On projective plane curves whose complements have finite non-abelian fundamental groups

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§0. Introduction

In this paper, we present new examples of singular projective plane curves $C \subset \mathbb{P}^2$ defined over the complex number field $\mathbb{C}$ such that the topological fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is non-abelian and finite.

The first example of a curve with this property is the three cuspidal quartic curve discovered by Zariski in [11]. Let $C_0 \subset \mathbb{P}^2$ be a quartic curve defined over $\mathbb{C}$ whose singular locus consists of 3 cusps. It is known [2] that any three cuspidal quartic is projectively isomorphic to the curve defined by

$$x^2y^2 + y^2z^2 + z^2x^2 - 2xyz(x+y+z) = 0.$$

The topological fundamental group $\pi_1(\mathbb{P}^2 \setminus C_0)$ is isomorphic to the binary 3-dihedral group

$$\widetilde{D}_3 := \langle \ \alpha, \beta \ \mid \ \alpha^2 = \beta^3 = (\alpha\beta)^2 \ \rangle$$

of order 12 (cf. [11] [4; Chapter 4, §4]). In [1], Abhyankar studied the complement of the three cuspidal quartic over an algebraically closed field k of arbitrary characteristics. He showed that, if char $k \neq 2, 3$, then the tame fundamental group of the complement is isomorphic to $\widetilde{D}_3$.

It is only quite recent that we come to know other examples exist. In [3], Degtyarev has found an irreducible curve of degree 5 with three singular points of type A_4 such that the complement has a nonabelian fundamental group of order 320. After this, some infinite series of projective plane curves with this property have been constructed by Oka [8] and, independently, by the author [10]. However, the curves in these infinite series should be considered as offspring of the three cuspidal quartic, because they are obtained as pull-backs of C_0 by certain branched coverings of the plane. On the other hand, the curves we will construct in this paper can be regarded as siblings of the three cuspidal quartic, as is explained below.

From now on, we will work over the complex number field $\mathbb{C}$ exclusively. The fundamental group always means the topological fundamental group.

Let q and k be positive integers, and suppose that q is odd and > 1. Let $F(s,t,x)$ be the polynomial

$$F(s,t,x) \quad := \quad \frac{(s^2x+t^q)^2-(s^2+t^2)^q}{s^2} \ .$$

(Note that $(s^2x+t^q)^2-(s^2+t^2)^q$ is divisible by s^2.) We put weights to the variables as follows;

$$\deg s = k, \quad \deg t = k, \quad \deg x = (q-2)k.$$

Then F is weighted homogeneous of total degree $2(q-1)k$. Recall that a germ of curve singularity is called to be of type A_{q-1} if it is locally defined by $u^2 - v^q = 0$. The main result of this paper is as follows:

Theorem. *Let $S(\xi_0,\xi_1,\xi_2)$, $T(\xi_0,\xi_1,\xi_2)$ and $X(\xi_0,\xi_1,\xi_2)$ be general homogeneous polynomials of degree k, k and $(q-2)k$, respectively. Let $C(q,k) \subset \mathbb{P}^2$ be the curve of degree $2(q-1)k$ defined by the homogeneous equation*

$$F(\ S(\xi_0,\xi_1,\xi_2),\ T(\xi_0,\xi_1,\xi_2),\ X(\xi_0,\xi_1,\xi_2)\ ) \ = \ 0.$$

Then the singular locus of $C(q,k)$ consists of $(2q-3)k^2$ singular points of type A_{q-1}, and the fundamental group $\pi_1(\mathbb{P}^2 \setminus C(q,k))$ is a central extension of the dihedral group D_q of order $2q$ by the cyclic group $\mathbb{Z}/(k(q-1))$. In particular, $\pi_1(\mathbb{P}^2 \setminus C(q,k))$ is finite and non-abelian.

Actually, we are going to describe the structure of $\pi_1(\mathbb{P}^2 \setminus C(q,k))$ completely as a subgroup of $GL_2(\mathbb{C})$.

The curve $C(3,1)$ is the three cuspidal quartic. The curves $C(q,k)$ $(k \geq 2)$ can be obtained from $C(q,1)$ by the method of [10]. So the essentially new part of Theorem is the construction of $C(q,1)$ with q odd integer ≥ 5.

The idea of the construction stems from the following observation due to Zariski [11. The three cuspidal quartic curve is an irreducible component of a reducible sextic curve defined by an equation of the form $\phi_2^3 + \phi_3^2 = 0$, where ϕ_ν is a suitable homogeneous polynomial on $\mathbb{P}^2$ of degree ν, with the residual component being a double line.

Let p, q be integers > 1 such that $(p,q) = 1$, and k a positive integer. Let f and g be general homogeneous polynomials on $\mathbb{P}^2$ of degree qk and pk, respectively, and let $C_{f,g}$ be the curve defined by $f^p - g^q = 0$. We denote by G the group $\langle \alpha, \beta | \alpha^p = \beta^q \rangle$. In [10], it was shown that the fundamental group $\pi_1(\mathbb{P}^2 \setminus C_{f,g})$ is isomorphic to the group

$$\langle\ a, b, c \ \mid \ a^p = b^q = c, \ \ c^k = e\ \rangle,$$

which is obtained by putting one more relation $\alpha^{pk} = e$ on G. In particular, when $k = 1$, this group is isomorphic to the free product $\mathbb{Z}/(p) * \mathbb{Z}/(q)$ of cyclic groups of order p and q ([7], [5]). The singular locus of the curve $C_{f,g}$ consists of pqk^2 points at which $C_{f,g}$ is locally defined by $u^p + v^q = 0$. Inspired by the above observation of Zariski, we will consider a situation in which the curve $C_{f,g}$ degenerates into the union of a reduced curve C_1 and a non-reduced curve mC_2 with multiplicity $m \geq 2$. More precisely, we specialize f and g to $S^m X + T^q$ and $S^m Y + T^p$, respectively, where S, T, X and Y are general homogeneous polynomials on $\mathbb{P}^2$ of suitable degrees. It turns out that, in many cases, $\pi_1(\mathbb{P}^2 \setminus C_1)$ remains non-abelian. Moreover, there is a surjection from G to $\pi_1(\mathbb{P}^2 \setminus C_1)$, but the generators α and β acquire more relations than $\alpha^{pk} = e$ in $\pi_1(\mathbb{P}^2 \setminus C_1)$. In the particular case described in Theorem – that is, when $p = 2$ and $m = 2$ – the added relations are so imposing that the quotient $\pi_1(\mathbb{P}^2 \setminus C_1)$ becomes finite. Moreover, the curve C_1 does not acquire any other type of singularities than those of $C_{f,g}$.

The main tools of the calculation of $\pi_1(\mathbb{P}^2 \setminus C(q,k))$ are the weighted Zariski's hyperplane section theorem [10; Theorem 1], and the generalized Zariski-Van Kampen's theorem [9; Theorem 2]. The main theorem stated above is obtained by combining Propositions 2 and 3 in sections 3 and 4.

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Conventions

(i) Let α and β be paths in a topological space. We define the product of α and β in such a way that $\alpha\beta$ is defined if and only if the ending point of α coincides with the starting point of β.

(ii) In order to distinguish various affine spaces, which will appear during the course of the discussions, we write the affine coordinate(s) of the space at the bottom of the name of the space. For example, $\mathbf{A}^1_u$ denotes an affine line with an affine coordinate u (that is, a u-line), and $\mathbf{A}^2_{(u,v)}$ is the affine plane $\mathbf{A}^1_u \times \mathbf{A}^1_v$.

(iii) A loop in a topological space is represented by a continuous map from the circle $\{e^{i\theta} \in \mathbb{C} | \theta \in \mathbb{R}\}$ with the base point 1 and the counter-clockwise orientation. For example, instead of denoting a loop $[0,1] \to \mathbf{A}^2_{(u,v)}$ in such a way as $t \mapsto (u,v) = (e^{2\pi imt}, e^{2\pi int})$, we simply write *a loop* $(e^{im\theta}, e^{in\theta})$ *on* $\mathbf{A}^2_{(u,v)}$.

§1. The defining polynomial of the curve

We consider a more general situation than the one described in Theorem.

Let $F(s,t,x,y)$ be the polynomial

$$F(s,t,x,y) \quad = \quad \frac{(s^m x + t^q)^p - (s^m y + t^p)^q}{s^m}, \tag{1.1}$$

where p, q and m are integers > 1 such that $(p,q) = 1$. Let k and l be positive integers such that $pl \geq mk$ and $ql \geq mk$. We put weights to the variables as follows;

$$\deg s = k, \quad \deg t = l, \quad \deg x = ql - mk \quad \text{and} \quad \deg y = pl - mk. \tag{1.2}$$

Then the polynomial F becomes weighted homogeneous of total degree $pql - mk$. Let $S(\xi_0, \xi_1, \xi_2)$, $T(\xi_0, \xi_1, \xi_2)$, $X(\xi_0, \xi_1, \xi_2)$ and $Y(\xi_0, \xi_1, \xi_2)$ be *general* homogeneous polynomials of degree k, l, $ql - mk$ and $pl - mk$, respectively. Substituting s, t, x, y with S, T, X, Y, we get a homogeneous polynomial $F(S, T, X, Y)$ of degree $pql - mk$ in variables (ξ_0, ξ_1, ξ_2). Therefore, we get a projective plane curve

$$C \quad := \quad \{F(S,T,X,Y) = 0\} \quad \subset \quad \mathbb{P}^2$$

of degree $pql - mk$. This C is the main object of the investigation in this article. We shall give a group presentation of the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ in section 2. In section 3, we shall investigate the structure of this group under the assumptions $p = m = 2$ and $k = l$. In section 4, we shall study the singularities of C under the assumption $p = m = 2$.

§2. Group presentation of $\pi_1(\mathbb{P}^2 \setminus C)$

Let $U \subset \mathbb{A}^4_{(s,t,x,y)}$ be the complement of the hypersurface $\Sigma := \{F(s,t,x,y) = 0\}$. Then the multiplicative group $\mathbb{G}_m$ of non-zero complex numbers acts on U by the weights (1.2); that is,

$$\lambda \cdot (s,t,x,y) = (\ \lambda^k s\ ,\ \lambda^l t\ ,\ \lambda^{ql-mk} x\ ,\ \lambda^{pl-mk} y\ ) \quad \text{where} \quad \lambda \in \mathbb{G}_m, \tag{2.1}$$

because F is weighted homogeneous under the weights (1.2). Let

$$\rho \quad : \quad \pi_1(\mathbb{G}_m) \longrightarrow \pi_1(U)$$

be the natural homomorphism induced by this action. Let μ and ν be integers such that $\mu p + \nu q = 1$. (Recall that p and q are supposed to be prime to each other.)

Proposition 1. *The fundamental group $\pi_1(U)$ is isomorphic to*

$$\langle\ \alpha,\ \beta \mid \alpha^p = \beta^q,\ \alpha(\beta^\mu\alpha^\nu)^m = (\beta^\mu\alpha^\nu)^m\alpha,\ \beta(\beta^\mu\alpha^\nu)^m = (\beta^\mu\alpha^\nu)^m\beta\ \rangle,$$

and, under this isomorphism, the image of the counter-clockwise generator of $\pi_1(\mathbb{G}_m)$ by ρ is given by $\beta^{ql}(\beta^\mu\alpha^\nu)^{-mk}$ in terms of the generators α and β.

Proof. The plan of the proof is as follows. The first projection from U to the s-line $\mathbb{A}^1_s$ is a locally trivial fiber space over $\mathbb{A}^1_s \setminus \{0\}$. Moreover, a general fiber is homotopically equivalent to the space

$$\mathbb{A}^2_{(u,v)} \setminus E, \quad \text{where } E \text{ is the affine plane curve defined by } u^p - v^q = 0,$$

and hence its fundamental group is isomorphic to $\langle\alpha,\beta|\alpha^p=\beta^q\rangle$ (cf. [4; Chapter 4, §2]). The kernel of the surjection $\langle\alpha,\beta|\alpha^p=\beta^q\rangle\to\pi_1(U)$ induced from the inclusion of a general fiber into the total space U can be determined by [9; Theorem 2].

Let $\mathrm{pr}_1: U\to\mathbf{A}^1_s$ be the first projection. The fiber $\mathrm{pr}_1^{-1}(\sigma)$ over the point $s=\sigma$ will be denoted by U_σ. We put

$$U^o \quad := \quad U\setminus U_0.$$

Over $\mathbf{A}^1_s\setminus\{0\}$, the affine hypersurface Σ is defined by $(s^m x+t^q)^p-(s^m y+t^p)^q=0$. Hence there exists an isomorphism Φ from U^o to $(\mathbf{A}^1_s\setminus\{0\})\times\mathbf{A}^1_t\times(\mathbf{A}^2_{(u,v)}\setminus E)$ which makes the following diagram commutative;

$$\begin{array}{ccc} U^o & \xrightarrow[\Phi]{\sim} & (\mathbf{A}^1_s\setminus\{0\})\times\mathbf{A}^1_t\times(\mathbf{A}^2_{(u,v)}\setminus E) \\ \mathrm{pr}_1 \searrow & & \swarrow \text{ the first projection} \\ & \mathbf{A}^1_s\setminus\{0\}. & \end{array} \tag{2.2}$$

This Φ is given by

$$\begin{array}{ccc} (s,t,x,y) & \overset{\Phi}{\longmapsto} & (s,t,\ \ (s^m x+t^q, s^m y+t^p)) \\ (s,t,\ \ (u-t^q)/s^m,\ \ (v-t^p)/s^m) & \underset{\Phi^{-1}}{\longleftarrow\!\shortmid} & (s,t,\ \ (u,v)). \end{array}$$

We put

$$\Phi=(\phi_s,\phi_t,\phi_{uv}),\qquad \text{where}\quad \phi_s: U^o\to\mathbf{A}^1_s\setminus\{0\},\quad \phi_t: U^o\to\mathbf{A}^1_t,\quad \phi_{uv}: U^o\to\mathbf{A}^2_{(u,v)}\setminus E.$$

Then we have an isomorphism

$$\pi_1(U^o,b)\quad\cong\quad\pi_1(\mathbf{A}^1_s\setminus\{0\},\ \phi_s(b))\ \times\ \pi_1(\mathbf{A}^2_{(u,v)}\setminus E,\ \phi_{uv}(b)) \tag{2.3}$$

induced by Φ. As a base point b, we choose

$$b=(\varepsilon,\ 1,\ 1/\varepsilon,\ 0)\quad\in U_\varepsilon\subset U^o\subset U$$

where ε is a small positive real number. Then we have

$$\phi_s(b)=\varepsilon\quad\text{and}\quad\phi_{uv}(b)=(1+\varepsilon^{m-1},1).$$

We shall write down explicitly the isomorphisms

$$\begin{array}{rcll} \pi_1(\mathbf{A}^1_s\setminus\{0\},\ \phi_s(b)) & \cong & \mathbb{Z}, & \text{and} \\ \pi_1(\mathbf{A}^2_{(u,v)}\setminus E,\ \phi_{uv}(b)) & \cong & G\langle p,q\rangle, & \end{array}$$

where

$$G\langle p,q\rangle:=\langle\ \alpha,\beta\ \mid\ \alpha^p=\beta^q\ \rangle.$$

As usual, for $\pi_1(\mathbb{A}^1_s \setminus \{0\},\ \phi_s(b)) \cong \mathbb{Z}$, we take the homotopy equivalence class of the loop $s = \varepsilon e^{i\theta}$ on $\mathbb{A}^1_s \setminus \{0\}$ as a positive generator $1 \in \mathbb{Z}$.

For the second isomorphism, whose proof can be found in [4; Chapter 4, §2], we consider the second projection $p_v : \mathbb{A}^2_{(u,v)} \setminus E \to \mathbb{A}^1_v$. The fiber $p_v^{-1}(1)$ is, by the first projection, identified with the u-line $\mathbb{A}^1_u$ minus the p-th roots of unity. For each integer j, we put

$$a'_j \quad := \quad [\omega(j)\eta(j)\omega(j)^{-1}] \quad \in \quad \pi_1(p_v^{-1}(1), \phi_{uv}(b)),$$

where $\omega(j) : [0,1] \to p_v^{-1}(1)$ and $\eta(j) : [0,1] \to p_v^{-1}(1)$ are the paths given below. (Here $p_v^{-1}(1)$ is regarded as a Zariski open subset of the u-line.)

$$\begin{array}{lcccl} \omega(j) & : & t_1 & \longmapsto & u = (1+\varepsilon^{m-1})\exp(2\pi i j t_1/p), \\ \eta(j) & : & t_2 & \longmapsto & u = (1+\exp(2\pi i t_2)\varepsilon^{m-1})\exp(2\pi i j/p). \end{array}$$

These a'_j generate $\pi_1(p_v^{-1}(1), \phi_{uv}(b))$. Then the inclusion $p_v^{-1}(1) \hookrightarrow \mathbb{A}^2_{(u,v)} \setminus E$ induces a surjective homomorphism on the fundamental groups by [6; Lemma 1.5(C)], because $\mathbb{A}^1_v$ is simply connected. We denote by $a_j \in \pi_1(\mathbb{A}^2_{(u,v)} \setminus E, \phi_{uv}(b))$ the image of $a'_j \in \pi_1(p_v^{-1}(1), \phi_{uv}(b))$. We put

$$\begin{aligned} \alpha &:= a_{q-1}a_{q-2}\cdot\cdots\cdot a_1 a_0, \quad \text{and} \\ \beta &:= a_{p-1}a_{p-2}\cdots a_1 a_0. \end{aligned}$$

Then we get the identification

$$\begin{aligned} \pi_1(\mathbb{A}^2_{(u,v)} \setminus E, \phi_{uv}(b)) &= \langle a_j \ \ (j \in \mathbb{Z}) \ \mid \ a_{j+q} = a_j,\ \ a_{j+p} = \beta a_j \beta^{-1} \quad \text{for all } j \in \mathbb{Z} \ \ \rangle \\ &= \langle \alpha, \beta \ \mid \ \alpha^p = \beta^q \rangle = G\langle p, q\rangle, \end{aligned}$$

(cf. [4; Chapter 4, §2]). Note that we have

$$a_0 = \beta^\mu \alpha^\nu, \qquad \text{where} \quad \mu p + \nu q = 1, \tag{2.4}$$

(cf. [*ibid.*]). Combining these isomorphisms and (2.3), we have fixed an isomorphism

$$\pi_1(U^o, b) \quad \cong \quad \mathbb{Z} \ \times \ G\langle p, q\rangle. \tag{2.5}$$

Consider the fiber $\mathrm{pr}_1^{-1}(\varepsilon) = U_\varepsilon$, which contains the base point b. By (2.2) and (2.3), the image of the natural homomorphism $\pi_1(U_\varepsilon, b) \to \pi_1(U^o, b)$ induced by the inclusion can be identified with the subgroup $\{0\} \times G\langle p,q\rangle$ of $\mathbb{Z} \times G\langle p,q\rangle$ via (2.5). Note that, by [6; Lemma 1.5(C)], the natural homomorphism $\pi_1(U_\varepsilon, b) \cong G\langle p,q\rangle \to \pi_1(U, b)$ induced by the inclusion is surjective, because $\mathbb{A}^1_s$ is simply connected. We shall determine the kernel of this surjection.

Claim 1. *The open disk*

$$D = \{ \ (z, 1, 1/\varepsilon, 0) \ ; \ |z| < 2\varepsilon\} \quad \subset \quad \mathbb{A}^4_{(s,t,x,y)}$$

is contained in U.

Proof. On the hyperplane defined by $s = 0$, the hypersurface $\Sigma \cap \{s = 0\} \subset \mathbb{A}^3_{(t,x,y)}$ is defined by $pxt^{q(p-1)} - qyt^{p(q-1)} = 0$. The intersection point $(0, 1, 1/\varepsilon, 0)$ of D and $\{s = 0\}$ does not satisfy this equation. When $s \neq 0$, the hypersurface Σ is defined by $(s^m x + t^p)^q - (s^m y + t^q)^p = 0$. Putting $s = z$, $t = 1$, $x = 1/\varepsilon$ and $y = 0$, we get $(z^m/\varepsilon + 1)^q - 1 = 0$. Since $m \geq 2$, this equation does not have any roots in $0 < |z| < 2\varepsilon$, because ε is small enough. Hence we get $D \cap \Sigma = \emptyset$. $\square$

Claim 2. *The kernel of the surjective homomorphism $\pi_1(U_\varepsilon, b) \cong G\langle p, q\rangle \to \pi_1(U, b)$ is generated by the set*

$$\{ \ \gamma^{-1} a_0^{-m} \gamma a_0^m \ ; \ \gamma \in G\langle p, q\rangle\},$$

where $a_0 = \beta^\mu \alpha^\nu$ by (2.4).

In other words, $\pi_1(U, b)$ is isomorphic to the maximal quotient group of $G\langle p, q\rangle$ such that the image of a_0^m is contained in the center.

By this claim, the first assertion of Proposition 1 will be proved.

Proof. Let $\Delta := \{ \ s \ ; \ |s| < 2\varepsilon\} \subset \mathbb{A}^1_s$ be the open disk on the s-line with the origin 0 and the radius 2ε. Let U_Δ be $\mathrm{pr}_1^{-1}(\Delta)$. Since $\mathrm{pr}_1 : U \to \mathbb{A}^1_s$ is a locally trivial fiber space over $\mathbb{A}^1_s \setminus \{0\}$, U_Δ is a strong deformation retract of U. Therefore the inclusion $U_\Delta \hookrightarrow U$ induces an isomorphism $\pi_1(U_\Delta, b) \cong \pi_1(U, b)$. Thus it is enough to consider $\mathrm{Ker}\ (\pi_1(U_\varepsilon, b) \to \pi_1(U_\Delta, b))$.

Note that, by Claim 1, the projection $\mathrm{pr}_1 : U \to \mathbb{A}^1_s$ has a section

$$\sigma \ : \ s \ \longmapsto \ (s, 1, 1/\varepsilon, 0)$$

over Δ passing through the base point b. We put $\Delta^\times := \Delta \setminus \{0\}$. By the section σ, the group $\pi_1(\Delta^\times, \varepsilon)$ acts on $\pi_1(U_\varepsilon, b)$ in a natural way, because $\mathrm{pr}_1 : \mathrm{pr}_1^{-1}(\Delta^\times) \to \Delta^\times$ is a locally trivial fiber space. Let $\zeta \in \pi_1(\Delta^\times, \varepsilon)$ be the counter-clockwise generator of $\pi_1(\Delta^\times, \varepsilon) \cong \mathbb{Z}$, and let $\gamma \mapsto \zeta_*(\gamma)$ denote its monodromy action on $\gamma \in \pi_1(U_\varepsilon, b)$. By [9; Theorem 2], the kernel of $\pi_1(U_\varepsilon, b) \to \pi_1(U_\Delta, b)$ is generated by the set

$$\{ \ \gamma^{-1} \cdot \zeta_*(\gamma) \ ; \ \gamma \in \pi_1(U_\varepsilon, b)\}.$$

There is an isomorphism Ψ from $\mathrm{pr}_1^{-1}(\Delta^\times)$ to $\Delta^\times \times U_\varepsilon$ which makes the following diagram commutative;

$$\begin{tikzcd}
\mathrm{pr}_1^{-1}(\Delta^\times) \arrow[rr, "\sim", "\Psi"'] \arrow[dr, "\mathrm{pr}_1"'] & & \Delta^\times \times U_\varepsilon \arrow[dl, "\text{the first projection}"] \\
& \Delta^\times. &
\end{tikzcd}$$

This Ψ is given by

$$\begin{array}{ccc} (s,\ t,\ x,\ y) & \overset{\Psi}{\longmapsto} & (s,\ (\varepsilon,\ t,\ s^m x/\varepsilon^m,\ s^m y/\varepsilon^m)) \\ (s,\ t,\ \varepsilon^m x'/s^m,\ \varepsilon^m y'/s^m) & \underset{\Psi^{-1}}{\longleftarrow} & (s,\ (\varepsilon,\ t,\ x',\ y')). \end{array}$$

Hence the fiber structure of $\mathrm{pr}_1 : \mathrm{pr}_1^{-1}(\Delta^\times) \to \Delta^\times$ is trivial. However the motion of the base point causes a non-trivial monodromy action of $\pi_1(\Delta^\times)$ on $\pi_1(U_\varepsilon)$. We let the generator $\zeta \in \pi_1(\Delta^\times, \varepsilon)$ be represented by the loop $\varepsilon e^{i\theta}$ on $\Delta^\times$. The image of this loop by the morphism

$$\Delta^\times \xrightarrow{\sigma} \mathrm{pr}_1^{-1}(\Delta^\times) \xrightarrow{\Psi} \Delta^\times \times U_\varepsilon \xrightarrow{\mathrm{pr}_2} U_\varepsilon$$

is the loop $(\varepsilon, 1, e^{im\theta}/\varepsilon, 0)$ from b to b. Let $\widetilde{\zeta} \in \pi_1(U_\varepsilon, b)$ be the homotopy equivalence class of this loop. By Lemma below, the action $\gamma \mapsto \zeta_*(\gamma)$ is given by $\zeta_*(\gamma) = \widetilde{\zeta}^{-1} \cdot \gamma \cdot \widetilde{\zeta}$. We shall write down the image of $\widetilde{\zeta}$ by the isomorphism $\pi_1(U_\varepsilon, b) \cong G\langle p, q\rangle$. The image of the loop $(\varepsilon, 1, e^{im\theta}/\varepsilon, 0)$ on U_ε by $\phi_{uv}|_{U_\varepsilon} : U_\varepsilon \to \mathbb{A}^2_{(u,v)} \setminus E$ is the loop $(1 + \varepsilon^{m-1} e^{im\theta}, 1)$ on $\mathbb{A}^2_{(u,v)} \setminus E$, which is nothing but the loop $(\eta(0))^m$. This represents $a_0^m \in G\langle p, q\rangle$. Thus we get

$$\zeta_*(\gamma) = a_0^{-m} \gamma a_0^m,$$

and the proof of Claim 2 is completed. □

Lemma. *Let (V, b_V) and (W, b_W) be topological spaces with base points. Suppose that we are given a section $\sigma : V \to V \times W$ of the first projection $V \times W \to V$ such that $\sigma(b_V) = (b_V, b_W)$. Then the action of $\alpha \in \pi_1(V, b_V)$ on $\gamma \in \pi_1(W, b_W)$ induced by this section is given by $\gamma \mapsto \widetilde{\alpha}^{-1} \cdot \gamma \cdot \widetilde{\alpha}$, where $\widetilde{\alpha}$ is the image of α by the homomorphism $\pi_1(V, b_V) \to \pi_1(W, b_W)$ induced from the composition of the section $\sigma : V \to V \times W$ and the second projection $\mathrm{pr}_2 : V \times W \to W$.* □

Next, we will investigate the image of $\rho : \pi_1(\mathbb{G}_m) \to \pi_1(U)$ induced by the action (2.1). By Claim 1, the loop $(\varepsilon e^{i\theta}, 1, 1/\varepsilon, 0)$ on U^o represents an element of the kernel of the homomorphism $\pi_1(U^o, b) \to \pi_1(U, b)$ induced by the inclusion. By $\phi_{uv} : U^o \to \mathbb{A}^2_{(u,v)} \setminus E$, this loop is mapped to the loop $(1 + \varepsilon^{m-1} e^{im\theta}, 1)$, which represents $a_0^m \in \pi_1(\mathbb{A}^2_{(u,v)} \setminus E, \phi_{uv}(b)) \cong G\langle p, q\rangle$. The image of the class of this loop by $\phi_{s*} : \pi_1(U^o, b) \to \pi_1(\mathbb{A}^1_s \setminus \{0\}, \phi_s(b))$ is obviously the generator $1 \in \pi_1(\mathbb{A}^1_s \setminus \{0\}, \varepsilon) \cong \mathbb{Z}$. Thus the loop $(\varepsilon e^{i\theta}, 1, 1/\varepsilon, 0)$ represents $(1, a_0^m) \in \mathbb{Z} \times G\langle p, q\rangle$ via the isomorphism (2.5). Hence we have shown that

$$(1, a_0^m) \in \mathrm{Ker}\ (\pi_1(U^o, b) \to \pi_1(U, b)). \tag{2.6}$$

Since $U^o \subset U$ is invariant under the action (2.1) of $\mathbb{G}_m$, there is a natural homomorphism $\pi_1(\mathbb{G}_m) \to \pi_1(U^o)$, which factors ρ. In order to determine the image of $\pi_1(\mathbb{G}_m) \to \pi_1(U^o) \cong \mathbb{Z} \times G\langle p, q\rangle$, we choose another base point

$$c := (1, 0, R, 0) \in U^o,$$

where R is a positive real number large enough.

Remark. A path from c to b in U^o induces an isomorphism $\pi_1(U^o, c) \cong \pi_1(U^o, b)$, which depends on the homotopy class of the path. However, the image of $1 \in \mathbb{Z} \cong \pi_1(\mathbb{G}_m)$ in $\pi_1(U^o, c)$ always corresponds to the image of 1 in $\pi_1(U^o, b)$, whatever choice we may have made on the path, because the image of $\pi_1(\mathbb{G}_m) \to \pi_1(U^o)$ is contained in the center.

The orbit of the new base point $c = (1, 0, R, 0)$ by the loop $e^{i\theta}$ in $\mathbb{G}_m$ is the loop $(e^{ik\theta}, 0, Re^{i(ql-mk)\theta}, 0)$ in U^o, which is mapped by ϕ_s and ϕ_{uv} to the loops $e^{ik\theta}$ in $\mathbb{A}^1_s \setminus \{0\}$ and $(Re^{iql\theta}, 0)$ in $\mathbb{A}^2_{(u,v)} \setminus E$, respectively. The loop $e^{ik\theta}$ on $\mathbb{A}^1_s \setminus \{0\}$ represents $k \in \mathbb{Z} \cong \pi_1(\mathbb{A}^1_s \setminus \{0\})$. Note that, by the definition, the element $\beta \in G\langle p, q\rangle$ is, as an element of $\pi_1(\mathbb{A}^2_{(u,v)} \setminus E)$, represented by the loop $(e^{i\theta}(1+\varepsilon^{m-1}), 1)$ on $p_v^{-1}(1)$. The loop $(Re^{iql\theta}, 0)$ on $\mathbb{A}^2_{(u,v)} \setminus E$, which is supported on $p_v^{-1}(0)$, can be deformed to the loop $(e^{iql\theta}(1+\varepsilon^{m-1}), 1)$ on $p_v^{-1}(1)$, which represents β^{ql} in $\pi_1(\mathbb{A}^2_{(u,v)} \setminus E) \cong G\langle p, q\rangle$. Therefore, by Remark above, the image of $1 \in \mathbb{Z} \cong \pi_1(\mathbb{G}_m)$ in $\pi_1(U^o, b)$ is

$$(k, \beta^{ql}) \in \mathbb{Z} \times G\langle p, q\rangle \cong \pi_1(U^o, b).$$

Because of (2.6), the elements (k, β^{ql}) and $(0, \beta^{ql} a_0^{-mk})$ are mapped to the same element in $\pi_1(U)$. Therefore, when we regard $\pi_1(U)$ as a quotient of $\pi_1(U_\varepsilon) \cong G\langle p, q\rangle$, the image of $1 \in \mathbb{Z} \cong \pi_1(\mathbb{G}_m)$ in $\pi_1(U)$ by ρ is the image of $\beta^{ql} a_0^{-mk} \in G\langle p, q\rangle$. $\square$

Corollary. *The fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to the group*

$$\left\langle \alpha, \beta \;\middle|\; \begin{array}{cc} \alpha^p = \beta^q, & \beta^{ql} = (\beta^\mu \alpha^\nu)^{mk} \\ \alpha(\beta^\mu\alpha^\nu)^m = (\beta^\mu\alpha^\nu)^m \alpha, & \beta(\beta^\mu\alpha^\nu)^m = (\beta^\mu\alpha^\nu)^m \beta \end{array} \right\rangle.$$

Proof. Note that the affine hypersurface Σ is reduced. Therefore, by [10; Theorem 1], when both of $\deg x$ and $\deg y$ are positive, $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to the cokernel of the natural homomorphism $\rho : \pi_1(\mathbb{G}_m) \to \pi_1(U)$, and thus we get the group presentation of $\pi_1(\mathbb{P}^2 \setminus C)$ above from Proposition 1.

In the case when $\deg x = pl - mk = 0$ or $\deg y = ql - mk = 0$, we cannot conclude $\pi_1(\mathbb{P}^2 \setminus C) \cong \operatorname{Coker} \rho$ by applying [10; Theorem 1] directly. However, this isomorphism still holds in this case, as is shown below. Interchanging p and q if necessary, we may assume that $pl - mk > 0$ and $ql - mk = 0$. Let $\mathrm{pr}_4 : U \to \mathbb{A}^1_y$ be the projection to the y-line. If $\eta \neq 0$, then the fiber $\mathrm{pr}_4^{-1}(\eta)$ is isomorphic to $\mathrm{pr}_4^{-1}(1)$. Indeed, the morphism $(s, t, x, 1) \mapsto (s\eta^{-1/m}, t, x\eta, \eta)$ gives an isomorphism from $\mathrm{pr}_4^{-1}(1)$ to $\mathrm{pr}_4^{-1}(\eta)$. Hence $\mathrm{pr}_4 : U \to \mathbb{A}^1_y$ is a locally trivial fiber space over $\mathbb{A}^1_y \setminus \{0\}$. On the other hand, it is easy to see that the hypersurface $\Sigma \cap \{y = 0\} = \{F(s, t, x, 0) = 0\}$ in $\mathbb{A}^3_{(s,t,x)}$ is reduced. Therefore, by [9; Theorem 1], we see that the inclusion $\mathrm{pr}_4^{-1}(\eta) \hookrightarrow U$ induces an isomorphism $\pi_1(\mathrm{pr}_4^{-1}(\eta)) \cong$

$\pi_1(U)$ for $\eta \neq 0$. On the other hand, the action (2.1) of $\mathbb{G}_m$ on U leaves $\mathrm{pr}_4^{-1}(\eta)$ invariant because of $\deg y = 0$. Hence we have a commutative diagram

$$\begin{tikzcd} & \pi_1(\mathrm{pr}_4^{-1}(\eta)) \arrow[dd, "\wr", no head, equal] \\ \pi_1(\mathbb{G}_m) \arrow[ur] \arrow[dr] & \\ & \pi_1(U). \end{tikzcd} \qquad \text{for} \qquad \eta \neq 0 \tag{2.7}$$

Suppose that the polynomial Y, which is of degree 0 in this case, is a constant $\eta_0 \neq 0$. By [10; Theorem 1], $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to Coker $(\pi_1(\mathbb{G}_m) \to \pi_1(\mathrm{pr}_4^{-1}(\eta_0)))$ because of $\deg s > 0$, $\deg t > 0$ and $\deg x > 0$. Hence, by the diagram (2.7), we obtain $\pi_1(\mathbb{P}^2 \setminus C) \cong$ Coker $(\pi_1(\mathbb{G}_m) \to \pi_1(U))$. $\square$

§3. Structure of $\pi_1(\mathbb{P}^2 \setminus C)$ in the dihedral case

The dihedral case in the title of this section is the case when $p = m = 2$. In this case, q is an odd integer $q = 2r + 1$ with $r \geq 1$.

Proposition 2. *Suppose that $p = m = 2$, and $k = l$. Then the fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to a central extension of the dihedral group D_q of order $2q$ by the cyclic group $\mathbb{Z}/(k(q-1))$. In particular, $\pi_1(\mathbb{P}^2 \setminus C)$ is non-abelian and finite.*

In this section, we use the following notation;

$$e(\alpha) := \exp(2\pi i \alpha).$$

We consider the dihedral group D_q as a subgroup of $PGL_2(\mathbb{C})$ generated by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} e(1/q) & 0 \\ 0 & 1 \end{bmatrix}.$$

Proof. Note that, in the case $p = 2$ and $q = 2r + 1$, we have $a_0 = \beta^{-r}\alpha$ by (2.4). Therefore, by Corollary of Proposition 1, the group $\pi_1(\mathbb{P}^2 \setminus C)$ is isomorphic to

$$H(q;k) := \left\langle \alpha, \beta \;\middle|\; \begin{array}{ccl} \alpha^2 = \beta^q, & \beta^{qk} = (\beta^{-r}\alpha)^{2k} & \cdots \text{Relations (1)} \\ \alpha(\beta^{-r}\alpha)^2 = (\beta^{-r}\alpha)^2\alpha, & \beta(\beta^{-r}\alpha)^2 = (\beta^{-r}\alpha)^2\beta & \cdots \text{Relations (2)} \end{array} \right\rangle.$$

Note that when $k = 1$, the relations (2) follows automatically from the relations (1).

Claim. *The order of $H(q;k)$ is at most $2q(q-1)k$.*

Proof. Note that in the present situation, we have $\deg y = 0$. Let $\rho_\nu : \pi_1(\mathbb{G}_m) \to \pi_1(U)$ be the homomorphism induced by the action of $\mathbb{G}_m$ on U with weights $(\nu, \nu, (q-2)\nu, 0)$ on the variables (s, t, x, y). Then $H(q;k) \cong$ Coker ρ_k holds for all $k \geq 1$, as is shown in the proof of Corollary. We see that $\rho_k = \rho_1 \circ \sigma_k$, where $\sigma_k : \pi_1(\mathbb{G}_m) \to \pi_1(\mathbb{G}_m)$ is induced from the morphism $\lambda \mapsto \lambda^k$. Hence we have an exact sequence

$$\text{Coker } \sigma_k \longrightarrow \text{Coker } \rho_k \longrightarrow \text{Coker } \rho_1 \longrightarrow \{1\}.$$

Consequently the order of $H(q;k)$ is at most k times the order of $H(q;1)$. Therefore, it is enough to show that the order of $H(q;1)$ is at most $2q(q-1)$. In $H(q;1)$, we have $\alpha^2 = \beta^q = (\beta^{-r}\alpha)^2$, and hence

$$\beta^r\alpha = \alpha\beta^{-r}. \tag{3.1}$$

Any integer n can be written as $nq - 2nr$. Therefore we have

$$\beta^n\alpha = (\beta^q)^n(\beta^r)^{-2n}\alpha = (\alpha^2)^n\alpha(\beta^{-r})^{-2n} = \alpha^{2n+1}\beta^{2nr}.$$

We also have $\beta^n\alpha^{-1} = \alpha^{2n-1}\beta^{2nr}$. By repeating these transformations, every word in the letters $\alpha^{\pm 1}$ and $\beta^{\pm 1}$ can be transformed into a word of the type $\alpha^M\beta^N$. Since $\alpha^2 = \beta^q$, we can assume that $M = 0$ or 1. Therefore the order of $H(q;1)$ is at most 2 times the order of β in $H(q;1)$. From the relation $\beta^{-r}\alpha\beta^{-r}\alpha = \alpha^2 = \beta^{2r+1}$, we get $\beta^{3r+1} = \alpha\beta^{-r}\alpha$. Thus

$$\beta^r = \alpha\beta^{-(3r+1)}\alpha = \alpha\beta^{-r}\beta^{-(2r+1)}\alpha = \alpha\beta^{-r}\alpha^{-1}.$$

This implies

$$\beta^{rq} = \alpha\beta^{-rq}\alpha^{-1} = \alpha(\alpha^{-2})^r\alpha^{-1} = \alpha^{-2r} = \beta^{-rq}.$$

Thus $\beta^{2rq} = \beta^{(q-1)q} = e$. $\square$

Consider the following two matrices in $GL_2(\mathbb{C})$;

$$A = \begin{pmatrix} 0 & e(\frac{q}{4rk}) \\ e(\frac{q}{4rk}) & 0 \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} e(\frac{1}{2rk} + \frac{r}{q}) & 0 \\ 0 & e(\frac{1}{2rk} - \frac{r}{q}) \end{pmatrix}.$$

It is easy to check that

$$A^2 = B^q, \quad B^{qk} = (B^{-r}A)^{2k}, \quad \text{and} \quad (B^{-r}A)^2 = cI_2 \quad \text{where} \quad c = e(\frac{1}{2rk}).$$

In particular, $(B^{-r}A)^2$ is contained in the center of $GL_2(\mathbb{C})$. Hence there exists a homomorphism

$$\varphi : H(q;k) \longrightarrow GL_2(\mathbb{C}),$$

which maps α to A and β to B. Let $P : GL_2(\mathbb{C}) \to PGL_2(\mathbb{C})$ be the natural projection. Then the image of $P \circ \varphi$, which is generated by $P(A)$ and $P(B)$, is the dihedral group D_q. Moreover, Ker $P \cap \text{Im } \varphi$ contains a cyclic group of order $2rk$ generated by $(B^{-r}A)^2 = cI_2$. Hence the order of $H(q;k)$ is at least $2rk$ times the order $2q$ of D_q. Combining this with Claim, we see that the order of $H(q;k)$ is exactly $2q(q-1)k$, and φ is injective. The homomorphism $P \circ \varphi$ gives $H(q;k)$ a structure of the central extension of D_q by the cyclic group $\mathbb{Z}/(k(q-1))$. $\square$

§4. Singularities of the curve C in the dihedral case

Proposition 3. *Suppose that $p = m = 2$. Then the singular locus of C consists of $(2ql - 3k)l$ points of type A_{q-1}.*

Proof. Let τ be a complex parameter, and we denote by $C_\tau \subset \mathbb{P}^2$ the curve of degree $2(ql-k)$ defined by $F(S,T,X,\tau Y) = 0$. Thus the curve C we have been treating is C_1. We prove Proposition 3 in the following way; first, we investigate the singular points of C_0, and then we see what happens to these singular points when τ becomes a non-zero small number ε.

The curve C_0 is defined by the equation

$$X(S^2X + 2T^q) = 0.$$

It has two irreducible components. We put

$$C_0(1) = \{X = 0\}, \qquad \text{and} \qquad C_0(-1) = \{S^2X + 2T^q = 0\}.$$

Consider the linear pencil

$$D_\lambda = \{S^2X + \lambda T^q = 0\}_{\lambda \in \mathbb{P}^1}$$

of curves of degree ql spanned by the curves $\{S^2X = 0\}$ and $\{T^q = 0\}$. We have $C_0(-1) = D_2$. The base locus of this pencil

$$Bs = \{S = T = 0\} \cup \{X = T = 0\}$$

consists of $kl + (ql - 2k)l$ distinct points. (Recall that S, T and X are general.) By Bertini's theorem, except for finite values $0, \infty, \lambda_1, \ldots, \lambda_M$ of λ, the members D_λ are non-singular outside Bs. Since T is general, we may assume that none of $\lambda_1, \ldots, \lambda_M$ coincides with 2. Hence $C_0(-1)$ is non-singular outside Bs. The intersection points of $C_0(-1)$ and $C_0(1)$ are also contained in Bs. Since X is general, $C_0(1)$ is non-singular. Hence we have $\operatorname{Sing} C_0 \subset Bs$. On the other hand, it is easy to see that C_0 is singular at each point of Bs. Thus we get

$$\operatorname{Sing} C_0 = \{S = T = 0\} \cup \{X = T = 0\}.$$

Let $P \in \operatorname{Sing} C_0$ be an intersection point of the curves $\{S = 0\}$ and $\{T = 0\}$.

Claim 1. *There exist a small open neighborhood $\Delta \subset \mathbb{P}^2$ of P and a small positive real number r such that C_ε has only one singular point in Δ for any $\varepsilon \in \mathbb{C}$ satisfying $|\varepsilon| < r$. Moreover, the singular point is of type A_{q-1}.*

Proof. Since S, T, X and Y are chosen generally, the curves $\{S=0\}$ and $\{T=0\}$ intersect transversely at P, and both of X and Y are non-zero at P. Let (u,v) be local analytic coordinates of $\mathbb{P}^2$ around P such that the curves $\{S=0\}$ and $\{T=0\}$ are defined by $u=0$ and $v=0$, respectively. Then the defining equation of C_ε locally around P is of the form

$$\frac{(au^2+v^q)^2-(\varepsilon bu^2+v^2)^q}{u^2} = a^2u^2+2av^q-\varepsilon b(u^2F(u,v,\varepsilon)+qv^{2(q-1)})=0,$$

where a and b are holomorphic functions of (u,v) corresponding to X and Y, respectively, satisfying $a(P)\neq 0$ and $b(P)\neq 0$, and $F(u,v,\varepsilon)$ is a holomorphic function of (u,v,ε) such that $F(0,0,0)=0$. Since $a(P)\neq 0$, there exist holomorphic functions $\phi(u,v,\varepsilon)$ and $\psi(u,v,\varepsilon)$, defined locally around $(u,v,\varepsilon)=(0,0,0)$, such that

$$\begin{aligned}\phi(u,v,\varepsilon)^2 &= a(u,v)^2-\varepsilon b(u,v)F(u,v,\varepsilon), \quad \text{and}\\ \psi(u,v,\varepsilon)^q &= -2a(u,v)+\varepsilon qb(u,v)v^{q-2}.\end{aligned}$$

We put

$$\widetilde{u}=u\phi(u,v,\varepsilon) \quad \text{and} \quad \widetilde{v}=v\psi(u,v,\varepsilon).$$

Then C_ε is defined by

$$\widetilde{u}^2-\widetilde{v}^q=0.$$

Note that

$$\det\begin{pmatrix}\dfrac{\partial\widetilde{u}}{\partial u}(0,0,0) & \dfrac{\partial\widetilde{u}}{\partial v}(0,0,0)\\ \dfrac{\partial\widetilde{v}}{\partial u}(0,0,0) & \dfrac{\partial\widetilde{v}}{\partial v}(0,0,0)\end{pmatrix} \neq 0,$$

because of $\phi(0,0,0)\neq 0$ and $\psi(0,0,0)\neq 0$. This implies that there exist a small open neighborhood Δ of P on $\mathbb{P}^2$ and a small positive real number r such that $(\widetilde{u},\widetilde{v})$ are local analytic coordinates in Δ when $|\varepsilon|<r$. Since the only singular point of $C_\varepsilon=\{\widetilde{u}^2-\widetilde{v}^q=0\}$ in Δ is the origin P and it is of type A_{q-1}, we have completed the proof of Claim 1. □

Let $Q\in \operatorname{Sing} C_0$ be an intersection point of the curves $\{X=0\}$ and $\{T=0\}$.

Claim 2. *There exist a small open neighborhood $\Delta\subset\mathbb{P}^2$ of Q and a small positive real number r such that, when ε satisfies $0<|\varepsilon|<r$, then C_ε has exactly 2 singular points in Δ, and each of them is of type A_{q-1}.*

Proof. Since S, T X and Y are chosen generally, the curves $\{X=0\}$ and $\{T=0\}$ intersect transversely at Q, and both of S and Y are non-zero at Q. Let (u,v) be local analytic coordinates of $\mathbb{P}^2$ around Q such that the curves $\{X=0\}$ and $\{T=0\}$ are defined by $u=0$ and $v=0$, respectively. Since S is non-zero at Q, the defining equation of C_ε locally around Q is of the form

$$(au+v^q)^2-(\varepsilon b+v^2)^q=0,$$

where a and b are holomorphic functions corresponding to S^2 and S^2Y, respectively, such that $a(Q) \neq 0$ and $b(Q) \neq 0$. Since $b(Q) \neq 0$, there exists a holomorphic function $b^{1/2}$ defined locally around Q. We put

$$\widetilde{u} = \frac{au + v^q}{b^{q/2}} \quad \text{and} \quad \widetilde{v} = \frac{v}{b^{1/2}}.$$

Then there is an open neighborhood $\Delta \subset \mathbb{P}^2$ of Q such that $(\widetilde{u}, \widetilde{v})$ is a local analytic coordinate system on Δ. In Δ, C_ε is defined by

$$\widetilde{u}^2 - (\varepsilon + \widetilde{v}^2)^q = 0.$$

Hence, if ε is small enough and non-zero, the singular points of C_ε in Δ consists of the two points

$$(\widetilde{u}, \widetilde{v}) = (0, \sqrt{-\varepsilon}) \quad \text{and} \quad (0, -\sqrt{-\varepsilon}),$$

and both of them are of type A_{q-1}. $\square$

By Claims 1 and 2, all singular points of C_ε are of type A_{q-1} when ε is non-zero and small enough. The number of them is

$$\deg S \cdot \deg T + 2 \deg X \cdot \deg T = (2ql - 3k)l.$$

The locus of all $\tau \in \mathbb{C}$ such that Sing C_τ consists of $(2ql - 3k)l$ points of type A_{q-1} is then Zariski open dense in $\mathbb{C}$. Since S, T X and Y are chosen generally, we can conclude that $\tau = 1$ is contained in this locus. $\square$

Now we can compute the geometric genus g of the curve C in Proposition 3. It is given by

$$g = 1 + 3k + 2k^2 - 3l - 4kl + 2l^2 - 6lr - 5klr + 6l^2r + 4l^2r^2,$$

where $r = (q-1)/2$. In particular, the geometric genus of the curve $C(q,k)$ in Theorem is given by

$$g = 1 - 6kr + k^2r + 4k^2r^2.$$

Hence, except for the case $(q,k) = (3,1)$ of the three cuspidal plane quartic, the curve $C(q,k)$ is not rational.

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