LIFTS OF DIFFEOMORPHISMS OF THE ORBIT SPACE FOR A FINITE REFLECTION GROUP

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Abstract. For a finite reflection group W acting on a real vector space V it is proved that any diffeomorphism of the orbit space V/W is liftable to a diffeomorphism of V. The analogous statement is proved for the induced action of W on the complexification $V \otimes \mathbf{C}$ of V and biholomorphic maps of $V \otimes \mathbf{C}$ onto itself preserving the discriminant locus.

0. Introduction

Let W be a finite reflection group acting on n-dimensional real vector space V. By the Chevalley theorem (see [3]) the algebra $\mathbf{R}[V]^W$ of W-invariant polynomials is a free algebra with n homogeneous algebraically independent generators. Let $p^1, \dots, p^n$ be such generators and $p = (p^1, \dots, p^n) : V \to \mathbf{R}^n$ the corresponding map. The image $P(V)$ is a closed subset of $\mathbf{R}^n$ homeomorphic to the orbit space V/W. The bijective map $f : p(V) \to p(V)$ is called a diffeomorphism of the orbit space V/W if it can be extended to a C^∞-diffeomorphism of a neghborhood of $p(V)$ into $\mathbf{R}^n$. It is easily seen that this definition is independent of the choice of generators p^i.

The main aim of this paper is to prove that each diffeomorphism f of V/W can be lifted to a diffeomorphism of V and it is analytic whenever f is analytic (Main Theorem 3.1). We prove also the local version of this result and extend it to a smooth family of diffeomorphisms of V/W, in particular, to a Lie group of diffeomorphisms of V/W. Besides, it is proved that a smooth vector field on $p(V)$, tangent to the discriminant locus of p, is liftable to V.

Furthermore, we consider the corresponding problems for the extended action of W on the complexification $V_{\mathbf{C}}$ of V. In this case the map p induces the biholomorphic map of the orbit space $V_{\mathbf{C}}/W$ onto $\mathbf{C}^n$ and a diffeomorphism of $V_{\mathbf{C}}/W$ is a biholomorphic map $V_{\mathbf{C}}/W \to V_{\mathbf{C}}/W$ which preserves the discriminant locus of p. It is proved that each diffeomorphism of $V_{\mathbf{C}}/W$ can be lifted to a biholomorphic map of $V_{\mathbf{C}}$ onto itself and this statement is true for a smooth or holomorphic family of diffeomorphisms, in particular, for a Lie group of diffeomorphisms.

Finally, the above results imply that the structure of Frobenius manifold on V/W and $V_{\mathbf{C}}/W$ defined by Dubrovin [4] has no nontrivial global and local automorphisms both in the real and in the complex case.

Section 1 contains the summary of known facts on finite reflection groups.

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In Section 2 we prove the existence of lifts under the map p for some class of smooth curves, containing all analytic curves, in V/W . This result is one of the principal tools in the proof of Main theorem 3.1.

In Section 3 we prove Main Theorem 3.1 and its natural consequences Theorems 3.6, 3.7, 3.8, and Corollary 3.9.

In Section 4 we prove the existence of lifts for some holomorphic curves in $p(V_{\mathbf{C}})$ to $V_{\mathbf{C}}$ and prove the version of Main Theorem 3.1 in the complex case (Theorem 4.1).

In Section 5 we prove that each global or local automorphism of the Frobenius manifold V/W or $V_{\mathbf{C}}/W$ is trivial.

Throughout the paper a smooth map means C^∞-map.

1. Preliminaries

In this section we formulate some standard notions and facts of the theory of finite reflection groups (see, for example, [2] and [6]), which will be used later.

Let V be a real n-dimensional vector space V endowed with a positive definite inner product $\langle x, y\rangle$. A reflection of V is a linear operator s which sends some nonzero vector $\alpha \in V$ to $-\alpha$ and fixes pointwise the hyperplane H_α orthogonal to α. We may write $s = s_\alpha$.

A finite set Φ of nonzero vectors of V such that

(1) $\Phi \cap \mathbf{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$;
(2) $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$

is called a root system in V and the vectors belonging to Φ are called roots. A finite reflection group W is a finite subgroup of $O(V)$ generated by reflections s_α ($\alpha \in \Phi$) and Φ is called the root system of W. It is known that the reflections s_α ($\alpha \in \Phi$) exhaust all reflections in W.

A subset Δ of Φ called a simple system if Δ is a basis for the $\mathbf{R}$-span of Φ in V such that each $\alpha \in \Phi$ is a linear combination of Δ with all all nonzero coefficients of the same sign. The vectors of Δ are called simple roots. If Δ is a simple system and $w \in W$, $w\Delta$ is also a simple system. This action of W on the set of simple systems is simply transitive.

The subset Π of Φ consisting of roots which are linear combinations of Δ with non-negative coefficients is called a positive system and the roots belonging to Π are called positive roots.

1.1. Theorem. *For any finite reflection group W a simple system Δ exists and W is generated by the reflections s_α ($\alpha \in \Delta$).*

For each simple system Δ and $\alpha \in \Delta$ define the open subspace

$$A_\alpha := \{\lambda \in V | \langle \lambda, \alpha\rangle > 0\},$$

the open cone

$$C := \cap_{\alpha\in\Delta} A_\alpha,$$

called a chamber, and a closed cone $\bar{C}$ which is the closure of C. The hyperplanes H_α ($\alpha \in \Delta$) are called walls of the chamber C. It is known that $\bar{C}$ is a fundamental domain for the action of W on V, i.e. each $\lambda \in V$ is conjugate under W to one and only one point of $\bar{C}$.

For any $w \in W$ wC is the chamber for the simple system $w\Delta$ with the corresponding fundamental domain $w\bar{C}$ and the walls wH_α. The simple transitive action of W on the set of simple systems induces the simple transitive action of W on the set of chambers. The chambers are characterized topopogically as the connected components of $V \setminus \cup_{\alpha\in\Pi} H_\alpha$.

1.2. Theorem. *If U is any subset of V, the subgroup of W fixing U pointwise is generated by those reflections which it contains.*

There is the following theorem due to Chevalley (see [2], [3]).

1.3. Theorem. *Let $R = \mathbf{R}[V]^W$ be a subalgebra of $\mathbf{R}[V]$ consisting of W-invariant polynomials. Then R is generated, as an $\mathbf{R}$-algebra, by n homogeneous, algebraically independent polynomials $p^1, \dots, p^n$ of positive degree.*

Let $p^1, \dots, p^n$ be the set of generators of the algebra R of degrees $d_1, \dots, d_n$, respectively. In the sequel we suppose that $d_1 \le \dots \le d_n$. It is known that the degrees $d_1, \dots, d_n$ are invariants of the group W. Consider the map $p = (p_1, \dots, p_n) : V \to \mathbf{R}^n$, and the matrix valued function G on V with entries $g^{ij} := \langle dp^i, dp^j \rangle^*$, where $\langle x, y\rangle^*$ means the dual inner product. Since $g^{ij} \in R$ there is a unique polynomial $\tilde{g}^{ij} \in \mathbf{R}[\mathbf{R}^n]$ such that $g^{ij} = \tilde{g}^{ij}(p_1, \dots, p_n)$. Denote by $\tilde{G}$ the matrix valued function on $\mathbf{R}^n$ with the entries $\tilde{g}^{ij}$ and by $\tilde{M}_{i_1 \dots i_k}$ $(i_1 < \dots, < i_k)$ its minor formed by the rows and the columns of $\tilde{G}$ with the numbers $i_1, \dots, i_k$, by M the determinant of the matrix $G = (g^{ij})$, and by $\tilde{M}$ the determinant of the matrix $\tilde{G}$.

Let x^i $(i = 1, \dots, n)$ be some orthogonal coordinates in V,

$$Jm := \left(\frac{\partial p^i}{\partial x^j} \right)$$

the jacobian matrix of p, and $J = \det(Jm)$. It is easily seen that $G = Jm(Jm)^t$, where $(Jm)^t$ denote the transpose matrix of Jm, and then $M = J^2$.

A polynomial $f \in \mathbf{R}[V]$ is called anti-invariant if for all $w \in W$ $f \circ w = \det w f$. For each root α define a linear form l_α as follows: $l_\alpha(x) = \langle \alpha, x\rangle$ $(x \in V)$.

1.4. Theorem. *(a) For each positive system* Π

$$J = c \prod_{\alpha\in\Pi} l_\alpha,$$

where c is a nonzero constant.

(b) A polynomial f is anti-invariant iff it is a product of J and a W-invariant polynomial. In particular, J is an anti-invariant polynomial.

A point $x \in V$ is called regular if it belongs to some chamber C and critical if it belongs to a wall of some chamber. By Theorem 1.4. the set V_{reg} of regular points coincides with the set $\{x \in V | J(x) \neq 0\}$. The set of critical values of the map p, i.e. $D := p(V \setminus V_{reg}) \subset \mathbf{R}^n$, is called the discriminant locus. Recall the following theorem due to Shephard and Todd [13].

1.5. Theorem. *Let V be an n-dimensional real vector space and W a finite subgroup of $GL(V)$ acting naturally on the algebra $\mathbf{R}[V]$. Suppose the algebra $\mathbf{R}[V]^W$ of W-invariant polynomials is generated by n algebraically independent homogeneous* *polynomials* p^i $(\deg p^i = d_i)$. *Then* W *is generated by the reflections it contains and* $|W| = d_1 \cdots d_n$, *where* $|W|$ *is the order of the group* W.

Theorem 1.5 means that the finite reflection group W is determined uniquely by its ring of W-invariant polynomials.

Formulate now the following partial case of the Procesi-Schwarz theorem [8] for compact Lie groups.

1.6. Theorem. *The map* $p : V \to \mathbf{R}^n$ *is proper. The image* $p(V)$ *of the map* p *is a semialgebraic subset defined by the inequalities*

$$\tilde{M}_{i_1 \dots i_k} \geq 0 \quad (k = 1, \dots, n; \quad i_1 < \cdots < i_k)$$

and it is homeomorphic to the orbit space V/W *equipped with the quotient topology.*

By Theorem 1.6 the discriminant locus D is the intersection of $p(V)$ with the set of zeros of the polynomial $\tilde{M}$. Remark that the choice of generators p^i does not affect the structure of the semialgebraic subset $p(V)$ because the elements of any system of generators of R are polynomials in p^i.

In the sequel we use the following results of Luna, Schwarz, and Solomon.

Let $k = \mathbf{R}, \mathbf{C}$. Suppose a group G acts linearly on k^n and this action is completely reducible. Then by the classical Hilbert theorem the algebra $k[k^n]^G$ of G-invariant polynomials on k^n has a finite system $p^1, \dots, p^m$ of generators. Put $p = (p^1, \dots, p^m) : k^n \to k^m$. Let U be an open subset of k^n and $\mathfrak{A}(U)$ the algebra of analytic functions on U.

1.7. Theorem. *(see [7])*

1. *For each* G*-invariant function* $f \in \mathfrak{A}(\mathbf{R}^n)$ *there exists an open subset* U *containing* $p(k^n)$ *and* $g \in \mathfrak{A}(U)$ *such that* $f = g \circ p$;
2. *For each* G*-invariant function* $f \in \mathfrak{A}(\mathbf{C}^n)$ *there exists a function* $g \in \mathfrak{A}(\mathbf{C}^m)$ *such that* $f = g \circ p$.

1.8. Theorem. *(see [12]) Let* G *be a compact Lie group acting orthogonally on* $\mathbf{R}^n$ *and* f *a smooth* G*-invariant function on* $\mathbf{R}^n$. *Then there exists a smooth function* g *on* $\mathbf{R}^m$ *such that* $f = g \circ p$.

1.9. Theorem. *(see [14]) Consider the action of a finite reflection group* W *on a real vector space* V *or the induced action on the complexification* $V \otimes \mathbf{C}$ *of* V. *Then each* W*-invariant differential form on* V *(or on* $V \otimes \mathbf{C}$*) with polynomial coefficients is the pull-back via the map* p *of some differential form with polynomial coefficients on* $\mathbf{R}^n$ *(or on* $V \otimes \mathbf{C}/W = \mathbf{C}^n$*).*

2. Lifts of curves in the orbit space of a finite reflection group

Consider a smooth (analytic) curve $c : \mathbf{R} \to \mathbf{R}^n$ such that $c(\mathbf{R}) \subset p(V)$. A smooth (analytic) curve $\tilde{c} : \mathbf{R} \to V$ is called a lift of c if $p \circ \tilde{c} = c$. The aim of this Section is to prove the existence of lifts under the map p for some class of smooth curves, containing all analytic curves, in $p(V)$. Our proof is a generalization to an arbitrary finite reflection group the solution of this problem given in [1] for A_n.

Let Φ be a root system in V and W the corresponding finite reflection group. Denote by V_1 the $\mathbf{R}$-span of Φ in V. Obviously W acts trivially on the orthogonal complement V_2 of V_1 in V and Φ is a root system in V_1 for the restriction of the action of W to V_1.

2.1. Lemma. $\mathbf{R}[V]^W = \mathbf{R}[V_1]^W \otimes \mathbf{R}[V_2]$.

The proof is obvious.

By Lemma 2.1 one can suppose that the set of homogeneous generators $p^1, \dots, p^n$ of $\mathbf{R}[V]^W$ is the union of the set of homogeneous generators of the algebra $\mathbf{R}[V_1]^W$ and the basis of the dual space V_2^* of V_2. Hence, $d_1 = \dots = d_m = 1$ and $d_{m+1} = 2$, where $m = \dim V_2$, and $p(V) = \mathbf{R}^m \oplus p_1(V_1)$, where $p_1 := (p^{m+1}, \dots, p^n) : V_1 \to \mathbf{R}^{n-m}$.

Consider a smooth (analytic) curve $c(t) = (c_1(t), \dots, c_n(t))$ in $p(V)$. It is clear that it has a lift in V iff the curve $c'(t) = (c_{m+1}(t), \dots, c_n(t)$ has a lift in V_1.

Thus, the lifting problem is reduced to the case when the $\mathbf{R}$-span of Φ coincides with V and $d_1 = 2$. Moreover, under this assumption one can choose $p^1(x) = \langle x, x\rangle$. Then we have

$$\text{(2.1)} \qquad \text{For} \quad a = (a_1, \dots, a_n) \in p(V), \quad p^1(a) = 0 \quad \text{implies} \quad a = 0.$$

First we study the existence of a local lift, i.e. we consider the smooth (analytic) curve $t \to c(t) \in p(V)$ defined near $t_0 \in \mathbf{R}$ and look for its lift in V. Without loss of generality one can suppose that $t_0 = 0$.

We consider the following three cases: (1) $c(0) \in p(V) \setminus D$; (2) $c(0) = 0$; (3) $c(0) \in D$ and $c(0) \neq 0$.

2.2. Lemma. Lifting outside of discriminant locus. *The restriction of p to each chamber C is an analytic diffeomorphism of C onto $p(V) \setminus D$. Then each smooth (analytic) curve $c(t)$ such that $c(0) \in p(V) \setminus D$ has a unique lift $\tilde{c}(t)$ near 0 such that $\tilde{c}(0) \in C$.*

Proof. The restriction of p to C is a bijective map of C to $p(V) \setminus D$ and the jacobian J is nonzero on C. This implies the statement of the lemma. $\square$

For any real-valued smooth function f defined near 0 in $\mathbf{R}$ the multiplicity $m(f)$ at 0 is a supremum of all integers p such that $f(t) = t^p g(t)$ near 0 for a smooth function g. If $m(f) < \infty$ then $f(t) = t^{m(f)} g(t)$, where $g(0) \neq 0$. It is easily checked that $m(f) \geq p$ iff f and its derivatives $f', \dots, f^{(p-1)}$ vanish at 0. A smooth nonzero function f defined near $t_0 \in \mathbf{R}$ is called flat at t_0 if f and all its derivatives at t_0 vanish. For a smooth curve $c(t)$ in V defined near 0 we write $m(c) \geq r$ if for each linear form ω on V $m(\omega \circ c) \geq r$.

Suppose $c(t)$ is a smooth curve in V defined near 0 and $m(c) \geq r$. Since $\deg p^i = d_i$, $m(p^i \circ c) \geq r d_i$.

Let $t \to c(t) \in p(V)$ be a smooth curve in $\mathbf{R}^n$ defined near 0. Denote by $\tilde{M}_{i_1 \dots i_k, j_1 \dots j_k}$ $(i_1 < \dots i_k, j_1 < \dots < j_k)$ the minor of the matrix $\tilde{M}$ formed by the rows with the numbers $i_1, \dots, i_k$ and by the columns with the numbers $j_1, \dots, j_k$.

2.3. Multiplicity Lemma. *Suppose the $\mathbf{R}$-span of Φ coincides with V and $p^1(x) = \langle x, x\rangle$. Let $t \to c(t) \in p(V)$ be a smooth curve in $\mathbf{R}^n$ defined near 0 and $c^i = p^i \circ c$. Then the following conditions are equivalent:*

1. $m(c^i) \geq r d_i$ *for* $1 \leq i \leq n$;
2. $m(\tilde{M}_{i_1 \dots i_k, j_1 \dots j_k} \circ c) \geq r(d_{i_1} + \dots + d_{i_k} + d_{j_1} + \dots + d_{j_k} - 2k)$ *for all* $1 \leq k \leq n$, $i_1 < \dots < i_k$, *and* $j_1 < \dots < j_k$;
3. $m(c^1) \geq 2r$.

Proof. (1) implies (2): By assumption $c^i(t) = t^{rd_i} c'_i(t)$ for smooth c'_i. Consider p^i as a graded indeteminate of degree d_i. Then $\deg \tilde{g}^{ij} = d_i + d_j - 2$ and it can be easily checked that all summands of the minor $\tilde{M}_{i_1 \dots i_k, j_1 \dots j_k}$ have degree $r(d_{i_1} + \dots + d_{i_k} + j_1 + \dots + j_k - 2k)$. Then $m(\tilde{M}_{i_1 \dots i_k, j_1 \dots j_k} \circ c) \geq r(d_{i_1} + \dots + d_{i_k} + d_{j_1} + \dots + d_{j_k} - 2k)$.

(2) implies (3) since $\tilde{M}_{1,1} = 4p^1$.

(3) implies (1): If $r = 0$ we are done. Let $r > 0$. By (2.1) $c^1(0) = 0$ implies $c(0) = 0$. Therefore $m(c^2), \dots, m(c^n) \geq 1$ and we have $c^i(t) = t^{m_i} c^i_{m_i}(t)$ for $i = 1, \dots, n$, where $m_1, \dots, m_n$ are positive integers, $c^i_{m_i}$ are smooth functions, and $m_1 = 2r$. We may assume that for each i either $m_i = m(c^i) < \infty$ or $m_i \geq rd_i$. Suppose now that for some $i > 1$ we have $m_i = m(c^i) < rd_i$. Then we put

$$m := \min(r, \frac{m_2}{d_2}, \dots, \frac{m_n}{d_n}) < r.$$

Consider the following continuous curve in $\mathbf{R}^n$ for $t \geq 0$:

$$c_{(m)}(t) := (t^{2r-2m} c^1_{2r}(t), t^{m_2 - md_2} c^2_{m_2}(t), \dots, t^{m_n - md_n} c^n_{m_n}(t)).$$

Since for each $x \in V$ and $t \in \mathbf{R}$ $p^i(tx) = t^{d_i} p^i(x)$, $c_{(m)}(t) \in p(V)$ for each $t > 0$. By Theorem 1.6 $p(V)$ is a closed set in $\mathbf{R}^n$, then $c_{(m)}(0) \in p(V)$. Since $m < r$, $c^1_{2r}(0) = 0$ and then by (2.1) $c_{(m)}(0) = 0$. In particular, for all i such that $m_i = md_i$ $c^i_{m_i}(0) = 0$. Then $m(c^i) > m_i$. This contradiction concludes the proof. □

Let one of the conditions (1), (2), and (3) of Multiplicity Lemma 2.3 be satisfied for the curve $c(t)$. Then $c^i(t) = t^{rd_i} c_{i,rd_i}(t)$ $(i = 1, \dots, n)$, where c_{i,rd_i} is a smooth function. Consider the smooth curve

$$c_{(r)}(t) := (c_{1,2r}(t), c_{2,rd_2}(t), \dots, c_{n,rd_n}(t)). \tag{2.2}$$

It is clear that $c_{(r)}(t) \in p(V)$ and $\tilde{c}(t) = t^r \tilde{c}_{(r)}(t)$ is a lift of $c(t)$ near 0 iff $\tilde{c}_{(r)}(t)$ is a lift of $c_{(r)}(t)$ near 0.

Let Π be a positive system, $x \in V \setminus V_{reg}$, and W_x the isotropy group of x. By Theorem 1.2 W_x is a finite reflection group generated by the set of reflections $\{s_\alpha | \alpha \in \Pi, s_\alpha(x) = x\}$. Suppose Φ_x be a root system of W_x and Π_x is the corresponding positive system. Put

$$V_x := \cap_{\alpha \in \Pi_x} H_\alpha.$$

Then by Lemma 2.1

$$\mathbf{R}[V]^{W_x} = \mathbf{R}[V'_x]^{W_x} \otimes \mathbf{R}[V_x],$$

where V'_x is the orthogonal complement of the subspace V_x in V.

Denote by $q^1, \dots, q^n$ a set of algebraically independent homogeneous generators of the algebra $\mathbf{R}[V]^{W_x}$ of W_x-invariant polynomials on V, where $q^1, \dots, q^m$ is the basis of the dual space of V_x, and put

$$q := (q^1, \dots, q^n) : V \to \mathbf{R}^n.$$

Since $\mathbf{R}[V]^W \subset \mathbf{R}[V]^{W_x}$, for each $i = 1, \dots, n$ there is a unique polynomial $P^i \in \mathbf{R}[\mathbf{R}^n]$ such that $p^i = P^i(q^1, \dots, q^n)$. Put

$$P = P_x := (P^1, \dots, P^n) : \mathbf{R}^n \to \mathbf{R}^n.$$

2.4. Reduction Lemma. [1]

(1) *The map* $P = P_x : \mathbf{R}^n \to \mathbf{R}^n$ *is a local analytic diffeomorphism in a neighborhood of* $q(x) \in \mathbf{R}^n$*;*

(2) *If* $c(t)$ *is a smooth (analytic) curve near* 0 *such that* $c(0) = p(x)$*, it has a unique smooth (analytic) lift* $\tilde{c}(t)$ *to* $\mathbf{R}^n$ *under the map* P *such that* $\tilde{c}(0) = q(x)$*;*

(3) *The set of lifts of the curve* c *near* 0 *is the union of sets of lifts for the curves* $P_x^{-1} \circ c$ *near* 0 $(x \in p^{-1}(c(0)))$.

Proof. Let e_i $(i = 1, \dots, n)$ be an orthonormal basis of V and x^i the corresponding coordinates. Denote by $J(p)$, $J(q)$, and $J(P)$ the jacobians of the maps p, q, and P with respect to these coordinates, respectively. It is sufficient to prove that $J(P)(q(x)) \neq 0$.

Since $p = q \circ P$ we have

$$J(p) = J(q)(J(P) \circ q). \tag{2.3}$$

By definition $J(p)$, $J(q)$, and $J(P)$ are polynomials in $x^1, \dots, x^n$. Hence, by Theorem 1.4

$$J(P) \circ q = \frac{J(p)}{J(q)}.$$

Denote by Φ_x the root system of W_x and put $\Pi_x := \Pi \cap \Phi_x$. Therefore,

$$J(p) = c \prod_{\alpha \in \Pi} l_\alpha \quad \text{and} \quad J(q) = c' \prod_{\alpha \in \Pi_x} l_\alpha,$$

where c, c' are nonzero constants. Then

$$J(P) = \frac{c}{c'} \prod_{\alpha \in \Pi \setminus \Pi_x} l_\alpha.$$

Since for each $\alpha \in \Pi \setminus \Pi_x$ $l_\alpha(x) \neq 0$, $J(P)(q(x)) \neq 0$. This proves (1).

(2) and (3) follow from (1) immediately. □

Remark. If x is a critical point distinct from 0 the order $|W_x|$ of the group W_x is strictly less than the order $|W|$ of the group W. Therefore, in this case Reduction Lemma 2.4 reduces problem of the existence of local lift for W to the same problem for the finite reflection group W_x of strictly less order.

Now we consider the class of the smooth curves $c(t)$ defined near t_0 satisfying the following condition (A):

There exists an integer $k \leq n$ such that

(A_1) $\operatorname{rank} \tilde{G} \circ c \leq k$ in some neighborhood of t_0;

(A_2) There is a nonzero minor $\tilde{M}_{i_1 \dots i_k, j_1 \dots j_k} \circ c$ which is not flat at t_0.

Note that the condition (A) is satisfied for any analytic curve c.

It is easily seen that the condition (A) is preserved under the reduction of the lifting problem by Lemma 2.1 and when we replace the curve c by the curve $c_{(r)}$ using Multiplicity Lemma 2.3.

[1] This lemma was proved with D.V.Alekseevsky.

Lemma 2.5. *The condition (A) is preserved under the reduction of the lifting problem by Reduction Lemma 2.4 to the group W_x.*

Proof. Without loss of generality we may assume that $t_0 = 0$. Let $Jm(P)$ be the jacobian matrix of the map P and $\tilde{G}_x = (\tilde{g}_x^{ij})$ the matrix constructed from the set of generators $q^1, \ldots, q^n$. Then $\tilde{G} = Jm(P)\tilde{G}_x(Jm(P))^t$. Since $J(P)(x) \neq 0$, $\operatorname{rank} \tilde{G}_x \circ c \leq k$ near 0, i.e. the condition (A_1) is satisfied for the group W_x.

Denote by $\tilde{M}_{x;i_1\ldots i_k,j_1\ldots j_k}$ the minors of the matrix $\tilde{G}_x$ formed by the rows with numbers $i_1, \ldots, i_k$ and by the columns with numbers $j_1, \ldots, j_k$. Then there is a minor $\tilde{M}_{x;i_1\ldots i_k,j_1\ldots j_k}$ which is not flat at 0. Actually, assume that, for all $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$, the minors $\tilde{M}_{x;i_1\ldots i_k,j_1\ldots j_k} \circ c$ either vanish identically near 0 or are flat at 0. It is known (see, for example, [5]) that each p-minor of the product of two matrices is equal to a sum of products of p-minors for factors. Therefore, all minors of the matrix $\tilde{G} \circ c$ of order k either vanish or are flat at 0. This contradiction proves the condition (A_2) for the group W_x. □

2.6.Theorem. *Suppose the $\mathbf{R}$-span of Φ coincides with V and $p_1(x) = \langle x, x \rangle$. Let c be a smooth (analytic) curve defined near t_0 which lies in $p(V)$, $x \in p^{-1}(c(t_0))$ and satisfies the condition (A). There is a unique smooth (analytic) lift $\tilde{c}$ of c near t_0 such that $\tilde{c}(t_0) = x$. All lifts of c near t_0 differ by transformations belonging to W.*

Proof. Without loss of generality we may assume that $t_0 = 0$. We point out the following algorithm in 3 steps to get the lift $\tilde{c}$.

*Step 1.*If $c(0) \in p(V) \setminus D$, there is a unique local lift through each $x \in p^{-1}(c(0))$ by Lemma 2.2.

Step 2. If $c(0) = p(x)$, where $c(0) \in D$ and $c(0) \neq 0$, the existence of a lift for c near 0 by Reduction Lemma 2.4 reduces to the existence of local lift for the group W_x such that $|W_x| < |W|$ and for the curve uniquely determined by c. By Lemma 2.5 for this reduction the condition (A) remains true and we reduce the lifting problem to the case $c(0) = 0$.

Step 3. Let $c(0) = 0$. Reduce the lifting problem to the case when the $\mathbf{R}$-span of Φ coincides with V and $p^1(x) = \langle x, x \rangle$ by Lemma 2.1. For this reduction the condition (A) remains true.

Consider the curve $c_{(r)}(t)$ defined by (2.2). Since the condition (A) is true for the curve $c_{(r)}$ we reduce the lifting problem to the case $c(0) \neq 0$. Then we can proceed in step 1 or 2.

By Lemmas 2.1, 2.2, 2.3, and 2.4, the lifts obtained by the above algorithm give all lifts of c near 0 and they differ by transformations belonging to W. This concludes the proof. □

2.7. Theorem. *Let $c(t) = (c^i(t))$ $(t \in \mathbf{R})$ be a smooth (analytic) curve lying into $p(V)$ and satisfying the condition (A) for each $t \in \mathbf{R}$. Then for each $x \in p^{-1}(0)$ there is a smooth (analytic) lift $\tilde{c}(t)$ $(t \in \mathbf{R})$ in V such that $\tilde{c}(0) = x$. All lifts of $c(t)$ differ by transformations belonging to W.*

Proof. By Theorem 2.6 one can cover $\mathbf{R}$ by a family U_s of open intervals such that there is a lift $\tilde{c}_s(t)$ of c for each of these intervals. We may suppose that for U_s containing 0 $\tilde{c}_s(0) = x$ and for each two overlapping U_{s_1} and U_{s_2} there is a point $t_0 \in U_{s_1} \cap U_{s_2}$ such that $\tilde{c}_{s_1}(t_0) = \tilde{c}_{s_2}(t_0)$. Then we can glue the lift $\tilde{c}$ from this family of lifts. This follows from the following statement:

Let $\tilde{c}_1$ and $\tilde{c}_2$ be two lifts of the curve c on an interval (a, b) of $\mathbf{R}$ and $\tilde{c}_1(t_0) = \tilde{c}_1(t_0)$ for some $t_0 \in (a, b)$. Then $\tilde{c}_1 = \tilde{c}_2$.

Actually, by Theorem 2.6 the set $\{t|\tilde{c}_1(t) = \tilde{c}_2(t)\}$ is open and closed simultaneously and then it equals (a, b). The uniqueness of the lift $\tilde{c}(t)$ follows from the same statement. $\square$

Note that by Theorem 2.7 any analytic curve c has a global analytic lift $\tilde{c}$.

3. Proof of Main Theorem

By Theorem 1.6 $p(V)$ is a closed subset of $\mathbf{R}^n$ and its interior is equal to $p(V) \setminus D$. A smooth (analytic) diffeomorphism of $V/W = p(V)$ is a bijective map $f : p(V) \to p(V)$ which can be extended to a smooth (analytic) diffeomorphism of some neighborhood of $p(V)$ into V. The aim of this Section is to prove the following

3.1. Main Theorem. *Let f be a smooth (analytic) diffeomorphism of V/W and (C, C') a pair of chambers. Then there is a unique smooth (analytic) diffeomorphism $F : V \to V$ such that $F(C) = C'$ and the following diagramm*

$$\begin{array}{ccc} V & \xrightarrow{F} & V \\ {\scriptstyle p}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ V/W & \xrightarrow{f} & V/W \end{array}$$

is commutative.

Remarks. 1. Since D is the boundary of $p(V)$ each diffeomorphism f of V/W induces the diffeomorphism of the interior $p(V) \setminus D$ onto itself and then $p(D) = D$.

2. Let F be a diffeomorphism of V of Main Theorem 3.1 and $w_1, w_2 \in W$. Then the statement of Main Theorem 3.1 is true for the diffeomorphism $w_1 \circ F \circ w_2$ of V and the pair of chambers $(w_2^{-1}(C), w_1(C'))$. Hence, it is sufficient to prove Main Theorem 3.1 for $C' = C$.

3. By Lemma 2.1 it is sufficient to prove Theorem 3.1 in the case when the $\mathbf{R}$-span of Φ is equal to V. Further we suppose that this condition is satisfied and $p^1(x) = \langle x, x\rangle$ $(x \in V)$.

Since for each $x \in V$ the tangent space T_x to V at x is canonically isomorphic to V the polynomial p^i define on T_x a unique symmetric d_i-form, i.e. the tensor $P_i(x)$ of type $(0, d_i)$ on T_x. Then the map $x \to P_i(x)$ is an analytic symmetric tensor field of type $(0, d_i)$ on V. By Theorem 1.5 the tensor fields P_i $(i = 1, \dots, n)$ define on V, as an affine space, the structure of W-module up to a choice of 0. In particular, $g(V) := P_1$ is the Riemannian metric of the Euclidean space V.

The restriction of map p to a chamber C is a diffeomorphism of C onto $p(V) \setminus D$. Then for a pair of chambers (C, C') there is a unique diffeomorphism $F_{C,C'} : C \to C'$ such that $p \circ F_{C,C'} = f \circ p$ on C. Consider the pull-back $F^*_{C,C'}P_i$ of the restriction of P_i to C'. Since the polynomial p^i is W-invariant, $F^*_{C,C'}P_i$ does not depend on C'. Denote by $P^*_{i,0}$ the symmetric tensor field of type $(0, d_i)$ on V_{reg} such that its restriction to each chamber C equals $F^*_{C,C'}P_i$. In particular, $g^*_0(V) := P^*_{1,0}$ is a flat Riemannian metric on V_{reg}. By construction the tensor field $P^*_{i,0}$ is W-invariant.

The proof of Theorem 3.1 is based on the following

3.2. Theorem. *There is a unique smooth W-invariant tensor field P_i^* of type $(0, d_i)$ $(i = 1, \dots, n)$ on V such that the restriction of P_i^* to V_{reg} coincides with $P_{i,0}^*$. The tensor field P_i^* is analytic whenever the diffeomorphism f is analytic.*

Let us fix some orthonormal basis of the Euclidean space V and the corresponding coordinates x^j. Then each tensor field P_i is defined by its components $P_{i,j_1 \dots j_m}$, where $m = d_i$, with respect to the coordinates x^j. By definition $P_{1,jk} = \delta_{jk}$. In just the same way we define the tensor fields on $\mathbf{R}^n$ by their components with respect to the natural coordinates of $\mathbf{R}^n$. Denote by P^i the tensor field on V of type $(m, 0)$ corresponding to P_i under the canonical isomorphism of the Euclidean space V with its dual space V^*.

First we prove the following three lemmas.

Let $TV = V \oplus V$ be the tangent bundle of V with the induced action of W and $\pi : TV \to V$ the natural projection. We identify a function ϕ on V with the corresponding function $\phi \circ \pi$ on TV.

3.3.Lemma. *Suppose ϕ is a smooth (analytic) W-invariant real-valued function on TV which vanishes on the restriction of the tangent bundle TV to $V \setminus V_{reg}$. Then M divides ϕ in the ring $\mathfrak{F}(TV)$ of smooth functions on TV, i.e. there is a smooth real-valued function ψ on TV such that $\phi = \psi M$. If ϕ is analytic, ψ is analytic as well.*

Proof. Let Δ be a simple system and Π the corresponding positive system for W. Since ϕ vanishes on each hyperplane $\pi^{-1}(H_\alpha)$ $(\alpha \in \Pi)$ of $TV = V \oplus V$, l_α divides ϕ in the ring $\mathfrak{F}(TV)$. Then, by Theorem 1.4 (a) there is a smooth function η on TV such that $\phi = \eta J$ and by Theorem 1.4 (b) the function η is anti-invariant, i.e. for each $w \in W$ $\eta \circ w^* = \det(w)\eta$, where w^* is a natural extension of w to TV.

Let $\alpha \in \Phi$. Then $\eta \circ s_\alpha^* = -\eta$. This forces that η vanishes on the hyperplane $\pi^{-1}(H_\alpha)$ and, therefore, J divides η in the ring $\mathfrak{F}(TV)$.

Finally, by Theorem 1.4 $M = J^2$ divides the function ϕ in the ring $\mathfrak{F}(TV)$.

Since M is a polynomial, the function ψ is analytic whenever the function ϕ is analytic. This concludes the proof. □

3.4. Lemma. *The function $h = \tilde{M} \circ f \circ p$ equals kM, where k is a W-invariant smooth positive function on V which is analytic whenever the diffeomorphism f is analytic.*

Proof. Since h is a W-invariant non-negative smooth function on V which vanishes on $V \setminus V_{reg}$, by Lemma 3.3 $h = kM$, where k is a smooth non-negative function on V.

The function $M = \tilde{M} \circ p$ is W-invariant, therefore k is also W-invariant. By Theorem 1.8 $k = \tilde{k} \circ p$, where $\tilde{k}$ is a smooth function on $\mathbf{R}^n$. It is clear that $\tilde{k}$ is positive on $p(V) \setminus D$.

Replace in the above reasoning f by f^{-1}. Then one has $\tilde{M} \circ f^{-1} \circ p = k_1 M$, where $k_1 = \tilde{k}_1 \circ p$ and $\tilde{k}_1$ is a smooth function on $\mathbf{R}^n$ which is positive on $p(V) \setminus D$. Hence, $\tilde{k}(\tilde{k}_1 \circ f) = 1$ on $p(V) \setminus D$ and, therefore, on $p(V)$. Thus, $k = \tilde{k} \circ p$ is a positive function on V.

For the analytic diffeomorphism f the proof is the same but we use Theorem 1.7 instead of Theorem 1.8. □

3.5. Lemma. *Let $b = (b_{jk})$ be the symmetric tensor field of type $(0,2)$ on $\mathbf{R}^n$ such that the components b_{jk} are equal to the entries of the adjoint $\tilde{G}^*$ of the matrix $\tilde{G}$. Then the pull-back p^*b of b equals $Mg(V)$.*

Proof. The statement of the lemma follows from the evident identity

$$\tilde{G}^{-1} = \tilde{M}^{-1}\tilde{G}^*.$$

□

Proof of Theorem 3.2. Let b be the tensor of Lemma 3.5. Denote by b^* the pull-back of b via the map $f \circ p$ and put $u(\tau) = b^*(\tau, \tau)$ for each tangent vector τ at $x \in V$. It is clear that b^* is a smooth (analytic) W-invariant function on TV.

Suppose $x \in V \setminus V_{reg}$. Consider the straight line $a(t) = x + \tau t$ in V near 0, where we identify the tangent vector τ with the corresponding vector of V, and its image $c := f \circ p \circ a$ under the map $f \circ p$ in $p(V)$. By definition the tangent vector $\frac{d}{dt}a(0)$ of $a(t)$ at 0 equals τ and then by the definition of pull-back

$$u(\tau) = b\left(\frac{d}{dt}c(0), \frac{d}{dt}c(0)\right).$$

Since the curve $p \circ a(t)$ is analitic it satisfies the conditions of Theorem 2.6. It is clear that these conditions are invariant under the diffeomorphism f. Then the curve $c(t)$ satisfies these conditions also and there is a lift $\tilde{c}(t)$ of $c(t)$ near 0. This implies the equality

$$u(\tau) = (p^*b)\left(\frac{d}{dt}\tilde{c}(0), \frac{d}{dt}\tilde{c}(0)\right).$$

Since $c(0) \in D$ and by Lemma 3.5 $p^*b = Mg(V)$, $u(\tau) = 0$. By Lemma 3.3 there is a smooth function v on TV such that $u = kMv$, where k is the function on V defined by Lemma 3.4.

The restriction of v to the tangent space T_x is a quadratic form on T_x and, hence, v defines the W-invariant smooth symmetric tensor field $g^*(V)$ of type $(0,2)$ on V. By Lemma 3.4 $g^*(V)$ coincides with $g_0^*(V)$ on V_{reg}. It is easily checked that $g^*(V)$ is non-degenerate on V .

Since the Riemannian metric $g^*(V)$ is flat there is a local isometry I of $(V, g^*(V))$ to $(V, g(V))$ such that $I(0) = 0$. We may suppose that I coincides with the isometry $F_{C,C} : C \to C$ on the intersection of the domain of I and C. Then on this intersection the pull-back I^*P_i coincides with P_i^*. By definition P_i is a W-invariant parallel tensor field on the Riemannian manifold $(V, g(V))$. Then I^*P_i is the W-invariant parallel tensor field on the flat Riemannian manifold $(V, g^*(V))$. Denote by P_i^* the parallel extension of I^*P_i to the whole of V. By construction the tensor field P_i^* is W-invariant and its restriction to V_{reg} equals $P_{0,i}^*$.

It remains to note that all tensor fields of the above proof are analytic whenever the diffeomorphism f is analytic. □

Proof of Main Theorem 3.1. Since the Riemannian metric $g^*(V)$ is flat it defines the Euclidean structure in some neighborhood U of 0. Therefore, the parallel tensor field P_i^* defines a polynomial $p^{*,i}$ in U such that $p^{*,i}(0) = 0$ and then by Theorem 1.5 U is isomorphic to a neighborhood U' of 0 of W-module V.

Fix such an isomorphism $I : U \to U'$. By definition, for each i $p^{*,i} = p^i \circ I$ on U, i.e. $p \circ I = f \circ p$ on U. Moreover, I establishes a one-to-one correspondence

$C \to C'$ between chambers such that $I(C \cap U) = C' \cap U'$ and, hence, I coincides with $F_{C,C'}$ on $C \cap U$. Therefore, one can glue the smooth diffeomorphism F_0 of $U \cup V_{reg}$ onto $U' \cup V_{reg}$ from I and the maps $F_{C,C'}$.

We prove now that F_0 has a smooth extension F to the whole of V. Since $U \cup V_{reg}$ is a dense subset of V this extension is unique, whenever it exists, and $F \circ p = p \circ f$. Suppose V_0 is the maximal open subset of V for which such an extension, denoted again by F, exists. Since F is an isometry of V_0 into the Euclidean manifold $(V, g(V))$ it is a diffeomorphism. We claim that $V_0 = V$ and $F(V_0) = V$.

Suppose $V_0 \neq V$ and $x \notin F(V_0)$. The curve $a(t) = p(tx)$ ($t \in \mathbf{R}$) is an analytic curve in $p(V)$ and, therefore, it satisfies the conditions of Theorem 2.7. Since these conditions are preserved under the diffeomorphism f^{-1} the curve $f^{-1} \circ a$ has a global smooth lift $\tilde{a}$ such that $\tilde{a}(0) = 0$. By Theorem 2.6 we may assume that $\tilde{a}$ coincides with the curve $t \to F_0^{-1}(tx)$ near 0 since it is a lift of the curve $f^{-1} \circ a$ near 0.

Consider the set of intervals $(0, t)$ such that $\tilde{a}((0, t)) \subset V_0$ and let $(0, t_0)$ be the maximal of them. Consider a local isometry I_1 of some neighborhood U_1 of the point $\tilde{a}(t_0)$ in the Riemannian manifold $(V, g^*(V))$ into the Euclidean space $(V, g(V))$ such that $I_1(\tilde{a}(t)) = tx$ for all $t \in (0, t_0)$. Then $I_1(\tilde{a}(t_0)) = t_0 x$ and we may assume that for all i $I_1^* P_i = P_i^*$ on U_1. But this means that I_1 coincides with F on $V_0 \cap U_1$ and, therefore, F_0 has a smooth extension to $V_0 \cup U_1 \neq V_0$. This implies that either $V_0 = V$ or $F(V_0) = V$.

Suppose $V = V_0$. Then $F \circ \tilde{a}$ is a global lift of the curve $p \circ a$ which coincides with a near 0 by construction. Therefore, $F \circ \tilde{a} = a$ and, in particular, $F(\tilde{a}(1)) = x$. This means that $x \in F(V)$. This contradiction implies $F(V) = V$.

Suppose $F(V_0) = V$. Then V_0 is a complete open Riemannian submanifold of $(V, g^*(V))$. But it is possible only if $V_0 = V$. This concludes the proof in the smooth case.

It remains to note that if f is analytic diffeomorphism, the diffeomorphism F is analytic as well. □

Now we indicate the local version of Theorem 3.1.

3.6. Theorem. *Let f be a diffeomorphism of a connected open neighborhood U of $y \in p(V) \subset \mathbf{R}^n$ into $\mathbf{R}^n$ such that $f(U \cap p(V)) \subset p(V)$ and $f(D \cap U) \subset D$. Then for each $x \in V$ such that $p(x) = y$, there is a neighborhood $\tilde{U}$ of x and a diffeomorphism F of $\tilde{U}$ into V such that $p \circ F = f \circ p$ on $\tilde{U}$. If (C, C') is a pair of chambers, the diffeomorphism F is determined uniquely by the condition $F(U \cap C) \subset C'$.*

Proof. Note that Lemma 3.3 is true for functions ϕ and ψ defined in an open subset of V. Therefore, the local version of Lemma 3.4 is also true. Then the proof of Theorem 3.6 coincides with first part of the proof of Main Theorem 3.1. □

3.7. Theorem. *Let Y be a smooth (analytic) manifold and f_y ($y \in Y$) a smooth (analytic) family of smooth (analytic) diffeomorphisms of V/W. Then for each pair of chambers (C, C') there is a unique smooth (analytic) family F_y of smooth (analytic) diffeomorphisms of V such that for each $y \in Y$ $F_y(C) = C'$ and $p \circ F_y = f_y \circ p$. In particular, let Y be a Lie group and f_y an action of Y on $p(V)$ by diffeomorphisms of V/W. Then for $C = C'$ the above family F_y gives an action of the Lie group Y on V independent of C.*

Proof. It is sufficient to consider the case $C' = C$. Note that Lemma 3.3 is true for a smooth family of functions ϕ_y ($y \in Y$) satisfying the conditions of this lemma and the corresponding family ψ_y is smooth. Denote by $g^*(V)_y$ the smooth family of Riemannian metrics on V corresponding to the family of diffeomorphisms f_y by Theorem 3.2.

From the proof of Theorem 3.1 it is easily seen that for $C = C'$ the exponential map for the Riemannian manifold $(V, g^*(V))$ at 0 $\exp_0 : V = T_0V \to V$ is equal to F^{-1}. Therefore, for the smooth family $g^*(V)_y$ the corresponding family F_y of diffeomorphisms of V is smooth as well.

If Y is a Lie group acting on V/W, it is easily to check that the corresponding family F_y defines an action of the Lie group Y on V.

In the analytic case the proof is the same. □

Now we prove the following analog of the Zakalyukin theorem [15] on the lift of a vector field, tangent to the discriminant locus, for the real case.

A function or a vector field on $V/W = p(V)$ is called smooth (analytic) if it has a smooth (analytic) extension to a neighborhood of $p(V)$.

3.8. Theorem. *Let X be a smooth (analytic) vector field on $p(V)$, tangent to the discriminant locus. Then there is a unique smooth (analytic) vector field $\tilde{X}$ on V such that $p_*\tilde{X} = X$. The map $X \to \tilde{X}$ is an isomorphism of the Lie algebra of smooth (analytic) vector fields on $p(V)$ onto the Lie algebra of smooth (analytic) W-invariant vector fields on V.*

Proof. Since X defines a local flow in a neghborhood of each $x \in p(V)$ and this flow preserves the discriminant locus, the proof of the existence and uniqueness of $\tilde{X}$ is analogous to one of Theorem 3.7 for a Lie group of transformations of $p(V)$ using Theorem 3.6 instead of Theorem 3.1.

Let $\tilde{X}$ be a smooth (analytic) W-invariant vector field on V. Since $\tilde{X}$ is invariant under each reflection s_α $(\alpha \in \Phi)$ it is tangent to the hyperplane H_α. This implies that the vector field $X = p_*\tilde{X}$ is tangent to the discriminant locus. This concludes the proof. □

3.9 Corollary. *Let $\mathfrak{F}(p(V))$ be the ring of smooth (analytic) functions on $p(V)$. The Lie algebra of smooth (analytic) vector fields on $p(V)$, tangent to the discriminant locus, is a free $\mathfrak{F}(p(V))$-module with n generators*

$$\sum_{j=1}^{n} \tilde{g}^{ij}\partial_j \quad (i = 1, \ldots, n).$$

Proof. It is evident that the inner product on V defines a one-to-one correspondence between the space of W-invariant smooth (analytic) vector fields and the space of W-invariant smooth (analytic) 1-forms on V. By Theorem 1.8 each smooth W-invariant function g on V can be represented as $h(p^i)$, where h is a smooth function on $\mathbf{R}^n$. By Theorem 1.7 the same assertion is true for an analytic W-invariant function g and a function h which is analytic in some neighborhood of $p(V)$. Using these theorems one can easily extend the proof of Theorem 1.9 (see [14]) about the W-invariant polynomial forms to smooth (analytic) forms. Therefore, each smooth (analytic)W-invariant form is a sum of forms $g(p^j)dp^i$, where g is a smooth function in $\mathbf{R}^n$ (an analytic function in some neighborhood of $p(V)$). Hence, each smooth (analytic) W-invariant vector field on V is a sum of vector fields $g(p^j)\,\mathrm{grad}\,p^i$. It remains to note that $p_*\,\mathrm{grad}\,p^i = \tilde{g}^{ij}\partial_j$. This concludes the proof. □

4. Main Theorem in the complex case

Consider the natural action of W on the complexification $V_{\mathbf{C}} = V \otimes \mathbf{C}$ of V and the corresponding subalgebra $R_{\mathbf{C}}$ of $\mathbf{C}[V_{\mathbf{C}}]$ of W-invariant polynomials. Then $p^1, \dots, p^n$ are again homogeneous generators of the algebra $R_{\mathbf{C}}$ and

$$p_{\mathbf{C}} = (p^1, \dots, p^n) : V_{\mathbf{C}} \to \mathbf{C}^n$$

is a bijective map. The complexification $\langle x, y\rangle_{\mathbf{C}}$ of the inner product $\langle x, y\rangle$ of V is a complex orthogonal Euclidean metric on $V_{\mathbf{C}}$.

In just the same way as for the real case we define the set $V_{\mathbf{C},reg}$ of regular points of $V_{\mathbf{C}}$, the discriminant locus $D_{\mathbf{C}}$, the matrix valued function $G_{\mathbf{C}} = (g^{ij}_{\mathbf{C}})$ on $V_{\mathbf{C}}$, the matrix valued function $\tilde{G}_{\mathbf{C}} = (\tilde{g}^{ij}_{\mathbf{C}})$ on $V_{\mathbf{C}}/W = \mathbf{C}^n$, the determinant $M_{\mathbf{C}}$ of the matrix G, and the determinant $M_{\mathbf{C}}$ of the matrix $\mathbf{G}$. For the sake of simplicity we usually drop the subscript $\mathbf{C}$ in $p_{\mathbf{C}}$ and $\langle x, y\rangle_{\mathbf{C}}$ till the end of this Section.

Let $f : \mathbf{C}^n \to \mathbf{C}^n$ be a biholomorphic map. A pair (x, x') of points of $V_{\mathbf{C}}$ is called f-compatible if $f \circ p(x) = p(x')$. In this Section we prove the following analog of Main Theorem 3.1.

4.1. Theorem. *Let $f : \mathbf{C}^n \to \mathbf{C}^n$ be a biholomorphic map such that $f(D) = D$. Then, for each pair (x, x') of regular f-compatible points of $V_{\mathbf{C}}$, there exists a unique biholomorphic map $F : V_{\mathbf{C}} \to V_{\mathbf{C}}$ such that $F(x) = x'$ and $p \circ F = f \circ p$.*

Since Lemma 2.1 is true for the algebra $R_{\mathbf{C}}$ it is sufficient to consider the case when $\mathbf{C}$-span of Φ coincides with $V_{\mathbf{C}}$, $d_1 = 2$, and $p^1(x) = \langle x, x\rangle$.

To prove Theorem 4.1 we need to lift some holomorphic curves in $V_{\mathbf{C}}/W$ to $V_{\mathbf{C}}$ but, unfortunately, only the part of results of Section 2 remains true in this case. More precisely, the map p is biholomorphic in a neighborhood of every point $x \in V_{\mathbf{C}}$ and Reduction Lemma 2.4 is true for the biholomorphic map $P : \mathbf{C}^n \to \mathbf{C}^n$. Thus we have a holomorphic lift of a holomorphic curve lying outside of discriminant, all these lifts differ by a transformation belonging to W, and the problem of local lifting of a holomorphic curve, passing through x, such that $x \in D$ and $x \neq 0$ reduces to the case when $x = 0$. Theorems 2.6 and 2.7 are not true in the complex case in general and we need to find the appropriate substitute.

As in Section 3 we consider the holomorphic tensor fields P_i of type $(0, d_i)$ $(i = 1, \dots, n)$ on $V_{\mathbf{C}}$ which are uniquely defined by the polynomials p^i. Since the map p is biholomorphic in a neighborhood of every regular point of $V_{\mathbf{C}}$, for each pair (x, x') of f-compatible points of V_{reg} the maps p and f define a biholomorphic map $F_{x,x'}$ of a neighborhood U_x of x onto a neighborhood of x' such that $p \circ F_{x,x'} = f \circ p$ in U_x. Hence, there are unique W-invariant holomorphic tensor fields $P^*_{i,0}$ of type $(0, d_i)$ on $V_{\mathbf{C},reg}$ such that for each pair (x, x') of regular f-compatible points of $V_{\mathbf{C}}$ $(f \circ p)_*(P^*_{i,0}(x)) = p_*(P_i(x'))$.

First we recall some facts we need further.

Denote by $\mathfrak{F}(V)^W$ the ring of W-invariant holomorphic functions on V, by $\mathfrak{X}(V)^W$ the Lie algebra of W-invariant holomorphic vector fields on V, by $\mathfrak{F}(V_{\mathbf{C}}/W)$ the ring of holomorphic functions on $\mathbf{C}^n = V_{\mathbf{C}}/W$, and by $\mathfrak{X}(V_{\mathbf{C}}/W)$ the Lie algebra of holomorphic vector fields on $V_{\mathbf{C}}/W$, tangent to the discriminant locus D.

There is the following theorem due to Zakalyukin (see [15]).

4.2. Theorem. *The map p induces the isomorphism of the Lie algebra $\mathfrak{X}(V)^W$ onto the Lie algebra $\mathfrak{X}(V_{\mathbf{C}}/W)$. The Lie algebra $\mathfrak{X}(V)^W$ is a free $\mathfrak{F}(V)^W$-module* *with the generators* $\operatorname{grad} p^i$ $(i = 1, \dots, n)$ *and the Lie algebra* $\mathfrak{X}(V_{\mathbf{C}}/W)$ *is a free* $\mathfrak{F}(V_{\mathbf{C}}/W)$*-module with the generators*

$$g^i = p_* \operatorname{grad} p^i = \sum_{j=1}^{n} \tilde{g}^{ij} \partial_j \quad (i = 1, \dots, n),$$

where $\partial_i = \frac{\partial}{\partial z^i}$ *and* z^i *are the natural coordinates on* $\mathbf{C}^n$*.*

This theorem implies, in particular, that 0 is a unique common zero point of all g^i and therefore $f(0) = 0$.

Let $x \in V_{\mathbf{C}}$. Put

$$V_x := \cap_{\alpha \in \Pi_x} H_\alpha,$$

where Π_x is a positive system for the isotropy group W_x of x. Then V_x is equal to the $\mathbf{C}$-span of the system of vectors $\operatorname{grad} p^i(x)$ $(i = 1, \dots, n)$ and by Theorem 4.2 the restriction of the map $p_* : T_x V_{\mathbf{C}} \to T_{p(x)} \mathbf{C}^n$ to $V_x \subset V_{\mathbf{C}} = T_x V_{\mathbf{C}}$ is a monomorphism.

4.3. Theorem. *There is a unique holomorphic* W*-invariant tensor field* P_i^* *of type* $(0, d_i)$ $(i = 1, \dots, n)$ *on* V *such that the restriction of* P_i^* *to* V_{reg} *coincides with* $P_{i,0}^*$*. In particular,* $g^*(V_{\mathbf{C}}) := P_1^*$ *is a holomorphic flat Riemannian metric on* $V_{\mathbf{C}}$*.*

Proof. Note that in the complex case Lemma 3.3 is true for holomorphic functions ϕ and ψ, Lemma 3.4 is true for a nonzero holomorphic function k, and Lemma 3.5 is true for a holomorphic tensor field b_{ij}. As in the proof of Theorem 3.2 put $b^* := (f \circ p)^* b$ and $b^*(\tau) := b^*(\tau, \tau)$ for each tangent vector τ at $x \in V_{\mathbf{C}}$. It is clear that $b^*(\tau)$ is a W-invariant holomorphic function on $TV_{\mathbf{C}}$.

Let $x = 0$. By definition b vanishes at 0, then $b^*(\tau) = 0$ for each tangent vector τ at 0.

Let $x \in V_{\mathbf{C}} \setminus V_{\mathbf{C},reg}$, $x \neq 0$, and let (x, x') be a pair of f-compatible points of $V_{\mathbf{C}}$. Consider the tangent spaces $T_x V_x$ and $T_{x'} V_{x'}$ of the subspaces V_x and $V_{x'}$ at points x and x', respectively. By Theorem 4.2 $(f \circ p)_*(T_x V_x) = p_*(T_{x'} V_{x'})$.

Note that Reduction lemma 2.4 is true in the complex case. Apply this lemma to the point x and denote by V'' the orthogonal complement of V_x. Since V_x is the set of all fixed points of W_x the action of W_x is trivial on V_x, V'' is an invariant subspace of this action, and W_x has no fixed nonzero points in V''. Fix an orthonormal basis (e_i) in $V_{\mathbf{C}}$ such that $e_1, \dots, e_m$ is a basis of the subspace V_x and denote by z^i the corresponding coordinates in $V_{\mathbf{C}}$. By Lemma 3.1 for the complex case choose the homogeneous generators $q^1, \dots, q^n$ of $\mathbf{C}[V_{\mathbf{C}}]^{W_x}$ so that $q^1, \dots, q^m$ is the dual basis of V_x^* and $q^{m+1}, \dots, q^n$ are homogeneous generators of the algebra of W_x-invariant polynomials on V''. Then

$$\frac{\partial q^i}{\partial z^j} = \delta_j^i \quad (i, j = 1, \dots, m)$$

and the other partial derivatives $\frac{\partial q^i}{\partial z^j}$ vanish on the subspace P_x. Since $p = P \circ q$, $p_*(\tau) = p_*(\tau')$, where $\tau = \tau' + \tau''$, $\tau \in P_x$, and $\tau'' \in P''$. Therefore, by Theorem 4.2 $(f \circ p)_*(\tau)$ lies in the $\mathbf{C}$-span of the vectors $g^i(p(x'))$ and there is a tangent vector τ_1 at x' such that $p_*(\tau_1) = \tau$. Hence, by Lemma 3.5 for the complex case $b^*(\tau) = M(x') g(V)(\tau_1, \tau_1) = 0$.

The rest of the proof coincides with one of Theorem 3.2. $\square$

Now we prove the following local version of Theorem 4.1.

4.4. Theorem. *Let f be a biholomorphic map of a connected open neighborhood U of $y \in \mathbf{C}^n$ onto some neighborhood of $y' \in \mathbf{C}^n$ such that $f(y) = y'$ and $f(U \cap D) \subset D$. Then, for $x, x' \in V_{\mathbf{C}}$ such that $p(x) = y$ and $p(x') = y'$ there exists a neighborhood $\tilde{U}$ of $x \in V_{\mathbf{C}}$ and a biholomorphic map F of $\tilde{U}$ onto some neighborhood of $x' \in V_{\mathbf{C}}$ such that $p \circ F = f \circ p$ on $\tilde{U}$ and $F(x) = x'$. The map F is uniquely defined by the condition $F(x_1) = x_1'$, where $x_1, x_1' \in V_{\mathbf{C},reg}$, $p(x_1) \in U$, and $f \circ p(x_1) = p(x_1')$.*

Proof. If the biholomorphic map f of Theorem 4.1 is defined only in a neighborhood U of $y \in \mathbf{C}^n$ the above constructions and results, including Theorem 4.3, remain true for some neighborhood of each $x \in V_{\mathbf{C}}$ such that $p(x) = y$. Then there exists a biholomorphic isometry F of a neighborhood $\tilde{U}$ of x in the Riemannian manifold $(V_{\mathbf{C}}, g^*(V_{\mathbf{C}}))$ onto some neighborhood of x' in the Riemannian manifold $(V_{\mathbf{C}}, g(V_{\mathbf{C}}))$ such that $F(x) = x'$. We may choose this isometry so that $F^*P_i = P_i^*$ on $\tilde{U}$ and then $p \circ F = f \circ p$ on $\tilde{U}$. The last statement of the proof is obvious. $\square$

Now we can prove the following substitute of Theorem 2.6.

4.5. Lemma. *Let $c(t)$ be a holomorphic curve in $V_{\mathbf{C}}$ defined near 0 such that $c(0) = 0$. Then the curve $c_1 = f \circ p \circ c$ has a holomorphic lift $\tilde{c}_1$ near 0. All such lifts differ by tranformations belonging to W.*

Proof. If $c(t) \equiv 0$ we are done.

Let $c(t) \not\equiv 0$. By Theorem 4.4 $\tilde{c}_1 = F \circ c$ gives the lift of the curve c_1.

Suppose r is a maximal integer such that $\tilde{c}_1(t) = t^r \tilde{a}_{(r)}(t)$, where $\tilde{a}_{(r)}(t)$ is a holomorphic curve in $V_{\mathbf{C}}$. Since $c(0) = 0$, $r > 0$. Let $c_1 = (c_1^i)$. Then $m(c_1^i) \geq rd_i$, i.e. there are holomorphic functions $a_{(r)}^i(t)$ $(i = 1, \ldots, n)$ such that $c_1^i(t) = t^{rd_i} a_{(r)}(t)$. It is clear that the existence of the holomorphic lift for the curve c_1 near 0 is equivalent to one for the curve $a_{(r)} = (a_{(r)}^i)$ and $\tilde{a}_{(r)}(t)$ is a holomorphic lift of $a_{(r)}$ near 0.

By the choice of r $\tilde{a}_{(r)}(0) \neq 0$. If $\tilde{a}_{(r)}(0) \in V_{\mathbf{C},reg}$, all lifts of the curve $a_{(r)}$ differ by transformations belonging to W. Otherwise, apply the version of Reduction Lemma for the comlex case to the curve $a_{(r)}$. Then, for each $x \in p^{-1}(a_{(r)}(0))$, we get the corresponding lifting problem for the group W_x such that $|W_x| < |W|$ and some holomorphic curve $a'_{(r)}$ such that $a'_{(r)}(0) = 0$. Moreover, we know that there is $w \in W$ such that $w \circ \tilde{c}_1$ is a holomorphic lift of $a'_{(r)}$ near 0. Therefore, the last statement of the lemma is proved by induction on the order $|W|$ of W. $\square$

4.6. Theorem. *Let the map f satisfy the conditions of Theorem 4.1 and let $c(t)$ $(t \in \mathbf{C})$ be a holomorphic curve in $V_{\mathbf{C}}$. Then there exists a global holomorphic lift $\tilde{c}(t)$ of the curve $f \circ p \circ c$ and all such lifts differ by transformations belonging to W.*

Proof. The proof follows from the versions of Lemmas 2.2 and 2.5 for the complex case and Lemma 4.5. Actually, by the above lemmas there are local holomorphic lifts of $f \circ p \circ c$ near each $t \in \mathbf{C}$ and one can choose these lifts so that they are glued to a global lift. Since local lifts are defined up to transformations belonging to W the same is true for a global lift. $\square$

Proof of Theorem 4.1. First we prove that for each $x \in V_{\mathbf{C}}$ there is a unique maximal geodesic curve $c(t)$ $(t \in \mathbf{C})$ in the Riemannian space $(V_{\mathbf{C}}, g^*(V_{\mathbf{C}}))$ such that $c(0) = 0$ and $c(1) = x$.

Let (x, x') be a pair of f-compatible points of $V_{\mathbf{C}}$ and F a biholomorphic map defined by Theorem 4.4. Consider the complex straight line $a(t) = tx'$ $(t \in \mathbf{C})$ and the holomorphic lift $\tilde{a}$ of the curve $f^{-1} \circ p \circ a$ defined by Theorem 4.6. We may suppose that $\tilde{a}$ coincides with the curve $F^{-1} \circ a$ near 0 and this condition determines $\tilde{a}$ uniquely. By definition $\tilde{c}$ is a geodesic curve of the Riemannian space $(V_{\mathbf{C}}, g^*(V))$ near 0 and by the uniqueness of holomorphic extension c is a geodesic curve of this Riemannian manifold in the whole of its domain. Obviously there is $w \in W$ such that $w \circ \tilde{a}(1) = x$, i.e. replacing F by $F \circ w^{-1}$ we may suppose that $\tilde{a}(1) = x$. Therefore, $c = \tilde{a}$ is a maximal geodesic curve which is unique by construction.

Let us fix some biholomorphic map F defined by Theorem 4.4. For each $x \in V_{\mathbf{C}}$ denote by $\tilde{c}$ the lift of the curve $f \circ p \circ c$, where c is the defined above for $x \in V_{\mathbf{C}}$ geodesic curve of $(V_{\mathbf{C}}, g^*(V_{\mathbf{C}}))$, which coincides with $F \circ c$ near 0. The curve $\tilde{c}$ is a determined uniquely complex straight line of the Euclidean space $(V_{\mathbf{C}}, g(V_{\mathbf{C}}))$. Let $\tilde{F}$ be a map $x \mapsto \tilde{c}(1)$. By construction $\tilde{F}$ is a well defined bijective map of $V_{\mathbf{C}}$ onto itself which coincides with F in the domain of F. By the definitions of geodesics and the tensor fields P_i^*, $p \circ \tilde{F} = f \circ p$. Then, for each $x \in V_{\mathbf{C},reg}$, $\tilde{F}$ coincides with $F_{x,\tilde{F}(x)}$ near x and, therefore, is biholomorphic near x.

Let $x \in V_{\mathbf{C}} \setminus V_{reg}$. Consider the local isometry I of a neighborhood of x onto a neighborhood of $F_1(x)$. Since the set $V_{\mathbf{C},reg}$ is connected we may suppose that I coincides with $\tilde{F}$ at regular points. But this means that $\tilde{F}$ is biholomorphic in a neighborhood of x. Since $\tilde{F}$ is a bijective map of $V_{\mathbf{C}}$ onto itself which is biholomorphic near each $x \in V_{\mathbf{C}}$, it is biholomorphic.

The above construction implies that for each pair (x, x') of regular f-compatible points of $V_{\mathbf{C}}$ there is a biholomorphic map $\tilde{F}$ such that $\tilde{F}(x) = x'$ and this map coincides with $F_{x,x'}$ near x. Hence, by the uniqueness of holomorphic extension the map $\tilde{F}$ is unique. This concludes the proof. □

4.7. Theorem. *Let Y be a connected smooth (real analytic, complex) manifold, $y_0 \in Y$, and f_y $(y \in Y)$ a smooth (analytic, complex analytic) family of biholomorphic maps of $\mathbf{C}^n$ onto itself such that for each $y \in Y$ $f_y(D) = D$. Then for each pair (x, x') of regular f_{y_0}-compatible points of $V_{\mathbf{C}}$ there is a unique smooth (real analytic, complex analytic) family F_y of biholomorphic maps of $V_{\mathbf{C}}$ onto itself such that $F_{y_0}(x) = x'$ and $p \circ F_y = f_y \circ p$. In particular, let Y be a Lie group and f_y an action of G on $V_{\mathbf{C}}$ by biholomorphic maps of $\mathbf{C}^n$ onto itself. Then, for y_0 equal to the identity element of Y and $x' = x$, the above family F_y is a Lie group of diffeomorphisms of V independent of x.*

Proof. Let T be a fundamental domain of the action of W on $V_{\mathbf{C}}$ containing x'. There is a continuous map $h : Y \to T$ such that $h(y_0) = x'$ and for each $y \in Y$ $(x, h(y))$ is a pair of f_y-compatible points of $V_{\mathbf{C}}$. By Theorem 4.1 there is a unique biholomorphic map F_y such that $p \circ F_y = f_y \circ p$ and $F_y(x) = h(y)$. It is easily checked that the family F_g satisfies the statements of the theorem. □

5. Automorphisms of some Frobenius manifolds

For an irreducible finite reflection group W acting on a real vector space V Dubrovin constructed the natural structure of Frobenius manifold on $V_{\mathbf{C}}/W = \mathbf{C}^n$ (see [4]). It is determined by two complex orthogonal flat Riemannian metrics on $V_{\mathbf{C}}/W$. One of them is given by the tensor field $\tilde{g}^{ij}$, defined in Section 4, and it is non-degenerate outside of the discriminant locus D. Another metric η, which gives a flat structure on $V_{\mathbf{C}}/W$, was defined by Saito, Sekuguchi, and Yano (see [11]) by direct calculations for all irreducible finite reflection groups except E_7 and E_8. The general construction of η for an arbitrary irreducible finite reflection group is given in [9] and [10]. The metric η posesses flat coordinates which are given by an appropriate choice of generators p^i of the algebra $\mathbf{R}[V]^W$.

By Dubrovin [4] the above two metrics form the so-called flat pencil of metrics which defines a unique structure of Frobenius manifold on $V_{\mathbf{C}}/W$. Note that the components of the above metrics are polynomials with real coefficients and, therefore, there is the corresponding structure on $V/W = p(V) \subset \mathbf{R}^n$. Further we consider both $V_{\mathbf{C}}/W$ and V/W with the above structures and call them Frobenius manifolds. An automorphism of the Frobenius manifold V/W ($V_{\mathbf{C}}/W$) is a smooth diffeomorphism of V/W (a biholomorphic map of $V_{\mathbf{C}}/W$ onto itself) which preserves the above flat pencil of metrics.

5.1. *The Frobenius manifolds $V_{\mathbf{C}}/W$ and V/W have no nontrivial automorphisms.*

Proof. Let f be an automorphism of the Frobenius manifold V/W. Consider the pull-back of the metrics on V/W, forming a flat pencil, via the map p. The pull-back of the first of them is the Euclidean metric $g(V)$ and it defines the structure of Euclidean space on V. The pull-back of the second one is a flat metric on V which is non-degenerate on V_{reg}. The flat coordinates of this metric are the compositions of the flat coordinates of the metric η with p and then they give some system of generators of $\mathbf{R}[V]^W$. Therefore, by Theorem 1.5 the above pull-backs of metrics determine uniquely the structure of W-module on V.

Consider the lift F of the automorphism f given by Theorem 3.1. The definition of the automorphism f implies that F is an automorphism of the W-module V and, hence, f is the identity map of V/W.

Note that each automorphism f of the Frobenius manifold $V_{\mathbf{C}}/W$ preserves the set of degenerate points of the metric $\tilde{g}^{ij}$, i.e. the discriminant locus D. Then the statement of the theorem for $V_{\mathbf{C}}/W$ follows from Theorem 4.1. □

Note that using Theorems 3.6 and 4.4, i.e. the local versions of Theorems 3.1 and 4.1, one can prove that the Frobenius manifolds V/W and $V_{\mathbf{C}}/W$ have no nontrivial local automorphisms.

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