EVERY HOLOMORPHIC SYMPLECTIC MANIFOLD ADMITS A KÄHLER METRIC

Andrey N. Todorov

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Str. 26
D-5300 Bonn 3

Institute of Mathematics
P.O.Box 373
Sofia
1090 Bulgaria

MPI/SFB 85-43

EVERY HOLOMORPHIC SYMPLECTIC MANIFOLD ADMITS A KÄHLER METRIC

Andrey N. Todorov

§ 0. Introduction

Definition. Suppose that X is a compact simply connected complex manifold such that 1) on X there exists a closed holomorphic two form $\omega_X(2,0)$ such that at each point $\chi \in X$ $\omega_X(2,0)$ defines a non-degenerate skew symmetric matrix, i.e. for $\forall \chi \in X$ and $\omega_X(2,0)|_U = \sum \omega_{\alpha\beta}\, dz^\alpha \wedge dz^\beta$, where U is an open neighborhood of χ,

$$\det \omega_{\alpha\beta}(z) \in \Gamma(U, \mathcal{O}_U^*)$$

2) $\dim_{\mathbb{C}} H^2(X, \mathcal{O}_X) = 1.$

Remark. Condition 2) is equivalent to the fact that on X we have a unique up to a constant closed holomorphic two form on X, $\omega_X(2,0)$.

In this article we prove the following theorem.

<u>Theorem.</u> Every holomorphic symplectic manifold admits a Kähler metric.

The two main points of the proof are: a) On a holomorphic symplectic manifold there exists a real closed two form $\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2}$, where $\omega^{1,1}$ (the (1,1) part of ω) at each point is positive definite and $\omega^{2,0} = \partial\alpha^{1,0}$. The construction of ω is done by checking the conditions of theorem 38 in the beautiful paper of R. Harvey and B. Lawson, see [6]. From the existence of the form ω we prove,following Bogomolov that there exists a non-singular family $\chi \to U$ of symplectic holomorphic manifolds with $\dim_{\mathbb{C}} U = \dim_{\mathbb{C}} M^2(X,\mathbb{C}) - 2$ in the Kuranishi family of X

Again from the existence of ω and Moishezon-Nakai criterium we get that in U we can find an open and everywhere dense subset W such that each point $t \in W$ correspond to a Kähler symplectic manifold. Now we come to the second main point.

b) Using the Yau's solution of Calabi conjecture we can construct the so called isometric deformations. From the isometric deformation we construct two families over the unit disc D, $\chi \to D$ and $\chi' \to D$, where all fibres $X' \to D$ are Kähler manifolds and more over these two families are isomorphic over some disc $D_1 \subset D$. Now from local Torelli theorem and lemma due to D. Burus Rapoport and Siu we can conclude that the two families are isomorphic. X is contained in $\chi \to D$.

We are following the lines of Siu in [7]. (See also [8].)

The construction of the real closed two form ω such that $\omega^{1,1}$ is positive definite everywhere is based on the idea of D. Sullivan to use Hahn-Banach theorem.

This article was finished during may stay in the Max-Planck-Insitut für Mathematik in Bonn. I want to thank the Max-Planck-Institut für excellent working conditions.

Theorem. Every symplectic holomorphic manifold admits a Kähler metric.

Prooof: The proof is based on the following lemma due to R. Harvey and B. Lawson Jr. See [6].

§1. Lemma 1. Let X be a holomorphic symplectic manifold, then X admits a real closed two form $\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2}$ such that a) $\omega^{1,1}$ is positive definite at each point $\chi \in X$ b) $\omega^{2,0} = \partial\alpha^{1,0}$, $\omega^{0,2} = \bar{\partial}\alpha^{0,1}$, $\alpha^{0,1} = \overline{\alpha^{1,0}}$.

Proof of Lemma 1: In [6] R. Harvey and B. Lawson proved:

Theorem. Suppose X is a compact complex manifold, then X admits a real closed two form $\omega = \omega^{2,0} + \omega^{1,1} + \omega^{0,2}$ with a) $\omega^{1,1}$ positive definite everywhere b) $\omega^{2,0} = \partial\alpha^{1,0}$ for some $1,0$ form α if and only if X does not support a (non-trivial) positive, d-closed current which is the bi-dimension $(1,1)$ component of a boundary.

So we need to check that if X is a holomorphic symplectic manifold then X satisfies the condition of the theorem of R. Harvey and B. Lawson Jr..

Let $\mu = \sqrt{-1}\Sigma \quad \eta^{i\bar{j}} \dfrac{\partial}{\partial z^i} \wedge \dfrac{\partial}{\overline{\partial z^j}}$ be an exact real $(1,1)$ positive current on X. Since on X we have a closed holomorphic form $\omega_X(2,0)$ which is non-degenerate at each point $\chi \in X$, so we can repeat the arguments of Darboux lemma (See [1]) and we will get a local coordinate system $(z^1,\dots,z^n,\dots,z^{2n})$ such that locally $\omega_X(2,0) = \sum_{i=1}^{n} dz^i \wedge dz^{i+n}$. Now we get immediately from $\mu(1,1)$-current, an exact $(n-1,\ n-1)$ curren η in the following way:

$$\eta = \mu \wedge (\underbrace{\omega_X^*(2,0)\wedge\dots\wedge\omega_X^*(2,0)}_{n-1}\ \wedge\underbrace{\overline{\omega_X^*(2,0)}\wedge\dots\wedge\overline{\omega_X^*(2,0)}}_{n-1})$$

where

$$\omega_X^*(2,0) = \sum_{i=1}^{n} \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^{i+n}}$$

$\alpha=\eta\ \ s(\omega_X(2,0)^n\wedge\omega_X(0,2))^n$ has distribution coefficients, where

$$\omega_X(2,0)^n = \omega_X(2,0)\wedge\dots\wedge\omega_X(2,0)\ \ (n\text{-times})$$

so $\qquad \alpha = d\gamma$

where γ is also real 1-form on X. We can write

$$\gamma = \beta + \overline{\beta}$$

where β is a $(1,0)$-form on X. Since η is of type $(1,1)$, it follows

$$\alpha = \overline{\partial}\beta + \partial\overline{\beta} \quad \text{and} \quad \overline{\partial}\overline{\beta} = 0$$

So from $\overline{\partial}\overline{\beta} = 0$ it follows that $\overline{\beta} \in H^1(X,0_X)$.

<u>Proposition 1.1.</u> $H^1(X,0_X) = 0$, where X is a holomorphic symplectic manifold.

Proof: Since $\pi_1(X) = 0$ it follows that Pic X is a discrete set. This is a standard fact. We know that $H^1(X,0_X)$ can be interpreted as the tangent space to the Picard variety of X. Since it is a discrete set we get that $H^1(X,0_X) = 0$.

Q.E.D.

Now since $\overline{\beta} \in H^1(X,0_X) = 0 \Rightarrow \overline{\beta} = \overline{\partial}\sigma$ for some (0,0)-current on X. Hence

$$\alpha = \sqrt{-1}\partial\overline{\partial}\tau \ ; \text{ where } \ \tau = \sqrt{-1}(\overline{\sigma} - \sigma)$$

The positivety of the (1,1)-current on X implies that τ is a plurisubharmonic function on X. By the compactness of X and the maximum principle we get that $\tau \equiv$ const. So

$$\alpha = \partial\overline{\partial} \text{ const} \equiv 0$$

Q.E.D.

So we have proved the following fact:

Fact 1. Suppose that η is positive (1,1) current and $\eta = d\varphi$, then $\eta = 0$.

Fact 2. Suppose that η is a positive closed (1,1) current (form) and $\eta = (d\alpha)_{(1,1)}$ (i.e. η is a (1,1) component of a boundary), then $\eta \equiv 0$.

<u>Proof</u>: Since $d\eta = 0$ and $\eta = \bar{\partial}\alpha^{1,0} + \partial\alpha^{0,1} \Rightarrow \partial\bar{\partial}\alpha^{1,0} = -\bar{\partial}\partial\alpha^{1,0} = 0$. From the regularity of the $\bar{\partial}$ operator we get that $\partial\alpha^{1,0}$ is a holomorphic two form on X. From the fact that $\omega_X(2,0)$ is a closed non-degenerate form at each point we get by easy calculations that

$$(*) \qquad 0 < \int_M \partial\alpha^{1,0} \wedge \overline{\partial\alpha^{1,0}} \wedge \underbrace{\omega_X(2,0)\wedge\ldots\wedge\omega_X(2,0)}_{n-1} \wedge \underbrace{\overline{\omega_X(2,0)}\wedge\ldots\wedge\overline{\omega_X(2,0)}}_{n-1}$$

where $2n = \dim_{\mathbb{C}} X$. On the other hand we have by Stoke's theorem

$$(**) \qquad 0 = \int_X d(\alpha^{1,0} \wedge \overline{\partial\alpha^{1,0}} \wedge \underbrace{\omega_X(2,0)\wedge\ldots\wedge\omega_X(2,0)}_{n-1} \wedge \underbrace{\overline{\omega_X}(2,0)\wedge\ldots\wedge\overline{\omega_X}(2,0)}_{n-1})$$

So from (*) and (**) we get that $\partial\alpha^{1,0} \equiv 0$. So from here we get that

$$\eta = d\alpha$$

From fact 1 we ge that $\eta \equiv 0$.

Q.E.D:

From fact 1 and fact 2 it follows that the conditions of the theorem of R. Harvey and Bl. Lawson Jr. are fulfilled. So lemma 1 is proved.

§2. Lemma 2. Let ω be the real closed 2-form constructed in lemma 1, then

a) ω defines a non-zero class $[\omega]$ in $H^2(X,\mathbb{R})$

b) there exists a real closed (1,1) form Θ such that $[\Theta] \equiv [\omega]$ in $H^2(X,\mathbb{R})$.

Proof.

Proposition 2.1. $\int_X \underbrace{\omega\wedge\ldots\wedge\omega}_{2n} > 0 \qquad 2n = \dim_{\mathbb{C}} X$

Proof. We need to compute:

$$\underbrace{(\omega^{2,0}+\omega^{1,1}+\omega^{0,2})\wedge\ldots\wedge(\omega^{2,0}+\omega^{1,1}+\omega^{0,2})}_{2n} =$$

$$= \underbrace{\omega\wedge\ldots\wedge\omega}_{2n} = \underbrace{\omega^{2,0}\wedge\ldots\wedge\omega^{2,0}}_{n}\wedge\underbrace{\omega^{0,2}\wedge\ldots\wedge\omega^{0,2}}_{n} +$$

$$+ \sum_{K=1}^{2n-1} c_K \omega_K^{2,0}\wedge\overline{\omega_K^{2,0}}\wedge\underbrace{\omega^{1,1}\wedge\ldots\wedge\omega^{1,1}}_{2n-2k} + \underbrace{\omega^{1,1}\wedge\ldots\wedge\omega^{1,1}}_{2n}$$

where $\omega_K^{2,0} = \underbrace{\omega^{2,0} \wedge \ldots \wedge \omega^{2,0}}_{K}$ and C_K is a positive integer.

From the following lemma proved in [4]:

<u>Lemma</u>. Let η be a primitive form of type (p,q), then

$$*\eta = \frac{i^{p-q}}{(2n-p-q)}\,(-1)^{\frac{(p+q)(p+q+1)}{2}}\; L^{2n-p-q}\,\eta$$

we get that $*\omega_K^{2,0} = \omega^{1,1} \wedge \ldots \wedge \omega^{1,1} \wedge \omega_K^{2,0}$, where $*$ is the Hodge star operator with respect to the metric induced on X by $\omega^{1,1}$.

So we get:

$$\int_X \omega\wedge\ldots\wedge\omega = \int_X \omega_n^{2,0} \wedge \bar{\omega}_n^{2,0} + \sum \int_X c_K \omega_K^{2,0} \wedge *\overline{\omega_K^{2,0}} + \int_X \underbrace{\omega^{1,1}\wedge\ldots\wedge\omega^{1,1}}_{2n}$$

$= \| \omega_n^{2,0} \|^2 + \sum c_K \| \omega_K^{2,0} \|^2 + \mathrm{vol}(X) > 0,$ where the norm is taken with respect to the metric induced by $\omega^{1,1}$.

Q.E.D.

From poposition 2.1. it follows immediately that ω defines a non-zero class $[\omega]$ in $H^2(X,\mathbb{R})$, indeed if $\omega = d\eta$, then

$$0 = \int_X d(\eta \wedge \underbrace{\omega\wedge\ldots\wedge\omega}_{2n-1}) = \int_X \underbrace{\omega\wedge\omega\wedge\ldots\wedge\omega}_{2n}$$

and we get a contradiction. This proves part a) of lemma 2.

Proof of b): Let $\Theta = \omega^{1,1} - \bar\partial\alpha^{1,0} - \partial\alpha^{0,1}$. Remember that $\omega = \partial\alpha^{1,0} + \omega^{1,1} + \overline{\partial\alpha^{1,0}} = \partial\alpha^{1,0} + \omega^{1,1} + \overline{\partial\alpha^{0,1}}$. Now clearly:

$$d\Theta = \partial\omega^{1,1} - \partial\bar\partial\alpha^{1,0} + \bar\partial\omega^{1,1} - \bar\partial\partial\alpha^{0,1} = 0$$

and $$\Theta - \omega = \omega^{1,1} - \bar\partial\alpha^{1,0} - \omega^{1,1} - \partial\alpha^{1,0} - \bar\partial\alpha^{0,1} =$$

$$= d(\alpha^{1,0} + \alpha^{0,1}).$$

Q.E.D.

Corollary 2.2. $\dim H^1_{\mathbb{C}}(\Omega^1) > 0.$

Proof: This follows from 2.1. b).

Q.E.D.

§3.Lemma 3. Let $[\varphi]$ be a non zero element of $H^2(X,\mathbb{R})$ then $[\varphi] = c\omega_X(2,0) + [\varphi^{1,1}] + \bar c\omega_X(2,0)$, where $\omega_X(2,0)$ is the everywhere non-degenerate closed holomorphic form on X and φ is a real closed form of type $(1,1)$.

Proof: From de Rham theorem it follows that $[\varphi]$ can be realized as a real closed 2-form φ. Let

$$(*) \qquad \varphi = \varphi^{2,0} + \varphi^{1,1} + \varphi^{0,2},\ \varphi^{0,2} = \overline{\varphi^{2,0}},\varphi^{1,1} = \overline{\varphi^{1,1}}$$

From $d\varphi = 0$ we obtain:

$$\partial\varphi^{2,0} = \bar\partial\varphi^{0,2} = 0, \ \overline{\partial\varphi^{2,0}} + \partial\varphi^{1,1} = \partial\varphi^{0,2} + \overline{\partial\varphi^{1,1}} = 0.$$

Let us denote by $B_{1,1}$ currents of type (1,1) that are components of boundaries and by $Z_{1,1}$ d-closed currents of bidimension (1,1). Let $\alpha \in Z_{1,1} \cap B_{1,1}$, then since $\alpha = d\eta$

$$\langle\varphi,\alpha\rangle = \langle\varphi,d\eta\rangle = \langle d\varphi,\eta\rangle = 0$$

So $\varphi \in (Z_{1,1} \cap B_{1,1})^{\perp}$, where $(Z_{1,1} \cap B_{1,1})^{\perp}$ is the anhilator of $Z_{1,1} \cap B_{1,1}$ in $\varepsilon^2(X)_R$. R. Harvey and B. Lawson proved in [ ] that

$$B^{\perp}_{1,1} = \{\omega \in \varepsilon^{1,1}(X)_R \mid d\omega = 0\}$$

$$Z^{\perp}_{1,1} = \{\omega \in \varepsilon^{1,1}(X)_R \mid \omega = (d\alpha)^{1,1}, \alpha \in E^{1,0} + E^{0,1}\}$$

$\varepsilon^K(X)_R$ are C^∞ K-forms on X.

Even more they proved that $(Z_{1,1} \cap B_{1,1})^{\perp} = B^{\perp}_{1,1} + Z^{\perp}_{1,1}$ is a closed subspace in $\varepsilon^{1,1}(X)_R$. So since

$$(**) \qquad \varphi^{1,1} \in B^{\perp}_{1,1} + Z^{\perp}_{1,1}$$

it follows that for some α, $\varphi^{1,1} - (d\alpha)^{1,1}$ is a closed form So we get:

$$(***) \quad \partial\varphi^{1,1} - \partial\bar\partial\alpha^{1,0} = \bar\partial\varphi^{1,1} - \bar\partial\partial\alpha^{0,1} = 0$$

From (*) and (**) we get

$$(****) \qquad \bar\partial\varphi^{2,0} + \partial\bar\partial\alpha^{1,0} = \partial\varphi^{0,2} + \bar\partial\partial\alpha^{0,1} = 0$$

So

$$\bar\partial(\varphi^{2,0} - \partial\alpha^{1,0}) = 0$$

From here we get that $\varphi^{2,0} - \partial\alpha^{1,0}$ is a holomorphic closed two form on X. Since there is a unique up to a constant holomorphic closed two form, which is non-degenerate at each point $\chi \in X$ we get that:

$$\varphi^{2,0} - \partial\alpha^{1,0} = c\omega_X(2,0)$$

So from here and (**) we get that

$$\varphi - d(\alpha^{1,0} + \alpha^{0,1}) = c\omega_X(2,0) + (\varphi^{1,1} - \partial\alpha^{0,1} - \bar\partial\alpha^{1,0}) + \bar c\omega_X(0,2).$$

So

$$[\varphi] = c\omega_X(2,0) + \Theta^{1,1} + \bar c\omega_X(0,2)$$

where $\Theta^{1,1} = \varphi^{1,1} - (d\alpha)^{1,1}$ and $d(\Theta^{1,1}) = 0$

Q.E.D.

<u>Corollary 3.1.</u> If $[\omega] \in H^2(X,\mathbb{C}) = H^2(X,\mathbb{R}) \otimes \mathbb{C}$, then

$$[\omega] = a\omega_X(2,0) + \omega^{1,1} + b\omega_X(0,2),$$

where $d\omega^{1,1} = 0$ and $a,b \in \mathbb{C}$.

Let ω be the form constructed on X in lemma 1 and let $\Theta^{1,1}$ be a closed $(1,1)$ form that represents the non-zero class $[\omega] \in H^2(X,\mathbb{R})$ and let $C \subset X$ be an irreducible complex analytic subspace in X, then if $\dim_{\mathbb{C}} C = k$

$$\int_C [\Theta]^k > 0 \text{ , where } [\Theta^k] = \underbrace{[\Theta \wedge \dots \wedge \Theta]}_{k}$$

Proof: From chapter 1 of [ ] we know that:

$$\int_C [\Theta]^k = \int_{C \setminus \mathrm{Sing}\, C} [\Theta]^k$$

So from here we may suppose that C is a non-singular submanifold. Repeating the calculations and the arguments in Proposition 2.1. we get that:

$$\underbrace{\omega \wedge \dots \wedge \omega}_{k}\big|_C = \omega_K^{2,0} \wedge \overline{\omega_K^{2,0}} + \sum_{i=1}^{K-1} C_i \omega_i^{2,0} \wedge \overline{\omega_i^{2,0}} \wedge \underbrace{\omega^{1,1} \wedge \dots \wedge \omega^{1,1}}_{k-2i}$$

$+ \omega^{1,1} \wedge \dots \wedge \omega^{1,1}$ $(C_i \in \mathbb{R},\ C_i > 0)$. So using the same notations as in 2.1. we get

$$\int_C \varphi^k = \| \omega^{2,0} \|^2 + \sum C_i \| \omega^{2,0} \|^2 + \mathrm{vol}\ (C) > 0$$

here $\| \omega_i^{2,0} \|^k$ and $\mathrm{vol}(C)$ are with respect to the metric induced on C by the hermitian metric defined by $\omega^{1,1}$. So from $\int_C \varphi^k > 0$ it follows that C is not homological to zero. Indeed if $C = \partial B$, then by Stoke's theorem:

$\int_C \varphi^k = \int d(\varphi^k) = 0$ and we will get a contradiction. Now it

is easy to see that $\int\limits_C \theta^k = \int\limits_C \varphi^k > 0$.

Q.E.D.

§ 5. Local deformation theory of symplectic manifolds. (Bogomolov's theory).

First we will make some remarks.

Remark 5.1. a) The closed holomorphic non-degenerate two form $\omega_X(2,0)$ induces an isomorphism:

$$i_{\omega_X(2,0)} : \Theta_X \xrightarrow{\sim} \Omega^1$$

and so we get an isomorphism:

$$i_{\omega_X(2,0)} : H^1(X,\Theta_X) \xrightarrow{\sim} H^1(X,\Omega^1)$$

b) The "small" deformations of the complex structure on X given by a differential form

$$\varphi_t = \sum \varphi^{\,i}_{\,j}(t)\ d\bar{z}^j \otimes \frac{\partial}{\partial z^i} \in \Gamma(X,\Omega^{0,1} \otimes \Theta)$$

Using the isomorphism i_ω we get that:

$$i_{\omega_{(2,0)}}\varphi_t = \tilde{\varphi}_t \in \Gamma(X,\Omega^{0,1} \otimes \Omega^{1,0})$$

c) We have a bracket operation [,] on $\Gamma(X, \Omega^{0,1} \otimes \Theta)$ coming from a bracket operation on $\Gamma(X,\Theta)$. See [ ] i_ω transforms the bracket operation [,] into a corresponding bracket operation [,] on $\Gamma(X,\Omega^{0,1} \otimes \Omega^{1,0})$. Bogomolov proved the following lemma:

Lemma 5.2 (Bogomolov [3]). Suppose $\omega_1, \omega_2 \in \Gamma(X,\Omega^{0,1} \otimes \Omega^{1,0})$ and either

$$\partial\omega_1 = 0,\ \partial\omega_2 = 0 \text{ or } d\omega_1 = d\omega_2 = 0$$

Then we have respectively:

$$\partial[\omega_1,\omega_2] = 0 \quad \text{or} \quad d[\omega_1,\omega_2] = 0.$$

d) We know from Kodaira-Spencer deformation theory that first order infinitesimal deformations of a complex structure are contained in $H^1(X,\Theta)$ and so if X is a symplectic manifold then in $H^1(X,\Omega^1) \xrightarrow{\sim} H^1(X,\Theta)$.

Definition. Let $H^1_d(X,\Omega^1) = \{\omega \in H^1(X,\Omega^1) \mid d\omega = 0$ and ω defines a non-zero class $[\omega]$ in $H^2(X,\mathbb{C})\}$. Let $H^1_d(X,\Theta) = i^{-1}_{\omega_X(2,0)}\ (H^1_d(X,\Omega^1))$.

Remark 5.3. From lemma 3 it follows that $\dim H^1_d(X,\Omega^1) = \dim_{\mathbb{C}} H^2(X,\mathbb{C}) - 2 = b_2 - 2$.

Lemma 5 (Bogomolov). There are no obstructions for deformations of the complex structures on X that corresponds to the elements of $H^1_d(X,\Theta) \cong H^1_d(X,\Omega^1)$.

Remark 5.4. Lemma 5 can be stated into the following manner: There exists a subfamily $\begin{matrix} X \hookrightarrow & \mathcal{X} \\ \downarrow & \downarrow \\ 0 \in & U \end{matrix}$ of the Kuranishi family of X with the following properties:

1) U is a non-singular manifold and $\dim_{\mathbb{C}} U = b_2 - 2$.

2) The tangent space $T_{0,U} \cong H^1_d(X,\Theta)$ (In a natural manner.)

<u>Proof:</u> Let $\omega_1, \omega_2 \in H^1_d(X,\Omega^1)$. So we suppose that $d\omega_1 = d\omega_2 = 0$. If we prove that for every 3-dimensional cycle $\Gamma \in H_3(X,\mathbb{Z})$

$$\int_\Gamma [\omega_1,\omega_2] = 0$$

then we will have

$$[\omega_1,\omega_2] = 0 \quad \text{in} \quad H^2_d(X,\Omega^1) \subset H^3(X,\mathbb{C})$$

We can realize each cycle Γ of some basis $\{\ell_1,\ldots,\ell_{b_3}\}$ in $H_3(X,\mathbb{Z})$ by a number of three dimensional real manifolds $\Gamma_1,\ldots,\Gamma_3$ such that $\Gamma_i \cap \Gamma_j = \emptyset$. This follows immediately from Thom's results and since

$$\dim_{\mathbb{R}} \Gamma = 3 < \frac{1}{2} \dim_{\mathbb{R}} X \geq 8. \quad \text{See [3].}$$

Notice that $H_1(\Gamma_i,\mathbb{Z}) = 0$ since $H_1(X,\mathbb{Z}) = 0$ and so $H_2(\Gamma_i,\mathbb{Z}) = 0$

(Poincare duality) Bogomolov proved in [3] the following fact.

Proposition 5.5. For each cycle $[\Gamma]$ in $H_3(X,\mathbb{Z})$ we can find non-singular three dimensional compact manifolds Γ_i, realizing $[\Gamma]$ and those Γ_i fulfill the following conditions:

a) $H_1(\Gamma,\mathbb{Z}) = H_2(\Gamma_i,\mathbb{Z}) = 0$

b) for each Γ_i there exists a small neighborhood $U(\Gamma_i)$, where $U(\Gamma_i)$ is a Stein manifold and $H^2(U(\Gamma_i),\mathbb{R}) = 0$.

Proof: See [3]. Let me outline the proof of Bogomolov.

Step 1. If J is the complex structure operator on X, then if for each $\chi \in T_i$ $JT_\chi(\Gamma_i)$ is transversal to $T_\chi(\Gamma_i)$, then we can construct $U(\Gamma_i)$ that fulfills condition b.

Step 2. By small deformation of Γ_i we can find Γ_i such that for each $\chi \in \Gamma_i$ $JT_\chi(\Gamma_i)$ is transversal to $T_\chi(\Gamma_i)$. For the details see [3].

Q.E.D.

Let $U(\Gamma_i)$ are the small Stein neighborhood of Γ_i constructed by 5.5. Let $\omega_1,\omega_2 \in H^1_d(X,\Omega^1)$, i.e. $d\omega_1 = d\omega_2 = 0$, then $\omega_1|\Gamma(U_i)$ & $\omega_2|U(\Gamma_i)$ are zero elements in $H^2(U(\Gamma_i),\mathbb{R})$, since $H^2(U(\Gamma_i),\mathbb{R}) = 0$.

Proposition 5.6. (Bogomolov [3]). Let U be a Stein manifold and let ω be a (p,q) form $(p,q > 1)$ and $d\omega = 0$ and $[\omega] \equiv 0$ in $H^{p+q}(U)$, then $\omega = \partial\bar{\partial}\varphi$ for some form φ.

From 5.6. we get that $\omega_{1|U(\Gamma)} = \partial\bar{\partial}\varphi_i^1$, $\omega_{2|U(\Gamma_i)} = \partial\bar{\partial}\varphi_i^2$ Now we can continue φ_i^1 and φ_i^2 as form to $\tilde{\varphi}_i^1$ and $\tilde{\varphi}_i^2$ on the whole X. Let $\omega_1' = \omega_1 - \partial\bar{\partial}\tilde{\varphi}_i^1$, $\omega_2' = \omega_2 - \partial\bar{\partial}\tilde{\varphi}_i^2$. So

a) $\omega'_{1|U(\Gamma_i)} = \omega'_{2|U(\Gamma_i)} \equiv 0$ b) $\omega_1' \sim \omega_1$, $\omega_2' \sim \omega_2$. Clearly

$$[\omega_1', \omega_2']_{|U(\Gamma_i)} \equiv 0 \qquad \text{and}$$

we have proved that

$$\int_\Gamma [\omega_1', \omega_2'] = 0$$

We have proved that the bracket operation defines thus trivial element in $H^3(X,\mathbb{C})$. From here we will get that the first obstruction vanish. For this we will need the following Proposition.

<u>Proposition 5.7.</u> Let X be a symplectic holomorphic manifold and let U be a Stein submanifold in X and let ω be a d-closed form of type $(1,2)$ and $[\omega] = 0$ in $H^3(X,\mathbb{C})$ and $\omega_{|U} \equiv 0$, then there exists a form φ such that:

a) $\partial\varphi \equiv 0$ b) $\bar{\partial}\varphi = \omega$ and c) $\varphi_{|U} = 0$.

<u>Proof.</u> Since ω represents zero in $H^3(X,\mathbb{C})$ and ω is of type $(1,2)$ we get that $\omega = d\alpha^{0,2} + d\beta^{1,1}$, where $\bar{\partial}\alpha^{0,2} = \partial\beta^{1,1} \equiv 0$.

So we get that $\alpha^{0,2} \in H^2(X,0_X) \cong \mathbb{C}\{\omega_X(0,2)\}$. Since $\dim H^2(X,0_X)=1$. If $\alpha^{0,2}$ a non-zero element in $H^2(X,0_X)$, then

$$\omega_X(0,2) = \alpha^{0,2} + \bar{\partial}\mu^{0,1}$$

Since $d\omega_X(0,2) \equiv 0$ we get that

$$\partial\alpha^{0,2} = \bar{\partial}\partial\mu^{0,1}$$

and so

$$\omega = \partial\alpha^{0,2} + \bar{\partial}\beta^{1,1} = \partial\bar{\partial}\mu^{0,1} + \bar{\partial}\beta^{1,1} = \bar{\partial}(\beta^{1,1} + \partial\mu^{0,1})$$

Let $\varphi = \beta^{1,1} + \partial\mu^{0,1}$, then $\partial\varphi = \partial\beta^{1,1} + \partial\partial\mu^{0,1} = \partial\beta^{1,1} = 0$ and $\bar{\partial}\varphi = \omega$. So we have proved a) and b). The condition c) follows immediately from Proposition 5.6. So in this case 5.7. is proved.

If $\alpha^{0,2}$ is zero in $H^2(X,0_X)$, then

$$\alpha^{0,2} = \bar{\partial}\xi^{0,1}$$

Since $\omega = \bar{\partial}\beta^{1,1} + \partial\alpha^{0,2} + \partial\bar{\beta}^{1,1} + \partial\bar{\partial}\xi^{1,1} - \bar{\partial}\beta^{1,1} - \bar{\partial}\partial\xi^{0,1} = \bar{\partial}(\beta^{1,1} - \partial\xi^{0,1})$.
If we define $\varphi = \beta^{1,1} - \partial\xi^{0,1}$ we will get that conditions a) and b) are fulfilled. From 5.6. we will get what we need.

Q.E.D.

Next we have to prove the triviality of all obstructions. So the deformation is given by the following formal power series

$$\omega(t) = \omega_1 t + \omega_2 t^2 + \dots + \omega_N t^N + \cdots$$

such that

1) $\partial\omega = 0$ 2) $\bar{\partial}\omega + \frac{1}{2}[\omega,\omega] = 0$ 3) $\omega_{i|U(\Gamma_j)} = 0$

From 2) we get that:

$$(*) \qquad \bar{\partial}\omega_N = \frac{1}{2}\sum_{i=1}^{N-1}[\omega_i,\omega_{N-i}]$$

Proposition 5.7. says that we can solve (*) by the argument we used before we proved 5.7. First we find ω_2, then ω_3 and so on.

So we have proved that all the obstructions vanish. From Kuranishi existence theorem we can conclude that there exists a family of non-isomorphic holomorphic symplectic manifolds $\chi \to U$, where

1) $\dim_{\mathbb{C}} U = \dim H^1_d(X,\Omega^1)$

2) $T_{0,U} \cong H^1_d(X,\Omega^1)$

3) $\begin{array}{ccc} X & \hookrightarrow & \chi \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & U \end{array}$

This finished the proof of Lemma 5.

Q.E.D.

From now on we will fix the family $\pi:\chi \to U$ constructed in lemma 5.

§ 6. Lemma.

In U (may be after shrinking) there exists an open and everywhere dense subset $U' \subset U$ such that if $t \in U'$, then $X_t = \pi^{-1}(t)$ is a Kähler manifold.

Proof: In order to prove lemma 6 we will define the period map

$$p: U \to \mathbb{P}(H^2(X,\mathbb{Z}) \otimes \mathbb{C})$$

Since $\pi: \chi \to U$ as C^∞ manifold is diffeomorphic to $U \times X$, if we fix a basis $(\gamma_1,\ldots,\gamma_{b_2})$ of $H_2(X,\mathbb{Z})$ in one fibre of

$$\pi: \chi \to U$$

then $(\gamma_1,\ldots,\gamma_{b_2})$ will be a basis of all fibres of $\pi: \chi \to U$. From now onw let us fix the basis $(\gamma_1,\ldots,\gamma_{b_2})$ of $H_2(X,\mathbb{Z})$. Notice that from lemma 3 we get a well defined Hodge structure on $H^2(X,\mathbb{R})$.

Definition 6.1. The period map

$$p: U \to \mathbb{P}(H^2(X,\mathbb{Z}) \otimes \mathbb{C})$$

is defined in the following way:

$$p(t) = (\ldots, \int_{\gamma_i} \omega_t(2,0), \ldots)$$

where $\omega_t(2,0)$ is the holomorphic non-degenerate closed form on $X_t = \pi^{-1}(t)$.

Now let us fix some notations:

$$\dim_{\mathbb{C}} X = 2n$$

We will suppose that $\int\limits_{X_t} \omega_t(2,0)^n \wedge \omega_t(0,2)^n = 1$ and $\omega_X(2,0) = \omega_0(2,0)$.

Definition 6.2. For every $\alpha \in H^2(X,\mathbb{C})$ we will define

$$q(\alpha) = \frac{n}{2} \int_X (\omega_0(2,0) \wedge \omega_0(0,2))^{n-1} \wedge \alpha^2 +$$

$$+ (1-n)\left(\int\limits_X \omega_0(2,0)^{n-1} \wedge \omega_0(0,2)^n \wedge \alpha\right) \cdot \left(\int \omega_0(2,0)^n \wedge \omega_0(0,2)^{n-1} \wedge \alpha\right).$$

Proposition 6.3. a) The quadratic form $q(\alpha)$ is a non-degenerate quadratic form. It is defined over $H^2(X,\mathbb{Z})$. b) Let Ω be a subvariety in $\mathbb{P}(H^2(X,\mathbb{C}))$ defined by

$$q(\alpha) = 0 \quad q(\alpha + \bar{\alpha}) > 0$$

then $p(U) \subset \Omega$ and the period map $p: U \to \Omega$ is a local isomorphism (See [2]).

Proof: The proof of b) is very easy. It is carried out in [2]. The proof of a) is based on the following sublemma.

Sublemma 6.3.1. The classes of cohomologies $[\omega]$ of the forms that are constructed in lemma 1 form an open and convex cone in $H^{1,1}(X,\mathbb{R}) \subset H^2(X,\mathbb{R})$.

<u>Remark:</u> From lemma 3 we know that $\dim H^{1,1}(X,\mathbb{R}) = b_2 - 2$ and that we have a well defined Hodge structure of weight two on $H^2(X,\mathbb{R}) \otimes \mathbb{C}$ and so $H^{1,1}(X,\mathbb{R})$ is a well defined subspace in $H^2(X,\mathbb{R})$ i.e. $H^{1,1}(X,\mathbb{R}) = \{\omega \in H^2_{DR}(X,\mathbb{R}) \mid \omega$ is a closed form of type 1.1.$\}$.

<u>Proof:</u> Let $\gamma_1,\dots,\gamma_{b_2-2}$ be a basis of $H^{1,1}(X,\mathbb{R})$, then if N is a positive real big enough number and $E_1,\dots,E_{b_2-2}$ are positive real small enough numbers, then

$$N\omega + \sum_{i=1}^{b_2-2} E_i\alpha_i \in H^{1,1}(X,\mathbb{R})$$

will fulfill the properties of a closed two form stated in lemma 1. So 6.3.1. is proved.

Q.E.D.

Notice that we have:

$$q(\mathrm{Re}\ \omega_0(2,0)) > 0 \qquad q(\mathrm{Im}\ \omega_0(2,0)) > 0$$ and if ω fulfills the conditions in lemma 1, then

$$q(\omega) = \frac{n}{2}\int_X (\omega_0(2,0) \wedge \omega_0(0,2))^{n-1} \wedge \omega^2 =$$

$$= \frac{n}{2}\int_X (\omega_0(2,0) \wedge \omega_0(0,2))^{n-1} \wedge \partial\alpha^{1,0} \wedge \overline{\partial}\alpha^{0,1} + \frac{n}{2}\int_X (\omega_0{}^{(2,0)} \wedge \omega_0(0,2)^{n-1} \wedge \omega^{1,1} \wedge \omega^{1,1}$$

and clearly $q(\omega) = q(\Theta^{1,1}) > 0$, where $\Theta^{1,1} = [\omega]$ by lemma 2. Now proposition 6.3. a) follows from 6.3.1. For the proof of the fact that q is defined over $\mathbb{Z}$, see [2] .

Q.E.D.

<u>Definition 6.4.</u> $K(X) \overset{\mathrm{def}}{=} \{\omega \in H^{1,1}(X,\mathbb{R}) \mid \int_{C_K} \omega^K > 0$ for any K-dimensional complex analytic cycle $C_K \subset X\}$. We will call $K(X)$ the Kähler cone of X. Lemma 3 shows that $H^{1,1}(X,\mathbb{R})$ is a well defined subspace in $H^2(X,\mathbb{R})$ and $\dim_{\mathbb{R}} H^{1,1}(X,\mathbb{R}) = b_2 - 2$. From lemma 1 and 2 we get that $K(X)$ is a non-empty set.

If we repeat the arguments of 6.3.1. we will get

<u>Proposition 6.6</u>. $K(X)$ is an open convex cone in $H^{1,1}(X,\mathbb{R})$.

From local Torelli theorem it follows that we may suppose that $U \subset \Omega \subset \mathbb{P}(H^2(X,\mathbb{C}))$. Remember that $\chi \to U$ was the family constructed in lemma 5 $\dim_{\mathbb{C}} U = \dim_{\mathbb{C}} \Omega$.

Let $W \overset{\mathrm{def}}{=} \cup K(X_t) \subset H^2(X,\mathbb{R})$. From the definition of W it follows that W is an open set in $H^2(X,\mathbb{R})$. Here we used also continuity argument.

Let $W(\mathbb{Q}) \overset{\mathrm{def}}{=} \{\ell \in W \mid \ell \in H^2(X,\mathbb{Q})\}$. Clearly $W(\mathbb{Q})$ is an open everywhere dense subset in W.

<u>6.7.</u> From the definition of $W(\mathbb{Q})$ we get that if $\ell \in W(\mathbb{Q})$ there exists $t \in U$ such that $\ell \in K(X_t)$. From Nakai-Moishezon criterium we get that X_t is an algebraic variety. So since $W(\mathbb{Q})$ is an everywhere dense subset in W we get immediately that the points $t \in U$ for which $W(\mathbb{Q}) \cap K(X_t) \neq \emptyset$ is an everywhere dense subset in U. So we get that in U we have an everywhere dense subset $U"$ such that if $t \in U$, then X_t is a projective algebraic. Now lemma 6 follows from Kodaira theorem, which says that Kählerian property is an open property. Q.E.D.

Remark. From lemma 6, i.e. since there exists an open and everywhere dense subset, it follows immediately that the quadratic form q defined by 6.2. has singnature $(3, b_2 - 3)$. For the proof of this see [ ].

§ 7. Review of Isometric Deformations

Definition 7.1. A Kähler metric $(g_{\alpha\overline{\beta}})$ an holomorphic symplectic manifold will be called Calabi-Yau metric if

$$\text{Ricci } (g_{\alpha\overline{\beta}}) = \overline{\partial}\partial \log \det (g_{\alpha\overline{\beta}}) \equiv 0$$

The existence of Calabi-Yau metric follows from the deep work of Yau. The Calabi-Yau metric $(g_{\alpha\overline{\beta}})$ induces a covariant differenciation ∇ on $\Lambda^2 T^*X \otimes \mathbb{C}$. See [10].

Lemma 7.2. $\mathrm{Re}\, \omega_X(2,0)$, $\mathrm{Im}\, \omega_X(2,0)$ and $\mathrm{Im}\, (g_{\alpha\overline{\beta}})$ are parallel sections of $\Gamma(X, \Lambda^2 T^*X)$ with respect to ∇.

Proof: See [9]. This is the so called Bochner principle.

Q.E.D.

Suppose that $*$ is the Hodge star operator with respect to Calabi-Yau metric and

$$\int \mathrm{Re}\omega_X(2,0) \wedge * \mathrm{Re}\omega_X(2,0) = \int \mathrm{Im}\, \omega_X(2,0) \wedge * \mathrm{Im}\, \omega_X(2,0) = \int \mathrm{Im}(g_{\alpha\overline{\beta}}) \wedge * \mathrm{Im}\, g_{\alpha\overline{\beta}} = 1.$$

$\mathrm{Re}\, \omega_X(2,0)$, $\mathrm{Im}\, \omega_X(2,0)$, $\mathrm{Im}\, (g_{\alpha\overline{\beta}})$ define a three dimensional subspace $E_X(L)$ in $\Gamma(X, \Lambda^2 T^*X)$ and since

$$\operatorname{Re}\,\omega_X(2,0),\ \operatorname{Im}\,\omega_X(2,0) \quad \text{and} \quad \operatorname{Im}\, g_{\alpha\bar{\beta}}$$

are harmonic forms with respect to the Calabi-Yau metric we may consider $E_X(L)$ as a three dimensional subspace in $H^2(X,\mathbb{R})$. It is easy to see that

$$q_{|E_X(L)} \text{ is positive definite. See [9].}$$

Let $\gamma = a \operatorname{Re}\,\omega_X(2,0) + b \operatorname{Im}\,\omega_X(2,0) + c \operatorname{Im}\,(g_{\alpha\bar{\beta}})$, where $a,b,c \in \mathbb{R}$ and $a^2 + b^2 + c^2 = 1$. Since $\gamma \in E_X(L)$, then $\nabla\gamma \equiv 0$.

Locally γ can be written in the following way

$$\gamma = \sum \gamma_{\mu\sigma} \quad d\chi^{\mu} \wedge dx^{\nu}$$

If $\sum g_{\tau\mu}\, d\chi^{\mu} \otimes dx^{\nu}$ is the Riemannian Ricci flat metric on X defined by the Calabi-Yau metric $(g_{\alpha\bar{\beta}})$ on X, then

$$J(\gamma) = (J(\gamma)^{\alpha}_{\beta}) \overset{\mathrm{def}}{=} (\sum_{\tau} g^{\alpha\tau}\gamma_{\tau\beta}) \in \Gamma(X, T^* \otimes T)$$

<u>Lemma 7.3.</u> a) $J(\gamma)$ defines a new integrable complex structure on X.

b) γ is an imaginary part of a Calabi-Yau metric with respect to the new complex structure $J(\gamma)$. The Calabi-Yau metric defined by γ and $J(\gamma)$ is equivalent as a Riemannian metric to the Calabi-Yau metric $g_{\alpha\bar{\beta}}$, that we started with.

c) Suppose $(X;\gamma_1,\ldots,\gamma_{b_2})$ is a marked Hyper-Kählerian manifold and suppose that $p(X;\gamma_1,\ldots,\gamma_{b_2}) = \chi_0 \in \Omega \subset \mathbb{P}(H^2(X,\mathbb{C}))$. There is one to one map defined by the period map between the complex structures $J(\gamma)$ on X, where

$$\gamma = a \operatorname{Re} \omega_X(2,0) + b \operatorname{Im} \omega_X(2,0) + c \operatorname{Im}(g_{\alpha\overline{\beta}}),\ a^2+b^2+c^2 = 1,\ a,b,c \in \mathbb{R}$$

and the points of the non-singular plane quadric

$$\mathbb{P}(E_X(L) \otimes \mathbb{C}) \cap \Omega = \mathbb{P}^1_{X_0}(L)$$

For the proof see [ ].

<u>Remark</u> a) Notice that $J(\operatorname{Im} g_{\alpha\overline{\beta}})$ is the original complex structure on X, so from c) we get that $\chi_0 \in \mathbb{P}^1_{x_0}(L)$.

§ 8. <u>Construction of a special family of Kähler manifolds.</u>

<u>Definition 8.1.</u> $N \overset{\text{def}}{=} \{$all three dimensional subspaces $E \subset H^2(X,\mathbb{R}) \mid E$ is spanned by $\operatorname{Re} \omega_X(2,0), \operatorname{Im} \omega_X(2,0)$ and φ, where $\varphi \in K(X)\}$. $K(X)$ is defined in 6.4. Now we suppose that $K(X)$ is spanned by all $\omega \in \Gamma(X,\Lambda^2 T^*X)$, where ω are constructed by lemma 1.

N as a subset is diffeomorphic to $E_x \times K(X)$, where $E_X = \{\operatorname{Re} \omega_X(2,0), \operatorname{Im} \omega_X(2,0)\} \subset H^2(X,\mathbb{R})$ so N is an open subset in $H^2(X,\mathbb{R})$.

<u>Remark 8.2.</u> a) We introduce in 6.2 a quadratic form q.

See 6.2. Then this qudratic form has a signature $(3,b_2 - 3)$. This was proved by Beauville. See [2]. Now let $<,>$ the scalar product defined by q on $H^2(X,\mathbb{R})$.

b) From the definition of N it follows that N is the union of all three dimensional subspaces $E \subset H^2(X,\mathbb{R})$ which have the following two properties:

1) $<,>$ on E is strictly positive

2) E contains E_X, where E_X is spanned by $(\mathrm{Re}\ \omega_X(2,0), \mathrm{Im}\ \omega_X(2,0))$, so E_X is a fixed subspace in $H^2(X,\mathbb{R})$:

In [ ] the following Proposition is proved:

<u>Proposition 8.3.</u> There is one to one map between the points of Ω and all oriented two planes in $H^2(X,\mathbb{R})$ on which $<,>$ is positive.

Let $N(\mathbb{Q}) = \{\ell \in N \mid \ell \in H^2(X,\mathbb{Q})\}$. Clearly since N is an open subset in $H^2(X,\mathbb{R})$, then $N(\mathbb{Q})$ is an everywhere dense subset in N. By the continuity argument we can choose $L \in N(\mathbb{Q})$ such that if $L = a\ \mathrm{Re}\ \omega_X(2,0) + b\ \mathrm{Im}\ \omega_X(2,0) + c\ \omega(\omega$ is constructed in lemma 1, i.e. $\omega^{1,1}$ is positive definite), then

a) If E_t is the orthogonal two dimensional plane to L in the three dimensional space $E = \{\mathrm{Re}\ \omega_X(2,0), \mathrm{Im}\ \omega_X(2,0), \omega\} \subset N(\mathbb{Q})$ then $t \in \Omega$ which corresponds to E_t by 8.3. belongs to U.

b) If $\omega_t^{1,1}$ is the $(1,1)$ component of the form ℓ, with respect to the complex structure X_t, then $\omega_t^{1,1}$ is positive everywhere.

From lemma 2, lemma 4 und Nakai-Moishezon criterium we get X_t is an algebraic manifold, so $t \in U'$ defined by lemma 7. Let $g_{\alpha\overline{\beta}}(t)$ be the Yau metric on X_t which corresponds to ℓ. Now we can define the isometric deformation $\chi \to S^2$ of X_t with respect to $g_{\alpha\overline{\beta}}(t)$, so this family is mapped by the period map p onto $\mathbb{P}(E \otimes \mathbb{C}) \cap \Omega$ according to 7.3. Notice that $\mathbb{P}(E \otimes \mathbb{C}) \cap \Omega$ is a non-singular plane curve of degree two, contained in Ω. (See 7.3.) On the other hand from the definition of E, i.e. $E \subset N(\mathbb{Q})$ and 8.2.,it follows that

$$U \cap (\mathbb{P}(E \otimes \mathbb{C}) \cap \Omega) = D$$

is an open disc. Notice that the point $p(X,\gamma_1,\ldots,\gamma_{b_2}) = \chi_0 \in D$,

$$\chi_0 \in U \cap (\mathbb{P}(E \otimes \mathbb{C}) \cap \Omega)$$

In [ ] it is proved that for Kähler holomorphic symplectic manifolds we have an everywhere dense subset of algebraic one in Ω and all of them are in $H_L \cap \Omega$, where

$$H_L =: \{u \in \Omega \mid \langle u,L\rangle = 0\}$$

and L are vectors in $H^2(X,\mathbb{Q})$. Now since $H_L \cap (\mathbb{P}(E \otimes \mathbb{C}) \cap \Omega) \neq \emptyset$ we get on D an everywhere dense subset of algebraic holomorphic symplectic manifolds, so from here we get that

$$D' = U' \cap D \quad (U' \text{ is defined by lemma } 6)$$

is an open and everywhere dense subset in D.

Over $D = \mathbb{P}^1(E \otimes \mathbb{C}) \cap U$ we have two families $\pi : \chi_D \to D$ which is obtained by restriction $\chi \to U$ on D and a family $\chi'_D \to D$ which is obtained by isometric deformation. From local Torelli theorem and lemma 9 our theorem follows, since we can prove that the families $\chi_D \overset{\pi}{\to} D$ and $\chi'_D \overset{\pi'}{\to} D$ are isomorphic. Indeed remember that $\exists t \in D$ such that $\pi^{-1}(t) \cong (\pi^1)^{-1}(t)$. From local Torelli theorem we conclude, that we can find a small open disk $D_1 \subset D$ and $t \in D_1$ such that there exists a biholomorphic mapping
$$f : \begin{array}{ccc} \chi_1 & \to & \chi'_1 \\ \downarrow & & \downarrow \\ D_1 & = & D_1 \end{array}$$
and f induces the identity on $H^2(X,\mathbb{R})$. From lemma 9 it follows that f can be prolonged to an isomorphism on the boundary of the disc D_1 in D. So from here we get that local Torelli gives us that the two families are isomorphic on an open subset of D. Lemma 9 says that they are isomorphic on a closed subset, so f must be an isomorphism. So we need to proof lemma 9 and the theorem will be proved.

§ 9

Lemma 9. (Siu, Burns und Rapoport) Let $\pi : \chi \to U$ and $\pi' : \chi' \to U$ be two holomorphic families of symplectic manifolds with a complex manifold U as a parameter space so that both are diffeomorphically identified with a trivial family $U \times X \to U$. Let τ be an isomorphism of $H^2(X,\mathbb{C})$ which is compatible with the quadratic form defined in [ ]. Let $X_s = \pi^{-1}(s)$ and $X'_s = \pi'^{-1}$ for $s \in U$. Let s_0 be a point of U and let A be a subset of U such that s_0 is an accumulation point of A. Assume the following two conditions.

(i) X_{s_0} is Kähler

(ii) For $s \in A$ the two symplectic manifolds X_s and X'_s are biholomorphic under a map f_s which induces τ on $H^2(X,\mathbb{C})$.

Then X_{s_0} and X'_{s_0} are biholomorphic. See [7] .

Proof: From lemma 1 we know that there exists a real d closed 2-form ω on the underlying differentiable structure X such that (1,1)-component $\omega^{1,1}$ of ω with respect to the complex structure of X'_{s_0} is positive definite at every point of X'_{s_0}. By continuity arguments there exists an open neighborhood W of s_0 in U such that for $s \in W$ the (1,1)-component $\omega^{1,1}(s)$ of $\omega(S)$ with respect to the complex structure X'_s is positive definite at every point of X'_s.

Since X_{s_0} is assumed Kähler, (after shrinking W if necessary) we have for every $s \in W$ a Kähler form $\Theta(s)$ which depends smoothly on s. Let η be a positive definite (1,1)-form on W. The collection of (1,1) forms $\omega^{1,1}(s)$ on X'_s, $\Theta(s)$ and η, define a Hermitian metric H on $\chi \times_W \chi'$. Let H be the (1,1) form on $\chi \times_W \chi'$ associated to the Hermitian metric H. Then the pullback of H to the submanifold $X_s \times X'_s$ of $\chi \times_W \chi'$ is equal to

$$\Theta(s) + \omega^{1,1}(s)$$

where for notational complicity we use $\omega^{1,1}(s)$ and $\Theta(s)$ to denote also their pullbacks under the projections form $X_s \times X'_s$ to X_s and X'_s respectively.

For $s \in W \cap A$ let $\Gamma_s \subset X_s \times X'_s$ be the graph of the holomorphic map $f_s : X_s \to X'_s$. We want to compute the volume (Γ_s) with repect to H on $\chi \times_W \chi'$ and to show that it is bounded in s as s approaches s_0 so that we can apply Bishop's theorem to conclude the convergence of the subvariety Γ_s in $\chi \times_W \chi'$ as s approaches s_0.

Proposition 4.1. Vol $(\Gamma_s) < C$ for every $s \in A$.

Proof: It is easy to see that:

$$\mathrm{Vol}\ (\Gamma_s) = \int_{X_s} \underbrace{(f_s^* \omega^{1,1}(s) + \Theta(s)) \wedge \ldots \wedge (f_s^* \omega^{1,1}(s) + \Theta(s))}_{2n}$$

Let $$\varphi(s) \overset{\mathrm{def}}{=} \int_{X_s} (f_s^* \omega(s) + \Theta(s)) \wedge \ldots \wedge (f_s^* \omega(s) + \Theta(s))$$

We will prove that the following inequalities hold:

$$\mathrm{Vol}\ (\Gamma_s) \leq \varphi(s) < C$$

First we will show that $\varphi(s) < C$. Indeed

$$\varphi(s) = \int_{X_s} (\tau[\omega(s)] + [\Theta(s)] \wedge \ldots \wedge (\tau[\omega(s)] + [\Theta(s)])$$

So from here it follows that on W

$\varphi(S) < C.$

So we need to prove that

$$(*) \qquad \mathrm{vol}\,(\Gamma_s) \leq \varphi(s).$$

Proof of the inequality (*):

Let $f_s^*\omega(s) = \omega^{2,0}(s) + \omega^{1,1}(s) + \omega^{0,2}(s)$, then

$$(4.1.1.) \qquad \underbrace{(\omega^{2,0}(s)+\omega^{1,1}(s)+\omega^{0,2}(s)+\Theta(s))\wedge\ldots\wedge(\omega^{2,0}(s)+\omega^{1,1}(s)+\omega^{0,2}(s)+\Theta(s))}_{2n} =$$

$$\underbrace{\omega^{2,0}\wedge\ldots\wedge\omega^{2,0}}_{n}\wedge\underbrace{\overline{\omega^{2,0}}\wedge\ldots\wedge\overline{\omega^{2,0}}}_{n}+\sum c_K\omega_K^{2,0}\wedge\omega_K^{0,2}\wedge\underbrace{(\omega^{1,1}(s)+\Theta(s))\wedge\ldots\wedge(\omega^{1,1}(s)+\Theta(s))}_{2n-2k}$$

$$+ \underbrace{(\omega^{1,1}(s) + \Theta(s))\wedge\ldots\wedge(\omega^{1,1}(s) + \Theta(s))}_{2n}, \quad \text{where} \quad c_K \in \mathbb{Z}, c_K > 0$$

and $\omega_K^{2,0}(s) = \underbrace{\omega^{2,0}(s)\wedge\ldots\wedge\omega^{2,0}(s)}_{k}.$

Notice that $*\omega^{2,0} = \omega_k^{2,0}\wedge\underbrace{(\omega^{1,1}(s) + \Theta(s))\wedge\ldots\wedge(\omega^{1,1}(s)+\Theta(s))}_{2n-2k}$

where $*$ is the Hodge operator with respect to the Hermitian metric H, where $\operatorname{Im} H = \omega^{1,1}(s) + \Theta(s)$ on $X_s \times X'_s$. So by integrating 4.1.1. we get

4.1.2. $$\varphi(s) = \|\omega_n^{2,0}(s)\|^2 + \sum c_k \|\omega_k^{2,0}(s)\|^2 + \operatorname{vol}(\Gamma_s)$$

So from 4.1.2. we get that

$$\operatorname{vol}(\Gamma_s) \leq \varphi(s) < C$$

Q.E.D.

For a subvariety Z of pure codimension in a complex manifold X, we denote by $[Z]$ the current on G defined by Z. Now we invoke Bishop's theorem [7] and conclude that for some subsequence $\{s_\nu\} \subset A$ converging to s_0 the current $[\Gamma_{s_\nu}]$ over $\chi \times_W \chi'$ converges weakly to a current on $\chi \times_W \chi'$ of the form $\sum_{i=1}^{k} m_i[\Gamma^i]$, where m_i is a positive integer and Γ^i is an irreducible subvariety of complex dimension $2n$ on $X_{s_0} \times X'_{s_0}$;

For any closed 4n-current Θ on $X_s \times X_s$, define a linear map

$$\Theta_* : H^*(X_s, \mathbb{C}) \to H^*(X'_s, \mathbb{C})$$

of cohomology rings as follows. A cohomology class defined by a closed p-form α on X_s is mapped by Θ_* to the cohomology class defined by the closed p-current

$$(pr_2)_*(\Theta \wedge (pr_1)^*\alpha) \quad \text{on} \quad X'_{s_1}$$

where pr_1 and pr_2 are respectively by projections of $X_s \times X'_s$ onto the first and second factors and $(pr_2)_*$ and $(pr_1)^*$ mean respectively the corresponding pushforward and pullback maps. By reversing the rules of X_s and X'_s, we define analogously a linear map

$$\Theta_*: H^*(X'_x, \mathbb{C}) \to H^*(X_s, \mathbb{C})$$

The map $[\Gamma_s]_*$ defined by the 4n-current $[\Gamma_s]$ in $X_s \times X'_s$ clearly agrees with the map from $H^*(X_s, \mathbb{C})$ to $H^*(X'_s, \mathbb{C})$ defined by f_s. Since f_s defines an isomorphism on $H^2(X, \mathbb{Z})$, by passing to limit along the subsequence $\{s_0\}$ we conclude that $(\sum_{i=1}^{m} m_i[\Gamma_k^i])_*$ is an isomorphism of $\Lambda\, H^2(X, \mathbb{Z})$.

Let $\omega_0(2n,0) = \underbrace{\omega_0(2,0) \wedge \ldots \wedge \omega_0(2,0)}_{2n}$ be the non-zero holomorphic $2n$ form, which has no zeroes and no poles on X_{s_0}. Since $(\sum_{i=1}^{K} m_i[\Gamma^i])_*$ is an isomorphism of $\Lambda H^2(X, \mathbb{C})$ it follows that the 2n-current

$$(pr_2)_*(\Sigma m_i\ [\Gamma^i] \wedge (pr_1)^*\ \omega_0(2n, 0))$$

on X'_{s_0} (which is automatically a holomorphic 2n-form on X'_{s_0}) can not be zero. Hence there must be some Γ^j which is projected both onto X_{s_0} and X'_{s_0}. There can only be one such Γ^j and moreover, $m_j = 1$ and its projection maps onto X_{s_0} and onto X'_{s_0} are both of degree one, because both $(\sum_{i=1}^{K} m_i[\Gamma^i])_*$ and $(\sum_{i=1}^{K} m_i[\Gamma^i])^*$ must leave fixed the class in $H^0(X,\mathbb{C})$ which is defined by the function on X with constant values. This particular Γ^j must be projected biholomorphically onto both X_{s_0} and X'_{s_0}, then the following two holomorphic 2n-forms, where $\omega'_0(2n,0)$ is a non-zero 2n-holomorphic form on X'_0: $(p_{2_1})_*([\Gamma^j] \wedge (p_{2_2})^*(\omega'_0(2n,0)) =$

$$= (p_{2_1})_*(\sum m_i [\Gamma^i] \wedge (p_{2_2})^*(\omega'_0(2n,0))$$

$$(p_{2_2})_*([\Gamma^j] \wedge (p_{2_1})^* \omega_0(2n,0)] = (p_{2_2})_*(\sum_{i=1}^{K} m_i[\Gamma^i] \wedge (p_{2_1})^* \omega_0(2k,0))$$

on X_0 and X'_0 respectively cannot be identically zero due to the fact that both $(\sum_{i=1}^{K} m_i[\Gamma^i])^*$ and $(\sum_{i=1}^{K} m_i[\Gamma^i])_*$ are isomorphisms of $\Lambda H^2(X,\mathbb{C})$ and therefore both holomorphic 2n-forms are nowhere zero. This can happen only when Γ^j are projected biholomorphically onto both X_{s_0} and X'_{s_0}. Hence X_{s_0} and X'_{s_0} are biholomorphic.

Q.E.D.

REFERENCES

[1] Abraham and Marsden "Foundations of Mechanics" The Benjamin/Cummings Publ. Company, Readings Mass. 1981.

[2] A. Beauville, "Variétés Kählerianes dont la primiére class de Chern est nulle "J. of Diff. Geometry" 18 (1983), 755-782.

[3] F.A. Bogomolov " Kählerian varieties with trivial canonical class", Preprint, I.H.E.S. February 1981.

[4] S.S. Chern "Complex Manifolds" Chicago University, 1957.

[5] Ph.A. Griffiths "Principle of Algebraic Geometry" John Wiley & Sons, New-York Toronto 1978.

[6] R. Harvey and H. Blaine Lawson, Jr.,"Intrinsic characterization of Kähler manifolds", Inventions Math. 74(1983) 169-198.

[7] Y.T. Siu, "Every K3 surface is Kähler", Inventions Math. 73 (1983), 139-µ50.

[8] A.N. Todorov, "Application of the Kähler-Einstein-Calabi-Yau metric to moduli of K3 surfaces Inventiones Math. 61 (1980), 251-265.

[9] A.N. Todorov, "Moduli of Hyper-Kählerian Algebraic Manifolds" SFB-MPI preprints 85-38.

[10] S.T. Yau, "On the Ricci curvature of a compact Kähler manifold and the complex Monge-Amper equation, I" Comm. Pure and Applied Math. 31(1978),229-411.