Orbits of Hurwitz action for coverings of a sphere with two special fibers

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Branched coverings of a 2-dimensional sphere (a complex projective line $\mathbb{P}^1$) by an orientable surface were investigated by Hurwitz in his two papers [1] and [2]. If we fix the degree d of the coverings and the number k of the critical values then the space of coverings form a covering space $\mathcal{Z}$ of the configuration space W_k of k points in $\mathbb{P}^1$. The fundamental group of W_k – the braid group of a sphere with k strings – acts on the fiber of the covering. This is the Hurwitz action of the braid group on the coverings with given critical values. The orbits of this action are in one-one correspondence with the connected components of $\mathcal{Z}$.

Consider a particular covering $f : X \to \mathbb{P}^1$. To each value in $\mathbb{P}^1$ correspond several points in the fiber. The map f has some local multiplicity greater than or equal to 1 in a small neighbourhood of each point. The value is critical if some multiplicity is greater than one. The multiplicities form a *branching data* of the critical value. The critical value is *simple* if one point in its preimage has multiplicity 2 and other points have multiplicity 1. Otherwise the critical value is *special.* The set of the critical values of f form the *discriminant* D of f. We choose a base point p outside D and we put $F = f^{-1}(p)$. A closed path α in $\mathbb{P}^1 - D$ issued from p determines a *monodromy permutation* $\mu_\alpha \in Aut(F)$. We get a monodromy homomorphism $\mu : \pi_1(\mathbb{P}^1 - D, p) \to Aut(F)$. The image of μ is the *monodromy group* of f (more precisely of $f|(X - f^{-1}(D)) : (X - f^{-1}(D)) \to (\mathbb{P}^1 - D)$). $Aut(F)$ is isomorphic to the group Σ_d of permutations on d letters and the monodromy group of f, as a subgroup of Σ_d, is defined up to conjugation.

The Hurwitz action preserves the branching data of the critical values, up to a permutation of the critical values, so we can restrict the investigation to the part of $\mathcal{Z}$ with the given branching data of all critical values. Hurwitz proved that there is only one orbit in the generic case, when all critical values are simple. If the coverings have only one special critical value then there is still only one orbit corresponding to the given branching data of the special critical value (see [3]). The Hurwitz action also preserves the monodromy group of the covering, which is equal to the whole group Σ_d in the above cases.

We prove in this paper that this last invariant classifies the coverings with two special critical values.

Theorem 1 *Let $\mathcal{Z}$ be the space of connected coverings $f : X \to \mathbb{P}^1$ of degree d and genus g which have given branching data over two special critical values, have no other special critical values, and have a fixed monodromy group. Then $\mathcal{Z}$ is connected.*

A special case of this theorem, when the coverings are totally ramified over 0 and over ∞, is contained in the recent paper of Looijenga [4]. The present work is a direct application of Looijenga's result and of his technique.

A similar theorem is not true for coverings with more than two special critical values. Some new invariants of the Hurwitz action are needed to distinguish the orbits of the Hurwitz action in this case.

By a covering we shall mean a connected, orientable branched covering of $\mathbb{P}^1$ or of a closed disk embedded in $\mathbb{P}^1$.

Definition. Coverings $f_1 : X_1 \to U_1$ and $f_2 : X_2 \to U_2$ are equivalent if there exists a homeomorphism $h : X_1 \to X_2$ and a homeomorphism $g : U_1 \to U_2$ such that $gf_1 = f_2h$.

Let $f : X \to U$ be a branched covering of degree d. We choose a base point p outside the discriminant of f and we put $F = f^{-1}(p)$. By a *simple arc* we shall mean an embedded interval connecting p with a point of the discriminant that does not meet the discriminant along the way. A simple arc α determines up to isotopy (relative p and the discriminant) a simple loop based at p, turning clockwise around the point of the discriminant. The beginning and the end points of the liftings of this loop define the monodromy permutation $\mu_\alpha \in Aut(F)$. μ_α is a transposition if and only if the point of the discriminant is a simple critical value of f. A collection of simple arcs that do not meet outside p will be called an *arc system*. It is a *complete arc system* if it connects p to all points of the discriminant. Loops around members of a complete arc system form a basis of $\pi_1(U-D,p)$, hence their monodromy permutations generate the monodromy group of f. Directions of departures of such collection of arcs determine a clockwise order of arcs around p (a cyclic clockwise order if p is an interior point of U).

We recall the Riemann-Hurwitz formula. The branching index of a point of multiplicity m equals $m-1$. Thus the branching index of a critical value w_i, i.e. the sum of branching indices of all points above w_i, is equal to d-(the number of points in $f^{-1}(w_i)$). The total branching index b of f is the sum of branching indices of all its critical values. The Riemann-Hurwitz formula says: $\chi(X) = d\chi(U) - b$. If $U = \mathbb{P}^1$ then the genus of X satisfies $2g = b - 2d + 2$. If U is a disk and X has s boundary components then the genus of X satisfies $2g = b - d + 2 - s$.

Lemma 2 *Let $f_1 : X_1 \to U_1$ and $f_2 : X_2 \to U_2$ be coverings of degree d with U_1 homeomorphic to U_2. Let $\alpha_1, ..., \alpha_k$ be a complete clockwise oriented arc system in*

U_1, based at p_1, and let $\beta_1, ..., \beta_k$ be a complete clockwise oriented arc system in U_2, based at p_2. We assume that the points p_1 and p_2 are both interior points or both are boundary points. Suppose that for a suitable bijection $\sigma : F_1 = f_1^{-1}(p_1) \to F_2 = f_2^{-1}(p_2)$ we have $\mu(\alpha_i) = \mu(\beta_i)$ for $i = 1, \ldots, k$. Then the coverings are equivalent.

Proof: Choose a homeomorphism $g : U_1 \to U_2$ such that $g(\alpha_i) = \beta_i$ for $i = 1, \ldots, k$. For a fixed point $a \in F_1$ choose $g(a) = \sigma(a) \in F_2$. For $x \in X_1$ which does not lie over a critical value choose a path τ_x connecting a with x. Lift the path $f_1(\tau_x)$ to the covering X_2 from the point $g(a)$. Let $g(x)$ be the end of the lifted path. The map is well defined, because of the monodromy conditions, it is a homeomorphism and can be extended uniquely to the points lying over the points of the discriminant.$\|$

Proposition 3 *Coverings $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d with no special critical values outside 0 and ∞ and with the same branching data over 0 and ∞ are equivalent.*

Proof: Functions f and g are rational functions and branching data determine multiplicities of zeros and poles of the functions. Composing f and g with fractional linear transformations we get equivalent coverings for which ∞ lies over the regular value 1. Thus $f = (\prod_{i=1}^m (z - a_i)^{p_i})/(\prod_{i=m+1}^k (z - a_i)^{p_i})$ and $g = (\prod_{i=1}^m (z - b_i)^{p_i})/(\prod_{i=m+1}^k (z - b_i)^{p_i})$ where $\sum_{i=1}^m p_i = \sum_{i=m+1}^n p_i = d$ and $a_i \neq a_j$ and $b_i \neq b_j$ for $i \neq j$. The space of k-tuples $a_1, \ldots, a_k$ for which f has special critical values outside 0 and ∞ form a complex hypersurface in the space of all k-tuples of distinct points. Thus we can find a path $f_t = (\prod_{i=1}^m (z - a_i(t))^{p_i})/(\prod_{i=m+1}^k (z - a_i(t))^{p_i})$ such that $a_i(0) = a_i$ and $a_i(1) = b_i$ for $i = 1, \ldots, k$ and f_t has no special critical values outside 0 and ∞ for $0 \leq t \leq 1$. By the Riemann-Hurwitz formula f_t has k critical values which move along a path in the configuration space W_k. The motion of the critical values different from ∞ takes place in some bounded domain B. We can fix a base point p outside B. Choose a complete arc system $\alpha_{1,t}, ..., \alpha_{k,t}$ for f_t which changes continuously with t. The points in $F_t = f_t^{-1}(p)$ also change continuously with t and the liftings of loops corresponding to the arc system always connect the same (corresponding) pairs of points in F_t. Therefore the arc system $\alpha_{1,1}, ..., \alpha_{k,1}$, corresponding to g, has the same monodromy as the arc system corresponding to f. Proposition 3 follows by Lemma 2. $\|$

Corollary 3.1 *Let $f_1 : X_1 \to U_1$ and $f_2 : X_2 \to U_2$ be coverings of degree d, such that X_1, X_2, U_1 and U_2 are disks, f_1 and f_2 have tha same branching data over one special critical value and have no other special critical values. Then f_1 and f_2 are equivalent.*

Proof: Composing with a homeomorphism we may assume that the special critical values are 0, and that U_1 and U_2 do not contain ∞. We can extend the coverings to coverings of $\mathbb{P}^1$ by $\mathbb{P}^1$ which are totally ramified over ∞ and have no other new critical values. There exist homeomorphisms g and h of $\mathbb{P}^1$ such that $gf_1 = f_2h$. In particular $g(0) = 0$, $g(\infty) = \infty$ and $g(U_1)$ is a disk containing all critical values of f_2 outside ∞. We can isotop $g(U_1)$ onto a big disk, containing U_2, by an isotopy fixed on the discriminant of f_2. Then we can isotop the big disk onto U_2 by an isotopy fixed on the discriminant of f_2. Composition of g with the two isotopies lifts to the required equivalence.||

Lemma 4 (The Riemann existence theorem) *Let $\sigma_1, \dots, \sigma_k$ be a sequence of permutations in Σ_d which generate a transitive subgroup of Σ_d. Let $\alpha_1, \dots, \alpha_k$ be a clockwise oriented sequence of simple arcs in a disk $U \in \mathbb{P}^1$, which meet only at their common end p. Then there exists a connected covering $f : X \to U$ such that $\alpha_1, \dots, \alpha_k$ form a complete arc system for f and for $i = 1, \dots, k$ we have $\mu_{\alpha_i} = \sigma_i$ for a suitable enumeration of the points in $F = f^{-1}(p)$. The number of the boundary components of X is equal to the number of disjoint cycles in $\sigma = \sigma_1 \dots \sigma_k$. The genus of X is determined by the Riemann-Hurwitz formula. If σ is trivial then f can be completed to a covering of $\mathbb{P}^1$ with no new critical values.*

Remark 1. The topological construction of the covering is obvious. The fact that a branched covering has a complex structure is due to Riemann. Because of that one could formulate the results of this paper in terms of algebraic curves and rational functions instead of topological coverings.

Corollary 4.1 *Let $f : X \to U$ be a covering of degree d of a disk by a disk with one special critical value. Then there exists a complete clockwise oriented arc system $\alpha_0, \alpha_1, \dots, \alpha_m$ based at p and an enumeration of points in $F = f^{-1}(p)$ such that*

1. *μ_{α_0} is a product of disjoint cycles $\sigma_1 = (1, \dots, p_1)$, $\sigma_2 = (p_1 + 1, \dots, p_2)$, $\dots$, $\sigma_{m+1} = (p_m + 1, \dots, d)$,*
2. *$p_1 \leq (p_2 - p_1) \leq \dots \leq (d - p_m)$,*
3. *$\mu_{\alpha_i} = (1, p_i + 1)$ for $i = 1, \dots, m$.*

Proof: Order the multiplicities of the points over the special critical value of f in the increasing order and choose cycles σ_i of the corresponding length, as in the claim of the corollary. The monodromy sequence μ_{α_0}, μ_{α_1}, $\dots$, μ_{α_m} generates a transitive subgroup of Σ_d. Therefore there exists a covering $f_1 : X_1 \to U_1$ with such data. The product of the monodromy sequence is equal to a d-cycle $(1, \dots, d)$, hence X_1 has one boundary component. We find from the Riemann-Hurwitz formula that X_1 has genus 0. Thus X_1 is a disk and f_1 has only one special critical value with the same branching data as the special critical value of f. By Corollary 3.1 there exists a homeomorphism $g : U_1 \to U$ which takes the arc system in U_1 onto a complete arc system in U with the required monodromies.||

Lemma 5 *Let $f : X \to \mathbb{P}^1$ be a covering with no special critical values outside* 0 *and* ∞. *Then there exist two embedded disjoint disks U_0 and U_∞ in $\mathbb{P}^1$ such that* $0 \in U_0$, $\infty \in U_\infty$, *and $f^{-1}(U_0)$ and $f^{-1}(U_\infty)$ are disks.*

Proof: X is connected, hence $X - f^{-1}(\infty)$ is also connected. Let V_0 be a disk neighbourhood of 0 containing no other critical values. Then $f^{-1}(V_0)$ is a union of m disjoint disks. Choose a base point $p_0 \in \partial V_0$. The monodromy $\mu_{\partial V_0}$ is a product of m disjoint cycles. Since X_1 is connected there exists a simple arc α in $\mathbb{P}^1 - V_0 - \infty$, based at p_0, such that $\mu_\alpha = (a, b)$ with a and b in different cycles of $\mu_{\partial V_0}$. If we adjoin a neighbourhood of α to V_0 we get a disk V_1 such that $f^{-1}(V_1)$ is a union of $m-1$ disks. After $m-1$ steps we get a disk U_0 such that $f^{-1}(U_0)$ is a disk. Now $X - f^{-1}(U_0)$ is connected and $\mathbb{P}^1 - U_0$ has only one special critical value ∞. We can repeat the previous argument and find the required disk U_∞.||

A pair of disks U_0, U_∞ obtained in the last lemma will be called a *partition* of a covering f. Thus a *partition* of f is a pair of disks in $\mathbb{P}^1$ such that their preimages are also disks and all critical values of f outside the disks are simple.

The following result was proven by Looijenga in [4].

Lemma 6 (Looijenga) *Let $f : X \to \mathbb{P}^1$ be a connected covering of degree d that is totally ramified over* 0 *and* ∞ *with no other special critical values. Suppose that the genus of X is positive. Then there exists a disk B in $\mathbb{P}^1 - \{0, \infty\}$ containing all other critical values of f such that for a base point $p \in \partial B$, the monodromy group of f over B is a single transposition (a', a''). Moreover, if σ is the monodromy of a simple loop in $\mathbb{P}^1 - int(B)$ around* 0, *based at p, then $a'' = \sigma^r(a')$ for some divisor r of d and f factorizes through the covering $z \in \mathbb{P}^1 \mapsto z^r \in \mathbb{P}^1$.*

Exactly the same argument gives the following corollary for partitions.

Corollary 6.1 *Let $f : X \to \mathbb{P}^1$ be a covering of degree d with no special critical values outside* 0 *and* ∞. *Suppose that the genus of X is positive. Let U_0, U_∞ be a partition of f. Then there exists a disk B in $\mathbb{P}^1 - U_0 - U_\infty$ containing all critical values of f outside U_0 and U_∞ such that for a base point $p \in \partial B$, the monodromy group of f over B is a single transposition (a, b). Moreover, if σ is the monodromy of a simple loop in $\mathbb{P}^1 - int(B)$ around U_0, based at p, then $b = \sigma^r(a)$ for some divisor r of d.*

If σ is a d-cycle and (a, b) is a transposition then there exists a positive integer r, $r \leq d/2$, such that $b = \sigma^r(a)$ or $a = \sigma^r(b)$. r is called the *mesh* of (a, b) with respect to σ. If α is a simple arc and if μ_α is a transposition then the *mesh* of α is the mesh of μ_α. The number r in the corollary is the *mesh* of the partition U_0, U_∞.

Remark 2. It follows from the proof of Looijenga that the mesh r of the partition depends only on the partition, not on the choice of B. r is equal to the minimal mesh of a simple arc in $\mathbb{P}^1 - U_0 - U_\infty$ with respect to $\mu_{\partial U_0}$. The mesh of any other simple arc in $\mathbb{P}^1 - U_0 - U_\infty$ is divisible by r.

Proposition 7 *Let $f : X \to \mathbb{P}^1$ be a covering of degree d with no special critical values outside* 0 *and* ∞ *and with genus of X positive. Then there exists a complete, clockwise oriented arc system $\alpha_0, \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_k, \gamma_\infty, \gamma_1, \ldots, \gamma_n$ based at p and an enumeration of points in $F = f^{-1}(p)$ such that*

1. *μ_{α_0} is a product of disjoint cycles $\sigma_1 = (1, 2, \ldots, p_1)$, $\sigma_2 = (p_1 + 1, \ldots, p_2)$, ..., $\sigma_{m+1} = (p_m + 1, \ldots, d)$,*
2. *$p_1 \leq (p_2 - p_1) \leq \ldots \leq (d - p_m)$,*
3. *$\mu_{\alpha_i} = (1, p_i + 1)$ for $i = 1, \ldots, m$,*
4. *$\mu_{\beta_i} = (1, r + 1)$ for $i = 1, \ldots, k$ and some divisor r of d,*
5. *μ_{γ_∞} is a product of disjoint cycles $\tau_1 = (1, d, d - 1, \ldots, q_1 + 2)$, $\tau_2 = (q_1 + 1, q_1, \ldots, q_2 + 2)$, ..., $\tau_{n+1} = (q_n + 1, q_n, \ldots, 2)$,*
6. *$(d - q_1) \leq (q_1 - q_2) \leq \ldots \leq q_n$,*
7. *$\mu_{\gamma_i} = (1, q_i + 1)$ for $i = 1, \ldots, n$,*
8. *the length of each cycle σ_i and τ_i is divisible by r.*

The monodromy group of f is generated by the d-cycle $\sigma = (1, \ldots, d)$ and the transposition $(1, r + 1)$. f factorizes through the covering $z \in \mathbb{P}^1 \mapsto z^r \in \mathbb{P}^1$. The number r is unique, determined by the monodromy group of f, and the equivalence class of f is uniquely determined by the degree, genus, branching data of 0 *and* ∞ *and the monodromy group.*

Proof: Choose a partition U_0, U_∞ for f with the smallest possible mesh r. Fix a base point $p \in \partial U_0$ and a complete arc system $\alpha_0, \alpha_1, \ldots, \alpha_m$ in U_0, based at p, as in Corollary 4.1. Now choose a disk B as in Corollary 6.1 and a complete arc system $\beta_1, \ldots, \beta_k$ in B based at a point $p_1 \in \partial B$. Now we isotop the arc system in B along a path connecting p_1 with p so that all the arcs form one arc system based at p. By Corollary 6.1 the monodromy μ_{β_i} is a fixed transposition (a, b) with mesh r with respect to $\sigma = \mu_{\partial U_0} = (1, \ldots, d)$. The monodromy is a homomorphism. If we replace a simple arc β by an arc β' which goes first along a closed path α and then proceeds along β then the loop around β' is equal $\alpha\beta\alpha^{-1}$. Therefore $\mu_{\beta'} = \mu_\alpha \mu_\beta \mu_\alpha^{-1}$. If we rotate all arcs β_i around U_0 (first counterclockwise around U_0 and then along β_i) their monodromy gets conjugate by σ and after a suitable number of turns the monodromy $\mu_{\beta_i} = (1, r + 1)$. We now turn to the disk U_∞. Choose a simple path δ which meets the arc system only at p, follows the arc β_k in the clockwise order around p, and connects p with a point of ∂U_∞. We can isotop U_∞ along δ and assume that U_∞ meets the arc system only at p. ∂U_∞ is a loop which goes around the arc system $\alpha_0, \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_k$ in a counterclockwise direction, when treated as a clockwise boundary of U_∞. $\mu_{\partial U_\infty}$ is a d-cycle, therefore k is even, $k > 0$ because X has a positive genus, and $\mu_{\partial U_\infty} = (d, d-1, \dots, 1)$. Reversing the enumeration in Corollary 4.1 we can find a complete arc system $\gamma_\infty, \gamma_1, \dots, \gamma_n$ in U_∞, based at p, which has the required monodromy, after a renumbering. But this renumbering does not change the d-cycle $\sigma^{-1} = \mu_{\partial U_\infty}$. Thus the renumbering is a conjugation by a power of σ. If we rotate the whole arc system in U_∞ around the boundary ∂U_∞ a suitable number of times, we produce the same renumbering of the monodromy. Now the complete arc system $\alpha_0, \alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_k, \gamma_\infty, \gamma_1, \dots, \gamma_n$ in $\mathbb{P}^1$ has the required monodromy.

We have to prove that the length of each cycle σ_i and τ_i is divisible by r. Suppose that the length of some cycle σ_i (the case of τ_i is similar) is equal to s and is not divisible by r. There are at least two such cycles so we may assume $s \leq d/2$. By Corollary 3.1 and Lemma 4 we can choose a new complete arc system in U_0, called again $\alpha_0, \alpha_1, \dots, \alpha_m$, with the monodromy as in Corrollary 4.1 but with the last cycle σ_{m+1} of length s. Then $\mu_{\alpha_m} = (1, d-s+1)$. We replace β_1 by an arc β, which turns once clockwise around α_m and proceeds along β_1. β precedes α_m in the clockwise order around p and $\mu_\beta = (r+1, d-s+1)$. Consider a simple loop λ based at p which goes clockwise around arcs $\alpha_0, \alpha_1, \dots, \alpha_{m-1}$. λ bounds a disk V and μ_λ is a product of two disjoint cycles $(1, \dots, d-s)$ and $(d-s+1, d)$. It follows from the construction in the proof of Lemma 3.1 that $f^{-1}(V)$ is a disjoint union of two disks. The monodromy μ_β connects the two cycles of μ_λ, therefore a simple loop around V and β bounds a disk W such that $f^{-1}(W)$ is a disk. So W and U_∞ form a new partition for f. $\sigma' = \mu_{\partial W} = (1, \dots, r, d-s+1, \dots, d-1, d, r+1, \dots, d-s)$. Arcs α_m and β_2 are outside W and U_∞ and have mesh r and $r+s$ with respect to σ'. By Remark 2 the mesh of the new partition must be smaller than r, which contradicts the choice of the partition. So all cycles σ_i and τ_i have length divisible by r. In particular each p_i and q_i is divisible by r. Clearly the monodromy of each arc in the arc system is a product of powers of σ and conjugates of $(1, r+1)$ by powers of σ. Thus σ and $(1, r+1)$ generate the monodromy group of f. The monodromy group determines uniquely the number r. The monodromy sequence in the statement of the proposition is uniquely determined by r and by the branching data of 0 and ∞. By Lemma 2 the equivalence class of f is determined by d, genus of X, the branching data and the monodromy group.

Consider the covering $h : z \in \mathbb{P}^1 \mapsto z^r \in \mathbb{P}^1$. f factorizes through h if for every closed path α in $X - f^{-1}(D)$ the lifting of $f(\alpha)$ by h is a closed path. Let ω be a loop in $\mathbb{P}^1 - \{0, \infty\}$ which turns r times around 0. A loop α in $\mathbb{P}^1$ lifts to a closed loop in X if its monodromy fixes a point and it lifts to a closed loop in $\mathbb{P}^1$ if it is a power of ω in $\mathbb{P}^1 - \{0, \infty\}$. Every loop in $\mathbb{P}^1 - discriminant(f)$ is a product of loops around the arcs of the complete arc system. A loop around γ_∞ is a product of the others, so we may ignore it. μ_{α_0} moves each point by 1 modulo r and the monodromy of any other arc of the arc system leaves all points fixed modulo r. Thus μ_α may fix a point only if α winds tr times around 0, i.e. α is a power of ω in $\mathbb{P}^1 - \{0, \infty\}$. It follows that f factorizes through h.||

Proof of Theorem 1. The objects in Theorem 1 were not described precisely so we shall start with some definitions. Two coverings f_1 and f_2 define the same point in $\mathcal{Z}$ if they have the same critical values and for a fixed complete arc system they have identical monodromies, up to renumbering. Thus they are equivalent (see Definition) with g equal to the identity and with h — a complex isomorphism for the complex structures induced by f_1 and f_2. A point in $\mathcal{Z}$ is mapped onto the set of its critical values — a point in W_k. For a point w in W_k and a fixed complete arc system there is a finite number of possible monodromies and hence a finite number of points in $\mathcal{Z}$ over w. We define a topology in $\mathcal{Z}$ as follows. For a fixed covering f we have a fixed discriminant w and for a fixed arc system Γ we have a fixed monodromy sequence. For w_1 very near w there is a unique, up to isotopy relative p and w_1, arc system Γ_1 very near Γ. To Γ_1 we assign the same monodromy sequence as to Γ. By Lemma 4 and the definitions this defines a unique point in $\mathcal{Z}$ over w_1. In this way to a small neighbourhood N of w corresponds a subset M of $\mathcal{Z}$ which covers N in a one-one way. This is a neighbourhood of f in $\mathcal{Z}$. This topology defines $\mathcal{Z}$ as a finite covering of W_k. Two equivalent coverings f_1 and f_2 lie in the same connected component of $\mathcal{Z}$. Indeed the homeomorphism g (see Definition) is isotopic to the identity in $\mathbb{P}^1$. (If g fixes two points 0 and ∞ we can find an isotopy relative to these two points.) Restriction of the isotopy to the discriminant of f_1 defines a path in W_k. The lifting of this path to $\mathcal{Z}$ from f_1 connects f_1 and f_2 in $\mathcal{Z}$ (as in the proof of Proposition 3). Covering with only two special critical values is equivalent , after a composition with a homeomorphism, to a covering with no special critical values outside 0 and ∞. The theorem follows from Proposition 7.||

Example.

Consider two triples of permutations in Σ_8:

$\sigma_1 = (1,2,3)(5,6,7,8)$, $\sigma_2 = (1,2)(3,4,5)$, $\sigma_3 = (\sigma_1\sigma_2)^{-1} = (8,7,6,5,4,2,3)$,

and $\tau_1 = (1,2,3)(5,6,7,8)$, $\tau_2 = (7,8)(3,4,5)$, $\tau_3 = (\tau_1\tau_2)^{-1} = (8,6,5,4,2,1,3)$.

Each triple has the trivial product. Each triple is determined by its first two permutations and it generates the whole group Σ_8. Let us fix three arcs α_1, α_2, α_3 in $\mathbb{P}^1$ meeting only at their common end point p. By Lemma 4 each triple defines a connected covering of $\mathbb{P}^1$. The coverings have the same degree and genus, the same branching data and the same monodromy group. We shall prove that the coverings are not equivalent. In the first pair σ_1 is a product of a 3-cycle and a 4-cycle. σ_2 is a product of a transposition, which belongs to the 3-cycle of σ_1, and a 3-cycle. So the first pair is not conjugate to the second pair, in which the transposition belongs to the 4-cycle of the first permutation.

Consider now the Hurwitz action. Since the branching datas of different critical values are different we should consider paths in W_3 which do not permute the points. The group of these paths is the pure braid group of a sphere with three strings. It is generated by a full (360°) rotation around the first two critical values. By the definition of the topology in $\mathcal{Z}$ the monodromy μ' of the new covering has old values on the new arc system $\beta_1 = \alpha_1\alpha_2\alpha_1\alpha_2^{-1}\alpha_1^{-1}$ and $\beta_2 = \alpha_1\alpha_2\alpha_1^{-1}$. Thus $\mu'_{\beta_i} = \mu_{\alpha_i}$ for $i = 1, 2$. Since μ' is a homomorphism we get $\mu'_{\alpha_1} = \sigma_2^{-1}\sigma_1\sigma_2$ and $\mu'_{\alpha_2} = \sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1\sigma_2$. So the new monodromy pair is conjugated to the original pair. We never get the second pair by the Hurwitz action so the coverings belong to different orbits and are not equivalent.

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