THE CENTRAL SERIES FOR PEIFFER COMMUTATORS IN GROUPS WITH OPERATORS

by

Hans Joachim Baues and Daniel Conduché

H.J. Baues
Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3
Deutschland

D. Conduché
Université de Rennes I
U.E.R. Mathématiques et
Informatique
Campus de Beaulieu
35042 Rennes Cedex
France

MPI/87-49

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The central series of groups and Lie algebras play a fundamental rôle in the algebraic topology of simply connected spaces, see for example [8], [18]. These concepts, however, do not lend themselves to deal with non simply connected spaces; and topologists have felt for a long time that there is a lack of methods for studying spaces with non trivial fundamental group.

The first two groups

$$\begin{tikzcd} G_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & G_0 \arrow[l] \end{tikzcd}$$

of a simplicical group are equivalent to a pre-crossed module

$$d_1 : \text{kernel } (d_0) \to G_0$$

where kernel (d_0) is a group with operators in G_0. Therefore a pre-crossed module corresponds exactly to that low dimensional part of a space which gives a presentation of the fundamental group.

In this paper we study the central series in a pre-crossed module defined by Peiffer commutators. Our principal objective is an extension of the following classical result of Witt [23].

Theorem A: *The quotients of the central series of a group form a Lie algebra which is a free Lie algebra provided the group is free.*

Theorem A is a basic tool in group theory and topology. For example, it is used by Curtis [8] as a starting point for his results on reduced simplicial groups and simply connected spaces. The following generalization of theorem A allows an extension of such results to spaces which are not simply connected.

Theorem B: *The quotients of the Peiffer central series form a partial Lie algebra which is a free partial Lie algebra provided the pre-crossed module is free with a free group of operators.*

Most of the paper is devoted to the proof of Theorem B. In case the group of operators is trivial the Peiffer central series is the same as the central series of a group and then Theorem B coincides with Witt's theorem above.

First applications of Theorem B are contained in [2], [3], and [7]. The restriction of theorem B to the second quotient of the

Peiffer central series was obtained in [2] by a geometric argument and is used there for the construction of minimal algebraic models of 4-dimensional CW-complexes. Moreover, we derive from Theorem B small algebraic models of 3-types by the methods in [6], see [3]. These examples show that Theorem B can be used for the construction of small algebraic models of spaces which avoid the redundant complexity in a simplicial group.

The authors want to acknowledge the support of the University of Rennes and the Max-Planck-Institut für Mathematik, Bonn.

A.M.S. Classification: 20F; secondary: 1713; SSP; 57M

Key Words: Pre crossed modules; Peiffer commutators; commutators Peiffer (lower) central series; (lower) central series; Partial Lie algebra; Lie algebra.

§1 The Peiffer central series

We fix some notation on commutators and Peiffer commutators and we introduce the Peiffer central series of a pre crossed module. Such (pre) crossed modules arise naturally from presentation of groups and from relative homotopy groups.

In this paper the group structure $+, -, 0$ of a *group* M is written additively though addition $+$ in M needs not to be abelian. The element 0 denotes the neutral element in M. An N-*group* M is given by an action of the group N on M denoted by x^α for $x \in M$, $\alpha \in N$. We have

$$\begin{cases} (x+y)^\alpha = x^\alpha + y^\alpha, & (-x)^\alpha = -x^\alpha \\ x^{\alpha+\beta} = (x^\alpha)^\beta, & x^0 = x \end{cases}$$

for $x,y \in M$, $\alpha,\beta \in N$. A *pre crossed module* $\partial : M \to N$ is a group homomorphism together with an action of N on M satisfying

$$\partial(x^\alpha) = -\alpha + \partial(x) + \alpha. \tag{1.1}$$

This is a *crossed module* if in addition

$$x^{\partial y} = -y + x + y \quad . \tag{1.2}$$

A *map* $f : \partial \to \partial'$ between pre crossed modules is a pair of homomorphisms $f = (m : M \to M', n : N \to N)$ with $\partial' m = n\partial$ and $m(x^{\alpha}) = (mx)^{n\alpha}$.

For a set Z let $<Z>$ be the *free group* generated by Z. The *free N-group* generated by Z is the free group $<Z \times N>$ generated by the product set $Z \times N$; the action is given on generators by

$$(1.3) \qquad (x,\alpha)^{\beta} = (x,\alpha + \beta)$$

for $x \in Z$, $\alpha,\beta \in N$. Moreover, $M \to N$ is a free *pre crossed module* if and only if M is a free N-group. In this case the associated crossed module (see (1.15) below) is a *free crossed module*.

(1.4) Example: Let $G = <a_1,\ldots a_n \mid R_1,\ldots,R_m>$ be a *presentation* of the group G with generators a_i and relations R_j. Then we have

$$G = \text{cokernel } (\partial : M \to N)$$

where ∂ is the following pre crossed module. The group $N = <Z_1>$ is the free group generated by the set $Z_1 = \{a_1,\ldots,a_n\}$ and the group M is the free N-group generated by the set $Z_2 = \{R_1,\ldots,R_m\}$. The homomorphism ∂ carries the generator (x,α) in $M = <Z_2 \times N>$ to $-\alpha + R + \alpha \in N$ where R is the relation given by $x \in Z_2$. Whence ∂ is a free pre crossed module on a free group N.

(1.5) Example: Let (X,A) be a pair of topological spaces with basepoint $* \in A$. Then the relative homotopy group $\pi_2(X,A)$ with the boundary homomorphisms

$$\partial_{X,A} : \pi_2(X,A) \rightarrow \pi_1(A)$$

is a crossed module by the usual action of $\pi_1(A)$ on $\pi_2(X,A)$, (such a result even holds in any cofibration category, see (II.7.16) in [1]). This example was considered by J.H.C. Whitehead in [22] where he introduced the notion of a crossed module, see also [19]. Whitehead proved that $\partial_{X,A}$ is actually a "free" crossed module provided X is obtained from A by attaching 2-cells. See also (1.19) below.

We denote the *action via conjugation* in a group M by

$$(1.6) \qquad x^y = -y + x + y \qquad (x,y \in M).$$

Hence we define the *commutator* by

$$(1.7) \qquad (x,y) = -x -y + x + y = -x + x^y.$$

The group M is abelian if $(x,y) = 0$ for all $x,y \in M$. In a pre crossed module $\partial : M \rightarrow N$ we have the *Peiffer commutator*

$$(1.8) \qquad \langle x,y\rangle = -x -y + x + y^{\partial x} \quad .$$

Thus $\partial : M \to N$ is a crossed module if and only if $\langle x,y\rangle = 0$ for all $x,y \in M$. Peiffer commutators are studied, for example in [12], [15], [16], [17], [19] and [20]. They are highly connected with deep problems like the Whitehead question and the Andrews-Curtis conjecture, see for example [5], [16].

We have the following identities for commutators and Peiffer commutators respectively.

(1.9) Witt Hall identities for commutators

Let $x,y,z \in M$, $\alpha \in N$

(W1) $$(x,y)^{\alpha} = (x^{\alpha},y^{\alpha})$$

(W2) $$\begin{aligned}(x,y+z) &= (x,z) - z + (x,y) + z \\ &= (x,z) + (x,y) + ((x,y),z)\end{aligned}$$

(W3) $$\begin{aligned}(x+y,z) &= -y + (x,z) + y + (y,z) \\ &= (x,z) + ((x,z),y) + (y,z)\end{aligned}$$

(W4) $$(x,y) + (y,x) = 0, \quad (x,x) = 0$$

(W5) $$((x,y),z^{x}) + ((z,x),y^{z}) + ((y,z),x^{y}) = 0$$

(W6) $$\begin{aligned}&((x,y),z) + ((y,z),x) + ((z,x),y) = \\ &(y,x) + (z,x) + (z,y)^{x} + (x,y) + (x,z)^{y} \\ &+ (y,z)^{x} + (x,z) + (z,x)^{y}\end{aligned}$$

Here (W5), (W6) are actually consequences of (W2), (W3) and (W4), see [14].

(1.10) Identities for Peiffer commutators

Let $x,y,z \in M$, $\alpha \in N$

(P1) $$\langle x,y\rangle^{\alpha} = \langle x^{\alpha},y^{\alpha}\rangle$$

(P2) (a) $$\langle x,y+z\rangle = \langle x,z\rangle - z^{\partial x} + \langle x,y\rangle + z^{\partial x}$$

(b) $$= \langle x,z\rangle + \langle x,y\rangle + \langle\langle x,y\rangle,z^{\partial x}\rangle$$

(P3) (a) $$\langle x+y,z\rangle = -y + \langle x,z\rangle + y + \langle y,z^{\partial x}\rangle$$

(b) $$= \langle x,z\rangle + \langle\langle x,z\rangle,y\rangle + \langle y,z^{\partial x}\rangle$$

(P4) Let $k \in M$ with $\partial k = 0$ then

(a) $$\langle k,x\rangle = (k,x)$$

(b) $$\langle k,x\rangle + \langle x,k\rangle = -k + k^{\partial x}$$

(P5) (a) $$-\langle x,y\rangle = -y^{\partial x} + \langle x,-y\rangle + y^{\partial x}$$

(b) $$= \langle x,-y\rangle + \langle\langle x,-y\rangle,y^{\partial x}\rangle$$

(c) $$= -x + \langle -x,y^{\partial x}\rangle + x$$

(d) $$= \langle -x,y^{\partial x}\rangle + \langle\langle -x,y^{\partial x}\rangle,x\rangle$$

Here (P5) follows from (P2) and (P3).

A N-*subgroup* $K \subset M$ is a subgroup satisfying $k^{\alpha} \in K$ for $k \in K$ and $\alpha \in N$. For subgroups K_0, K_1 of M let

(1.11) $$K_0 + K_1 \subset M$$

be the subgroup generated by elements $k_0 + k_1$ with $k_0 \in K_0$, $k_1 \in K_1$. Similarly let $(K_0,K_1) \subset M$ and $<K_0,K_1> \subset M$ be the subgroups generated by commutators (k_0,k_1) and by Peiffer commutators $<k_0,k_1>$ respectively. In case K_0 and K_1 are N-subgroups then $K_0 + K_1$, (K_0,K_1) and $<K_0,K_1>$ are N-subgroups. If K_0 and K_1 are normal subgroups then also $K_0 + K_1$ and (K_0,K_1) are normal.

(1.12) Definition: The (lower) *central series* $\Gamma_n = \Gamma_n(M)$,

$$\Gamma_{n+1} \subset \Gamma_n \subset \ldots \subset \Gamma_2 \subset \Gamma_1 = M,$$

of the group M is defined inductively by

$$\Gamma_n = \sum_{i+j=n} (\Gamma_i,\Gamma_j)$$

where the sum is defined as in (1.11). By (1.9) we have $\Gamma_n = (\Gamma_{n-1},M)$.

Similarly we introduce the following

(1.13) Definition: The (lower) *Peiffer central series* $P_n = P_n(M \to N)$,

$$P_{n+1} \subset P_n \subset \ldots \subset P_2 \subset P_1 = M$$

of the pre crossed module $\partial : M \to N$ is defined inductively by

$$P_n = \sum_{i+j=n} \langle P_i, P_j \rangle \quad .$$

Clearly $\partial(P_n) = 0$ for $n \geq 2$. It follows from (P5) that P_n is a normal subgroup of M. We will prove in (2.11) that $P_n = \langle P_{n-1}, M \rangle + \langle M, P_{n-1} \rangle$.

The group $\Gamma_2(M) = (M,M)$ is the *commutator subgroup* of M and the quotient

$$M^{ab} = M/\Gamma_2(M) \tag{1.14}$$

is the *abelianization* of M. The group $P_2(\partial) = \langle M,M \rangle$ is the *Peiffer subgroup* of the pre crossed module $\partial : M \to N$ and the homorphism

$$M^{cr} = M/P_2(\partial) \xrightarrow{\partial^{cr}} N, \tag{1.15}$$

induced by ∂, is the *crossed module associated to* the pre crossed module ∂, (we also write $\partial^{cr} = \partial$). Let $\{x\} \in M^{cr}$ be the coset represented by $x \in M$. Clearly the action of N on M^{cr} is given by $\{x\}^{\alpha} = \{x^{\alpha}\}$.

(1.16) Remark: Suppose that $\partial = 0$ is the trivial homorphism. Then the central series and the Peiffer central series of M coincide, that is $P_n(M \to N) = \Gamma_n(M)$.

Next we consider the connection of free pre crossed modules with 2-dimensional CW-complexes. Let $\partial : M \to N$ be a free pre crossed module on a free group $N = \langle Z_1 \rangle$ so that $M = \langle Z_2 \times N \rangle$ is the free N-group generated by a set Z_2. Let

(1.17)
$$X^1 = \bigvee_{Z_1} S^1$$

be a one point union of 1-spheres $S^1 = S^1_x$, $x \in Z_1$. It is well known that the fundamental group $\pi_1(X^1)$ is the free group $\langle Z_1 \rangle$. Let

(1.18)
$$f : \bigvee_{Z_2} S^1 \longrightarrow X^1$$

be a map which induces the composition $\pi_1(f) : \langle Z_2 \rangle \subset M \xrightarrow{\partial} N = \langle Z_1 \rangle$ on fundamental groups. The mapping cone

(1.19)
$$X = C_f = X^1 \cup_f \bigvee_{Z_2} CS^1$$

of f is a *2-dimensional CW-complex associated to* ∂. Here CS^1 is the cone of S^1 or equivalently a 2-disk. By a result of J.H.C. Whitehead [22] we have isomorphic crossed modules

(1.20)
$$\begin{tikzcd} M^{cr} \arrow[r, "\partial^{cr}"] \arrow[d, equal] & N \arrow[d, equal] \\ \pi_2(X,X^1) \arrow[r, "\partial"] & \pi_1(X^1), \end{tikzcd}$$

see (1.5) and (1.15). The isomorphism carries an element $x \in Z_2$ to the element in $\pi_2(X,X^1)$ represented by $(CS^1_x, S^1_x) \to (X,X^1)$. Clearly (1.20) yields an isomorphism of cokernels

$$\pi = \operatorname{cok}(\partial^{cr}) = \operatorname{cok}(\partial) = \pi_1(X). \tag{1.21}$$

We shall use the following commutative diagram which is completely determined by ∂.

(1.22)

$$\begin{tikzcd}
 & M \arrow[d, "p"] \arrow[dr, "\partial"] & & & \\
\ker(\partial^{cr}) \arrow[r, phantom, "\subset"] \arrow[d, "\cong"'] & M^{cr} \arrow[r, "\partial^{cr}"] \arrow[d, "h_2"] & N \arrow[r, two heads, "q"] \arrow[d, "h_1"] & \pi & \\
K \arrow[r, phantom, "\subset"] & C_2 \arrow[r, "d_2"] & C_1 \arrow[r, "d_1"] & C_0 \arrow[r, "\epsilon"] & \mathbb{Z}
\end{tikzcd}$$

Here $C_0 = \mathbb{Z}[\pi]$ is the group ring of π and C_n is the free π-module generated by Z_n, $(n = 1,2)$. The maps p and q are the quotient maps. The map h_2 is the q-equivariant homomorphism with $h_2(x) = x$ for $x \in Z_2$. The function h_1 is uniquely determined by

$$\begin{cases} h_1(y) = y & \text{for} \quad y \in Z_1 \\ h_1(a+b) = h_1(a)^{q(b)} + h_1(b) & \text{for} \quad a,b \in N \end{cases} \tag{1.23}$$

The second equation says that X_1 is a q-*crossed homomorphism*. The bottom row of (1.22) is an exact sequence of π-modules and of π-equivariant homomorphisms. In fact, d_2 is the unique map which extends the diagram commutatively and d_1 is given by $d_1(y) = 1 - [q(y)]$. Here $[\alpha] \in \mathbb{Z}[\pi]$ is the generator for $\alpha \in \pi$. The map ϵ is the augmentation with $\epsilon[\alpha] = 1$. The map h_2 induces an isomorphism of π-modules

$$(1.24) \qquad h_2 : \ker(\partial^{cr}) \xrightarrow{\cong} K = \ker(d_2)$$

Moreover, the kernel of h_2 is the commutator subgroup of M^{cr} so that

$$(1.25) \qquad C = (M^{cr})^{ab} = C_2.$$

Compare [22] or VI, §1 in [1]. Diagram (1.22) has the following geometrical interpretation. Let $p : \hat{X} \to X$ be the *universal covering* of X. Then the bottom row of (1.22) coincides with the *cellular chain complex of* $\hat{X}$ with

$$(1.26) \qquad C_n = H_n(\hat{X}^n, \hat{X}^{n-1}).$$

Now $h_n = hp_*^{-1}$ is given by the Hurewicz map h. In particular, the isomorphism (1.24) coincides with the well known *Hurewicz isomorphism*

$$(1.27) \qquad h_2 = hp_*^{-1} : \pi_2(X) \cong \pi_2(\hat{X}) \cong H_2(\hat{X}) .$$

We derive from (1.22) the

(1.28) Lemma: *The group* K *is a direct summand of the free abelian group* C.

The π-module K, however, needs not to be a direct summand of the π-module C.

§ 2 Iterated brackets

In this section we describe some results on "iterated brackets" obtained by forming commutators and Peiffer commutators respectively. In particular, a kind of Jacobi identity for Peiffer commutators is proved.

Commutators $(\, , \,)$ and Peiffer commutators $< \, , \, >$ in M are two different binary operations $M \times M \to M$. This leads to the following definitions.

(2.1) Definition: A *magma* is a pair $(M,[,])$ where M is a set and where $[\, , \,] : M \times M \to M$ is a function which carries a pair of elements $(x,y) \in M \times M$ to $[x,y] \in M$. For a set X let $B(X)$ be the *free* magma generated by X. The elements of $B(X)$ are the iterated brackets

$$[x_1,\dots,x_n]_c$$

of length $n \geq 1$, $x_1 \in X$. Here c is the "type" of the bracket. See [4].

(2.2) Definition: A *double magma* is a triple $(M,<,>,(,))$ where M is a set and where $< \, , \, >$ and $(\, , \,)$ are functions $M \times M \to M$. For a set X let $B^2(X)$ be the *free* double magma generated by X. The elements of $B^2(X)$ are iterated double brackets of length $n \geq 1$ which we denote by

$$<x_1,\dots,x_n)_c, \quad x_i \in X .$$

These are obtained inductively by forming brackets either by $(\, , \,)$ or by $< \, , \, >$. For $n = 1$ we set $<x_1>_c = x_1$. The brackets

$$(x_1,\dots,x_n)_c, \quad \text{resp. } <x_1,\dots,x_n>_c,$$

denote brackets of length n inductively formed only by $(\, , \,)$, resp. by $< \, , \, >$. This corresponds to the two inclusions of $B(X)$ in $B^2(X)$.

For a pre crossed module $\partial : M \to N$ we have the canonical map between double magmas

$$(2.3) \qquad B^2(M) \to M$$

which extends the identity of M and which clearly carries $(\, , \,)$ to a commutator and carries $< \, , \, >$ to a Peiffer commutator. We denote the image of a bracket $<x_1,\dots,x_n>_c \in B^2(M)$ by the same symbol.

A double bracket in $B^2(M)$ is *special* if the outside bracket is of the form $< \, , \, >$. For example

$$(2.4) \qquad <(x_1,<x_2,x_3>),x_4>$$

is a special double bracket. By definition in (1.13) the group $P_n = P_n(M \to N)$ is generated by all iterated Peiffer brackets $<x_1,\ldots,x_n>_c$ of length n, $x_i \in M$. This follows inducitively from (P2)(b) and (P3)(b). Moreover we get

(2.5) Proposition: *All special brackets of length* n *are elements of* P_n.

We prove this in (2.14) below. Let

$$(2.6) \qquad Q_n \subset M$$

be the subgroup of M generated by all double brackets of length n. We derive from (2.5) the

(2.7) Corollary: $Q_n = P_n + \Gamma_n(M)$

Proof: The lemma holds for $n \leq 2$. Assume that it holds for $n < m$. Then we get

$$Q_m = P_m + \sum_{i+j=m} (Q_i, Q_j)$$

by definition of Q_m and by (2.5). Here we have

$$\begin{aligned}(Q_i,Q_j) &= (P_i+\Gamma_i,P_j+\Gamma_j)\\ &= (P_i,Q_j)+(\Gamma_i,P_j)+(\Gamma_i,\Gamma_j)\\ &= \langle P_i,Q_j\rangle+\langle P_j,\Gamma_i\rangle+(\Gamma_i,\Gamma_j)\\ &\subset P_m+\Gamma_m\end{aligned}$$

□

The commutator in M induces the functions

(2.8)
$$\begin{tikzcd} \Gamma_n \times \Gamma_m \arrow[r, "{(\ ,\ )}"] \arrow[d, phantom, "\cap"] & \Gamma_{n+m} \arrow[d, phantom, "\cap"] \\ Q_n \times Q_m \arrow[r, "{(\ ,\ )}"] & Q_{m+n}. \end{tikzcd}$$

Moreover, the Peiffer commutator yields by (2.5) the functions

(2.9)
$$\begin{tikzcd} P_n \times P_m \arrow[r, "{<\ ,\ >}"] \arrow[d, phantom, "\cap"] & P_{n+m} \arrow[d, phantom, "\parallel"] \\ Q_n \times Q_m \arrow[r, "{<\ ,\ >}"] & P_{m+n}. \end{tikzcd}$$

For the proof of (2.5) we need a lemma which is a kind of a Jacobi identity. To this end we introduce the following notation. For $x,y,z \in M$ let

(2.10) $$\Lambda_k(x,y,z) \subset M, \qquad k \geq 3,$$

be the subgroup generated by all Peiffer brackets $<x_1,\dots,x_n>_c$ with the following properties (a), (b), and (c).

(a) $n \geq k$,

(b) $x_i \in \{\pm x^\alpha, \pm y^\beta, \pm z^\gamma : \alpha,\beta,\gamma \in \partial M\}$ for $i = 1,\dots,n$,

(c) there exist $i_1, i_2, i_3 \in \{1,\dots,n\}$ with $x_{i_1} = \pm x^\alpha$, $x_{i_2} = \pm y^\beta$, $x_{i_3} = \pm z^\gamma$.

(2.11) Lemma: *For* $x,y,z \in M$ *we have the equations*

(1) $<x,(y,z)> = <<x,y>,z> - <<x,z>,y> + \lambda$

(2) $<(x,y),z> - <<x,y>,z> + <x,<y,z>> =$
$\qquad - <<y,x>,z> + <y,<x,z>> + \lambda'$

with $\lambda,\lambda' \in \Lambda_4(x,y,z)$.

(3) *For* $n \geq 2$ *we have* $P_n = <M,P_{n-1}> + <P_{n-1},M>$.

(2.12) Remark: If $\partial y = 0$ we have $(y,z) = <y,z>$ so that (2.11)(1) is a *Jacobi identity* for Peiffer brackets in this case. On the other hand if $\partial x = 0$ we have $(x,y) = <x,y>$ so that the first two terms of (2.11)(2) cancel yielding the equation

$$<x,<y,z>> = -<<y,x>,z> + <y,<x,z>> + \lambda' .$$

This equation with $\partial x = 0$ is equivalent to equation (2.11)(1) with $\partial y = 0$ since we can exchange x and y.

□

Proof of (2.11): In the following the elements λ_i $(i = 1,\dots,10)$ lie in $\Lambda_4(x,y,z)$. We use the linearity rules in (1.10) which yield an expansion $(\bar{z} = -z,\ \bar{y} = -y)$

(4) $<x, - y - z + y + z> = -<x,\bar{z}> - <x,\bar{y}>$
$+ <x,\bar{z}> + <x,\bar{y}> + \Delta + \lambda_1$
where $\Delta = <<x,y>,z> - <<x,z>,y>$,

In (4) the commutator of $<x,\bar{z}>$ and $<x,\bar{y}>$ is an element in Λ_4 since

(5) $(<x,\bar{z}>,<x,\bar{y}>) = <<x,\bar{z}>>,<x,\bar{y}>>$.

Whence (2.11) (1) is proved if we check (4). For this we get by (P2)(a)

(6) $<x,(y,z)> = <x,(\bar{y} + \bar{z}) + (y + z)>$
$= <x,y + z> - (y + z)^{\partial x} + <x,\bar{y} + \bar{z}> + (y + z)^{\partial x}$.

Here we use (P2)(b) twice and we get

(7) $<x,y + z> = <x,z> + <x,y> + <<x,y>,z^{\partial x}>$,

(8) $<x,\bar{y} + \bar{z}> = <x,\bar{z}> + <x,\bar{y}> + <<x,\bar{y}>,\bar{z}^{\partial x}>$.

In (7) we can replace $<x,z>$ and $<x,y>$ by the formulas (see (P5)(a))

(9) $<x,z> = -z^{\partial x} - <x,\bar{z}> + z^{\partial x}$

(10) $<x,y> = -y^{\partial x} - <x,\overline{y}> + y^{\partial x}$

We first use (9). Whence we get by (b)...(a) the equations

(11) $<x,(y,z)> = -z^{\partial x} - <x,\overline{z}> + z^{\partial x} + <x,y> +$

$<<x,y>,z^{\partial x}> - z^{\partial x} - y^{\partial x} +$

$<x,\overline{z}> + <x,\overline{y}> + <<x,\overline{y}>,\overline{z}^{\partial x}> + y^{\partial x} + z^{\partial x}$

By (P4)(a) we get

(12) $<x,(y,z)> = A + <<x,y>,z^{\partial x}> + <<x,\overline{y}>,\overline{z}^{\partial x}> + \lambda_2$

with

$$\begin{cases} A = -z^{\partial x} - <x,\overline{z}> + z^{\partial x} + <x,y> - z^{\partial x} - y^{\partial x} + B \\ B = <x,\overline{z}> + <x,\overline{y}> + y^{\partial x} + z^{\partial x} \end{cases}$$

Here we have by (P4)(a)

(13) $<x,y> - z^{\partial x} = -z^{\partial x} + <x,y> + <<x,y>,-z^{\partial x}>$

so that

(14) $A = -z^{\partial x} - <x,\overline{z}> + <x,y> + <<x,y>,-z^{\partial x}> - y^{\partial x} + B$

(15) $$\begin{cases} = -z^{\partial x} - <x,\overline{z}> - y^{\partial x} - <x,\overline{y}> + B \\ \qquad + <<x,y>,-z^{\partial x}> + \lambda_3. \end{cases}$$

In (15) we use (10) and (P4)(a).

As in (13) we get

$y^{\partial x} + <x,\overline{z}> = <x,\overline{z}> + y^{\partial x} - <<x,\overline{z}>,y^{\partial x}>$

so that by (15) and (P4)(a)

(16) $$\begin{cases} A = -z^{\partial x} - y^{\partial x} - <x,\overline{z}> - <x,\overline{y}> + B \\ \qquad +<<x,y>,-z^{\partial x}> + <<x,\overline{z}>,y^{\partial x}> + \lambda_4 \end{cases}$$

Now (5) and the definiiton of B yield by (12)

$$(17)\quad \begin{cases} \langle x,(y,z)\rangle = \langle\langle x,y\rangle,z^{\partial x}\rangle + \langle\langle x,\bar{y}\rangle,\bar{z}^{\partial x}\rangle \\ \qquad\qquad + \langle\langle x,y\rangle,-z^{\partial x}\rangle + \langle\langle x,\bar{z}\rangle,y^{\partial x}\rangle + \lambda_5. \end{cases}$$

By (P5)(b) we have

$$(18)\quad \langle x,(y,z)\rangle = \langle\langle x,\bar{y}\rangle,\bar{z}^{\partial x}\rangle + \langle\langle x,\bar{z}\rangle,y^{\partial x}\rangle + \lambda_6$$

Here one gets

$$(19)\quad \langle\langle x,\bar{y}\rangle,\bar{z}^{\partial x}\rangle = \langle\langle x,y\rangle,z^{\partial x}\rangle + \lambda_7$$

$$(20)\quad \langle\langle x,\bar{z}\rangle,y^{\partial x}\rangle = -\langle\langle x,z\rangle,y^{\partial x}\rangle + \lambda_8$$

These equation follow from (P5) and (P3)(b).

Next we have

$$(21)\quad \langle\langle x,y\rangle,z^{\partial x}\rangle = \langle\langle x,y\rangle,z\rangle + \lambda_9$$

For this consider $z^{\partial x} = z + (z,x) + \langle x,z\rangle$ so that by (P2)(b) we get

$$(22)\quad \langle\langle x,y\rangle,z^{\partial x}\rangle = \langle\langle x,y\rangle,z\rangle + \langle\langle x,y\rangle,(z,x)\rangle + \lambda_{10}$$

Here we have $\langle\langle x,y\rangle,(z,x)\rangle \in \Lambda_4$ by (18). By (18), (19), (20) and (21) equation (1) and (2.11) is proved. Equation (2) in (2.11) can be proved in the same way. Next we prove equation (3). Equivalent to (18) we have

$$(23)\quad \langle\langle x,z\rangle,y\rangle = \langle x,(y^{\partial x},-z)\rangle - \langle\langle x,-y^{-\partial x}\rangle,z^{\partial x}\rangle + \lambda_6'$$

We use this formula for the proof of

(24) $<P_{i+k},P_j> \subset <P_{i+j},P_k> + <P_i,P_{j+k}>$
where $j \geq 2$ and $j \geq k$.

Let $x \in P_i$, $y \in P_j$, $z \in P_k$, then
$<<x,-y^{-\partial x}>,z^{\partial x}> \in <P_{i+j},P_k>$
and for $j \geq 2$ we have $(y^{-\partial x},-z) = <y^{-\partial x},-z>$ so that
$<x,(y^{-\partial x},z)> \in <P_i,P_{j+k}>$.

It remains to consider λ_6' in (23) or equivalently λ_6 in (18). The calculus from (6) to (18) shows that λ_6 is created by commuting terms in P_{i+j+k} with other terms, which gives elements in $<P_{i+j+k},P_\ell>$ with $\ell \geq j$ or $\ell \geq k$, and by terms of the form
$<<x,\bar{y}>,<x,\bar{z}>> \in <P_{i+j},P_{i+k}>$.
So if $j \geq 2$ and $j \geq k$ we get the inclusion (24).

We now can use (24) for the proof of (3). If $n \geq m \geq 2$ we have $<P_n,P_m> = <P_m,P_n>$ since $\partial P_n = \partial P_m = 0$. Now take $j = n$, $i + k = m$ in (24). An easy induction on m shows

(25) $<P_n,P_m> \subset <P_1,P_{n+m-1}> + <P_{n+m-1},P_1>$
Now (1.13) yields (3).

□

(2.13) Corollary: *Consider elements* $x,y \in M$ *satisfying one of the conditions (1), (2) or (3) respectively.*

(1) $x \in P_i$, $y \in \Gamma_j$
(2) $x \in \Gamma_i$, $y \in P_j$
(3) $x \in \Gamma_i$, $y \in \Gamma_j$.

Then $<x,y> \in P_{i+j}$.

Proof: We prove (1) inductively. For this consider $y = (a,b)$ with $a \in \Gamma_r$, $b \in \Gamma_s$ $(r + s = j)$. Then (2.11)(1) shows

$$(4) \quad <x,(a,b)> = <<x,a>,b> - <<x,b>,a> + \lambda$$

Here $\lambda \in P_{i+j}$ by the induction hypothesis. Also $<<x,a>,b>$ and $<<x,b>,a>$ are elements in P_{i+j} by the induction hypothesis. This proves (2.13) for x,y in (1). Similarly we get (2.13) for x,y in (2). Now consider x,y in (3). Then again $y = (a,b)$ as above and we get (4). By induction on $i + j$ we know that $<x^\alpha,a^\beta>$, $<a^\beta,x^\alpha>$, $<a^\beta,b^\gamma>$, $<b^\gamma,a^\beta>$, $<x^\alpha,b^\gamma>$, and $<b^\gamma,x^\alpha>$ are elements in P_{i+r}, P_{r+s} and P_{i+s} respectively. Whence (1) and (2) show that the terms on the right hand side of (4) are elements in P_{i+j}.

□

(2.14) Proof of (2.5): The result is clear for $n = 2$. Now assume (2.5) holds for $n < m$. Then we have to show that $<x,y> \in P_m$ provided $x \in Q_i$, $y \in Q_j$, $i + j = m$. Now (2.7) shows (since $i < m, j < m$) that $Q_i = P_i + \Gamma_i$ and $Q_j = P_j + \Gamma_j$. Thus it is enough to prove $<x,y> \in P_m$ for the three cases (1), (2) and (3) in (2.13). Whence (2.13) yields the result.

□

§3 Partial Lie algebras and the quotients of the Peiffer central series

The properties of the Peiffer central series lead to the notion of a "partial Lie algebra" which generalizes the classical notion of a Lie algebra.

The group $A \otimes B$ denotes the tensor product of abelian groups A, B.

(3.1) Definition: A *Lie algebra* $(L,[\,,\,])$ is given by an abelian group L and by a homorphism $[\,,\,] : L \otimes L \to L$ such that for $x,y,z \in L$ the equations

(1) $[x,x] = 0$ and

(2) $[[x,y],z] = [x,[y,z]] - [y,[x,z]]$

hold. Let Lie be the category of Lie algebras; maps in Lie are homorphisms f with $f[x,y] = [fx,fy]$.

We introduce the following generalization of Lie algebras.

(3.2) Definition: A *partial Lie algebra* $(R,L,<,>)$ is an abelian group R, a subgroup L and a homomorphism $<,> : R \otimes R \to L$ such that the following properties are satisfied $(x,y,z \in R)$.

(1) $<x,x> = 0$ for $x \in L$.

(2) $<<x,y>,z> = <x,<y,z>> - <y,<x,z>>$ for $y \in L$.

Let <u>par</u> <u>Lie</u> be the category of partial Lie algebras; maps are homorphisms $f : R \to R'$ with $f(L) \subset L'$ and $f<x,y> = <fx,fy>$.

Clearly, a partial Lie algebra $(R,L,<,>)$ is the same as a Lie algebra provided $R = L$. Moreover, the subgroup $(L,<,>)$ of a partial Lie algebra is always a Lie algebra. The following example of a Lie algebra is classical.

(3.3) Example: Let M be a group and let

$$L_M = \bigoplus_{n \geq 1} (\Gamma_n/\Gamma_{n+1}) \tag{1}$$

with $\Gamma_n = \Gamma_n(M)$ be given by the quotients in the lower central series of M. The commutator (2.8) induces an homomorphism

$$(\Gamma_n/\Gamma_{n+1}) \otimes (\Gamma_m/\Gamma_{m+1}) \longrightarrow \Gamma_{n+m}/\Gamma_{n+m+1}$$

and whence an homorphism

$$[,] : L_M \otimes L_M \longrightarrow L_M \tag{2}$$

It is well known that $(L_M,[,])$ is a Lie algebra. This follows readily from the Witt Hall identities (1.9). Clearly $M^{ab} = \Gamma_1/\Gamma_2$, so that we have the canonical injection

$$(3) \qquad i : M^{ab} \rightarrowtail L_M$$

of abelian groups.

Similarly as in this example we obtain below a partial Lie algebra by the quotients of the Peiffer central series. Let $\partial : M \to N$ be a pre crossed module and let $P_n = P_n(\partial)$ be defined as in (1.13). Let

$$(3.4) \qquad \pi = \text{cokernel } (\partial : M \to N).$$

Then the kernel of the associated crossed module

$$\partial^{cr} : M^{cr} = P_1/P_2 \to N$$

is a π-module. Also the abelianization

$$(3.5) \qquad C = (P_1/P_2)^{ab} = M/Q_2$$

is a π-module by $\{x\}^{(\alpha)} = \{x^\alpha\}$ for $x \in M$, $\alpha \in N$. Moreover, we show

(3.6) Lemma: *For* $n \geq 2$ *the quotient groups* Q_n/Q_{n+1} *and* P_n/P_{n+1} *are* π*-modules.*

Proof: Let $x,y \in P_n$. We have to show $(x,y) \in P_{n+1}$. In fact, since $\partial x = 0$ we have

$$(x,y) = \langle x,y\rangle \in P_{2n} \subset P_{n+1}.$$

Moreover, let $\alpha \in N$ with $\alpha = \partial x$, $x \in M$. Then we get for $y \in P_n$ the element

$$\langle x,y\rangle = -x - y + x + y^{\alpha} \in P_{n+1} .$$

Whence in P_n/P_{n+1} we have

$$y^{\alpha} = y$$

since P_n/P_{n+1} is abelian. This shows that the action of N on P_n induces an action of π on P_n/P_{n+1}. A similar proof is avaiable for Q_n/Q_{n+1}.

□

There is the following basic example of a partial Lie algebra.

(3.7) Example: For the pre crossed module $\partial : M \to N$ let $K \subset C$ be the π-submodule given by the image of $\ker(\partial^{cr})$ in C, see (3.5). Let

$$(1) \qquad \begin{cases} R_{\partial} = C \oplus \bigoplus\limits_{n \geq 2} P_n/P_{n+1} \\ \\ L_{\partial} = K \oplus \bigoplus\limits_{n \geq 2} P_n/P_{n+1} \end{cases}$$

be given by the quotients of the Peiffer central series. The Peiffer bracket (2.9) induces a bilinear map $(n,m \geq 1)$

$$<,> : (P_n/P_{n+1}) \times (P_m/P_{m+1}) \longrightarrow P_{n+m}/P_{n+m+1}$$

and whence a homomorphism

$$<,> : R_\partial \otimes R_\partial \to L_\partial \ . \tag{2}$$

Here we set $<\{x\},\{y\}> = \{<x,y>\}$ for $x \in P_n$, $y \in P_m$ with $\{x\} \in C$ for $n = 1$ and $\{x\} \in P_n/P_{n+1}$ for $n > 1$. Moreover, we have the injection of pairs of abelian groups

$$i : (C,K) \rightarrowtail (R_\partial, L_\partial) \ . \tag{3}$$

(3.8) Proposition: $(R_\partial, L_\partial, <,>)$ *in* (3.7) *is a partial Lie algebra.*

Proof: In fact, (3.2)(1) is satisfied since for $\partial x = 0$ we have $<x,x> = (x,x) = 0$ for $x \in M$. Moreover, (3.2)(2) is satisfied since we can apply (2.11) and (P4)(a).

□

We point out that the second equation in (2.11) yields no further "universal" equation for $<,>$ in (2), see (2.12). Clearly, by (3.6) the abelian group R_∂ and L_∂ are as well π-modules and the bracket (2) satisfies

$$\langle a,b\rangle^{\alpha} = \langle a^{\alpha},b^{\alpha}\rangle \tag{3.9}$$

for $a,b \in R_{\partial}$, $\alpha \in \pi$, see (P1).

§4 Free partial Lie algebras

A well known theorem of Witt [23] shows that the quotients of the central series of a free group form a free Lie algebra. In this section we describe our main result on the Peiffer central series which generalizes the theorem of Witt.

We have the forgetful functor ϕ from the category $\underline{\text{Lie}}$ of Lie algebras to the category $\underline{\text{Ab}}$ of abelian groups. Let $L : \underline{\text{Ab}} \to \underline{\text{Lie}}$ be the left adjoint of ϕ. Hence we have the natural inclusion of abelian groups

$$(4.1) \qquad C \rightarrowtail L(C)$$

where $L(C)$ is called the *free Lie algebra* generated by C. For the construction of $L(C)$ we use the

(4.2) Definition: The *free non associative algebra* $A(C)$ is inductively defined by

$$(1) \qquad \begin{cases} A^1(C) = C \\ A^r(C) = \bigoplus_{i+j=r} A^i(C) \otimes A^j(C) \\ A(C) = \bigoplus_{i\geq 1} A^i(C) \end{cases}$$

For each bracket $c = (\ldots)_c$ of length n we have the inclusion

(2) $$i_c : C^{\otimes n} = C \otimes \ldots \otimes C \rightarrowtail A^n(C)$$

such that $A^n(C) = \bigoplus_c C^{\otimes n}$ is the direct sum over all c.

Let I_L be the homogeneous two sided ideal in $A(C)$ generated by the elements

$$\begin{cases} x \otimes x \\ (x \otimes y) \otimes z - x \otimes (y \otimes z) + y \otimes (x \otimes z) \end{cases}$$

for $x,y,z \in A(C)$. Then $L(C)$ is the quotient

(4.3) $$L(C) = A(C)/I_L.$$

The Lie bracket in $L(C)$ is induced by the multiplication $\otimes$ on $A(C)$ and the inclusion (4.1) is induced by $A^1(C) \rightarrowtail A(C)$. Let

(4.4) $$L^n(C) \subset L(C)$$

be the image of $A^n(C)$ in $L(C)$. Clearly $L(C)$ is the direct sum of all $L^n(C)$, $n \geq 1$, with $L^1(C) = C$. In a similar way we obtain free partial Lie algebras. For this we use the forgetful functor ϕ from the category $\underline{\underline{\text{par}}}\ \underline{\underline{\text{Lie}}}$ of partial Lie algebras to the category $\underline{\underline{\text{Ab}}}_2$ of pairs of abelian groups. Objects in $\underline{\underline{\text{Ab}}}_2$ are pairs (C,K) where K is a subgroup of the abelian group C and morphisms $f : (C,K) \to (C',K')$ are homorphisms

$f : C \to C'$ with $f(K) \subset K'$. The forgetful functor ϕ carries $(R, L, <,>)$ to the pair (R, L). Let $R : \underline{\underline{Ab}}_2 \to \underline{\underline{par}}\ \underline{\underline{Lie}}$ be the left adjoint of ϕ. Hence we have the natural inclusion of pairs of abelian groups

$$(4.5) \qquad (C,K) \rightarrowtail (R(C,K), L(C,K), <,>) = R(C,K)$$

where $R(C,K)$ is also called the *free partial Lie algebra* generated by (C,K). We construct $R(C,K)$ by the quotient

$$(4.6) \qquad R(C,K) = A(C)/I_R.$$

Here I_R is the homogeneous two sided ideal in $A(C)$ generated by the elements

$$\begin{cases} y \otimes y \\ (x \otimes y) \otimes z - x \otimes (y \otimes z) + y \otimes (x \otimes z) \end{cases}$$

for $x,y,z \in A(C)$ with

$$y \in K \oplus \bigoplus_{n \geq 2} A^n(C) = A(C,K).$$

Let $L(C,K)$ be the image of $A(C,K)$ in $R(C,K)$. The bracket

$$<,> : R(C,K) \otimes R(C,K) \to L(C,K)$$

is induced by the multiplication $\otimes$ on $A(C)$. Let

(4.7) $$R^n(C,K) \subset R(C,K)$$

be the image of $A^n(C)$ in $R(C,K)$. This yields for $n = 1$ the inclusion (4.5) of pairs of abelian groups. Clearly $R(C,C) = L(C,C) = L(C)$ in case $K = C$. Moreover, $R(C,K)$ is the direct sum of all $R^n(C,K)$, $n \geq 1$, with $R^1(C,K) = C$ and $L(C,K)$ is the direct sum of K and of all $R^n(C,K)$, $n \geq 2$.

For a group M the inclusion (3.3)(3) induces a surjective homomorphism

(4.8) $$\bar{i} : L(M^{ab}) \twoheadrightarrow L_M$$

between Lie algebras. A classical theorem of Witt [23] may be stated as

(4.9) Theorem: *If* M *is a free group then* $\bar{i} : L(M^{ab}) \cong L_M$ *is an isomorphism.*

In particular $L^n(M^{ab}) \cong \Gamma_n/\Gamma_{n+1}$ for $\Gamma_n = \Gamma_n(M)$. Compare also [4]. We now extend this result to the case of the Peiffer central series. In fact, for a pre crossed module $\partial : M \to N$ the inclusion (3.7)(3) induces a surjective homomorphism

(4.10) $$\bar{i} : (R(C,K),L(C,K)) \twoheadrightarrow (R_\partial,L_\partial)$$

between partial Lie algebras. This homomorphism is equivariant with respect to the action of π . Here the action of π on C induces an action on R(C,K) since (C,K) is a pair of π-modules and since R is a functor. Clearly, in R(C,K) equation (3.9) is satisfied.

We now are ready to formulate the main result of this paper.

(4.11) Theorem: *If N is a free group and if* $\partial : M \to N$ *is a free pre crossed module then* $\bar{I} : R(C,K) \twoheadrightarrow R_\partial$ *is an isomorphism. In particular, the map*

$$\bar{I}_n : R^n(C,K) \twoheadrightarrow P_n/P_{n+1} \quad (n \geq 2),$$

defined by (4.10), *is an isomorphism.*

Clearly, for $\partial = 0$ the theorem holds by the result of Witt, see (1.16). We prove theorem (4.11) in (7.14) below.

Now we consider the case $n = 2$ in (4.11). Let A be an abelian group and let $\gamma : A \to \Gamma(A)$ be the universal quadratic map in the sense of J.H.C. Whitehead [21]. We obtain a natural homomorphism

$$(4.12) \qquad \tau : \Gamma(A) \longrightarrow A \otimes A$$

by the condition $\tau\gamma(a) = a \otimes a$, $a \in A$. If A is free abelian τ is an injection onto the symmetric part of $A \otimes A$. The inclusion $K \subset C$ in (4.11) yields the injection

$$\tau : \Gamma(K) \longrightarrow K \otimes K \subset C \otimes C.$$

One readily verifies that there is a natural isomorphism

(4.13) $$R^2(C,K) = (C \otimes C)/\tau\Gamma K.$$

For $K = C$ this corresponds to the classical isomorphism

(4.14) $$L^2(C) = C \wedge C = C \otimes C/\tau\Gamma C$$

where $C \wedge C$ is the exterior product. The restriction of theorem (4.9) in degree 2 yields the well known isomorphism

(4.15) $$C \wedge C = L^2(C) \cong \Gamma_2(M)/\Gamma_3(M)$$

where M is a free group and where $C = M^{ab}$. This is a special case of the isomorphism

(4.16) $$(C \otimes C)/\tau\Gamma K = R^2(C,K) \cong P_2(\partial)/P_3(\partial)$$

given by (4.13) and $\bar{i}_2$ in (4.11). The isomorphism (4.16) was originally proved by a geometric argument in [2] where it is used to construct minimal algebraic models of 4-dimensional CW-complexes. An algebraic application of the isomorphism (4.16) is given in [7].

§5 Basic Peiffer commutators

We describe an additive basis in a free partial Lie algebra and we use this basis for a collecting process with respect to Peiffer commutators.

It is well known that the free Lie algebra $L(C)$ is a free abelian group provided C is free abelian. A basis of the group $L(C)$ can be obtained as follows. Let Z be a basis of C and let

$$\bar{i} : B(Z) \longrightarrow (L(C),[,])$$

be the map between magmas induced by the inclusion $i : Z \subset C \subset L(C)$.

(5.1) Definition: A subset $b(Z)$ of $B(Z)$ is called a *set of basic commutators on* Z if the restriction $\bar{i} : b(Z) \to L(C)$ of $\bar{i}$ above is a basis of the free abelian group $L(C)$.

The basis theorem of M. Hall [10] shows that such sets of basic commutators exist. In particular, the following inductive procedure yields such a set.

(5.2)(a) Example: We construct a subset $b^n(Z)$ inductively as follows. For $n = 1$ let $b^1(Z) = Z$. We choose well ordering $<$ of Z. Now assume $b^r = b^r(Z)$ is defined for $r < n$ and assume the union

$$U^{n-1} = b^1 \cup b^2 \cup \dots \cup b^{n-1}$$

is well ordered, $n \geq 2$. Then $b^n(Z)$ is the set of all brackets $[x,y]$ with the following properties (a),(b), and (c).

(a) $x \in b^i(Z)$, $y \in b^j(Z)$ and $i + j = n$.

(b) $x > y$

(c) If $x = [x',x'']$ then $x'' \leq y$.

Now we choose a well ordering of $U^n = U^{n-1} \cup b^n(Z)$ which extends the one of U^{n-1} and which satisfies $U^{n-1} < b^n(Z)$. The union $b(Z)$ of all $b^n(Z)$, $n \geq 1$, is a set of basic commutators in the sense of (5.1), see [11].

(5.2)(b) Example:

The construction of a set of basic commutators in (5.2)(a) can be slightly generalized by using a grading on the set of generators Z. Let $Z = \{Z_i : i \geq 1\}$ be a graded set. Then we define $b^n(Z) = b^n$ inductively as follows. We choose a well ordering $<$ of Z_1 and let $b^1(Z) = Z_1$. Assume $b^r(Z)$ is defined for $r < n$ and that

$$U^{n-1} = b^1 \cup \dots \cup b^{n-1}$$

is well ordered, $n \geq 2$. Then

$$b^n(Z) = Z_n \cup b_0^n.$$

Here b_0^n is the set of all brackets $[x,y]$ with the properties (a), (b), (c) in (5.2)(a). Now we choose a well ordering of $U^n = U^{n-1} \cup b^n(Z)$ which extends the one of U^{n-1} and such that $U^{n-1} < b^n(Z)$. The union $b(Z)$ of all $b^n(Z)$, $n \geq 1$, is a set of basic commutators in the sense of (5.1). We prove this as follows.

Proof of (5.2)(b): Let Z' be the disjoint union of all sets Z_i, $i \geq 1$, and of an element t. Let L' be the free Lie algebra generated by Z' and let L be the free Lie algebra generated by the disjoint union of all Z_i, $i \geq 1$. We have a Lie algebra injection

$$\varphi : L \rightarrowtail L'$$

which carries $z \in Z_i$ to the bracket

$$\varphi(z) = [\ldots[z,t],\ldots],t]$$

of length i, see chap II §2 and in [4]. Now $b(Z')$, defined as in (5.2)(a), is a basis of L' which via φ contains $b(Z)$ defined in (5.2)(b).

$\square$

Next we consider an additive basis in a free partial Lie algebra. In fact, we will show that the free partial Lie algebra $R(C,K)$ is a free abelian group if C is free abelian and if K is a direct summand of C. Let Z be a basis of C which contains a basis Z_K of K. Then we have the map

$$\bar{1} : B(Z) \longrightarrow (R(C,K),<,>)$$

between magmas induced by $Z \subset C \subset R(C,K)$.

(5.3) Definition: A subset $b(Z,Z_K)$ of $B(Z)$ is called a set of *basic Peiffer commutators* on (Z,Z_K) if the restriction $\bar{1} : b(Z,Z_K) \to R(C,K)$ of $\bar{1}$ above is a basis of the free abelian group $R(C,K)$. Let $b^n(Z,Z_K)$ be the subset of $b(Z,Z_K)$ of all brackets of length n. The following inductive procedure yields such a set of basic Peiffer commutators.

(5.4) Example: Let $Z_R = Z - Z_K$. We construct a subset $b^n(Z,Z_K)$ of $B(Z)$ inductively as follows. For $n = 1$ let $b^1(Z,Z_K) = Z$. We choose a well ordering of Z with $Z_R < Z_K$. Now assume $b^r = b^r(Z,Z_K)$ is defined for $r < n$ and assume the union

$$U^{n-1} = b^1 \cup b^2 \cup \dots \cup b^{n-1}$$

is well ordered, $n \geq 2$. Then $b^n(Z,Z_K)$ is the set of all brackets $[x,y]$ with the following properties (a)...(d).

(a) $x \in b^i$, $y \in b^j$ with $i + j = n$.

(b) $x > y$ if $x \notin Z_R$ and $y \notin Z_R$.

(c) If $x = [x',x'']$ with $x' \notin Z_R$ and $x'' \notin Z_R$ then $x'' \leq y$.

(d) If $x \in Z_R$ and if $y = [y',y'']$ then $y' \in Z_R$.

Now we choose a well ordering of $U^n = U^{n-1} \cup b^n$ which extends the one of U^{n-1} and which satisfies $U^{n-1} < b^n$. The set $b(Z,Z_K)$ is the union of all $b^n(Z,Z_K)$. In particular, we get

(e) $b^2(Z,Z_K) = [Z_R,Z] \cup [Z_K,Z_R] \cup \{[x,y] : x,y \in Z_K, x > y\}$

(f) $b^3(Z,Z_K) = [Z_R,[Z_R,Z]] \cup [[Z,Z_R],Z] \cup [[Z_R,Z_K],Z]$
$\cup \{[[z,w],v] : z,v,w \in Z_K, z > w \leq v\}$

We point out that for $Z = Z_K$ this definition of $b(Z,Z_K)$ coincides with the one in (5.2) since $Z_R = \phi$ is the empty set in this case.

(5.5) **Theorem:** *The set* $b(Z,Z_K)$ *constructed in* (5.4) *is a set of basic Peiffer commutators, see* (5.3). *In particular,* $b^n(Z,Z_K)$ *is a basis of the free abelian group* $R^n(C,K)$, *see* (4.7).

In case $C = K$ this result coincides with the basis theorem for free Lie rings of M. Hall [10]. We prove (5.5) in (6.12) below; in addition the next lemma gives an explicit procedure to express any element of $R(C,K)$ as a linear combination of basic Peiffer brackets.

(5.6) Lemma: $b^n(Z,Z_K)$ *generates* $R^n(C,K)$, $n \geq 1$.

Proof: We say that an element of $R^n(C,K)$ is in "standard form" if it is a linear combination of elements in $b^n(Z,Z_K)$. Clearly (5.6) is true for $n = 1$. Now suppose (5.6) is true for all degrees $1,\ldots,n-1$. We shall reduce

$$\lambda = \sum_k t_k \langle y_k, z_k\rangle \in R^n(C,K) \tag{1}$$

to standard form by a canonical process which will seen to leave λ unchanged if λ is in standard form.

First step: Let

$$y_k = \sum_i a_{ik}u_{ik}, \quad z_k = \sum_j b_{jk}v_{jk} \tag{2}$$

be standard forms where the u's and v's are in $b(Z,Z_K)$. Put

$$\lambda = \sum_{i,j,k} t_k a_{ik} b_{jk} \langle u_{ik}, v_{jk}\rangle. \tag{3}$$

Second step: If $u \notin Z_R$ and $v \notin Z_R$ put

(4) $\langle u,v\rangle = 0$ if $u = v$

(5) $\langle u,v\rangle = -\langle v,u\rangle$ if $u < v$

(6) $\langle u,v\rangle = \langle u,v\rangle$ if $u > v$

Third step: If $u = \langle z,w\rangle$ with $z \notin Z_R$ and $w \notin Z_R$, and if $u > v$ for $v \notin Z_R$, put

(7) $$\langle u,v\rangle = \langle\langle z,w\rangle,v\rangle \quad \text{if} \quad w \leq v$$

(8) $$\langle u,v\rangle = -\langle\langle w,v\rangle,z\rangle + \langle\langle z,v\rangle,w\rangle \quad \text{if} \quad w > v.$$

Fourth step: If $u \in Z_R$ and $v = \langle x,y\rangle$ put

(9) $$\langle u,v\rangle = \langle u,\langle x,y\rangle\rangle \quad \text{if} \quad x \in Z_R$$

(10) $$\langle u,v\rangle = \langle\langle u,x\rangle,y\rangle - \langle\langle u,y\rangle,x\rangle \quad \text{if} \quad x \notin Z_R$$

Fifth step: Return to the first step and repeat the process until nothing but linear combinations of elements in $b^n(Z,Z_K)$ remain.

One can check that the above process terminates. From the definition in (5.4) it is clear that expressions λ in standard form are left unaltered by the canonical process above, and that expressions left unaltered are in standard form.

$\square$

We derive from (5.5) and (4.11) a "basis theorem" which generalizes P. Hall's basis theorem for collecting processes in free groups, see for example page 343 in [13].

Let $M \to N$ be a free pre crossed module on a free group $N = \langle Z_1 \rangle$ and assume $M = \langle Z_2 \times N \rangle$ is the free N-group generated by the set Z_2. By (1.28) we can choose a basis Z of the free abelian group C which extends a basis Z_K of K. Moreover, we choose an injection

$$(5.7) \qquad i : (Z, Z_K) \longrightarrow (M, \ker \partial)$$

for which $h_2 pi$ is the inclusion $Z \subset C$, see (1.22) and (1.25). The map i induces the map

$$\bar{i} : B(Z) \longrightarrow (M, <,>)$$

between magmas. With the notation in (5.3) we get the following "basis theorem for Peiffer commutators".

(5.8) **Theorem:** *The map*

$$q\bar{i} : b^n(Z, Z_K) \longrightarrow P^n(\partial) \twoheadrightarrow P^n(\partial)/P^{n+1}(\partial)$$

(given by $\bar{i}$ *above) is a basis of the free abelian group* $P^n(\partial)/P^{n+1}(\partial)$, $n \geq 2$.

The following corollary is an easy consequence since $P_2 \subset \ker(\partial) \twoheadrightarrow K$ is a short exact sequence.

(5.9) Corollary: *Any element* $W \in \ker(\partial)$ *has a unique representation* $(n \geq 1)$

$$W = \left[\sum_{i=1}^{n} \sum_{b \in b(n)}^{<} n_b \cdot b \right] + V_{n+1}$$

where $V_{n+1} \in P_{n+1}(\partial)$. *The sum is defined via an ordering* $<$ *of the set* $b(n)$ *with* $b(1) = Z_K$ *and* $b(n) = b^n(Z, Z_K)$ *for* $n \geq 2$, *see* (5.3). *Only a finite number of integers* n_b *are non trivial and the integers* n_b *are uniquely determined by* W *and by the ordering* $<$.

§6 The enveloping Lie algebra of a partial Lie algebra

In this section we introduce ∂-Lie algebras and we use them to define enveloping Lie algebras. We show that in the free case the canonical map to enveloping Lie algebra is injective. This as well leads to a proof of the basis theorem (5.5) above for free partial Lie algebras.

(6.1) Definition: A *∂-Lie algebra* is a Lie algebra $(L,[\,,\,])$ endowed with a group homomorphism $\partial : L \to L$ satisfying $\partial[x,y] = 0$ for $x,y \in L$. Let $\underline{\partial\text{-Lie}}$ be the category of ∂-Lie algebras. Morphisms are maps f in $\underline{\text{Lie}}$ with $f\partial = \partial f$, see (3.1).

We obtain free ∂-Lie algebras as follows. Let $\underline{\text{Pair (Ab)}}$ be the category of pairs in $\underline{\text{Ab}}$. Objects are homomorphism $d : C_2 \to C_1$ between abelian groups and morphisms $f : d \to d'$ are pairs $f = (f_2,f_1)$ of homomorphisms with $d'f_2 = f_1\, d$. We have the forgetful functor $\phi : \underline{\partial\text{-Lie}} \longrightarrow \underline{\text{Pair (Ab)}}$ which carries $(L,[\,,\,],\partial)$ to ∂. Let $L : \underline{\text{Pair (Ab)}} \to \underline{\partial\text{-Lie}}$ be the left adjoint of ϕ. Hence one has the natural map in $\underline{\text{Pair (Ab)}}$

(6.2) $$d \xrightarrow{i} L(d), \quad d : C_2 \longrightarrow C_1,$$

where $L(d)$ is also called the *free ∂-Lie algebra* generated by d. Using the free Lie algebra in (4.1) we have

(6.3) $$L(d) = (L(C_1 \oplus C_2),\partial)$$

with ∂ given by the commutative diagram

$$\begin{tikzcd}
C_2 \arrow[r, "i_2"] \arrow[d, "d"'] & L(C_1 \oplus C_2) \arrow[r, "P_2"] \arrow[d, "\partial"] & C_2 \arrow[d, "d"] \\
C_1 \arrow[r, "i_1"] & L(C_1 \oplus C_2) & C_1 \arrow[l, "i_1"']
\end{tikzcd}$$

Let $i = (i_2,i_1) : d \to L(d)$ be given by the inclusions i_2,i_1 and let p_2 be the unique Lie algebra map which extends the projection $C_1 \oplus C_2 \to C_2$. Here the Lie bracket on C_2 is trivial. One readily checks that $i = (i_2,i_1)$ has the universal property of a free ∂-Lie algebra as in (6.2).

For a ∂-Lie algebra $(L,[\,,\,],\partial)$ we define a homorphism

(6.4) $$\begin{cases} <\,,\,> : L \oplus L \longrightarrow \ker(\partial) \quad \text{by} \\ <x,y> = [x,y] - [\partial x,y] = [x - \partial x,y] \end{cases}$$

(6.5) Lemma: *The triple* $(L,\ker(\partial),<,>)$ *is a partial Lie algebra, see* (3.2).

Proof: For $\partial x = 0$ we get

(1) $$<x,x> = [x,x] - [0,x] = 0.$$

Moreover, for $\partial y = 0$ we get

(2) $$<<x,y>,z> = <x,<y,z>> - <y,<x,z>>.$$

In fact we have

$$\begin{aligned} <<x,y>,z> &= [<x,y>,z] \\ &= [[x,y],z] - [[\partial x,y],z] \\ <x,<y,z>> &= [x,<y,z>] - [\partial x,<y,z>] \\ &= [x,[y,z]] - [\partial x,[y,z]] \\ -<y,<x,z>> &= -[y,<x,z>] \\ &= -[y,[x,z]] + [y,[\partial x,z]] \end{aligned}$$

Now the Jacobi identity for $[[x,y],z]$ and $[[\partial x,y],z]$ respectively shows that (2) is satisfied.

□

Now let ϕ : $\underline{\partial\text{-Lie}} \rightarrow \underline{\text{par}}\ \underline{\text{Lie}}$ be the functor from ∂-Lie algebras to partial Lie algebras given by (6.5) and let

(6.6) $$\mathcal{L} : \underline{\text{par}}\ \underline{\text{Lie}} \longrightarrow \underline{\partial\text{-Lie}}$$

be the left adjoint of ϕ. Hence we have for a partial Lie algebra $R = (R,L,<,>)$ a ∂-Lie algebra $(\mathcal{L}(R),[,],\partial)$ endowed with the natural (adjunction) map

$$(6.7) \qquad i : (R,L,<,>) \longrightarrow (\mathcal{L}(R),\ker(\partial),<,>)$$

between partial Lie algebras. We call the map i the *enveloping Lie algebra* of the partial Lie algebra R.

(6.8) Lemma: *The enveloping Lie algebra exists.*

Proof: $\mathcal{L}(R)$ is a quotient of $L(d)$ where $d : R \longrightarrow R/L$ is the quotient map. This is clear since the following diagram commutes

$$\begin{tikzcd}
L \arrow[r] \arrow[d, phantom, "\cap"] & \ker(\partial) \arrow[d, phantom, "\cap"] \\
R \arrow[r, "i"] \arrow[d, "d"'] & L(R) = \mathcal{L}(d)/I \arrow[d, "\partial"] \\
R/L \arrow[r, dashed, "i_1"] & L(R)
\end{tikzcd}$$

The map i_1 is induced by i since $i(L) \subset \ker(\partial)$. Moreover, I is the two sided ideal in $L(d)$ generated by $i<x,y> - [ix,iy] + [\partial ix,iy]$, $x,y \in R$.

□

As an example we consider the free partial Lie algebra $R(C,K)$ in (4.5). In this case we get the quotient map $d : C \to C/K$ and the free ∂-Lie algebra $L(d) = (L(C/K \oplus C),\partial)$. Moreover, the inclusion of pairs

$$i : (C,K) \longrightarrow (L(d), \ker \partial)$$

induces a unique map

(6.9) $$\bar{i} : R(C,K) \longrightarrow (L(C/K \oplus C), \ker \partial, <,>)$$

between partial Lie algebras since $R(C,K)$ is free. It is clear that this map is the enveloping Lie algebra of $R(C,K)$ with

(6.10) $$L(C/K \oplus C) = \mathfrak{L}R(C,K)$$

since $\mathfrak{L}R$ is the composition of two adjoint functors.

(6.11) Theorem: *Let* C *be free abelian and let* K *be a direct summand of* C. *Then the map* $\bar{i}$ *in* (6.9) *is injective.*

Proof: By definition in (4.7) and (4.4) we have the restriction (see (6.4))

(1) $$\bar{i} : R^n(C,K) \longrightarrow L^n(C/K \oplus C)$$

which is clearly the inclusion $C \subset C/K \oplus C$ for $n = 1$. It is enough to show that (1) is injective for $n \geq 2$ or equivalently that

(2) $$\bar{i} : L(C,K) \longrightarrow L(C/K \oplus C)$$

is injective. Let Z_K be a basis of K and let $Z = Z_K \cup Z_R$ be a basis of C. A *Lie generator of the first kind* in $L(C,K)$ is either an element of Z_K or an element

$$(3) \qquad <x_1,<x_2,\ldots<x_n,y>\ldots> \in L(C,K)$$

with $x_k \in Z_R$, $1 \leq k \leq n$ and $y \in Z$. Moreover, a *Lie generator of the second kind* in $L(C,K)$ is an element of the form

$$(4) \qquad <\ldots<<z,y_1>,y_2>,\ldots,y_n> \in L(C,K)$$

where $y_i \in Z_R$, $1 \leq i \leq n$, and where z is a Lie generator of the first kind. For $n = 0$ the elements in (4) coincide with the elements in (3).

(5) **Lemma:** *The set* $\overline{Z}$ *of Lie generators of the second kind generates* $L(C,K)$ *as a Lie algebra.*

Proof of (5): We consider

$$L[\overline{Z}] \subset L(C,K) = K \oplus \bigoplus_{n \geq 2} R^n(C,K).$$

Here $L[\overline{Z}]$ denotes the Lie subalgebra in $L(C,K)$ generated by $\overline{Z}$. Clearly $K \subset L[\overline{Z}]$. Moreover $R^2(C,K) \subset L[\overline{Z}]$. In fact, $R^2(C,K)$ is additively generated by $<z',z''>$ where $z',z'' \in Z$. Now we have three cases

(a) $z' \in Z_K$ and $z'' \in Z$, (b) $z' \in Z_R$ and $z'' \in Z$,

(c) $z' \in Z_K$ and $z'' \in Z_R$. Here (a) yields a Lie bracket in $L[\overline{Z}]$, (b) yields a first kind generator and (c) yields a second kind generator in $\overline{Z}$. We now assume that $R^k(C,K) \subset L[\overline{Z}]$ holds for $2 \leq k < n$ and we consider the additive generators

$$<u,v> = <z^{(1)},\ldots,z^{(n)}>_C \in R^n(C,K)$$

where $z^{(i)} \in Z$, $1 \leq i \leq n$. If $u \notin Z_R$ and $v \notin Z_R$ then $<u,v>$ is a Lie bracket of elements $u,v \in L[\overline{Z}]$ and whence an element in $L[\overline{Z}]$. Now suppose $v \in Z_R$ and
$u \in R^{n-1}(C,K) \subset L[\overline{Z}]$. Then either $u \in \overline{Z}$ and $<u,v> \in \overline{Z}$ by (4) or

$$u = \sum_{k=1}^{p} <u_k',u_k''> \tag{6}$$

with u_k', $u_k'' \in L[\overline{Z}]$ so that $<u,v>$ is a linear combination of the elements

$$<<u_k',u_k''>,v> = <u_k',<u_k'',v>> - <u_k'',<u_k',v>> \tag{7}$$

Here $<u_k'',v>$, $<u_k',v>$ are elements in $L[\overline{Z}]$ by induction whence also $<u,v> \in L[\overline{Z}]$ by (7). Finally suppose $u \in Z_R$ and $v \in R^{n-1}(C,K) \subset L[\overline{Z}]$. then we get $v \in \overline{Z}$ or

$$v = \sum_{k=1}^{p} <v_k',v_k''> \tag{8}$$

with $v'_k, v''_k \in L[\overline{Z}]$. In the latter case we have

$$(9) \qquad \langle u,\langle v'_k,v''_k\rangle\rangle = \langle\langle u,v'_k\rangle,v''_k\rangle + \langle v'_k,\langle u,v''_k\rangle\rangle$$

with $\langle u,v'_k\rangle$, $\langle u,v''_k\rangle \in L[\overline{Z}]$ by induction so that $\langle u,v\rangle \in L[\overline{Z}]$ by (9). Moreover, for $v \in \overline{Z}$, v of the first kind, we get $\langle u,v\rangle \in \overline{Z}$ by (3) and for $v \in \overline{Z}$, v of the second kind, we have $v = \langle v',v''\rangle$ with $v'' \in Z_R$ and $v' \in \overline{Z}$ by (4). We now use the Jacobi identity for $\langle u,\langle v',v''\rangle\rangle$ as in (9). Then clearly $\langle v',\langle u,v''\rangle\rangle \in L[\overline{Z}]$. Moreover $\langle\langle u,v'\rangle,v''\rangle \in L[\overline{Z}]$ since we can apply the same argument as in (6) above where we replace u by $\langle u,v'\rangle$ and v by v''. This completes the proof of (5).

□

For the proof of (6.11) we use the next lemma, see Chap. II §2 no 9 in [4].

(10) **Lemma:** *Let* X *be a set,* S *a subset of* X *and let* $L(X)$ *be the free Lie algebra generated by* X*. Then the subset* $T = T(X,S)$ *of all elements*

$$[s_1,[s_2,\ldots[[s_n,x]\ldots]], \quad n \geq 0,$$

with $s_j \in S$, $1 \leq j \leq n$, *and* $x \in X - S$ *generate a Lie subalgebra* $L[T]$ *of* $L(X)$ *which is actually a free Lie algebra generated by* T*, that is* $L[T] = L(T)$.

(11) **Addendum:** *By the antisymmetry of the Lie bracket we can replace* T *in* (10) *by the set* $T' = T'(X,S)$ *of all elements*

$$[\dots,[[x,s_n],s_{n-1}],\dots,s_1], \quad n \geq 0,$$

with $s_j \in S$, $1 \leq j \leq n$, *and* $x \in X - S$.

Now we continue the proof of (6.11). The free abelian group $C/K \oplus C$ has the basis

$$W = Z_K \cup Z_R \cup Z_R' \quad . \tag{12}$$

Here Z_R' is the set of all elements $\bar{x} = x - \partial x \in C/K \oplus C$, $x \in Z_R$. Recall that ∂ is the projection $\partial : C \longrightarrow C/K$. The map

$$\bar{I} : L(C,K) \longrightarrow L(C/K \oplus C) = L(W)$$

carries a Lie generator of the first kind to the element

$$\bar{I}\langle x_1,\dots,\langle x_n,y\rangle\dots\rangle = [\bar{x}_1,\dots,[\bar{x}_n,y]\dots] \quad , \tag{13}$$

see (6.4). This element is an element in $T = T(W,Z_R')$, see (10). Moreover since $Z_R \subset T$ we get the inclusion

$$\bar{I}(L(C,K)) \subset L(T) \subset L(W). \tag{14}$$

Here we use (9) and (10). Let $T' = T'(T,Z_R)$ be given by T and (11). Then we get the inclusions

$$(15) \qquad \bar{I}(L(C,K)) \subset L(T') \subset L(T) \subset L(W)$$

where $\bar{I}(\bar{Z})$, see (5), is a subset of T'. In fact, $\bar{I} : \bar{Z} \to T'$ is a bijection. Whence

$$(16) \qquad \bar{I} : L(C,K) \longrightarrow L(T')$$

is an isomorphism. This completes the proof of (6.11), see (2).

□

Theorem (5.5) is a corollary of (6.11)

(6.12) Proof of (5.5): We introduce the graded set $V = (V_n : n \geq 1)$ as follows. Let $V_1 = Z_K$ and let V_n be the set of all Lie generators of the second kind in $R^n(C,K)$, $n \geq 2$. We claim that

$$b^n(V) = b^n(Z,Z_K), \quad n \geq 2.$$

Here the left hand side is defined by (5.2)(b) and the right hand side is the set defined in (5.4). We see that the equation holds simply by comparing the definitions of both sides. Now (5.2)(b) implies the proposition of (5.5).

□

§7 ∂-Lie algebras associated to pre crossed modules and the proof of theorem (4.11)

A pre crossed module $\partial : M \to N$ yields the semidirect product $N \times M$ of groups. The Lie algebra $L_{N\times M}$, given by the lower central series of $M \times N$, is actually a ∂-Lie algebra and the canonical map $M \to N \times M$ induces a map between partial Lie algebras $R_\partial \longrightarrow (L_{N\times M},\partial)$. This map is the crucial ingredient in our proof of theorem (4.11).

Let $\partial : M \to N$ be a pre crossed module. From the N-group M we derive the *semi direct product* $N \times M$. this is the product set $N \times M$ with the group structure $+$ given by $((\alpha,x),(\beta,y) \in N \times M)$

(7.1) $$(\alpha,x) + (\beta,y) = (\alpha + \beta, x^\beta + y).$$

We have the split short exact sequence of groups

(7.2) $$\begin{tikzcd} M \arrow[r, tail, "i"] & N \times M \arrow[r, shift left, "p"] & N \arrow[l, shift left, "s"] \end{tikzcd}$$

with $i(x) = (0,x)$, $p(\alpha,x) = \alpha$ and $s(\alpha) = (\alpha,0)$. Since ∂ is a pre crossed module we also have the induced homomorphism

(7.3) $$\bar{\partial} : N \times M \longrightarrow N$$

with $\bar{\partial}(\alpha,x) = \alpha + \partial(x)$. One readily checks by (1.1) that $\bar{\partial}$ actually is a homomorphism. We now observe that i carries a Peiffer commutator $<x,y>$ to a sum of commutators, in fact

$$i<x,y> = (ix,iy) + (iy,s\partial x). \tag{7.4}$$

We check this by the equations

$$\begin{aligned} i<x,y> &= i(-x - y + x + y^{\partial x}) \\ &= (0,-x) + (0,-y) + (0,x) + (0,y^{\partial x}) \\ &= (0,-x) + (0,-y) + (0,x) + (0,y) \\ &\quad -(0,y) - (\partial x,0) + (0,y) + (\partial x,0) \end{aligned}$$

in $N \times M$. Equation (7.4) implies that the map i carries the Peiffer central series $P_n(\partial)$ to the central series $\Gamma_n(N \times M)$. Whence i induces the map

$$i_* : R_\partial \longrightarrow L_{N\times M} \tag{7.5}$$

on the quotients of these central series, see (3.3) and (3.7). Since Γ_1/Γ_2 is abelian the map $i : P_1/P_2 \longrightarrow \Gamma_1/\Gamma_2$ actually factors over $C = (P_1/P_2)^{ab}$ so that i_* in (7.5) is well defined. We use the map $\bar{\partial}$ in (7.3) for the definiton of the following composition ∂

$$\begin{tikzcd} L_{N\times M} \arrow[r, "p_1"] \arrow[d, "\partial"'] & (N \times M)^{ab} \arrow[d, "(s\bar{\partial})^{ab}"] \\ L_{N\times M} \arrow[r, "i_1"] & (N \times M)^{ab} \end{tikzcd} \tag{7.6}$$

Here i_1 and p_1 are the inclusion and projection for $\Gamma_1/\Gamma_2 = (N \propto M)^{ab}$, see (3.3). Clearly $\partial[x,y] = 0$ for $x,y \in L_{N\propto M}$ since $p_1[x,y] = 0$. Whence $(L_{N\times M},[,],\partial)$ is a natural *∂-Lie algebra associated to* $\partial : M \to N$, see (6.1). Moreover, equation (7.4) implies

(7.7) Lemma: *The map* i_* *in* (7.5) *is a map between partial Lie algebras, see* (3.8) *and* (6.5).

Proof: It is clear that $i_* L_\partial \subset \ker(\partial)$ by the definition of K in (3.7). Moreover, we have by (7.4) and by definition of the Lie bracket in $L_{N\times M}$ the following equations. Let

$$\xi = \{x\} \in P_n/P_{n+1}, \eta = \{y\} \in P_m/P_{m+1}.$$

(1) $$i_*\langle\xi,\eta\rangle = i_*\{\langle x,y\rangle\} = \{i\langle x,y\rangle\}$$

$$= \{(ix,iy) + (iy,s\partial x)\}$$

Here we have $\partial x = 0$ for $n \geq 2$ since $\partial P_2 = 0$. Whence we get for $n \geq 2$

(2) $$i_*\langle\xi,\eta\rangle = [i_*\xi,i_*\eta]$$

and we get for $n = 1$

(3) $$i_*\langle\xi,\eta\rangle = [i_*\xi,i_*\eta] + [i_*\eta,\{s\partial x\}]$$

with

$$(4) \qquad (s\partial x) = (s\bar{\partial} ix) = (s\bar{\partial})_* \, i_*\xi = \partial i_*\xi$$

Here the last equation holds by (7.6) since $n = 1$. Now (2) and (3) show by definition of ∂ in (7.6)

$$(5) \qquad i_*\langle\xi,\eta\rangle = [i_*\xi, i_*\eta] - [\partial i_*\xi, i_*\eta]$$

since $\partial i_*\xi = 0$ for $n \geq 2$. This shows by (6.4) that

$$i_*\langle\xi,\eta\rangle = \langle i_*\xi, i_*\eta\rangle.$$

□

We now consider the free case. Let $\partial : M \to N$ be a free pre crossed module on a free group $N = \langle Z_1\rangle$ with $M = \langle Z_2 \times N\rangle$, see (1.3) and (1.18). In this case the semi direct product $N \times M$ is the free group

$$(7.8) \qquad N \propto M = \langle Z_1 \cup Z_2\rangle$$

generated by the disjoint union $Z_1 \cup Z_2$. Let $C_n' = C_n \otimes_{\mathbb{Z}[\pi]} \epsilon^*\mathbb{Z}$ be the free abelian group generated by Z_n $(n = 1,2)$, see (1.22), and let

$$(7.9) \qquad d_2' = d_2 \otimes 1 : C_2' \longrightarrow C_1'$$

be induced by d_2 in (1.22). One readily checks by the theorem of Witt, see (4.9), that (7.6) is the free ∂-Lie algebra generated by d_2', see (6.3), that is

(7.10) $$L_{N \propto M} \cong L(C_1' \oplus C_2') \quad .$$

Moreover, the following diagram of maps between partial Lie algebras commutes.

(7.11) $$\begin{tikzcd} R(C,K) \arrow[r, "\bar{j}"] \arrow[d, "\bar{i}"'] & L(C_1' \oplus C_2') \arrow[d, "\cong"] \\ R_\partial \arrow[r, "i_*"] & L_{N \propto M} \end{tikzcd}$$

Here i_* is the map in (7.5) and $\bar{i}$ is the map in (4.10). Moreover, $\bar{j}$ is the unique map between partial Lie algebras which extends

(7.12) $$j : (C_2,K) \xrightarrow{q} (C_2',K') \subset (L(C_1' \oplus C_2'),\ker \partial)$$

where $K' = \text{kernel } (d_2')$ and where q is the quotient map; recall that $C_2 = C$ by (1.25).

We are now ready for the proof of theorem (4.11) which states that $\bar{i}$ in (7.11) is an isomorphism. This result is an easy consequence of the following two lemmas.

(7.13) Lemma: *Suppose that* $\partial : M \to N$ *is a free pre crossed module on a free group* N *and suppose* ∂ *is surjective. Then* $\bar{I}$ *in (7.11) is injective.*

Proof: The surjectivity of ∂ implies that q in (7.12) is the identity and that $d_2 = d_2'$ in (7.9) is surjective. Whence $C_1' = C_1 = C_2/K$. This shows that $\bar{j}$ in (7.11) coincides with the injective map in (6.9) and (6.11). Whence the commutativity of (7.11) shows that $\bar{I}$ in (7.11) is injective.

□

(7.14) Lemma: *Is is enough to prove theorem (4.11) in case* $\partial : M \to N$ *is surjective.*

Clearly by (7.13) this completes the proof of (4.11).

For the proof of lemma (7.14) we need the classical notion of a Schreier system.

(7.15) Definition: Let $F = \langle X \rangle$ be the free group generated by the set X and let H be a normal subgroup of F. A *Schreier system* for the cosets of H in F is a set S of elements in F with the following three properties:

(1) Each coset of F/H contains exactly one element of S.

(2) The neutral element is in S, that is $0 \in S$.

(3) If the reduced word $\varepsilon_1 X_{i_1} + \dots + \varepsilon_k X_{i_k} (\varepsilon_i \in \{+1,-1\}, x_j \in X)$ is an element in S then so is every shorter word

$\varepsilon_h x_{i_h} + \dots + \varepsilon_j x_{i_j}$ for $1 \leq h \leq j \leq k$.

The following lemma can be found for example in chapter 7, §2 [11].

(7.16) Lemma of Schreier: *For each normal subgroup* H *of* $F = \langle X\rangle$ *there exists a Schreier system* S. *Moreover, a Schreier system* S *for the cosets of* H *in* F *yields a basis* X_S *of the free group* $H = \langle X_S\rangle$. *The set* X_S *consists of all reduced words* $y + x - \tilde{S}(y + x)$ *in* F *with* $y \in S$, $x \in X$ *and* $y + x - \tilde{S}(y + x) \neq 0$. *Here* $\tilde{S}$ *is the function from* F *to* S *which takes the element* $f \in F$ *to the representative* $\tilde{S}(f) = (f + H) \cap S \in S$ *in the coset* $f + H$.

(7.17) Proof of (7.14): Let ∂ be given as in (4.11), let Z_1 be a basis of the free group $N = \langle Z_1\rangle$, and let $Z_2 \subset M$ be a basis of the free pre crossed module $\partial : M \to N$. We consider the homomorphism

(1) $$p_2 : F = \langle Z_1 \cup Z_2\rangle \longrightarrow \langle Z_1\rangle = N$$

where $Z_1 \cup Z_2$ is the disjoint union of sets. Here p_2 is the identity on Z_1 and p_2 is trivial on Z_2, $p_2 Z_2 = 0$. Let t be the section of p_2 defined by the inclusion $Z_1 \subset Z_1 \cup Z_2$. Moreover, let H' be a normal subgroup of N and let

(2) $$H = p_2^{-1}(H')$$

be the inverse image in F. A Schreier system S' for the cosets of H' in N is carried by t to a Schreier system $S = t(S')$ for the cosets of H in F. Such a Schreier system S yields a basis $(Z_1 \cup Z_2)_S$ of H by use of $\tilde{S}$ in (7.16). We have for $s \in S$, $x \in Z_2$

(3) $$\begin{aligned} \tilde{S}(s + x) &= ((s + x) + H) \cap S \\ &= (s + H) \cap S \quad \text{by} \quad (2) \\ &= s \end{aligned}$$

Whence we get the set Z_2' of all elements

(4) $$s + x - \tilde{S}(s + x) = s + x - s$$

of the basis $(Z_1 \cup Z_2)_S$. Now assume $s \in S$ and $x \in Z_1$. Then we have $s = t(s')$ and

(5) $$s + x - \tilde{S}(s + x) = t(s' + x - \tilde{S}'(s' + x))$$

where $s' + x - \tilde{S}'(s' + x)$ is a reduced word in N and an element in the basis $Z_1' = (Z_1)_{S'}$ of H'. Clearly

(6) $$(Z_1 \cup Z_2)_S = Z_1' \cup Z_2'$$

by (4) and (5) and t is the identity on Z_1'. Whence we have the epimorphism

(7) $$p_2' : H = \langle Z_1' \cup Z_2' \rangle \longrightarrow \langle Z_1' \rangle = H'$$

defined in the same way as p_2 in (1) and, in fact, p_2' is the restriction of p_2 in (1).

Now let $H' = \partial M$ be given by $\partial : M \longrightarrow N$. We construct a free pre crossed module $\delta : G \longrightarrow H'$ where δ is surjective and we construct a map $(m,n) : \delta \longrightarrow \partial$ between pre crossed modules,

(8) $$\begin{tikzcd} G \arrow[r, "m", "\cong"'] \arrow[d, "\delta"] & M \arrow[d, "\partial"] \\ \partial M = H' \arrow[r, tail, "n"] & N \end{tikzcd} ,$$

such that m is an isomorphism and such that n is the inclusion $\partial M \subset N$. As the definition of Peiffer brackets in M involves only the action of the group ∂M we see that (m,n) induces an isomorphism

(9) $$(m,n)_* : P_n(\delta) \xrightarrow{\cong} P_n(\partial)$$

of the groups in the Peiffer central series. Whence (m,n) induces an isomorhism of the quotient groups P_n/P_{n+1} as well; this implies the proposition in (7.14) since $\bar{I}$ is natural and since (m,n) induces an isomorphism of pairs

(10) $$(C,K)_\delta \xrightarrow{\cong} (C,K)_\partial$$

Here $(C,K)_\partial = (C,K)$ is defined by ∂ as in (4.11). Whence the proof of (7.14) is complete by the following construction of diagram (8).

Let $G = \langle Z_2' \times H' \rangle$ and consider the commutative diagram

(11)
$$\begin{tikzcd}
\langle Z_1' \cup Z_2' \rangle = H \arrow[rr, tail, "\partial"] \arrow[dd, bend right=40, "\bar{\delta}"'] & & F = \langle Z_1 \cup Z_2 \rangle \arrow[dd, bend left=40, "\bar{\partial}"] \\
\langle Z_2' \times H' \rangle = G \arrow[u, tail, "i"] \arrow[d, "\delta"] \arrow[rr, "m"] & & M = \langle Z_2 \times N \rangle \arrow[u, tail, "i"] \arrow[d, "\partial"] \\
\langle Z_1' \rangle = H' = \partial M \arrow[rr, tail, "n"'] & & N = \langle Z_1 \rangle
\end{tikzcd}$$

where j and n are the inclusions. Moreover, the homomorphism $\bar{\partial}$ is defined by ∂ as in (7.3), see (7.8), that is $\bar{\partial}x = x$ for $x \in Z_1$ and $\bar{\partial}x = \partial x$ for $x \in Z_2$. The map $\bar{\partial}$ carries H to ∂M since

(12)
$$\bar{\partial}(H) = \bar{\partial}(p_2^{-1}\partial M) = \partial M$$

by (4), (5) and (6). Therefore the restriction $\bar{\delta}$ of ∂ in (11) is defined. We consider the inclusion i given on generators by $i(x,\alpha) = -\alpha + x + \alpha$, $(x,\alpha) \in Z_2 \times N$ or $(x,\alpha) \in Z_2' \times H'$. This corresponds by (7.8) to the inclusion i in (7.2). We claim that j induces an isomorhism m such that (11) commutes where $\delta = \bar{\delta}i$. Here we use the result in (7) that p_2' is the restriction of p_2 in (2). Therefore we know

(13)
$$iG = \text{kernel } p_2' = \text{kernel } p_2 = iM.$$

Clearly δ in (11) is surjective and $\delta = \overline{\delta}i$ is a pre crossed module by definition of i. The isomorphism m is n-equivariant since we have chosen the Schreier system S by S = tS′. This completes the construction of diagram (8).

□

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H.J. Baues
Max-Planck-Institut für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3
Deutschland

D. Conduché
Université de Rennes I
U.E.R. Mathématiques et
Informatique
Campus de Beaulieu
35042 Rennes Cedex

France