A pseudo-differential calculus on non-commutative phase space

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Abstract

We express non-commutative quantum mechanics as a Weyl pseudo-differential calculus on double phase space $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$, which is intertwined with the standard Weyl calculus using a family of partial isometries of $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$ indexed by $\mathcal{S}(\mathbb{R}^n)$. This allows us to reduce the study of non-commutative quantum mechanics to that of conventional Weyl calculus. In particular we easily obtain spectral results for the operators arising in non-commutative quantum mechanics.

1 Introduction

Traditional quantum mechanics is based on the canonical commutation relations

$$[\widehat{X}_\alpha, \widehat{X}_\beta] = [\widehat{P}_\alpha, \widehat{P}_\beta] = 0 \ \ , \ \ [\widehat{X}_\alpha, \widehat{P}_\beta] = i\hbar\delta_{\alpha\beta} \tag{1}$$

for $1 \leq \alpha, \beta \leq n$. Setting $\widehat{Z}_\alpha = \widehat{X}_\alpha$ if $1 \leq \alpha \leq n$ and $\widehat{Z}_\alpha = \widehat{P}_{\alpha-n}$ if $n+1 \leq \alpha \leq 2n$, these relations can be rewritten

$$[\widehat{Z}_\alpha, \widehat{Z}_\beta] = i\hbar j_{\alpha\beta} \ \text{ for } \ 1 \leq \alpha, \beta \leq 2n \tag{2}$$

where $J = (j_{\alpha\beta})_{1\leq\alpha,\beta\leq 2n}$ is the standard symplectic matrix $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ (0 and I are the zero and unity $n \times n$ matrices). We now make the following observation. In traditional quantum mechanics one traditionally chooses to represent explicitly the canonical commutation rules (1)–(2) by imposing that $\widehat{X}_\alpha$ and $\widehat{P}_\alpha$ are the operators defined by $\widehat{X}_\alpha =$ *multiplication by* x_α and $\widehat{P}_\alpha = -i\hbar\partial_{x_\alpha}$; both operators are viewed as acting on functions defined on $\mathbb{R}^n$. Of course, this choice (which is suggested by historical reasons) is not the only possible; for instance we could as well define the "Bopp shifts" [6]

$$\widetilde{X}_\alpha = x_\alpha + \tfrac{1}{2}i\hbar\partial_{p_\alpha} \ \ , \ \ \widetilde{P}_\alpha = p_\alpha - \tfrac{1}{2}i\hbar\partial_{x_\alpha} \tag{3}$$

where $\widehat{X}_\alpha$ and $\widehat{P}_\alpha$ now act on functions defined on phase space. Indeed, in two recent papers de Gosson [11] and de Gosson and Luef [15] have shown that this approach is useful for the reformulation of deformation quantization in terms of the Moyal product, and for the study of generalized "Landau operators". Now, the study of noncommutative field theories and their connections with quantum gravity [8, 20, 23] has led physicists to consider more general commutation relations of the type

$$[\widetilde{X}_\alpha, \widetilde{X}_\beta] = \theta_{\alpha\beta} \ , \ [\widetilde{P}_\alpha, \widetilde{P}_\beta] = \eta_{\alpha\beta} \ , \ [\widetilde{X}_\alpha, \widetilde{P}_\beta] = i\hbar\delta_{\alpha\beta} \tag{4}$$

where $\Theta = (\theta_{\alpha\beta})_{1\leq\alpha,\beta\leq n}$ and $N = (\eta_{\alpha\beta})_{1\leq\alpha,\beta\leq n}$ are antisymmetric matrices measuring the non-commutativity in the position and momentum variables. Writing $\widetilde{Z}_\alpha = \widetilde{X}_\alpha$ if $1 \leq \alpha \leq n$ and $\widetilde{Z}_\alpha = \widetilde{P}_{\alpha-n}$ if $n+1 \leq \alpha \leq 2n$ these relations are equivalent to

$$[\widetilde{Z}_\alpha, \widetilde{Z}_\beta] = i\hbar\omega_{\alpha\beta} \tag{5}$$

where $\Omega_{\Theta,N} = (\omega_{\alpha\beta})_{1\leq\alpha,\beta\leq 2n}$ is the $2n \times 2n$ antisymmetric matrix defined by

$$\Omega_{\Theta,N} = \begin{pmatrix} \hbar^{-1}\Theta & I \\ -I & \hbar^{-1}N \end{pmatrix}. \tag{6}$$

Since $\det \Omega_{\Theta,N} = \det(I + \hbar^{-2}\Theta N)$ the matrix $\Omega_{\Theta,N}$ is invertible as soon as $\hbar^{-2}\Theta N$ is sufficiently small (we will give a precise statement in Subsection 4.1); this requirement is physically meaningful: see the discussions in [5, 7]. Two of us have have investigated in detail the features of the "non-commutative quantum mechanics" determined by the commutation relations (5) in recent papers [1, 2].

It turns out that when $\Omega_{\Theta,N}$ is invertible we can use it to define a symplectic (non-Kählerian) structure on phase space $\mathbb{R}^{2n}$; the discussion above then suggests that we represent the operators $\widetilde{X}_\alpha$ and $\widetilde{P}_\alpha$ by the following generalization of (3).

$$\widetilde{X}_\alpha = x_\alpha + \tfrac{1}{2}i\hbar\partial_{p_\alpha} + \tfrac{1}{2}i\sum\nolimits_\beta \theta_{\alpha\beta}\partial_{x_\beta} \tag{7}$$

$$\widetilde{P}_\alpha = p_\alpha - \tfrac{1}{2}i\hbar\partial_{x_\alpha} + \tfrac{1}{2}i\sum\nolimits_\beta \eta_{\alpha\beta}\partial_{p_\beta}; \tag{8}$$

which we find convenient to write in compact form as

$$\widetilde{Z} = z + \tfrac{1}{2}i\hbar\Omega_{\Theta,N}\partial_z; \tag{9}$$

notice that these operators reduce to (3) when $\Theta = N = 0$. These "quantization rules" lead us to consider pseudo-differential operators formally defined by

$$\widetilde{A}_\omega = a(\widetilde{Z}) = a(z + \tfrac{1}{2}i\hbar\Omega\partial_z) \tag{10}$$

where Ω is an arbitrary antisymmetric invertible matrix; such a matrix defines a symplectic form ω on $\mathbb{R}^{2n}$:

$$\omega(z, z') = z \cdot \Omega^{-1} z'$$

which coincides with the standard symplectic form σ when $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

In this article we will show that:

- The formal definition (10) can be made rigorous, and that the Weyl symbol of the operators $\widetilde{A}_\omega$;
- The operators $\widetilde{A}_\omega$ are intertwined with the usual Weyl operators $\widehat{A}$ using a family of partial isometries $\psi \longmapsto U_{s,\phi}\psi$ of $L^2(\mathbb{R}^n)$ in $L^2(\mathbb{R}^{2n})$ parametrized by $\phi \in S(\mathbb{R}^n)$;
- The spectral properties of the operators $\widetilde{A}_\omega$ can be recovered from those of $\widehat{A}$ using these intertwining relations; in particular the consideration of Shubin's classes of globally hypoelliptic symbols will allow us to state a very precise result when $\widehat{A}$ is formally self-adjoint.
- The matrix $\Omega_{\Theta,N}$ is invertible (and hence defines a symplectic form $\omega = \omega_{\Theta,N}$) under a certain condition of "smallness" of the entries of the matrices Θ and N (this was proven in former work by two of us and our collaborators in [5, 7].

In a sense our results show that the study of noncommutative quantum mechanics is reduced to that of standard quantum mechanics provided that one works in a double phase space.

Remark 1 *In a recent paper two of us pointed out the relevance of Sjöstrand classes for deformation quantization, which relies on the fact that Sjöstrand classes are members of the family of modulation spaces. The results in this investigation extend to the present setting. Our approach to this problem will closely tie the previous setting with the symplectic structure of the double phase space $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. The main consequences are that the Sjöstrand's classes are Banach algebras with respect to twisted convolution and that its is spectral invariant; these facts will be exploited in a forthcoming work.*

Notation 2 *The generic point of $T^*\mathbb{R}^n = \mathbb{R}^{2n}$ is denoted by $z = (x, p)$ and that of $T^*\mathbb{R}^{2n} = \mathbb{R}^{4n}$ by (z, ζ). The standard symplectic form σ on $\mathbb{R}^{2n}$ is defined by $\sigma(z, z') = p \cdot x' - p' \cdot x$ and the corresponding symplectic group is denoted $\mathrm{Sp}(2n, \sigma)$. Given an arbitrary symplectic form ω on $\mathbb{R}^{2n}$ we denote by $\mathrm{Sp}(2n, \omega)$ the corresponding symplectic group.*

We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^n$; its dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions. The scalar *product of two functions* $\psi, \phi \in L^2(\mathbb{R}^n)$ *is denoted by* $(\psi|\phi)$ *and that of* $\Psi, \Phi \in L^2(\mathbb{R}^{2n})$ *by* $((\Psi|\Phi))$. *The corresponding norms are written* $||\psi||$ *and* $|||\Psi|||$.

2 Phase Space Weyl Operators

In this section we show how to define a Weyl-type pseudodifferential calculus on a symplectic space $(\mathbb{R}^{2n}, \omega)$ where ω is an arbitrary symplectic form (with constant coefficients) on $\mathbb{R}^{2n}$.

Let us begin by giving a short review of the main definitions and properties from standard Weyl calculus as exposed (with fluctuating notation) in for instance [9, 10, 17, 21, 22, 24].

2.1 Standard Weyl calculus

Given a function $a \in \mathcal{S}(\mathbb{R}^{2n})$ the Weyl operator $\widehat{A}$ with symbol a is defined by:

$$\widehat{A}\psi(x) = \left(\tfrac{1}{2\pi\hbar}\right)^n \iint_{\mathbb{R}^{2n}} e^{\frac{i}{\hbar}p\cdot(x-y)} a(\tfrac{1}{2}(x+y), p)\psi(y)dydp \tag{11}$$

for $\psi \in \mathcal{S}(\mathbb{R}^n)$; here $\hbar$ is a positive constant which is identified with Planck's constant h divided by 2π in quantum mechanics. This definition makes sense for more general symbols a provided that the integral interpreted in some "reasonable way" (oscillatory integral, for instance) when a is in a suitable symbol class, for instance the Hörmander classes $S^m_{\rho,\delta}$, or the global Shubin spaces $H\Gamma^{m_1,m_0}_\rho$ (which will be defined later in this article). We refer to the existing literature for these well-known facts. A better definition is, no doubt, the following:

$$\widehat{A}\psi = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} F_\sigma a(z_0)\widehat{T}(z_0)\psi dz_0 \tag{12}$$

because it immediately makes sense for arbitrary symbols $a \in \mathcal{S}'(\mathbb{R}^n)$; in the formula above F_σ is the symplectic Fourier transform:

$$F_\sigma a(z_0) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\sigma(z_0,z)} a(z)dz \tag{13}$$

which extends into an automorphism $\mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ and $\widehat{T}(z_0)$ is the Heisenberg–Weyl operator $\mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ formally defined by

$$\widehat{T}(z_0) = e^{-\frac{i}{\hbar}\sigma(\widehat{z},z_0)} \quad \textit{with} \quad \widehat{z} = (x, -i\hbar\partial_x); \tag{14}$$

the action of $\widehat{T}(z_0)$ on $\psi \in \mathcal{S}(\mathbb{R}^n)$ is given by the explicit formula

$$\widehat{T}(z_0)\psi(x) = e^{\frac{i}{\hbar}(p_0\cdot x - \frac{1}{2}p_0\cdot x_0)}\psi(x - x_0) \tag{15}$$

if $z_0 = (x_0, p_0)$. The Weyl correspondence, written $a \overset{\text{Weyl}}{\longleftrightarrow} \widehat{A}$ or $\widehat{A} \overset{\text{Weyl}}{\longleftrightarrow} a$, between an element $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and the Weyl operator it defines is bijective; in fact the Weyl transformation is one-to-one from $\mathcal{S}'(\mathbb{R}^{2n})$ onto the space $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^{2n}))$ of continuous maps $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ (see e.g. Maillard [19], Wong [24]). This can be proven using Schwartz's kernel theorem and the fact that the Weyl symbol a of the operator $\widehat{A}$ is related to the distributional kernel of that operator by the formula

$$a(x,p) = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} p_0 \cdot y} K(x + \tfrac{1}{2}y, x - \tfrac{1}{2}y) dy \tag{16}$$

where the integral is interpreted as the distributional bracket

$$\langle e^{-\frac{i}{\hbar} p_0 \cdot (\cdot)}, K(x + \tfrac{1}{2}(\cdot), x - \tfrac{1}{2}(\cdot)) \rangle$$

(which is essentially a Fourier transform) when $K \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$. Conversely (cf. (11)) the kernel K is expressed in terms of the symbol a by the formula

$$K(x,y) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{\frac{i}{\hbar} p \cdot (x-y)} a(\tfrac{1}{2}(x+y), p) dp.$$

Assume that the product $\widehat{A}\widehat{B}$ exists (this is the case for instance if $\widehat{B} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n)$); then the Weyl symbol c of $\widehat{C} = \widehat{A}\widehat{B}$ and its symplectic Fourier transform c_σ are given by

$$c(z) = \left(\tfrac{1}{4\pi\hbar}\right)^{2n} \iint_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} e^{\frac{i}{2\hbar} \sigma(u,v)} a(z + \tfrac{1}{2}u) b(z - \tfrac{1}{2}v) du dv \tag{17}$$

$$F_\sigma c(z) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} e^{\frac{i}{2\hbar} \sigma(z,z')} F_\sigma a(z - z') F_\sigma b(z') dz'. \tag{18}$$

The first of the formulas above is often written

$$c = a \star_\hbar b \tag{19}$$

especially in the context of deformation quantization [3, 4] where the operation $\star_\hbar$ is called the Moyal star-product.

Two important properties of Weyl operators are the following:

- The operator $\widehat{A} \overset{\text{Weyl}}{\longleftrightarrow} a$ is formally self-adjoint if and only the symbol a is real; more generally the symbol of the formal adjoint of an operator with Weyl symbol a is its complex conjugate $\overline{a}$;

- The property of symplectic covariance: let $\mathrm{Mp}(2n, \sigma)$ be the metaplectic group, that is the unitary representation of the double cover of $\mathrm{Sp}(2n, \sigma)$. To every $s \in \mathrm{Sp}(2n, \sigma)$ thus corresponds, via the natural projection $\pi : \mathrm{Mp}(2n, \sigma) \longrightarrow \mathrm{Sp}(2n, \sigma)$, two operators $\pm S \in \mathrm{Mp}(2n, \sigma)$, and we have $S^{-1}\widehat{A}S \overset{\text{Weyl}}{\longleftrightarrow} a \circ s$. This property is *characteristic* of the Weyl pseudo-differential calculus (see Stein [22], Wong [24]).

A related object is the cross-Wigner transform $W(\psi,\phi)$ of $\psi,\phi\in\mathcal{S}(\mathbb{R}^n)$ (see e.g. [9, 10]); it is defined by

$$W(\psi,\phi)(z)=\left(\tfrac{1}{2\pi\hbar}\right)^n\int_{\mathbb{R}^n}e^{-\frac{i}{\hbar}p\cdot y}\psi(x+\tfrac{1}{2}y)\overline{\phi(x-\tfrac{1}{2}y)}dy \tag{20}$$

(it is thus, up to a constant, the Weyl symbol of the operator with kernel $\psi\otimes\overline{\phi}$). We note, for further use, that $W(\psi,\phi)$ can alternatively be defined by the formula

$$W(\psi,\phi)(z)=\left(\tfrac{1}{\pi\hbar}\right)^n(\widehat{T}_{\mathrm{GR}}(z)\psi|\phi) \tag{21}$$

where $\widehat{T}_{\mathrm{GR}}(z)$ is the Grossmann–Royer operator:

$$\widehat{T}_{\mathrm{GR}}(z_0)\psi(x)=e^{\frac{2i}{\hbar}p_0\cdot(x-x_0)}\psi(2x_0-x). \tag{22}$$

Formula (21) allows us to define $W(\psi,\phi)$ when $\psi,\phi\in\mathcal{S}'(\mathbb{R}^n)$. Following property is important, and is sometimes taken as the definition of $\widehat{A}$:

$$(\widehat{A}\psi|\phi)=\int_{\mathbb{R}^{2n}}a(z)W(\psi,\phi)(z)dz. \tag{23}$$

Also note that the cross-Wigner transform satisfies the Moyal identity

$$((W(\psi,\phi)|W(\psi',\phi')))=\left(\tfrac{1}{2\pi\hbar}\right)^n(\psi|\psi')(\phi|\phi'). \tag{24}$$

2.2 Definition of the operators $\widetilde{A}_\omega$

In what follows Ω denotes an arbitrary (real) invertible antisymmetric $2n\times 2n$ matrix. The formula

$$\omega(z,z')=z\cdot\Omega^{-1}z'=-\Omega^{-1}z\cdot z' \tag{25}$$

defines a symplectic form on $\mathbb{R}^{2n}$; notice that ω coincides with the standard symplectic form σ when $\Omega=J$.

Let us introduce the following variant of the symplectic Fourier transform: if $a\in\mathcal{S}(\mathbb{R}^{2n})$ we set

$$F_\omega a(z)=\left(\tfrac{1}{2\pi\hbar}\right)^n|\det\Omega|^{-1/2}\int_{\mathbb{R}^{2n}}e^{-\frac{i}{\hbar}\omega(z,z')}a(z')dz'; \tag{26}$$

the presence of the inverse square root $|\det\Omega|^{-1/2}$ ensures us that F_ω extends into a unitary automorphism of $L^2(\mathbb{R}^{2n})$: $|||F_\omega a|||=|||a|||$. This unitarity most easily follows from the second formula (38) in Proposition 6 below, or from the observation that F_ω is related to the usual unitary Fourier transform F on $\mathbb{R}^{2n}$ by the formula

$$Fa(z)=|\det\Omega|^{1/2}F_\omega a(-\Omega z). \tag{27}$$

The symplectic Fourier transform F_ω extends into a continuous automorphism of $\mathcal{S}'(\mathbb{R}^{2n})$ in the usual way by defining $F_\omega a$ for $a \in \mathcal{S}'(\mathbb{R}^{2n})$ by the formula $\langle F_\omega a, b\rangle = \langle a, F_\omega b\rangle$ for all $b \in \mathcal{S}(\mathbb{R}^{2n})$ (or, alternatively, by using the relation (27) above). Note that when $\Omega = J$ we have $F_\omega = F_\sigma$ since $\det J = 1$. Using (27) together with the usual Fourier inversion formula shows that F_ω is involutive, that is

$$F_\omega F_\omega a = a. \tag{28}$$

We will also need the operators

$$\widetilde{T}_\omega(z_0) : \mathcal{S}'(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$$

defined by the formula

$$\widetilde{T}_\omega(z_0)\Psi(z) = e^{-\frac{i}{\hbar}\omega(z,z_0)}\Psi(z - \tfrac{1}{2}z_0). \tag{29}$$

These operators satisfy the same commutation relations as the usual Heisenberg–Weyl operators $\widehat{T}(z_0)$ when $\omega = \sigma$. In fact, a straightforward computation shows that

$$\widetilde{T}_\omega(z_0 + z_1) = e^{-\frac{i}{2\hbar}\omega(z_0,z_1)}\widetilde{T}_\omega(z_0)\widetilde{T}_\omega(z_1) \tag{30}$$

$$\widetilde{T}_\omega(z_0)\widehat{T}(z_1) = e^{\frac{i}{\hbar}\omega(z_0,z_1)}\widetilde{T}_\omega(z_1)\widetilde{T}_\omega(z_0). \tag{31}$$

Let us now define the operators $\widetilde{A}_\omega$.

Proposition 3 *Let $a \in \mathcal{S}'(\mathbb{R}^{2n})$ and $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$. The operator $\widetilde{A}_\omega : \mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$ defined by*

$$\widetilde{A}_\omega\Psi = \left(\tfrac{1}{2\pi\hbar}\right)^n (\det\Omega)^{-1/2}\left\langle F_\omega a(\cdot), \widetilde{T}_\omega(\cdot)\Psi\right\rangle \tag{32}$$

that is formally by

$$\widetilde{A}_\omega\Psi = \left(\tfrac{1}{2\pi\hbar}\right)^n (\det\Omega)^{-1/2}\int_{\mathbb{R}^{2n}} F_\omega a(z)\widetilde{T}_\omega(z)\Psi dz \tag{33}$$

is continuous $\mathcal{S}(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}(\mathbb{R}^{2n})$ and its Weyl symbol is given by

$$\widetilde{a}_\omega(z,\zeta) = a\left(z - \tfrac{1}{2}\Omega\zeta\right) \tag{34}$$

and we have $\widetilde{a}_\omega \in \mathcal{S}'(\mathbb{R}^{2n}\oplus\mathbb{R}^{2n})$. When $a = 1$ the operator $\widetilde{A}_\omega$ is the identity on $\mathcal{S}(\mathbb{R}^{2n})$.

Proof. Since $\widetilde{T}_\omega(z)\Psi \in \mathcal{S}(\mathbb{R}^{2n})$ for every z and $F_\omega a \in \mathcal{S}'(\mathbb{R}^{2n})$ the operator $\widetilde{A}_\omega$ is well-defined. We have, setting $u = z - \frac{1}{2}z_0$,

$$\begin{aligned}\widetilde{A}_\omega\Psi(z) &= \left(\tfrac{1}{2\pi\hbar}\right)^n |\det\Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\omega a(z_0)\widetilde{T}_\omega(z_0)\Psi(z)dz_0 \\ &= \left(\tfrac{1}{2\pi\hbar}\right)^n |\det\Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\omega a(z_0)e^{-\frac{i}{\hbar}\omega(z,z_0)}\Psi(z-\tfrac{1}{2}z_0)dz_0 \\ &= \left(\tfrac{2}{\pi\hbar}\right)^n |\det\Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\omega a[2(z-u)]e^{\frac{2i}{\hbar}\omega(z,u)}\Psi(u)du\end{aligned}$$

hence the kernel of $\widetilde{A}_\omega$ is given by the formula

$$K(z,u) = \left(\tfrac{2}{\pi\hbar}\right)^n |\det\Omega|^{-1/2} F_\omega a[2(z-u)]e^{\frac{2i}{\hbar}\omega(z,u)}.$$

It follows from formula (16) that the symbol $\widetilde{a}_\omega$ is given by

$$\begin{aligned}\widetilde{a}_\omega(z,\zeta) &= \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\zeta\cdot\zeta'}K(z+\tfrac{1}{2}\zeta', z-\tfrac{1}{2}\zeta')d\zeta' \\ &= \left(\tfrac{2}{\pi\hbar}\right)^n |\det\Omega|^{-1/2} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\zeta\cdot\zeta'}F_\omega a(2\zeta')e^{-\frac{2i}{\hbar}\omega(z,\zeta')}d\zeta'\end{aligned}$$

that is, using the obvious relation

$$\zeta\cdot\zeta' + 2\omega(z,\zeta') = \omega(2z-\Omega\zeta,\zeta')$$

together with the change of variables $z' = 2\zeta'$,

$$\begin{aligned}\widetilde{a}_\omega(z,\zeta) &= \left(\tfrac{1}{\pi\hbar}\right)^{2n} |\det\Omega|^{-1/2} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\omega(2z-\Omega\zeta,\zeta')}F_\omega a(2\zeta')d\zeta' \\ &= \left(\tfrac{1}{2\pi\hbar}\right)^{2n} |\det\Omega|^{-1/2} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\omega(z-\frac{1}{2}\Omega\zeta,z')}F_\omega a(z')d\zeta'.\end{aligned}$$

Formula (34) immediately follows using the Fourier inversion formula (28). That $\widetilde{A}_\omega = I$ when $a = 1$ immediately follows from the fact that $F_\omega a = (2\pi\hbar)^{2n}\delta$ where δ is the Dirac measure on $\mathbb{R}^{2n}$. The continuity statement follows from the fact that $\widetilde{A}_\omega$ is a Weyl operator. ∎

Two immediate consequences of this result are:

Corollary 4 *(i) The operator $\widetilde{A}_\omega$ defined by (33) is formally self-adjoint if and only if a is real. The formal adjoint $\widetilde{A}_\omega^*$ of $\widetilde{A}_\omega$ is obtained by replacing a with its complex conjugate $\overline{a}$. (ii) The symbol $\widetilde{c}$ of $\widetilde{C}_\omega = \widetilde{A}_\omega\widetilde{B}_\omega$ is given by $\widetilde{c}_\omega(z,\zeta) = c\left(z-\frac{1}{2}\Omega\zeta\right)$ where $c = a\star_\hbar b$ is the Weyl symbol of the operator $\widehat{C} = \widehat{A}\widehat{B}$.*

Proof. (i) The property is obvious since $\widetilde{A}_\omega$ is formally self-adjoint if and only if its Weyl symbol $\widetilde{a}_\omega$ is real, that is if and only if a itself is real. Similarly, the Weyl symbol of $\widetilde{A}_\omega^*$ is the function

$$(z,\zeta) \longmapsto \overline{c\left(z-\tfrac{1}{2}\Omega\zeta\right)}.$$

(ii) The property is an immediate consequence of the definition of $\widetilde{C}_\omega$ since $a \star_\hbar b \overset{\text{Weyl}}{\longleftrightarrow} \widehat{A}\widehat{B}$. ∎

2.3 Symplectic transformation properties

Let ω be the symplectic form (25) on $\mathbb{R}^{2n}$. The symplectic spaces $(\mathbb{R}^{2n}, \omega)$ and $(2n, \sigma)$ are symplectomorphic (as are all symplectic spaces with same dimension); ω and σ having constant coefficients they are even linearly symplectomorphic. That is, there exists a linear automorphism s of $\mathbb{R}^{2n}$ such that $s^*\omega = \sigma$ that is

$$\omega(sz, sz') = \sigma(z, z') \tag{35}$$

for all $(z, z') \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$. [$s$ is sometimes called the "Seiberg–Witten map" in the physical literature; its existence is of course mathematically a triviality, because all symplectic structures with constant coefficients are linearly isomorphic (see e.g. de Gosson [10], §1.1.2)]. Identifying the automorphism s with its matrix in the canonical basis, the relation (35) is equivalent to the matrix equality

$$\Omega = sJs^T. \tag{36}$$

Such a symplectomorphism $s : (\mathbb{R}^{2n}, \omega) \longrightarrow (\mathbb{R}^{2n}, \sigma)$ is by no means unique; we can in fact replace it by any automorphism $s' = ss_\sigma$ where $s_\sigma \in \mathrm{Sp}(2n, \sigma)$; note however that the determinant is an invariant because $\det s' = \det s \det s_\sigma = \det s$ since $\det s_\sigma = 1$.

We are going to see that the study of the operators $\widetilde{A}_\omega$ is easily reduced to the case where $\omega = \sigma$, the standard symplectic form on $\mathbb{R}^{2n}$. This result is closely related to the symplectic covariance of Weyl operators under metaplectic conjugation as we will see below.

For s a linear automorphism of $\mathbb{R}^{2n}$ we define the operator

$$M_s : \mathcal{S}'(\mathbb{R}^{2n}) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$$

by the formula

$$M_s\Psi(z) = \sqrt{|\det s|}\Psi(sz). \tag{37}$$

Clearly M_s is unitary: we have $|||M_s\Psi||| = |||\Psi|||$ for all $\Psi \in L^2(\mathbb{R}^{2n})$.

Notation 5 *When $\Omega = J$ we write $\widetilde{T}(z_0) = \widetilde{T}_\sigma(z_0)$ and $\widetilde{A} = \widetilde{A}_\sigma$.*

Proposition 6 *Let $s : (\mathbb{R}^{2n}, \omega) \longrightarrow (\mathbb{R}^{2n}, \sigma)$ be a linear symplectomorphism. (i) We have the conjugation formulas*

$$M_s\widetilde{T}_\omega(z_0) = \widetilde{T}(s^{-1}z_0)M_s \ , \ M_sF_\omega = F_\sigma M_s \tag{38}$$

$$M_s\widetilde{A}_\omega = \widetilde{A'}M_s \text{ with } a'(z) = a(sz). \tag{39}$$

(ii) When s is replaced by an automorphism $s' = ss_\sigma$ with $s_\sigma \in \mathrm{Sp}(2n,\sigma)$ then $\widetilde{A'}$ is replaced by the operator

$$\widetilde{A''} = M_{s_\sigma}\widetilde{A'}M_{s_\sigma}^{-1} \tag{40}$$

where $M_{s_\sigma}\Psi(z) = \Psi(s_\sigma z)$.

Proof. (i) Since $\omega(sz, z_0) = \sigma(z, s^{-1}z_0)$ we have for all $\Psi \in \mathcal{S}(\mathbb{R}^{2n})$,

$$\begin{aligned}
M_s\left[\widetilde{T}_\omega(z_0)\Psi\right](z) &= \sqrt{|\det s|}e^{-\frac{i}{\hbar}\omega(sz,z_0)}\Psi(sz - \tfrac{1}{2}z_0) \\
&= \sqrt{|\det s|}e^{-\frac{i}{\hbar}\sigma(z,s^{-1}z_0)}\Psi(s(z - \tfrac{1}{2}s^{-1}z_0)) \\
&= e^{-\frac{i}{\hbar}\sigma(z,s^{-1}z_0)}M_s\Psi(z - \tfrac{1}{2}s^{-1}z_0) \\
&= \widetilde{T}(s^{-1}z_0)M_s\Psi(z)
\end{aligned}$$

which is equivalent to the first equality (38). We have likewise

$$\begin{aligned}
M_sF_\omega a(z) &= \sqrt{|\det s|}F_\omega a(sz) \\
&= \left(\tfrac{1}{2\pi\hbar}\right)^n|\det\Omega|^{-1/2}\sqrt{|\det s|}\int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\omega(sz,z')}a(z')dz' \\
&= \left(\tfrac{1}{2\pi\hbar}\right)^n|\det\Omega|^{-1/2}\sqrt{|\det s|}\int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\sigma(z,s^{-1}z')}a(z')dz' \\
&= \left(\tfrac{1}{2\pi\hbar}\right)^n|\det\Omega|^{-1/2}|\det s|\int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\sigma(z,z'')}M_sa(z'')dz'
\end{aligned}$$

hence the second equality (38) because

$$|\det\Omega|^{-1/2}|\det s| = 1 \tag{41}$$

in view of the equality (36). To prove the equality $M_s\widetilde{A}_\omega = \widetilde{A'}M_s$ it suffices to use the relations (38) together with definition (33) of $\widetilde{A}_\omega$:

$$\begin{aligned}
M_s\widetilde{A}_\omega &= \left(\tfrac{1}{2\pi\hbar}\right)^{2n}|\det\Omega|^{-1/2}\int_{\mathbb{R}^{2n}} F_\omega a(z)M_s\widetilde{T}_\omega(z)dz \\
&= \left(\tfrac{1}{2\pi\hbar}\right)^{2n}|\det\Omega|^{-1/2}\int_{\mathbb{R}^{2n}} F_\omega a(z)\widetilde{T}(s^{-1}z)M_sdz;
\end{aligned}$$

performing the change of variables $z \longmapsto sz$ we get, using again (41), and noting that $|\det s|^{-1/2}M_s a(z) = a(sz)$,

$$\begin{aligned}
M_s\widetilde{A}_\omega &= \left(\tfrac{1}{2\pi\hbar}\right)^{2n} |\det\Omega|^{-1/2}|\det s| \int_{\mathbb{R}^{2n}} F_\omega a(sz)\widetilde{T}(z)M_s dz \\
&= \left(\tfrac{1}{2\pi\hbar}\right)^{2n} \int_{\mathbb{R}^{2n}} F_\omega a(sz)\widetilde{T}(z)M_s dz \\
&= \left(\tfrac{1}{2\pi\hbar}\right)^{2n} |\det s|^{-1/2} \int_{\mathbb{R}^{2n}} M_s F_\omega a(z)\widetilde{T}(z)M_s dz \\
&= \left(\tfrac{1}{2\pi\hbar}\right)^{2n} |\det s|^{-1/2} \int_{\mathbb{R}^{2n}} F_\sigma M_s a(z)\widetilde{T}(z)M_s dz \\
&= \left(\tfrac{1}{2\pi\hbar}\right)^{2n} \int_{\mathbb{R}^{2n}} F_\sigma (a\circ s)(z)\widetilde{T}(z)M_s dz \\
&= \widetilde{A'}M_s.
\end{aligned}$$

(ii) To prove formula (40) it suffices to note that

$$\begin{aligned}
M_{s'}\widetilde{A}_\omega &= (M_{s'}M_s^{-1})M_s\widetilde{A}_\omega \\
&= M_{s_\sigma}(\widetilde{A'}M_s) \\
&= (M_{s_\sigma}\widetilde{A'}M_{s_\sigma}^{-1})M_{s_\sigma}M_s \\
&= (M_{s_\sigma}\widetilde{A'}M_{s_\sigma}^{-1})M_{s'}.
\end{aligned}$$

That we have $M_{s_\sigma}\Psi(z) = \Psi(s_\sigma z)$ is clear since $\det s_\sigma = 1$. ■

We note that formula (40) can be interpreted in terms of the symplectic covariance property of Weyl calculus. To see this, let us equip the double phase space $\mathbb{R}^{2n}\oplus\mathbb{R}^{2n}$ with the symplectic structure $\sigma^\oplus = \sigma\oplus\sigma$. In view of formula (34) with $\Omega = J$ the Weyl symbols of operators $\widetilde{A''}$ and $\widetilde{A'}$ are, respectively

$$\widetilde{a'}(z,\zeta) = a\left[s(z-\tfrac{1}{2}J\zeta)\right] \quad , \quad \widetilde{a''}(z,\zeta) = a\left[s'(z-\tfrac{1}{2}J\zeta)\right]$$

and hence, using the identities $s^{-1}s' = s_\sigma \in \mathrm{Sp}(2n,\sigma)$ and $s_\sigma J = J(s_\sigma^T)^{-1}$,

$$\widetilde{a''}(z,\zeta) = a'\left[s_\sigma(z-\tfrac{1}{2}J(s_\sigma^T)^{-1}\zeta)\right] = \widetilde{a'}(s_\sigma z,(s_\sigma^T)^{-1}\zeta).$$

Let now m_{s_σ} be the automorphism of $\mathbb{R}^{2n}\oplus\mathbb{R}^{2n}$ defined by

$$m_{s_\sigma}(z,\zeta) = (s_\sigma^{-1}z, s_\sigma^T\zeta);$$

formula (40) can thus be restated as

$$\widetilde{A''} = M_{s_\sigma}\widetilde{A'}M_{s_\sigma}^{-1} \ \text{ with } \ a'' = a'\circ m_{s_\sigma}^{-1} \tag{42}$$

Recall now (see for instance [10], Chapter 7) that each automorphism s of $\mathbb{R}^{2n}$ induces an element m_s of $\mathrm{Sp}(4n,\sigma^\oplus)$ defined by $m_s(z,\zeta) = (s^{-1}z, s^T\zeta)$

and that m_s is the projection of the metaplectic operator $M_s \in \mathrm{Mp}(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, \sigma^{\oplus})$ (with $\sigma^{\oplus} = \sigma \oplus \sigma$) defined by (37). Formulas (42) and (42) thus reflect the symplectic covariance property of Weyl calculus mentioned at the end of Subsection 2.1.

We finally note that if we equip $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ with the symplectic form $\omega^{\oplus} = \omega \oplus \omega$, the symplectomorphism $s : (\mathbb{R}^{2n}, \omega) \longrightarrow (\mathbb{R}^{2n}, \sigma)$ induces a natural symplectomorphism

$$s \oplus s : (\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, \omega^{\oplus}) \longrightarrow (\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, \sigma^{\oplus}).$$

3 The Intertwining Property

In this section we show that the operators $\widetilde{A}_{\omega}$ can be intertwined with the standard Weyl operator $\widehat{A}$ using an infinite family of partial isometries U_{ϕ} of $L^2(\mathbb{R}^n)$ (depending Ω) on closed subspaces $\mathcal{H}_{\phi}$ of $L^2(\mathbb{R}^{2n})$.

3.1 The partial isometries $U_{s,\phi}$

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be such that $||\phi|| = 1$ (L^2 norm). In [15] two of us have studied the linear mapping $U_{\phi} : \mathcal{S}(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^{2n})$ defined by the formula

$$U_{\phi}\psi = (2\pi\hbar)^{n/2} W(\psi, \phi) \tag{43}$$

where $W(\psi, \phi)$ is the cross-Wigner distribution (20). We can thus take the formula

$$U_{\phi}\psi(z) = \left(\tfrac{2}{\pi\hbar}\right)^{n/2} (\widehat{T}_{\mathrm{GR}}(z)\psi|\phi) \tag{44}$$

as an equivalent definition of U_{ϕ}; recall that $\widehat{T}_{\mathrm{GR}}(z)$ is the Grossmann–Royer transform (22).

Proposition 7 *(i) For every if $\phi \in \mathcal{S}(\mathbb{R}^n)$ the mapping $U_{\phi} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{2n})$ extends into a mapping*

$$U_{\phi} : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^{2n})$$

whose restriction to $L^2(\mathbb{R}^n)$ is an isometry onto a closed subspace $\mathcal{H}_{\phi}$ of $L^2(\mathbb{R}^{2n})$. (ii) The inverse of U_{ϕ} is given by the formula $\psi = U_{\phi}^{-1}\Psi$ with

$$\psi(x) = \left(\tfrac{2}{\pi\hbar}\right)^{n/2} \int_{\mathbb{R}^{2n}} \Psi(z_0)\widehat{T}_{GR}(z_0)\phi(x)dz_0 \tag{45}$$

and the adjoint U_{ϕ}^* of U_{ϕ} is given by the formula

$$U_{\phi}^*\Psi = \left(\tfrac{2}{\pi\hbar}\right)^{n/2} \int_{\mathbb{R}^{2n}} \Psi(z_0)\widehat{T}_{GR}(z_0)\phi dz_0. \tag{46}$$

(iii) The operator $P_{\phi} = U_{\phi}U_{\phi}^*$ is the orthogonal projection of $L^2(\mathbb{R}^{2n})$ onto the Hilbert space $\mathcal{H}_{\phi}$.

Proof. In view of Moyal's identity (24) the operator U_ϕ extends into an isometry of $L^2(\mathbb{R}^n)$ onto a subspace $\mathcal{H}_\phi$ of $L^2(\mathbb{R}^{2n})$:

$$((U_\phi\psi|U_\phi\psi')) = (\psi|\psi').$$

The subspace $\mathcal{H}_\phi$ is closed, being homeomorphic to $L^2(\mathbb{R}^n)$. The inversion formula (45) is verified by a direct calculation: let us set

$$\chi(x) = \left(\tfrac{2}{\pi\hbar}\right)^{n/2}\int_{\mathbb{R}^{2n}}\Psi(z_0)\widehat{T}_{\mathrm{GR}}(z_0)\phi(x)dz_0$$

and choose an arbitrary function $\alpha\in\mathcal{S}(\mathbb{R}^{2n})$. We have

$$\begin{aligned}(\chi|\alpha) &= \left(\tfrac{2}{\pi\hbar}\right)^{n/2}\int_{\mathbb{R}^{2n}}\Psi(z_0)(\widehat{T}_{\mathrm{GR}}(z_0)\phi|\alpha)dz_0\\ &= (2\pi\hbar)^{n/2}\int_{\mathbb{R}^{2n}}\Psi(z_0)\overline{W(\alpha,\phi)}(z_0)dz_0\\ &= \int_{\mathbb{R}^{2n}}U_\phi\psi(z_0)\overline{U_\phi\alpha(z_0)}dz_0\\ &= (\psi|\alpha)\end{aligned}$$

hence $\chi=\psi$ which proves (45); formula (46) for the adjoint follows since $U_\phi^*U_\phi$ is the identity on $L^2(\mathbb{R}^n)$. (iii) We have $P_\phi = P_\phi^*$ and $P_\phi P_\phi^* = P_\phi$ hence P_ϕ is an orthogonal projection. Since $U_\phi^*U_\phi$ is the identity on $L^2(\mathbb{R}^n)$ the range of U_ϕ^* is $L^2(\mathbb{R}^n)$ and that of P_ϕ is therefore precisely $\mathcal{H}_\phi$. ■

Remark 8 *The union of the ranges of the partial isometries U_ϕ viewed as mappings defined on $\mathcal{S}'(\mathbb{R}^n)$ is in a sense a rather small subset of $\mathcal{S}'(\mathbb{R}^{2n})$ even when ϕ runs over all of $\mathcal{S}'(\mathbb{R}^n)$; this is a consequence of Hardy's theorem on the concentration of a function and its Fourier transform (de Gosson and Luef [13, 14]), and is related to a topological formulation of the uncertainty principle (de Gosson [12]). We will come back to these notions in the framework of noncommutative quantum mechanics in a forthcoming publication.*

In [15]) it was shown that the partial isometries U_ϕ can be used to intertwine the operators $\widetilde{A}=\widetilde{A}_\sigma$ with the usual Weyl operators with same symbol; we reproduce the proof for convenience:

Proposition 9 *Let $\widetilde{T}(z_0)=\widetilde{T}_\sigma(z_0)$ and $\widehat{T}(z_0)$ be the Heisenberg–Weyl operator (14)–(15). We have the following intertwining properties:*

$$U_\phi\widehat{T}(z_0) = \widetilde{T}(z_0)U_\phi \text{ and } U_\phi^*\widetilde{T}(z_0) = \widehat{T}(z_0)U_\phi^* \tag{47}$$

$$\widetilde{A}U_\phi = U_\phi\widehat{A} \text{ and } U_\phi^*\widetilde{A} = \widehat{A}U_\phi^*. \tag{48}$$

Proof. Making the change of variable $y = y' + x_0$ in the integral in the right-hand side of (20) we get

$$U_\phi \left[\widehat{T}(z_0)\psi\right](z) = e^{-\frac{i}{\hbar}\sigma(z,z_0)} U_\phi\psi(z - \tfrac{1}{2}z_0)$$

which is precisely (47). On the other hand we have

$$U_\phi \widehat{A}\psi = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} F_\sigma a(z_0) U_\phi[\widehat{T}(z_0)\psi] dz_0.$$

and hence, in view of (47),

$$U_\phi \widehat{A}\psi = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^{2n}} F_\sigma a(z_0)[\widetilde{T}(z_0) U_\phi\psi] dz_0$$

which is the first equality (48). To prove the second equality (48) it suffices to apply the first to $U_\phi^* \widetilde{A} = (\widetilde{A}^* U_\phi)^*$. ∎

Let us generalize this result to the case of an arbitrary operator $\widetilde{A}_\omega$.

Proposition 10 *Let ω be a symplectic form (25) on $\mathbb{R}^{2n}$ and s a linear automorphism such that $s^*\omega = \sigma$. The mappings $U_{s,\phi} : \mathcal{S}(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^{2n})$ defined by the formula:*

$$U_{s,\phi} = M_s^{-1} U_\phi \tag{49}$$

are partial isometries $L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$, in fact isometries on a closed subspace $\mathcal{H}_{s,\phi}$ of $L^2(\mathbb{R}^{2n})$, and we have

$$\widetilde{A}_\omega U_{s,\phi} = U_{s,\phi}\widehat{A'} \;\; \text{and} \;\; U_{s,\phi}^* \widetilde{A}_\omega = \widehat{A'} U_{s,\phi}^* \tag{50}$$

where $\widehat{A'} \overset{Weyl}{\longleftrightarrow} a \circ s$.

Proof. We have, using the first formula (48),

$$\begin{aligned}
\widetilde{A}_\omega U_{s,\phi} &= M_s^{-1} \widetilde{A'} M_s (M_s^{-1} U_\phi) \\
&= M_s^{-1} (\widetilde{A'} U_\phi) \\
&= M_s^{-1} U_\phi \widehat{A'} \\
&= U_{s,\phi} \widehat{A'};
\end{aligned}$$

the equality $U_{s,\phi}^* \widetilde{A}_\omega = \widehat{A'} U_{s,\phi}^*$ is proven in a similar way. That $U_{s,\phi}$ is a partial isometry is obvious since U_ϕ is a a partial isometry and M_s is unitary. ∎

Let us make explicit the change of the mapping s:

Proposition 11 *Let s and s' be linear automorphisms of $\mathbb{R}^{2n}$ such that $s^*\omega = s'^*\omega = \sigma$. We have*

$$U_{s',\phi}\psi = U_{s,S_\sigma\phi}(S_\sigma\psi) \tag{51}$$

where $S_\sigma \in \mathrm{Mp}(2n, \sigma)$ is such that $\pi(S_\sigma) = s^{-1}s'$.

Proof. The relation $s^*\omega = s'^*\omega = \sigma$ implies that $s_\sigma = s^{-1}s' \in \mathrm{Sp}(2n, \sigma)$. We have $M_{s'} = M_{ss_\sigma} = M_{s_\sigma} M_s$ and hence

$$U_{s',\phi} = M_{s'}^{-1} U_\phi = M_s^{-1} M_{s_\sigma}^{-1} U_\phi.$$

Now, taking into account definition (43) of U_ϕ in terms of the cross-Wigner transform and the fact that $\det s_\sigma = 1$ we have

$$\begin{aligned} M_{s_\sigma}^{-1} U_\phi \psi(z) &= (2\pi\hbar)^{n/2} W(\psi, \phi)(s_\sigma^{-1} z) \\ &= (2\pi\hbar)^{n/2} W(S_\sigma \psi, S_\sigma \phi)(z) \\ &= (2\pi\hbar)^{n/2} U_{S_\sigma \phi}(S_\sigma \psi)(z) \end{aligned}$$

hence formula (51). ∎

3.2 Action on orthonormal bases

Let us prove the following important result that shows that orthonormal bases of $L^2(\mathbb{R}^n)$ can be used to generate orthonormal bases of $L^2(\mathbb{R}^{2n})$ using the mappings $U_{s,\phi}$:

We begin by showing that thew result holds for U_ϕ. The general case will readily follow.

Proposition 12 *Let $(\phi_j)_j$ be an arbitrary orthonormal basis of $L^2(\mathbb{R}^n)$; the vectors $\Phi_{j,k} = U_{\phi_j}\phi_k$ form an orthonormal basis of $L^2(\mathbb{R}^{2n})$.*

Since the U_{ϕ_j} are isometries the vectors $\Phi_{j,k}$ form an orthonormal system. It is thus sufficient to show that if $\Psi \in L^2(\mathbb{R}^{2n})$ is orthogonal to the family $(\Phi_{j,k})_{j,k}$ (and hence to all the spaces $\mathcal{H}_{\phi_j}$) then $\Psi = 0$. Assume that $(\Psi|\Phi_{jk})_{L^2(\mathbb{R}^{2n})} = 0$ for all indices j, k. Since we have

$$(\Psi|\Phi_{jk}) = (\Psi|U_{\phi_j}\phi_k) = (U_{\phi_j}^*\Psi|\phi_k)$$

it follows that $U_{\phi_j}^*\Psi = 0$ for all j since $(\phi_j)_j$ is a basis; using the anti-linearity of U_ϕ in ϕ we have in fact $U_\phi^*\Psi = 0$ for all $\phi \in L^2(\mathbb{R}^n)$. Let us show that this property implies that we must have $\Psi = 0$. Recall (formula (46)) that the adjoint of the wavepacket transform U_ϕ^* is given by

$$U_\phi^*\Psi = \left(\tfrac{2}{\pi\hbar}\right)^{n/2} \int_{\mathbb{R}^{2n}} \Psi(z_0) \widehat{T}_{\mathrm{GR}}(z_0)\phi dz_0$$

where $\widehat{T}_{\mathrm{GR}}(z_0)$ is the Grossmann–Royer operator. Let now ψ be an arbitrary element of $\mathcal{S}(\mathbb{R}^n)$; we have, using definition 21 of the cross-Wigner transform,

$$\begin{aligned} (U_\phi^*\Psi|\psi) &= \left(\tfrac{2}{\pi\hbar}\right)^{n/2} \int_{\mathbb{R}^{2n}} \Psi(z)(\widehat{T}_{\mathrm{GR}}(z)\phi|\psi) dz \\ &= (2\pi\hbar)^{n/2} \int_{\mathbb{R}^{2n}} \Psi(z) W(\psi, \phi)(z) dz. \end{aligned}$$

Let us now view $\Psi \in L^2(\mathbb{R}^{2n})$ as the Weyl symbol of an operator $\widehat{A}_\Psi$. In view of formula (23) we have

$$(2\pi\hbar)^{n/2} \int_{\mathbb{R}^{2n}} \Psi(z) W(\psi, \phi)(z) dz = (\widehat{A}_\Psi \psi | \phi)$$

and the condition $U_\phi^* \Psi = 0$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ is thus equivalent to $(\widehat{A}_\Psi \psi|\phi)_{L^2} =$ for all $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$. It follows that $\widehat{A}_\Psi \psi = 0$ for all ψ and hence $\widehat{A}_\Psi = 0$. Since the Weyl correspondence is one-to-one we must have $\Psi = 0$ as claimed.

Corollary 13 *Let $(\phi_j)_j$ be an arbitrary orthonormal basis of $L^2(\mathbb{R}^n)$; the vectors $\Phi_{j,k} = U_{s,\phi_j}\phi_k$ form an orthonormal basis of $L^2(\mathbb{R}^{2n})$.*

Proof. We have, by definition, $U_{s,\phi} = M_s^{-1} U_\phi$. The result follows since M_s is unitary. (Alternatively one could have used formula (51) to prove the statement). ■

4 Spectral Properties of the Operators $\widetilde{A}_\omega$

We begin by studying the standard case $\Omega = J$; as before we then use the notation $\widetilde{A}_\omega = \widetilde{A}$. The extension to the general case is rather straightforward using again the reduction result in Proposition 6.

4.1 The case $\Omega = J$

Proposition 12 and its Corollary are the keys to the following general spectral result, which shows how to obtain the eigenvalues and eigenvectors of $\widetilde{A}$ from those of $\widehat{A}$:

Proposition 14 *The following properties hold true: (i) The eigenvalues of the operators $\widehat{A}$ and $\widetilde{A}$ are the same; (ii) Let ψ be an eigenvector of $\widehat{A}$: $\widehat{A}\psi = \lambda\psi$. Then $\Psi = U_\phi \psi$ satisfies $\widetilde{A}\Psi = \lambda\Psi$; in particular, if $\Psi \neq 0$ it is an eigenvector of $\widetilde{A}$ corresponding to the same eigenvalue. (iii) Conversely, if Ψ is an eigenvector of $\widetilde{A}$ then $\psi = U_\phi^* \Psi$ is an eigenvector of $\widehat{A}$ corresponding to the same eigenvalue.*

Proof. (i) That every eigenvalue of $\widehat{A}$ also is an eigenvalue of $\widetilde{A}$ is clear: if $\widehat{A}\psi = \lambda\psi$ for some $\psi \neq 0$ then

$$\widetilde{A}(U_\phi \psi) = U_\phi \widehat{A}\psi = \lambda U_\phi \psi$$

and $\Psi = U_\phi \psi \neq 0$; this proves at the same time that $U_\phi \psi$ is an eigenvector of $\widehat{A}$ because U_ϕ has kernel $\{0\}$. (ii) Assume conversely that $\widetilde{A}\Psi = \lambda\Psi$ for $\Psi \in L^2(\mathbb{R}^{2n})$, $\Psi \neq 0$, and $\lambda \in \mathbb{R}$. For every ϕ we have

$$\widehat{A} U_\phi^* \Psi = U_\phi^* \widetilde{A} \Psi = \lambda U_\phi^* \Psi$$

hence λ is an eigenvalue of $\widehat{A}$ and ψ an eigenvector if $\psi = U_\phi^* \Psi \neq 0$. We have $U_\phi \psi = U_\phi U_\phi^* \Psi = P_\phi \Psi$ where P_ϕ is the orthogonal projection on the range $\mathcal{H}_\phi$ of U_ϕ. Assume that $\psi = 0$; then $P_\phi \Psi = 0$ for every $\phi \in \mathcal{S}(\mathbb{R}^n)$, and hence $\Psi = 0$ in view of Proposition 12. ■

Let us now consider the case of general operators $\widetilde{A}_\omega$. It follows from Proposition 14 that:

Corollary 15 *(i) The eigenvalues of $\widetilde{A}_\omega$ are the eigenvalues of the Weyl operator $\widehat{A'} \overset{\mathrm{Weyl}}{\longleftrightarrow} a \circ s$ are the same; (ii) Let ψ be an eigenvector of $\widehat{A}$: $\widehat{A}\psi = \lambda\psi$. Then $\Psi = U_{s,\phi}\psi$ satisfies $\widetilde{A}_\omega \Psi = \lambda \Psi$; (iii) Conversely, if Ψ is an eigenvector of $\widetilde{A}$ then $\psi = U_\phi^* \Psi$ is an eigenvector of $\widehat{A}$ corresponding to the same eigenvalue.*

Let us now specialize our discussion to the case where the Weyl symbol of $\widehat{A}$ belongs to a very convenient space of symbols. Shubin has introduced in [21] very convenient "global" symbol classes $H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ where $m_0, m_1 \in \mathbb{R}$ and $0 < \rho \leq 1$. Introducing the multi-index notation $\alpha = (\alpha_1, ..., \alpha_{2n}) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_{2n}$, and $\partial_z^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} \partial_{y_1}^{\alpha_{n+1}} \cdots \partial_{y_n}^{\alpha_{2n}}$, we have by definition $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ if:

- *We have $a \in C^\infty(\mathbb{R}^{2n})$;*
- *There exist constants $R, C_0, C_1 \geq 0$ and, for every $\alpha \in \mathbb{N}^n$, $|\alpha| \neq 0$, a constant $C_\alpha \geq 0$ such that for $|z| \geq R$ we have the estimates*

$$C_0 |z|^{m_0} \leq |a(z)| \leq C_1 |z|^{m_1} \quad , \quad |\partial_z^\alpha a(z)| \leq C_\alpha |a(z)| |z|^{-\rho|\alpha|}. \tag{52}$$

A simple but typical example is the following: the function a defined by $a(z) = \frac{1}{2}|z|^2$ is in $H\Gamma_1^{2,2}(\mathbb{R}^{2n})$, the same applies, more generally to $a(z) = \frac{1}{2}Mz \cdot z$ when M is a real positive definite matrix.

The interest of these symbol classes comes from the following result:

Proposition 16 *Let $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ be real, and $m_0 > 0$. Then the formally self-adjoint operator $\widehat{A}$ with Weyl symbol a has the following properties: (i) $\widehat{A}$ is essentially self-adjoint and has discrete spectrum in $L^2(\mathbb{R}^n)$; (ii) There exists an orthonormal basis of eigenfunctions $\phi_j \in \mathcal{S}(\mathbb{R}^n)$ ($j = 1, 2, ...$) with eigenvalues $\lambda_j \in \mathbb{R}$ such that $\lim_{j\to\infty} |\lambda_j| = \infty$.*

For a proof we refer to Shubin [21], Chapter 4; the essential property that there is a basis of eigenfunctions belonging to $\mathcal{S}(\mathbb{R}^n)$ is due to the global hypoellipticity of operators with Weyl symbol in $H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$:

$$\psi \in \mathcal{S}'(\mathbb{R}^n) \text{ and } \widehat{A}\psi \in \mathcal{S}(\mathbb{R}^n) \text{ implies } \psi \in \mathcal{S}(\mathbb{R}^n)$$

(global hypoellipticity is thus a stronger property than that of the usual hypoellipticity, familiar from the (micro)local analysis of pseudodifferential operators). Let us apply this result to the operators $\widetilde{A}_\omega$. We will need the following elementary result that says that the symbol classes $H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ are invariant under linear changes of variables:

Lemma 17 *Let $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ with $m_0 > 0$. For every linear automorphism s of $\mathbb{R}^{2n}$ we have $a \circ s \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$.*

Proof. Set $a'(z) = a(sz)$; clearly $a' \in C^\infty(\mathbb{R}^{2n})$. We now note that there exist $\lambda, \mu > 0$ such that $\lambda|z| \leq |sz| \leq \mu|z|$ for all $z \in \mathbb{R}^n$. Since $m_0 > 0$ it follows that

$$C_0'|z|^{m_0} \leq |a'(z)| \leq C_1^{'}|z|$$

with $C_0' = C_0\lambda^{m_0}$ and $C_1^{'} = C_1\mu^{m_1}$. Next, we observe that for every $\alpha \in \mathbb{N}^n$, $|\alpha| \neq 0$, there exists $B_\alpha > 0$ such that $|\partial_z^\alpha a'(z)| \leq B_\alpha|\partial_z^\alpha a(sz)|$ (this is easily seen by induction on $|\alpha|$ and using the chain rule); we thus have

$$|\partial_z^\alpha a'(z)| \leq C_\alpha B_\alpha|a'(z)||sz|^{-\rho|\alpha|} \leq C_\alpha'|a'(z)||z|^{-\rho|\alpha|}$$

with $C_\alpha' = B_\alpha C_\alpha \lambda|^{-\rho|\alpha|}$. Hence $a' \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$. ■

Proposition 18 *Let $a \in H\Gamma_\rho^{m_1,m_0}(\mathbb{R}^{2n})$ be real, and $m_0 > 0$. Then: (i) The operator $\widetilde{A}$ has discrete spectrum $(\lambda_j)_{j\in\mathbb{N}}$ with $\lim_{j\to\infty}|\lambda_j| = \infty$. (ii) The eigenfunctions of $\widetilde{A}$ are given by $\Phi_{jk} = U_{s,\phi_j}\phi_k$ where the ϕ_j are the eigenfunctions of the operator $\widehat{A}$ with Weyl symbol a. (iii) We have $\Phi_{jk} \in \mathcal{S}(\mathbb{R}^{2n})$ and the Φ_{jk} form an orthonormal basis of $\mathcal{S}(\mathbb{R}^n)$.*

Proof. It is an immediate consequence of Proposition 16 using Lemma 17. ■

Appendix: The Case $\Omega = \Omega_{\Theta,N}$

We now specialize our discussion to the physically interesting case where $\Omega = \Omega_{\Theta,N}$ with

$$\Omega = \begin{pmatrix} \hbar^{-1}\Theta & I \\ -I & \hbar^{-1}N \end{pmatrix} \tag{53}$$

discussed in the Introduction. In order to apply the theory exposed in the previous sections to systems associated with such a matrix, we have to find conditions that ensure the invertibility of this matrix. We will follow closely the exposition in [5, 7]. Before we proceed, let us recall a few notions about the Pfaffian of an antisymmetric matrix. It turns out that the determinant of such a matrix Ω can always be written as the square of a polynomial in the entries of Ω. This polynomial is called the Pfaffian $\mathrm{Pf}(\Omega)$ of the matrix Ω. Thus, by definition:

$$[\mathrm{Pf}(\Omega)]^2 = \det\Omega. \tag{54}$$

It follows that the Pfaffian is nonvanishing only for $2n \times 2n$ antisymmetric matrices, in which case it is a polynomial of degree exactly n. It immediately follows from (54) that the Pfaffian has the following properties:

$$\operatorname{Pf}(s\Omega s^T) = \det(s) \operatorname{Pf}(\Omega) \tag{55}$$

and

$$\operatorname{Pf}(\Omega^T) = (-1)^n \operatorname{Pf}(\Omega) \ , \ \operatorname{Pf}(\lambda\Omega) = \lambda^n \operatorname{Pf}(\Omega).$$

Moreover, for an arbitrary $n \times n$ matrix M we have:

$$\operatorname{Pf}\begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} = (-1)^{n(n-1)/2} \det M. \tag{56}$$

Let $(\omega_{\alpha\beta})$ $(\alpha, \beta = 1, \cdots, 2n)$ denote the elements of Ω. The Pfaffian of Ω can be obtained from the following recursive formula:

$$\operatorname{Pf}(\Omega) = \sum_{\alpha=2}^{2n} (-1)^{\alpha} \omega_{1,\alpha} \operatorname{Pf}(\Omega_{\hat{1},\hat{\alpha}}) \tag{57}$$

where $\Omega_{\hat{1},\hat{\alpha}}$ denotes the matrix Ω with both the first and α-th rows and columns removed.

Proposition 19 *let us assume that*

$$\zeta = \max\left\{\theta_{ij}\eta_{kl}/\hbar^2 \quad , 1 \leq i < j \leq n \ , \ 1 \leq k < l \leq n\right\} < 1. \tag{58}$$

Then: (i) we have $\det \Omega \neq 0$ *and the sign of* $\operatorname{Pf}(\Omega)$ *is given by*

$$\operatorname{sign}\left[\operatorname{Pf}(\Omega)\right] = (-1)^{n(n-1)/2}. \tag{59}$$

(ii) A matrix s such that $sJs^T = \Omega$ has positive determinant: $\det s > 0$.

Proof. (i) From (53) and (57) we get:

$$\operatorname{Pf}(\Omega) = \sum_{i=2}^{n} (-1)^i \frac{\theta_{1i}}{\hbar} \operatorname{Pf}(\Omega_{\hat{1},\hat{i}}) + (-1)^{n+1} \operatorname{Pf}(\Omega_{\hat{1},\widehat{n+1}}). \tag{60}$$

A term which is independent of the elements of Θ and N can only be found in $(-1)^{n+1} \operatorname{Pf}(\Omega_{\hat{1},\widehat{n+1}})$. Suppose that $n \geq 3$. If we apply the recursive formula (57) again we obtain a term of the form $(-1)^{n+1}(-1)^n \operatorname{Pf}(A_2)$ where A_2 is obtained from Ω by removing the 1st, 2nd, $(n+1)$-th and $(n+2)$-th rows and columns. After i steps we obtain a term $(-1)^{n+1}(-1)^n \cdots (-1)^{n+2-i} \operatorname{Pf}(A_i)$ where A_i is obtained from Ω by removing the 1st, 2nd,..., i-th, and $(n+1)$th,

$(n+2)$th,..., $(n+i)$th rows and columns. We terminate this process when $i = n-2$. We thus obtain:

$$(-1)^{n+1}(-1)^n \cdots (-1)^4 \,\mathrm{Pf} \begin{pmatrix} 0 & \frac{\theta_{n-1,n}}{\hbar} & 1 & 0 \\ \frac{\theta_{n,n-1}}{\hbar} & 0 & 0 & 1 \\ -1 & 0 & 0 & \frac{\eta_{n-1,n}}{\hbar} \\ 0 & -1 & \frac{\eta_{n,n-1}}{\hbar} & 0 \end{pmatrix} \tag{61}$$

$$= \left(\frac{\theta_{n-1,n}\eta_{n-1,n}}{\hbar^2} - 1\right)(-1)^{\sum_{i=4}^{n+1} i}.$$

And thus the term independent of the elements of Θ and N is $(-1)^{n(n-1)/2}$. We leave to the reader the simple task of verifying that this result also holds when $n = 2$. Let us now turn to the θ and η dependent terms. We resort to the definition of the Pfaffian:

$$\mathrm{Pf}(\Omega) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} sgn(\sigma) \Pi_{i=1}^n \omega_{\sigma(2i-1),\sigma(2i)}, \tag{62}$$

where S_{2n} is the symmetric group and $sgn(\sigma)$ is the signature of the permutation σ. Moreover, we use the following notation. If $n = 2$, for instance, then we consider the permutations of the set $\{1, 2, 3, 4\}$. Suppose that in the string $\Pi_{i=1}^n \omega_{\sigma(2i-1),\sigma(2i)}$ we pick k elements of the matrix $\hbar^{-1}\Theta$, p elements of the matrix $\hbar^{-1}N$ and l elements of the matrix I or $-I$. Then, of course:

$$k + l + p = n. \tag{63}$$

If we pick l elements from I or $-I$, then the remaining $k + p$ terms can only be taken from Ω when $2l$ lines and rows have been eliminated. In particular, we remove l lines and rows from $\hbar^{-1}\Theta$. That leaves us with $(n-l-1)(n-l)/2$ non-vanishing independent parameters in $\hbar^{-1}\Theta$. Each time we choose one of the latter for our string $\Pi_{i=1}^n \omega_{\sigma(2i-1),\sigma(2i)}$, we have to eliminate another 2 lines and 2 columns. So if we pick k elements out of the $(n-l-1)(n-l)/2$ non-vanishing independent elements of $\hbar^{-1}\Theta$, we remove $2k$ lines and columns. We are left with $(n-l-2k-1)(n-l-2k)/2$ non-vanishing independent elements. But this is only possible if we have

$$2k \le n - l. \tag{64}$$

A similar argument leads to the inequality

$$2p \le n - l. \tag{65}$$

Now, (64) and (65) are only compatible with (63) if:

$$k = p = \frac{n-l}{2}. \tag{66}$$

This means that in each string we have exactly the same number of elements of $\hbar^{-1}\Theta$ and $\hbar^{-1}N$. This proves that:

$$\mathrm{Pf}(\Omega) = (-1)^{n(n-1)/2} + P_{[n/2]}, \tag{67}$$

where $P_{[n/2]}$ is a homogeneous polynomial of degree $[n/2]$ (the integral part of $n/2$) in the dimensionless variables $\theta_{ij}\eta_{kl}/\hbar^2$ with $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$. Let σ' be the permutation which yields the contribution $(-1)^{n(n-1)/2}$ to the Pfaffian and let $S'_{2n} := S_{2n} \backslash \{\sigma'\}$. We thus have:

$$\left|P_{[n/2]}\right| = \left|\frac{1}{2^n n!}\sum_{\sigma \in S'_{2n}} sgn(\sigma)\Pi_{i=1}^{n}\omega_{\sigma(2i-1),\sigma(2i)}\right| \tag{68}$$

$$\leq \frac{1}{2^n n!}\sum_{\sigma \in S'_{2n}} \Pi_{i=1}^{n}\left|\omega_{\sigma(2i-1),\sigma(2i)}\right|. \tag{69}$$

If a string $\Pi_{i=1}^{n}\omega_{\sigma(2i-1),\sigma(2i)}$ contains k elements of $\hbar^{-1}\Theta$ and k elements of $\hbar^{-1}N$, then

$$\Pi_{i=1}^{n}\left|\omega_{\sigma(2i-1),\sigma(2i)}\right| \leq \zeta^k < \zeta, \tag{70}$$

where we used $\zeta < 1$. Since there are $n! - 1 < n!$ elements in S'_{2n}, we conclude that:

$$\left|P_{[n/2]}\right| < \frac{\zeta}{2^n} < 1. \tag{71}$$

This proves our claim. (ii) We have

$$\det(s)\,\mathrm{Pf}(J) = \mathrm{Pf}(\Omega); \tag{72}$$

from (56) we get $\mathrm{Pf}(J) = (-1)^{n(n-1)/2}$ and hence, by (72), $\det s > 0$ as claimed. ∎

Remark 20 *Writing s in block-matrix form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ the condition $sJs^T = \Omega$ is equivalent to the relations*

$$AB^T - BA^T = \hbar^{-1}\Theta \ \ , \ \ CD^T - DC^T = \hbar^{-1}N \ \ , \ \ AD^T - BC^T = I.$$

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