To Yuri Ivanovich Manin
on the occasion of his 70^{th}
birthday

RANK 2 VECTOR BUNDLES ON IND-GRASSMANNIANS

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1. Introduction

The simplest example of an ind-grassmannian is the infinite projective space $\mathbf{P}^\infty$. The Barth-Van de Ven-Tyurin (BVT) Theorem, proved more than 30 years ago [BV], [T], [Sa1] (see also a recent proof by A. Coanda and G. Trautmann, [CT]), claims that any vector bundle of finite rank on $\mathbf{P}^\infty$ is isomorphic to a direct sum of line bundles. In the last decade natural examples of infinite flag varieties (or flag ind-varieties) have arisen as homogeneous spaces of locally linear ind-groups, [DPW], [DiP]. In the present paper we concentrate our attention to the special case of ind-grassmannians, i.e. to inductive limits of grassmannians of growing dimension. If V is a countable dimensional vector space, where $\bigcup\limits_{n>k} V^n = V$, then the ind-variety $\mathbf{G}(k;V) = \lim\limits_{\to} G(k;V^n)$ (or simply $\mathbf{G}(k;\infty)$) of k-dimensional subspaces of V is of course an ind-grassmannian: this is the simplest example beyond $\mathbf{P}^\infty = \mathbf{G}(1;\infty)$. A significant difference between $\mathbf{G}(k;V)$ and a general ind-grassmannian $\mathbf{X} = \lim\limits_{\to} G(k_i;V^{n_i})$ defined via a sequence of closed immersions

$$G(k_1;V^{n_1}) \xrightarrow{\varphi_1} G(k_2;V^{n_2}) \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{m-1}} G(k_m;V^{n_m}) \xrightarrow{\varphi_m} \dots, \tag{1}$$

is that in general the morphisms φ_m can have arbitrary degrees. We say that the ind-grassmannian $\mathbf{X}$ is *twisted* if $\deg\varphi_m > 1$ for infinitely many m, and that $\mathbf{X}$ is *linear* if $\deg\varphi_m = 1$ for almost all m.

Conjecture 1.1. *Let the ground field be $\mathbb{C}$ and $\mathbf{E}$ be a vector bundle of rank $r \in \mathbf{Z}_{>0}$ on an ind-grasmannian $\mathbf{X} = \lim\limits_{\to} G(k_m;V^{n_m})$, i.e. $\mathbf{E} = \lim\limits_{\leftarrow} E_m$, where $\{E_m\}$ is an inverse system of vector bundles of (fixed) rank r on $G(k_m;V^{n_m})$. Then*

- (i) *$\mathbf{E}$ is semisimple: it is isomorphic to a direct sum of simple vector bundles on $\mathbf{X}$, i.e. vector bundles on $\mathbf{X}$ with no non-trivial subbundles;*
- (ii) *for $m \gg 0$ the restriction of each simple bundle $\mathbf{E}$ to $G(k_m,V^{n_m})$ is a homogeneous vector bundle;*
- (iii) *each simple bundle $\mathbf{E}'$ has rank 1 unless $\mathbf{X}$ is isomorphic $\mathbf{G}(k;\infty)$ for some k: in the latter case $\mathbf{E}'$, twisted by a suitable line bundle, is isomorphic to a simple subbundle of the tensor algebra $T^\cdot(\mathbf{S})$, $\mathbf{S}$ being the tautological bundle of rank k on $\mathbf{G}(k;\infty)$;*
- (iv) *each simple bundle $\mathbf{E}$ (and thus each vector bundle of finite rank on $\mathbf{X}$) is trivial whenever $\mathbf{X}$ is a twisted ind-grassmannian.*

The BVT Theorem and Sato's theorem about finite rank bundles on $\mathbf{G}(k;\infty)$, [Sa1], [Sa2], as well as the results in [DP], are particular cases of the above conjecture. The purpose of the present note is to prove Conjecture 1.1 for vector bundles of rank 2, and also for vector bundles of arbitrary rank r on linear ind-grassmannians $\mathbf{X}$.

In the 70's and 80's Yuri Ivanovich Manin taught us mathematics in (and beyond) his seminar, and the theory of vector bundles was a reoccuring topic (among many others). In 1980, he asked one of us (I.P.) to report on A. Tyurin's paper [T], and most importantly to try to understand this paper. The present note is a very preliminary progress report.

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2. Notation and Conventions

The ground field is $\mathbb{C}$. Our notation is mostly standard: if X is an algebraic variety, (over $\mathbb{C}$), $\mathcal{O}_X$ denotes its structure sheaf, Ω^1_X (respectively T_X) denotes the cotangent (resp. tangent) sheaf on X under the assumption that X is smooth etc. If F is a sheaf on X, its cohomologies are denoted by $H^i(F)$, $h^i(F) := \dim H^i(F)$, and $\chi(F)$ stands for the Euler characteristic of F. The Chern classes of F are denoted by $c_i(F)$. If $f : X \to Y$ is a morphism, f^* and f_* denote respectively the inverse and direct image functors of $\mathcal{O}$-modules. All vector bundles are assumed to have finite rank. The dual of a sheaf of $\mathcal{O}_X$-modules F (as well as the the dual of a vector space) we denote by the superscript $^\vee$. Under an *embedding* of smooth varieties $i : X \to Y$ we always understand a closed immersion. Furthermore, in what follows for any ind-grassmannian $\mathbf{X}$ defined by (1), no embedding φ_i is an isomorphism.

We fix a finite dimensional space V and denote by X the grassmannian $G(k;V)$ for $k < \dim V$. In the sequel we write sometimes $G(k;n)$ indicating simply the dimension of V. Below we will often consider (parts of) the following diagram of flag varieties:

(2)

$$\begin{tikzcd}[column sep=small]
 & & Z := Fl(k-1,k,k+1;V) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & Y := Fl(k-1,k+1;V) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & & X := G(k;V), \\
Y^1 := G(k-1;V) & & Y^2 := G(k+1;V) &
\end{tikzcd}$$

under the assumption that $k+1 < \dim V$. Moreover we reserve the letters X, Y, Z for the varieties in the above diagram. By S_k, S_{k-1}, S_{k+1} we denote the tautological bundles on X,Y and Z, whenever they are defined (S_k is defined on X and Z, S_{k-1} is defined on Y^1, Y and Z, etc.). By $\mathcal{O}_X(i)$, $i \in \mathbf{Z}$, we denote the isomorphism class (in the Picard group $\operatorname{Pic} X$) of the line bundle $(\Lambda^k(S_k^\vee))^{\otimes i}$, where Λ^k stands for the k^{th} exterior power (in this case maximal exterior power as $\mathrm{rk} S_k^\vee = k$). The Picard group of Y is isomorphic to the direct product of the Picard groups of Y^1 and Y^2, and by $\mathcal{O}_Y(i,j)$ we denote the isomorphism class of the line bundle $p_1^*(\Lambda^{k-1}(S_{k-1}^\vee))^{\otimes i} \otimes_{\mathcal{O}_Y} p_2^*(\Lambda^{k+1}(S_{k+1}^\vee))^{\otimes j}$.

If $\varphi : X = G(k;V) \to X' := G(k;V')$ is an embedding, then $\varphi^*\mathcal{O}_{X'}(1) \simeq \mathcal{O}_X(d)$ for some $d \in \mathbf{Z}_{\geq 0}$: by definition d is the *degree* $\deg \varphi$ of φ. We say that φ is linear if $\deg \varphi = 1$. By a *projective subspace* (in particular a *line*, i.e. a 1-dimensional projective subspace) of X we mean a linearly embedded projective space into X. It is well known that all such are Schubert varieties of the form $\{V^k \in X | V^{k-1} \subset V^k \subset V^t\}$ or $\{V \in X | V^i \subset V^k \subset V^{k+1}\}$, where V^k is a variable k-dimensional subspace of V, and V^{k-1}, V^{k+1}, V^t, V^i are fixed subspaces of V of respective dimensions $k-1$, $k+1$, t, i. (In what follows we will automatically assume that a given finite dimensional space written as V^t has dimension t). In other words, all projective subspaces of X are of the form $G(1;V^t/V^{k-1})$ or $G(k-i, V^{k+1}/V^i)$. Note also that $Y = Fl(k-1,k+1;V)$ is the variety of lines in $X = G(k;V)$.

3. The linear case

We consider the cases of linear and twisted ind-grassmannians separately. In the case of a linear ind-grassmannian, we show that Conjecture 1.1 is a straightforward corollary of existing results combined with the following proposition. We recall, [DP], that a *standard extension* of grassmannians is an embedding of the form

$$G(k;V) \to G(k+a; V \oplus \hat{W}), \quad \{V^k \subset \mathbb{C}^n\} \mapsto \{V^k \oplus W \subset V \oplus \hat{W}\}, \tag{3}$$

where W is a fixed a-dimensional subspace of a finite dimensional vector space $\hat{W}$.

Proposition 3.1. *Let $\varphi : X = G(k;V) \to X' := G(k';V')$ be an embedding of degree 1. Then φ is a standard extension, or φ factors through a standard extension $\mathbb{P}^r \to G(k';V')$ for some r.*

Proof. We assume that $k \le n-k$, $k \le n'-k'$, where $n = \dim V$ and $n' = \dim V'$, and use induction on k. For $k=1$ the statement is obvious as the image of φ is a projective subspace of $G(k';V')$ and hence φ is a standard extension. Assume that the statement is true for $k-1$. Since $\deg \varphi = 1$, φ induces an embedding $\varphi_Y : Y \to Y'$, where $Y = Fl(k-1,k+1;V)$ is the variety of lines in X and $Y = Fl(k'-1,k'+1;V')$ is the variety of lines in X'. Moreover, clearly we have a commutative diagram of natural projections and embeddings

$$\begin{tikzcd}
& Z \arrow[rrr, "\varphi_Z"] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & & Z' \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & \\
Y \arrow[rrr, bend right, "\varphi_Y"'] & & X \arrow[rrr, bend right, "\varphi"'] & Y' & & X',
\end{tikzcd}$$

where $Z := Fl(k-1,k,k+1;V)$ and $Z' := Fl(k'-1,k',k'+1;V')$.

We claim that there is an isomorphism

$$\varphi_Y^*\mathcal{O}_{Y'}(1,1) \simeq \mathcal{O}_Y(1,1). \tag{4}$$

Indeed, $\varphi_Y^*\mathcal{O}_{Y'}(1,1)$ is determined up to isomorphism by its restriction to the fibers of p_1 and p_2 (see diagram (2)), and therefore it is enough to check that

$$\varphi_Y^*\mathcal{O}_{Y'}(1,1)_{|p_1^{-1}(V^{k-1})} \simeq \mathcal{O}_{p_1^{-1}(V^{k-1})}(1), \tag{5}$$

$$\varphi_Y^*\mathcal{O}_{Y'}(1,1)_{|p_2^{-1}(V^{k+1})} \simeq \mathcal{O}_{p_2^{-1}(V^{k+1})}(1) \tag{6}$$

for some fixed subspaces $V^{k-1} \subset V$, $V^{k+1} \subset V$. Note that the restriction of φ to the projective subspace $G(1;V/V^{k-1}) \subset X$ is simply an isomorphism of $G(1;V/V^{k-1})$ with a projective subspace of X', hence the map induced by φ on the variety $G(2;V/V^{k-1})$ of projective lines in $G(1;V/V^{k-1})$ is an isomorphism with a grassmannian of 2-dimensional subspaces of an appropriate quotient space of V'. Note furthermore that $p_1^{-1}(V^{k-1})$ is nothing but the variety of lines $G(2;V/V^{k-1})$ in $G(1;V/V^{k-1})$, and that the image of $G(2;V/V^{k-1})$ under φ is nothing but $\varphi_Y(p_1^{-1}(V^{k-1}))$. This shows that the restriction of $\varphi_Y^*\mathcal{O}_{Y'}(1,1)$ to $G(2;V/V^{k-1})$ is isomorphic to the restriction of $\mathcal{O}_Y(1,1)$ to $G(2;V/V^{k-1})$, and we obtain (5). The isomorphism (6) follows from a very similar argument.

The isomorphism (4) leaves us with two alternatives:

$$\varphi_Y^*\mathcal{O}_{Y'}(1,0) \simeq \mathcal{O}_Y \text{ or } \varphi_Y^*\mathcal{O}_{Y'}(0,1) \simeq \mathcal{O}_Y, \tag{7}$$

or

$$\varphi_Y^*\mathcal{O}_{Y'}(1,0) \simeq \mathcal{O}_Y(1,0) \text{ or } \varphi_Y^*\mathcal{O}_{Y'}(1,0) \simeq \mathcal{O}_Y(0,1). \tag{8}$$

Let (7) hold, more precisely let $\varphi_Y^*\mathcal{O}_{Y'}(1,0) \simeq \mathcal{O}_Y$. Then φ_Y maps each fiber of p_2 into a single point in Y' (depending on the image in Y^2 of this fiber), say $((V')^{k'-1} \subset (V')^{k'+1})$, and moreover the space $(V')^{k'-1}$ is constant. Thus φ maps X into the projective subspace $G(1;V'/(V')^{k'-1})$

of X'. If $\varphi_Y^*\mathcal{O}_{Y'}(0,1) \simeq \mathcal{O}_Y$, then φ maps X into the projective subspace $G(1;(V')^{k'+1})$ of X'. Therefore, the Proposition is proved in the case (7) holds.

We assume now that (8) holds. It is easy to see that (8) implies that φ induces a linear embedding φ_{Y^1} of $Y^1 := G(k-1;V)$ into $G(k'-1;V')$ or $G(k'+1;V')$. Assume that $\varphi_{Y^1} : Y^1 \to (Y')^1 := G(k'-1;V')$ (the other case is completely similar). Then, by the induction assumption, φ_{Y^1} is a standard extension or factors through a standard extension $\mathbb{P}^r \to (Y')^1$. If φ_{Y^1} is a standard extension corresponding to a fixed subspace $W \subset \hat{W}$, then $\varphi_{Y^1}^* S_{k'-1} \simeq S_{k-1} \oplus (W \otimes_{\mathbb{C}} \mathcal{O}_{Y^1})$ and we have a vector bundle monomorphism

$$0 \to \pi_1^* p_1^* \varphi_{Y^1}^* S_{k'-1} \to \pi_2^* \varphi^* S_{k'}. \tag{9}$$

By restricting (9) to the fibers of π_1 we see that the quotient line bundle $\pi_2^*\varphi^*S_{k'}/\pi_1^*p_1^*\varphi_{Y^1}^*S_{k'-1}$ is isomorphic to $S_k/S_{k-1} \otimes \pi_1^*p_1^*\mathcal{L}$, where $\mathcal{L}$ is a line bundle on Y^1. Applying π_{2*} we obtain

$$0 \to W \otimes_{\mathbb{C}} \mathcal{O}_X \to \pi_{2*}(\pi_2^*\varphi^*S_{k'}) = \varphi^*S_{k'} \to \pi_{2*}((S_k/S_{k-1}) \otimes \pi_1^*p_1^*\mathcal{L}) \to 0. \tag{10}$$

Since $\mathrm{rk}\varphi^*S_{k'} = k'$ and $\dim W = k'-k$, $\mathrm{rk}\pi_{2*}((S_k/S_{k-1}) \otimes \pi_1^*p_1^*\mathcal{L}) = k$, which implies immediately that $\mathcal{L}$ is trivial. Hence (10) reduces to $0 \to W \otimes_{\mathbb{C}} \mathcal{O}_X \to \varphi^*S_{k'} \to S_k \to 0$, and thus

$$\varphi^*S_{k'} \simeq S_k \oplus (W \otimes_{\mathbb{C}} \mathcal{O}_X) \tag{11}$$

as there are no non-trivial extensions of S_k by a trivial bundle. Now (11) implies that φ is a standard extension.

It remains to consider the case when φ_{Y^1} maps Y^1 into a projective subspace $\mathbb{P}^s$ of $(Y')^1$. Then $\mathbb{P}^s$ is of the form $G(1;V'/(V')^{k'-2})$ for some $(V')^{k'-2} \subset V'$, or of the form $G(k'-1;(V')^{k'})$ for some $(V')^{k'} \subset V'$. The second case is clearly impossible because it would imply that φ maps X into the single point $(V')^{k'}$. Hence $\mathbb{P}^s = G(1;V'/(V')^{k'-2})$ and φ maps X into the grassmannian $G(2;V'/(V')^{k'-2})$ in $G(k';V')$. Let S_2' be the rank 2 tautological bundle on $G(2;V'/(V')^{k'-2})$. Then its restriction $S'' := \varphi^*S_2'$ to any line l in X is isomorphic to $\mathcal{O}_l \oplus \mathcal{O}_l(-1)$, and we claim that this implies one of the two alternatives:

$$S'' \simeq \mathcal{O}_X \oplus \mathcal{O}_X(-1) \tag{12}$$

or

$$S'' \simeq S_2 \text{ and } k=2, \text{ or } S'' \simeq (V \otimes_{\mathbb{C}} \mathcal{O}_X)/S_2 \text{ and } k = n-k = 2. \tag{13}$$

Let $k \geq 2$. The evaluation map $\pi_1^*\pi_{1*}\pi_2^*S'' \to \pi_2^*S''$ is a monomorphism of the line bundle $\pi_1^*\mathcal{L} := \pi_1^*\pi_{1*}\pi_2^*S''$ into π_2^*S'' (here $\mathcal{L} := \pi_{1*}\pi_2^*S''$). Restricting this monomorphism to the fibers of π_2 we see immediately that $\pi_1^*\mathcal{L}$ is trivial when restricted to those fibers and is hence trivial. Therefore $\mathcal{L}$ is trivial, i.e. $\pi_1^*\mathcal{L} = \mathcal{O}_Z$. Push-down to X yields

$$0 \to \mathcal{O}_X \to S'' \to \mathcal{O}_X(-1) \to 0, \tag{14}$$

and hence (14) splits as $\mathrm{Ext}^1(\mathcal{O}_X(-1), \mathcal{O}_X) = 0$. Therefore (12) holds. For $k=2$, there is an additional possibility for the above monomorphisms to be of the form $\pi_1^*\mathcal{O}_Y(-1,0) \to \pi_2^*S$ (or of the form $\pi_1^*\mathcal{O}_Y(0,-1) \to \pi_2^*S$ if $n-k=2$) which yields the option (13).

If (12) holds, φ maps X into an appropriate projective subspace of $G(2;V'/(V')^{k'-2})$ which is then a projective subspace of X', and if (13) holds, φ is a standard extension corresponding to a zero dimensional space W. The proof is now complete. □

We are ready now to prove the following theorem.

Theorem 3.2. *Conjecture 1.1 holds for any linear ind-grassmannian* $\mathbf{X}$.

Proof. Assume that $\deg \varphi_m = 1$ for all m, and apply Proposition 3.1. If infinitely many φ_m's factor through respective projective subspaces, then $\mathbf{X}$ is isomorphic to $\mathbf{P}^\infty$ and the BVT Theorem implies Conjecture 1.1. Otherwise, all φ_m's are standard extensions of the form (3). There are two alternatives: $\lim\limits_{m\to\infty} k_m = \lim\limits_{m\to\infty} n_m - k_m = \infty$, or one of the limits $\lim\limits_{m\to\infty} k_m$ or $\lim\limits_{m\to\infty} n_m - k_m$ equals l for some $l \in \mathbf{N}$. In the first case the claim of Conjecture 1.1 is proved in [DP]: Theorem 4.2. In the second case $\mathbf{X}$ is isomorphic to $\mathbf{G}(l;\infty)$, and therefore Conjecture 1.1 is proved in this case by E. Sato in [Sa2]. □

4. Auxilary results

In order to prove Conjecture 1.1 for rank 2 bundles $\mathbf{E}$ on a twisted ind-grassmannian $\mathbf{X} = \varinjlim G(k_m; V^{n_m})$, we need to prove that the vector bundle $\mathbf{E} = \varprojlim E_m$ of rank 2 on $\mathbf{X}$ is trivial, i.e. that E_m is a trivial bundle on $G(k_m; V^{n_m})$ for each m. From this point on we assume that none of the grassmannians $G(k_m; V^{n_m})$ is a projective space, as for a twisted projective ind-space Conjecture 1.1 is proved in [DP] for bundles of arbitrary rank r.

The following known proposition gives a useful triviality criterion for vector bundles of arbitrary rank on grassmannians.

Proposition 4.1. *A vector bundle E on $X = G(k;n)$ is trivial iff its restriction $E_{|l}$ is trivial for every line l in $G(k;n)$, $l \in Y = Fl(k-1,k+1;n)$.*

Proof. We recall the proof given in [P]. It uses the well known fact that the Proposition holds for any projective space, [OSS, Theorem 3.2.1]. Let first $k = 2$, $n = 4$, i.e. $X = G(2;4)$. Since E is linearly trivial, $\pi_2^* E$ is trivial along the fibers of π_1 (we refer here to diagram (2)). Moreover, $\pi_{1*}\pi_2^* E$ is trivial along the images of the fibers of π_2 in Y. These images are of the form $\mathbb{P}^1_1 \times \mathbb{P}^1_2$, where $\mathbb{P}^1_1$ (respectivelly $\mathbb{P}^1_2$) are lines in $Y^1 := G(1;4)$ and $Y^2 := G(3;4)$. The fiber of p_1 is filled by lines of the form $\mathbb{P}^1_2$, and thus $\pi_{1*}\pi_2^* E$ is linearly trivial, and hence trivial along the fibers of π_1. Finally the lines of the form $\mathbb{P}^1_1$ fill Y^1, hence $p_{1*}\pi_{1*}\pi_2^* E$ is also a trivial bundle. This implies that $E = \pi_{2*}\pi_1^* p_1^*(p_{1*}\pi_{1*}\pi_2^* E)$ is also trivial.

The next case is the case when $k = 2$ and n is arbitrary, $n \geq 5$. Then the above argument goes through by induction on n since the fiber of p_1 is isomorphic to $G(2; n-1)$. The proof is completed by induction on k for $k \geq 3$: the base of p_1 is $G(k-1;n)$ and the fiber of p_1 is $G(2; n-1)$. □

If $C \subset N$ is a smooth rational curve in an algebraic variety N and E is a vector bundle on N, then by a classical theorem of Grothendieck, $E_{|C}$ is isomorphic to $\bigoplus_i \mathcal{O}_C(d_i)$ for some $d_1 \geq d_2 \geq \cdots \geq d_{\mathrm{rk}E}$. We call the ordered rk$E$-tuple $(d_1, \ldots, d_{\mathrm{rk}E})$ *the splitting type* of $E_{|C}$ and denote it by $\mathbf{d}_E(C)$. If $N = X = G(k;n)$, then the lines on N are parametrized by points $l \in Y$, and we obtain a map

$$Y \to \mathbf{Z}^{\mathrm{rk}E} \ : \ l \mapsto \mathbf{d}_E(l).$$

By semicontinuity (cf. [OSS, Ch.I, Lemma 3.2.2]), there is a dense open set $U_E \subset Y$ of lines with minimal splitting type with respect to the lexicographical ordering on $\mathbf{Z}^{\mathrm{rk}E}$. Denote this minimal splitting type by $\mathbf{d}_E$. By definition, $U_E = \{l \in Y |\ \mathbf{d}_E(l) = \mathbf{d}_E\}$ is the set of *non-jumping* lines of E, and its complement $Y \setminus U_E$ is the proper closed set of *jumping* lines.

A coherent sheaf F over a smooth irreducible variety N is called *normal* if for every open set $U \subset N$ and every closed algebraic subset $A \subset U$ of codimension at least 2 the restriction map $F(U) \to F(U \smallsetminus A)$ is surjective. It is well known that a rank-1 normal torsion-free sheaf F is reflexive, i.e. $F^{\vee\vee} = F$. Therefore, by [OSS, Ch.II, Theorem 2.1.4] F is necessarily a line bundle (see [OSS, Ch.II, 1.1.12 and 1.1.15]).

Theorem 4.2. *Let E be a rank r vector bundle of splitting type $\mathbf{d}_E = (d_1, ..., d_r)$, $d_1 \geq ... \geq d_r$, on $X = G(k;n)$. If $d_s - d_{s+1} \geq 2$ for some $s < r$, then there is a normal subsheaf $F \subset E$*

of rank s with the following properties: over the open set $\pi_2(\pi_1^{-1}(U_E)) \subset X$ the sheaf F is a subbundle of E, and for any $l \in U_E$

$$F_{|l} \simeq \bigoplus_{i=1}^{s} \mathcal{O}_l(d_i).$$

Proof. It is similar to the proof of Theorem 2.1.4 of [OSS, Ch.II]. Consider the vector bundle $E' = E \bigotimes \mathcal{O}_X(-d_s)$ and the evaluation map $\Phi : \pi_1^*\pi_{1*}\pi_2^*E' \to \pi_2^*E'$. The definition of U_E implies that $\Phi_{|\pi_1^{-1}(U_E)}$ is a morphism of constant rank s and that its image $\mathrm{Im}\Phi \subset \pi_2^*E'$ is a subbundle of rank s over $\pi_1^{-1}(U_E)$. Let $M := \pi_2^*E'/\mathrm{im}\Phi$, let $T(M)$ be the torsion subsheaf of M, and $F' := \ker(\pi_2^*E' \to M' := M/T(M))$. Consider the singular set $\mathrm{Sing}\, F'$ of the sheaf F' and set $A := Z \smallsetminus \mathrm{Sing}\, F'$. By the above, A is an open subset of Z containing $\pi_1^{-1}(U_E)$ and $f = \pi_{2|A} : A \to B := \pi_2(A)$ is a submersion with connected fibers.

Next, take any point $l \in Y$ and put $L := \pi_1^{-1}(l)$. By definition, $L \simeq \mathbb{P}^1$, and we have

$$T_{Z/X\,|L} \simeq \mathcal{O}_L(-1)^{\oplus(n-2)}, \tag{15}$$

where $T_{Z/X}$ is the relative tangent bundle of Z over X. The construction of the sheaves F' and M implies that for any $l \in U_E$: $F'^{\vee}_{|L} = \oplus_{i=1}^{s}\mathcal{O}_L(-d_i+d_s)$, $M'_{|L} = \oplus_{i=s+1}^{r}\mathcal{O}_L(d_i-d_s)$. This, together with (15) and the condition $d_s - d_{s+1} \geq 2$, immediately implies that $H^0(\Omega^1_{A/B} \otimes F'^{\vee} \otimes M'_{|L}) = 0$. Hence $H^0(\Omega^1_{A/B} \otimes F'^{\vee} \otimes M'_{|\pi_1^{-1}(U_E)}) = 0$, and thus, since $\pi_1^{-1}(U_E)$ is dense open in Z, $\mathrm{Hom}(T_{A/B}, \mathcal{H}om(F', M'_{|A})) = H^0(\Omega^1_{A/B} \otimes F'^{\vee} \otimes M'_{|A}) = 0$. Now we apply the Descent Lemma (see [OSS, Ch.II, Lemma 2.1.3]) to the data $(f_{|\pi_1^{-1}(U_E)} : \pi_1^{-1}(U_E) \to V_E,\ F'_{|\pi_1^{-1}(U_E)} \subset E'_{|\pi_1^{-1}(U_E)})$. Then $F := (\pi_{2*}F') \otimes \mathcal{O}_X(-d_s)$ is the desired sheaf. □

5. The case rk$\mathbf{E} = 2$

In what follows, when considering a twisted ind-grassmannian $\mathbf{X} = \lim\limits_{\to} G(k_m; V^{n_m})$ we set $G(k_m; V^{n_m}) = X_m$. Theorem 4.2 yields now the following corollary.

Corollary 5.1. *Let $\mathbf{E} = \lim\limits_{\leftarrow} E_m$ be a rank 2 vector bundle on a twisted ind-grassmannian $\mathbf{X} = \lim\limits_{\to} X_m$. Then there exists $m_0 \geq 1$ such that $\mathbf{d}_{E_m} = (0,0)$ for any $m \geq m_0$.*

Proof. Note first that the fact that $\mathbf{X}$ is twisted implies

$$c_1(E_m) = 0,\ m \geq 1. \tag{16}$$

Indeed, $c_1(E_m)$ is nothing but the integer corresponding to the line bundle $\Lambda^2(E_m)$ in the identification of $\mathrm{Pic}\, X_m$ with $\mathbf{Z}$. As $\mathbf{X}$ is twisted, $c_1(E_m) = \deg\varphi_m \deg\varphi_{m+1} \ldots \deg\varphi_{m+k} c_1(E_{m+k+1})$ for any $k \geq 1$, in other words $c_1(E_m)$ is divisible by larger and larger integers and hence $c_1(E_m) = 0$ (cf. [DP, Lemma 3.2]). Suppose that for any $m_0 \geq 1$ there exists $m \geq m_0$ such that $\mathbf{d}_{E_m} = (a_m, -a_m)$ with $a_m > 0$. Then Theorem 4.2 applies to E_m with $s = 1$, and hence E_m has a normal rank-1 subsheaf F_m such that

$$F_{m|l} \simeq \mathcal{O}_l(a_m) \tag{17}$$

for a certain line l in X_m. Since F_m is a torsion-free normal subsheaf of the vector bundle E, the sheaf F_m is a line bundle, i.e. $F_m \simeq \mathcal{O}_{X_m}(a_m)$. Therefore we have a monomorphism:

$$0 \to \mathcal{O}_{X_m}(a_m) \to E_m, \quad a_m \geq 1. \tag{18}$$

This is clearly impossible. In fact, this monomorphism implies in view of (16) that any rational curve $C \subset X_m$ of degree $\delta_m := \deg\varphi_1 \cdot \ldots \cdot \deg\varphi_{m-1}$ has splitting type $\mathbf{d}_{E_m}(C) = (a'_m, -a'_m)$, where $a'_m \geq a_m\delta_m \geq \delta_m$. Hence, by semiconinuity, any line $l \in X_1$ has splitting type $\mathbf{d}_{E_1}(l) = (b, -b)$, $b \geq \delta_m$. Since $\delta_m \to \infty$ as $m_0 \to \infty$, this is a contradiction. □

We now recall some standard facts about the Chow rings of $X_m = G(k_m; V^{n_m})$, (see, e.g., [F, 14.7]):

(i) $A^1(X_m) = \mathrm{Pic}(X_m) = \mathbb{Z}[\mathbb{V}_m]$, $A^2(X_m) = \mathbb{Z}[\mathbb{W}_{1,m}] \oplus \mathbb{Z}[\mathbb{W}_{2,m}]$, where $\mathbb{V}_m, \mathbb{W}_{1,m}, \mathbb{W}_{2,m}$ are the following Schubert varieties: $\mathbb{V}_m := \{V^{k_m} \in X_m | \ \dim(V^{k_m} \cap V_0^{n_m-k_m}) \geq 1$ for a fixed subspace $V_0^{n_m-k_m-1}$ of $V^{n_m}\}$, $\mathbb{W}_{1,m} := \{V^{k_m} \in X_m | \dim(V^{k_m} \cap V_0^{n_m-k_m-1}) \geq 1$ for a fixed subspace $V_0^{n_m-k_m-1}$ in $V^{n_m}\}$, $\mathbb{W}_{2,m} := \{V^{k_m} \in X_m | \ \dim(V^{k_m} \cap V_0^{n_m-k_m+1}) \geq 2$ for a fixed subspace $V_0^{n_m-k_m+1}$ of $V^{n_m}\}$;

(ii) $[\mathbb{V}_m]^2 = [\mathbb{W}_{1,m}] + [\mathbb{W}_{2,m}]$ in $A^2(X_m)$;

(iii) $A_2(X_m) = \mathbb{Z}[\mathbb{P}^2_{1,m}] \oplus \mathbb{Z}[\mathbb{P}^2_{2,m}]$, where the projective planes $\mathbb{P}^2_{1,m}$ (called *α-planes*) and $\mathbb{P}^2_{2,m}$ (called *β-planes*) are respectively the Schubert varieties $\mathbb{P}^2_{1,m} := \{V^{k_m} \in X_m | \ V_0^{k_m-1} \subset V^{k_m} \subset V_0^{k_m+2}$ for a fixed flag $V_0^{k_m-1} \subset V_0^{k_m+2}$ in $V^{n_m}\}$, $\mathbb{P}^2_{2,m} := \{V^{k_m} \in X_m | \ V_0^{k_m-2} \subset V^{k_m} \subset V_0^{k_m+1}$ for a fixed flag $V_0^{k_m-2} \subset V_0^{k_m+1}$ in $V^{n_m}\}$;

(iv) the bases $[\mathbb{W}_{i,m}]$ and $[\mathbb{P}^2_{j,m}]$ are dual in the standard sense that $[\mathbb{W}_{i,m}] \cdot [\mathbb{P}^2_{j,m}] = \delta_{i,j}$.

Lemma 5.2. *There exists $m_1 \in \mathbf{Z}_{>0}$ such that for any $m \geq m_1$ one of the following holds:*

(1) $c_2(E_{m|\mathbb{P}^2_{1,m}}) > 0$, $c_2(E_{m|\mathbb{P}^2_{2,m}}) \leq 0$,

(2) $c_2(E_{m|\mathbb{P}^2_{2,m}}) > 0$, $c_2(E_{m|\mathbb{P}^2_{1,m}}) \leq 0$,

(3) $c_2(E_{m|\mathbb{P}^2_{1,m}}) = 0$, $c_2(E_{m|\mathbb{P}^2_{2,m}}) = 0$.

Proof. According to (i), for any $m \geq 1$ there exist $\lambda_{1m}, \lambda_{2m} \in \mathbf{Z}$ such that

$$c_2(E_m) = \lambda_{1m}[\mathbb{W}_{1,m}] + \lambda_{2m}[\mathbb{W}_{2,m}]. \tag{19}$$

Moreover, (iv) implies

$$\lambda_{jm} = c_2(E_{m|\mathbb{P}^2_{j,m}}), \quad j = 1, 2. \tag{20}$$

Next, (i) yields:

$$\varphi_m^*[\mathbb{W}_{1,m+1}] = a_{11}(m)[\mathbb{W}_{1,m}] + a_{21}(m)[\mathbb{W}_{2,m}], \quad \varphi_m^*[\mathbb{W}_{2,m+1}] = a_{12}(m)[\mathbb{W}_{1,m}] + a_{22}(m)[\mathbb{W}_{2,m}], \tag{21}$$

where $a_{ij}(m) \in \mathbb{Z}$. Consider the 2×2-matrix $A(m) = (a_{ij}(m))$ and the column vector $\Lambda_m = (\lambda_{1m}, \lambda_{2m})^t$. Then, in view of (iv), the relation (21) gives: $\Lambda_m = A(m)\Lambda_{m+1}$. Iterating this equation and denoting by $A(m,i)$ the 2×2-matrix $A(m) \cdot A(m+1) \cdot ... \cdot A(m+i)$, $i \geq 1$, we obtain

$$\Lambda_m = A(m,i)\Lambda_{m+i+1}. \tag{22}$$

The twisting condition $\varphi_m^*[\mathbb{V}_{m+1}] = \deg \varphi_m[\mathbb{V}_m]$ together with (ii) implies: $\varphi_m^*([\mathbb{W}_{1,m+1}] + [\mathbb{W}_{2,m+1}]) = (\deg \varphi_m)^2([\mathbb{W}_{1,m}] + [\mathbb{W}_{2,m}])$. Substituting (21) into the last equality, we have: $a_{11}(m) + a_{12}(m) = a_{21}(m) + a_{22}(m) = (\deg \varphi_m)^2, \quad m \geq 1$. This means that the column vector $v = (1,1)^t$ is the eigenvector of the matrix $A(m)$ with the eigenvalue $(\deg \varphi_m)^2$. Hence, it is the eigenvector of $A(m,i)$ with the eigenvalue $d_{m,i} = (\deg \varphi_m)^2 (\deg \varphi_{m+1})^2 ... (\deg \varphi_{m+i})^2$:

$$A(m,i)v = d_{m,i}v. \tag{23}$$

Notice that the entries of $A(m)$, $m \geq 1$, are nonnegative integers (in fact, from the definition of Schubert varieties $\mathbb{W}_{j,m+1}$ it follows quickly that $\varphi_m^*[\mathbb{W}_{j,m+1}]$ is an effective cycle on X_m, so that (21) and (iv) give $0 \leq \varphi_m^*[\mathbb{W}_{i,m+1}] \cdot [\mathbb{P}^2_{j,m}] = a_{ij}(m)$); hence also the entries of $A(m,i)$, $m, i \geq 1$, are nonnegative integers). Besides, clearly $d_{m,i} \to \infty$ as $i \to \infty$ for any $m \geq 1$. This, together with (22) and (23), implies that, for $m \gg 1$, λ_{1m} and λ_{2m} cannot both be nonzero and have the same sign. This together with (20) is equivalent to the statement of the Lemma. ◻

In what follows we denote the α-planes and the β-planes on $X = G(2;4)$ respectively by $\mathbb{P}^2_\alpha$ and $\mathbb{P}^2_\beta$.

Proposition 5.3. *There exists no rank 2 vector bundle E on the grassmannian $X = G(2;4)$ such that:*

(a) $c_2(E) = a[\mathbb{P}^2_\alpha], \quad a > 0$,

(b) $E_{|\mathbb{P}^2_\beta}$ *is trivial for a generic β-plane $\mathbb{P}^2_\beta$ on X.*

Proof. Now assume that there exists a vector bundle E on X satisfying the conditions (a) and (b) of the Proposition. Fix a β-plane $P \subset X$ such that

$$E_{|P} \simeq \mathcal{O}_P^{\oplus 2}. \tag{24}$$

As X is the grassmannian of lines in $\mathbb{P}^3$, the plane P is the dual plane of a certain plane $\tilde{P}$ in $\mathbb{P}^3$. Next, fix a point $x_0 \in \mathbb{P}^3 \smallsetminus \tilde{P}$ and denote by S the variety of lines in $\mathbb{P}^3$ which contain x_0. Consider the variety $Q = \{(x,l) \in \mathbb{P}^3 \times X \mid x \in l \cap \tilde{P}\}$ with natural projections $p : Q \to S : (x,l) \mapsto \mathrm{Span}(x, x_0)$ and $\sigma : Q \to X : (x,l) \mapsto l$. Clearly, σ is the blowing up of X at the plane P, and the exceptional divisor $D_P = \sigma^{-1}(P)$ is isomorphic to the incidence subvariety of $P \times \tilde{P}$. Moreover, one easily checks that $Q \simeq \mathbb{P}(\mathcal{O}_S(1) \oplus T_S(-1))$, so that the projection $p : Q \to S$ coincides with the structure morphism $\mathbb{P}(\mathcal{O}_S(1) \oplus T_S(-1)) \to S$. Let $\mathcal{O}_Q(1)$ be the Grothendieck line bundle on Q such that $p_*\mathcal{O}_Q(1) = \mathcal{O}_S(1) \oplus T_S(-1)$. Using the Euler exact triple on Q

$$0 \to \Omega^1_{Q/S} \to p^*(\mathcal{O}_S(1) \oplus T_S(-1)) \otimes \mathcal{O}_Q(-1) \to \mathcal{O}_Q \to 0, \tag{25}$$

we find the p-relative dualizing sheaf $\omega_{Q/S} := \det(\Omega^1_{Q/S})$:

$$\omega_{Q/S} \simeq \mathcal{O}_Q(-3) \otimes p^*\mathcal{O}_S(2). \tag{26}$$

Set $\mathcal{E} := \sigma^* E$. By construction, for each $y \in S$ the fiber $S_y = p^{-1}(y)$ is a plane such that $l_y = S_y \cap D_Y$ is a line, and, by (24),

$$\mathcal{E}_{|l_y} \simeq \mathcal{O}_{l_y}^{\oplus 2}. \tag{27}$$

Furthermore, $\sigma(S_y)$ is an α-plane in X, hence by condition (a) of the Proposition we obtain that $\mathcal{E}_{|S_y}$ is a stable vector bundle for $a > 1$, and that $\mathcal{E}_{|S_y}$ fits into an exact triple $0 \to \mathcal{O}_{S_y} \to \mathcal{E}_{|S_y} \to \mathcal{I}_{z,S_y} \to 0$ for a certain point $z \in S_y$ for $a = 1$. In both cases it is well known (and immediately verified) that (27) implies

$$h^1(\mathcal{E}_{|S_y}(-1)) = h^1(\mathcal{E}_{|S_y}(-2)) = a, \quad h^1(\mathcal{E}_{|S_y} \otimes \Omega^1_{S_y}) = 2a + 2, \tag{28}$$

$$h^i(\mathcal{E}_{|S_y}(-1)) = h^i(\mathcal{E}_{|S_y}(-2)) = h^i(\mathcal{E}_{|S_y} \otimes \Omega^1_{S_y}) = 0, \quad i \neq 1$$

(see [OSS, p.285]). Consider the sheaves of $\mathcal{O}_S$-modules

$$E_{-1} := R^1p_*(\mathcal{E} \otimes \mathcal{O}_Q(-2) \otimes p^*\mathcal{O}_S(2)), \quad E_0 := R^1p_*(\mathcal{E} \otimes \Omega^1_{Q/S}), \quad E_1 := R^1p_*(\mathcal{E} \otimes \mathcal{O}_Q(-1)). \tag{29}$$

The equalities (28) together with semicontinuity imply that E_{-1}, E_1 and E_0 are locally free $\mathcal{O}_S$-modules, and $\mathrm{rk}(E_{-1}) = \mathrm{rk}(E_1) = a$, and $\mathrm{rk}(E_0) = 2a + 2$. Moreover,

$$R^ip_*(\mathcal{E} \otimes \mathcal{O}_Q(-2)) = R^ip_*(\mathcal{E} \otimes \Omega^1_{Q/S}) = R^ip_*(\mathcal{E} \otimes \mathcal{O}_Q(-1)) = 0 \tag{30}$$

for $i \neq 1$. Note that $\mathcal{E}^\vee \simeq \mathcal{E}$ as $c_1(\mathcal{E}) = 0$ and $\mathrm{rk}\mathcal{E} = 2$. Furthermore, (26) implies that the nondegenerate pairing (p-relative Serre duality) $R^1p_*(\mathcal{E} \otimes \mathcal{O}_Q(-1)) \otimes R^1p_*(\mathcal{E}^\vee \otimes \mathcal{O}_Q(1) \otimes \omega_{Q/S}) \to R^2p_*\omega_{Q/S} = \mathcal{O}_S$ can be rewritten as $E_1 \otimes E_{-1} \to \mathcal{O}_S$, thus giving an isomorphism

$$E_{-1} \simeq E_1^\vee. \tag{31}$$

Similarly, since $\mathcal{E}^\vee \simeq \mathcal{E}$ and $\Omega^1_{Q/S} \simeq T_{Q/S} \otimes \omega_{Q/S}$, p-relative Serre duality yields a nondegenerate pairing $E_0 \otimes E_0 = R^1p_*(\mathcal{E} \otimes \Omega^1_{Q/S}) \otimes R^1p_*(\mathcal{E} \otimes \Omega^1_{Q/S}) = R^1p_*(\mathcal{E} \otimes \Omega^1_{Q/S}) \otimes R^1p_*(\mathcal{E}^\vee \otimes T_{Q/S} \otimes \omega_{Q/S}) \to R^2p_*\omega_{Q/S} = \mathcal{O}_S$. Therefore E_0 is self-dual, i.e. $E_0 \simeq E_0^\vee$, and in particular $c_1(E_0) = 0$.

Now, let J denote the fiber product $Q \times_S Q$ with projections $Q \overset{pr_1}{\leftarrow} J \overset{pr_2}{\rightarrow} Q$ such that $p \circ pr_1 = p \circ pr_2$. Put $F_1 \boxtimes F_2 := pr_1^* F_1 \otimes pr_2^* F_2$ for sheaves F_1 and F_2 on Q, and consider the standard $\mathcal{O}_J$-resolution of the structure sheaf $\mathcal{O}_\Delta$ of the diagonal $\Delta \hookrightarrow J$:

$$(32) \qquad 0 \to \mathcal{O}_Q(-1) \otimes p^*\mathcal{O}_S(2) \boxtimes \mathcal{O}_Q(-2) \to {\Omega^1}_{Q/S}(1) \boxtimes \mathcal{O}_Q(-1) \to \mathcal{O}_J \to \mathcal{O}_\Delta \to 0.$$

Twist this sequence by the sheaf $(\mathcal{E} \otimes \mathcal{O}_Q(-1)) \boxtimes \mathcal{O}_Q(1)$ and apply the functor $R^i pr_{2*}$ to the resulting sequence. In view of (29) and (30) we obtain the following monad for $\mathcal{E}$:

$$(33) \qquad 0 \to p^*E_{-1} \otimes \mathcal{O}_Q(-1) \xrightarrow{\lambda} p^*E_0 \xrightarrow{\mu} p^*E_1 \otimes \mathcal{O}_Q(1) \to 0, \qquad \ker(\mu)/\mathrm{im}(\lambda) = \mathcal{E}.$$

Put $R := p^*h$, where h is the class of a line in S. Furthermore, set $H := \sigma^* H_X$, $[\mathbb{P}_\alpha] := \sigma^*[\mathbb{P}^2_\alpha]$, $[\mathbb{P}_\beta] := \sigma^*[\mathbb{P}^2_\beta]$, where H_X is the class of a hyperplane section of X (via the Plücker embedding), and respectively, $[\mathbb{P}^2_\alpha]$ and $[\mathbb{P}^2_\beta]$ are the classes of an α- and β-plane. Note that, clearly, $\mathcal{O}_Q(H) \simeq \mathcal{O}_Q(1)$. Thus, taking into account the duality (31), we rewrite the monad (33) as:

$$(34) \qquad 0 \to p^*E_1^\vee \otimes \mathcal{O}_Q(-H) \xrightarrow{\lambda} p^*E_0 \xrightarrow{\mu} p^*E_1 \otimes \mathcal{O}_Q(H) \to 0, \qquad \ker(\mu)/\mathrm{im}(\lambda) \simeq \mathcal{E}.$$

As a next step, we are going to express all Chern classes of the sheaves in the monad (34) in terms of a. We start by writing down the Chern polynomials of the bundles $p^*E_1 \otimes \mathcal{O}_Q(H)$ and $p^*E_1^\vee \otimes \mathcal{O}_Q(-H)$ in the form:

$$(35) \qquad c_t(p^*E_1 \otimes \mathcal{O}_Q(H)) = \prod_{i=1}^{a}(1 + (\delta_i + H)t), \quad c_t(p^*E_1^\vee \otimes \mathcal{O}_Q(-H)) = \prod_{i=1}^{a}(1 - (\delta_i + H)t),$$

where δ_i are the Chern roots of the bundle p^*E_1. Thus

$$(36) \qquad cR^2 = \sum_{i=1}^{a} \delta_i^2, \quad dR = \sum_{i=1}^{a} \delta_i.$$

for some $c, d \in \mathbb{Z}$. Next we invoke the following easily verified relations in $A^\cdot(Q)$:

$$(37) \qquad H^4 = RH^3 = 2[pt], \quad R^2H^2 = R^2[\mathbb{P}_\alpha] = RH[\mathbb{P}_\alpha] = H^2[\mathbb{P}_\alpha] = RH[\mathbb{P}_\beta] = H^2[\mathbb{P}_\beta] = [pt],$$
$$[\mathbb{P}_\alpha][\mathbb{P}_\beta] = R^2[\mathbb{P}_\beta] = R^4 = R^3H = 0,$$

where $[pt]$ is the class of a point. This, together with (36), gives:

(38)
$$\sum_{1\le i<j\le a} \delta_i^2\delta_j^2 = \sum_{1\le i<j\le a} (\delta_i^2\delta_j + \delta_i\delta_j^2)H = 0, \sum_{1\le i<j\le a} \delta_i\delta_j H^2 = \frac{1}{2}(d^2 - c)[pt], \sum_{1\le i\le a} (\delta_i + \delta_j)H^3 = 2(a-1)d[pt].$$

Note that, since $c_1(E_0) = 0$,

$$(39) \qquad c_t(p^*E_0) = 1 + bR^2t^2$$

for some $b \in \mathbb{Z}$. Furthermore,

$$(40) \qquad c_t(\mathcal{E}) = 1 + a[\mathbb{P}_\alpha]t^2$$

by the condition of the Proposition. Substituting (39) and (40) into the polynomial $f(t) := c_t(\mathcal{E})c_t(p^*E_1 \otimes \mathcal{O}_Q(H))c_t(p^*E_1^\vee \otimes \mathcal{O}_Q(-H))$, we have $f(t) = (1 + a[\mathbb{P}_\alpha]t^2)\prod_{i=1}^{a}(1 - (\delta_i + H)^2t^2)$. Expanding $f(t)$ in the variable t and using (36)-(38), we obtain:

$$(41) \qquad f(t) = 1 + (a[\mathbb{P}_\alpha] - cR^2 - 2dRH - aH^2)t^2 + e[pt]t^4,$$

where

$$(42) \qquad e = -3c - a(2d + a) + (a-1)(a + 4d) + 2d^2.$$

Next, the monad (34) implies $f(t) = c_t(p^*E_0)$. A comparison of (41) with (39) yields

$$(43) \qquad c_2(\mathcal{E}) = a[\mathbb{P}_\alpha] = (b + c)R^2 + 2dRH + aH^2,$$

$$(44) \qquad e = c_4(p^*E_0) = 0.$$

The relation (44) is the crutial relation which enables us to express the Chern classes of all sheaves in the monad (34) just in terms of a. More precisely, (43) and (37) give $0 = c_2(\mathcal{E})[\mathbb{P}_\beta] = 2d+a$, hence $a = -2d$. Substituting these latter equalities into (42) we get $e = -a(a-2)/2-3c$. Hence $c = -a(a-2)/6$ by (44). Since $a = -2d$, (36) and the equality $c = -a(a-2)/6$ give $c_1(E_1) = -a/2, \quad c_2(E_1) = (d^2-c)/2 = a(5a-4)/24$. Substituting this into the standard formulas $e_k := c_k(p^*E_1 \otimes \mathcal{O}_Q(H)) = \sum_{i=0}^{2} \binom{a-i}{k-i} R^i H^{k-i} c_i(E_1), \quad 1 \le k \le 4$, we obtain

$$e_1 = -aR/2 + aH, \quad e_2 = (5a^2/24 - a/6)R^2 + (a^2-a)(-RH+H^2)/2, \tag{45}$$

$$e_3 = (5a^3/24 - 7a^2/12 + a/3)R^2H + (-a^3/4 + 3a^2/4 - a/2)RH^2 + (a^3/6 - a^2/2 + a/3)H^3,$$

$$e_4 = (-7a^4/144 + 43a^3/144 - 41a^2/72 + a/3)[pt].$$

It remains to write down explicitely $c_2(p^*E_0)$: (37), (43) and the relations $a = -2d$, $c = -a(a-2)/6$ give $a = c_2(\mathcal{E})[\mathbb{P}_\alpha] = b + c$, hence

$$c_2(E_0) = b = (a^2+4a)/6 \tag{46}$$

by (39).

Our next and final step will be to obtain a contradiction by computing the Euler characteristic of the sheaf $\mathcal{E}$ and two different ways. We first compute the Todd class $\mathrm{td}(T_Q)$ of the bundle T_Q. From the exact triple dual to (25) we find $c_t(T_{Q/S}) = 1 + (-2R+3H)t + (2R^2 - 4RH + 3H^2)t^2$. Next, $c_t(Q) = c_t(T_{Q/S})c_t(p^*T_Q)$. Hence $c_1(T_Q) = R+3H$, $c_2(T_Q) = -R^2+5RH+3H^2$, $c_3(T_Q) = -3R^2H + 9H^2R$, $c_4(T_Q) = 9[pt]$. Substituting into the formula for the Todd class of T_Q, $\mathrm{td}(T_Q) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 - \frac{1}{720}(c_1^4 - 4c_1^2c_2 - 3c_2^2 - c_1c_3 + c_4)$, where $c_i := c_i(T_Q)$ (see, e.g., [H, p.432]), we get:

$$\mathrm{td}(T_Q) = 1 + \frac{1}{2}R + \frac{3}{2}H + \frac{11}{12}RH + H^2 + \frac{1}{12}HR^2 + \frac{3}{4}H^2R + \frac{3}{8}H^3 + [pt]. \tag{47}$$

Next, by the condition of Proposition $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = a[\mathbb{P}_\alpha]$, $c_3(\mathcal{E}) = c_4(\mathcal{E}) = 0$. Substituting this into the general formula for the Chern character of a vector bundle F,

$$\mathrm{ch}(F) = \mathrm{rk}(F) + c_1 + (c_1^2 - 2c_2)/2 + (c_1^3 - 3c_1c_2 - 3c_3)/6 + (c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4)/24,$$

$c_i := c_i(F)$ (see, e.g., [H, p.432]), and using (47), we obtain by the Riemann-Roch Theorem for $F = \mathcal{E}$

$$\chi(\mathcal{E}) = \frac{1}{12}a^2 - \frac{23}{12}a + 2. \tag{48}$$

In a similar way, using (45), we obtain:

$$\chi(p^*E_1 \otimes \mathcal{O}_Q(H)) + \chi(p^*E_1^\vee \otimes \mathcal{O}_Q(-H)) = \frac{5}{216}a^4 - \frac{29}{216}a^3 - \frac{1}{54}a^2 + \frac{113}{36}a. \tag{49}$$

Next, in view of (46) and the equality $c_1(E_0) = 0$ the Riemann-Roch Theorem for E_0 easily gives

$$\chi(p^*E_0) = \chi(E_0) = -\frac{1}{6}a^2 + \frac{4}{3}a + 2. \tag{50}$$

Together with (48) and (49) this yields

$$\Phi(a) := \chi(p^*E_0) - (\chi(\mathcal{E}) + \chi(p^*E_1 \otimes \mathcal{O}_Q(H)) + \chi(p^*E_1^\vee \otimes \mathcal{O}_Q(-H))) = -\frac{5}{216}a(a-2)(a-3)(a-\frac{4}{5}).$$

The monad (34) implies now $\Phi(a) = 0$. The only positive integer roots of the polynomial $\Phi(a)$ are $a = 2$ and $a = 3$. However, (48) implies $\chi(\mathcal{E}) = -\frac{3}{2}$ for $a = 2$, and (50) implies $\chi(p^*E_0) = \frac{9}{2}$ for $a = 3$. This is a contradiction as the values of $\chi(\mathcal{E})$ and $\chi(p^*E_0)$ are integers by definition. $\square$

We need a last piece of notation. Consider the flag variety $Fl(k_m-2,k_m+2;V)$. Any point $u=(V^{k_m-2},V^{k_m+2})\in Fl(k_m-2,k_m+2;V)$ determines a standard extension

$$i_u:\ X=G(2;4)\hookrightarrow X_m, \tag{51}$$

$$W^2\mapsto V^{k_m-2}\oplus W^2\subset V^{k_m+2}=V^{k_m-2}\oplus W^4, \tag{52}$$

where $W^2\in X=G(2;W^4)$ and an isomorphism $V^{k_m-2}\oplus W^4\simeq V^{k_m+2}$ is fixed (clearly i_u does not depend on the choice of this isomorphism). We clearly have isomorphisms of Chow groups

$$i_u^*:\ A^2(X_m)\xrightarrow{\sim}A^2(X),\quad i_{u*}:\ A_2(X)\xrightarrow{\sim}A_2(X_m), \tag{53}$$

where the flag variety $Y_m:=Fl(k_m-1,k_m+1;V^{n_m})$ (respectively, $Y:=Fl(1,3;4)$) is the set of lines in X_m (respectively, in X).

Theorem 5.4. *Let $\mathbf{X}=\lim\limits_{\to}X_m$ be a twisted ind-grassmannian. Then any vector bundle $\mathbf{E}=\lim\limits_{\leftarrow}E_m$ on $\mathbf{X}$ of rank 2 is trivial, and hence Conjecture 1.1(iv) holds for vector bundles of rank 2.*

Proof. Fix $m\geq\max\{m_0,m_1\}$, where m_0 and m_1 are as in Corollary 5.1 and Lemma 5.2. For $j=1,2$, let $E^{(j)}$ denote the restriction of E_m to a projective plane of type $\mathbb{P}^2_{j,m}$, $T^j\simeq Fl(k_m-j,k_m+3-j,V^{n_m})$ be the variety of planes of the form $\mathbb{P}^2_{j,m}$ in X_m, and $\Pi^j:=\{\mathbb{P}^2_{j,m}\in T^j|\ E_{m|\mathbb{P}^2_{j,m}}$ is properly unstable (i.e. not semistable)$\}$. As semistability is an open condition, Π^j is a closed subset of T^j.

(i) Assume that $c_2(E^{(1)})>0$. Then, since $m\geq m_1$, Lemma 5.2 implies $c_2(E^{(2)})\leq 0$.

(i.1) Suppose that $c_2(E^{(2)})=0$. If $\Pi^2\neq T^2$, then for any $\mathbb{P}^2_{2,m}\in T^2\smallsetminus\Pi^2$ the corresponding bundle $E^{(2)}$ is semistable, hence $E^{(2)}$ is trivial as $c_2(E^{(2)})=0$, see [DL, Prop. 2.3,(4)]. Thus, for a generic point $u\in Fl(k_m-2,k_m+2;V^{n_m})$, the bundle $E=i_u^*E_m$ on $X=G(2;4)$ satisfies the conditions of Proposition 5.3, which is a contradiction.

We therefore assume $\Pi^2=T^2$. Then for any $\mathbb{P}^2_{2,m}\in T^2$ the corresponding bundle $E^{(2)}$ has a maximal destabilizing subsheaf $0\to\mathcal{O}_{\mathbb{P}^2_{2,m}}(a)\to E^{(2)}$. Moreover $a>0$. In fact, otherwise the condition $c_2(E^{(2)})=0$ would imply that $a=0$ and $E^{(2)}/\mathcal{O}_{\mathbb{P}^2_{2,m}}=\mathcal{O}_{\mathbb{P}^2_{2,m}}$, i.e. $E^{(2)}$ would be trivial, in particular semistable. Hence

$$\mathbf{d}_{E^{(2)}}=(a,-a). \tag{54}$$

Since any line in X_m is contained in a plane $\mathbb{P}^2_{2,m}\in T^2$, (54) implies $\mathbf{d}_{E_m}=(a,-a)$ with $a>0$ for $m>m_0$, contrary to Corollary 5.1.

(i.2) Assume $c_2(E^{(2)})<0$. Since $E^{(2)}$ is not stable for any $\mathbb{P}^2_{2,m}\in T^2$, its maximal destabilizing subsheaf $0\to\mathcal{O}_{\mathbb{P}^2_{2,m}}(a)\to E^{(2)}$ clearly satisfies the condition $a>0$, i.e. $E^{(2)}$ is properly unstable, hence $\Pi^2=T^2$. Then we again obtain a contradiction as above.

(ii) Now we assume that $c_2(E^{(2)})>0$. Then, replacing $E^{(2)}$ by $E^{(1)}$ and vice versa, we arrive to a contradiction by the same argument as in case (i).

(iii) We must therefore assume $c_2(E^{(1)})=c_2(E^{(2)})=0$. Set $D(E_m):=\{l\in Y_m|\ \mathbf{d}_{E_m}(l)\neq(0,0)\}$ and $D(E):=\{l\in Y|\ \mathbf{d}_E(l)\neq(0,0)\}$. By Corollary 5.1, $\mathbf{d}_{E_m}=(0,0)$, respectively, $\mathbf{d}_E(l)=(0,0)$ for a generic line $l\in Y$. Then by deformation theory [B], $D(E_m)$ (respectively, $D(E)$) is an effective divisor on Y_m (respectively, on Y). Hence, $\mathcal{O}_Y(D(E))=p_1^*\mathcal{O}_{Y^1}(a)\otimes p_2^*\mathcal{O}_{Y^2}(b)$ for some $a,b\geq 0$, where p_1, p_2 are as in diagram (2). Note that each fiber of p_1 (resp., of p_2) is a plane $\tilde{\mathbb{P}}^2_\alpha$ dual to some α-plane $\mathbb{P}^2_\alpha$ (respectively, a plane $\tilde{\mathbb{P}}^2_\beta$ dual to some β-plane $\mathbb{P}^2_\beta$). Thus, setting $D(E_{|\mathbb{P}^2_\alpha}):=\{l\in\tilde{\mathbb{P}}^2_\alpha|\ \mathbf{d}_E(l)\neq(0,0)\}$, $D(E_{|\mathbb{P}^2_\beta}):=\{l\in\tilde{\mathbb{P}}^2_\beta|\ \mathbf{d}_E(l)\neq(0,0)\}$, we obtain $\mathcal{O}_{\tilde{\mathbb{P}}^2_\alpha}(D(E_{|\mathbb{P}^2_\alpha}))=\mathcal{O}_Y(D(E))_{|\tilde{\mathbb{P}}^2_\alpha}=\mathcal{O}_{\tilde{\mathbb{P}}^2_\alpha}(b)$, $\quad\mathcal{O}_{\tilde{\mathbb{P}}^2_\beta}(D(E_{|\mathbb{P}^2_\beta}))=\mathcal{O}_Y(D(E))_{|\tilde{\mathbb{P}}^2_\beta}=\mathcal{O}_{\tilde{\mathbb{P}}^2_\beta}(a)$. Now if $E_{|\mathbb{P}^2_\alpha}$ is semistable, a theorem of Barth [OSS, Ch. II, Theorem 2.2.3] implies that $D(E_{|\mathbb{P}^2_\alpha})$ is a divisor of degree $c_2(E_{|\mathbb{P}^2_\alpha})=a$ on $\mathbb{P}^2_\alpha$. Hence $a=c_2(E^{(1)})=0$ for a semistable $E_{|\mathbb{P}^2_\alpha}$. If

$E_{|\mathbb{P}^2_\alpha}$ is not semistable, it is unstable and the equality $\mathbf{d}_E(l) = (0,0)$ yields $\mathbf{d}_{E_{|\mathbb{P}^2_\alpha}} = (0,0)$. Then the maximal destabilizing subsheaf of $E_{|\mathbb{P}^2_\alpha}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2_\alpha}$ and, since $c_2(E_{|\mathbb{P}^2_\alpha}) = 0$, we obtain an exact triple $0 \to \mathcal{O}_{\mathbb{P}^2_\alpha} \to E_{|\mathbb{P}^2_\alpha} \to \mathcal{O}_{\mathbb{P}^2_\alpha} \to 0$, such that $E_{|\mathbb{P}^2_\alpha} \simeq \mathcal{O}^{\oplus 2}_{\mathbb{P}^2_\alpha}$ is semistable, a contradiction. This shows that $a = 0$ whenever $c_2(E^{(1)}) = c_2(E^{(2)}) = 0$. Similarly, $b = 0$. Therefore $D(E_m) = \emptyset$, and Proposition 4.1 implies that E_m is trivial. Therefore $\mathbf{E}$ is trivial as well. $\square$

In [DP] Conjecture 1.1 (iv) was proved not oly when $\mathbf{X}$ is a twisted projective ind-space, but also for finite rank bundles on special twisted ind-grassmannians defined through certain homogeneous embeddings φ_m. These include embeddings of the form

$$G(k;n) \to G(kw;na)$$

$$V^k \subset V \mapsto V^k \otimes W \subset V \otimes W,$$

where W is a fixed a-dimensional subspace, or of the form

$$G(k;n) \to G\left(\frac{k(k+1)}{2}; n^2\right)$$

$$V^k \subset V \mapsto S^2(V^k) \subset V \otimes V.$$

More precisely, Conjecture 1.1 (iv) was proved in [DP] for twisted ind-grassmannians whose defining embeddings are homogeneous embeddings satisfying some specific numerical conditions relating the degrees $\deg \varphi_m$ with the pairs of integers (k_m, n_m). There are many twisted ind-grassmannians for which those conditions are not satisfied. For instance, this applies to the ind-grassmannians defined by iterating each of the following embeddings:

$$G(k;n) \to G\left(\frac{k(k+1)}{2}; \frac{n(n+1)}{2}\right)$$

$$V^k \subset V \mapsto S^2(V^k) \subset S^2(V),$$

$$G(k;n) \to G\left(\frac{k(k-1)}{2}; \frac{n(n-1)}{2}\right)$$

$$V^k \subset V \mapsto \Lambda^2(V^k) \subset \Lambda^2(V).$$

Therefore the resulting ind-grassmannians $\mathbf{G}(k,n,S^2)$ and $\mathbf{X}(k,n,\Lambda^2)$ are examples of twisted ind-grassmannians for which Theorem 5.4 is new.

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