lift of $\mathcal{D}_N(g_{2\ell})$ to the principal pin$^-$ bundle over $\mathcal{Z}\times\mathcal{Z}$; the sign will be normalized by assertion (3). Then we have $\mathcal{P}_N(g_{2\ell})^2=\pm\mathcal{P}_M(g_\ell)\times\mathcal{P}_M(g_\ell)$ so $\mathcal{P}_N(g_{2\ell})^{2\ell}=\mathcal{P}_M(g_\ell)^\ell\times\mathcal{P}_M(g_\ell)^\ell=1$ so $N(M)$ admits a natural pin$^-$ structure. This proves the first assertion.

Let $c_m:\operatorname{Clif}^-(m)\to\operatorname{End}(\Delta_m)$ be the canononical spin representation discussed in Lemma 2.6. Since $((c_m(\xi_1)\boxtimes c_m(\xi_2))^2=-|\xi_1^2|-|\xi_2^2|$, $c_m\boxtimes c_m$ defines a representation of $\operatorname{Clif}^c(2m)$; for dimensional reasons this representation is irreducible and can be identified with c_{2m}. This proves the second assertion.

Let $\mathcal{F}(\xi_1,\xi_2):=(\xi_2,\xi_1)\in O(2m)$ interchange the factors of $\mathbb{R}^m\times\mathbb{R}^m$. Let $\mathcal{F}_1$ be a lift of $\mathcal{F}$ from $O(2m)$ to $\operatorname{Pin}^-(2m)$. Let e_i be an orthonormal basis for $\mathbb{R}^m$. Then $e_i^1:=e_i\oplus 0$ and $e_i^2:=0\oplus e_i$ form an orthonormal basis for $\mathbb{R}^m\oplus\mathbb{R}^m=\mathbb{R}^{2m}$. Reflection in the hyperplane defined by $(e_i^1-e_i^2)/\sqrt{2}$ interchanges e_i^1 and e_i^2 and preserves e_j^ν for $i\neq j$. Consequently $\mathcal{F}_1=2^{-m/2}\Pi_i(e_i^1-e_i^2)$. This shows $\mathcal{F}_1^2=-1$ if $m\equiv 1$ mod 4 and $\mathcal{F}_1^2=+1$ if $m\equiv 3$ mod 4. Let $\mathcal{F}_2$ give the action of $\mathcal{F}_1$ on the spin bundle $\Delta(\mathcal{Z}\times\mathcal{Z})$; we used c_{2m} to define the operator and thus we use $\tilde{c}_{2m}$ to define the bundles; see §2 for details. Let $\mathcal{F}_3(v_1\otimes v_2\otimes z):=v_2\otimes v_1\otimes a_3 z$. Since $C(g_{2\ell})=(C(g_\ell)\otimes 1\otimes 1)\mathcal{F}_2$, to complete the proof of (3) we must show that $\mathcal{F}_2=\epsilon\mathcal{F}_3$ for suitably chosen ϵ. By equation (2.10), $\mathcal{F}_2 c_{2m}(\xi_1,\xi_2)=c_{2m}(\xi_2,\xi_1)\mathcal{F}_2$. By equation (3.3), $a_3a_2=a_1a_3$ and $a_3a_1=a_2a_3$ so $\mathcal{F}_3c_{2m}(\xi_1,\xi_2)=c_{2m}(\xi_2,\xi_1)\mathcal{F}_3$. Since c_{2m} is irreducible, $\mathcal{F}_2=\epsilon\mathcal{F}_3$ for some $\epsilon\in\mathbb{C}$. Since $a_3^2=1$, $\mathcal{F}_3^2=1$. If $m\equiv 3$ mod 4, then $\mathcal{F}_1^2=1$ so $\mathcal{F}_2^2=1$ and $\epsilon^2=1$. If $m\equiv 1$ mod 4, then $\mathcal{F}_1^2=-1$ so $\mathcal{F}_2^2=-1$ and $\epsilon^2=-1$. Since we can replace ϵ by $-\epsilon$ by changing the sign of $C(g_{2\ell})$, the third assertion follows; we use this choice of sign to normalize the pin structure chosen in assertion (1). □

3.5 Equivariant computation of eta invariant. Let $\mathcal{Z}_X$ be the principal $\mathbb{Z}_\ell$ bundle defined by a $\mathbb{Z}_\ell$ structure f on a manifold X which admits a (s)pin^c structure with associated flat determinant line bundle given by $f^*\rho_b$; we assume $\mathcal{Z}_X$ orientable. We may decompose $L^2(\Delta(\mathcal{Z}_X))=\oplus_\lambda E(\lambda,\mathcal{Z}_X)$ into the eigenspaces of the Dirac operator on $\mathcal{Z}_X$. The action $\mathcal{C}$ of $\mathbb{Z}_\ell$ on $L^2(\Delta(\mathcal{Z}_X))$ commutes with the Dirac operator so the eigenspaces $E(\lambda,\mathcal{Z}_X)$ are representation spaces for $\mathbb{Z}_\ell$. We decompose $E(\lambda,\mathcal{Z}_X)=\oplus_s E_s(\lambda,\mathcal{Z}_X)$ where $\mathcal{C}(g_\ell)=\rho_s(g_\ell)$ on $E_s(\lambda,\mathcal{Z}_X)$. We may identify $E_s(\lambda, \mathcal{Z}_X)$ with the corresponding eigenspace of the Dirac operator on X with coefficients in the representation ρ_s. Let $\{\lambda_{i,s}, \psi_{i,s}\}$ for $0 \le s < \nu$ and $i \in \mathbb{N}$ be an equivariant spectral resolution for the Dirac operator on $\mathcal{Z}_X$. The $\psi_{i,s}$ are a complete orthonormal basis for $L^2(\Delta(\mathcal{Z}_X))$ so that $\psi_{i,s} \in E_s(\lambda_{i,s}, \mathcal{Z}_X)$. Then

$$\eta(X, \rho_s)(z) = \Sigma_\lambda \operatorname{sign}(\lambda)|\lambda|^{-z} \dim E_s(\lambda, \mathcal{Z}_X) = \Sigma_i \operatorname{sign}(\lambda_{i,s})|\lambda_{i,s}|^{-z}. \tag{3.6}$$

We now come to the main analytic result of this paper.

3.7 Theorem. *Let $\mathcal{Z}$ be the principal $\mathbb{Z}_\ell$ bundle defined by a $\mathbb{Z}_\ell$ structure f on an orientable odd dimensional manifold M. Assume M admits a spin^c structure with associated flat determinant line bundle given by $f^*\rho_b$. If $m \equiv 3$ mod 4, let $\beta = 0$; if $m \equiv 1$ mod 4, let $\beta = \ell/2$.*

(1) *If $u = 2v - b + \beta$, $\eta(N, \rho_u) = \eta(M, \rho_v) - \eta(M, \rho_{v+\ell/2})$ in $\mathbb{R}/\mathbb{Z}$.*
(2) *If $u = 2v - b + \beta + 1$, $\eta(N, \rho_u) = 0$ in $\mathbb{R}/\mathbb{Z}$.*
(3) *If there are no harmonic spinors on $\mathcal{Z}$, these equalities hold in $\mathbb{R}$.*

Proof. We apply equation (3.6) and work equivariantly to compute the eta invariant. Let $\{\mu_{i,s}, \phi_{i,s}\}$ be an equivariant spectral resolution of the Dirac operator on $\mathcal{Z}$. Let $E(i,s,j,t) := \phi_{i,s} \otimes \phi_{j,t} \otimes \mathbb{C}^2$. Then $L^2(\Delta(\mathcal{Z} \times \mathcal{Z})) = \oplus_{i,s,j,t} E(i,s,j,t)$ and

$$P(\mathcal{Z} \times \mathcal{Z})(\phi_{i,s} \otimes \phi_{j,t} \otimes v) = \phi_{i,s} \otimes \phi_{j,t} \otimes (\mu_{i,s} a_1 + \mu_{j,t} a_2) v. \tag{3.8}$$

If $(i,s) \ne (j,t)$, the 4 dimensional space $\mathcal{E} := E(i,s,j,t) \oplus E(j,t,i,s)$ is invariant under the action of both the Dirac operator and the group $\mathbb{Z}_{2\ell}$. We compute

$$\begin{aligned} &\mathcal{C}(g_{2\ell})(\phi_{i,s} \otimes \phi_{j,t} \otimes v_1 \oplus \phi_{j,t} \otimes \phi_{i,s} \otimes v_2) \\ &= \rho_{-b+\beta+2s}(g_{2\ell})\phi_{i,s} \otimes \phi_{j,t} \otimes a_3 v_2 \oplus \rho_{-b+\beta+2t}(g_{2\ell})\phi_{j,t} \otimes \phi_{i,s} \otimes a_3 v_1. \end{aligned} \tag{3.9}$$

A word of explaination is in order. The action of $\mathbb{Z}_\ell$ on the principal spin^c bundle of $\mathcal{Z}$ is given by $e^{\pi\sqrt{-1}b/\ell}\mathcal{P}_M(g_\ell)$; we must undo this complex factor in computing the $\mathbb{Z}_{2\ell}$ action; this creates the factor of $\rho_{-b}(g_{2\ell})$. The factor of $\rho_\beta(g_{2\ell})$ comes from the correction factor of ϵ relating $\mathcal{F}_2$ and $\mathcal{F}_3$ as was discussed in Lemma 3.4. The remaining factors come from the parameters s and t. Let

$$\begin{aligned} &v_\varepsilon := \phi_{i,s} \otimes \phi_{j,t} \otimes v + \varepsilon \phi_{j,t} \otimes \phi_{i,s} \otimes a_3 v, \\ &\mathcal{C}(g_{2\ell}) v_\varepsilon = \rho_{-b+\beta+t+s}(g_{2\ell}) v_\varepsilon \text{ if } \varepsilon = \rho_{t-s}(g_{2\ell}), \\ &\mathcal{C}(g_{2\ell}) v_\varepsilon = \rho_{-b+\beta+t+s+\ell}(g_{2\ell}) v_\varepsilon \text{ if } \varepsilon = \rho_{t-s+\ell}(g_{2\ell}). \end{aligned}$$

Thus this gives the equivariant decomposition of $\mathcal{E}$. Let $\mathcal{E}_\varepsilon$ be the span of the v_ε for $v \in \mathbb{C}^2$. We use equation (3.8) and the commutation relations of equation (3.3) to see that the Dirac operator maps v_ε to $((\mu_{i,s}a_1 + \mu_{j,t}a_2)v)_\varepsilon$. Since the a_i are trace free, this operator has eigenvalues which occur in with opposite signs. If $(\mu_{i,s}, \mu_{j,t}) \ne (0,0)$ these eigenvalues are non-zero and cancel in the calculation of the eta invariant. If $(\mu_{i,s}, \mu_{j,t}) = (0,0)$, there are two 0 eigenvalues which do not contribute to the $\mathbb{R}/\mathbb{Z}$ valued invariant.

The argument given above shows that to compute $\eta(N,\rho)$, we may suppose $i=j$ and $s=t$. The eigenvalues of $a_3=(a_1+a_2)/2^{1/2}$ are ± 1. Let $a_3 v_\pm = \pm v_\pm$ and let $\psi_{\pm,i,s} := \phi_{i,s}\otimes\phi_{i,s}\otimes v_\pm$. Then $C(g_{2\ell})\psi_{\pm,i,s} = \pm\rho_{-b+\beta+2s}(g_{2\ell})\psi_{\pm,i,s}$, and

$$P(\mathcal{Z}\times\mathcal{Z})\psi_{\pm,i,s} = \mu_{i,s}\phi_{i,s}\otimes\phi_{i,s}\otimes(a_1+a_2)v = \pm 2^{1/2}\mu_{i,s}\psi_{\pm,i,s}.$$

The normalizing constant of $2^{1/2}$ plays no role in the eta invariant when we evaluate at $z=0$ and may be ignored. Let $u=-b+\beta+2v$. The eigenfunctions $\psi_{+,i,s}$ correspond to the representation ρ_u precisely when $s=v$; the eigenvalues $\mu_{i,v}$ then give rise to $\eta(M,\rho_v)$ in the calculation of $\eta(N,\rho_u)$. Since

$$-\rho_{-b+\beta+2s}(g_{2\ell}) = \rho_{-b+\beta+2(s-\ell/2)}(g_{2\ell}),$$

the eigenfunctions $\psi_{-,i,s}$ correspond to the representation ρ_u when $s=v+\ell/2$; the eigenvalues $-\mu_{i,v+\ell/2}$ then give rise to $\dim E_{v+\ell/2}(0,\mathcal{Z})-\eta(M,\rho_{v+\ell/2})$ in the calculation of $\eta(N,\rho_u)$; we must correct for the sign of the zero eigen values but this plays no role mod $\mathbb{Z}$. This proves assertion (1). Assertion (2) follows since there are no equivariant eigenspaces of this form corresponding to ρ_u if $u=-b+\beta+2v+1$. If there are no harmonic spinors, the 0 spectrum plays no role and the identities of assertions (1) and (2) hold in $\mathbb{R}$. □

One can also study even dimensional twisted products; we refer to the thesis of Barrera-Yanez [5] for the proof of the following result.

3.10 Theorem (Barrera-Yanez). *Let $\mathcal{Z}$ be the principal $\mathbb{Z}_\ell$ bundle defined by a $\mathbb{Z}_\ell$ structure f on a non-orientable even dimensional manifold M. Assume M admits a pin^c structure with associated flat determinant line bundle given by $f^*\rho_b$ and that $\mathcal{Z}$ is orientable. Use equation (1.5) to define an action of $\mathbb{Z}_{2\ell}$ on $\mathcal{Z}\times\mathcal{Z}$ and let $N:=(\mathcal{Z}\times\mathcal{Z})/\mathbb{Z}_{2\ell}$.*

(1) *If $\ell=2$, then N is a non-orientable manifold with $\pi_1(N)=\mathbb{Z}_4$ which admits a canonical pin^c structure with associated determinant line bundle given by ρ_1.*
   - i) *If $u=2s-b+m/2$, then $\eta(N,\rho_u)=\eta(M,\rho_s)$ in $\mathbb{R}/\mathbb{Z}$.*
   - ii *If $u=2s-b+1+m/2$, then $\eta(N,\rho_u)=0$ in $\mathbb{R}/\mathbb{Z}$.*

(2) *If $\ell>2$, then N is a non-orientable manifold with $\pi_1(N)=\mathbb{Z}_{2\ell}$ which admits a canonical pin$^-$ structure.*
   - i) *If $u=2s-b+m/2+\ell/4$, then $\eta(N,\rho_u)=\eta(M,\rho_s)+\eta(M,\rho_{s+\ell/4})$ in $\mathbb{R}/\mathbb{Z}$.*
   - ii) *If $u=2s+1-b+m/2+\ell/4$, then $\eta(N,\rho_u)=0$ in $\mathbb{R}/\mathbb{Z}$.*

(3) *If there are no harmonic spinors on $\mathcal{Z}$, these equalities hold in $\mathbb{R}$.*

There is another twisted product formula that is useful.

3.11 Theorem. *Let $\mathcal{Z}$ be the principal $\mathbb{Z}_\ell$ bundle defined by a $\mathbb{Z}_\ell$ structure f on a non-orientable even dimensional manifold M. Assume M admits a pin^c structure with associated flat determinant line bundle given by $f^*\rho_b$, that $\mathcal{Z}$ is orientable, and that $\mathcal{Z}$ has no harmonic spinors. Let U be an even dimensional spin manifold* *which admits a $\mathbb{Z}_\ell$ spin action. Give $U(M) := U \times_{\mathbb{Z}_\ell} \mathcal{Z}$ the natural $\mathbb{Z}_\ell$ structure and pin^c structure with associated flat determinant line bundle given by $f^*\rho_b$.*

(1) *The $\mathbb{Z}_\ell$ action on U induces representations $\rho_U^\pm$ on the kernel of the half spin operators. Decompose $\rho_U^+ - \rho_U^- = \Sigma_s n_s \rho_s$. Then we have $\eta(U(M), \rho_u) = \Sigma_s n_s \eta(M, \rho_{u-s})$.*

(2) *Give $U = S^1 \times S^1$ the spin structure with trivial principal spin bundle. Let $\ell \geq 4$. The map $g_\ell : u \mapsto -u$ defines a spinor action of $\mathbb{Z}_\ell$ on U and $\eta(U(M), \rho_u) = \eta(M, \rho_{u-\ell/4}(\rho_0 - \rho_{\ell/2}))$.*

Proof. Let ω_U be the normalized orientation of U; $c(\omega_U)$ anti-commutes with P_U and $c(\omega_U) = \pm 1$ on $\ker P_U^\pm$. We can decompose $\Delta(U \times \mathcal{Z}) = \Delta(U) \otimes \Delta(\mathcal{Z})$. Under this decomposition the Dirac operator on $U \times \mathcal{Z}$ takes the form $P_U \otimes 1 + c(\omega_U) \otimes P_M$. The action of $\mathbb{Z}_\ell$ is the tensor product of the two actions and commutes with $c(\omega_U)$. Let $\{\phi_{i,s}, \lambda_{i,s}\}$ and $\{\psi_{j,t}, \mu_{j,t}\}$ be an equivariant spectral resolution of the Dirac operators on U and on $\mathcal{Z}$. We have $c(\omega_U)\phi_{i,s} \in E_s(-\lambda_{i,s}, P_U)$. If $\lambda_{i,s} \neq 0$, let

$$E(i,s,j,t) := span\{\phi_{i,s} \otimes \psi_{j,t}, c(\omega_U)\phi_{i,s} \otimes \psi_{j,t}\}.$$

Then $\mathbb{Z}_\ell$ acts diagonally on $E(i,s,j,t)$. Since the eigenvalues of P on $E(i,s,j,t)$ are $\pm(\lambda_{i,s}^2 + \mu_{j,t}^2)^{1/2}$, these spaces play no role in the computation of the equivariant eta invariant. If $\phi_{i,s} \in \ker(P_U^\pm)$, then

$$P(\phi_{i,s} \otimes \psi_{j,t}) = \pm \mu_{j,t} \phi_{i,s} \otimes \psi_{j,t}.$$

We set $s + t = u$ and sum to derive the desired formula. We assume $\mathcal{Z}$ has no harmonic spinors to ensure that $U \times \mathcal{Z}(M)$ has no harmonic spinors to avoid difficulties with the zero eigenspace.

Give $U = S^1 \times S^1$ the spin structure s which has trivial principal spin bundle. Paradoxically, s is often called the *non-trivial* spin structure since $[U, s]$ generates $MSpin_2 = \mathbb{Z}_2$. The spinor bundle on U is a trivial bundle of complex dimension 2. The kernels of $P_U^\pm$ are 1 dimensional and g_ℓ acts by $\pm\sqrt{-1}$ on $\ker(P_U^\pm)$. The second assertion now follows from the first. ☐

§4 Spherical space forms and spherical space form bundles

In this section, we establish some technical results we will use in §5 and §6 to study metrics of positive scalar curvature. In Lemma 4.3, we express the eta invariant of lens spaces and lens space bundles in terms of Dedekind sums; in Lemma 4.4, we use the results of §3 to compute the eta invariant for the associated twisted products. Lemma 4.5 is a technical Lemma we will use to prove Lemma 4.8 which gives the order of the range of the eta invariant.

4.1 Definition (Lens spaces and lens space bundles). If $\vec{a} = (a_1, ..., a_k)$ is a collection of odd integers, let $\tau(\vec{a}) := \rho_{a_1} \oplus ... \oplus \rho_{a_k}$ define a fixed point free representation from $\mathbb{Z}_\ell$ to the unitary group $U(k)$. Let

$$L^{2k-1}(\ell; \vec{a}) := S^{2k-1}/\tau(\vec{a})(\mathbb{Z}_\ell)$$

be the resulting lens space. Let $H^{\otimes 2} \oplus (k-1)1$ be the Whitney sum of the tensor square of the complex Hopf line bundle with $(k-1)$ copies of the trivial complex line bundle over the sphere S^2. Let $\lambda \in S^1$ act by multiplication by λ^{a_ν} on the ν^{th} summand. This action restricts to a fixed point free action of $\mathbb{Z}_\ell$ on the sphere bundle $S(H^{\otimes 2} \oplus (k-1)1)$. Let

$$X^{2k+1}(\ell;\vec{a}) := S(H^{\otimes 2} \oplus (k-1)1)/\tau(\vec{a})(\mathbb{Z}_\ell);$$

this a bundle over S^2 with fiber $L^{2k-1}(\ell;\vec{a})$.

4.2 Definition (Notational Conventions). Let $\mathcal{G}_L^m$ and $\mathcal{G}_X^m$ be the free Abelian groups generated by the lens spaces and by the lens space bundles of dimension m. The eta invariant is additive and extends to these groups.

(1) If k is even, let $\Psi_L(\vec{a};\lambda) = \lambda^{-|\vec{a}|/2}\det(I - \tau(\vec{a})(\lambda))$.

(2) If k is odd, let $\Psi_L(\vec{a};\lambda) = \lambda^{-(|\vec{a}|+1)/2}\det(I - \tau(\vec{a})(\lambda))$.

(3) If $\lambda \neq 1$, let $\mathcal{F}_L(\vec{a};\lambda) = \Psi_L(\vec{a};\lambda)^{-1}$. If $\lambda = 1$, let $\mathcal{F}_L(\vec{a};\lambda) = 0$.

(4) Let $\mathcal{F}_X(\vec{a};\lambda) = (1+\lambda^{a_1})(1-\lambda^{a_1})^{-1}\mathcal{F}_L(\vec{a};\lambda)$.

(5) Define $\mathcal{B} : \mathcal{G}_L^m \to \mathcal{G}_L^{m+4}$ and $\mathcal{B} : \mathcal{G}_X^m \to \mathcal{G}_X^{m+4}$ by
$L^m(\ell;\vec{a}) \mapsto L^{m+4}(\ell;\vec{a},1,1) - 3L^{m+4}(\ell;\vec{a},1,3)$, and
$X^m(\ell;\vec{a}) \mapsto X^{m+4}(\ell;\vec{a},1,1) - 3X^{m+4}(\ell;\vec{a},1,3)$.

(6) Let $K_L^1 := L^1(\ell;1)$ and let $K_L^{4j+1} := \mathcal{B}K_L^{4j-3} \in \mathcal{G}_L^{4j+1}$.

(7) Let $K_X^3 := X^3(\ell;1)$ and let $K_X^{4j+3} := \mathcal{B}K_X^{4j-1} \in \mathcal{G}_X^{4j+3}$.

(8) Let $\psi := \Psi_L(1,1) = \rho_{-1}(\rho_0 - \rho_1)^2$ and let $\delta := \rho_0 - \rho_{\ell/2}$.

(9) Since $\Psi_L(j\cdot 1, j\cdot 3) \in R_0(\mathbb{Z}_\ell)^{2j} = \psi^{2j}R(\mathbb{Z}_\ell)$, we can choose ϕ_j so that $\phi_j\psi^j = \Psi_L(j\cdot 1, j\cdot 3)$.

We refer to [9,10] for the proof of the following result; the assertions concerning the eta invariant are based on results of Donnelly [12].

4.3 Lemma. *Let $\Sigma_\lambda := \Sigma_{\lambda\in\mathbb{Z}_\ell}$, and let $\tilde{\Sigma}_\lambda := \Sigma_{\lambda\in\mathbb{Z}_\ell,\lambda\neq 1}$.*

(1) *For $m \geq 3$, $L^m(\ell;\vec{a})$ and $X^m(\ell;\vec{a})$ admit metrics of positive scalar curvature.*

(2) *If k is even, then $L^{2k-1}(\ell;\vec{a})$ and $X^{2k+1}(\ell;\vec{a})$ admit spin structures.*

(3) *If k is odd, then $L^{2k-1}(\ell;\vec{a})$ and $X^{2k+1}(\ell;\vec{a})$ have spin^c structures with determinant line bundle given by ρ_1.*

(4) *We have $\eta(L^{2k-1}(\ell;\vec{a}),\rho) = \ell^{-1}\tilde{\Sigma}_\lambda Tr(\rho)\mathcal{F}_L(\vec{a};\lambda) \in \mathbb{Q}$.*

(5) *We have $\eta(X^{2k+1}(\ell;\vec{a}),\rho) = \ell^{-1}\tilde{\Sigma}_\lambda Tr(\rho)\mathcal{F}_X(\vec{a};\lambda) \in \mathbb{Q}$.*

The manifolds $N(L^m(\ell;\vec{a}))$ and $N(X^m(\ell;\vec{a}))$ admit metrics of positive scalar curvature for $m \geq 3$. Thus the formula of Lichnerowicz [25] shows there are no harmonic spinors. The following is an immediate consequence of Lemma 4.3 and of Theorem 3.7.

4.4 Lemma.

(1) *If* $m \equiv 3$ *mod* 4, *then* $\eta(N(L^m(\ell;\vec{a})),\rho_{2v}) = \eta(L^m(\ell;\vec{a}),\delta\rho_v)$.

(2) *If* $m \equiv 3$ *mod* 4, *then* $\eta(N(X^m(\ell;\vec{a})),\rho_{2v-1}) = \eta(X^m(\ell;\vec{a}),\delta\rho_v)$.

(3) *If* $m \equiv 1$ *mod* 4, *then* $\eta(N(L^m(\ell;\vec{a})),\rho_{2v-1+\ell/2}) = \eta(L^m(\ell;\vec{a}),\delta\rho_v)$.

(4) *If* $m \equiv 1$ *mod* 4, *then* $\eta(N(X^m(\ell;\vec{a})),\rho_{2v+\ell/2}) = \eta(X^m(\ell;\vec{a}),\delta\rho_v)$.

(5) *Otherwise* $\eta(N(L^m(\ell;\vec{a})),\rho_u) = 0$ *and* $\eta(N(X^m(\ell;\vec{a})),\rho_u) = 0$.

We shall need the following technical result.

4.5 Lemma. *Let* $\ell \geq 4$, $\rho \in R(\mathbb{Z}_\ell)$, $\vec{a} = (a_1,...,a_k)$, *and* $\vec{b} = (b_1,...,b_{k-1})$.

(1) *For* $k \equiv 0$ *mod* ℓ, $\ell^{-1}\Sigma_\lambda\lambda^k = 1$; *this vanishes otherwise.*

(2) *We have* $\eta(K_L^1,\delta) \equiv \eta(K_X^3,\delta) \equiv 1/2$ *mod* $\mathbb{Z}$.

(3) *If* $\gamma \in R_0(\mathbb{Z}_\ell)^{k+1}$, *then* $\ell^{-1}\tilde{\Sigma}_\lambda \operatorname{Tr}(\gamma(\lambda))\Pi_\nu(1-\lambda^{a_\nu})^{-1} \in \mathbb{Z}$,
$\eta(L^{2k-1}(\ell;\vec{a}),\gamma) \in \mathbb{Z}$, *and* $\eta(X^{2k-1}(\ell;\vec{b}),\gamma) \in \mathbb{Z}$.

(4) *We have* $\eta(L^{2k-1}(\ell;\vec{a},1) - 3L^{2k-1}(\ell;\vec{a},3),\rho) = \eta(L^{2k-1}(\ell;\vec{a},3),\psi\rho)$.

(5) *We have* $\eta(X^{2k-1}(\ell;\vec{b},1) - 3X^{2k-1}(\ell;\vec{b},3),\rho) = \eta(X^{2k-1}(\ell;\vec{b},3),\psi\rho)$.

(6) *If* $Y \in \mathcal{G}_L^m$ *or if* $Y \in \mathcal{G}_X^m$, *then* $\eta(B^jY,\phi_j\rho) = \eta(Y,\rho)$.

(7) *If* $j \geq 1$, *then* $\eta(K_L^{4j+1},\psi\delta\rho) \in \mathbb{Z}$ *and* $\eta(K_X^{4j+3},\psi\delta\rho) \in \mathbb{Z}$.

(8) *We have* $\eta(K_L^{4j+1},\phi_j\delta) \equiv \eta(K_X^{4j+3},\phi_j\delta) \equiv \frac{1}{2}$ *mod* $\mathbb{Z}$.

(9) *If* $X^5 := X^5(\ell;1,1) - 3X^5(\ell;1,3)$, *then* $\eta(X^5,\delta\rho) \equiv 0$ *mod* $\mathbb{Z}$ *and*
$\eta(X^5,\delta\rho_{\ell/4}) \equiv 1$ *mod* $2\mathbb{Z}$.

Proof. The first assertion follows from the orthogonality relations. We compute

$$\begin{aligned}\eta(K_L^1,\delta) &= \ell^{-1}\tilde{\Sigma}_\lambda\lambda(1-\lambda^{\ell/2})(1-\lambda)^{-1}\\ &= \ell^{-1}\tilde{\Sigma}_\lambda\lambda(1+\lambda+...+\lambda^{\ell/2-1})\\ &= -\ell^{-1}(\ell/2) + \ell^{-1}\Sigma_\lambda\lambda(1+\lambda+...+\lambda^{\ell/2-1}) = -1/2.\end{aligned}$$

We use the identity $\delta^2 = 2\delta$ to compute:

$$\begin{aligned}\eta(K_X^3,\delta) &= (2\ell)^{-1}\tilde{\Sigma}_\ell\lambda(1+\lambda)(1-\lambda^{\ell/2})^2(1-\lambda)^{-2}\\ &= (2\ell)^{-1}\tilde{\Sigma}_\lambda\lambda(1+\lambda)(1+\lambda+...+\lambda^{\ell/2-1})^2\\ &= -\ell/4 + (2\ell)^{-1}\Sigma_\lambda\lambda(1+\lambda)(1+\lambda+...+\lambda^{\ell/2-1})^2\\ &= -\ell/4 + 1/2.\end{aligned}$$

The second assertion now follows since $\ell \geq 4$. We set $\gamma_1 = \Pi_{1\leq\nu\leq k}(\rho_0 - \rho_{a_\nu})$. Then $\gamma_1 R_0(\mathbb{Z}_\ell) = R_0(\mathbb{Z}_\ell)^{k+1}$ so $\gamma = \gamma_1\epsilon$ for some $\epsilon \in R_0(\mathbb{Z}_\ell)$. Thus

$$\ell^{-1}\tilde{\Sigma}_\lambda \operatorname{Tr}(\gamma(\lambda))\Pi_\nu(1-\lambda^{a_\nu})^{-1} = \ell^{-1}\tilde{\Sigma}_\lambda \operatorname{Tr}(\epsilon(\lambda)).$$

Since $\operatorname{Tr}(\epsilon(1)) = 0$, we may replace $\tilde{\Sigma}_\lambda$ by Σ_λ and use (1) to see $\ell^{-1}\Sigma_\lambda \operatorname{Tr}(\epsilon(\lambda)) \in \mathbb{Z}$; this proves the first part of (3); the remaining parts now follow from Lemma 4.3. Assertions (4) and (5) follow from the identities

$$\begin{aligned}\mathcal{F}_L(\vec{a},1)(\lambda) - 3\mathcal{F}_L(\vec{a},3)(\lambda) &= \psi(\lambda)\mathcal{F}_L(\vec{a},3)(\lambda),\\ \mathcal{F}_X(\vec{a},1)(\lambda) - 3\mathcal{F}_X(\vec{a},3)(\lambda) &= \psi(\lambda)\mathcal{F}_X(\vec{a},3)(\lambda).\end{aligned} \tag{4.6}$$

Assertion (6) follows from equation (4.6) since $\phi_j\psi^j\mathcal{F}_-(\vec{a},j1,j3) = \mathcal{F}_-(\vec{a})$ for L and X. We use equation (4.6) to see that for any $\gamma \in R(\mathbb{Z}_\ell)$:

$$\text{(4.7)}\qquad \begin{aligned}\eta(K_L^{4j+1},\gamma) &= \eta(L^{4j+1}(\ell;(j+1)1,j3),\psi^j\gamma)\\ \eta(K_X^{4j+3},\gamma) &= \eta(X^{4j+3}(\ell;(j+1)1,j3),\psi^j\gamma).\end{aligned}$$

We apply equation (4.7) to $\gamma = \psi\delta\rho$. Since $\psi^{j+1}\rho\delta \in R_0(\mathbb{Z}_\ell)^{2j+3}$, assertion (7) follows from assertion (3). Assertion (8) follows from assertion (2) and assertion (6). Since $\lambda^{\ell/2} = \lambda^{3\ell/2}$ for $\lambda \in \mathbb{Z}_\ell$, we have

$$\begin{aligned}\eta(X^5,\delta\rho_{\ell/4}) &= \eta(X^5(\ell;1,3),\delta\rho_{\ell/4}\psi)\\ &= \ell^{-1}\tilde\Sigma_\lambda\lambda^{\ell/4}\lambda(1+\lambda)(1-\lambda^{\ell/2})/(1-\lambda^3)\\ &= \ell^{-1}\tilde\Sigma_\lambda\lambda^{\ell/4}(\lambda+\lambda^2)(1+\lambda^3+\ldots+\lambda^{3\ell/2-3})\\ &= -1+\ell^{-1}\Sigma_\lambda\lambda^{\ell/4}(\lambda+\lambda^2)(1+\lambda^3+\ldots+\lambda^{3\ell/2-3}).\end{aligned}$$

By (1), $\ell^{-1}\Sigma_\lambda\lambda^\nu = 0$ unless ℓ divides ν. The powers of λ that appear in this sum are $\nu = \ell/4+1+3k$ or $\nu = \ell/4+2+3k$ for $0 \le k < 3\ell/2$. Since $\nu \ge 1$ and since $3\ell/2-3+\ell/4+2 < 2\ell$, we need only consider $\nu = \ell$ so $1 = \ell-\ell/4-3k$ or $2 = \ell-\ell/4-3k$. This is not possible as $\ell-\ell/4-3k = 3(-k+\ell/4)$ and 3 does not divide 1 or 2. Thus $\eta(X^5,\rho_{\ell/4}) = -1$. We can decompose any $\rho = n_1\rho_{\ell/4}+\gamma$ for $\gamma \in R_0(\mathbb{Z}_\ell)$ and thus

$$\eta(X^5,\delta\rho) = -n_1 + n_2\eta(X^5(\ell;1,3),\delta\gamma\psi).$$

Since $\gamma\delta\psi \in R_0(\mathbb{Z}_\ell)^4$, $\eta(X^5,\delta\gamma\psi) \in \mathbb{Z}$. □

The Poincare dual A^* of an Abelian group A is the group of homomorphisms from A to $\mathbb{R}/\mathbb{Z}$. Thus, for example, $\mathbb{Z}_\ell^* = \mathbb{Z}/\ell\mathbb{Z}$.

(1) Let $\eta^*(M) \in R(\mathbb{Z}_\ell)^*$ be the homomorphism $\eta^*(M) : \rho \mapsto \eta(M,\delta\rho) \in \mathbb{Q}/\mathbb{Z}$.
(2) For $k \ge 1$, let $\mathcal{L}_{2k-1}(\ell) := \operatorname{span}_{\mathbb{Z}}\{\eta^*(L^{2k-1}(\ell;a_1,\ldots,a_k))\} \subseteq R(\mathbb{Z}_\ell)^*$.
(3) For $k \ge 1$, let $\mathcal{X}_{2k+1}(\ell) := \operatorname{span}_{\mathbb{Z}}\{\eta^*(X^{2k+1}(\ell;a_1,\ldots,a_k))\} \subseteq R(\mathbb{Z}_\ell)^*$.

4.8 Lemma. *Let $j \ge 0$.*

(1) *We have $2^{j+1} \le |\mathcal{L}_{2j+1}(2)|$.*
(2) *If $\ell \ge 4$, then $2^{j+1} \le |\mathcal{L}_{4j+1}(\ell)| \le |\mathcal{L}_{4j+3}(\ell)|$.*
(3) *If $\ell \ge 4$, then $2^{j+1} \le |\mathcal{X}_{4j+3}(\ell)| \le |\mathcal{X}_{4j+5}(\ell)|$.*

Proof. We use Lemma 4.3 to prove the first assertion: $\eta^*(RP^{2j+1},\rho_0) = \pm 2^{-j-1}$. Let $\sigma_{2j} := \rho_{-1}(\rho_0-\rho_1)$ and let $\sigma_{2j-1} := (\rho_0-\rho_1)$. Let $\vec{a} = (a_1,\ldots,a_k)$ for $k \ge 1$. Then

$$\text{(4.9)}\qquad \begin{aligned}&\sigma_k\mathcal{F}_L(\vec{a},1) = \mathcal{F}_L(\vec{a}) \text{ so } \eta(L^{2k+1}(\ell;\vec{a},1),\rho\sigma_k) = \eta(L^{2k-1}(\ell;\vec{a}),\rho);\\ &\sigma_k\mathcal{F}_X(\vec{a},1) = \mathcal{F}_X(\vec{a}) \text{ so } \eta(X^{2k+3}(\ell;\vec{a},1),\rho\sigma_k) = \eta(X^{2k+1}(\ell;\vec{a}),\rho).\end{aligned}$$

The map $\rho \mapsto \sigma_k\rho$ induces a dual map $\sigma_k^* : R(\mathbb{Z}_\ell)^* \to R(\mathbb{Z}_\ell)^*$. By equation (4.9), $\sigma_k^*\eta^*(L^{2k+1}(\ell;\vec{a},1)) = \eta^*(L^{2k-1}(\ell;\vec{a}))$ and $\sigma_k^*\eta^*(X^{2k+3}(\ell;\vec{a},1)) = \eta^*(X^{2k+1}(\ell;\vec{a}))$ so

$$\begin{aligned}&\mathcal{L}_{2k-1}(\ell) \subset \sigma_k^*\mathcal{L}_{2k+1}(\ell),\ |\mathcal{L}_{2k-1}(\ell)| \le |\mathcal{L}_{2k+1}(\ell)|,\\ &\mathcal{X}_{2k+1}(\ell) \subset \sigma_k^*\mathcal{X}_{2k+3}(\ell),\ \text{and } |\mathcal{X}_{2k+1}(\ell)| \le |\mathcal{X}_{2k+3}(\ell)|.\end{aligned}$$

By Lemma 4.5 (2), we have $|\mathcal{L}_1(\ell)| \geq 2$ and $|\mathcal{X}_3(\ell)| \geq 2$. Since $\psi = \Psi_L(1,1)$, a similar argument shows ψ^* is surjective so

$$|\mathcal{L}_{4k+1}(\ell)| \leq |\mathcal{L}_{4k-3}(\ell)| \cdot |\ker(\psi^*) \cap \eta^* \mathcal{L}_{4k+1}(\ell)|, \text{ and}$$
$$|\mathcal{X}_{4k+3}(\ell)| \leq |\mathcal{X}_{4k-1}(\ell)| \cdot |\ker(\psi^*) \cap \eta^* \mathcal{X}_{4k+3}(\ell)|.$$

By Lemma 4.5 (7), $\eta^* K_L^{4k+1} \in \ker(\psi^*)$ and $\eta^* K_X^{4k+3} \in \ker(\psi^*)$; by Lemma 4.5 (8), these elements have order at least 2. Assertion (2) now follows by induction. □

§5 Exotic metrics of positive scalar curvature

5.1 Twisted bordism groups. We generalize the equivariant pin bordism groups defined in §2.14. Let ξ be a real vector bundle over the classifying space $B\pi$ of a finite group π. The equivariant twisted bordism group $MSpin_m(B\pi, \xi)$ consists of triples (M, f, s) where f is a π structure on a manifold M of dimension m and where s is a spin structure on the bundle $T(M) \oplus f^*(\xi)$; we assume M is closed but not necessarily connected. We impose the equivalence relation $[(M, f, s)] = 0$ if there exists a compact manifold N (which need not be connected) with boundary M so the structures (f, s) extend over N. The group structure is defined by disjoint union. If $\pi_1(M) = \pi$, we shall give M the canonical π structure.

If the Stiefel Whitney classes w_1 and w_2 of ξ and $\tilde{\xi}$ agree, then the groups $MSpin_m(B\pi, \xi)$ and $MSpin_m(B\pi, \tilde{\xi})$ agree; thus only $w_i(\xi)$ for $i = 1, 2$ are relevant. If $w_1(\xi) = 0$ and if $w_2(\xi) = 0$, then $MSpin_m(B\pi, \xi) = MSpin_m(B\pi)$. If $w_1^2(\xi) + w_2(\xi) = 0$, then ξ admits a pin$^-$ structure and there is a natural map from $MSpin_m(B\pi, \xi)$ to $MPin_m^-(B\pi)$. If ξ admits a spin^c structure or a pin^c structure, there is a natural map from $MSpin_m(B\pi, \xi)$ to $MSpin_m^c(B\pi)$ or $MPin_m^c(B\pi)$; we use these maps to extend the eta invariant to this setting.

Note that not every pair of cohomology classes (u_1, u_2) for $u_i \in H^i(B\pi; \mathbb{Z}_2)$ can be realized as the first two Stiefel Whitney classes of a vector bundle ξ. Nevertheless, there is a generalization of the twisted spin bordism groups defined above which associates an Abelian group to every such pair (u_1, u_2) which is isomorphic to $MSpin_m(B\pi, \xi)$ if $(u_1, u_2) = (w_1(\xi), w_2(\xi))$. We refer to Stolz [35] for further details.

Suppose $\pi = \mathbb{Z}_n$ is cyclic. We take ξ trivial if n is odd. If n is even, let x generate $H^1(B\mathbb{Z}_n; \mathbb{Z}_2) = \mathbb{Z}_2$ and let y generate $H^2(B\mathbb{Z}_n; \mathbb{Z}_2) = \mathbb{Z}_2$. If $n \equiv 0 \mod 4$, then $x^2 = 0$; if $n \equiv 2 \mod 4$, then $x^2 = y$. We define real bundles ξ_i over $B\mathbb{Z}_n$ by requiring

$$w_1(\xi_0) = 0,\ w_2(\xi_0) = 0, \quad w_1(\xi_1) = 0,\ w_1(\xi_1) = y$$
$$w_1(\xi_2) = x,\ w_2(\xi_2) = 0, \quad w_1(\xi_3) = x,\ w_2(\xi_3) = y.$$

For example, we could take ξ_0 to be the trivial real line bundle, ξ_2 to be the real line bundle defined by $\rho_{n/2}$, ξ_1 to be the real 2-plane bundle defined by the complex representation ρ_1, and $\xi_3 = \xi_1 \oplus \xi_2$.

Let $\pi_1(M) = \mathbb{Z}_n$. Assume that the universal cover of M admits a spin structure. There exists a structure s on M; that $[(M, f, s)] \in MSpin_m(B\mathbb{Z}_n, \xi)$. If $n \equiv 0 \mod 4$:

0) We take $\xi = \xi_0$ if $w_1(M) = 0$ and $w_2(M) = 0$; M admits a spin structure.

1) We take $\xi = \xi_1$ if $w_1(M) = 0$ and $w_2(M) \neq 0$; M admits a spin^c structure with determinant line bundle given by ρ_1.

2) We take $\xi = \xi_2$ if $w_1(M) \neq 0$ and $w_2(M) = 0$; M admits a pin$^-$ structure.

3) We take $\xi = \xi_3$ if $w_1(M) \neq 0$ and $w_2(M) \neq 0$; M admits a pin^c structure with determinant line bundle given by ρ_1.

If n is odd, M is spin and we take $\xi = \xi_0$. If $n \equiv 2 \mod 4$, we take $\xi = \xi_3$ and give M a pin^c structure in 2) and we take $\xi = \xi_2$ and give M a pin$^-$ structure in 3).

Give $U = S^1 \times S^1$ the spin structure with associated trivial principal spin bundle. Let $\nu \equiv 0 \mod 4$. If Y is an even dimensional pin$^-$ manifold with a $\mathbb{Z}_\nu$ structure which carries the orientation, let $U(Y) := U \times_{\mathbb{Z}_\nu} \mathcal{Z}(Y)$ where $\mathcal{Z}(Y)$ is the associated principal $\mathbb{Z}_\nu$ bundle over Y. We have that:

a) $\mathbb{RP}^{4k} \in MSpin(B\mathbb{Z}_2, \xi_3)$, $\mathbb{RP}^{4k+2} \in MSpin(B\mathbb{Z}_2, \xi_2)$,

b) $L^{4k+1}(\ell; -) \in MSpin_{4k+1}(B\mathbb{Z}_\ell, \xi_1)$, $L^{4k+3}(\ell; -) \in MSpin_{4k+3}(B\mathbb{Z}_\ell, \xi_0)$,

c) $X^{4k+1}(\ell; -) \in MSpin_{4k+1}(B\mathbb{Z}_\ell, \xi_0)$, $X^{4k+3}(\ell; -) \in MSpin_{4k+3}(B\mathbb{Z}_\ell, \xi_1)$,

d) $N(L^m(\ell; -)) \in MSpin_{2m}(B\mathbb{Z}_{2\ell}, \xi_2)$, $N(X^m(\ell; -)) \in MSpin_{2m}(B\mathbb{Z}_{2\ell}, \xi_2)$,

e) $UN(L^m(\ell; -)) \in MSpin_{2m+2}(B\mathbb{Z}_{2\ell}, \xi_2)$, and
$UN(X^m(\ell; -)) \in MSpin_{2m+2}(B\mathbb{Z}_{2\ell}, \xi_2)$.

5.2 Twisted geometrical bordism groups. The group ${}^+MSpin_m(B\pi, \xi)$ is defined similarly. It consists of quadruples (M, f, s, g) where (M, f, s) is as above and where g is a metric of positive scalar curvature on M. We impose the equivalence relation ${}^+[(M, f, s, g)] = 0$ if there exists a compact manifold N (which need not be connected) with boundary M so the structures (f, s) extend over N and so that the metric on M extends as a metric of positive scalar curvature over N which is product near M. Again, the group structure is defined by disjoint union. If $[(M, f, s, g_1)] = [(M, f, s, g_2)]$ in ${}^+MSpin_m(B\pi, \xi)$, the metrics g_1 and g_2 are said to be *geometrically bordant.*

We say that two metrics of positive scalar curvature g_i on M are *concordant* if there exists a metric g on $M \times [0, 1]$ which has positive scalar curvature, which is product near the boundary, and which restricts to the given metrics at $M \times i$ for $i = 0, 1$. Let $\mathcal{R}(M)$ be the space of metrics of positive scalar curvature on M and let $\mathcal{M}(M) := \mathcal{R}(M)/\text{Diff}(M)$ be the associated moduli space. Two metrics which are in the same arc component of $\mathcal{R}(M)$ are necessarily concordant; it is not known if the converse holds.

A special case of the following Theorem for ξ orientable was proved by Botvinnik and Gilkey [7, 8]; it uses work of Gromov and Lawson [20, 21], Rosenberg [28], Rosenberg and Stolz [29], and Schoen and Yau [31]. The extension to the nonorientable setting is entirely straightforward and is therefore omitted.

5.3 Theorem. *Let π be a finite group. Let ρ be a virtual representation of π and let ξ be a real vector bundle over the classifying space $B\pi$. If m is even, assume that ξ is non-orientable and that ξ admits a pin^c structure. If m is odd, assume that ξ admits a spin^c structure and that ρ has virtual dimension 0. Let M be a connected closed manifold of dimension $m \geq 5$ with $\pi_1(M) = \pi$. Let f be the* *canonical π structure on M. Assume there exists a spin structure s on $T(M) \oplus f^*\xi$ so $[(M, f, s)] \in MSpin_m(B\pi, \xi)$.*

(1) *Suppose there exists a closed manifold M_1 which admits a metric g_1 of positive scalar curvature so that $[(M, f, s)] = [(M_1, f_1, s_1)]$ in $MSpin_m(B\pi, \xi)$; M_1 need not be connected. Then M admits a metric of positive scalar curvature g so that $[(M, s, f, g)] = [(M_1, s_1, f_1, g_1)]$ in ${}^+MSpin_m(B\pi, \xi)$.*
(2) *Let $[(M_2, f_2, s_2, g_2)] = 0$ in ${}^+MSpin_m(B\pi, \xi)$. Then $\eta(M_2, \rho) = 0$ in $\mathbb{R}$.*
(3) *Suppose that there exists $[(M_3, f_3, s_3, g_3)]$ in ${}^+MSpin_m(B\pi, \xi)$ such that $\eta(M_3, f_3, s_3, g_3, \rho) \neq 0$ in $\mathbb{R}$. Suppose that M admits a metric of positive scalar curvature. Then $\mathcal{M}(M)$ has an infinite number of components and there exists a countable family of metrics g_i of positive scalar curvature on M which are not geometrically bordant and which are not concordant.*

To apply Theorem 5.3, we must construct manifolds which admit metrics of positive scalar curvature and which have non-vanishing eta invariant. If σ is a group homomorphism from G to H, we have natural maps

$$\sigma_B : BG \to BH,\ \sigma_R : RH \to RG, \text{ and}$$
$$\sigma_M : MSpin_m(BG, \sigma_B^*\xi) \to MSpin_m(BH, \xi).$$

(When discussing the case m is even, we shall need to assume that both ξ and $\sigma_B^*\xi$ are non-orientable). Inequivalent spin^c structures on ξ_H are parametrized by complex line bundles; there exists a suitable linear representation ρ^ϵ which reflects choice of the determinant line bundle on $\sigma_B^*\xi$ so that:

$$\eta(\sigma_M(M), \rho) = \eta(M, \rho^\epsilon \sigma_R(\rho)). \tag{5.4}$$

For example, let σ be the natural surjective map from $\mathbb{Z}_{2\ell}$ to $\mathbb{Z}_\ell$. Then $\sigma_B^*\xi_1 = \xi_0$ and we take $\rho^\epsilon = \rho_1$. We can use equation (5.4) to reduce the existence of non-trivial eta invariant to a corresponding question concerning cyclic groups in many instances.

We begin with the odd dimensional case:

5.5 Lemma. *Let $m \geq 5$ be odd, let $n \geq 2$, and let $i = 0, 1$. If $i = 0$ and if $m \equiv 1$ mod 4 or if $i = 1$ and if $m \equiv 3$ mod 4, assume $n \geq 3$. Then there exists $[(M, f, s, g)]$ in ${}^+MSpin_m(B\mathbb{Z}_n, \xi_i)$ and $\rho \in R_0(\mathbb{Z}_n)$ so that $\eta(M, \rho) \neq 0$.*

Proof. This was proved in [7, 8]. We sketch the proof briefly. Let m be odd and let $n = ab$ where a and b are coprime and let σ be the natural inclusion of $\mathbb{Z}_a$ in $\mathbb{Z}_n$. Then σ_R is surjective. Thus by equation (5.4), we may suppose without loss of generality that $n = p^\nu$ is a non-trivial prime power. Suppose n is odd. The lens space $L^m(n; \vec{a})$ admit spin structures. We use a suitable generalization of Lemma 4.3 (4) to compute $\eta(L^m(n; \vec{a}), \Psi_L(\vec{a})) = (n-1)/n \neq 0$. Suppose $n = \ell$ is a non-trivial power of 2. We use the same argument if $m \equiv 3$ mod 4 and $\xi = \xi_0$ or if $m \equiv 1$ mod 4 and if $\xi = \xi_1$. If $m \equiv 3$ mod 4 and $\xi = \xi_1$ or if $m \equiv 1$ mod 4 and if $\xi = \xi_0$, we compute

$$\eta(X^m(\ell; \vec{a}), \Psi_L(\vec{a})(\rho_0 - \rho_{a_1})) = \ell^{-1}\tilde{\Sigma}(1 + \lambda^{a_1}) = \ell^{-1}(\ell - 2)$$

so this is non-trivial for $\ell > 2$. □

The following is now immediate

5.6 Theorem. *Let X be an orientable manifold of odd dimension $m \geq 5$ with non-trivial cyclic fundamental group $\mathbb{Z}_n$ whose universal cover is spin and which admits a metric of positive scalar curvature. If $m \equiv 3$ mod 4 and if $w_2(X) \neq 0$ or if $m \equiv 1$ mod 4 and if $w_2(X) = 0$, assume $n \geq 3$. Then $\mathcal{M}(X)$ has an infinite number of components and there exists a countable family of metrics g_i of positive scalar curvature on X which are not geometrically bordant and which are not concordant.*

There are suitable generalizations of this theorem to manifolds with other finite fundamental groups; we restrict to spin manifolds for the sake of simplicity and we refer to [7, 8] for the proof of the following result.

5.7 Theorem. *Let X be a spin manifold of odd dimension $m \geq 5$ with non-trivial finite fundamental group π which admits a metric of positive scalar curvature. If $m \equiv 1$ mod 3, assume π contains an element g which is not conjugate to g^{-1}. Then $\mathcal{M}(X)$ has an infinite number of components and there exists a countable family of metrics g_i of positive scalar curvature on X which are not geometrically bordant and which are not concordant.*

For the remainder of this section, we shall be interested in the case m even and M non orientable. We first take π cyclic.

5.8 Lemma. *Let $m \geq 6$ be even, let n be even, and let $i = 2, 3$. If $i = 2$ and if $m \equiv 0$ mod 4 or if $i = 3$ and if $m \equiv 2$ mod 4, assume $n \equiv 0$ mod 4. Then there exists $[(M, f, s, g)]$ in ${}^+MSpin_m(B\mathbb{Z}_n, \xi_i)$ and $\rho \in R(\mathbb{Z}_n)$ so that $\eta(M, \rho) \neq 0$.*

Proof. We may assume without loss of generality that $n = \ell$ is a power of 2. If $\ell = 2$, we take $M = \mathbb{RP}^m$ and use [15, Theorem 3.3] to see $\eta(\mathbb{RP}^m, \rho_0) = \pm 2^{-(m+2)/2} \neq 0$. We therefore suppose $\ell \geq 4$.

Let $m = 2(2k-1)$. We consider the following cases

(1) If $\xi = \xi_2$ and if $2k - 1 \equiv 3$ mod 4, let $\epsilon = 0$, let $v(u) = u$, and let $Y = L$.
(2) If $\xi = \xi_3$ and if $2k - 1 \equiv 3$ mod 4, let $\epsilon = 1$, let $v(u) = u + 1$, and let $Y = X$.
(3) If $\xi = \xi_2$ and if $2k - 1 \equiv 1$ mod 4, let $\epsilon = 0$, let $v(u) = u - \ell/4$, and let $Y = X$.
(4) If $\xi = \xi_3$ and if $2k - 1 \equiv 1$ mod 4, let $\epsilon = 1$, let $v(u) = u - \ell/4 + 1$, and let $Y = L$.

Let σ be the natural projection from $\mathbb{Z}_{2\ell}$ to $\mathbb{Z}_\ell$. We use equation (5.4) and Lemma 4.4 to see

$$\begin{aligned}\eta(\sigma_M N(Y^{2k-1}(\ell;\vec{a})), \rho_u) &= \eta(N(Y^{2k-1}(\ell;\vec{a})), \rho_{2u+\epsilon}) \\ &= \eta(Y^{2k-1}(\ell;\vec{a}), \rho_{v(u)}\delta).\end{aligned} \tag{5.9}$$

Since $\eta(N(Y^{2k-1}(\ell;\vec{a})), \rho_w)$ is supported on the representations $w \equiv 2u + \epsilon$ mod 2, we use Lemma 4.10 to see the eta invariant in equation (5.9) is non-trivial.

If M is an $m-2$ dimensional pin$^-$ manifold with a $\mathbb{Z}_{2\ell}$ structure, then we use Theorem 3.11 (2) to dimension shift. We use equation (5.9) and compute

$$\begin{aligned}\eta(\sigma_M UN(Y^{2k-1}(\ell;\vec{a})), \rho_u) &= \eta(UN(Y^{2k-1}(\ell;\vec{a})), \rho_{2u+\epsilon}) \\ &= \eta(N(Y^{2k-1}(\ell;\vec{a})), \rho_{2u+\epsilon+\ell/2}(\rho_0 - \rho_\ell)) \\ &= \eta(Y^{2k-1}(\ell;\vec{a}), \rho_{v(u)+\ell/4}\delta^2) \\ &= 2\eta(Y^{2k-1}(\ell;\vec{a}), \rho_{v(u)+\ell/4}\delta).\end{aligned}$$

Since we are working in $\mathbb{R}/\mathbb{Z}$, the additional factor of 2 plays no role and we use Lemma 4.4 to see these invariants are non-trivial. □

The following Theorem is a consequence of the discussion given above; the case $m \equiv 0 \bmod 4$ has also been derived by Barrera-Yanez [5] using a different method to establish the non-triviality of the eta invariant.

5.10 Theorem. *Let X be a non orientable manifold of even dimension $m \geq 6$ with cyclic fundamental group $\mathbb{Z}_n$ whose universal cover is spin and which admits a metric of positive scalar curvature. If $m \equiv 0$ mod 4 and if $w_2(X) \neq 0$ or if $m \equiv 2$ mod 4 and if $w_2(X) = 0$, assume $n \equiv 0$ mod 4. Then $\mathcal{M}(X)$ has an infinite number of components and there exists a countable family of metrics g_i of positive scalar curvature on X which are not geometrically bordant and which are not concordant.*

We can generalize these results to certain other finite groups. For the sake of simplicity, we will work with pin^- structures on manifolds of dimension $m \equiv 2$ mod 4; there are other theorems of this type for the other cases but they are more complicated to state. Again, we begin by constructing manifolds with non-vanishing eta invariants. Let ρ_0 be the trivial representation of π.

5.11 Lemma. *Let $m = 4k+2 \geq 6$. Let ξ be the real line bundle over the classifying space of a finite group π defined by a non-trivial representation Ξ from π to $\mathbb{Z}_2$. There exists $[(M, f, s, g)]$ in ${}^+MSpin^-_m(B\pi, \xi)$ so that $\eta(M, \rho_0) \neq 0$.*

Proof. Suppose that π contains an element g of order 2 so that $\Xi(g) = -1$. The map $g_2 \to g$ defines an embedding $\sigma : \mathbb{Z}_2 \to \pi$ and $\sigma_M[\mathbb{RP}^m]$ belongs to $MSpin_m(B\pi, \xi)$. We use [15] to see $\eta(\sigma_M \mathbb{RP}^m, \rho_0) = \eta(\mathbb{RP}^m, \rho_0) \neq 0$.

Suppose π contains an element g of order $\ell \geq 4$ so that $\Xi(g) = -1$. Give $Y^{4j+6} := U^{2j} N(L^3(\ell; 1, 1))$ the natural pin^- structure and $\mathbb{Z}_{2\ell}$ structure. The map $\sigma : g_{2\ell} \mapsto g$ defines a map $\sigma : \mathbb{Z}_{2\ell} \to \pi$. Then

$$\begin{aligned}
&\eta(\sigma_M U^{2j} N(L^3(\ell; 1, 1)), \rho_0) = \eta(N(L^3(\ell; 1, 1)), \rho_{j\ell}(\rho_0 - \rho_\ell)^{2j}) \\
&= \eta(L^3(\ell; 1, 1), \rho_{j\ell/2}\delta^{2j+1}) = 2^{2j-1}\ell^{-1}\tilde{\Sigma}\lambda^{j\ell/2+1}(1 - \lambda^{\ell/2})^2/(1 - \lambda)^2 \\
&= 2^{2j-1}\ell^{-1}\tilde{\Sigma}\lambda^{j\ell/2+1}(1 + \lambda + \lambda^2 + \ldots\lambda^{\ell/2-1})^2 \\
&= 2^{2j-1}\ell^{-1}(-\ell^2/4 + \Sigma\lambda^{j\ell/2+1}(1 + \ldots + \ell/2\lambda^{\ell/2-1} + \ldots + \lambda^{\ell/2-2}) \\
&= \pm 2^{2j-1}\ell/4. \quad \square
\end{aligned}$$

The following theorem is now immediate.

5.12 Theorem. *Let X be a pin^- manifold of dimension $m = 4k+2 \geq 6$ with finite fundamental group π. Assume X admits a metric of positive scalar curvature. Then $\mathcal{M}(X)$ has an infinite number of components. Furthermore there exists a countable family of metrics g_i of positive scalar curvature on X which are not geometrically bordant and which are not concordant.*

§6 The Gromov-Lawson conjecture

Let M be a spin manifold of dimension $m \equiv 0$ mod 4. Let $\hat{A}(M) \in \mathbb{Z}$ be the index of the Dirac operator on M; by the index theorem, we can compute $\hat{A}(M)$ as the integral of a polynomial in the Pontrjagin classes of M so $\hat{A}(M)$ is independent of the metric on M and of the spin structure which is chosen on M. If M admits a metric of positive scalar curvature, there are no harmonic spinors on M by the Lichnerowicz formula [25] and thus $\hat{A}(M) = 0$. Consequently, if $\hat{A}(M) \neq 0$, then M does not admit a metric of positive scalar curvature. The Kummer surface

$$K^4 := \{\langle z \rangle \in \mathbb{C}P^3 : z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$$

is an algebraic surface which admits a spin structure with $\hat{A}(K^4) = 2$. Thus K^4 does not admit a metric of positive scalar curvature. If $m \equiv 0 \bmod 4$, $\hat{A}$ is $\mathbb{Z}$ valued. We can define a $\mathbb{Z}_2$ valued index if $m \equiv 1$ or if $m \equiv 2 \bmod 8$. Let P_s be the Dirac operator defined by a spin structure s on a manifold of dimension m. If $m \equiv 1 \bmod 8$, let $\hat{A}(M,s) \in \mathbb{Z}_2$ be the mod 2 reduction of $\dim(\ker(P_s))$. If $m \equiv 2 \bmod 8$, then $\dim(\ker(P_s))$ is even and we let $\hat{A}(M,s) = \dim(\ker(P_s))/2 \in \mathbb{Z}_2$. We set $\hat{A} = 0$ for other values of m. Then $\hat{A}(M,s)$ depends on the spin structure s but not on the Riemannian metric. For example, the circle S^1 admits two spin structures s_i. If s_1 defines the trivial principal spin bundle and s_2 defines the non-trivial principal spin bundle, $\hat{A}(S^1, s_1) \neq 0$ and $\hat{A}(S^1, s_2) = 0$. We extend $\hat{A}$ to the groups $MSpin_m(B\mathbb{Z}_\ell, \xi_i)$ for $i = 1, 2, 3$ by defining $\hat{A}(M) = \hat{A}(\mathcal{Z}(M))$.

If M is simply connected, the spin structure is unique and we drop the dependence upon s. The $\hat{A}$ genus vanishes if M admits a metric of positive scalar curvature. Stolz [34] has shown that the converse holds in the simply connected case if $m \geq 5$.

6.1 Theorem. *Let M be a spin manifold of dimension $m \geq 5$. Then M admits a metric of positive scalar curvature $\iff$ $\hat{A}(M) = 0$.*

If the fundamental group π of a spin manifold M is non-trivial, Rosenberg [28] has defined an element $\alpha(M)$ generalizing the A-roof genus which takes values in the K theory of the reduced C^* algebra $C_r^*(\pi)$. If M is not spin, but the universal cover of M is spin, then α extends suitably; if M admits a metric of positive scalar curvature, then $\alpha(M) = 0$. The Gromov-Lawson-Rosenberg conjecture is that this is the only obstruction to the existence of a metric of positive scalar curvature if $m \geq 5$. We refer to Rosenberg and Stolz [29] for a general discussion of this conjecture.

The fundamental group of M is crucial in this subject. We refer to [9, 10] for the proof of the following theorem:

6.2 Theorem. *Let M be an orientable manifold of dimension $m \geq 5$ with cyclic fundamental group whose universal cover $\tilde{M}$ is a spin manifold.*

(1) *If M is spin, then M admits a metric of positive scalar curvature if and only if $\hat{A}(M,s) = 0$ for every spin structure s on M.*
(2) *If M is not spin, then M admits a metric of positive scalar curvature if and only if $\hat{A}(\tilde{M}) = 0$.*

In this section, we will establish a special case of the Gromov-Lawson conjecture in the non-orientable setting.

6.3 Theorem. *Let M be a non-orientable of dimension $m = 4k+2 \geq 6$ with cyclic fundamental group which admits a pin^- structure. Then M admits a metric of positive scalar curvature if and only if $\hat{A}(\tilde{M}) = 0$.*

By a theorem of Kwasik and Schultz [23], the Gromov-Lawson-Rosenberg Conjecture is true for a finite group π if and only if it is true for all Sylow subgroups of π. Thus we can work one prime at a time. The odd primes are covered by Theorem 6.2 so we assume $\pi_1(M) = \mathbb{Z}_\ell$. The case $\ell = 2$ is covered by work of Rosenberg and Stolz [29] so we assume $\ell \geq 4$. The rest of this section is devoted to consideration of this case. We must first establish some additional technical results.

Theorem 5.3 reduces the question of constructing a metric of positive scalar curvature on M to a question in equivariant bordism. Our next step is to reduce to a question in connective k theory. Let $\mathbb{HP}^2$ be quaternion projective space with the usual homogeneous metric and let $\mathbb{HP}^2 \to E \to B$ be a fiber bundle where the transition functions are the group of isometries $PSp(3)$ of $\mathbb{HP}^2$. Since $\mathbb{HP}^2$ is simply connected, the projection $p: E \to B$ induces an isomorphism of fundamental groups; any $\mathbb{Z}_\ell$ structure on E arises from a $\mathbb{Z}_\ell$ structure on B. Let $T_m(B\mathbb{Z}_\ell, \xi_i)$ be the subgroup of $MSpin_m(B\mathbb{Z}_\ell, \xi_i)$ generated by manifolds E arising in this fashion. Let

$$to_m(B\mathbb{Z}_\ell, \xi_i) := MSpin_m(B\mathbb{Z}_\ell, \xi_i)/T_m(B\mathbb{Z}_\ell, \xi_i).$$

Let $to_m^+(B\mathbb{Z}_\ell, \xi_i)$ be the image of the subgroup generated by classes $[(M, s, f)]$ where M admits a metric of positive scalar curvature. Let $ko_m(B\mathbb{Z}_\ell, \xi_i)$ be the twisted connective K-theory groups. If M has a $\mathbb{Z}_\ell$ structure, let $\mathcal{Z}(M)$ be the associated $\mathbb{Z}_\ell$ principal bundle.

6.4 Lemma.

(1) *We have $to_m(B\mathbb{Z}_\ell, \xi_i) \approx ko_m(B\mathbb{Z}_\ell, \xi_i)$.*

(2) *If $\ell > 2$, then $|ko_{8k+2}(B\mathbb{Z}_\ell, \xi_2)| \leq 2^{2k+3}$ and $|ko_{8k+6}(B\mathbb{Z}_\ell, \xi_2)| \leq 2^{2k+2}$.*

(3) *If m is odd, if $\rho \in R_0(\mathbb{Z}_\ell)$, and if $i = 0, 1$, then the map $M \to \eta(M, \rho)$ extends to homomorphisms η_ρ from $to_m(B\mathbb{Z}_\ell, \xi_i)$ to $\mathbb{R}/\mathbb{Z}$. If $m \equiv 3$ mod 8, if $i = 0$, and if ρ is real, we can extend η_ρ to take values in $\mathbb{R}/2\mathbb{Z}$.*

(4) *If m is even, if $\rho \in R(\mathbb{Z}_\ell)$, and if $i = 2, 3$, then the map $M \to \eta(M, \rho)$ extends to homomorphisms η_ρ from $to_m(B\mathbb{Z}_\ell, \xi_i)$ to $\mathbb{R}/\mathbb{Z}$. If $m \equiv 2$ mod 8, if $i = 2$, and if ρ is real, we can extend η_ρ to take values in $\mathbb{R}/2\mathbb{Z}$.*

(5) *If $\ell > 2$, $\hat{A}$ extends to a surjective homomorphism from $to_{2+8k}(B\mathbb{Z}_\ell; \xi_2)$ to $\mathbb{Z}_2$.*

(6) *Let $m = 4k+2 \geq 6$. To prove Theorem 6.3, it suffices to show that*
$to_m^+(B\mathbb{Z}_\ell, \xi_2) = \ker(\hat{A}) \cap to_m(B\mathbb{Z}_\ell, \xi_2)$.

Proof. The first assertion follows from results of Stolz [33] and is a crucial link between the geometry of $\mathbb{HP}^2$ fibrations and some powerful methods of algebraic topology. The second assertion follows from [9, Theorem 1.5]. It is based on a calculation using the Adams spectral sequence. We use Lemma 2.15 to extend the eta invariant to $MSpin_m(\mathbb{Z}_\ell, \xi_i)$. Let E be the total space of a $\mathbb{HP}^2$ fibration. Botvinnik and Gilkey [9] showed that E admits a metric g so that $\eta(E, \rho) = 0$ in $\mathbb{R}$ and so that g has positive scalar curvature. The extension of the eta invariant and the $\hat{A}$ genus to connective K theory now follows. Let $M = N(S^1)$. Then $\mathcal{Z}(M) = S^1 \times S^1$ has the trivial (i.e. non-bounding) spin structure. Since the dimension of the kernel of the Dirac operator is 2, $\hat{A}(N(S^1)) = \hat{A}(S^1 \times S^1) = 1$ and $\hat{A}$ is surjective if $m = 2$. Let B^8 be the Bott manifold; this is a simply connected spin manifold with $\hat{A}(B^8) = 1$. The Cartesian product $N(S^1) \times B^8$ inherits natural pin$^-$ and $\mathbb{Z}_\ell$ structures. We use the multiplicative nature of the $\hat{A}$ genus to complete the proof of assertion (4) by checking

$$\hat{A}(N(S^1) \times (B^8)^j) = \hat{A}(N(S^1))\hat{A}(B^8)^j = 1.$$

The elements of $T_m(B\mathbb{Z}_\ell, \xi_2)$ can be represented by manifolds that admit metrics of positive scalar curvature. If assertion (5) holds, then we can represent any element of $\ker(\hat{A}) \cap MSpin_m(B\mathbb{Z}_\ell, \xi_2)$ by a manifold that admits a metric of positive scalar curvature. We then use Theorem 5.3 to establish the Gromov-Lawson conjecture in this case. □

Proof of Theorem 6.3. Let $n = 2^\nu \geq 4$. Since $\hat{A}$ is non-trivial, we use Lemma 6.4 to see

$$|\ker(\hat{A}) \cap to_{8k+2}(B\mathbb{Z}_n, \xi_2)| \leq 2^{2k+2} \text{ and } |to_{8k+6}(B\mathbb{Z}_n, \xi_2)| \leq 2^{2k+2}.$$

By Lemma 6.4, to prove Theorem 6.3, we must show

$$|to^+_{8k+2}(B\mathbb{Z}_n, \xi_2)| \geq 2^{2k+2} \text{ and } |to^+_{8k+6}(B\mathbb{Z}_n; \xi_2)| \geq 2^{2k+2}. \tag{6.5}$$

Suppose first $n = 4$. We use Lemmas 4.3 and 4.4 to compute

$$\eta(N(\mathbb{RP}^{4k+1}), \rho_0) = \eta(\mathbb{RP}^{4k+1}, \delta) = 2^{-2k-1}, \text{ and}$$
$$\eta(N(\mathbb{RP}^{4k+3}), \rho_0) = \eta(\mathbb{RP}^{4k+3}, \delta) = 2^{-2k-2}.$$

By Lemma 6.4, $N \to \eta(N, \rho_0)$ extends to a map from $to_m(B\mathbb{Z}_4, \xi_2)$ to $\mathbb{R}/2\mathbb{Z}$ for $m = 8k + 2$ and to $\mathbb{R}/\mathbb{Z}$ for $m = 8k + 6$. Thus $N(\mathbb{RP}^{4k+1})$ and $N(\mathbb{RP}^{4k+3})$ are elements of order at least 2^{2k+2} in $to^+_{8k+2}(B\mathbb{Z}_4, \xi_2)$ and $ko^+_{8k+6}(B\mathbb{Z}_4, \xi_2)$ so the estimate of equation (6.5) holds.

Now suppose $n = 2\ell$ for $\ell \geq 4$. Let $\mathcal{T}_m(L, 2\ell)$ and $\mathcal{T}_m(X, 2\ell)$ be the subspaces of $to_m(B\mathbb{Z}_{2\ell}, \xi_2)$ spanned by the images of $N(L^m(\ell; *))$ and $N(X^m(\ell; *))$ respectively. The map $M \mapsto \eta^*(M) \in R(\mathbb{Z}_{2\ell})^*$ defined in §3 extends to these two spaces with disjoint supports; the relevant parities in Lemma 4.4 are opposite. Thus the eta invariant decouples and we may use Lemma 4.8 to see

$$|\eta^* to^+_m(B\mathbb{Z}_{2\ell}, \xi_2)| \geq |\eta^*(\mathcal{T}_m(L, 2\ell))| \cdot |\eta^*(\mathcal{T}_m(X, 2\ell))| = |\mathcal{L}_m(2\ell)| \cdot |\mathcal{X}_m(2\ell)| \text{ so}$$
$$|\eta^* to^+_{8j+2}(B\mathbb{Z}_{2\ell}, \xi_2)| \geq 2^{2j+1} \text{ and } |\eta^* to^+_{8j+6}(B\mathbb{Z}_{2\ell}, \xi_2)| \geq 2^{2j+2}.$$

This shows that estimate (6.5) holds if $m = 8j + 6$; to obtain the desired estimate if $m = 8j + 2$, we need only show:

$$|\ker(\eta^*) \cap to^+_{8j+2}(B\mathbb{Z}_{2\ell}, \xi_2)| \geq 2.$$

We will use the refined eta invariant $\eta_0(N) := \eta(N, \rho_0) \in \mathbb{R}/2\mathbb{Z}$ to detect the kernel of η^*. Let $X^5 := X^5(\ell; 1, 1) - 3X^5(\ell; 1, 3)$. By Lemma 4.8, $\eta^*(N(X^5)) = 0$ and $\eta_0(N(X^5)) = \eta(X^5, \delta\rho_{\ell/4}) = 1$. Thus $N(X^5)$ is a non-trivial element of order at least 2 in $\ker(\eta^*)$. Give the manifold B^8 the trivial $\mathbb{Z}_{2\ell}$ structure. By Theorem 3.11, $\eta(N(X^5) \times B^8) = \eta(N(X^5))\hat{A}(B^8)^j = \eta(N(X^5))$ and the general case now follows. □

We can draw some consequences from the discussion given above. We use the result of Stolz cited above to identify ko_m with to_m; let $ko_m^+(\mathbb{Z}_\ell, \xi_i)$ be the subgroup generated by the manifolds that admit metrics of positive scalar curvature. By Theorems 6.2 and 6.3, if $m \geq 5$

$$\begin{aligned} ko_m^+(\mathbb{Z}_\ell, \xi_0) &= \ker(\hat{A}) \cap ko_m(\mathbb{Z}_\ell, \xi_0) \\ ko_m^+(\mathbb{Z}_\ell, \xi_1) &= \ker(\hat{A}) \cap ko_m(\mathbb{Z}_\ell, \xi_1) \\ ko_m^+(\mathbb{Z}_\ell, \xi_2) &= \ker(\hat{A}) \cap ko_m(\mathbb{Z}_\ell, \xi_2) \text{ if } m \equiv 2 \bmod 4. \end{aligned}$$

In [9, 10] we showed the eta invariant and the $\hat{A}$ genus provided the characteristic numbers of the connective K theory groups $ko_m(\mathbb{Z}_\ell, \xi_i)$ for $i = 0, 1$. We can generalize this result to ξ_2 if $m \equiv 2 \bmod 4$.

6.6 Theorem. *Let $k \geq 1$.*

(1) *Let $x \in ko_{8k+2}(B\mathbb{Z}_{2\ell}, \xi_2)$. If $\hat{A}(x) = 0$, if $\eta^*(x) = 0$, and if $\eta_0(x) = 0$, then $x = 0$.*
(2) *Let $x \in ko_{8k+6}(B\mathbb{Z}_{2\ell}, \xi_2)$. If $\eta^*(x) = 0$, then $x = 0$.*
(3) *Let $i = 0$ or $i = 1$; if $4k + 2 \equiv 2$ mod 8 and if $\ell = 2$, assume $i = 0$. The map $M \mapsto N(M)$ extends to a homomorphism from $ko_{2k+1}^+(B\mathbb{Z}_\ell, \xi_i)$ to $ko_{4k+2}(B\mathbb{Z}_{2\ell}, \xi_2)$.*

Proof. The first two assertions follow from the proof of Theorem 6.3. Lemma 4.4 expresses $\eta^*(N(M^{2k+1}))$ in terms of $\eta^*(M^{2k+1})$. By Lemma 6.4, the eta invariant extends to connective K theory so $[M] = 0$ in $ko_{2k+1}(B\mathbb{Z}_\ell, \xi_i)$ implies $\eta^*(M) = 0$ so $\eta^* N(M) = 0$; this shows $M \mapsto N(M)$ extends to connective K theory if $4k + 2 \equiv 6$ mod 8. Let $4k + 2 \equiv 2$ mod 8. We suppose M admits a metric of positive scalar curvature. This implies $\hat{A}(N(M)) = 0$ so to complete the proof we must show $\eta_0(M) = 0$. We apply the identities of Theorem 3.7 to the case $u = 0$. We have $\eta_0 = 0$ if $u - b + \ell/2$ is odd which handles the cases $i = 0$ and $\ell = 2$ and $i = 1$ and $\ell > 2$.

If $i = 0$ and if $\ell > 2$, then $\eta_0(N(M)) = \eta(M, \rho_{\ell/4} - \rho_{-\ell/4})$. There are two fundamental representations $\Delta^\pm$ of the complex Clifford algebra $\text{Clif}^c(2k+1)$ which may be distinguished by the identity $\Delta^\pm(\omega_m) = \pm 1$. Since $2k + 1 \equiv 1$ mod 4, we have $\bar{\omega}_m = -\omega_m$. Consequently complex conjugation defines a conjugate linear isomorphism

$$E(\lambda, P(\rho_s)) \approx E(-\lambda, P(\rho_{-s}));$$

it is crucial at this point that we are dealing with a spin structure not with a spin^c structure. Since there are no harmonic spinors, $\eta(M, \rho_s) = -\eta(M, \rho_{-s})$. Thus

$$\begin{aligned} \eta_0(N(M)) &= \eta(M, \rho_{\ell/4} - \rho_{-\ell/4}) = \eta(M, \rho_{\ell/4} - \rho_0) - \eta(M, \rho_{-\ell/4} - \rho_0) \\ &= 2\eta(M, \rho_{\ell/4} - \rho_0). \end{aligned}$$

Since $[M] = 0$ in connective K theory and since the eta invariant extends to connective K theory, $\eta(M, \rho_{\ell/4} - \rho_0) \in \mathbb{Z}$ and thus $\eta_0(N(M)) \in 2\mathbb{Z}$.

This shows the map $M \mapsto N(M)$ is well defined map in connective K theory. We complete the proof by showing it is a group homomorphism. Let $M = M_1 \dot{\sqcup} M_2$ be the disjoint union of two manifolds M_i. Let

$$X := (\mathcal{Z}_1 \times \mathcal{Z}_2) \dot{\sqcup} (\mathcal{Z}_2 \times \mathcal{Z}_1) \text{ and } N_3 := X/\mathbb{Z}_{2\ell}.$$

Then $N(M) = N(M_1) \dot{\sqcup} N(M_2) \dot{\sqcup} N_3$. We may choose an orientation of X so the flip which interchanges the two pieces preserves the orientation. Then $\mathbb{Z}_{2\ell}$ acts on X by orientation preserving isometries so N_3 is orientable and the pin$^-$ structure on N_3 is a spin structure. The action of the orientation form anti-commutes with the Dirac operator in even dimensions so $E(\lambda, P_{\rho_s}) \approx E(-\lambda, P_{\rho_s})$. Since there are no harmonic spinors, this shows $\eta(N_3, \rho_s) = 0$ and hence $[N_3] = 0$ in $ko_m(\mathbb{Z}_\ell, \xi_2)$. □

References

[1] M. F. Atiyah, R. Bott, and R. Shapiro, *Clifford modules*, Topology **3 supl** (1964), 3–38.

[2] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry I, II, III*, Math. Proc. Cambr. Phil. Soc. **77** (1975) 43–69; **78** (1975) 405–432; **79** (1976) 71–99.

[3] A. Bahri, M. Bendersky, D. Davis, and P. Gilkey, *The complex bordism of groups with periodic cohomology*, Trans. AMS **316** (1989), 673–687.

[4] A. Bahri and P. Gilkey, *The eta invariant, Pin^c bordism, and equivariant Spin^c bordism for cyclic 2-groups*, Pacific Journal **128** (1987), 1–24.

[5] E. Barrera-Yanez, *The eta invariant, connective K-theory, and equivariant spin bordism*, Ph. D. thesis (University of Oregon) 1997.

[6] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete; 3. Folge, Bd. 10., Springer-Verlag, Berlin, New York, 1987.

[7] B. Botvinnik and P. Gilkey, *The eta invariant and metrics of positive scalar curvature*, Math. Ann. **302** (1995), 507–517.

[8] ———, *Metrics of positive scalar curvature on spherical space forms*, Canadian Math Journal **48** (1996), 64–80.

[9] ———, *The Gromov-Lawson-Rosenberg conjecture: the twisted case*, preprint.

[10] B. Botvinnik, P. Gilkey, and S. Stolz, *The Gromov-Lawson-Rosenberg conjecture for groups periodic cohomology*, preprint IHES/M/94/62.

[11] T. Branson and P. Gilkey, *Residues of the eta function for an operator of Dirac Type*, J. Functional Analysis **108** (1992), 47–87.

[12] H. Donnelly, *Eta invariant for G spaces*, Indiana Univ. Math. J. **27** (1978), 889–918.

[13] V. Giambalvo, *pin and pin^c cobordism*, Proc. Amer. Math. Soc. **39** (1973), 395–401.

[14] P. Gilkey, *The eta invariant and the K-theory of odd dimensional spherical space forms*, Invent. Math. **76** (1984), 421–453.

[15] ———, *The eta invariant for even dimensional pinc manifolds*, Adv. in Math. **58** (1985), 243–284.

[16] ———, *The eta invariant and equivariant spin^c bordism for spherical space form groups*, Can. J. Math. **XL** (1988), 392–428.

[17] ———, *The eta invariant and the equivariant unitary bordism of spherical space form groups*, Compositio Math. **65** (1988), 33–50.

[18] ———, *The Geometry of Spherical Space Form Groups*, ISBN 9971-50-927-X, World Scientific, Singapore, 1989.

[19] ———, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index theorem (2nd edition)*, ISBN 0-8493-7874-4, CRC Press, Boca Raton, Florida, 1994.

[20] M. Gromov and H. B. Lawson, Jr., *The classification of simply connected manifolds of positive scalar curvature*, Annals of Math. **111** (1980), 423–434.

[21] ______, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Publ. Math. I.H.E.S. (1983), no. 58, 83–196.
[22] M. Kreck and S. Stolz, *Nonconnected moduli spaces of positive sectional curvature metrics*, *J. Amer. Math. Soc.* **6** (1993), 825–850.
[23] S. Kwasik and R. Schultz, *Positive scalar curvature and periodic fundamental groups*, Comment. Math. Helv. **65** (1990), 271–286.
[24] ______, *Fake spherical spaceforms of constant positive scalar curvature*, Comment. Math. Helv **71** (1996), 1–40.
[25] A. Lichnerowicz, *Spineurs harmoniques*, C. R. Acad. Sci. Paris **257** (1963), 7–9.
[26] W. J. Oledzki, *On the eta invariant of Pin operator on some exotic 4 dimensional projective space*, Comp. Math. **78** (1991), 1–27.
[27] ______, *Exotic involutions of Low-dimensional spheres*, preprint.
[28] S. Rosenberg, *C^*-algebras, positive scalar curvature, and the Novikov Conjecture, III*, Topology **25** (1986), 319–336.
[29] J. Rosenberg and S. Stolz, *Manifolds of positive scalar curvature*, Algebraic Topology and its Applications (G. Carlsson, R. Cohen, W.-C. Hsiang, and J. D. S. Jones, eds.), M. S. R. I. Publications, vol. 27, Springer, New York, 1994, pp. 241–267.
[30] ______, *A "stable" version of the Gromov-Lawson conjecture*, Contemp. Math. **181** (1995), 405–418.
[31] R. Schoen and S.-T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. **28** (1969), 159–183.
[32] R. Schultz, *Positive scalar curvature and odd order Abelian fundamental groups*, (to appear Proceedings of the AMS).
[33] S. Stolz, *Splitting certain MSpin module spectra*, Topology **33** (1994), 159–180.
[34] ______, *Simply connected manifolds of positive scalar curvature*, Ann. of Math. **136** (1992), 511–540.
[35] ______, *Concordance classes of positive scalar curvature metrics*, (in preparation).

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