MAPS BETWEEN SPACES WHOSE COHOMOLOGY ARE FINITELY GENERATED POLYNOMIAL ALGEBRAS

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Abstract. We classified homotopy classes of maps between p–completed spaces whose cohomology are finitely generated polynomial algebras with Weyl groups of orders prime to p.

0. INTRODUCTION

The aim of this paper is to apply the program from [1] to study maps between spaces whose cohomology with F_p–coefficients are finitely generated polynomial algebras concentrated in even degrees. The starting point was an attempt to generalize one result of Hubbuck (see [7] Theorem 1.1.). The plan of work will follow closely that of [3] and [12].

Let X be a space whose cohomology with F_p–coefficients is a finitely generated polynomial algebra concentrated in even degrees. Let T be a torus. For a torus T, the solutions in T of $t^{p^n} = 1$ make up a subgroup $T(n)$; let $T(\infty) = \bigcup_n T(n)$. Let $W \subset Aut(T(\infty))$ be a finite subgroup. Then W acts on a classifying space $BT(\infty)$ and therefore also on $H^*(BT,F_p)$.

We say that X has a maximal torus T and a Weyl group W if there is a map $i : BT \longrightarrow X$ which satisfies

$$H^*(X,F_p) = H^*(BT(\infty),F_p)^W .$$

We shall call $i : BT \longrightarrow X$ a structure map for X.

We assume throughout that X, X' are p–completed spaces, whose cohomology with F_p–coefficients are finitely generated polynomial algebras concentrated in even degrees. We assume that X and X' have maximal tori and Weyl groups; T, T' are their maximal tori, $i : BT \to X$ and $i' : BT' \to X'$ are structure maps and W and W' are their Weyl groups. We shall denote by Y_p the p–completion of Y. Let us observe that $i : BT \to X$ induces a unique map, which we denote also by $i : (BT)_p \to X$ because X is p–complete.

Now we shall state our main results.

THEOREM 1. Assume that p *does not divide the orders of* W *and* W'. *Then for any map* $f : X \to X'$ *there is a map* $\Upsilon : (BT)_p \to (BT')_p$ *such that the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & X' \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i'} \\ (BT)_p & \xrightarrow{\ \Upsilon\ } & (BT')_p \end{array}$$

commutes up to homotopy. Moreover we have:

a) *if* $\Upsilon' : (BT)_p \to (BT')_p$ *is such that* $i \circ f$ *is homotopic to* $\Upsilon' \circ i'$ *then there is* $w \in W'$ such that $w \circ \Upsilon'$ *is homotopic to* Υ,
b) *for any* $w \in W$ *there is* $w' \in W'$ *such that* $\Upsilon \circ w$ *is homotopic to* $w' \circ \Upsilon$.

The group W acts on $T(\infty)$, hence W acts also on $\pi_2((BT(\infty))_p) = \pi_1(T) \otimes Z_p$, and consequently on $\pi_1(T) \otimes R$ for any Z_p–module R.

DEFINITION 1. Let R *be a* Z_p*–algebra. We say that a homomorphism of* R*–modules*

$$\varphi : \pi_1(T) \otimes R \longrightarrow \pi_1(T') \otimes R$$

is admissible if for any $w \in W$ *there is* $w' \in W'$ *such that* $\varphi \circ w = w' \circ \varphi$.
We say that two admissible maps φ *and* ψ *from* $\pi_1(T) \otimes R$ *to* $\pi_1(T') \otimes R$ *are equivalent if there is* $w \in W'$ *such that* $w \circ \varphi = \psi$.

It is clear that the relation defined above is an equivalence relation on the set of admissible maps from $\pi_1(T) \otimes R$ to $\pi_1(T') \otimes R$. We shall denote by $\mathrm{Ahom}_R(T,T')$ the set of equivalence classes of admissible maps from $\pi_1(T) \otimes R$ to $\pi_1(T') \otimes R$.

Let us notice that the map $\pi_1(\Upsilon)$ induced by Υ from Theorem 1 on fundamental groups is admissible for $R = Z_p$. This map is unique up to the action of W', so any map $f : X \longrightarrow X'$ determines uniquely an equivalence class of $\pi_1(\Upsilon)$ in $\mathrm{Ahom}_{Z_p}(T,T')$ which we shall denote by $\chi(f)$.

THEOREM 2. *Let us assume that* p *does not divide the orders of* W *and* W'. *Then the natural map*

$$\chi : [X,X'] \longrightarrow \mathrm{Ahom}_{Z_p}(T,T')$$

is bijective.

For any space X we set

$$H^*(X,Q_p) := H^*(X,Z_p) \otimes Q ,$$

where Q_p is a field of p–adic numbers.

THEOREM 3. *Let us assume that* p *does not divide the orders of* W *and* W'. *Then the natural map*

$$\phi : [X,X'] \longrightarrow \mathrm{Hom}(H^*(X',\mathbb{Q}_p), H^*(X,\mathbb{Q}_p))$$

is injective.

We denote by $K^0(,R)$ the 0^{th}–term of complex K–theory with R–coefficients. Let $\mathcal{O}_R$ be the set of operations in $K^0(,R)$. The functor $K^0(,R)$ is equipped with the natural augmentation $K^0(,R) \longrightarrow R$. Let $\mathrm{Hom}_{\mathcal{O}_R}(K^0(X',R),K^0(X,R))$ be the set of R–algebra homomorphisms which commute with the action of $\mathcal{O}_R$ and augmentations.

THEOREM 4. *If* p *does not divides the orders of* W *and* W′, *then the natural map*

$$\phi : [X,X'] \longrightarrow \mathrm{Hom}_{\mathcal{O}_{Z_p}}(K^0(X',Z_p),K^0(X,Z_p))$$

is bijective.

The result from [15] about the homotopy uniqueness of classifying spaces and Theorem 2 suggest that the homotopy category of spaces whose cohomology are finitely generated polynomial algebras over F_p should be equivalent to some algebraic category. Below we make this hope more precise in the special case considered in this paper. But first we give a definition.

Let V be a vector space or a free Z_p–module. One says that an endomorphism s of V is a generalized reflection if id –s has rank 1. A group $W \subset GL(V)$ is a generalized reflection group if it is generated by generalized reflections.

Let M be a finitely generated, free Z_p–module and let $W \subset GL_{Z_p}(M)$ be a finite generalized reflection group. We shall view the inclusion $W \subset GL_{Z_p}(M)$ as a representation $\rho : W \longrightarrow GL_{Z_p}(M)$.

We shall define a category $P\,Ref_p$ in the following way. The objects of the category $P\,Ref_p$ are representations $\rho : W \longrightarrow GL_{Z_p}(M)$ described above such that p does not divide the order of W. It rests to define morphisms in this category. If $\theta : W \longrightarrow GL(M)$ and $\theta' : W' \longrightarrow GL(M')$ are two objects of $PRef_p$, we say that a homomorphism of Z_p–modules $f : M \longrightarrow M'$ is admissible if for each $w \in W$ there is $w' \in W'$ such that $f \circ w = w' \circ f$. We say that two admissible homomorphisms f and g from M to M′ are equivalent if there is $w \in W'$ such that $f = w' \circ g$. We shall denote by $Hom(\theta,\theta')$ the set of equivalence classes of admissible homomorphisms from M to M′. The set $Hom(\theta,\theta')$ is the set of morphisms from θ to θ' in the category $PRef_p$. The category $PRef_p$ is equipped with the product defined in the following way:

$$(\theta : W \longrightarrow GL(M)) \oplus (\theta';W' \longrightarrow GL(M')) = \theta \oplus \theta' : W \times W' \longrightarrow GL(M \oplus M').$$

The product of morphisms is defined in the obvious way.

We denote by $HPol_p$ the category whose objects are p–completed spaces X such that their cohomology with F_p–coefficients are finitely generated polynomial algebras. We assume further that any X in $HPol_p$ has a maximal torus and a Weyl group and that p does not divide the order of the Weyl group of X. Morphisms in $H\,Pol_p$ are homotopy classes of maps. The category $H\,Pol_p$ also has products define in an obvious way.

THEOREM 5. *There is an equivalence of categories*

$$R : PRef_p \longrightarrow HPol_p$$

with products.

If we drop out the assumption that p does not divide the orders of W and W′ we get weaker results.

THEOREM 6. *In Theorems* 1,2,3 *and* 4 *we can drop out the assumption* "p *does not divide the order of* W' " *if* $X' = (BG)_p$, *where* G *is a connected, compact Lie group.*

THEOREM 7. *For any* $f : X \longrightarrow X'$ *there is a map* $\Upsilon : (BT)_p \longrightarrow (BT')_p$ *such that the diagrams*

$$\begin{array}{ccc} K^0(X',Z_p) & \xrightarrow{f^*} & K^0(X,Z_p) \\ \downarrow i'^* & & \downarrow i^* \\ K^0((BT')_p,Z_p) & \xrightarrow{\Upsilon^*} & K^0((BT)_p,Z_p) \end{array}$$

and

$$\begin{array}{ccc} H^*(X',\mathbb{Q}_p) & \xrightarrow{f^*} & H^*(X,\mathbb{Q}_p) \\ \downarrow i'^* & & \downarrow i^* \\ H^*((BT')_p,\mathbb{Q}_p) & \xrightarrow{\Upsilon^*} & H^*((BT)_p,\mathbb{Q}_p) \end{array}$$

are commutative.

a) *If* $\Upsilon' : (BT)_p \longrightarrow (BT')_p$ *is such that* $i'^* \circ f^* = \Upsilon'^* \circ i^*$ *then there is* $w \in W'$ *such that* $w \circ \Upsilon'$ *is homotopic to* Υ.
b) *For any* $w \in W$ *there is* $w' \in W'$ *such that* $\Upsilon \circ w$ *is homotopic to* $w' \circ \Upsilon$.

COROLLARY 8. *Let us assume that the natural representation of* W *on* $\pi_1(T) \otimes \mathbb{Q}_p$ *is irreducible. Then there is a finite number of self-maps* $I_1,\dots,I_n$ *of* X *such that for any* $f : X \longrightarrow X$ *there is* k *for which* $f \circ I_k$ *is an Adams* ψ^α*–map i.e. the map induced by* $f \circ I_k$ *on* $H^{2i}(X,\mathbb{Q}_p)$ *is a multiplication by* α^i. *The number* n *is smaller or* *equal to a number of elements of* Aut(W)/Inn(W) *which preserve the natural representation of* W *on* $\pi_1(T) \otimes \mathbb{Q}_p$.

Example. (see also [3])
Let $X = BSU(n)_p$. *The Weyl group of* SU(n) *is* Σ_n. *If* $n \neq 6$ *then* $\mathrm{Aut}\,\Sigma = \mathrm{Inn}\,\Sigma_n$ *and for* $n = 6$ *the outer automorphism does not preserve the natural representation of* Σ_6 *on* $\pi_1(T) \otimes \mathbb{Q}_p$. *This implies that the self-maps of* $BSU(n)_p$ *are Adams* ψ^{α}*–maps.*

We point out that Corollary 8 can be view as a generalization of a result of Hubbuck (see [7] Theorem 1.1.) The example is a special case of the result of Hubbuck. However, it concerns maps between p–completed spaces $BSU(n)_p$ while Hubbuck is dealing with classical spaces BG. ▫

Let us notice that a homomorphism $\tilde{f}_* : \pi_1(T) \otimes Z_p \longrightarrow \pi_1(T') \otimes Z_p$ from Theorem 7 induced by $\tilde{f}$ is admissible. An equivalence class of $\tilde{f}_*$ in $\mathrm{Ahom}_{Z_p}(T,T')$ we shall denote by $\chi(f)$.

THEOREM 9. *Let* f *and* g *be two maps from* X *to* X′. *Then the following conditions are equivalent:*

a) $\chi(f) = \chi(g)$ in $\mathrm{Ahom}_{Z_p}(T,T')$;
b) $K^0(f,Z_p) = K^0(g,Z_p)$;
c) $H^*(f,\mathbb{Q}_p) = H^*(g,\mathbb{Q}_p)$.

1. THE LANNES T FUNCTOR FOR SPACES WHOSE COHOMOLOGY ARE FINITELY GENERATED POLYNOMIAL ALGEBRAS

In this section we shall compute the cohomology of the mapping space map(BV,X) and its connected component $\mathrm{map}_f(BV,X)$ where V is an elementary abelian p–group and X is a p–complete space whose cohomology is a finitely generated polynomial algebra over F_p. We assume that X has a maximal torus T and a Weyl group W.

Let us suppose that

$$H^*(X,F_p) = H^*(BT,F_p)^W .$$

The map $f\colon BV \longrightarrow X$ induces a map $f^* : H^*(X,F_p) \longrightarrow H^*(BV,F_p)$. It follows from [2] Proposition 1.10 and the fact that $H^*(X,F_p)$ is concentrated in even degrees that there is $g^* : H^*(BT,F_p) \longrightarrow H^*(BV,F_p)$ such that $f^* = g^* \circ i^*$ where $i^* : H^*(X,F_p) \longrightarrow H^*(BT,F_p)$ is the inclusion induced by a structure map $i : BT \longrightarrow X$.

We recall that for a torus T, the solutions in T of $t^p = 1$ make up a subgroup T(1). The map g^* is induced by a homomorphism $\varphi : V \longrightarrow T(1)$. This follows from [8] Theorem 0.4. Let $\Lambda_f : V \otimes T(1)^* \longrightarrow F_p$ be an adjoint map of φ. The group W acts on $\mathrm{Hom}(V \otimes T(1)^*, F_p)$ through its action on $T(1)^*$. Let W_f be the isotropy subgroup of Λ_f.

PROPOSITION 1.1. *Let* X *be a* p*–complete space whose cohomology with* F_p*–coefficients is a finitely generated polynomial algebra over* F_p *concentrated in even degrees. We assume that* X *has a maximal torus* T *and a Weyl group* W. *Let* V *be an elementary abelian* p*–group and let* $f : BV \longrightarrow X$ *be any map. Then we have an isomorphism*

$$H^*(map_f(BV,X);F_p) = P^{W_f}$$

where $P = H^*(BT,F_p)$.

PROOF: For a vector space U over F_p let us denote by $P(U)$ the polynomial algebra on U, by $\Lambda(U)$ the exterior algebra on U and by $A(U)$ the symmetric algebra on U divided by the ideal generated by all polynomials $x^p - x$ for $x \in U$. The polynomial $x^p - x$ splits completely over F_p. Hence we have an isomorphism of F_p–algebras $A(U) = \bigoplus_{a \in U^*} F_p$. We point out that $A(U)$ is concentrated in degree zero.

Let us notice that we have the following natural identifications

$$P = H^*(BT,F_p) = P(T(1)^*)$$

and

$$H^*(BV,F_p) = P(V^*) \otimes \Lambda(\beta^{-1}V^*).$$

It follows from Corollary 2 in [4] that for any unstable A_p–algebra M and any A_p–algebra homomorphism $f : P((Z/p)^*) \longrightarrow M \otimes H^*(BZ/p,F_p)$ we have

$$f(t^*) = m_{t*} \otimes 1 + m_{v*} \otimes v^*.$$

This implies that we have a natural isomorphism

$$\Phi_M : Hom_{unA_p}(P(T(1)^*); M \otimes H^*(BV)) \approx$$
$$Hom_{unA_p}(A(V \otimes T(1)^*) \otimes P(T(1)^*);M).$$

where $Hom_{unA_p}(\, , \,)$ is in the category of unstable A_p–algebras. If $f(t^*) =$

$m_{t*} \otimes 1 + \sum_{v* \in V*} m_{v*} \otimes v^*$ then $\Phi_M(f)([v \otimes t^*] \otimes 1) = \sum_{v* \in V*} m_{v*} \cdot v^*(v)$

and $\Phi_M(f)(1 \otimes t^*) = m_{t*}$.

Hence it follows that

$$(*) \qquad T_V(P) = A(V \otimes T(1)^*) \otimes P.$$

If $M = F_p$ then we have an isomorphism
$\phi_{F_p} : \mathrm{Hom}(P(T(1)^*), H^*(BV)) \approx \mathrm{Hom}(A(V \otimes T(1)^*), F_p)$. The group W acts on $P(T(1)^*)$ through its action on $T(1)^*$ hence W acts also on $A(V \otimes T(1)^*)$ through the action on $T(1)^*$. The isomorphism $(*)$ implies that

$$(**) \qquad T_V(P^W) = (A(V \otimes T(1)^*) \otimes P)^W$$

(see [4] Proposition 3).

Let $f^* : H^*(X,F_p) \longrightarrow H^*(BV,F_p)$ be the map induced by f on cohomology. Let $\lambda : T_V(H^*(X,F_p)) \longrightarrow F_p$ be the adjoint map of f^* and let $\bar{\lambda} : T_V(P) \longrightarrow F_p$ be the adjoint map of g^*. The restriction of $\bar{\lambda}$ to $V \otimes T(1)^*$ is equal to Λ_f.

It follows from [5] 2.3 Theorem and the equality $(**)$ that

$$H^*(\mathrm{map}_f(BV,X),F_p) \approx T_V(H^*(X,F_p)) \underset{T_V^0(H*(X,F_p))}{\otimes} F_p \approx (A \otimes P)^W \underset{A^W}{\otimes} F_p$$

where $A = A(V \otimes T(1)^*)$.

If $V^* \otimes T(1) = \coprod W/W'$, as a W–set then $A \approx \otimes F_p[W/W']$ as a W–module. For any $W' \subset W$, $F_p[W/W']^W \approx F_p$. The maps $\bar{\lambda}$ and λ induce $\tilde{\lambda} : A \longrightarrow F_p$

and $\tilde{\lambda} : A^W = \oplus F_p \longrightarrow F_p$. The algebra homomorphism $\tilde{\lambda}$ is the identity on one's of F_p's and it is zero on all others. The fact that λ restricts to λ_f on $V \otimes T(1)^*$ implies that $\tilde{\lambda}$ is the identity on $F_p[W/W_f]^W$. Hence we have the following isomorphisms

$$(A \otimes P)^W \underset{A^W}{\otimes} F_p \approx (F_p[W/W_f] \otimes P)^W \underset{F_p}{\otimes} F_p \approx P^{W_f} . \qquad \square$$

2. MAPS FROM BP TO X

Let M be a finitely generated, free Z_p–module. Let $W \subset GL_{Z_p}(M)$ be a finite generalized reflection group. The action of W on M extends to the action of W on $M \otimes \mathbb{Q}$. The lattice M in $M \otimes \mathbb{Q}$ is invariant therefore W acts also on $M \otimes \mathbb{Q}/_M$. Observe that $M \otimes \mathbb{Q}/_M = T(\infty)$ for some torus T. From the action of W on $T(\infty)$ we can recover the original action of W on M if we take the induced action of W on $(H^2(BT(\infty);Z_p))^*$. Hence if $W \subset GL_{Z_p}(M)$ is a finite generalized reflection group then W can be realized as a subgroup of $\mathrm{Aut}(T(\infty))$.

PROPOSITION 2.1. *Let* $W \subset GL_{Z_p}(M)$ *be a finite generalized reflection group which we consider as a subgroup of* $\mathrm{Aut}(T(\infty))$. *Let us assume that* p *does not divide the order of* W. *If* P *is a finite* p*–group then any map* $f : BP \longrightarrow (B(T(\infty)\tilde{\times}W))_p$ *is induced by a homomorphism* $\varphi : P \longrightarrow T(\infty)\tilde{\times}W$.

We were informed that a similar result was also known to W. Dwyer.
This proposition is an analog of the theorem of Dwyer and Zabrodsky (see [6] 1.1. Theorem). The proof will follow closely the proof of the Dwyer and Zabrodsky theorem contained in [13], which depends very much on [9].
Let us set $G = T(\infty) \tilde{\times} W$.

LEMMA 2.2. *Let* $V = \mathbb{Z}/p$, *let* $\varphi : V \longrightarrow G$ *be a homomorphism, let* G_0 *be the centralizer of* $\mathrm{im}\varphi$ *in* G *and let* $\varphi_0 : V \longrightarrow G_0$ *be the map induced by* φ . *Then the map*

$$\mathrm{map}_{B\varphi_0}(BV,(BG_0)_p) \longrightarrow \mathrm{map}_{B\varphi}(BV,(BG)_p)$$

is a homotopy equivalence.

PROOF: It follows from Proposition 1.1 that
$H^*(\mathrm{map}_{B\varphi}(BV,(BG)_p),F_p) \approx P^{W_0}$ where $P \approx H^*(BT,F_p)$ and $W_0 = G_0/T(\infty)$ is the isotropy subgroup of $\varphi : V \longrightarrow T(\infty)$. In the same way we get
$H^*(\mathrm{map}_{B\varphi_0}(BV,(BG_0)_p),F_p) = P^{W_0}$. Hence the map considered by us is a homotopy equivalence. □

LEMMA 2.3. *Let* P *be a* p*–group, let* $\mathbb{Z}/p = V$ *be a subgroup of the center of* P . *Let* $\varphi : V \longrightarrow G$ *be a homomorphism, let* G_0 *be the centralizer of* $\mathrm{im}\,\varphi$ *in* G *and let* $\varphi_0 : V \longrightarrow G_0$ *be the induced homomorphism. Let*

$$[BP,(BG)_p](B\varphi) = \{f \in [BP,(BG)_p] : f_{|BV} \sim B\varphi\}$$

and let $[BP,(BG_0)_p](B\varphi_0)$ *be defined in an analogous way. Then the inclusion map* $i : G_0 \longrightarrow G$ *induces a bijection*

$$(*) \qquad [BP,(BG_0)_p](B\varphi_0) \longrightarrow [BP,(BG)_p](B\varphi) \ .$$

PROOF: We have a fibration $BV \longrightarrow BP \longrightarrow B(P/V)$. Let
$BV \longrightarrow BV \longrightarrow E(P/V)$ be a fibration induced by $\mathrm{pr} : E(P/V) \longrightarrow B(P/V)$. Then P/V acts on BV through maps homotopics to the identity and BV is a model for BV. It follows from Lemma 2.2 that the map

$$\mathrm{map}_{P/V}(E(P/V),\mathrm{map}_{B\varphi_0}(BV,(BG_0)_p) \longrightarrow \mathrm{map}_{P/V}(E(P/V),\mathrm{map}_{B\varphi}(BV,(BG)_p)$$

is a homotopy equivalence. The induced map on π_0 is the map (*). This finishes the proof. □

LEMMA 2.4. (see [14] 1.5. Lemma) *Let* $\varphi : L \longrightarrow K$ *be a simplicial map. Let* $V_0^{\varphi}(L,X)$ *be the subspace of the space* map.(L,X) *of pointed maps from* L *to* X *consisting of maps* $f : L \longrightarrow X$ *such that* $f_{|\varphi^{-1}(k)} \sim *$ *for every* $k \in K$. *Let* $\mathrm{map}_*(\varphi^{-1}(k),X)$ *be the path component of the constant map in the space of pointed maps* $\mathrm{map.}(\varphi^{-1}(k),X)$. *Let us assume that for every* $k \in K$, *the space* $\mathrm{map}_*(\varphi^{-1}(k),X)$ *is weakly homotopy equivalent to* *. *Then* φ *induces a weak homotopy equivalence*

$$\varphi^* : \mathrm{map.}(K,X) \xrightarrow{\approx} V_0^{\varphi}(L,X) \ .$$

PROOF OF PROPOSITION 2.1: Let us assume that $P = Z/p$. It follows from [2] Proposition 1.10 that $f^* : H^*(BG,F_p) \longrightarrow H^*(BP,F_p)$ factors through $H^*(BT(\infty),F_p)$. But any morphism $H^*(BT(\infty),F_p) \longrightarrow H^*(BT,F_p)$ is of the form $B\varphi$ (see [8] Theorem 0.4). Hence f is induced by a homomorphism.

Let us suppose that any map $f : BP \longrightarrow (BG)_p$ is induced by a homomorphism if the order of P is less or equal to p^{n-1}.

Let the order of P be equal to p^n and let $f : BP \longrightarrow (BG)_p$ be a map. Let $V = Z/p$ be contained in the center of P and let $i : V \longrightarrow P$ be the inclusion.

Assume that the composition

$$BV \xrightarrow{Bi} BP \xrightarrow{f} X$$

is null homotopic. We want to show that f is homotopic to $f_1 \circ Bpr$ where $pr : P \longrightarrow P/_V$ is the natural homomorphism and $f_1 : B(P/_V) \longrightarrow X$ is a map. First we show that the space $map_*(BV,X)$ is weakly contractible. This space is p–local because BV and X are p–local. Let $map_{const}(BV,X)$ be the connected component containing a constant map of $map(BV,X)$. It follows from Proposition 1.1 that

$$H^*(map_{const}(BV,X),F_p) = H^*(BT(\infty),F_p)^W .$$

The last group is of course $H^*(X,F_p)$. Hence the evaluation map $map_{const}(BV,X) \longrightarrow X$ is a weak homotopy equivalence and consequently the space $nap_*(BV,X)$ is weakly contractible. Lemma 2.4 implies that f is homotopic to $f_1 \circ Bpr$. By the inductive assumption f_1 is induced by a homomorphism.

Let us suppose that foi is induced by a homomorphism $\varphi : V \longrightarrow G$ and $\varphi(V) \neq 0$. Let G_0 be the centralizer of $\varphi(V)$ in G . It follows from Lemma 2.3 that up to homotopy there is a unique map $f_0 : BP \longrightarrow (BG_0)_p$ such that $BP \xrightarrow{f_0} (BG_0)_p \longrightarrow (BG)_p$ is homotopic to f and f_0 restricted to BV is induced by φ. Let $\rho : G_0 \longrightarrow G_0/\varphi(V)$ be the natural projection. The composition

$$BV \longrightarrow BP \xrightarrow{f_0} (BG_0)_p \xrightarrow{(B\rho)_p} (BG_0/\varphi(V))_p$$

is null–homotopic hence $(B\rho)_p \circ f_0$ factors uniquely as

$$BP \xrightarrow{Bpr} B(P/V) \xrightarrow{f_1} B(G_0/\varphi(V))_p .$$

This follows from the previous discussion.

One has the homotopy pullback

$$\begin{array}{ccc} BP & \xrightarrow{\ f_0\ } & (B(G_0)_p \\ \Big\downarrow{\scriptstyle Bpr} & & \Big\downarrow{\scriptstyle (B\rho)_p} \\ B(P/V) & \xrightarrow{\ f_1\ } & (B(G_0/\varphi(V)))_p \end{array}$$

because $\varphi(V)$ is contained in the center of G_0. By the inductive assumption f_1 is induced by a homomorphism $\varphi_1 : P/V \longrightarrow G_0/\varphi(V)$. We have a pullback of groups

$$\begin{array}{ccc} P & \xrightarrow{\ \psi\ } & G_0 \\ \Big\downarrow{\scriptstyle pr} & & \Big\downarrow{\scriptstyle \rho} \\ P/V & \xrightarrow{\ \varphi_1\ } & G_0/\varphi(V) \end{array} .$$

After applying the functor $(B\ )_p$ we get a homotopy pullback

$$\begin{array}{ccc} BP & \xrightarrow{\ (B\psi)_p\ } & (BG_0)_p \\ \Big\downarrow{\scriptstyle Bpr} & & \Big\downarrow{\scriptstyle (B\rho)_p} \\ B(P/V) & \xrightarrow{\ (B\varphi_1)_p\ } & B(G_0/\varphi(V))_p \end{array} .$$

The map f_0 is homotopic to $(B\psi)_p$ hence f is homotopic to $(B\rho)_p \circ (B\psi)_p$. □

COROLLARY 2.5. *Let* T' *by any torus. Then any map* $BT'(\infty) \longrightarrow (BG)_p$ *is induced by a homomorphism.*

This follows directly from Proposition 2.1.

3. PROOFS

We shall need a result from [15].

PROPOSITION 3.1. (*see* [15] *pages* 1 *and* 8) *Let* $W \subset \mathrm{Aut}(T(\infty))$ *be a finite generalized reflection group. Assume that* p *does not divide the order of* W. *Let* X *be a* p*–complete space such that there is an isomorphism*

$$(*) \qquad H^*(X,F_p) = H^*(BT(\infty),F_p)^W$$

of A_p*–algebras. Then there is a map* $i : BT(\infty) \longrightarrow X$ *which realizes the isomorphism* (*). *Moreover for any* $w \in W$, $i \circ w$ *is homotopic to* i.

PROOF OF THEOREM 1:
It follows from Proposition 3.1 that we can assume that $X \approx (B(T(\infty) \tilde{\times} W))_p$ and $X' \simeq (B(T'(\infty) \tilde{\times} W'))_p$. It follows from Corollary 2.5 that $f \circ i$ is induced by a homomorphisms $\varphi : T(\infty) \longrightarrow T'(\infty)$. We set $\tilde{f} = (B\varphi)_p$.

The proof of the point a) is the same as the proof of Theorem 1.7 in [1]. The point b) follows from a). □

PROOF OF THEOREM 3:
Let $f,g : X \longrightarrow X'$ be two maps such that $H^*(f,\mathbb{Q}_p) = H^*(g,\mathbb{Q}_p)$. Let $i : BT_p \longrightarrow X$ be the map induced by an inclusion of a maximal torus. Proposition 3.1 and Corollary 2.5 imply that $f \circ i$ and $g \circ i$ are induced by two homomorphisms $\varphi,\psi : T(\infty) \longrightarrow T'(\infty) \tilde{\times} W'$. The Chern character $ch : K^0(BT'(\infty),Z_p) \longrightarrow H^*(BT'(\infty),\mathbb{Q}_p)$ is injective for any torus T'. It is also injective for the space $B(T'(\infty) \tilde{\times} W')$. For a finite group π let $R(\pi)$ be its complex representation ring. The group $R(T(\infty)) := \varprojlim_n R(T(n))$ is mapped injectively into $K^0(BT(\infty),Z_p)$. Hence we have

$$R(\varphi) = R(\psi) : R(T'(\infty) \tilde{\times} W') \longrightarrow R(T(\infty)).$$

where $R(T',\infty) \tilde{\times} W) := \varprojlim_n R(T'(n) \tilde{\times} W')$.

We must show that φ and ψ are conjugate homomorphisms. For each subgroup $S = Z/p^\infty$ of $T(\infty)$ the restrictions of φ and ψ to S are conjugate by some element of W. The fact that W is finite implies that φ and ψ are conjugate. Hence $f \circ i$ and $g \circ i$ are homotopic. It follows from [11] Theorem 1 that f and g are homotopic. □

PROOF OF THEOREM 2:
We set $\chi(f) = \pi_1(\tilde{f})$ where $\tilde{f}$ is the map from Theorem 1. The injectivity of χ follows from Theorem 3. The surjectivity is obvious. □

PROOF OF THEOREM 4:
The fact that ψ is injective follows from Theorem 3 and the injectivity of Chern character. The proof of surjectivity is the same as in Theorem 4 in [12]. □

PROOF OF THEOREM 5:
It follows from [15] (see Proposition 3.1 in this paper) that R is an essential surjection. Theorem 2 implies that the functor R is faithful and full. □

PROOF OF THEOREM 6:
This follows from the fact that any map from $BT(\infty)$ to $(BG)_p$ is induced by a homomorphism, what is an immediate consequence of [6] 1.1 Theorem.

PROOF OF THEOREM 7:
We would like to construct $\tilde{f} : (BT)_p \longrightarrow (BT')_p$ such that the following diagram

(*)
$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X' \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i'} \\ (BT)_p & \xrightarrow{\quad \tilde{f} \quad} & (BT')_p \end{array}$$

is homotopy commutative where i and i′ are structure maps. However, we do not know how to do it. So we shall proceed in the following way. It follows from [10] theorem 4.1. that there is $\Phi : K^0(BT')_p, Z_p) \longrightarrow K^0(BT)_p, Z_p)$ such that $\Phi \circ i'^* = i^* \circ f^*$. Let us notice that Φ commutes with operations in $K^0(\ , Z_p)$ and augmentations (see [10] pages 326 and 327). It follows from [12] lemma 2.1 that there is $\Upsilon : (BT)_p \longrightarrow (BT')_p$ such that $\Upsilon^* = \Phi$. Using Chern character and passing to cohomology with Q_p–coefficients we get that the diagram (∗) commutes after applying $H^*(\ , Q_p)$. Point a) follows in the same way as Theorem 1.7 in [1]. Point b) follows from a). □

PROOF OF COROLLARY 8: If the natural representation of W on $\pi_1(T) \otimes Q_p$ is irreducible then $\pi_1(\Upsilon) : \pi_2((BT)_p) \longrightarrow \pi_2((BT)_p)$ is an isomorphism or a trivial map. The correspondence $w \longrightarrow w'$ from Theorem 7 point b) is then an isomorphism. The rest is obvious. □

PROOF OF THEOREM 9:
The proof is the same as the proof of Theorem 1.7 in [12]. □

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