Max-Planck-Institut für Mathematik Bonn

On classical tensor categories attached to the irreducible representations of the general linear supergroups $GL(n \mid n)$

by

Thorsten Heidersdorf
Rainer Weissauer

Max-Planck-Institut für Mathematik
Preprint Series 2019 (14)

On classical tensor categories attached to the irreducible representations of the general linear supergroups $GL(n \mid n)$

Thorsten Heidersdorf
Rainer Weissauer

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Universität Heidelberg
Mathematisches Institut
Im Neuenheimer Feld 288
69120 Heidelberg
Germany

MPIM 19-14

ON CLASSICAL TENSOR CATEGORIES ATTACHED TO THE IRREDUCIBLE REPRESENTATIONS OF THE GENERAL LINEAR SUPERGROUPS $GL(n|n)$

TH. HEIDERSDORF, R. WEISSAUER

Abstract. We study the quotient of $\mathcal{T}_n = Rep(GL(n|n))$ by the tensor ideal of negligible morphisms. If we consider the full subcategory $\mathcal{T}_n^+$ of $\mathcal{T}_n$ of indecomposable summands in iterated tensor products of irreducible representations up to parity shifts, its quotient is a semisimple tannakian category $Rep(H_n)$ where H_n is a pro-reductive algebraic group. We determine the connected derived subgroup $G_n \subset H_n$ and the groups $G_\lambda = (H_\lambda)^0_{der}$ corresponding to the tannakian subcategory in $Rep(H_n)$ generated by an irreducible representation $L(\lambda)$. This gives structural information about the tensor category $Rep(GL(n|n))$, including the decomposition law of a tensor product of irreducible representations up to summands of superdimension zero. Some results are conditional on a hypothesis on 2-torsion in $\pi_0(H_n)$.

Contents

2010 *Mathematics Subject Classification*: 17B10, 17B20, 17B55, 18D10, 20G05.

1. Introduction

1.1. **Semisimple quotients.** The categories of finite dimensional representations $\mathcal{T}_n$ of the general linear supergroups $GL(n|n)$ over an algebraically closed field k of characteristic zero are abelian tensor categories, where representations in this article are always are understood to be algebraic. However, contrary to the classical case of the general linear groups $GL(n)$, these categories are not semisimple. Whereas the tensor product $V \otimes V$, $V \simeq k^{n|n}$, is completely reducible, this is no longer true for the tensor product $\mathbb{A} = V \otimes V^\vee$. Indeed $\mathbb{A}$ defines the indecomposable adjoint representation of $GL(n|n)$, hence admits a trivial one dimensional subrepresentation defined by the center and a trivial one dimensional quotient representation defined by the supertrace. In contrast to the classical case the supertrace is trivial on the center, and $\mathbb{A}$ is indecomposable with three irreducible Jordan-Hoelder factors $1, S^1, 1$ with the superdimensions $1, -2, 1$ respectively defined by the filtration $\mathfrak{z} \subseteq \mathfrak{sl}(n|n) \subseteq \mathfrak{gl}(n|n)$, where $\mathfrak{z}$ denotes the center of $\mathfrak{gl}(n|n)$.

Although the irreducible representations of $GL(n|n)$ can be classified by highest weights similarly to the classical case, this implies that the tensor product of irreducible representations is in general far from being completely reducible. In fact Weyl's unitary trick fails in the superlinear setting. While the structure of $\mathcal{T}_n$ as an abelian category is now well understood [BS12a], its monoidal structure remains mysterious.

The perspective of this article is that in order to restore parts of the classical picture two finite dimensional representations M and M' of $GL(n|n)$ should not be distinguished, if there exists an isomorphism

$$M \oplus N \cong M' \oplus N'$$

where N and N' are negligible modules. Here we use the notion that a finite dimensional module is said to be negligible if it is a direct sum of indecomposable modules whose superdimensions are zero. A typical example of a negligible module is the indecomposable adjoint representation $\mathbb{A}$. To achieve this we divide our category $\mathcal{T}_n$ by the tensor ideal $\mathcal{N}$ [AK02] of negligible morphisms. The quotient is a semisimple abelian tensor category. By a fundamental result of Deligne it is equivalent to the representation category of a pro-reductive supergroup G^{red} [Hei14].

Taking the quotient of a non-semisimple tensor category by objects of categorial dimension 0 has been studied in a number of different cases. A well-known example is the quotient of the category of tilting modules by the negligible modules (of quantum dimension 0) in the representation category of the Lusztig quantum group $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is a semisimple Lie algebra over k [AP95] [BK01]. The modular categories so obtained have been studied extensively in their applications to the 3-manifold invariants of Reshetikhin-Turaev. In [Ja92] Jannsen proved that the category of numerical motives as defined via algebraic correspondences modulo numerical equivalence is an abelian semisimple category. It was noted by André and Kahn [AK02] that taking numerical equivalence amounts to taking the quotient by the negligible morphisms. Jannsen's theorem has been generalized to a categorical setting by [AK02]. In particular they study quotients of tannakian categories by the ideal of negligible morphisms. Recently Etingof and Ostrik [EO18] studied semisimplifactions with an emphasis on finite tensor categories.

A general study of $Rep(G)/\mathcal{N}$, where G is a supergroup scheme, was initiated in [Hei14] where in particular the reductive group G^{red} given by $Rep(G^{red}) \simeq Rep(GL(m|1))/\mathcal{N}$ was determined. This example is rather special since $Rep(GL(m|1))$ has tame representation type. For $m, n \geq 2$ the problem of classifying irreducible representations of G^{red} is wild [Hei14]. Therefore one should not study the entire quotient $\mathcal{T}_n/\mathcal{N}$, but rather pass to a suitably small tensor subcategory in $\mathcal{T}_n$.

1.2. **The Tannaka category $\overline{\mathcal{T}}_n$.** In this article we work with the tensor subcategory generated by the irreducible representations of positive superdimension in the following sense. Recall that an irreducible representation $L(\lambda)$ of $GL(n|n)$, defined by some integrable highest weight λ, can be replaced by a parity shift X_λ of $L(\lambda)$ so that the superdimension $\mathrm{sdim}(X_\lambda)$ becomes ≥ 0. This is of course ambiguous for irreducible representations of $GL(n|n)$ with $\mathrm{sdim}(L) = 0$, but these representations are negligible in the sense that we want to get rid of them. We therefore consider only objects that are retracts of iterated tensor products of irreducible representations $L(\lambda)$ of $GL(n|n)$ satisfying $\mathrm{sdim}(L(\lambda)) \geq 0$. The tensor category thus obtained will be baptized $\mathcal{T}_n^+$. The tensor subcategory $\mathcal{T}_n^+$ of $Rep(GL(n|n))$ has more amenable properties than the full category $Rep(GL(n|n))$. To motivate this, let us compare it with the tensor category of finite dimensional algebraic representations $Rep(G)$ of an arbitrary algebraic group G over k. In this situation the tensor subcategory generated by irreducible representations is semisimple[1] and can be identified with the tensor category of the maximal reductive quotient of G. The tensor category $\mathcal{T}_n^+$ however is not a semisimple tensor category in general. To make it semisimple we proceed as follows:

Let $\overline{\mathcal{T}}_n$ denote the quotient category of $\mathcal{T}_n^+$ obtained by killing the negligible morphisms in the maximal tensor ideal $\mathcal{N}$ and hence in particular all neglegible objects, i.e. $\overline{\mathcal{T}}_n \cong \mathcal{T}_n^+/\mathcal{N}$. In order to analyze these categories, we work inductively using the cohomological tensor functors $DS : \mathcal{T}_n \to \mathcal{T}_{n-1}$ of [HW14]. We show in lemma 5.4 that DS induces a tensor functor $DS : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$.

[1] In [KrW15, p.231, 1.22ff] it was forgotten to mention the important passage to the tensor subcategory generated by simple objects. The corresponding statement is false without it as kindly pointed out by Y. André.

Theorem 1.1. *(1) The categories $\overline{\mathcal{T}}_n$ are semisimple Tannakian categories $\overline{\mathcal{T}}_n$, i.e $\overline{\mathcal{T}}_n \cong Rep(H_n)$ where H_n is a projective limit of reductive groups over k.*

(2) From DS one can construct a k-linear tensor functor between the quotient categories

$$\eta : \mathcal{T}_n^+/\mathcal{N} \to \mathcal{T}_{n-1}^+/\mathcal{N}.$$

These functors $\eta = \eta_n$ induce embeddings of affine group schemes $H_{n-1} \hookrightarrow H_n$. Furthermore $\eta : Rep(H_n) \to Rep(H_{n-1})$ can be identified with the restriction functor (with respect to this embedding) and is induced by the functor DS on objects.

If $X_\lambda \in \mathcal{T}_n^+$ is an irreducible maximal atypical representation, we denote by H_λ the reductive group corresponding to the tensor subcategory $< X_\lambda > \simeq Rep(H_\lambda)$ generated by the image of X_λ in $\overline{\mathcal{T}}_n$. The derived group of its connected component will be baptized G_λ. In order to determine the group H_n we essentially have to identify each H_λ for maximal atypical λ. We first determine the derived groups $G_n \subseteq H_n$ of their connected components H_n^0 which basically amounts to determine G_λ for each λ.

Any object X in $\overline{\mathcal{T}}_n$ can be viewed by the tannakian formalism as a representation of H_n. We denote by ω the fiber functor

$$\omega : (\overline{\mathcal{T}}_n, \otimes) \cong Rep_k(H_n) \to vec_k$$

which associates to an object X the underlying finite dimensional k-vector space of the representation associated to X. For the irreducible representation X_λ we use the notation V_λ simultaneously for the irreducible representation of H_λ as well as for the underlying vector space $\omega(X_\lambda)$. Note that $\dim(V_\lambda) = \mathrm{sdim}(X_\lambda)$. We distinguish two cases: Either X_λ is a weakly selfdual object (SD), i.e. $X_\lambda^\vee \cong Ber^r \otimes X_\lambda$ for some tensor power Ber^r of the Berezin determinant; or alternatively X_λ is not weakly selfdual (NSD). In the (SD) case V_λ carries a symmetric (the even (SD)-case) or antisymmetric pairing (the odd (SD)-case). The dual and the superdimension of X_λ can be easily expressed in terms of the weights λ or in terms of self-equivalences of the irreducible objects X_λ.

Theorem 1.2. *(Structure theorem for G_λ) $G_\lambda = SL(V_\lambda)$ if X_λ is (NSD). If X_λ is (SD) and $V_\lambda|_{G_{\lambda'}}$ is irreducible, $G_\lambda = SO(V_\lambda)$ respectively $G_\lambda = Sp(V_\lambda)$ according to whether X_λ is even respectively odd. If X_λ is (SD) and $V_\lambda|_{G_{\lambda'}}$ decomposes into at least two irreducibe representations, then $G_\lambda \cong SL(W)$ for $V_\lambda|_{G_{\lambda'}} \cong W \oplus W^\vee$.*

The group G_n can be understood from this in rather down to earth terms: For this let $X^+ = X^+(n)$ denote the set of highest weights of $GL(n|n)$ and $X_0^+(n)$ the subset of maximal atypical highest weights. For a simple equivalence relation on $X_0^+(n)$ (two irreducible modules M, N are equivalent if $M \cong N$ or $M^\vee \cong N$ holds after restriction to $SL(n|n)$) let $Y_0^+(n)$ denote the quotient of $X_0^+(n)$ by this equivalence relation. Then we have

Theorem 1.3. *(Structure theorem for G_n) There exists an isomorphism*

$$G_n \cong \prod_{\lambda \in Y_0^+(n)} G_\lambda$$

where G_λ is as in theorem 1.2.

The representations $V_\lambda = \omega(X_\lambda)$ of the group G_n corresponding to the irreducible representations X_λ of the group $GL(n|n)$ factorize over the quotient

$$pr_\lambda : \ G_n = \prod_{\lambda' \in Y_0^+(n)} G_{\lambda'} \twoheadrightarrow G_\lambda$$

and correspond to the standard representation of its quotient group G_λ on the vectorspace V_λ.

We conjecture that V_λ is always irreducible as a representation of G_λ. This would imply the following stronger structure theorem.

Conjecture 1.4. $G_\lambda = SL(V_\lambda)$ *resp.* $G_\lambda = SO(V_\lambda)$ *resp.* $G_\lambda = Sp(V_\lambda)$ *according to whether X_λ satisfies (NSD) respectively (SD) with either X_λ being even respectively odd.*

The ambiguity in the determination of G_λ is only due to the fact that we cannot exclude special elements with 2-torsion in $\pi_0(H_n)$. More precisely, under some assumptions on the weakly selfdual weight λ, the category $Rep(H_\lambda)$ might contain non-trivial one-dimensional representations which correspond to indecomposable representations $I \in \mathcal{T}_n^+$ with the following properties:

1. I is indecomposable in $\mathcal{T}_n^+$ with $\mathrm{sdim}(I) = 1$.
2. There exists an irreducible object L of $\mathcal{T}_n^+$ such that I occurs (with multiplicity one) as a direct summand in $L \otimes L^\vee$.
3. $L \otimes I \cong L \oplus N$ for some negligible object N.
4. $I^\vee \cong I$.
5. $I^* \cong I$.
6. $DS(I)$ is $\mathbf{1}$ plus some negligible object.

We claim that this implies $I \simeq \mathbf{1}$ in $\mathcal{T}_n^+$ (which would imply the conjecture), but are unable to prove this at the moment. For some remarks and special cases see appendix D.

1.3. **The Picard group of $\overline{\mathcal{T}}_n$.** In order to determine H_n from G_n, we need to determine the invertible elements in $Rep(H_n)$, i.e. the Picard group $Pic(H_n)$, or in down-to-eart terms, the character group of H_n. A first analysis of $Pic(\overline{\mathcal{T}}_n)$ can be found in section 12. We complete the determination of H_λ and H_n then in the partially conjectural *big picture* section 13. If a certain integer $\ell(\lambda)$ (defined in section 12) is non-zero, we show in section 13 that the groups H_λ are given by $GL(V_\lambda)$, $GSO(V_\lambda)$ and

$GSp(V_\lambda)$. This follows from the fact that the tensor powers of the determinant $det(X_\lambda) = \Lambda^{\mathrm{sdim}(X_\lambda)}$ generate a subgroup isomorphic to $Rep(GL(1))$ for $\ell(\lambda) \neq 0$, and that the character group of H_λ is therefore as large as possible. In the general case the difficult part is to rule out other 1-dimensional representations of H_λ, e.g. those that could come from a finite abelian subgroup. We call an indecomposable module V in $\mathcal{T}_n^+$ with $\mathrm{sdim}(V) = 1$ *special*, if $V^* \cong V$ and $H^0(V)$ contains $\mathbf{1}$. We then conjecture

Conjecture 1.5. *Every special module is trivial $V \cong \mathbf{1}$.*

We use this to calculate the determinant $\Lambda^{\mathrm{sdim}(X_\lambda)}(X_\lambda)$ of an irreducible representation X_λ. We prove in theorem 13.5 (assuming conjecture 1.5) that the determinant is always a Berezin power

$$\Lambda^{\mathrm{sdim}(X_\lambda)}(X_\lambda) \cong Ber^{\ell(\lambda)} \oplus \text{ negligible}$$

up to negligible modules. More generally we conjecture

Conjecture 1.6. *Any invertible object I in $\overline{\mathcal{T}}_n$ is represented in $\mathcal{T}_n$ by a power of the Berezin determinant.*

Under this strong conjecture, by theorem 13.11, the possible Tannaka groups H_λ are the following groups:

$$H_\lambda = SL(V_\lambda), GL(V_\lambda), Sp(V_\lambda), SO(V_\lambda), GSO(V_\lambda), GSp(V_\lambda).$$

This would in particular imply that the restriction of any irreducible representation of H_n to G_n stays irreducible.

Reformulating these statements for the category of representations of $GL(n|n)$, what we have achieved is

- a description of the decomposition law of tensor products of irreducible representations into indecomposable modules up to negligible indecomposable summands; and
- a classification (in terms of the highest weights of H_λ and H_μ) of the indecomposable modules of non-vanishing superdimension in iterated tensor products of $L(\lambda)$ and $L(\mu)$.

To determine this decomposition it suffices to know the Clebsch-Gordan coefficients for the classical simple groups of type A, B, C, D. Furthermore the superdimensions of the indecomposable summands are just the dimensions of the corresponding irreducible summands of the tensor products in $Rep_k(H_n)$. Without this, to work out any such decomposition is rather elaborate. For the case $n = 2$ see [HW15]. In fact the knowledge of the Jordan-Hölder factors usually gives too little information on the indecomposable objects itself. In the (NSD) and the odd (SD)-case it is enough for these two applications to know the connected derived group G_λ since the restriction of any irreducible representation of H_λ to G_λ stays irreducible. Therefore these results hold unconditionally in these cases. In the even (SD)-case we need the finer (but conjectural) results of section 13 to see that H_λ is connected. We refer the reader to example 9.7 and section 13 for some examples.

1.4. **Structure of the article.** Our main tool are the cohomological tensor functors $DS : \mathcal{T}_n \to \mathcal{T}_{n-1}$ of [HW14]. In the main theorem of [HW14, Theorem 16.1] we calculate $DS(L(\lambda))$. In particular $DS(L(\lambda))$ is semisimple and multiplicity free. We show in lemma 5.4 that DS induces a tensor functor $DS : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$ and by lemma 5.10 one can construct a tensor functor on the quotient categories

$$\eta : \mathcal{T}_n^+/\mathcal{N} \to \mathcal{T}_{n-1}^+/\mathcal{N}.$$

This seemingly minor observation is one of the crucial points of the proof since it allows us to determine the groups H_n and G_n inductively. We also stress that it is not clear whether DS naturally induces a functor between the quotients $\mathcal{T}_n/\mathcal{N}$ and $\mathcal{T}_{n-1}/\mathcal{N}$ on the level of morphisms. It is however compatible with the functor η on objects. The quotient $\mathcal{T}_n^+/\mathcal{N}$ is equivalent to the representation category $Rep(H_n)$ of finite-dimensional representations of a pro-reductive group. By a deep theorem 5.16 of Deligne the induced DS functor determines an embedding of algebraic groups $H_{n-1} \hookrightarrow H_n$ and the functor DS is the restriction functor with respect to this embedding.

Hence the main theorem of [HW14] tells us the branching laws for the representation V_λ with respect to the embedding $H_{n-1} \hookrightarrow H_n$. Our strategy is to determine the groups H_n or G_n inductively using the functor DS. For $n = 2$ we need the explicit results of [HW15] to give us the fusion rule between two irreducible representations and we describe the corresponding Tannaka group in lemma 9.2. Starting from the special case $n = 2$ we can proceed by induction on n. For this we use the embedding $H_{n-1} \to H_n$ along with the known branching laws and the classification of small representations due to Andreev, Elashvili and Vinberg [AVE67] which allows to determine inductively the connected derived groups $G_n = (H_n^0)_{der}$ for $n \geq 3$; see section 10. The passage to the connected derived group is forced due to our lack of knowledge about the connected components of H_n. On the other hand this means that we have to deal with the possible decomposition of V_λ when restricted to G_n. In order to determine G_n we first determine the connected derived groups G_λ corresponding to the tensor subcategory generated by the image of $L(\lambda)$ in $\overline{\mathcal{T}}_n$ in theorem 6.2. Roughly speaking the strategy of the proof is rather primitive: We use the inductively known situation for G_{n-1} to show that for sufficiently large n the rank and the dimension of G_λ is large compared to the dimension of V_λ, i.e. V_λ or any of its irreducible constituents in the restriction to G_λ should be *small* in the sense of [AVE67]. We refer to section 10 for more details on the proof.

The two final sections are devoted to the determination of $Rep(H_n)$. While section 12 is independent of the sections on the structure theorem, section 13 assumes the stronger conjectural structure theorem 11.1 for G_n.

We have outsourced a large number of technical (but necessary) results to the appendices A B C as to not distract the reader too much from the structure of the arguments. The other appendices D E F discuss mostly examples and evidences for our conjectures.

Most of the results discussed here for the general linear supergroups $GL(n|n)$ can be rephrased for representations of the general linear supergroups $GL(m|n)$ for $m \neq n$. This will be discussed elsewhere.

A part of the motivation for our computation of the Tannaka groups H_n comes from the relationship to the real algebraic supergroups $SU(2,2|N)$ for $N \leq 4$ which are covering groups of the super conformal groups $SO(2,4|N)$. The complexification $\mathfrak{g}$ of their Lie algebras are the complex Lie superalgebras $\mathfrak{sl}(4|N)$, whose finite dimensional representations are related to those of the Lie superalgebras $\mathfrak{gl}(n|n)$ for $n \leq 4$, as mentioned in the last section above. The complexification $\mathfrak{g}$ defines vector fields on four dimensional Minkowski superspace M and plays an important role for string theory and the AdS/CFT correspondence. We wonder whether there exist reasonable supersymmetric conformal field theories whose fields ψ are defined on M (or related spaces) and have their values not in a representation of $\mathfrak{g}$ of superdimension zero, but rather have values in maximal atypical basic representations V of $\mathfrak{g}$. The Feynman integrals of any such theory are computed from tensor contractions via superintegration and contractions between tensor products of the fields, and hence their values are influenced by the underlying rules of the tensor categories $\mathcal{T}_n$. If, for some mysterious physical reasons, in such a theory the contribution to the Feynman integrals from summands of superdimension zero would be relatively small by supersymmetric cancellations, hence to first order negligible in a certain energy range, a physical observer might come up with the impression that the underlying rules of symmetry were dictated by contractions imposed by the invariant theory of the quotient tensor categories $\overline{\mathcal{T}} = Rep(H_n)$, i.e. those tensor categories that are obtained by ignoring negligible indecomposable summands of superdimension zero. Since for $\mathfrak{gl}(n|n)$, besides $U(1)$, the smallest quotient groups of the tannakian groups H_n that arise for $\overline{\mathcal{T}} = Rep(H_n)$ and $n = 2,3,4$ are $SU(2)$ and $SU(3)$ (see the example 13.12 and also thereafter), where the latter two are related to the representations $V = S^1$ and $V = S^2$, we henceforth ask whether there may be any connection with the symmetry groups arising in the standard model of elementary particle physics. Of course this speculation is highly tentative. Fields with values in maximal atypical representations V very likely produce ghosts in the associated infinite dimensional representations of $\mathfrak{g}$. In other words, such field theories may a priori not be superunitary and it is unclear whether the passage to the cohomology groups for operators like DS or the Dirac operator H_D [HW14], breaking the conformal symmetry, would suffice to get rid of ghosts.

2. The superlinear groups

Let k be an algebraically closed field of characteristic zero. We adopt the notations of [HW14]. With $GL(m|n)$ we denote the general linear supergroup and by $\mathfrak{g} = \mathfrak{gl}(m|n)$ its Lie superalgebra. A representation ρ of $GL(m|n)$ is a representation of $\mathfrak{g}$ such that its restriction to $\mathfrak{g}_{\bar{0}}$ comes from an algebraic representation of $G_{\bar{0}} = GL(m) \times GL(n)$. We denote by $\mathcal{T} = \mathcal{T}_{m|n}$ the category of all finite dimensional representations with parity preserving morphisms.

2.1. **The category $\mathcal{R}$.** Fix the morphism $\varepsilon : \mathbb{Z}/2\mathbb{Z} \to G_{\bar{0}} = GL(n) \times GL(n)$ which maps -1 to the element $diag(E_n, -E_n) \in GL(n) \times GL(n)$ denoted ϵ_n. Notice that $Ad(\epsilon_n)$ induces the parity morphism on the Lie superalgebra $\mathfrak{gl}(n|n)$ of G. We define the abelian subcategory $\mathcal{R} = sRep(G, \varepsilon)$ of $\mathcal{T}$ as the full subcategory of all objects (V, ρ) in $\mathcal{T}$ with the property $p_V = \rho(\epsilon_n)$; here p_V denotes the parity morphism of V and ρ denotes the underlying homomorphism $\rho : GL(n) \times GL(n) \to GL(V)$ of algebraic groups over k. The subcategory $\mathcal{R}$ is stable under the dualities $^\vee$ and *. For $G = GL(n|n)$ we usually write $\mathcal{T}_n$ instead of $\mathcal{T}$, and $\mathcal{R}_n$ instead of $\mathcal{R}$. The irreducible representations in $\mathcal{R}_n$ are parametrized by their highest weight with respect to the Borel subalgebra of upper triangular matrices. A weight $\lambda = (\lambda_1, ..., \lambda_n \mid \lambda_{n+1}, \cdots, \lambda_{2n})$ of an irreducible representation in $\mathcal{R}_n$ satisfies $\lambda_1 \geq \ldots \lambda_n$, $\lambda_{n+1} \geq \ldots \lambda_{2n}$ with integer entries. The Berezin determinant of the supergroup $G = G_n$ defines a one dimensional representation Ber. Its weight is is given by $\lambda_i = 1$ and $\lambda_{n+i} = -1$ for $i = 1, .., n$. For each representation $M \in \mathcal{R}_n$ we also have its parity shifted version $\Pi(M)$ in $\mathcal{T}_n$. Since we only consider parity preserving morphisms, these two are not isomorphic. In particular the irreducible representations in $\mathcal{T}_n$ are given by the $\{L(\lambda), \Pi L(\lambda) \mid \lambda \in X^+\}$. The whole category $\mathcal{T}_n$ decomposes as $\mathcal{T}_n = \mathcal{R}_n \oplus \Pi\mathcal{R}_n$ [Bru03, Corollary 4.44].

2.2. **Kac objects.** We put $\mathfrak{p}_\pm = \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(\pm 1)}$ for the usual $\mathbf{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{(-1)} \oplus \mathfrak{g}_{(0)} \oplus \mathfrak{g}_{(1)}$. We consider a simple $\mathfrak{g}_{(0)}$-module as a $\mathfrak{p}_\pm$-module in which $\mathfrak{g}_{(1)}$ respectively $\mathfrak{g}_{(-1)}$ acts trivially. We then define the Kac module $V(\lambda)$ and the anti-Kac module $V'(\lambda)$ via

$$V(\lambda) = Ind_{\mathfrak{p}_+}^{\mathfrak{g}} L_0(\lambda) \ , \ V'(\lambda) = Ind_{\mathfrak{p}_-}^{\mathfrak{g}} L_0(\lambda)$$

where $L_0(\lambda)$ is the simple $\mathfrak{g}_{(0)}$-module with highest weight λ. The Kac modules are universal highest weight modules. $V(\lambda)$ has a unique maximal submodule $I(\lambda)$ and $L(\lambda) = V(\lambda)/I(\lambda)$ [Kac78, Proposition 2.4]. We denote by $\mathcal{C}^+$ the tensor ideal of modules with a filtration by Kac modules in $\mathcal{R}_n$ and by $\mathcal{C}^-$ the tensor ideal of modules with a filtration by anti-Kac modules in $\mathcal{R}_n$.

2.3. **Equivalence classes of weights.** Two irreducible representations M, N in $\mathcal{T}$ are said to be equivalent $M \sim N$, if either $M \cong Ber^r \otimes N$ or $M^\vee \cong Ber^r \otimes N$ holds for some $r \in \mathbb{Z}$. This obviously defines an equivalence relation on the set of isomorphism classes of irreducible representations of T. A self-equivalence of M is given by an isomorphism $f : M \cong Ber^r \otimes M$ (which implies $r = 0$ and f to be a scalar multiple of the identity) respectively an isomorphism $f : M^\vee \cong Ber^r \otimes M$. If it exists, such an isomorphism uniquely determines r and is unique up to a scalar and we say M is of type (SD). Otherwise we say M is of type (NSD). The isomorphism f can be viewed as a nondegenerate G-equivariant bilinear form

$$M \otimes M \to Ber^r \ ,$$

which is either symmetric or alternating. So we distinguish bitween the cases (SD$_\pm$). Let $Y^+(n)$ denote the set of equivalence classes of irreducible representations in $\mathcal{T}_n$.

2.4. **Negligible objects.** An object $M \in \mathcal{T}_n$ is called negligible, if it is the direct sum of indecomposable objects M_i in $\mathcal{T}_n$ with superdimensions $\mathrm{sdim}(M_i) = 0$. The collection of these objects forms an ideal. We denote the largest proper tensor ideal of $\mathcal{T}_n$ by $\mathcal{N}$. An object $X \in \mathcal{T}_n$ is isomorphic to zero in $\mathcal{T}_n/\mathcal{N}$ if and only if X is negligible.

Example 2.1. An irreducible representation has superdimension zero if and only if it is not maximal atypical, see section 3. The standard representation $V \simeq k^{n|n}$ has superdimension zero and therefore also the indecomposable adjoint representation $\mathbb{A} = V \otimes V^\vee$.

3. Weight and cup diagrams

3.1. **Weight diagrams and cups.** Consider a weight

$$\lambda = (\lambda_1, ..., \lambda_n | \lambda_{n+1}, \cdots, \lambda_{2n}).$$

Then $\lambda_1 \geq ... \geq \lambda_n$ and $\lambda_{n+1} \geq ... \geq \lambda_{2n}$ are integers, and every $\lambda \in \mathbb{Z}^{2n}$ satisfying these inequalities occurs as the highest weight of an irreducible representation $L(\lambda)$. The set of highest weights will be denoted by $X^+ = X^+(n)$. Following [BS12a] to each highest weight $\lambda \in X^+(n)$ we associate two subsets of cardinality n of the numberline $\mathbb{Z}$

$$\begin{aligned} I_\times(\lambda) &= \{\lambda_1, \lambda_2 - 1, ..., \lambda_n - n + 1\} \\ I_\circ(\lambda) &= \{1 - n - \lambda_{n+1}, 2 - n - \lambda_{n+2}, ..., -\lambda_{2n}\}. \end{aligned}$$

We now define a labeling of the numberline $\mathbb{Z}$. The integers in $I_\times(\lambda) \cap I_\circ(\lambda)$ are labeled by $\vee$, the remaining ones in $I_\times(\lambda)$ resp. $I_\circ(\lambda)$ are labeled by $\times$ respectively $\circ$. All other integers are labeled by $\wedge$. This labeling of the numberline uniquely characterizes the weight vector λ. If the label $\vee$ occurs r times in the labeling, then $r = atyp(\lambda)$ is called the *degree of atypicality* of λ. Notice $0 \leq r \leq n$, and for $r = n$ the weight λ is called *maximal atypical.* A weight is maximally atypical if and only if $\lambda_i = -\lambda_{n+i}$ for $i = 1, \ldots, n$ in which case we write

$$L(\lambda) = [\lambda_1, \ldots, \lambda_n] \ .$$

To each weight diagram we associate a cup diagram as in [BS11] [HW14]. The outer cups in a cup diagram define the sectors of the weight as in [HW14]. We number the sectors from left to right S_1, S_2, …, S_k.

3.2. **Important invariants.** The segment and sector structure of a weight diagram is completely encoded by the positions of the $\vee$'s. Hence any finite subset of $\mathbb{Z}$ defines a unique weight diagram in a given block. We associate to a maximal atypical highest weight the following invariants:

- the type (SD) resp. (NSD),
- the number $k = k(\lambda)$ of sectors of λ,
- the sectors $S_\nu = (I_\nu, K_\nu)$ from left to right (for $\nu = 1, ..., k$),
- the ranks $r_\nu = r(S_\nu)$, so that $\#I_\nu = 2r_\nu$,
- the distances d_ν between the sectors (for $\nu = 1, ..., k-1$),
- and the total shift factor $d_0 = \lambda_n + n - 1$.

If convenient, k sometimes may also denote the number of segments, but hopefully no confusion will arise from this.

A maximally atypical weight $[\lambda]$ is called basic if $(\lambda_1, ..., \lambda_n)$ defines a decreasing sequence $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0$ with the property $n - i \geq \lambda_i$ for all $i = 1, ..., n$. The total number of such *basic weights* in $X^+(n)$ is the Catalan number C_n. Reflecting the graph of such a sequence $[\lambda]$ at the diagonal, one obtains another basic weight $[\lambda]^*$. By [HW14, Lemma 21.4] a basic weight λ is of type (SD) if and only if $[\lambda]^* = [\lambda]$ holds. To every maximal atypical highest weight λ is attached a unique maximal atypical highest weight λ_{basic}

$$\lambda \mapsto \lambda_{basic}$$

having the same invariants as λ, except that $d_1 = \cdots = d_{k-1} = 0$ holds for λ_{basic} and the leftmost $\vee$ is at the vertex $-n+1$.

4. Cohomological tensor functors.

4.1. **The Duflo-Serganova functor.** We attach to every irreducible representation a sign. If $L(\lambda)$ is maximally atypical we put $\varepsilon(L(\lambda)) = (-1)^{p(\lambda)}$ for the parity $p(\lambda) = \sum_{i=1}^n \lambda_i$. For the general case see [HW14]. Now for $\varepsilon \in \{\pm 1\}$ define the full subcategories $\mathcal{R}_n(\varepsilon)$. These consists of all objects whose irreducible constituents X have sign $\varepsilon(X) = \varepsilon$. Then by [HW14, Corollary 15.1] the categories $\mathcal{R}_n(\varepsilon)$ are semisimple categories. Note that $\mathrm{sdim}(X) \geq 0$ holds for all irreducible objects $X \in \mathcal{R}_n(\varepsilon)$ in case $\varepsilon = 1$ and also for all irreducible objects $X \in \Pi\mathcal{R}_n(\varepsilon)$ in case $\varepsilon = -1$.

We recall some constructions from the article [HW14]. Fix the following element $x \in \mathfrak{g}_1$,

$$x = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \text{ for } y = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & & \dots & \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since x is an odd element with $[x,x]=0$, we get

$$2\cdot\rho(x)^2=[\rho(x),\rho(x)]=\rho([x,x])=0$$

for any representation (V,ρ) of G_n in $\mathcal{R}_n$. Notice $d=\rho(x)$ supercommutes with $\rho(G_{n-1})$. Then we define the cohomological tensor functor DS as

$$DS=DS_{n,n-1}:\mathcal{T}_n\to\mathcal{T}_{n-1}$$

via $DS_{n,n-1}(V,\rho)=V_x:=Kern(\rho(x))/Im(\rho(x))$.

In fact $DS(V)$ has a natural $\mathbf{Z}$-grading and decomposes into a direct sum of G_{n-1}-modules

$$DS(V,\rho)\ =\ \bigoplus_{\ell\in\mathbb{Z}}\ \Pi^\ell(H^\ell(V))\ ,$$

for certain cohomology groups $H^\ell(V)$. If we want to emphasize the $\mathbb{Z}$-grading, we also write this in the form

$$DS(V,\rho)\ =\ \bigoplus_{\ell\in\mathbb{Z}}\ H^\ell(V)[-\ell].$$

Theorem 4.1. [HW14, Theorem 16.1] *Suppose $L(\lambda)\in\mathcal{R}_n$ is an irreducible atypical representation, so that λ corresponds to a cup diagram*

$$\bigcup_{j=1}^r\ [a_j,b_j]$$

with r sectors $[a_j,b_j]$ for $j=1,...,r$. Then

$$DS(L(\lambda))\ \cong\ \bigoplus_{i=1}^r\ \Pi^{n_i}L(\lambda_i)$$

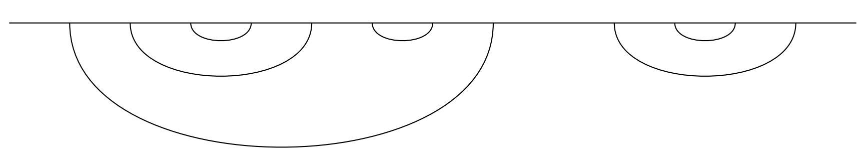
is the direct sum of irreducible atypical representations $L(\lambda_i)$ in $\mathcal{R}_{n-1}$ with shift $n_i\equiv\varepsilon(\lambda)-\varepsilon(\lambda_i)$ modulo 2. The representation $L(\lambda_i)$ is uniquely defined by the property that its cup diagram is

$$[a_i+1,b_i-1]\quad\cup\quad\bigcup_{j=1,j\neq i}^r\ [a_j,b_j]\ ,$$

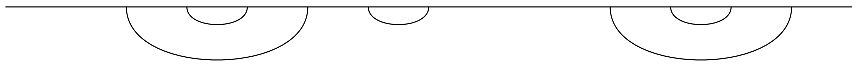
the union of the sectors $[a_j,b_j]$ for $1\leq j\neq i\leq r$ and (the sectors occuring in) the segment $[a_i+1,b_i-1]$.

In particular $DS(L(\lambda))$ is semisimple and multiplicity free.

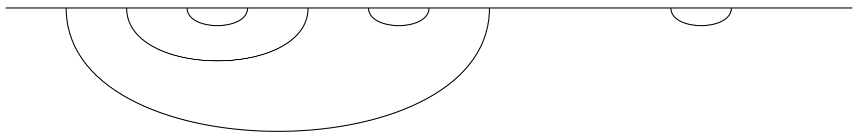
Example 4.2. Consider the (maximal atypical) irreducible representation $[7,7,4,2,2,2]$ of $GL(6|6)$ and $p(\lambda)=1$. Its associated cup diagram is



Hence the cup diagram has two sectors of rank 4 and 2 respectively with $d_0 = 7$ and $d_1 = 1$. Applying DS gives 2 irreducible representation, namely $[\lambda_1] = [7, 7, 4, 2, 2]$ with cup diagram



Then the parity is $p(\lambda_1) = 1 = p(\lambda)$. The second irreducible representation is $\Pi[7, 3, 1, 1, 1]$ (note the parity shift since $p(\lambda_2) \neq p(\lambda)$) with cup diagram



All in all $DS[7, 7, 4, 2, 2, 2] \cong [7, 7, 4, 2, 2] \oplus \Pi[7, 3, 1, 1, 1]$.

4.2. **The Hilbert polynomial.** Similarly to DS we can define the tensor functors $DS_{n,n-m} : \mathcal{T}_n \to T_{n-m}$ by replacing the x in the definition of DS by an x with m 1's on the antidiagonal. These functors admit again a $\mathbf{Z}$-grading. In particular we can consider the functor $DS_{n,0} : \mathcal{T}_n \to T_0 = svec_k$ with its decomposition $DS_{n,0}(V) = \bigoplus_{\ell \in \mathbb{Z}} D^{\ell}_{n,0}(V)[-\ell]$ for objects V in $\mathcal{T}_n$ and objects $D^{\ell}_{n,0}(V)$ in $svec_k$ where $D^{\ell}_{n,0}(V)[-\ell]$ is the object $\Pi^{\ell} D^{\ell}_{n,0}(V)$ concentrated in degree ℓ with respect to the $\mathbf{Z}$-gradation of $DS_{n,0}(V)$. For $V \in \mathcal{T}_n$ we define the Laurent polynomial

$$\omega(V, t) = \sum_{\ell \in \mathbb{Z}} \mathrm{sdim}(DS^{\ell}_{n,0}(V)) \cdot t^{\ell}$$

as the Hilbert polynomial of the graded module $DS^{\bullet}_{n,0}(V) = \bigoplus_{\ell \in \mathbb{Z}} DS^{\ell}_{n,0}(V)$. Since $\mathrm{sdim}(W[-\ell]) = (-1)^{\ell}\,\mathrm{sdim}(W)$ and $V = \bigoplus DS^{\ell}_{n,0}(V)[-\ell]$ holds, the formula

$$\mathrm{sdim}(V) = \omega(V, -1)$$

follows. For $V = Ber^i_n$

$$\omega(Ber^i_n, t) \ = \ t^{ni} \ .$$

For more details we refer the reader to [HW14, section 25].

5. Tannakian arguments

5.1. **The category $\mathcal{T}^+_n$.** Let $\mathcal{T}^+_n$ denote the full subcategory of $\mathcal{T}_n$, whose objects consist of all retracts of iterated tensor products of irreducible representations in $\mathcal{T}_n$ that are not maximal atypical and of maximal atypical irreducible representations in $\mathcal{R}_n(+1) \oplus \Pi\mathcal{R}_n(-1)$ for $\mathcal{R}_n(\pm 1)$ defined at the begining of section 4.1. Obviously $\mathcal{T}^+_n$ is a symmetric monoidal idempotent complete k-linear category closed under the $*$-involution. It contains all irreducible objects of $\mathcal{T}_n$ up to a parity shift. It contains the standard representation V and its dual $V^\vee$, and hence contains all mixed tensors [Hei14]. Furthermore all objects X in $\mathcal{T}_n^+$ satisfy condition $\mathtt{T}$ (see section 6 in [HW14]) and $\mathcal{T}_n^+$ is rigid. For this it suffices for irreducible $X \in \mathcal{T}_n^+$ that $X^\vee \in \mathcal{T}_n^+$. This is obvious since $X^\vee$ is irreducible with $\mathrm{sdim}(X^\vee) = \mathrm{sdim}(X) \geq 0$, and hence $X^\vee \in \mathcal{T}_n^+$.

5.2. **The ideal of negligible morphisms.** An ideal in a k-linear category $\mathcal{A}$ is for any two objects X, Y the specification of a k-submodule $\mathcal{I}(X,Y)$ of $Hom_{\mathcal{A}}(X,Y)$, such that for all pairs of morphisms $f \in Hom_{\mathcal{A}}(X,X')$, $g \in Hom_{\mathcal{A}}(Y,Y')$ $g\mathcal{I}(X',Y)f \ \subseteq \mathcal{I}(X,Y')$ holds. Let $\mathcal{I}$ be an ideal in $\mathcal{A}$. By definition $\mathcal{A}/\mathcal{I}$ is the category with the same objects as $\mathcal{A}$ and with

$$Hom_{\mathcal{A}/\mathcal{I}}(X,Y) = Hom_{\mathcal{A}}(X,Y)/\mathcal{I}(X,Y) \ .$$

An ideal in a tensor category is a tensor ideal if it is stable under $\mathbf{1}_C \otimes -$ and $- \otimes \mathbf{1}$ for all $C \in \mathcal{A}$. Let Tr be the trace. For any two objects A, B we define $\mathcal{N}(A,B) \subset Hom(A,B)$ by

$$\mathcal{N}(A,B) = \{f \in Hom(A,B) \mid \forall g \in Hom(B,A),\ Tr(g \circ f) = 0\}.$$

The collection of all $\mathcal{N}(A,B)$ defines a tensor ideal $\mathcal{N}$ of $\mathcal{A}$ [AK02].

Let $\mathcal{A}$ be a super tannakian category. An indecomposable object will be called negligible, if its image in $\mathcal{A}/\mathcal{N}$ is the zero object. By [Hei14] an object is negligible if and only if its categorial dimension is zero. Any super tannakian category is equivalent (over an algebraically closed field) to the representation category of a supergroup scheme by [Del02]. In that case the categorial dimension is the superdimension of a module. If $\mathcal{A}$ is a super tannakian category over k, the quotient of $\mathcal{A}$ by the ideal $\mathcal{N}$ of negligible morphisms is again a super tannakian category by [AK02], [Hei14]. More generally, for any pseudo-abelian full subcategory $\tilde{\mathcal{A}}$ in $\mathcal{A}$ closed under tensor products, duals and containing the identity element the following holds:

Lemma 5.1. *The quotient category $\tilde{\mathcal{A}}/\mathcal{N}$ is a semisimple super tannakian category.*

Proof. The quotient is a k-linear semisimple rigid tensor category by [AK05, Theorem 1 a)]. The quotient is idempotent complete by lifting of idempotents (or see [AK02, 2.3.4 b)] and by [AK02, 2.1.2] a k-linear pseudoabelian category is abelian. The Schur finiteness [Del02] [Hei14] is inherited from $\mathcal{A}$ to $\tilde{\mathcal{A}}/\mathcal{N}$. □

This in particular applies to the situation where $\tilde{\mathcal{A}}$ is the full subcategory of objects which are retracts of iterated tensor products of a fixed set of objects in $\mathcal{A}$. In particular for $\tilde{\mathcal{A}} = \mathcal{T}_n^+$ and $\mathcal{A} = \mathcal{T}_n$ this implies

Corollary 5.2. *The tensor functor $\mathcal{T}_n^+ \to \mathcal{T}_n^+/\mathcal{N}$ maps $\mathcal{T}_n^+$ to a semisimple super tannakian category $\overline{\mathcal{T}}_n := \mathcal{T}_n^+/\mathcal{N}$.*

Proposition 5.3. *The category $\overline{\mathcal{T}}_n$ is a tannakian category, i.e. there exists a pro-reductive algebraic k-groups H_n such that the category $\overline{\mathcal{T}}_n$ is equivalent as a tensor category to the category $Rep_k(H_n)$ of finite dimensional k-representations of H_n*

$$\overline{\mathcal{T}}_n \sim Rep_k(H_n) \ .$$

Proof. By a result of Deligne [Del90, Theorem 7.1] it suffices to show that for all objects X in $\mathcal{T}_n^+$ we have $\mathrm{sdim}(X) \geq 0$. We prove this by induction on n. Suppose we know this assertion for $\mathcal{T}_{n-1})$ already. Then all objects of $\mathcal{T}_{n-1}^+$ have superdimension ≥ 0 (for the induction start $n = 0$ our assertion is obvious). Notice that the tensor functor $DS : \mathcal{T}_n \to \mathcal{T}_{n-1}$ preserves superdimensions, hence for the induction step it suffices that DS maps $\mathcal{T}_n^+$ to $\mathcal{T}_{n-1}^+$.

Lemma 5.4. *The functors $DS_{n,n-m} : \mathcal{T}_n \to \mathcal{T}_{n-m}$ and $\omega_{n,n-m} : \mathcal{T}_n \to \mathcal{T}_{n-m}$ restrict to functors from $\mathcal{T}_n^+$ to $\mathcal{T}_{n-m}^+$. In particular*

$$DS : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+ \ .$$

Proof. Since $DS_{n,n-m}$ and ω_{n-m} preserve tensor products and idempotents, it suffices by the definition of $\mathcal{T}_n^+$ that $DS_{n-m}(X), \omega_{n-m}(X) \in \mathcal{T}_{n-m}^+$ holds for all irreducible objects X in $\mathcal{T}_n^+$. Now theorem 4.1 implies $DS(X) \in \mathcal{T}_{n-1}^+$ since any irreducible representation X maps to a semisimple representation $DS(X)$. [If X is irreducible but not maximal atypical, then all constituents of $DS(X)$ are irreducible and not maximal atypical. If $X \in \mathcal{T}_n^+$ is irreducible and maximal atypical, then all summands of $DS(X)$ are in $\mathcal{T}_{n-1}^+$.] This proves the claim for $DS(X)$, X irreducible. But then also for $DS_{n,n-m}(X)$, X irreducible, since then again $DS_{n,n-m}(X)$ is semisimple by proposition 8.1 in [HW14]. The same then also holds for $\omega_{n,n-m}(X) = H_{\overline{\partial}}(DS_{n-m}(X))$ by loc.cit. □

Corollary 5.5. *Negligible objects X in $\mathcal{T}_n^+$ map under DS to negligible objects in $\mathcal{T}_{n-1}^+$.*

Proof. We have shown $\mathrm{sdim}(Y) \geq 0$ for all objects Y in $\mathcal{T}_{n-1}^+$. Therefore $\mathrm{sdim}(DS(X)) = \mathrm{sdim}(X) = 0$ implies $sdim(Y_i) = 0$ for all indecomposable summands Y_i of $Y = DS(X)$, since $sdim(Y_i) \geq 0$. □

Remark 5.6. Since irreducible objects L satisfy condition `T` in the sense that $\overline{\partial}$ is trivial on $DS_{n,n-m}(L)$ [HW14, proposition 8.5], and since condition `T` is inherited by tensor products and retracts, all objects in $\mathcal{T}_n^+$ satisfy condition `T`. Hence [HW14, proposition 8.5] implies the following lemma.

Lemma 5.7. *On the category $\mathcal{T}_n^+$ the functor $H_D(.)$ is naturally equivalent to the functor $DS : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$. Similarly the functors $\omega_{n,n-m}(.) : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$ are naturally equivalent to $DS_{n,n-m}(.)$.*

Corollary 5.8. *$DS(X) = 0$ in $\mathcal{T}_{n-1}^+$ if and only if X is a projective object in $\mathcal{T}_n$.*

Proof. Any negligible maximal atypical object in $\mathcal{T}_n^+$ maps under DS to a negligible maximal atypical object in $\mathcal{T}_{n-1}^+$. Furthermore $DS(X) = 0$ for X in $\mathcal{T}_n^+$ implies that X is an anti-Kac object. If $X \neq 0$, then X^* is a Kac object in $\mathcal{T}_n^+$. Hence $H_D(X^*) = 0$. Since $X^* \in \mathcal{T}_n^+$ satisfies condition $\mathtt{T}$, this implies $DS(X^*) = 0$ and hence X^* is a Kac and anti-Kac object. The corollary follows since $\mathcal{C}^+ \cap \mathcal{C}^- = Proj$.

Corollary 5.9. *If $X \in \mathcal{T}_n^+$ and X is a Kac or anti-Kac object, then $X \in Proj$.*

Corollary 5.5 implies

Lemma 5.10. *The functor $DS : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$ gives rise to a k-linear exact tensor functor between the quotient categories*

$$\eta : \overline{\mathcal{T}}_n \to \overline{\mathcal{T}}_{n-1} \ .$$

Proof. We define the ideal $\mathcal{I}^0$ via

$$\mathcal{I}^0(X,Y) = \{f : X \to Y \mid f \text{ factorizes over a negligible object.}\}$$

Obviously $\mathcal{I}^0$ is a tensor ideal for $\mathcal{T}_n^+$. As for any tensor ideal $\mathcal{I}^0 \subset \mathcal{N}$ the quotient $\mathcal{T}_n^+/\mathcal{I}^0 =: \mathcal{A}_n^+$ becomes a rigid tensor category and $\mathcal{T}_n^+ \to \mathcal{T}_n^+/\mathcal{I}^0 = \mathcal{A}_n^+$ a tensor functor. Under this tensor functor an indecomposable object X in $\mathcal{T}_n^+$ maps to zero in the quotient $\mathcal{A}_n^+$ if and only if $\mathrm{sdim}(X) = 0$. Furthermore, since the tensor functor DS maps negligible objects of $\mathcal{T}_n^+$ to negligible objects of $\mathcal{T}_{n-1}^+$, the functor DS induces a k-linear tensor functor $DS' : \mathcal{A}_n^+ \to \mathcal{A}_{n-1}^+$. The category $\mathcal{A}_n^+$ is pseudoabelian since we have idempotent lifting in the sense of [Li13, Theorem 5.2] due to the finite dimensionality of the Hom spaces. By the definition of $\mathcal{A}_n^+$ and $\mathcal{T}_n^+$, the dimension of each object in $\mathcal{A}_n^+$ is a natural number and, contrary to $\mathcal{T}_n^+$, it does not contain any nonzero object that maps to an element isomorphic to zero under the quotient functor $\mathcal{A}_n^+ \to \mathcal{A}_n^+/\mathcal{N}$. Therefore $\mathcal{A}_n^+$ satisfies conditions d) and g) in [AK02, Theorem 8.2.4]. By [AK02, Theorem 8.2.4 (i),(ii)] this implies that $\mathcal{N}(\mathcal{A}_n^+)$ equals the radical $\mathcal{R}(\mathcal{A}_n^+)$ of $\mathcal{A}_n^+$; note that $\mathcal{N}(\mathcal{A}_n^+) = \mathcal{N}(\mathcal{T}_n^+)/\mathcal{I}^0$ and that $\mathcal{N}(A,A)$ is a nilpotent ideal in $End(A)$ for any A in $\mathcal{A}_n^+$ by assertion b) of [AK02, Theorem 8.2.4 (i),(ii)]. Since $\mathcal{N}$ always is a tensor ideal, $\mathcal{R}(\mathcal{A}_n^+)$ in particular is a tensor ideal. This allows to apply [AK02, Theorem 13.2.1] to construct a monoidal section $s_n : \mathcal{A}_n^+/\mathcal{N}(\mathcal{A}_n^+) \to \mathcal{A}_n^+$ for the tensor functor $\pi_n : \mathcal{A}_n^+ \to \mathcal{A}_n^+/\mathcal{N}(\mathcal{A}_n^+)$. The composite tensor functor

$$\eta := \pi_{n-1} \circ DS' \circ s_n$$

defines a k-linear tensor functor

$$\eta : \overline{\mathcal{T}}_n \to \overline{\mathcal{T}}_{n-1} \ .$$

Since DS' is additive and $\overline{\mathcal{T}}_n$ is semisimple, η is additive and hence exact. $\square$

Remark 5.11. The k-linear tensor functor $\pi_{n-1} \circ DS' : \mathcal{A}_n^+ \to \overline{\mathcal{T}}_{n-1}$ defines the tensor ideal $\mathcal{K}_n$ of $\mathcal{A}_n^+$ of morphisms annihilated by $\pi_{n-1} \circ DS'$. Obviously $\mathcal{K}_n \subseteq \mathcal{N}$. Let $\overline{\mathcal{A}}_n^+ = \mathcal{A}_n^+/\mathcal{K}_n$ be the quotient tensor category. Since $\mathcal{N}(\mathcal{A}_n^+) = \mathcal{R}(\mathcal{A}_n^+)$, for all simple objects S in $\mathcal{A}_n^+$ some given morphism $f \in Hom_{\mathcal{A}_n^+}(S, A)$ is in $\mathcal{N}(\mathcal{A}_n^+)(S, A)$ if and only if for all $g \in Hom_{\mathcal{A}_n^+}(S, A)$ the composite $g \circ f$ is zero [AK02, Lemma 1.4.9]. Indeed all endomorphism $f \in \mathcal{N}(\mathcal{A}_n^+)(S, S)$ are nilpotent, hence $DS'(f)$ is nilpotent and maps to zero in $\overline{\mathcal{T}}_{n-1}$. Therefore the image $\overline{f}$ of f in $\mathcal{N}(\overline{\mathcal{A}}_n^+)(S, S)$ is zero. Since S is a simple object, we conclude that the endomorphisms of S in $\overline{\mathcal{A}_n^+}$ are in $k \cdot id$, hence [AK02, Lemma 1.4.9] can be applied.

Corollary 5.12. *The functor $DS_{n,0} : \mathcal{T}_n^+ \to \mathcal{T}_0^+$ sends negligible morphisms to zero. The functor $DS_{n,1} : \mathcal{T}_n^+ \to \mathcal{T}_1^+$ satisfies $DS_{n,1}(\mathcal{N}) \subset \mathcal{N}$.*

Proof. The claim about $DS_{n,0}$ follows from the commutative diagram

$$\begin{tikzcd}
\mathcal{T}_n^+ \arrow[r, "DS"] \arrow[d] & \mathcal{T}_{n-1}^+ \arrow[r, "DS"] \arrow[d] & \dots \arrow[r] & \mathcal{T}_1^+ \arrow[r, "DS"] \arrow[d] & \mathcal{T}_0^+ \arrow[d, "="] \\
\mathcal{A}_n^+ \arrow[r, "DS'"] \arrow[d] & \mathcal{A}_{n-1}^+ \arrow[r, "DS'"] \arrow[d] & \dots \arrow[r] & \mathcal{A}_1^+ \arrow[r, "DS'"] \arrow[d, "="] & \mathcal{T}_0^+ \arrow[d, "="] \\
\mathcal{T}_n^+/\mathcal{N} \arrow[r, "\eta"] & \mathcal{T}_{n-1}^+/\mathcal{N} \arrow[r, "\eta"] & \dots \arrow[r] & \mathcal{T}_1^+/\mathcal{N} \arrow[r, "\eta"] & \mathcal{T}_0^+/\mathcal{N}
\end{tikzcd}$$

and the fact that η maps negligible morphisms to negligible morphisms and $\mathcal{T}_0^+ = \mathcal{A}_0^+ = \mathcal{T}_0^+/\mathcal{N}$. For the $\mathcal{T}_1^+$-case let $f : M \to M'$ be negligible with M, M' indecomposable in $\mathcal{T}_n^+$. Since $DS_{n,1}$ sends negligible objects to negligible objects, it is clear that the claim holds if either M or M' are negligible. So suppose that $\text{sdim}(M)$ and $\text{sdim}(M') \neq 0$. The image of M and M' in $\mathcal{T}_1^+$ is of the form

$$\bigoplus_i Ber^i \oplus \bigoplus_j P(Ber^j).$$

Since any morphism to or from $P(Ber^j)$ is negligible, it suffices to consider $DS_{n,1}(f)$ as a morphism in

$$Hom(\bigoplus_i Ber^i, \bigoplus_j Ber^j) = \bigoplus_i End(Ber^i).$$

Such a morphism is of the form $\sum_i \lambda_i id_{Ber^i}$ and so is either trivial or not negligible. We need to show that it is trivial. This follows from the $DS_{n,0}$-case. Indeed if there would be an i such that $DS_{n,1}(f) \in End(Ber^i)$ is not negligible, then its image under DS in $\mathcal{T}_0^+$ would not be negligble either, in contradiction to $DS_{n,0}(\mathcal{N}) = 0$. $\square$

Corollary 5.13. *The functor $DS_{n,0}$ induces a super fibre functor $DS : \mathcal{T}_n^+/\mathcal{N} \to \mathcal{T}_0^+ = svec$. It is isomorphic to the functor $(\eta \circ \ldots \circ \eta)$ defined by iterated application of η.*

Proof. Since $DS_{n,0}(\mathcal{N}) = 0$, we obtain an induced tensor functor $\mathcal{T}_n^+/\mathcal{N} \to \mathcal{T}_0^+$. Since these categories are semisimple, $DS_{n,0}$ is faithful and exact and is therefore a super fibre functor. Since any two fibre functors over an algebraically closed field are isomorphic, we obtain the required isomorphism of tensor functors. □

Corollary 5.14. *A nontrivial morphism $f : S \to A$ in $\mathcal{T}_n^+$ from an irreducible object S in $\mathcal{T}_n^+$ to an arbitrary object A in $\mathcal{T}_n^+$ is a split monomorphism if $DS_{n,0}(f)$ is nonzero. The converse also holds.*

Proof. $DS_{n,0}(f) \neq 0$ implies by corollary 5.12 that f is not negligible. Since any morphism $S \to A$ between two indecomposable objects S, A with $S \ncong A$ is negligible, this implies the claim. □

Remark 5.15. We do not know whether $DS(\mathcal{N}(\mathcal{T}_n^+)) \subseteq \mathcal{N}(\mathcal{T}_{n-1}^+)$ holds. If this were true for all n, then also $DS_{n,n-i}(\mathcal{N}(\mathcal{T}_n^+)) \subseteq \mathcal{N}(\mathcal{T}_{n-i}^+)$ would hold. We consider this a fundamental question in the theory. For $n = 1$ observe that $\mathcal{A}_1^+ = \mathcal{T}_1^+/\mathcal{N}$. Indeed $\mathcal{T}_1^+$ has only one proper tensor ideal $\mathcal{N} = \mathcal{I}^0$ as can be easily seen by looking at the maximal atypical objects Ber^i and $P(Ber^j)$ in $\mathcal{T}_1^+$. The tensor ideal $\mathcal{I}^0$ could be different from $\mathcal{N}$ for $n \geq 2$. With respect to the partial ordering on the set of tensor ideals given by inclusion, $\mathcal{I}^0$ is the minimal element in the fibre of the decategorification map of the thick ideal of indecomposable objects of superdimension 0 [Co18, Theorem 4.1.3]. The negligible morphisms are the largest tensor ideal in this fibre.

5.3. *DS* **as a restriction functor.** Recall from [Del90, Theorem 8.17] the following fundamental theorem on k-linear tensor categories: Suppose $\mathcal{A}_1, \mathcal{A}_2$ are k-linear abelian rigid symmetric monoidal tensor categories with $k \cong End_{\mathcal{A}_i}(\mathbf{1})$ as in loc. cit. Assume that all objects of $\mathcal{A}_i$ have finite length and all Hom-groups have finite k-dimension. Assume that k is a perfect field so that $\mathcal{A}_1 \otimes \mathcal{A}_2$ is again k-linear abelian rigid symmetric monoidal tensor categories with $k \cong End_{\mathcal{A}_i}(\mathbf{1})$ as in [Del90, 8.1]. Suppose

$$\eta : \mathcal{A}_1 \to \mathcal{A}_2$$

is an *exact tensor functor*. Then η is faithful [DM82, Proposition 1.19].

Theorem 5.16. [Del90, Theorem 8.17] *Under the assumptions above there exists a morphism*

$$\pi(\mathcal{A}_2) \to \eta(\pi(\mathcal{A}_1))$$

as in [Del90, 8.15.2] *such that η induces a tensor equivalence between the category $\mathcal{A}_1$ and the tensor category of objects in $\mathcal{A}_2$ equipped with an action of $\eta(\pi(\mathcal{A}_1))$, so that that natural action of $\pi(\mathcal{A}_2))$ is obtained via the morphism $\pi(\mathcal{A}_2) \to \eta(\pi(\mathcal{A}_1))$.*

Suppose $\omega : \mathcal{A}_2 \to Vec_k$ is fiber functor of $\mathcal{A}_2$, i.e. ω is an exact faithful tensor functor. Then $\mathcal{A}_2$ is a Tannakian category and $\mathcal{A}_2 \cong Rep_k(H)$ as a tensor category. If $\mathcal{A}_2 = Rep_k(H)$ is a Tannakian category for some affine group H over k, then $\pi(\mathcal{A}_2) = H$ by [Del90, Example 8.14 (ii)]. More precisely, an $\mathcal{A}_2$-group is the same as an affine k-group equipped with a H action, and here H acts on itself by conjugation. The forgetful functor ω of $Rep_k(G)$ to Vec_k is a fiber functor. By applying this fiber functor we obtain a fiber functor $\omega \circ \eta : \mathcal{A}_1 \to Vec_k$ for the tensor category $\mathcal{A}_1$. In particular $\mathcal{A}_1$ becomes a Tannakian category with Tannaka group $H' = \omega \circ \eta(\pi(\mathcal{A}_1))$. Furthermore, by applying η to the morphism $\pi(\mathcal{A}_2) \to \eta(\pi(\mathcal{A}_2))$ in $\mathcal{A}_2$, we get a morphism $\omega(\pi(\mathcal{A}_2)) \to (\omega\circ\eta)(\pi(\mathcal{A}_1))$ in the category of k-vectorspaces, which defines a group homomorphism

$$f : H' \to H$$

of affine k-groups inducing a pullback functor

$$Rep(H') \to Rep(H) \ ,$$

that gives back the functor $\eta : \mathcal{A}_1 \to \mathcal{A}_2$ via the equivalences $\mathcal{A}_1 = Rep_k(H')$ and $\mathcal{A}_2 = Rep_k(H)$ obtained from the fiber functors.

Lemma 5.17. [DM82, Proposition 2.21(b)] *The morphism $f : H' \to H$ thus obtained is a closed immersion if and only if every object Y of $\mathcal{A}_2$ is isomorphic to a subquotient of an object of the form $\eta(X), X \in \mathcal{A}_1$.*

The statements above will now be applied for the tensor functor

$$\eta : \mathcal{A}_1 \to \mathcal{A}_2$$

obtained from DS between the quotient categories $\mathcal{A}_1 = \mathcal{T}_n^+/\mathcal{N}$ and $\mathcal{A}_2 = \mathcal{T}_{n-1}^+/\mathcal{N}$. Notice that the assumptions above on k and $\mathcal{A}_i$ are satisfied so that $\mathcal{A}_2$ is a tannakian category with fiber functor ω giving an equivalence of tensor categories $\mathcal{A}_2 = Rep_k(H_{n-1})$. Obviously DS induces an *exact tensor functor* between the quotient categories, since DS is additive, maps negligible objects of $\mathcal{T}_n^+$ into negligible objects of $\mathcal{T}_{n-1}^+$ and since the categories $\mathcal{A}_i$ are semisimple. As in our case k is algebraically closed, we know that up to an isomorphism the group H_n only depends on $\mathcal{A}_1$ but not on the choice of a fiber functor. As explained above, this defines a homomorphism of affine k-groups

$$f : H_{n-1} \longrightarrow H_n \ .$$

Theorem 5.18. *The homomorphism $f : H_{n-1} \to H_n$ is injective and the functor $\eta : Rep_k(H_n) \to Rep_k(H_{n-1})$ induced by $DS : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$ can be identified with the restriction functor for the homomorphism f.*

Proof. By lemma 5.17 it suffices that every indecomposable Y in $\mathcal{T}_{n-1}^+$ with $\mathrm{sdim}(Y) > 0$ is a subobject of an object $DS(X), X \in \mathcal{T}_n^+$. By assumption Y is a retract of a tensor product of irreducible modules $L_i \in \mathcal{T}_{n-1}^+$. So it suffices that each L_i is a subobject of some object $DS(X_i), X_i \in \mathcal{T}_n^+$. We can assume that Y is not negligible and irreducible, hence maximal atypical and $Y = \Pi^r L(\lambda)$ for some r. Then $L(\lambda) = [\lambda] = [\lambda_1, ..., \lambda_{n-1}]$. By a twist with Berezin we may assume that $\lambda_{n-1} \geq 0$. Then we define $[\tilde{\lambda}] = [\lambda_1, ..., \lambda_{n-1}, 0]$ so that for $X = \Pi^r L(\tilde{\lambda})$ we get by theorem 4.1 and [HW14, Lemma 10.2] the assertion $DS(X) = Y \oplus$ other summands. Notice that by construction $X = \Pi^r L(\tilde{\lambda})$ is in $\mathcal{T}_n^+$. But this proves our claim. $\square$

In other words, the description of the functor DS on irreducible objects in $\mathcal{T}_n$ given by theorem 4.1 can be interpreted as branching rules for the inclusion

$$f : \ H_{n-1} \hookrightarrow H_n \ .$$

We will show later how this fact gives information on the groups H_n.

5.4. **Enriched morphism.** Now recall that the collection of cohomology functors $H^i : \mathcal{R}_n \to \mathcal{R}_{n-1}$ for $i \in \mathbb{Z}$ defines a tensor functor

$$H^\bullet : \mathcal{R}_n \to Gr^\bullet(\mathcal{R}_{n-1})$$

to the category of $\mathbb{Z}$-graded objects in $\mathcal{R}_{n-1}$. Using the parity shift functor Π, this functor can be extended to a tensor functor

$$H^\bullet : \mathcal{T}_n^+ \to Gr^\bullet(\mathcal{T}_{n-1}^+) \ ,$$

which induces a corresponding tensor functor on the level of the quotient categories

$$H^\bullet : \overline{\mathcal{T}}_n = \mathcal{T}_n^+/\mathcal{N} \to Gr^\bullet(\mathcal{T}_{n-1}^+/\mathcal{N})) = Gr^\bullet(\overline{\mathcal{T}}_{n-1}) \ .$$

Using the language of tannakian categories this induces an 'enriched' group homomorphism

$$f^\bullet : \ H_{n-1} \times \mathbb{G}_m \to H_n \ .$$

Its restriction to the subgroup $1 \times H_{n-1}$ is the homomorphism f from above.

5.5. **The involution τ.** Note that the category $\mathcal{T}_n^+$ is closed under $\vee$ and $*$ and hence is equipped with the tensor equivalence $\tau : X \mapsto (X^\vee)^*$. This tensor equivalence induces a tensor equivalence of $\overline{\mathcal{T}}_n = \mathcal{T}_n^+/\mathcal{N}$ and hence an automorphism $\tau = \tau_n$ of the group H_n. Since all objects of $\mathcal{T}_n^+$ satisfy property T [HW14, Section 6], the involution $*$ commutes with DS. Since this also holds for the Tannaka duality, we get a compatibility

$$(H_{n-1}, \tau_{n-1}) \hookrightarrow (H_n, \tau_n) \ .$$

5.6. **Characteristic polynomial.** By iteration the morphisms $f^\bullet$ successively define homomorphisms $H_{n-i} \times (\mathbb{G}_m)^i \to H_n$ and therefore we get a homomorphism in the case $i = n$

$$h : (\mathbb{G}_m)^n \to H_n \ .$$

This allows to define a characteristic polynomial, defined by the restriction $h^*(V_X)$ of the representation $V_X = \omega(X)$ of H to the torus $(\mathbb{G}_m)^n$

$$h_X(t) = \sum_\chi \dim(h^*(V_X)_\chi) \cdot t^\chi$$

where χ runs over the characters $\chi = (\nu_1, ..., \nu_n) \in \mathbb{Z}^n = \mathbb{X}^*((\mathbb{G}_m)^n)$. Of course $\omega(X,t) = h_X(t,...,t)$.

6. The structure of the derived connected groups G_n

6.1. **Setup.** We now consider the categories $\mathcal{T}_n^+$ for the cases $n \geq 4$. We compute the connected derived groups

$$G_X := (H_X^0)_{der}$$

for irreducible objects X in $\mathcal{T}_n^+$. The Tannaka group generated by the object $X_\lambda = \Pi^{|\lambda|} L(\lambda)$ for $|\lambda| = \sum_{i=1}^n \lambda_i$ will be denoted H_λ and we define

$$G_\lambda := (H_\lambda^0)_{der} \subseteq H_\lambda^0 \subseteq H_\lambda \ .$$

Finally define $V_\lambda \in Rep(H_\lambda)$ as the irreducible finite dimensional faithful representation (or the underlying vector space) of H_λ corresponding to X_λ.

A normalization. By twisting with a Berezin power we may assume that λ is a maximal atypical weight with the property $\lambda_n = 0$. We therefore make the assumption $\lambda_n = 0$.

A priori bounds. We distinguish two cases: Either X_λ is a weakly selfdual object (SD), i.e. $X_\lambda^\vee \cong Ber^r \otimes X_\lambda$ for some r; or alternatively X_λ is not weakly selfdual (NSD). In the (SD) case V_λ carries a symmetric or anti-symmetric pairing $<,>$ and we can define the orthogonal similitude group $GO(V_\lambda)$ and the symplectic similitude group $GSp(V_\lambda)$ as

$$GO(V_\lambda) = \{g \in GL(V_\lambda) \mid \ < gv, gv >= \mu(g) < v, v >, \ \forall v \in V_\lambda\},$$
$$GSp(V_\lambda) = \{g \in GL(V_\lambda) \mid \ < gv, gv >= \mu(g) < v, v >, \ \forall v \in V_\lambda\}$$

for the similitude character $\mu : GO(V_\lambda) \to k^*$ respectively $\mu : GSp(V_\lambda) \to k^*$. Note that $\dim(V_\lambda) = 2m$ is always even by lemma C.4. In the GSp-case $det(g) = \mu(g)^m$ and $GSp(V_\lambda)$ is connected. In the GO-case $(det(g))^2 = \mu(g)^{2m}$ and we have the sign character sgn on $GO(V_\lambda)$

$$sgn : g \mapsto \frac{g}{\mu(g)^m} \in \mu_2.$$

Then the connected component of 1 in $GO(V_\lambda)$ is denoted by $GSO(V_\lambda) = ker(sgn)$ and sits in an exact sequence

$$1 \longrightarrow GSO(V_\lambda) \longrightarrow GO(V_\lambda) \longrightarrow \mu_2 \longrightarrow 1.$$

Using these notations we obtain the following bounds for the groups H_λ. Whereas

$$H_\lambda \subseteq GL(V_\lambda)$$

in the case (NSD), we have

$$H_\lambda \subseteq GO(V_\lambda) \quad , \quad H_\lambda \subseteq GSp(V_\lambda)$$

in the case (NSD) for even resp. odd X_λ. In the case of a proper self duality $X_\lambda^\vee \cong X_\lambda$ the groups can be furthermore replaced by the subgroups $O(V_\lambda)$ resp. $Sp(V_\lambda)$.

6.2. **The structure theorem on G_λ.** Recall that two maximal atypical weights λ, μ are equivalent $\lambda \sim \mu$ if there exists $r \in \mathbf{Z}$ such that $L(\lambda) \cong Ber^r \otimes L(\mu)$ or $L(\lambda)^\vee \cong Ber^r \otimes L(\mu)$ holds. Another way to express this is to consider the restriction of the representations $L(\lambda)$ and $L(\mu)$ to the Lie superalgebra $\mathfrak{sl}(n|n)$. These restrictions remain irreducible and $\lambda \sim \mu$ holds if and only if $L(\lambda) \cong L(\mu)$ or $L(\lambda) \cong L(\mu)^\vee$ as representations of $\mathfrak{sl}(n|n)$. Let $X^+(n)$ be the set of dominant weights and let $Y^+(n)$ be the set of equivalence classes of dominant weights. Similarly let $X_0^+(n)$ denote the class of maximal atypical dominant weights and $Y_0^+(n)$ the set of corresponding equivalence classes. If we write $\lambda \in Y_0^+(n)$, we mean that $\lambda \in X_0^+(n)$ is some representative of the class in $Y_0^+(n)$ defined by λ. If $L(\lambda)^\vee \not\sim L(\lambda)$, $\lambda \in Y_0^+(n)$ is of type (NSD). Otherwise it is of type (SD), and there exists $r \in \mathbb{Z}$ such that $L(\lambda) \cong Ber^r \otimes L(\lambda)^\vee$. Hence there exists an equivariant nondegenerate pairing

$$L(\lambda) \times L(\lambda) \longrightarrow Ber^r \ .$$

This pairing is either symmetric (even) or antisymmetric (odd). The next lemma is proven in appendix B.

Lemma 6.1. *The selfdual representation $[\lambda] = [\lambda_1, \ldots, \lambda_{n_1}, 0]$ is even. Its parity shift $\Pi[\lambda]$ is odd.*

Theorem 6.2. *$G_\lambda = SL(V_\lambda)$ if X_λ is (NSD). If X_λ is (SD) and $V_\lambda|_{G_{\lambda'}}$ is irreducible, then $G_\lambda = SO(V_\lambda)$ respectively $G_\lambda = Sp(V_\lambda)$ according to whether X_λ is even respectively odd. If X_λ is (SD) and $V_\lambda|_{G_{\lambda'}}$ decomposes into at least two irreducibe representations, then $G_\lambda \cong SL(W)$ for $V_\lambda|_{G_{\lambda'}} \cong W \oplus W^\vee$.*

This theorem is proven in sections 7 - 10. Many examples can be found in section 9. We conjecture that a stronger version is true: V_λ should always stay irreducible. We refer to section 11 for a discussion of this case.

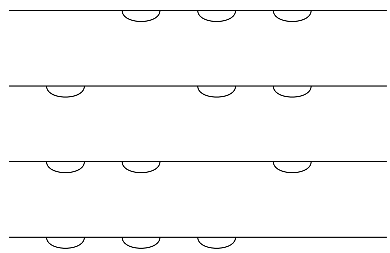
Remark 6.3. The (NSD) case is the generic case for $n \geq 4$. Since $SL(V_\lambda) \cong G_\lambda \subset GL(V_\lambda)$, all representations of H_λ stay irreducible upon restriction to G_λ. Hence the derived group sees already the entire tensor product decomposition into indecomposable representations up to superdimension zero. The same remark is true for a selfdual weight of symplectic type. In the orthogonal case we could have a decomposition of an irreducible representation of H_λ into two irreducible representations of G_λ since $O(V_\lambda)$ and $GO(V_\lambda)$ have two connected components.

Example 6.4. The smallest case for which V_λ could decompose when restricted to G_λ is the case $[\lambda] = [3, 2, 1, 0] \in \mathcal{T}_4^+$ with sector structure

Then $DS[3, 2, 1, 0]$ decomposes into four irreducible representations

$$L_1 = [3, 2, 1], L_2 = [3, 2, -1], L_3 = [3, 0, -1], L_4 = [1, 0, -1]$$

represented by the cup diagrams



Since $L_1 = Ber^2 L_4$ and $L_2 \cong L_3^\vee$ we have two equivalence classes

$$\{L_1, L_4\}, \{L_2, L_3\}.$$

In fact

$$V_{\lambda_1} \cong V_{\lambda_4} \cong st(SO(6))$$
$$V_{\lambda_2} \cong st(SL(6)), V_{\lambda_3} \cong st(SL(6))^\vee.$$

If $V_{[3,2,1,0]}$ does not decompose under restriction to $G_{[3,2,1,0]}$, then $G_\lambda \cong SO(24)$ and $V_\lambda \cong st(SO(24))$. If it decomposes $V_\lambda = W \oplus W^\vee$, then $G_\lambda \cong SL(12)$ and $V_\lambda \cong st(SL(12))$. Since $W \not\cong W^\vee$ this implies that the embedding $SO(6) \times SL(6) \to SL(12)$ gives the branching rules

$$W \mapsto st(SL(6)) \oplus st(SO(6))$$
$$W^\vee \mapsto st(SL(6))^\vee \oplus st(SO(6)).$$

6.3. **The structure theorem on G_n.** We now determine G_n.

Lemma 6.5. *Suppose a tannakian category $\mathcal{R}$ with Tannaka group H is $\otimes$-generated as a tannakian category by the union of two subsets V' and V''. Let H' and H'' be the Tannaka groups of the tannakian subcategories generated by V' respectively V''. Then there exists an embedding $H \hookrightarrow H' \times H''$ so that the composition with the projections is surjective.*

Proof. There are natural epimorphisms $\pi' : H \to H'$ and $\pi'' : H \to H''$ which induce a morphism $i : H \to H' \times H''$ so that the composition with the projections are π' and π''. It remains to show that i is injective. For this we can reduce to the case where H' and H'' are Tannaka groups of tannakian subcategories $\langle X' \rangle_\otimes$ and $\langle X'' \rangle_\otimes$ of selfdual objects X' and X'' of $\mathcal{R}$. Then $\mathcal{R} = \langle X' \oplus X'' \rangle_\otimes$. For the fiber functor ω the group H therefore acts faithful on $\omega(X' \oplus X'') = V' \oplus V'' = V$. The operation of H on V' factors over the quotient H' of H and the operation of H on V'' factors over the quotient H''. Hence the kernel of i acts trivially on V. Therefore the kernel of i is trivial by the faithfulness of V. □

We remark that the inclusion $H \hookrightarrow H' \times H''$ induces an inclusion $H^0 \hookrightarrow (H')^0 \times (H'')^0$ of the Zariski connected components and hence an inclusion of the corresponding adjoint groups $H^0_{ad} := (H^0)_{ad}$ and derived groups $G := H^0_{der} := (H^0)_{der}$

$$H^0_{ad} \hookrightarrow (H')^0_{ad} \times (H'')^0_{ad} \ ,$$

and

$$G \hookrightarrow G' \times G''$$

abbreviating $H^0_{der} \hookrightarrow (H')^0_{der} \times (H'')^0_{der}$.

We also need the following variant of Goursat's lemma.

Lemma 6.6. *Suppose H is a subgroup of the product $A \times B$ of two semisimple affine algebraic k-groups A and B, so that the projections to A and B are surjective. Then*

1. *If A and B are connected simple k-groups, then either $H_{ad} = A_{ad} \times B_{ad}$ or $H_{ad} \cong A_{ad} \cong B_{ad}$.*
2. *$H \cong A \times B$, if A and B are of adjoint type without common factor.*
3. *If A and B are connected, $H \cong A \times B$ if and only if $H_{ad} \cong A_{ad} \times B_{ad}$.*
4. *Suppose A is a connected semisimple group and B is a connected simple group. Let H be a proper subgroup H of $A \times B$, that surjects onto A and B for the projections. Then there exists a simple normal subgroup C of A, such that the image H/C of H in $(A/C) \times B$ is a proper subgroup of $(A/C) \times B$, if A is not a simple group.*

Proof. (1)-(3) are obvious. Part (4) can be reduced to the case of adjoint groups by part (3). So we may assume that B and A are groups of adjoint type. We now use the following fact. Any semisimple A group of adjoint type is isomorphic to the product $\prod_{i=1}^r A_i$ of its simple subgroups A_i. Its factors are the normal simple subgroups of A. These factors and hence this product decomposition is unique up to a permutation of the factors. Any nontrivial algebraic homomorphism of A to a simple group B is obtained as projection of A onto some factor A_i of the product decomposition composed with an injective homomorphism $A_i \to B$. Since $H \subseteq A \times B$ projects onto the first factor A and B is simple, and since H is a proper subgroup of the connected semisimple group $A \times B$, the kernel of the projection $p_A : H \to A$ is a finite normal and hence central subgroup of H. It injects into the center of B, hence is trivial. Thus $p_A : H \to A$ is an isomorphism so that H defines the graph of a group homomorphism $A \to B$. Since A is of adjoint type and therefore a product of simple groups $A \cong \prod_{i=1}^r A_i$, the kernel of the homomorphism $A \to B$ must be of the form $\prod_{i \neq j} A_i$. Unless A is simple, for $C = A_j$ and $j \neq i$ assertion (4) becomes obvious. $\square$

Corollary 6.7. *Let λ and μ be two maximal atypical weights and denote by $G_{\lambda,\mu}$ the connected derived group of the Tannaka group $H_{\lambda,\mu}$ corresponding to the subcategory in $\overline{\mathcal{T}}_n$ generated by $L(\lambda)$ and $L(\mu)$. If λ is not equivalent to μ,*

$$G_{\lambda,\mu} \cong G_\lambda \times G_\mu.$$

Proof. If G_λ and G_μ are not isomorphic, lemma 6.6 implies the claim. Otherwise $G_{\lambda,\mu} \cong G_\lambda \cong G_\mu$ (special case of lemma 6.6.1). We assume by induction that the statement holds for smaller n. By lemma A.5 there exist constituents $L(\lambda_i)$ of $DS(L(\lambda))$ and $L(\mu_j)$ of $DS(L(\mu)$ such that λ_i and μ_j are inequivalent maximal atypical weights (for $n > 2$) - a contradiction. For $n = 2$ we give an adhoc argument in section 9. □

Theorem 6.8. Structure Theorem for G_n. *The connected derived group G_n of the Tannaka group H_n of the category $\mathcal{T}_n^+$ is isomorphic to the product*

$$G_n \;\cong\; \prod_{\lambda \in Y_0^+(n)} G_\lambda \ .$$

Proof. This follows essentially from theorem 6.2, where the structure of the individual groups G_λ was determined. Using lemma 6.6, one reduces the statement of the theorem to a situation that involves only two inequivalent weights λ and μ: By part (3) of lemma 6.6 we may replace the derived groups by the adjoint groups. Then the assertion follows from part (4) of the lemma by induction on the number of factors reducing the assertion to the case of two groups G_λ, G_μ dealt with in corollary 6.7. □

Example 6.9. Consider the tensor product of two inequivalent representations $L(\lambda)$ and $L(\mu)$ of non-vanishing superdimension. Then

$$L(\lambda) \otimes L(\mu) = I \mod \mathcal{N}$$

for an indecomposable representation I. Indeed $L(\lambda)$ and $L(\mu)$ correspond to representations of the derived connected Tannaka groups G_λ and G_μ. Since G_λ and G_μ are disjoint groups in G_n, tensoring with $L(\lambda)$ and $L(\mu)$ corresponds to taking the external tensor product of these representations.

7. Proof of the structure theorem: Overview

We determine G_λ inductively using the k-linear exact tensor functor between the quotient categories of the representation categories

$$\eta : \overline{\mathcal{T}}_n \to \overline{\mathcal{T}}_{n-1}$$

constructed in lemma 5.10 with the help of $DS : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$. We remark that η is compatible with DS on objects not true for the functor DS : $\mathcal{T}_n \to \mathcal{T}_{n-1}$. Recall that the category $\overline{\mathcal{T}}_n$ is equivalent to the representation category of a pro-reductive group H_n. By a deep theorem of Deligne on tensor categories (theorem 5.16), one can use the functor $\eta : \overline{\mathcal{T}}_n \to \overline{\mathcal{T}}_{n-1}$ to construct an embedding of affine group schemes $H_{n-1} \to H_n$. By definition of H_λ, $L(\lambda)$ defines an irreducible faithful representation of H_λ which we denote by V_λ. By the main theorem on DS (theorem 4.1), the restriction of V_λ to the subgroup H_{n-1} is a multiplicity free representation. We assume by induction that theorem 6.2 and theorem 6.8 hold for H_{n-1} and G_{n-1}.

We have inclusions

$$G_{\lambda'} \hookrightarrow G_\lambda \hookrightarrow H_\lambda^0 \hookrightarrow H_\lambda$$

where $G_{\lambda'}$ denotes the image of the natural map $(H_{n-1}^0)_{der} \to G_\lambda = (H_\lambda^0)_{der}$. The restriction of V_λ to $G_{\lambda'}$ decomposes

$$V_\lambda \;\cong\; \bigoplus_{i=1}^{k} V_{\lambda_i}$$

where the V_{λ_i} are the irreducible representations in the category $\overline{\mathcal{T}}_{n-1}^{+}$ corresponding to the irreducible constituents $L(\lambda_i), i = 1, .., k,$ of $DS(L(\lambda))$. By induction we obtain

$$G_{\lambda'} \cong \prod_{\lambda_i/\sim} G_{\lambda_i}$$

where the G_{λ_i} are described in theorem 6.2.

In a first step we discuss the situation in the $n = 2$ and the $n = 3$ case as well as the Tannaka groups G_λ for $L(\lambda) = Ber^r \otimes [i, 0, \ldots, 0]$, $r, i \in \mathbf{Z}$. The $n = 2$-case is needed for the start of the inductive determination of G_n. In this case we can use the known tensor product decomposition between irreducible modules in $\mathcal{T}_2$ to determine G_2 and H_2. In order to get a clear induction scheme in the proof of the structure theorem, we need to rule out certain exceptional cases which can only occur for $n \leq 3$ and for the modules $Ber^r \otimes [i, 0, \ldots, 0]$. This will allow us to assume $n \geq 4$ in section 10.

In the next step we show that G_λ is simple. By induction all the V_{λ_i} are standard representations for simple groups of type A, B, C, D or $V_{\lambda_i}|_{G_{\lambda_i}} = W \oplus W^\vee$ for $G_{\lambda_i} \cong SL(W)$. The representation V_λ decomposes under restriction to G_λ in the form $W_1 \oplus \ldots \oplus W_s$ (we later show that s is at most 2). If we restrict these W_ν to $G_{\lambda'}$, they are meager representation of $G_{\lambda'}$ in the sense of definition 10.2. The crucial lemma 10.3 shows then that G_λ is simple. This allows us to use the classification of small representations due to Andreev-Elashvili-Vinberg.

Our aim is then to show that the dimension of the subgroup $G_{\lambda'}$ is large compared to the dimension of V_λ (given by the superdimension formula for $L(\lambda)$ in [HW14]) as in lemma 8.1 or corollary 8.2. A large rank and a large dimension of $G_{\lambda'}$ implies that the rank and the dimension of G_λ must be large, forcing V_λ to be a small representation of G_λ in the sense of lemma 8.1 and corollary 8.2. If we additionally know that G_λ is simple and that also $r(G_\lambda) \geq \frac{1}{2}(\dim(V_\lambda - 1)$, corollary 8.2 will immediately imply that G_λ is of type $SL(V_\lambda)$, $SO(V_\lambda)$ or $Sp(V_\lambda)$. However the strong rank estimate will not always hold and we will be in the less restrictive situation of lemma 8.1.

Here the (NSD) and the (SD) case differ considerably. In the (NSD) case each irreducible representation V_{λ_i} (corresponding to $L(\lambda_i)$ in $DS(L(\lambda))$) gives a distinct direct factor in the product $G_{\lambda'} \cong \prod_{\lambda_i/\sim} G_{\lambda_i}$ since all irreducible representations of $DS(L(\lambda))$ are inequivalent in the (NSD) case by lemma A.2. The dimension estimate for G_λ so obtained then implies that V_λ is a small representation. In the (SD) case however two representations $V_{\lambda_i}, V_{\lambda_j}$ will contribute the same direct factor $G_{\lambda_i} \simeq G_{\lambda_j}$ if $\lambda_i \sim \lambda_j$. This decreases the dimension and rank estimate of the subgroup $G_{\lambda'}$ in G_λ and therefore of G_λ.

To finish the proof we need to understand the restriction of V_λ to G_λ. The group of connected components acts transitively on the irreducible constituents $V_\lambda = W_1 \oplus \ldots \oplus W_s$ of the restriction to H_λ^0 and G_λ. Using that the decomposition of V_λ to H_{n-1} is multiplicity free in a weak sense (obtained from an analysis of the derivatives of $L(\lambda)$ in section A), we show finally in section 10.3, using Clifford-Mackey theory, that V_λ can decompose into at most $s = 2$ irreducible representations of G_λ.

8. Small representations

Our aim is to understand the Tannaka groups associated to an irreducible representation by means of the restriction functor $DS : \mathcal{T}_n^+ \to \mathcal{T}_{n-1}^+$. We have a formula for the superdimension of an irreducible representation [HW14] and we know inductively the ranks and dimensions of the groups arising for $k < n$. This gives strong restrictions about the groups in the $\mathcal{T}_n^+$-case due to the following list of small representations.

List of small representations. For a simple connected algebraic group H and a nontrivial irreducible representation V of H the following holds [AVE67]

Lemma 8.1. $\dim(V) = \dim(H)$ *implies that* V *is isomorphic to the adjoint representation of* H*. Furthermore, except for a finite number of exceptional cases,* $\dim(V) < \dim(H)$ *implies that* V *belongs to the regular cases*

- ***R.1*** $V \cong st, S^2(st), \Lambda^2(st)$ *or their duals in the* A_r*-case,*
- ***R.2*** $V = st$ *(the standard representation) in the* B_r, D_r*-case,*
- ***R.3*** $V \cong st$ *in the* C_r*-case,*
- ***R.4*** $V \hookrightarrow \Lambda^2(st)$ *in the* C_r*-case*

where the list of exceptional cases is

- ***E.1*** $\dim(V) = 20, 35, 56$ *for* $V = \Lambda^3(st)$ *and* A_r *in the cases* $r = 5, 6, 7$.
- ***E.2*** $\dim(V) = 4, 8, 16, 32, 64$ *for the spin representations of* B_r *in the cases* $r = 2, 3, 4, 5, 6$.
- ***E.3*** $\dim(V) = 8, 8, 16, 16, 32, 32, 64, 64$ *for the two spin representations of* D_r *in the cases* $r = 4, 5, 6, 7$.
- ***E.4*** $\dim(V) = 27, 27$ *for* E_6 *with* $\dim(E_6) = 78$ *(standard representation and its dual).*
- ***E.5*** $\dim(V) = 56$ *for* E_7 *with* $\dim(E_7) = 133$.
- ***E.6*** $\dim(V) = 7$ *for* G_2 *with* $\dim(G_2) = 14$.
- ***E.7*** $\dim(V) = 26$ *for* F_4 *with* $\dim(F_4) = 52$.

In particular $\dim(V) \geq r+2$ *holds, except for* $G = A_r$ *in the cases* $V \cong st$ *or* $V \cong st^\vee$.

Corollary 8.2. *Let* V *be an irreducible representation of a simple connected group* H *such that* $4 \leq \dim(V) < \dim(H)$ *and*

$$2r(H) \geq \dim(V) - 1$$

holds. Then H *is of type* A_r, B_r, C_r, D_r *and* $V = st$ *the standard representation of this group of dimension* $r+1,\ 2r+1,\ 2r,\ 2r$ *for* $r \geq 3,\ 2,\ 2,\ 2$ *respectively, or* $H = D_4$ *and* V *is one of the two 8-dimensional spin representations.*

Note that D_4 has an automorphism of order three so that the spin representations of D_4 can be obtained from the standard representation by a twist. From the classification in lemma 8.1 one also obtains

Lemma 8.3. *For a simple connected grous* H *with an irreducible root system of rank* r *we have* $\dim(H) \geq r(2r-1)$ *except for* $H \cong SL(n)$ *with* $\dim(H) = r(r+2)$. *Furthermore* $r \leq \dim(V)$ *holds for any nontrivial irreducible representation* V *of* H.

9. The cases $n = 2, 3$ and the S^i-case

In the next sections we determine the group G_n and the groups G_λ. Since we will determine these groups inductively starting from $n = 2$, we need to start with this case. We also discuss the $n = 3$ case separately since we have to rule out some exceptional low rank examples in the classification of [AVE67] in section 8.

Warm-up. Suppose $n = 1$. Then H_1 is the multiplicative group $\mathbb{G}_m$. Indeed the irreducible representations of it correspond to the irreducible modules $\Pi^i Ber^i$ for $i \in \mathbb{Z}$.

9.1. The case $n = 2$.

Suppose

$$X_i := \Pi^i([i,0])$$

for $i \geq 1$. Then $X_i^\vee \cong Ber^{1-i} \otimes X_i$, hence $X_1^\vee \cong X_1$. We use from [HW15] the fusion rule

$$[i,0] \otimes [j,0] \ = \ \text{indecomposable} \ \oplus \ \delta_i^j \cdot Ber^{i-1} \oplus \text{negligible}$$

for $1 \leq i \leq j$ together with $Ber^r \otimes [i,0] \cong [r+i, r]$ for all $r \in \mathbb{Z}$.

Lemma 9.1. *If* H_{X_i} *denotes the Tannaka group of* X_i, *then*

$$H_{X_i} \simeq \begin{cases} SL(2) & i = 1 \\ GL(2) & i \geq 2. \end{cases}$$

Proof. Since $H_1 \hookrightarrow H_2 \twoheadrightarrow H_{X_i}$ can be computed from DS we see that H_1 injects into $H = H_{X_i}$ and the two dimensional irreducible representation $V = V_{X_i}$ of H_{X_i} attached to X_i decomposes into

$$V|_{H_1} \;=\; det^{-1} \oplus det^i \;.$$

corresponding to $DS(X_i) \;=\; Ber^{-1} \oplus Ber^i$. If $H^0_{X_i} \cong \mathbb{G}_m$, the finite group $\pi_0(H)$ acts on H^0. By Mackey's theorem the stabilizer of the character Ber^{-1} has index two in H_{X_i} and acts by a character on V. Since the only automorphisms of $\mathbb{G}_m$ are the identity and the inversion, this would imply $i = 1$. Hence $V \otimes V$ would restrict to $\mathbb{G}_m$ with at least three irreducible constituents $det^{-2} \oplus det^2$ (corresponding to $Ber^{-2} \oplus Ber^2$) and a two dimensional module W with an action of $\pi_0(H)$ such that a subgroup of index two acts by a character. But $X_1^\vee \cong X_1$ implies that V is self dual, and hence W contains the trivial representation. This contradicts the fusion rule from above. Hence $H^0 \neq \mathbb{G}_m$ and the same argument as above shows that H^0 can not be a torus. Hence the rank r of each irreducible component of the Dynkin diagram of $(H^0_{der})_{sc}$ is $r \geq 1$ and hence $\dim(H) \geq 3$. By lemma 8.3 we know $r \leq \dim(V) = 2$ and accordingly $\dim(H) = 3$ by lemma 8.1. Therefore $(H^0_{der}) = SL(2)$ and $V|_{H^0_{der}}$ is the irreducible standard representation. Since H acts faithful on V

$$SL(2) \subseteq H \subseteq GL(2) \;.$$

Now we use $V^\vee \cong Ber^{i-1} \otimes V$, which implies $H = GL(2)$ for $i > 1$. Indeed $\Lambda^2(V)$ is the character Ber^{i-1} by the fusion rules above. For $i = 1$ the isomorphism $V^\vee \cong V$ implies that $det(V)$ is trivial on H, hence

$$H = SL(2)$$

in the case $i = 1$. □

9.2. **The H_2-case.** We discuss the Tannaka group generated by all irreducible representations. First consider the Tannaka group H of $\langle X_i, X_j \rangle_\otimes$ for some pair $j > i$. The derived groups of the Tannaka groups H' resp. H'' of $\langle X_i \rangle_\otimes$ and $\langle X_j \rangle_\otimes$ are $SL(2)$.

We claim that $H_{der} \cong H'_{der} \times H''_{der}$. If this were not the case, then $H_{der} \cong SL(2)$ (special case of lemma 6.6.1). But then the tensor product $X_i \otimes X_i$ considered as a representation of H corresponds to the tensor product of two standard representation and hence is a reducible representation with two irreducible factors. However this contradicts the fusion rules stated above. This implies $H_{der} \cong SL(2) \times SL(2)$ and hence $H_{ad} \cong H'_{ad} \times H''_{ad}$.

Now consider the Tannaka group H of $\langle X_{i_1}, ..., X_{i_k} \rangle_\otimes$ for $k > 2$. We claim that H is connected and that it is the product

$$H_{der} \;\cong\; \prod_{\nu=1}^{k} H_{der}(X_{i_\nu})$$

of the derived Tannaka groups of the $\langle X_{i_\nu} \rangle_\otimes$. This is an immediate consequence of lemma 6.6

So the Tannaka group H_2 of the category $\mathcal{T}_2^+/\mathcal{N}$ sits in an exact sequence

$$0 \to \lim_k \prod_{\nu=0}^{k-1} SL(2) \to H_2 \to \mathbb{G}_m \to 0 \ .$$

The derived group of H_2 is the projective limit of groups $SL(2)$ with a copy for each irreducible object $X_{\nu+1}$ for $\nu = 0, 1, 2, 3, ...$ The structure of the extension is now easily recovered from the following decription:

Lemma 9.2. *$H_2 \subset \prod_{\nu=0}^\infty GL(2)$ is the subgroup defined by all elements $g = \prod_{\nu=0}^\infty g_\nu$ in the product with the property $det(g_\nu) = det(g_1)^\nu$. The automorphism τ_2 is inner.*

We usually write $GL(2)_\nu$ for the ν-th factor of the product $\prod_{\nu=0}^\infty GL(2)$. Using the description of the last lemma, the torus $H_1 \cong \mathbb{G}_m$ embeds into H_2 as follows

$$H_1 \ni t \mapsto \prod_{\nu=0}^\infty diag(t^{\nu+1}, t^{-1}) \in H_2 \subset \prod_{\nu=0}^\infty GL(2)_\nu \ .$$

Defining $det(g) = det(g_1)$ for $g = \prod_{\nu=0}^\infty g_\nu$ in H_2, the representation of the quotient group $\mathbb{G}_m$ of H_2 defined by the Berezin determinant $Ber \in \mathcal{T}_2$, corresponds to the character $det(g)$ of the group H_2.

We continue with two special cases: The S^i-case for any n, and the case G_3.

9.3. **The S^i-case.** Consider the modules $X_i = \Pi^i([i, 0, 0])$ in $\mathcal{T}_3^+$. They have superdimension 3 for $i \geq 2$. Let H (or sometimes H_{X_i}) denote the associated Tannaka group and V the associated irreducible representation of H.

Lemma 9.3. *We have $H_{X_1} = SL(2)$ and $G_{X_i} \simeq SL(3)$ for any $i \geq 2$ and $H_{X_i} \simeq GL(3)$ for any $i \geq 3$.*

Proof. The natural map $H_2 \to H_3 \to H$ allows to consider V as a representation of H_2, and as such we get

$$V|_{H_2} \ \cong \ det^{-1} \oplus X_i$$

for $i \geq 2$ (here X_i on the right is the irreducible 2-dimensional standard representation of $GL(2)_{i-1}$, restricted to H_2). Hence $\dim(A) \geq 3$ for at least one simple factor A of H^0 and every irreducible summand W of $V|_A$ has dimension $\leq \dim(A)$. By lemma 8.1 therefore W either has dimension 3 and $A_{sc} = SL(3)$, $W = st$ or $W = st^\vee$, or $A_{sc} = SL(2)$ and $W = S^2(st)$. If H^0_{der} is not simple, we replace it by its simply connected cover and write $(H^0_{der})_{sc} = A_{sc} \times A'$ (where A' is a product of simple groups). The representation V is then an external tensor product

$$V = W \boxtimes W'$$

of irreducible representations W, W' of A_{sc} and A'. Since V is a faithful representation of H, the lift of V (again denoted V) to $(H^0_{der})_{sc}$ has finite kernel. Since it has finite kernel, $\dim(W) > 1$, $\dim(W') > 1$ holds. Hence $\dim(W) = 3$ implies $(H^0_{der}) = A$ and $V|_{H^0}$ and $V|_{H^0_{der}}$ remain irreducible by dimension reasons. If $A_{sc} = SL(2)$ and $W = S^2(st)$, the image of H_2 surjects onto H_{der}. This contradicts the fact that V is irreducible but $V|_{H_2}$ decomposes, and excludes the case $A_{sc} = SL(2)$. Hence

$$H^0_{der} \ \cong \ SL(3) \ .$$

Since H acts faithfully on V, we also have $H \subseteq GL(V) = GL(3)$. The restriction of V to H_2 has determinant $det^{-1} \cdot \det(X_i) \cong det^{-1} det^{i-1} = det^{i-2}$. Hence

$$H \ \cong \ GL(3)$$

for all $i \geq 3$. $\square$

For $j > i \geq 2$ let H denote the Tannaka group of $\langle X_i, X_j \rangle_\otimes$ and H', H'' the connected components of the Tannaka groups of $\langle X_i \rangle_\otimes$ resp. $\langle X_j \rangle_\otimes$. Then we claim

$$H^0_{der} \ \cong \ H'_{der} \times H''_{der} \ ,$$

since otherwise $H'_{der} \cong H''_{der}$ by lemma 6.6.1. But this is impossible since then the morphisms $H_2 \to H_3 \to H$ would induce the same morphisms $(H_2)_{der} \to H_{der} \to H'_{der}$ and $(H_2)_{der} \to H_{der} \to H''_{der}$, which contradicts theorem 4.1. Indeed the factor $SL(2)_{i-1}$ maps nontrivially to H'_{der} but trivially to H''_{der}. Since H acts faithfully on the representation associated to the object $X_i \oplus X_j$ on the other hand $H \subseteq GL(\omega(X_i)) \times GL(\omega(X_j))$.

The same arguments enable us to determine the connected derived groups for any $n \geq 3$:

Lemma 9.4. *The Tannaka group H of the modules $\Pi^i([i, 0, .., 0]) \in \mathcal{T}_n^+$ satisfies $H^0_{der} \cong SL(n)$ and $H \subseteq GL(n)$ for all $i \geq n-1$, and $H = GL(n)$ for all $i \geq n$. For $i < n-1$ we get $H^0_{der} \cong SL(\mathrm{sdim}(L_i))$.*

Proof. Indeed we have in H^0_{der} a simple component A of semisimple rank $r \geq n-1$ by induction. Obviously A contains $SL(n-1)$ and cannot be of Dynkin type A_r unless $A = SL(n)$ by lemma 8.1.

Notice that $\dim(A) \geq r(2r-1) \geq (n-1)(2n-3) > n$ or $\dim(A) \geq r(r+2) \geq (n-1)(2n) > n$, for $n \geq 3$ by lemma 8.3. The restriction of V decomposes into irreducible summands $W, W', ...$ of dimension $\dim(W) \leq n$, and the dimension of all these representations is $\leq r$. So the possible representations are listed in lemma 8.1. None of them has dimension $\leq r+1$ except for the case where A is of type A_r and $V \cong st$ or $V \cong st^\vee$. $\square$

9.4. **The $n = 3$-case.** We analyse the remaining $n = 3$-cases.

Lemma 9.5. *The derived connected group $G_3 = (H_3)^0_{der}$ of H_3 is*

$$G_3 \ \cong \ \prod_\lambda G_\lambda \ ,$$

where λ *runs over all* $\lambda = [\lambda_1, \lambda_2, 0]$ *with integers* λ_1, λ_2 *such that*

$$0 \leq 2\lambda_2 \leq \lambda_1$$

and $G_\lambda \cong 1, SL(2), SL(3), Sp(6), SL(6)$ *according to whether* λ *is* $0, [1,0,0]$ *or* $[2+\nu, 0, 0]$, *for* $\nu \geq 0$, *or* $\lambda = [2\lambda_2, \lambda_2, 0]$, *for* $\lambda_2 > 0$, *or* $0 < 2\lambda_2 < \lambda_1$.

Remark 9.6. We discuss the general case in the next section assuming $n \geq 4$. The assumption $n \geq 4$ is only relevant because we want to have a uniform behaviour regarding derivatives. Essentially all the arguments regarding simplicity of G_λ and Clifford-Mackey theory apply to the $n = 3$ case at hand. In the proof we discuss $[2,1,0]$ in detail and sketch the key inputs for the other cases.

Proof. Let us consider $X = \Pi([210])$. The associated irreducible representation Tannaka group $H = H_X$ admits an alternating pairing, hence H_X is contained in the symplectic group of this pairing

$$H_X \ \subseteq \ Sp(6) \ .$$

We claim that H^0_{der} is simple. If not, we replace it by its simply connected cover and write it as a product

$$(H^0_{der})_{sc} = G_1 \times G_2.$$

The faithful representation V_X of H_X has finite kernel when seen lifted to a representation of $(H^0_{der})_{sc}$. Therefore V_λ as a representation of $(H^0_{der})_s c$ is of the form $V_1 \boxtimes V_2$ with $\dim(V_i) > 1$. The representation V_λ restricts to the subgroup $SL(2) \times SL(2) = G_{\lambda'}$ as

$$V_\lambda|_{G_{\lambda'}} \cong 2 \cdot (st \boxtimes \mathbf{1}) \oplus (\mathbf{1} \boxtimes st).$$

This is easily seen using

$$DS(\Pi[2,1,0]) \cong \Pi[2,1] \oplus \Pi[2,-1] \oplus \Pi[0,-1].$$

Since $\Pi[2,1] \cong Ber^{-2} \otimes [0,-1]$ they both give a copy of the standard representation of the same $SL(2)$. Hence the restriction of V_λ to the first $SL(2)$-factor is of the form

$$V_\lambda|_{SL(2)} \cong 2st \oplus 2 \cdot \mathbf{1}$$

and

$$V_\lambda|_{SL(2)} \cong st \oplus 4 \cdot \mathbf{1}$$

for the second $SL(2)$-factor. Now consider the restriction to any of the two $SL(2)$-factors

$$V|_{SL(2)} = V_1|_{SL(2)} \otimes V_2|_{SL(2)}.$$

Since $dim(V_1) = 2$ and $dim(V_2) = 3$, their restriction to $SL(2)$ is either st or $2 \cdot \mathbf{1}$ for V_1 and $st \oplus \mathbf{1}$ or $3 \cdot \mathbf{1}$ for V_2. The Clebsch-Gordan rule for $SL(2)$ shows that $V|_{SL(2)} = V_1|_{SL(2)} \otimes V_2|_{SL(2)}$ is not possible, hence H^0_{der} must be simple. The image of H_2 in H contains two copies of $SL(2)$. Since H^0_{der} is not $SL(2) \times SL(2)$, we get $\dim(H^0_{der}) \geq 7$ and the representation V is small. Since V_λ restricted to the subgroup $SL(2) \times SL(2)$ has 3 summands of dimension 2 each, the restriction to H^0_{der} can decompose into at most 3 summands: either V_λ stays irreducible, or decomposes in the form $W \oplus W^\vee$ or in the form $W_1 \oplus W_2 \oplus W_3$ with $\dim(W_i) = 2$. But the latter implies $W_i \cong st$ for the standard representation of $SL(2)$. This would mean $rang(H^0_{der}) \leq 6$, a contradiction. The case $W \oplus W^\vee$ cannot happen either since the restriction of $W \oplus W^\vee$ to $SL(2) \times SL(2)$ would have an even number of summands. Therefore $V_\lambda|_{H^0_{der}}$ is irreducible. Since it is selfdual irreducible of dimension 6 and carries a symplectic pairing, we conclude from lemma 8.1 or lemma 8.2 that $H^0_{der} = Sp(6)$ and V is the standard representation. But then

$$H_X \ \cong \ Sp(6) \ .$$

Similarly consider $X = \Pi(Ber^{1-b} \otimes [2b, b, 0])$ for $b > 1$. Then $X^\vee \cong X$. Then either $H \subseteq O(6)$ or $H \subseteq Sp(6)$ for $H = H_X$. The image of H_2 in H contains $SL(2)^2$. Hence $\dim(H^0_{der}) \geq 6$ and $r \geq 2$. Furthermore $H^0_{der} \ncong SL(3)$. If $r = 2$, then we get a contradiction by Mackey's lemma. Hence $r \geq 3$ and the restriction of the 6-dimensional representation $V = \omega(X)$ of H to H^0_{der} remains irreducible. By the upper bound obtained from duality therefore the semisimple rank is $r = 3$. Hence V is a small irreducible representation of H^0_{der} of dimension 6. Hence by lemma 8.1 we get $H^0_{der} = SO(V)$ resp. $Sp(V)$, since $H^0_{der} \ncong SL(3)$. In the second case then $H = Sp(6)$. In the first case it remains to determine whether $H = SO(6)$ or $H = O(6)$.

Finally the case $X = \Pi^{a+b}([a, b, 0])$ for $a > b > 0$ and $a \neq 2b$. In this case $X^\vee \ncong Ber^\nu \otimes X$ for all $\nu \in \mathbb{Z}$. The image of H_2 in $H = H_X$ contains $SL(2)^3$, hence the restriction of $V = \omega(X)$ to H^0_{der} remains again irreducible and defines a small representation of dimension 6. This now implies $H^0_{der} = SL(6)$, since $SL(2)^3 \subset H^0_{der}$ now excludes the two cases $SO(6), Sp(6)$. On the other hand we know that $det(V)$ is nontrivial on the image of H_1, and hence

$$H_X \ \cong \ GL(6) \ .$$

The structure of G_3 follws from theorem 6.8. $\square$

Example 9.7. For $\Pi[2, 1, 0]$ the associated Tannaka group is $H_X = Sp(6)$. Furthermore X corresponds to the standard representation of $Sp(6)$ and decomposes accordingly. Hence

$$X \otimes X \ = \ I_1 \oplus I_2 \oplus I_3 \qquad \mathrm{mod}\ \mathcal{N}$$

with the indecomposable representations $I_i \in \mathcal{R}_3$ corresponding to the irreducible $Sp(6)$ representations $L(2, 0, 0)$, $L(1, 1, 0)$ and $L(0, 0, 0)$. Now consider the tensor product $I_1 \otimes I_1$. For I_1 corresponding to $L(2, 0, 0)$ it decomposes as

$$I_1 \otimes I_1 \ = \ \bigoplus_{i=1}^{6} J_i \quad \mathrm{mod}\ \mathcal{N}$$

with the 6 indecomposable representations J_i corresponding to the 6 irreducible $Sp(6)$-representations in the decomposition

$$L(2,0,0)^{\otimes 2} \ = \ L(4,0,0) \oplus L(3,1,0) \oplus L(2,2,0) \oplus L(2,0,0) \oplus L(1,1,0) \oplus \mathbf{1} \ .$$

In this way we obtain the tensor product decomposition up to superdimension 0 for any summand of nonvanishing superdimension in such an iterated tensor product. Furthermore these indecomposable summands are parametrized by the irreducible representation of $Sp(6)$. Although $n = 3$ and the weight $[2,1,0]$ are small, it is hardly possible to achieve this result by a brute force calculation.

10. Tannakian induction: Proof of the structure theorem

10.1. **Restriction to the connected derived group.** Recall that H_λ denotes the Tannaka group of the tensor category generated by X_λ and $V_\lambda = \omega(X_\lambda)$ is a faithful representation of H_λ. We have inclusions

$$G_{\lambda'} \hookrightarrow G_\lambda \hookrightarrow H_\lambda^0 \hookrightarrow H_\lambda$$

where $G_{\lambda'}$ denotes the image of the natural map $(H_{n-1}^0)_{der} \to G_\lambda = (H_\lambda^0)_{der}$. Similarly we denote by $H_{\lambda'}$ the image of H_n in H_λ. The restriction of V_λ to H_{n-1} (or $H_{\lambda'}$) decomposes

$$V_\lambda \ \cong \ \bigoplus_{i=1}^k \ V_{\lambda_i}$$

where V_{λ_i} are the irreducible representations in the category $Rep(H_{n-1})$ corresponding to the irreducible constituents $L(\lambda_i), i = 1,..,k$ of $DS(L(\lambda))$. To describe $G_{\lambda'}$ we use the structure theorem for $\mathcal{T}_{n-1}^+$ (induction assumption). Therefore it suffices to group the highest weight λ_i for $i = 1,..,k$ into equivalence classes. Using the structure theorem for the category $\mathcal{T}_{n-1}^+$ and theorem 4.1, we then obtain

$$G_{\lambda'} \cong \prod_{\lambda_i/\sim} G_{\lambda_i}$$

Again using the structure theorem for G_{n-1}, each V_{λ_i} is either irreducible on G_{λ_i} or it decomposes in the form $W_i \oplus W_i^\vee$ and $G_{\lambda_i} \cong SL(W)$. The groups G_{λ_i} are independent in case (NSD). For (SD) the only dependencies between them come from the equalities $G_{\lambda_{k+1-i}} = G_{\lambda_i}$ for $i = 1,...,k$ by section A. Using these strong conditions let us consider V_λ as a representation of H_λ^0. Since an irreducible representation of H_λ^0 is an irreducible representation of its derived group G_λ, the decomposition of V_λ into irreducible representation for the restriction to H_λ^0 resp. G_λ coincide. Let

$$V_\lambda \ = \ \bigoplus_{\nu=1}^s \ W_\nu$$

denote this decomposition. We then restrict each W_ν to $G_{\lambda'}$.

$$\begin{tikzcd}
V_\lambda \arrow[dd, mapsto] & H_\lambda \arrow[dd, dash] \arrow[dr, dash] & & \\
 & & G_\lambda \arrow[dd, dash] & \bigoplus_{\nu=1}^s W_\nu \arrow[dd, mapsto] \\
\bigoplus_{i=1}^k V_{\lambda_i} & H_{\lambda'} \arrow[dr, dash] & & \\
 & & G_{\lambda'} & \bigoplus_{l=1}^t W'_l
\end{tikzcd}$$

By induction each W'_l can be seen as the standard representation or its dual of a simple group of type A, B, C, D.

10.2. **Meager representations.** If we use by induction the structure theorem for G_{n-1}, we see that the representations W_i in $V_\lambda|_{G_\lambda}$ are meager in the sense below. We analyze in this section the implications of W_i to be meager.

Definition 10.1. A finite dimensional representation V of a reductive group H will be called small if $\dim(V) < \dim(H)$ holds.

Definition 10.2. A representation V of a semisimple connected group G will be called meager, if every irreducible constituent W of V factorizes over a simple quotient group of G and is isomorphic to the standard representation of this simple quotient group or isomorphic the dual of the standard representation for a simple quotient group of Dynkin type A, B, C, D.

If a representation V of H is small resp. meager, any subrepresentation of V is small resp. meager.

Suppose G' is a semisimple connected simply connected group and V is a faithful meager representation of G. Each irreducible constituent of V then factorizes over one of the projections $p_\mu : G' \to G'_\mu$. We then say that the corresponding constituent is of type μ.

Lemma 10.3. *Suppose V is an irreducible faithful representation of the semisimple connected group G of dimension ≥ 2. Suppose G' is a connected semisimple group and $\varphi : G' \to G$ is a homomorphism with finite kernel such that*

(1) The restriction $\varphi^*(V)$ of V to G' is meager and for fixed μ every (nontrivial) irreducible constituents of type μ in the restriction of V to G' has multiplicity at most 2.

(2) If an irreducible constituent W' occurs with multiplicity 2 for a type μ in $V|_{G'}$ (such a μ is called an exceptional type), then either

  (i) W' is the standard representation of $G_\mu \cong SL(2)$, or

  (ii) there is a unique type $\mu = \mu_2$ such that W' is the direct sum of the standard representation and its dual $W' = W \oplus W^\vee$ as a representation of the quotient $SL(W)$ of G' or

(iii) there is a unique type $\mu = \mu_0$ with $G'_\mu \cong Sp(W')$ or $(G'_\mu)_{sc} = Spin(W)$ such that the standard representation st of G'_μ occurs twice.

(3) No irreducible constituent of the restriction of $V|_{G'}$ is a trivial representation of G'.

(4) The semisimple group G' has at most one simple factor isomorphic to $SL(2)$. The index, if it occurs, will be denoted μ_1.

Under these assumptions G is a simple group or G' is a product of exceptional types.

Proof. We may replace G and G' by their simply connected coverings without changing our assumptions, so that we can assume that G and $G' = \prod_\mu G'_\mu$ decompose into a product of simple groups. Then V is not faithful any more, but has finite kernel. The restriction of the meager representation V to G' decomposes into the sum $\bigoplus_\mu J_\mu$ of representations J_μ such that J_μ is trivial on $\prod_{\lambda \neq \mu} G'_\lambda$

$$V|_{G'} = \bigoplus_\mu J_\mu \ ,$$

hence J_μ can be considered as a representation of the factor G'_μ of G'. Furthermore J_μ is either an irreducible representation of G'_μ, or the direct sum $J_\mu \cong W \oplus W^\vee$ (as a representation of $G'_\mu \cong Sl(W)$) by the assumption 1) and 2) or there exists a unique type μ of Dynkin type B, C, D where $J_\mu = st \oplus st$ for the standard representation st of this group G'_μ.

If G is not simple, $G = G_1 \times G_2$ is a product of groups and the irreducible representation V is a external tensor product

$$V = V_1 \boxtimes V_2$$

of irreducible representations V_1, V_2 of G_1 resp. G_2. Since V has finite kernel, $\dim(V_i) > 1$ holds. For each factor $G'_\mu \hookrightarrow G' = \prod_\mu G'_\mu$ consider the composed map

$$G'_\mu \to G_1 \times G_2 \ .$$

This map is either trivial, or defines an isogeny.

We claim that there exists at least one index μ such that both compositions $G'_\mu \to G_i$ with the projections $G \to G_i$ $(i = 1, 2)$ are nontrivial except when G' has only exceptional types. To prove the claim, suppose $G'_\mu \to G_2$ would be the trivial map. Then the restriction of V to $G'_\mu \subseteq G'$ is $V|G'_\mu = \dim(V_2) \cdot V_1|_{G'_\mu}$. Hence $\dim(V_2) \leq 2$, since otherwise we get a contradiction to assumption 1. Indeed, $V_1|_{G'_\mu}$ also contains at least one nontrivial irreducible constituent by assumption 3), and this constituent can occur at most with multiplicity two in $V|_{G'}$. Hence $\dim(V_2) \leq 2$. If $\dim(V_2) = 2$, then there exists a nontrivial irreducible constituent $I_\mu \subseteq V_1|_{G'_\mu}$ of G'_μ by assumption 3). Hence $V|_{G'_\mu}$ contains $I_\mu \oplus I_\mu$ both of type μ and we are in an exceptional type.

We assume now that $\{\mu\}$ is not an exceptional type. We may therefore choose μ so that both $G'_\mu \to G_i$ are nontrivial. Then

$$V|_{G'_\mu} = V_1|_{G'_\mu} \otimes V_2|_{G'_\mu}$$

is the tensor product of two nontrivial representations $V_1|_{G'_\mu}$ and $V_2|_{G'_\mu}$ of G'_μ. Since $V|_{G'}$ is a meager representation of G', all irreducible constituents of the restriction of $V|_{G'}$ to G'_μ are trivial representations of G'_μ except for at most two of them, which are standard representations up to duality. Since V_i are irreducible representations of G and V has finite kernel, the restriction of V to G'_ν has finite kernel. Hence both of the representations $V_i|_{G'_\mu}$ have finite kernel, hence contain an irreducible nontrivial representation of G'_μ. Otherwise the restriction $V|_{G'_\mu}$ would be trivial contradicting that $G'_\mu \to G_i$ is an isogeny for both $i = 1, 2$ and V_i both have finite kernel on G_i. For every nontrivial irreducible representations $I_1 \subseteq V_1|_{G'_\mu}$ and $I_2 \subseteq V_2|_{G'_\mu}$ of G'_μ the representation

$$I_1 \otimes I_2$$

only contains trivial representations and standard representations st up to duality by assumption 2). Since the trivial representation occurs at most once in the tensor product of two irreducible representations, this implies $I_1 \otimes I_2 \subseteq J_\mu \oplus 1 \subseteq st \oplus st^\vee \oplus 1$. Hence $\dim(I_1)\dim(I_2) \leq 1 + 2\cdot\dim(st) < 1 + 2\cdot\dim(st) + \dim(st)^2$. Hence $\min(\dim(I_\nu)) < 1 + \dim(st)$. In particular, the corresponding representation with minimal dimension, say I_1, has dimension $\leq \dim(st)$ and hence I_1 is a small representation of G'_μ. Since it satisfies $\dim(I_1) \leq \dim(st)$, it belongs to the list of lemma 8.2. Therefore I_1 is the standard representation of G'_μ or its dual, unless the group G'_μ is of Dynkin type D_4 and I_1 is a spin representation. In the first case, considering highest weights it is clear that $st \otimes I_2 \subseteq st \oplus st^\vee \oplus 1$ is impossible. In the remaining orthogonal case G'_μ of Dynkin type D_4, the representation $I_1 \otimes I_2$ must have dimension $\geq 8^2$. But this contradicts $\dim(I_1)\dim(I_2) \leq 1 + 2\cdot\dim(st) = 1 + 8 + 8 = 17$, and finally proves our assertion. ☐

Corollary 10.4. *In the situation of lemma 10.3, the restriction of the representation V to the group G' is multiplicity free unless G' contains an exceptional type (in which case the irreducible constituent has multiplicity 2). If G' has at least one non-exceptional type, then the restriction contains at least one constituent with multiplicity 1.*

Proof. If the restriction of V to G' contains an irreducible summand I of G' with multiplicity ≥ 2, then the restriction of I at least under one map $G'_\mu \to G$ contains a nontrivial constituent of G'_μ with multiplicity > 1. Hence the restriction of I contains J_μ by the assumption 1) and 2) of the main lemma such that $J_\mu \cong I_\mu \oplus I_\mu$ and we are in an exceptional type. ☐

Definition 10.5. Let G, G' be semisimple connected groups and $\varphi : G' \to G$ a homomorphism with finite kernel. The restriction of the irreducible representation V of G to G' is called *weakly multiplicity-free* if at least one irreducible constituent has multiplicity 1.

10.3. **Mackey-Clifford theory.** Let H be a reductive group and H^0 its connected component. We assume that G is the connected derived group of H^0. Let V be a finite dimensional irreducible faithful representation of H and let

$$V|_{H^0} = W_1 \oplus \cdots \oplus W_s$$

be the decomposition of V into irreducible summands (W_ν, ρ_ν) after restriction to H^0. The restriction of each W_ν to G remains irreducible (this follows from Schur's lemma and the fact that the image of H^0 in $GL(W_\nu)$ is generated by the image of G in $GL(W_\nu)$ and the image of the connected component of the center of H^0, whose image is in the center of $GL(W)$). By Clifford theory [Cl37] $\pi_0(H) = H/H^0$ acts on these subspaces W_ν for $\nu = 1, .., s$ permuting them transitively; i.e. $\rho_\nu(g) = \rho_1(hgh^{-1})$ for certain $h \in H$. We define the isotypic part of an irreducible W_ν to be the sum of all subrepresentations of $V|_{H^0}$ which are isomorphic to W_ν. Since $\pi_0(H)$ acts transitively on the W_ν, the multiplicity of each isotypic part is the same.

Representations (W_ν, ρ_ν) from different isotypic parts are pairwise non-isomorphic representations of H^0 (in our application later this also remains true for the restriction to G by the G'-multiplicity arguments). But $\rho_1(h_1 g h_1^{-1}) \cong \rho_1(h_2 g h_2^{-1})$ as representations of $g \in H^0$ (or $g \in G$) holds if $h_1^{-1} h_2 \in H^0$ (resp. $h = h_1^{-1} h_2 \in H^0$). Therefore the automorphism $int_h : H^0 \to H^0$ acts trivially and the W_ν are permuted transitively by $Out(H^0) = Aut(H^0)/Inn(H^0)$. If a finite group acts transitvely on a set X, this implies that the cardinality of the set divides the order of the group. Therefore

$$s \leq |Out(H^0)|.$$

If $H = H_\lambda$ is the Tannaka group of an irreducible maximal atypical module $L(\lambda) \in \mathcal{T}_n^+$ and $V = V_\lambda = \omega(L(\lambda))$ is the associated irreducible representation of H and $W_1, ..., W_s$ are the irreducible constituents of the restriction of V to H^0, then the following holds

Theorem 10.6. *Suppose that $L(\lambda)$ is not a Berezin twist of S^i for some i or its dual, and suppose $n \geq 4$. Then for $G = H_\lambda^0$ and $G' = G_{\lambda'}$ the irreducible representations $W_1, ..., W_s$ of G satisfy the conditions of lemma 10.3 and G' has at least one non-exceptional type μ. In particular G is a connected simple algebraic group and V is a weakly multiplicity free representation of H^0.*

Proof. The irreducibility and faithfulness is a tannakian consequence of the definitions. Condition 1) and 2) follow from induction on n and the classification of similar and selfdual derivatives λ_i of λ in section A. Condition 3) is seen as follows: The trivial representation of G' is attached to a derivative λ_μ of λ only if $L(\lambda)$ isomorphic to $S^i \otimes Ber^j$ for some $i \geq 1$ and some $j \in \mathbb{Z}$

by lemma C.3. Concerning condition 4): A factor G'_μ of G' of rank 1 (i.e. with derived group $SL(2)$) is attached to some derivative λ_μ of λ only if $L(\lambda) = S^1$ or λ has only two sectors, one sector S of rank 1 and the other sector S' corresponds to S^1 on the level $n-1$. In other words $\partial SS'$ resp. $S'\partial S$ gives S^1 and the corresponding group $SL(2)$, but not the other derivative unless $n \leq 3$. Hence by our assumptions, the group G' has at most one simple factor $SL(2)$. If an irreducible constituent of the restriction of V to G' has multiplicity 2, it comes from a derivative of type (SD). Hence if all types of G' are exceptional, all derivatives of $L(\lambda)$ would have to be selfdual. This can only happen for $n \leq 3$ by the analysis in section A. Hence lemma 10.3 and corollary 10.4 imply the last statement. $\square$

Theorem 10.7. *The simple group G is of type A, B, C, D and $W_1|_G$ is either the standard representation of G or its dual.*

Proof. We suppose that $L(\lambda)$ is not a Berezin twist of S^i for some i and suppose $n \geq 4$. We distinguish the cases NSD and SD. In the NSD-case we claim that we have

$$r(G_\lambda) \geq (\dim(V_\lambda) - 1)/2$$

and that for $n \geq 4$ and $\dim(V_\lambda) \geq 4$

$$\dim(G_\lambda) > \dim(V_\lambda)$$

holds (note that $\dim(V_\lambda) \leq 3$ for $n \geq 4$ implies $k = 1$ and $\dim(V_\lambda) = \dim(V_{\lambda_1})$). For all $i = 1, .., k$ the superdimension formula of [Wei10][HW14, Section 16] implies by lemma C.5 that

$$\dim(V_\lambda) \leq n \cdot \dim(V_{\lambda_i})/r_i$$

where $r_i = r(V_{\lambda_i}) \geq 1$ is the rank of λ_i. Obviously $\dim(G_{\lambda_i}) \leq \dim(G_\lambda)$.

Since we excluded the S^i-case, no V_{λ_i} has dimension 1 by lemma C.3. At most one of the representations V_{λ_i} is selfdual by lemma A.6. We make a case distinction on whether there exists one V_{λ_i} that splits in the form $W'_i \oplus (W'_i)^\vee$ upon restriction to $G_{\lambda'}$ or not. In the latter case we know $r(G_{\lambda_i}) \geq \frac{1}{2}\dim(V_{\lambda_i})$ by theorem 6.2 and the induction assumption. Now by proposition A.2 and the assumption (NSD) all λ_i in the derivative of λ are inequivalent for $i \neq j$. Hence we get

$$r(G_\lambda) \geq \sum_i r(G_{\lambda_i}) \geq \sum_i \frac{1}{2}\dim(V_{\lambda_i}) \geq \frac{1}{2}(\dim(V_\lambda)) \ .$$

Since $\dim(G_{\lambda_i}) \geq 3r(G_{\lambda_i})$, this implies $\dim(G_\lambda) \geq \frac{3}{2}(\dim(V_\lambda) - 1)$ and hence $\dim(G_\lambda) > \dim(V_\lambda)$ (note that we have at least one SL factor G_{λ_i} for which $r(G_{\lambda_i}) > \frac{1}{2}\dim(V_{\lambda_i})$). If V_λ splits $V_\lambda = W_1 \oplus \ldots \oplus W_s$ we may replace V_λ by any W_ν for an even better estimate. Therefore lemma 8.2 implies that V_λ (or W_ν) is the standard representation or its dual of a simple group of type A, B, C, D. If V_λ stays irreducible, then we obtain $G_\lambda \cong SL(V_\lambda)$ since V_λ is not self-dual.

If V_{λ_i} splits, $G_{\lambda_i} \cong SL(W_i)$ for $V_\lambda \cong W_i \oplus W_i^\vee$ by induction assumption. If the dimension of V_{λ_i} is $2d_i$, we then have $r(G_{\lambda_i}) = d_i - 1$ and therefore have to replace the estimate $r(G_{\lambda_i}) \geq \frac{1}{2}\dim(V_{\lambda_i})$ by the estimate $r(G_{\lambda_i}) \geq \frac{1}{2}(\dim(V_{\lambda_i} - 2))$. Since V_{λ_i} can only decompose if it is of type SD, $L(\lambda)$ has more than one sector. All the other $k-1 \geq 1$ derivatives $L(\lambda_i)$ are of type NSD and define inequivalent $SL(V_{\lambda_i})$. For each of these we obtain $r(G_{\lambda_j}) = \dim V_{\lambda_j} - 1$. Summing up we obtain

$$r(G_\lambda) \geq \sum_i r(G_{\lambda_i}) \geq \frac{1}{2}(\dim(V_{\lambda_i}) - 2) + \sum_{j \neq i} \dim(V_{\lambda_j}) - 1 \ .$$

This implies again the necessary estimates to apply lemma 8.2.

We now consider the SD-case. If V_λ decomposes

$$V_\lambda|_{G_\lambda} \cong W_1 \oplus \ldots \oplus W_s$$

then we can assume by reindexing that $\dim(W_1) \leq \frac{1}{s}\dim(V_\lambda)$. Note that $\dim(W_1) > 1$ follows from the induction assumption.

In the SD case we proceed as follows: We first show that V_λ or W_1 is small. Since we cannot prove the strong rank estimates for $r(G_\lambda)$ as in the NSD case, we work through the list of exceptional cases in lemma 8.1.

The list of superdimensions in the n=4 and n=5 case in sections 13 C.2 along with the induction assumption shows in these cases that V_λ is small. Therefore we can assume $n \geq 5$. We use the known formulas $\dim(SL(n)) = n^2 - 1$, $\dim SO(n) = \frac{n(n-1)}{2}$ and $\dim(Sp(2n)) = n(2n+1)$.

We recall from the analysis in lemma A.6 that $L(\lambda)$ can only have more than one selfdual derivative if it is completely unnested, i.e. it has n sectors of cardinality 2. In this case it has 2 selfdual derivatives coming from the left and rightmost sectors and, if n is odd, another derivative coming from the middle sectors. If λ is not of this form, then the unique weakly selfdual derivative comes from the middle sector (of arbitrary rank).

We want to show $\dim(G_\lambda) \leq \dim(V_\lambda)$. By induction G_{λ_i} is either $SO(V_{\lambda_i})$, $Sp(V_{\lambda_i})$, $SL(V_{\lambda_i})$ or $SL(W_i)$ for $V_{\lambda_i} = W_i' \oplus (W_i')^\vee$. We estimate the dimension of G_{λ_i} via $\sum \dim(G_{\lambda_i})$. We claim that we can assume that we have more than one sector because otherwise $\dim(V_\lambda) = \dim(V_{\lambda_1})$ implies that V_λ is small using the induction assumption. If V_{λ_1} is an irreducible representation of $G_{\lambda'}$ the claim is clear by induction assumption. If it splits $V_{\lambda_1} = W_1' \oplus (W_1')^\vee$, then $\dim(V_{\lambda_1}) < \dim(SL(W_1'))$ provided $\mathrm{sdim}(L(\lambda_1)) \geq 3$. Now $\mathrm{sdim}(L(\lambda_1)) = 2$ can only happen for $L(\lambda_i) \cong Ber^{\cdots} \otimes S^1$ (and then V_{λ_1} is an irreducible representation of $G_{\lambda'}$). We therefore assume $k > 1$. The worst estimate for the dimension is obtained if all V_{λ_i} split as $W_i' \oplus (W_i')^\vee$ and therefore $G_{\lambda_i} \cong SL(W_i)$. This case can only happen if either $n = 2$ or $n = 3$. For $n \geq 4$ the lowest estimate for the dimension of G_λ occurs if λ is completely unnested with 2 selfdual derivatives coming from the left and right sector and we have $n/2$ equivalence classes of derivatives (or

$\lfloor n/2 \rfloor + 1$ for odd n). The left and right sector then contribute a single $SL(W_1') = SL(W_k')$ and if n is even for all other derivatives $G_{\lambda_i} \cong SL(V_{\lambda_i})$ with $V_{\lambda_i} \sim V_{\lambda_{k-i}}$ and therefore same connected derived Tannaka group. If $n = 2l+1$ is odd the middle sector can contribute another derivative of type SD with Tannaka group $SL(W_{l+1}')$. The dimension estimate works as in the case above and we therefore ignore this case.

We show now that $\dim(G_\lambda) > \dim(V_\lambda)$ provided we have two SD derivatives coming from the left- and rightmost sector. Denote by d_i the dimension of V_{λ_i}. For $i = 1, k$ it is even $d_1 = 2d_1' = 2d_k'$ by lemma C.4. We then obtain for the dimension of $G_{\lambda'}$

$$\dim(G_{\lambda'}) = \frac{1}{2}((d_1')^2 - 1 + (d_k')^2 - 1) + \frac{1}{2} \sum_{j \neq 1,k} d_j^2 - 1.$$

It is enough to show $2 \dim V_{\lambda_i} < \dim G_{\lambda_i}$ for each i. The smallest possible superdimensions for a selfdual irreducible representation are $2, 4, 12, \ldots$. The $\dim = 2$ case can only happen for $L(\lambda_i) \cong Ber^{\cdots} \otimes S^1$ which is not possible by assumption. Hence $d_1' \geq 3$. This case occurs for $[2,1,0]$ for $n = 3$, $[2,2,0,0]$ for $n = 4$ and all their counterparts for larger n by appending zeros to the weight (e.g. $[2,1,0,0]$). These are not derivatives of a selfdual representation $L(\lambda)$ unless $L(\lambda)$ has one sector (which we excluded). Therefore we can assume $d_1' \geq 6$. Then

$$2 \ \dim(V_{\lambda_1}) = 4d_1' < (d_1')^2 - 1 = \dim(G_{\lambda_1}).$$

For the NSD derivatives we can exclude the case $d_i = 2$ since this only happens for $L(\lambda_i) \cong Ber^{\cdots} \otimes S^1$. For $d_i \geq 3$ we obtain $2d_i < d_i^2 - 1$, hence again $2\dim(V_{\lambda_i}) < \dim(G_{\lambda_i})$. Clearly this estimates also hold if we have more than $n/2$ equivalence classes of weights or if we have $SO(V_{\lambda_i})$ or $Sp(V_{\lambda_i})$ in case of $SL(W_i)$.

Hence $\dim(V_\lambda) < \dim(G_\lambda)$. If V_λ is an irreducible representation of G_λ, it is a small representation of G_λ and lemma 8.1 applies. If it decomposes $V_\lambda \cong W_1 \oplus \ldots \oplus W_s$, then each W_ν is an irreducible small representation of G_λ.

Assume first that $V_\lambda \cong W_1 \oplus \ldots \oplus W_s$ with $s \geq 3$ and $\dim(W_1) \leq \frac{1}{s}\dim(V_\lambda)$. Again the smallest rank estimate for the subgroup $G_{\lambda'}$ occurs for $n \geq 4$ if λ is completely unnested with 2 selfdual derivatives coming from the left and right sector and we have $n/2$ equivalence classes of derivatives (we assume here n even. In the odd case we can have another derivative from the middle sector. The estimate below still holds). Then

$$\begin{aligned} r(G_\lambda) \geq r(G_{\lambda'}) &\geq \frac{1}{2}(d_1/2 - 1 + d_k/2 - 1 + \sum_{j \neq 1,k} d_j - 1) \\ &= \frac{1}{2}(\dim(V_\lambda) - k - d_1/2 - d_k/2). \end{aligned}$$

In the completely unnested case this equals

$$\frac{1}{2}(n! - n - (n-1)!).$$

We need $r(G_\lambda) \geq \frac{1}{2}(\dim(V_\lambda) - 1)$ to apply lemma 8.2. We replace now V_λ by W_1 with $\dim(W_1) \leq 1/s \ \dim(V_\lambda)$. For $n \geq 4$ and $s \geq 2$ we obtain $n!/s - 1 \leq n! - n - (n-1)!$, hence lemma 8.2 can be applied to the irreducible representation W_1.

If λ is not completely unnested, it can have at most one SD derivative coming from the middle sector for $k = 2l+1$ odd. Then we obtain

$$\begin{aligned} r(G_\lambda) \geq r(G_{\lambda'}) &\geq \frac{1}{2}(d_{l+1}/2 - 1 + \sum_{j \neq l+1} d_j - 1) \\ &= \frac{1}{2}(\dim(V_\lambda) - k - d_{l+1}/2). \end{aligned}$$

As above we replace V_λ with W_1 with $\dim(W_1) \leq \frac{1}{s} V_\lambda$ and show $\dim(V_\lambda/s - 1) \leq \dim(V_\lambda) - k - d_{l+1}/2$. For $s = 2$ this is equivalent to $\dim(V_\lambda) \geq d_{l+1} + 2(k-1)$. This follows easily from $\dim(V_\lambda) = \dim(V_{\lambda_{l+1}}) \frac{n}{r_{l+1}}$ (lemma C.5). For $s > 2$ the estimates are even stronger. The cases where the SD derivative occurs and contributes $SO(V_{\lambda_{l+1}})$ or $Sp(V_{\lambda_{l+1}})$, or the case in which no SD derivative occurs, can be treated the same way.

We can therefore assume that either a) V_λ is an irreducible representation of G_λ or it splits in the form $V_\lambda = W \oplus W^\vee$. The analysis of small superdimensions in section C.2 shows that the possible superdimensions of weakly selfdual irreducible representations less than 129 are

$$1,\ 2,\ 6,\ 12,\ 20,\ 24,\ 30,\ 42,\ 56,\ 70,\ 72,\ 80,\ 90,\ 110,\ 112.$$

Except for the numbers 20 and 56 none of the exceptional dimensions in lemma 8.1 is equal to either the superdimension or half the superdimension of an irreducible weakly selfdual representation in $\mathcal{T}_n^+$. It is easy to exclude these two cases (see section C.2) since in this case V_λ or W would be either a symmetric or alternating square of a standard representation (which would give a contradiction to the induction assumption) or the irreducible representation of minimal dimension of E_7 which is impossible by rank estimates. $\square$

Theorem 10.8. *Either the restriction of V_λ to H^0 and G_λ is irreducible, or $G \cong SL(W)$ and $V|_G \cong W \oplus W^\vee$ for a vectorspace W of dimension ≥ 3. If $V|_G \cong W \oplus W^\vee$, then*

$$V_\lambda \cong Ind_{H_1}^H W$$

for a subgroup H_1 of index 2 between H^0 and H. In particular V_λ is an irreducible representation of G_λ if $L(\lambda)$ is not weakly selfdual.

Proof. As in the statement of theorem 10.6 we can assume that $n \geq 4$ and that $L(\lambda)$ is not a Berezin twist of S^i (or its dual) since these cases were already treated in section 9.

We claim that the representation $V|_{H^0} = W_1 \oplus \ldots \oplus W_s$ is multiplicity free. Since the restriction of V to G' is weakly multiplicity free, at least one irreducible constituent occurs only with multiplicity 1 for some (non-exceptional) μ. By Clifford theory the multiplicity of each isotypic part in the restriction of V to H^0 is the same (since π_0 acts transitively). If the multiplicity of each isotypic part would be bigger than 1, the restriction of V to G' could not be weakly multiplicity free. Therefore the multiplicity of each isotypic part is 1. Any W_ν restricted to G_λ is irreducible (restriction to the derived group). Since G_λ is a normal subgroup of H, H still operates transitively on the set $\{W_\nu|_{G_\lambda}\}$. Fix any $W_\nu|_{G_\lambda}$. Its H-orbit has s' elements where s' divides s and s/s' is the multiplicity of each $W_\nu|_{G_\lambda}$ in $V_\lambda|_{G_\lambda}$. Hence the argument from Clifford theory explained preceding theorem 10.6 shows

$$s' \leq |Out(G)| \ .$$

But a nontrivial outer automorphism of G that does not fix the isomorphism class of the standard representation W_1 of G exists only for the groups G of the Dynkin type A_r for $r \geq 2$. For the special linear groups $G = SL(\mathbb{C}^{r+1})$ the nontrivial representative in $Out(G)$ it is given by $g \mapsto g^{-t}$. The twist of the standard representation by this automorphism gives the isomorphism class of the dual standard representation $W_1^\vee$. This implies $s' = 1$ or $s' = 2$. If $s' = 2$, then $V_\lambda|_{G_\lambda} \cong W \oplus W^\vee$ where W is the standard representation of SL and $G_\lambda \cong SL(W)$. Since $V_\lambda|_{G_{\lambda'}}$ is weakly multiplicity free and $G_{\lambda'} \subset G_\lambda$, $V_\lambda|_{G_\lambda}$ is weakly multiplicity free as well. Accordingly $s/s' = 1$ and we also obtain $s = 1$ or 2. If $s = 2$, Clifford theory further implies that

$$V_\lambda \cong Ind_{H_1}^H W$$

for a subgroup H_1 of index 2 between H^0 and H. $\square$

Remark 10.9. Since $W \oplus W^\vee$ is selfdual, this implies in particular that V_λ can only decompose if $L(\lambda)$ is weakly selfdual. If V_λ decomposes, its restriction to $G_{\lambda'}$ is of the form $\bigoplus_i W_i \oplus W_i^\vee$. This leads to some restrictions on SD weights λ such that V_λ decomposes in the form $W \oplus W^\vee$. Consider for an instance the weakly selfdual weight $[n-1, n-2, \ldots, 1, 0]$ for odd $n = 2l+1$. Then V_λ can only decompose if the irreducible representation $V_{\lambda_{l+1}}$ associated to the middle derivative $L(\lambda_{l+1})$ decomposes upon restriction to $G_{\lambda'}$ in the form $W'_{l+1} \oplus (W'_{l+1})^\vee$.

11. A conjectural structure theorem

According to the structure theorem 6.2 we need to consider the case where $V_\lambda|_{G_\lambda} \cong W \oplus W^\vee$. We conjecture that this case does not happen. If this would be true, the following stronger variant of the structure theorem would hold.

Conjecture 11.1. *$G_\lambda = SL(V_\lambda)$ resp. $G_\lambda = SO(V_\lambda)$ resp. $G_\lambda = Sp(V_\lambda)$ according to whether X_λ satisfies (NSD) respectively (SD) with either X_λ being even respectively odd. The dimension of V_λ is even unless V_λ has dimension 1.*

11.1. **Applications.** The conjectural structure theorem would have the following consequences.

Corollary 11.2. *For given $L = L(\lambda)$ in $\mathcal{R}_n$ and $r \in \mathbb{Z}$ there can exist at most one summand M in $L \otimes (Ber^r \otimes L^\vee)$ with the property* $\mathrm{sdim}(M) = \pm 1$. *If it exists then $M \cong Ber^r$.*

Proof of the corollary. We can assume that L is maximal atypical. Then $\mathbf{1}$ is a direct summand of $L \otimes L^\vee$ and hence Ber^r is a direct summand of $L \otimes (Ber^r \otimes L^\vee)$. Hence it suffices to show that $\mathbf{1}$ is the unique summand M of $L \otimes L^\vee$ with $\mathrm{sdim}(M) = \pm 1$. Equivently it suffices to show that $V_\lambda \otimes V_\lambda^\vee$ contains no one-dimensional summand except $\mathbf{1}$. This now follows from conjecture 11.1 using the well known fact that $st \otimes st^\vee$ for the standard representation st of $SL(V), SO(V), Sp(V)$ contains only one summand of dimension 1. $\square$

Since the groups H_λ always satisfy $H_\lambda \subseteq GL(V_\lambda)$ resp. $H_\lambda \subseteq GO(V_\lambda)$ resp. $H_\lambda \subseteq GSp(V_\lambda)$ according to whether X_λ satisfies (NSD) respectively (SD) with either X_λ even respectively odd, another immediate consequence of conjecture 11.1 is

Proposition 11.3. *The groups H_λ/G_λ are abelian.*

11.2. **A criterion for irreducibility.** We analyze the consequences of $V_\lambda|_{G_\lambda} \cong W \oplus W^\vee$ further and show that this can happen only if there exists special indecomposable modules $I \neq \mathbf{1}$ of superdimension 1 in the tensor product $L(\lambda) \otimes L(\lambda)^\vee$.

If

$$V_\lambda|_{G_\lambda} \cong W \oplus W^\vee$$

decomposes,$G_\lambda = SL(W)$ is a maximal proper semisimple subgroup in $Sp(2m)$ resp. $SO(2m)$ depending on the parity of the underlaying pairing. This implies that H_λ is contained in the normalizer of G_λ in $GSp(2m)$ resp. $GO(2m)$. So H_λ is contained in

$$0 \to GL(W) \to G \to \mathbb{Z}/2\mathbb{Z} \to 0$$

and H_λ itself contains

$$0 \to SL(W) \to G \to \mathbb{Z}/2\mathbb{Z} \to 0$$

as a subgroup. The irreducible representation of H_λ on $V_\lambda = W \oplus W^\vee$ becomes $st \oplus (det^r \otimes st^\vee)$, when restricted to the subgroup $GL(W)$. Then it is easy to see that both

$$Sym^2(V_\lambda) \quad \text{and} \quad \Lambda^2(V_\lambda)$$

contain a unique one dimensional representation of H_λ. These one dimensional representations define two nondegenerate H_λ-equivariant pairings

$$V_\lambda \otimes V_\lambda \longrightarrow det^r$$

$$V_\lambda \otimes V_\lambda \longrightarrow \varepsilon \otimes det^r$$

for the nontrivial character $\varepsilon : H_\lambda \to \pi_0(H_\lambda) \cong \mathbb{Z}/2\mathbb{Z}$. One of these pairings has to be symmetric and the other one has to be skew symmetric. These H_λ-modules det^r and $\varepsilon \otimes det^r$ correspond to nonisomorphic indecomposable objects I of $\mathcal{T}_n^+$ (represented by ε maybe up to a parity shift) such that for I the following holds.

Lemma 11.4. Properties of I. *The module I has the following properties:*

1. *I is indecomposable in $\mathcal{T}_n^+$ with $\mathrm{sdim}(I) = 1$.*
2. *There exists an irreducible object L of $\mathcal{T}_n^+$. such that I occurs (with multiplicity one) as a direct summand in $L \otimes L^\vee$.*
3. *$L \otimes I \cong L \oplus N$ for some negligible object N.*
4. *$I^\vee \cong I$.*
5. *$I^* \cong I$.*
6. *$DS(I)$ is $\tilde{I} \oplus$ negligible for an indecomposable object $\tilde{I}$ concentrated in degree 0 of superdimension 0 satisfying $\tilde{I}^\vee = \tilde{I}$ and $\tilde{I}^* \cong \tilde{I}$. If we assume the stronger structure theorem for $n-1$ by induction, $DS(I)$ is $\mathbf{1}$ plus some negligible object.*

Proof. 1) is obvious. For 2) notice that for $L = L(\lambda)$ we have $L^\vee \cong Ber^{-r} \otimes L$ so that we get a nondegenerate pairing $L \otimes L^\vee \to I$. By the representation theory of the semidirect product G from above any one dimensional representation in $L \otimes L^\vee$ must have multiplicity one. In fact there occur exactly two nonisomorphic one dimensional summands, namely those corresponding to the pairings. This fact implies property 3) and 4). Indeed I is one of the two one dimensional retracts of $L \otimes L^\vee$. Since $(L \otimes L^\vee)^\vee \cong L \otimes L^\vee$, this implies $I^\vee \cong I$, and similarly $(L \otimes L^\vee)^* \cong L \otimes L^\vee$ implies $I^* \cong I$. Property 6) follows since I is selfdual of superdimension 1, and so its cohomology is concentrated in degree 0. By definition I is a retract of $L \otimes L^\vee$, so $DS(I)$ is a retract of $DS(L) \otimes DS(L)^\vee$. If we assume that the stronger structure theorem holds for $n-1$, the only summands of superdimension 1 in a tensor product $L(\lambda_i) \otimes L(\lambda_i)^\vee$ are Berezin powers by corollary 11.2, hence $DS(I) \cong \mathbf{1} \oplus N$. ∎

Conjecture 11.5. $I \simeq \mathbf{1}$.

We are unable to prove this result at the moment. For some special cases see the appendices D E. This conjecture immediately implies that V_λ stays irreducible under restriction to G_λ and therefore would prove the stronger version of the structure theorem.

12. The Picard group of $\overline{\mathcal{T}}_n$

We analyze the invertible elements in $Rep(H_n)$, i.e. $Pic(H_n)$, or in down-to-eart terms the character group of H_n.

12.1. **Invertible elements.** For a rigid symmetric k-linear tensor category $\mathcal{C}$ an object I of $\mathcal{C}$ is called invertible if $I \otimes I^\vee \cong \mathbf{1}$ holds. The tensor product of two invertible objects of $\mathcal{C}$ is an invertible object of $\mathcal{C}$. Let $Pic(\mathcal{C})$ denote the set of isomorphism classes of invertible objects of $\mathcal{C}$. The tensor product canonically turns $(Pic(\mathcal{C}, \otimes)$ into an abelian group with unit object $\mathbf{1}$, the Picard group of $\mathcal{C}$.

Suppose that the categorial dimension dim is an integer ≥ 0 for all indecomposable objects of $\mathcal{C}$. The objects of categorial dimension 0 define a thick tensor ideal of $\mathcal{C}$. An indecomposable object I of $\mathcal{C}$ is an invertible object in $\overline{\mathcal{C}} = \mathcal{C}/\mathcal{N}$ if and only if $\mathrm{sdim}(I) = 1$ holds. In fact $\dim(I) = 1$ implies $\dim(I^\vee) = 1$ and hence $\dim(I \otimes I^\vee) = 1$. Hence $I \otimes I^\vee \cong \mathbf{1} \oplus N$ for some negligible object N. Note that the evaluation morphisms $eval : I \otimes I^\vee \to \mathbf{1}$ splits since $\dim(I) \neq 0$.

12.2. $Pic(\overline{\mathcal{T}}_n)$ **and the determinant.** We are interested in the Picard group of the tensor category $\overline{\mathcal{T}}_n = \mathcal{T}_n^+/\mathcal{N}$. Since $\overline{\mathcal{T}}_n \sim Rep_k(H_n)$, to determine the Picard group $Pic(\overline{\mathcal{T}}_n)$ is tantamount to determine the character group of H_n. Hence $H_n^{ab} = H_n/G_n$ is determined by $Pic(\overline{\mathcal{T}}_n)$. The elements of $Pic(\overline{\mathcal{T}}_n)$ are represented by indecomposable objects $I \in \mathcal{T}_n^+$ with the property

$$I \otimes I^\vee \cong \mathbf{1} \oplus \text{ negligible } .$$

The category $\overline{\mathcal{T}}_n$ is generated by the images of the objects X_λ, where λ is a maximal atypical weight and X_λ is the irreducible module of highest weight λ in $\mathcal{T}_n^+$. Recall $\mathrm{sdim}(X_\lambda) \geq 0$. So we can define $det(X_\lambda) = \Lambda^{\mathrm{sdim}(X_\lambda)}(X_\lambda)$. Notice

$$\det(X_\lambda) \ = \ I_\lambda \ \oplus \ \text{negligible}$$

is the sum of a unique indecomposable module I_λ in $\mathcal{T}_n^+$ and a direct sum of negligible indecomposable modules in $\mathcal{T}_n^+$. Furthermore $I_\lambda^* \cong I_\lambda$ and $\mathrm{sdim}(I_\lambda) = 1$ holds, and if X_λ is selfdual, then I_λ is selfdual. In particular, $det(X_\lambda)$ in $\mathcal{T}_n^+$ has superdimension one, hence its image defines an invertible object of the representation category $\overline{\mathcal{T}}_n \sim Rep_k(H_n)$. By abuse of notation we also write

$$det(X_\lambda) \ \in \ Rep_k(H_n) \ .$$

12.3. **The invariant** $\ell(\lambda)$**.** As one easily shows, for any object X of $\mathcal{T}_n$

$$det(Ber^m \otimes X) = Ber^{m \cdot \mathrm{sdim}(X)} \otimes det(X) \ .$$

Hence to determine I_λ we may assume $\lambda_n = 0$. So let us assume this for the moment. Then, for a maximal atypical weight λ with the property $\lambda_n = 0$, let $S_1, ..., S_k$ denote its corresponding sectors, from left to right. If $i = 1, .., k-1$ let $d_i = dist(S_i, S_{i+1})$ denote the distances between these sectors and $r(S_i)$ denotes the rank of S_i, then $\sum_{i=1}^k r(S_i) = n$. Furthermore $d = \sum_{i=1}^k d_i = 0$ holds if and only if the weight λ is a basic weight. Recall, if we translate S_2 by shifting it d_1 times to the left, then shift S_3 translating it $d_1 + d_2$ to left and so on, we obtain a basic weight. This basic weight is called the *basic weight associated to* λ. The weighted total number of shifts necessary to obtain this associated basic weight by definition is the *integer*

$$\ell(\lambda) := \sum_{i=1}^{k} \mathrm{sdim}(X_{\lambda_i}) \cdot (\sum_{j<i} d_j)$$

where $L(\lambda_i) \in \mathcal{R}_{n-1}$ denote the irreducible representations associated to the derivatives $S_1 \ldots . \partial S_i \ldots . S_k$. By [Wei10] [HW14, Section 16] $\mathrm{sdim}(X_{\lambda_i}) = \frac{r_i}{n} \cdot \mathrm{sdim}(X_\lambda)$ holds for $r_\nu = r(S_\nu)$, which allows to rewrite this in the form

$$\ell(\lambda) = \frac{\mathrm{sdim}(X_\lambda) \cdot D(\lambda)}{n} \ ,$$

where $D(\lambda)$ is the *total number of left moves* needed to shift the support of the plot λ into the support of the associated basic plot λ_{basic}, i.e. the integer

$$D(\lambda) \ := \ \sum_{\nu=1}^{k} \ r_\nu \cdot (\sum_{\mu<\nu} d_\mu) \ .$$

Now, to remove our temporary assumption $\lambda_n = 0$ and hence to make the formulas above true unconditionally, we have to introduce the additional terms $d_0 = \lambda_n$ (for $\mu = 0$) in the formulas above. For further details on this see [HW14, Section 25]. We remark that in the following we also write $D(L)$ instead of $D(\lambda)$ for the irreducible representations $L = L(\lambda)$ and similarly $\ell(L)$ instead of $L(\lambda)$.

12.4. *Pic^0*. We return to indecomposable objects $I \in \mathcal{T}_n^+$ representing invertible objects of $\overline{\mathcal{T}}_n$.

Since $I \otimes I^\vee \cong \mathbf{1} \oplus$ negligible objects, we obtain

$$\omega(I,t)\omega(I^\vee,t) = \omega(I \otimes I^\vee, t) = 1.$$

Indeed, the functor ω annihilates negligible objects. For the Laurent polynamial $\omega(I,t)$ this now implies

$$\omega(I,t) = t^\nu$$

for some integer $\nu \in \mathbf{Z}$ which defines the degree $\nu(I) = \nu$. Obviously this degree $\nu(I)$ induces a homomorphism $Pic(\mathcal{R}_n) \to \mathbb{Z}$ of groups by $I \mapsto \nu = \nu(I) \in \mathbb{Z}$ and gives an exact sequence

$$0 \longrightarrow Pic^0(\overline{\mathcal{T}}_n) \longrightarrow Pic(\overline{\mathcal{T}}_n) \xrightarrow{\ \nu\ } \mathbb{Z}$$

with kernel $Pic^0(\overline{\mathcal{T}}_n)$. Clearly $\nu(Ber) = n$, hence the next lemma follows.

Lemma 12.1. *The intersection of $Pic^0(\overline{\mathcal{T}}_n)$ with the subgroup generated by $I = Ber$ is trivial.*

Lemma 12.2. *For any irreducible object L in $\mathcal{T}_n^+$ the invertible element $det(L) \in \mathcal{T}_n$ has the property*
$$\nu(det(L)) = \mathrm{sdim}(L) \cdot D(L) = \ell(L) \cdot n.$$
In particular, the image of the homomorphism ν is $n \cdot \mathbb{Z}$.

Proof. The functor $\omega : \mathcal{T}_n \to gr - vec_k$ is a tensor functor. Hence $\nu(\det(L)) = \nu(det(\omega(L))$. Hence
$$\nu(det(L)) = \sum_i i \cdot a_i \qquad (*)$$
for $\omega(L,t) = \sum_i a_i t^i$. By [HW14, Lemma 25.2] we obtain the formula $\omega(L,t^{-1}) = t^{-2D(\lambda)}\omega(L,t)$ and hence $\omega(L_{basic},t) = \omega(L_{basic},t^{-1})$, the latter because of $D(L_{basic}) = 0$. So the formula (*) implies
$$\nu(det(L_{basic})) = 0.$$
From $\omega(L,t) = t^{D(L)}\omega(L_{basic},t)$ and $\mathrm{sdim}(L_{basic}) = \mathrm{sdim}(L)$, again by (*) we therefore obtain for $\omega(L_{basic},t)$
$$\begin{aligned} t^{-D(\lambda)} \sum_i a_i t^i &= t^{\sum_i (i - D(\lambda)) a_i} \\ &= t^{\sum_i i a_i - D(\lambda)\,\mathrm{sdim}(L)} \\ &= \omega(det(L),t) t^{-D(\lambda)\,\mathrm{sdim}(L)}\ , \end{aligned}$$
and hence
$$\omega(det(L),t) = t^{D(\lambda)\,\mathrm{sdim}(L)} \cdot \omega(det(L_{basic},t)).$$
The second factor being t^0, the result follows. $\square$

Since $\omega(L(\lambda),t)t^{-D(\lambda)}$ is invariant under $t \mapsto t^{-1}$, we also obtain

Corollary 12.3. *$d\ log(\omega(L,t))|_{t=1} = D(L)$.*

Corollary 12.4. *We have $det(L) \otimes Ber^{-\ell(L)} \in Pic^0(\overline{\mathcal{T}}_n^+)$, i.e.*
$$det(L) \in Pic^0(\overline{\mathcal{T}}_n) \times Ber^{\mathbf{Z}}$$
for irreducible $L \in \mathcal{T}_n^+$.

Example 12.5. For $GL(2|2)$ we obtained (up to parity shifts) in [HW15] the formula $S^i \otimes S^i = Ber^{i-1} \oplus M$ for some module M of superdimension 3. Since $\mathrm{sdim}(S^i) = 2$, $det(S^i) = Ber^{i-1} \oplus$ negligible. Indeed for S^i we obtain $\ell([i,0]) = r_1 d_0 + r_2 d_1$ where r_i denotes the rank of the i-th sector. Clearly $r_1 = r_2 = 1$ and $d_0 = 0$ and $d_1 = i - 1$, hence $\ell([i,0]) = i - 1$.

13. The Picard group of $\overline{\mathcal{T}}_n$ and the group H_n

We discuss in this section the groups H_λ and H_n. We assume throughout that the stronger structure theorem 11.1 on G_λ and G_n holds (although some results hold without this assumption). Parts of this section are conjectural and hence the purpose of this section is to give the *big picture*.

13.1. **The groups H_λ for $\ell(\lambda) \neq 0$.** If the integer $\ell(\lambda)$ is non-zero, it is easier to determine the groups H_λ since they are as large as possible.

Lemma 13.1. *For $\ell(\lambda) \neq 0$, the Tannaka groups H_λ of X_λ are the following:*

(1) NSD: $H_\lambda = GL(\mathrm{sdim}(X_\lambda))$.
(2) SD, $\mathrm{sdim}(L(\lambda) > 0$: $H_\lambda = GSO(\mathrm{sdim}(X_\lambda))$.
(3) SD, $\mathrm{sdim}(L(\lambda) < 0$: $H_\lambda = GSp(\mathrm{sdim}(X_\lambda))$.

In each case the representation V_λ of H_λ coming from X_λ corresponds to the standard representation. In the GSO and GSp cases the similitude character is given by a Berezin power.

Proof. By corollary 12.4 we have

$$det(L) \otimes Ber^{-\ell(L)} \in Pic^0(\mathcal{T}_n^+),$$

or equivalently

$$det(L) \in Pic^0(\overline{\mathcal{T}}_n) \times Ber^{\mathbf{Z}}$$

for irreducible $L \in \mathcal{T}_n^+$. In particular the determinant powers of $L(\lambda)$ give a subcategory equivalent to $Rep(GL(1))$ in $Rep(H_\lambda)$. By section 6 and the structure theorem on the G_λ we have the estimates

$$SL(V_\lambda) \subseteq H_\lambda \subseteq GL(V_\lambda)$$

in the case (NSD) and

$$SO(V_\lambda) \subseteq H_\lambda \subseteq GO(V_\lambda),$$
$$Sp(V_\lambda) \subseteq H_\lambda \subseteq GSp(V_\lambda)$$

in the case (SD) for even respectively odd X_λ. The additional $GL(1)$ factor implies then immediately that we get $GL(V_\lambda)$ in the (NSD) case and the groups

$$GSp(\mathrm{sdim}(L(\lambda))) \cong (Sp(\mathrm{sdim}(L(\lambda))) \times \mathbf{G}_m)/\mathbf{Z}_2$$

in the odd SD case and either

$$GO(\mathrm{sdim}(L(\lambda))) \cong O(sdim(L(\lambda))) \times \mathbf{G}_m \quad \text{or}$$
$$GSO(\mathrm{sdim}(L(\lambda))) \cong (SO(\mathrm{sdim}(L(\lambda))) \times \mathbf{G}_m)/Z_2$$

(where Z_2 is diagonally embedded in the centres) in the even SD case. Notice that for the groups $GSO(2m)$ and $GSp(2m)$ the determinant character is given by $det = \mu^m$ for the similitude character. Therefore to show that μ and hence det is a Berezin power we may use the same argument as in section 6: Indeed the object I in $\mathcal{T}_n^+$ corresponding to μ (up to a parity shift) defines a nondegenerate pairing

$$L(\lambda) \otimes L(\lambda) \to I \ .$$

Since $L(\lambda)^\vee \cong Ber^{-r} \otimes L(\lambda)$ there exists on the other hand a nondegenerate (SD) pairing

$$L(\lambda) \otimes L(\lambda) \to Ber^r \ .$$

Therefore $Ber^{-r} \otimes I$ is an indecomposable constituent of $L(\lambda) \otimes L(\lambda)^\vee$ of superdimension ± 1. However by proposition 6.2 the module $L(\lambda) \otimes L(\lambda)^\vee$ contains a unique indecomposable constituent of superdimension ± 1, namely the trivial representation $\mathbf{1}$. In fact this is a well known property of the standard representations of the groups $SO(V)$ and $Sp(V)$. This implies $Ber^{-r} \otimes I \cong \mathbf{1}$ and therefore $\mu = Ber^r$. But this also implies that $H_\lambda \cong GSO(V_\lambda)$ instead of $GO(V_\lambda)$. □

13.2. **Special modules and determinants.** We now calculate the determinants $det(L(\lambda))$ under the assumption that special modules (see below) are trivial.

Consider modules $V \in \mathcal{T}_n^+$ with the property $V^* \cong V$ such that $\mathrm{sdim}(V) = 1$. Then, up to negligible summands, V has a unique indecomposable summand with nonvanishing superdimension. So we may assume that V is *indecomposable*.

Definition 13.2. An indecomposable module V in $\mathcal{T}_n^+$ with $\mathrm{sdim}(V) = 1$ will be called *special*, if $V^* \cong V$ and $H^0(V)$ contains $\mathbf{1}$ as a direct summand.

For special modules V the assumption $\mathrm{sdim}(V) = 1$ implies that special modules are maximal atypical modules, i.e. contained in $\mathcal{R}_n^0$. Furthermore for special modules

$$DS(V) \cong \mathbf{1} \oplus N$$

holds for some negligible module N, since $\mathrm{sdim}(DS(V)) = \mathrm{sdim}(V)$. Recall that the assumption $V \in \mathcal{T}_n^+$ implies that $H_D(V) = DS(V)$. Hence for special V also

$$H_D(V) = \mathbf{1} \oplus N \ .$$

Lemma 13.3. *Suppose $V \cong V^* \cong V^\vee$ and $DS(V) \cong \mathbf{1} \oplus N$ holds for some negligible module N. Then V is special.*

Proof. The assumptions imply that there exists a unique integer ν for which $H^\nu(V)$ is not a negligible module. Since $H^\nu(V)^\vee \cong H^{-\nu}(V^\vee)$, the assumption $V \cong V^\vee$ implies $\nu = 0$. Hence $H^0(V) = \mathbf{1} \oplus N$ for some negligible N. □

Conjecture 13.4. *Up to a parity shift, any special module V in $\mathcal{T}_n^+$ is isomorphic to the trivial module $\mathbf{1}$.*

Assuming this conjecture we now prove the following theorem. We refer to section 12 for the definition of the determinant and the integer $\ell(\lambda)$. In fact we would only have to prove the (NSD)-case of the following theorem as explained in lemma 13.7.

Theorem 13.5. *Assume that the derived connected groups G_λ are as in proposition 6.2 for the degrees $\leq n$ and assume conjecture 13.4. Then for any maximal atypical weight λ defining X_λ in $\mathcal{T}_n^+$, for $\lambda_n = 0$ the module*

$det(X_\lambda)$ satisfies

$$det(X_\lambda) \;=\; Ber^{\ell(\lambda)} \;\oplus\; negligible.$$

In particular, for $\lambda_n = 0$ we have $det(X_\lambda) = 0$ if (and only if) the maximal atypical weight weight λ is a basic weight.

Lemma 13.6. *If for all $i < n$ the last theorem holds for the categories $\mathcal{T}_i^+$, then for all X_λ in $\mathcal{T}_n^+$ we have $DS(I_\lambda) = Ber^{\ell(\lambda)}$.*

Proof of lemma 13.6. Since

$$det(A \oplus B) \cong det(A) \otimes det(B) \oplus negligible$$

holds using

$$\Lambda^N(A \oplus B) \cong \bigoplus_{p+q=N} \Lambda^p(A) \otimes \Lambda^q(B),$$

it is enough to show

$$\ell(\lambda) = \frac{sdim(X_\lambda)}{n} \cdot D(\lambda) = \sum_{i=1}^k \ell(\lambda_i) \ ,$$

using induction on n. By section 12

$$\ell(\lambda_i) = \frac{\mathrm{sdim}(X_{\lambda_i})}{n-1} \cdot D(\lambda_i) = \frac{\mathrm{sdim}(X_\lambda) \cdot r_i}{n(n-1)} \cdot D(\lambda_i) \ .$$

So it suffices to verify

$$(n-1) \cdot D(\lambda) \;=\; \sum_{i=1}^k \; r_i \cdot D(\lambda_i) \ .$$

Notice $D(\lambda_1) - D(\lambda) = r_2 + ... + r_k$. For $i \geq 2$ we get the slightly different formula

$$D(\lambda_i) \;-\; \big(D(\lambda) - \sum_{j<i} d_j\big) \;=\; -(n-1) + (r_i - 1) + 2 \cdot (r_{i+1} ... + r_k) \ .$$

Indeed, the term $-D(\lambda)$ is obtained from moving n labels to the left, where for λ_i the deleted point $a_i \in S_i$ is omitted, which gives the modification by $\sum_{j<i} d_j$ on the left side of the formula. The term $-(n-1)$ on the right side appears in the cases $i \geq 2$ only. It comes from the normalization condition for the last coordinate of the weight vectors. For $i \geq 2$ the last coordinate of λ_i is $-n+1$. It has to be normalized to $-(n-1)+1$ by a Berezin twist. Finally the $r_i - 1$ labels within ∂S_i, if compared to λ, need to be moved to the left by an additional distance $+1$, whereas the labels within S_j for $j > i$, if compared to λ, need to be moved by an additional distance $+2$. Hence

$$D(\lambda_i) = D(\lambda) \;-\; \sum_{j<i} d_j \;+\; \sum_{j=1}^{i} \varepsilon_{ij} \cdot r_j$$

for $\varepsilon_{ji} = -\varepsilon_{ij}$ and $\varepsilon_{ij} = 1$ for $i < j$. Since $\sum_i \sum_j \varepsilon_{ij} r_i r_j = 0$,

$$\sum_{i=1}^{k} r_i \cdot D(\lambda_i) = (\sum_{i=1}^{k} r_i) \cdot D(\lambda) - D(\lambda) = (n-1) \cdot D(\lambda) \ .$$

□

The last lemma 13.6 implies that

$$Ber^{-\ell(\lambda)} \otimes I_\lambda$$

is a special module. Hence $Ber^{-\ell(\lambda)} \otimes I_\lambda \cong \mathbf{1}$ by conjecture 13.4. This proves theorem 13.5.

We remark that it is sufficient to prove the determinant formula in the (NSD)-case as shown in the next lemma.

Lemma 13.7. *Assume that $det(L) = Ber^{\ell(\lambda)} \oplus$ negligible for L non-basic of type (NSD). Then the same formula for $det(L)$ is true for any maximal atypical L.*

Proof. Assume $det(L) = Ber^{\ell(\lambda)} \oplus$ negligible holds for (NSD) non-basic. Let us assume that $L(\lambda) = L(\lambda_1, \ldots, \lambda_n)$ is an arbitrary maximal atypical representation in $\mathcal{T}_n^+$. Choose a large number λ_0 and consider the maximal atypical irreducible representation

$$\tilde{L} = L(\tilde{\lambda}) = [\lambda_0, \lambda_1, \ldots, \lambda_n]$$

in $\mathcal{T}_{n+1}$. For large enough λ_0 this representation is of type NSD non-basic. For the NSD case the induction step from n to $n+1$ of section 6 works and shows $G_{\tilde{\lambda}} = SL(\mathrm{sdim}(X_{\tilde{\lambda}})$ and therefore $H_{\tilde{\lambda}} = GL(\mathrm{sdim}(X_{\tilde{\lambda}})$.

Assuming the NSD-case, we have

$$det(X_{\tilde{\lambda}}) \ = \ Ber^{\ell(\tilde{\lambda})} \oplus \text{ negligible}$$

Since the tensor functor DS commutes with det and furthermore since $det(A \oplus B) = det(A) \otimes det(B) \oplus$ negligible holds, we obtain

$$DS(det(\mathrm{sdim}(X_{\tilde{\lambda}}))) \ \cong \ \Pi Ber^{\ell(\tilde{\lambda})} \oplus \text{ negligible}$$

and

$$Ber^{\ell(\tilde{\lambda})} \oplus \text{ negligible } \cong \bigotimes_{i=1}^{r} det(L(\tilde{\lambda}_i)) \oplus \text{ negligible.}$$

Except for $L(\tilde{\lambda}_1) = L(\lambda)$ all other summands of $DS(\tilde{L})$ contain the number λ_0 which prevents them from being of basic type or NSD type. Hence their determinant is a Berezin power and we obtain

$$Ber^{\ell(\tilde{\lambda})} \oplus \text{ negligible} \cong \ det(L(\lambda)) \otimes \bigotimes_{i=2}^{r} Ber^{\ell(\tilde{\lambda}_i)} \oplus \text{ negligible.}$$

Hence $det(L(\lambda))$ is a Berezin power, and the computations of lemma 12.2 show $det(L(\lambda)) = Ber^{\ell(\lambda)}$. □

13.3. H_λ **and its character group.** In order to determine H_λ, we need to understand its character group, i.e. the invertible elements in $Rep(H_\lambda)$. In particular we would like to rule out a nontrivial group of connected components.

Conjecture 13.8. *Any invertible object I in $\overline{\mathcal{T}}_n$ is represented in $\mathcal{T}_n^+$ by a power of the Berezin determinant.*

Remark 13.9. The same assertion cannot hold for $\mathcal{R}_n$, since there exists a nontrivial extension V between $\mathbf{1}$ and S^1 with $\mathrm{sdim}(V) = -1$. Hence $V \otimes V^\vee \cong \mathbf{1} \oplus N$ for some object N in $\mathcal{R}_n$ of $\mathrm{sdim}(N) = 0$. Note that $End_{\mathcal{R}_n}(V) \to End_{\mathcal{R}_n}(\mathbf{1})$ has kernel of dimension at most one, since

$$Hom_{\mathcal{R}_n}(S^1, \mathbf{1}) = Hom_{\mathcal{R}_n}(\mathbf{1}, S^1) = 0.$$

Thus $End_{\mathcal{R}_n}(V) = End_{\mathcal{R}_n}(N) \oplus End_{\mathcal{R}_n}(\mathbf{1})$ and $End_{\mathcal{R}_n}(N) = k \cdot id_N$. Hence N is indecomposable and negligible.

Remark 13.10. From $I \otimes I^\vee \cong \mathbf{1}$ in $\overline{\mathcal{T}}_n$ we conclude $\dim(I) = 1$. Every character is the determinant of a faithful representation, but this representation might not be irreducible. If it is not, we can decompose it and can use $det(X_1 \oplus \ldots \oplus X_r) = det(X_1) \otimes \ldots \otimes det(X_r)$. However these X_i could come from indecomposable modules I_i in $\mathcal{T}_n^+$, and we don't have a formula for their determinants even under the assumption that special modules are trivial.

This conjecture allows us to determine the possible Tannaka groups.

Theorem 13.11. *The Tannaka groups H_λ of X_λ are the following:*

1. *NSD non-basic: $H_\lambda = GL(\mathrm{sdim}(X_\lambda))$.*
2. *NSD basic: $H_\lambda = SL(\mathrm{sdim}(X_\lambda))$.*
3. *SD, proper selfdual, $\mathrm{sdim}(L(\lambda) > 0$: $H_\lambda SO(\mathrm{sdim}(X_\lambda))$.*
4. *SD, proper selfdual, $\mathrm{sdim}(L(\lambda) < 0$: $H_\lambda = Sp(\mathrm{sdim}(X_\lambda))$.*
5. *SD, weakly selfdual, $\mathrm{sdim}(L(\lambda) > 0$: $H_\lambda = GSO(\mathrm{sdim}(X_\lambda))$.*
6. *SD, weakly selfdual, $\mathrm{sdim}(L(\lambda) < 0$: $H_\lambda = GSp(\mathrm{sdim}(X_\lambda))$.*

In each case the representation V_λ of H_λ coming from X_λ corresponds to the standard representation. In the GL case the determinant comes from a (nontrivial) Berezin power. In the GSO and GSp cases the similitude character is given by a Berezin power.

Note that a basic representation of SD type always satisfies $L \cong L^\vee$. In the (SD) case $\ell(\lambda) = 0$ if and only if $L(\lambda) \simeq L(\lambda)^\vee$. In the (NSD)-case $\ell(\lambda) = 0$ if and only if λ is basic.

Proof. For $\ell(\lambda) \neq 0$ we have seen this in lemma 13.1. Assume therefore $\ell(\lambda) = 0$. In the (SD)-case this implies $L(\lambda) \cong L(\lambda)^\vee$, and we get the a-priori estimates

$$SO(V_\lambda) \subseteq H_\lambda \subseteq O(V_\lambda),$$
$$Sp(V_\lambda) \subseteq H_\lambda \subseteq Sp(V_\lambda).$$

To distinguish between $O(V_\lambda)$ and $SO(V_\lambda)$ we need to rule out nontrivial indecomposable representations J satisfying $J^{\otimes 2} \cong \mathbf{1}$, $J \cong J^* \cong J^\vee$ and $J \not\cong \mathbf{1}$. But such a representation is special, therefore trivial and therefore $H_\lambda \cong SO(V_\lambda)$. In the (NSD) case every character is given by a Berezin power according to conjecture 13.8. Therefore a nontrivial character means $Pic(H_\lambda) \simeq \mathbf{Z}$ and therefore $H_\lambda \simeq GL(V_\lambda)$. But then $det(V_\lambda) \neq \mathbf{1}$, in contradiction to $det(X_\lambda) \simeq \mathbf{1} \oplus negligible$ according to theorem 13.5 for $\ell(\lambda) = 0$. □

The structure of H_n can now be recovered from the H_λ as in the $GL(2|2)$-case.

Example 13.12. The $GL(3|3)$-case. For $n = 3$ the structure theorem on the G_λ holds unconditionally. Here is a list of the nontrivial basic representations and their Tannaka groups. We automatically consider the possible parity shifted representation with positive superdimension here.

(1) $[2,1,0]$, , sdim $= 6$, $H_\lambda = Sp(6)$.
(2) $[1,1,0]$, sdim $= 3$, $H_\lambda = SL(3)$.
(3) $[2,0,0]$, sdim $= 3$, $H_\lambda = SL(3)$.
(4) $[1,0,0]$, sdim $= 2$, $H_\lambda = SL(2)$.

Twisting any of these with a nontrivial Berezin power gives the GL, GSO or GSp version. The appearing groups exhaust all possible Tannaka groups arising from an $L(\lambda)$.

Example 13.13. The $GL(4|4)$-case. Here the structure theorem for G_λ (and therefore the determination of H_λ) holds unconditional except for the case where $L(\lambda)$ is weakly selfdual with $[\lambda_{basic}] \neq [3,2,1,0]$ by the following lemma:

Lemma 13.14. *For the basic representations of (SD) type*

$$[3,1,1,0], [2,1,0,0], [2,2,0,0]$$

we have $I \cong \mathbf{1}$.

Proof. For $[2,2,0,0]$ this follows from appendix D and example D.5. It is enough to verify that $DS([2,2,0,0])$ does not contain a summand $L(\lambda_i)$ with $(\lambda_i)_{basic} = [2,1,0]$. The irreducible representations $[3,1,1,0]$ and $[2,1,0,0]$ have $k = 3$ sectors each. However V_λ can only decompose under the restriction to G_λ if k is even. Alternatively note that we have embedded subgroups $Sp(6)$ and $Sp(6) \times Sl(3)$ in $G_{[2,1,0,0]}$ and $G_{[3,1,1,0]}$ respectively which implies that G_λ cannot be $SL(3)$ or $SL(6)$. □

For $n = 4$ there are 14 maximal atypical basic irreducible representations in $\mathcal{R}_4$, the self dual representations

$$\mathbf{1} = [0,0,0,0], S^1 = [1,0,0,0], [2,1,0,0], [2,2,0,0], [3,1,1,0], [3,2,1,0]$$

of superdimension $1, -2, -6, 6, -12, 24$ and the representations

$$S^2 = [2,0,0,0], S^3 = [3,0,0,0], [3,1,0,0], [3,2,0,0]$$

of superdimension $3, -4, 8, -12$ and their duals

$$[1,1,0,0], [1,1,1,0], [2,1,1,0], [2,2,1,0] \ .$$

Here is a list of the nontrivial basic representations and their Tannaka groups. We automatically consider the possible parity shifted representation with positive superdimension here. Note that the result for the first example $[3,2,1,0]$ assumes that $G_\lambda \cong SO(24)$ (a consequence of the conjectural structure theorem 11.1).

(1) $[3,2,1,0]$, sdim $= 24$, $H_\lambda = SO(24)$.
(2) $[3,2,0,0]$, sdim $= 12$, $H_\lambda = SL(12)$.
(3) $[3,1,1,0]$, sdim $= 12$, $H_\lambda = Sp(12)$.
(4) $[3,1,0,0]$, sdim $= 8$, $H_\lambda = SL(8)$.
(5) $[3,0,0,0]$, sdim $= 4$, $H_\lambda = SL(4)$.
(6) $[2,2,1,0]$, sdim $= 12$, $H_\lambda = SL(12)$.
(7) $[2,2,0,0]$, sdim $= 6$, $H_\lambda = SO(6)$.
(8) $[2,1,1,0]$, sdim $= 8$, $H_\lambda = SL(8)$.
(9) $[2,1,0,0]$, sdim $= 6$, $H_\lambda = Sp(6)$.
(10) $[2,0,0,0]$, sdim $= 3$, $H_\lambda = SL(3)$.
(11) $[1,1,1,0]$, sdim $= 4$, $H_\lambda = SL(4)$.
(12) $[1,1,0,0]$, sdim $= 3$, $H_\lambda = SL(3)$.
(13) $[1,0,0,0]$, sdim $= 2$, $H_\lambda = SL(2)$.

Twisting any of these with a nontrivial Berezin power gives the GL, GSO or GSp version. The appearing groups exhaust all possible Tannaka groups arising from an $L(\lambda)$.

Theorem 4.1 implies the following branching rules (the lower index indicates the superdimensions up to a sign):

(1) $DS([3,2,1,0]_{24}) \cong [3,2,1]_6 \oplus [1,0,-1]_6 \oplus [3,0,-1]_6 \oplus [3,2,-1]_6$
(2) $DS([3,2,0,0]_{12}) \cong [3,2,0]_6 \oplus [1,-1,-1]_3 \oplus [3,-1,-1]_3$
(3) $DS([3,1,1,0]_{12}) \cong [3,1,1]_3 \oplus [3,1,-1]_6 \oplus [0,0,-1]_3$
(4) $DS([3,1,0,0]_8) \cong [3,1,0]_6 \oplus [0,-1,-1]_2$
(5) $DS([3,0,0,0]_4) \cong [3,0,0]_3 \oplus [-1,-1,-1]_1$
(6) $DS([2,2,0,0]_6) \cong [2,2,0]_3 \oplus [2,-1,-1]_3$
(7) $DS([2,1,0,0]_6) \cong [2,1,0]_6$
(8) $DS([2,0,0,0]_3) \cong [2,0,0]_3$
(9) $DS([1,0,0,0]_2) \cong [1,0,0]_2$
(10) $DS([1,1,1,1]_1) \cong [1,1,1]_1$

and $DS([n,0,0,0]_4) \cong [n,0,0]_3 \oplus [-1,-1,-1]_1$ for all $n \geq 4$. We also have to consider the dual representations in the cases (2), (4), (5) and (8).

Example 13.15. Consider $L(\lambda) = [6,6,1,1]$. It is weakly selfdual with dual $[1,1,-4,-4] = Ber^{-5} \otimes [6,6,1,1,]$. Its superdimension is 6. Since $\ell(\lambda) \neq 0$, the associated Tannakagroup is therefore $H_\lambda = GSO(V_\lambda) \simeq GSO(6)$. This does not depend on the conjecture $I \simeq \mathbf{1}$. Indeed $DS([6,6,1,1])$ does not contain an irreducible summand $L(\lambda_i)$ with $(\lambda_i)_{basic} = [2,1,0]$ and one can argue as in lemma 13.14.

Appendix A. Equivalences and derivatives

Recall that two weights λ, μ are equivalent $\lambda \sim \mu$ if there exists $r \in \mathbf{Z}$ such that $L(\lambda) \cong Ber^r \otimes L(\mu)$ or $L(\lambda)^\vee \cong Ber^r \otimes L(\mu)$ holds. We denote the equivalence classes of maximal atypical weights by $Y_0^+(n)$. The embedding $H_{n-1} \to H_n$ induces an embedding $G_{n-1} \to G_n$. Since inductively $G_{n-1} = \prod_{\lambda \in Y_0^+(n)} G_\lambda$, we need to understand the equivalence classes of weights and their behaviour under DS.

A.1. **Plots.** We use the notion of plots from [HW14, Section 13] to describe weight diagrams and their sectors. A plot λ is a map

$$\lambda : \mathbb{Z} \to \{\boxplus, \boxminus\}$$

such that the cardinality r of the fiber $\lambda^{-1}(\boxplus)$ is finite. Then by definition $r = r(\lambda)$ is the degree and $\lambda^{-1}(\boxplus)$ is the support of λ. The fiber $\lambda^{-1}(\boxplus)$ corresponds to those vertices of the weight diagram which are labeled by a $\vee$. An interval $I = [a, b]$ of even cardinality $2r$ and a subset K of cardinality of rank r defines a plot λ of rank r with support K. We consider formal finite linear combinations $\sum_i n_i \cdot \lambda_i$ of plots with integer coefficients. This defines an abelian group $R = \bigoplus_{r=0}^{\infty} R_r$ (graduation by rank r). In [HW14] we defined a derivation on R called derivative. Any plot can be written as a product of prime plots and we use the formula $\partial(\prod_i \lambda_i) = \sum_i \partial\lambda_i \cdot \prod_{j \neq i} \lambda_j$ to reduce the definition to the case of a prime plot λ. For prime λ let (I, K) be its associated sector. Then $I = [a, b]$. Then for prime plots λ of rank n with sector (I, K) we define $\partial\lambda$ in R by $\partial\lambda = \partial(I, K)$, $I = [a, b]$ with $\partial(I, K) = (I, K)' = (I', K')$ for $I' = [a+1, b-1]$ and $K' = I' \cap K$. The importance of ∂ is that it describes the effect of DS on irreducible representations according to theorem 4.1: If $L(\lambda)$ has sector structure $S_1 \ldots S_k$, $L(\lambda_i)$ has sector structure $S_1 \ldots \partial S_i \ldots S_k$.

A.2. **Duality.** If $L = L(\lambda)$ is an irreducible maximal atypical representation in $\mathcal{R}_n$, its weight λ is uniquely determined by its plot. Let $S_1 ... S_2 ... S_k$ denote the segments of this plot. Each segment S_ν has even cadinality $2r(S_\nu)$, and can be identified up to a translation with a unique basic weight of rank $r(S_\nu)$ and a partition in the sense of [HW14, Lemma 20.3]. For the rest of this section we denote the segment of rank $r(S_\nu)$ attached to the dual partition by S_ν^*, hoping that this will not be confused with the contravariant functor $*$. Using this notation, Tannaka duality maps the plot $S_1 .. S_2 ... S_k$ to the plot $S_k^* ... S_2^* .. S_1^*$ so that the distances d_i between S_i and S_{i+1} coincide with the distances between S_{i+1}^* and S_i^*. This follows from [HW14, proposition 20.1] and determines the Tannaka dual $L^\vee$ of L up to a Berezin twist.

A.3. **Equivalent weights.** Let λ be a maximal atypical highest weight in $X^+(n)$ with the sectors $S_1 .... S_k$. The constituents λ_i (for $i = 1, .., k$) of the derivative have the sector-structure $S_1 ... \partial S_i ... S_k$. Recall that two irreducible representations M, N in $\mathcal{T}_n$ are equivalent $M \sim N$, if either $M \cong Ber^r \otimes N$

or $M^\vee \cong Ber^r \otimes N$ holds for some $r \in \mathbb{Z}$. Assume that λ_i and λ_j are equivalent for $i \neq j$. Then

$$S_1...(\partial S_i)...S_j...S_k \ \sim \ S_1...S_i...(\partial S_j)...S_k$$

define equivalent weights of $\mathcal{T}^+_{n-1}$. Passing from $L(\lambda)$ to $Ber^i \otimes L(\lambda)$ involves a shift of the vertices in the weight diagram by i. We refer to this as the translation case. Applying the duality functor $L(\lambda) \mapsto L(\lambda)^\vee$ is described in terms of the cup diagram as a kind of reflection, see section **??**. We refer to this as the reflection case.

Lemma A.1. *For a maximal atypical weight λ assume that there exists an equivalence $\lambda_i \sim \lambda_j$ for some $i \neq j$ between two constituents λ_i, λ_j of the derivative of λ. Then $S_\nu \equiv S^*_{k+1-\nu}$ holds for all $\nu = 1,..,k$ and $d_{k-\nu} = d_\nu$ holds for all $\nu = 1,..,k$.*

Proof. 1) Translation case. We first discuss whether this equivalence can be achieved by a translation and show that this implies

$$r(S_\nu) = 1 \text{ for all } \nu,\ i = 1 \text{ and } j = k.$$

To prove this we first exclude $1 < i, j$. Indeed then the starting sector is S_1 in both cases and a translation equivalence different from the identity is impossible. Now assume $i = 1$. Again an equivalence is not possible unless $\partial S_1 = \emptyset$, since otherwise S_1 and ∂S_1 would be starting sectors of different cardinality and hence they can not be identified by a translation. So the only possibility could be $i = 1$ and $r(S_1) = 1$ (so that $\partial S_1 = \emptyset$). The equivalence of $S_2....S_k$ with $S_1...\partial S_j...S_k$ then implies $\partial S_j = \emptyset$, since both plots must have the same number of sectors. But then the only equivalence comes from a left shift by two. Hence it is not hard to see that this implies $r(S_\nu) = 1$ for all ν, $i = 1$ and $j = k$. Furthermore $d_1 = ... = d_k$ must hold. But then we see that this translation equivalence is also induced by a reflection equivalence.

2) Reflection case. Let us consider equivalences between $S_1...(\partial S_i)...S_j...S_k$ and $S_1...S_i...(\partial S_j)...S_k$ involving duality as in section A.2.

The case $r(S_i) > 1$. Notice that $r(S_i) > 1$ is equivalent to $\partial S_i \neq \emptyset$. Furthermore notice that $r(S_i) > 1$ implies $r(S_j) > 1$, since equivalent plots need to have the same number of sectors. To proceed let us temporarily ignore the distances between the different sectors S_ν; we write $\equiv$ to indicate this. Then for all $\nu \neq i, j, k+1-i, k+1-j$ we get

$$S^*_\nu \equiv S_{k+1-\nu}$$

(equality up to a shift). The easy case now is $j = k+1-i$, where we get the further condition (*)

$$S^*_i \equiv S_{k+1-i} \text{ and hence } \partial S^*_i \equiv \partial S_{k+1-i}\ .$$

We also then conclude

$$d_\nu = d_{k-\nu} \quad \text{for all} \quad \nu = 1,..,k\ .$$

We now show that the more complicated looking case $i \neq j$ and $i \neq k+1-j$, where we also have $j \neq k+1-i$, can not occur. In this case [HW14, proposition 20.1], implies, from comparing

$$... \quad \partial S_i \quad ... \quad S_{k+1-j} \quad ... \quad S_j \quad ... \quad S_{k+1-i} \quad ...$$

and the reflection of

$$... \quad S_i \quad ... \quad S_{k+1-j} \quad ... \quad \partial S_j \quad ... \quad S_{k+1-i} \quad ...$$

the following assertions

$$\partial S_i \equiv S^*_{k+1-i} \quad , \quad \partial S_j \equiv S^*_{k+1-j} \ ,$$
$$S_i \equiv S^*_{k+1-i} \quad , \quad S_j \equiv S^*_{k+1-j} \ .$$

However this is absurd, since it would imply $r(S_i) = r(S^*_{k+1-i}) = r(S_i) - 1$.

So now $r(S_i) = 1$. Then $\partial S_i = \emptyset$ and hence also $\partial S_j = \emptyset$ since the cardinality of sectors of equivalent plots coincide. First assume $j = k+1-i$. In the case of a reflection symmetry this implies

$$S_\nu \equiv S^*_{k+1-\nu} \quad \text{for all} \quad \nu \neq i, k+1-i \ .$$

Furthermore it implies

$$d_{k-\nu} = d_\nu \quad , \quad \nu = 1, ..., k \ .$$

This follows by comparing

$$S_1 \quad ...d... \quad \partial S_i \quad d_i \quad S_{i+1} \quad .... \quad S_{k+1-i} \quad ...d... \quad S_k$$

with the reflection of

$$S_1 \quad ...d... \quad S_i \quad d_i \quad S_{i+1} \quad ....S_{k-i} \quad \partial S_{k+1-i} \quad ...d... \quad S_k \ .$$

Then $d_1 = d_{k+1-i}, ..., d_{i-1} = d_{k-i+1}$, by a comparison of the lower left side and the upper right side, and then also $d_i = d_{k-i}$ and so on till $d_{k-i-1} = d_{i+1}$, but then also $d_{k-i} + d = d_i + d$ for $d = d_1 + ... + d_{i-1}$. Hence we conclude that $d_\nu = d_{k-\nu}$ holds for all $\nu = 1, ..., k$. Similarly we see $S_\nu \equiv S^*_{k+1-\nu}$ for $\nu \neq i, k+1-i$. But taking into account $r(S_i) = r(S_{k+1-i})$ the assertion $S_\nu \equiv S^*_{k+1-\nu}$ also holds for $\nu = i, k+1-i$.

Finally we want to show that we have now covered all case. This means that again for $r(S_i) = 1$ the case $j \neq k+1-j$ is impossible. To show this we can assume $min(i, k+1-i) < min(j, k+1-j)$ by reverting the role of i and j and we can then assume $i < k+1-i$ by left-right reflection. Then we have to compare the reflection of

$$S_1 \quad ...d... \quad \partial S_i S_{i+1} \quad ... \quad S_{k+1-j} \quad ... \quad S_j \quad ... \quad S_{k+1-i} \quad ...d.... \quad S_k$$

with

$$S_1 \quad ...d... \quad S_i S_{i+1} \quad ... \quad S_{k+1-j} \quad ... \quad \partial S_j \quad ... \quad S_{k+1-i} \quad ...d... \quad S_k \ .$$

We claim that an equivalence is not possible by a reflection! (We could easily reduce to the case where $i = 1$ by the way). In fact, by comparing the left side of the second plot with the right side of the first plot, then

$S_i \equiv S^*_{k+1-i}$ and the distance $d = d_1 + ... + d_{i-1}$ between S_1 and S_i must be the same as the distance $d_{k+1-i} + ... + d_{k-1}$ between S_{k+1-i} and S_k. However, by comparing the left side of the first plot with the right side of the second plot, then $S_{i+1} \equiv S^*_{k+1-i}$ and the distance $d + 2 + d_i$ between S_1 and S_{i+1} must be the same as the distance d between S_{k+1-i} and S_k. In fact this follows from the fact $\partial S_i = \emptyset$ and $\# S_i = 2r(S_i) = 2$. This implies $2 + d_i = 0$. A contradiction! □

From lemma A.1 we easily get

Proposition A.2. *Suppose for the k irreducible constituents $L(\lambda_i)$ of $DS(\lambda)$ there are two different integers $i, j \in \{1, ..., k\}$ such that $\lambda_i \sim \lambda_j$. Then there exists an integer r such that $L(\lambda)^\vee \cong Ber^r \otimes L(\lambda)$ holds. The converse also holds.*

Proof. By the last lemma we conclude $S_\nu \equiv S^*_{k+1-\nu}$ and $d_{k-\nu} = d_\nu$ for all sectors $S_\nu, \nu = 1, .., k$ of λ. By proposition [HW14, Proposition 20.1] or section A.2 this implies $L(\lambda)^\vee \cong Ber^r \otimes L(\lambda)$ for some integer r. □

Another conclusion of the considerations above is

Lemma A.3. *For fixed i between 1 and k the plot $S_1...\partial S_i...S_k$ can only be equivalent to at most one of the plot $S_1...(\partial S_i)...S_j...S_k$ for $j \neq i$.*

Corollary A.4. *Every equivalence class of the constituents λ_i of the derivative of λ can contain at most $s = 2$ representatives.*

Lemma A.5. *Suppose $\lambda, \tilde{\lambda}$ are maximal atypical weights in X_0^+ that are inequivalent $\lambda \not\sim \tilde{\lambda}$. Then there exist constituents $L(\lambda_i)$ of $DS(L(\lambda))$ and $L(\tilde{\lambda}_j)$ of $DS(L(\tilde{\lambda}))$ such that λ_i and $\tilde{\lambda}_j$ are inequivalent maximal atypical weights except for the case where $k = 2$ and $n = 2$.*

Proof. This is obvious if for λ and $\tilde{\lambda}$ the number of sectors is different. If these numbers coincide assume that $\partial S_1....S_k$ and $\tilde{S}_1 \partial\partial \tilde{S}_k$ are equivalent. Then λ and $\tilde{\lambda}$ are equivalent unless $r(S_1) = r(\tilde{S}_k) = 1$. Then go on and assume that $S_1 \partial S_2....S_k$ and $\tilde{S}_1...\partial \tilde{S}_{k-1} S_k$ are equivalent. Then λ and $\tilde{\lambda}$ are equivalent unless $r(S_2) = r(\tilde{S}_{k-1}) = 1$ and so on. Hence we can assume $r(S_i) = r(\tilde{S}_i)$ for all i. But then the assertion is immediate except for the case where $k = 2$. But then $n = r(S_1) + r(S_2) = 2$. □

A.4. **Selfdual derivatives.**

Lemma A.6. *Suppose the maximal atypical weight λ has a weakly selfdual derivative λ_i for some $i = 1, ..., k$. Then λ_i is the unique weakly selfdual derivative except in the case where λ is weakly selfdual and has equidistant sectors all of cardinality two.*

Proof. Suppose λ is a maximal atypical weight such that one of its derivatives λ_i is weakly selfdual. Let $S_1, ..., S_k$ denote the sectors of λ. Then there are the following cases

(1) $k = 2m+1$ is odd and $S^1, ..., \partial S_{m+1} ...., S_k$ is weakly selfdual.
(2) $S^1, ..., \partial S_\nu ...., S_k$ is weakly selfdual such that $\partial S_\nu = \emptyset$ and not of type 1).
(3) $S^1, ..., \partial S_\nu ...., S_k$ is weakly selfdual and we are in neither of the two cases above.

In the first case $\lambda_{m+1} = S^1, ..., \partial S_{m+1} ...., S_k$ is the unique selfdual derivative of λ. This immediately follows from lemma A.1. Furthermore, if $S^1, ..., \partial S_{m+1} ...., S_k$ is weakly selfdual, $S^1, ..., S_{m+1} ...., S_k$ is weakly selfdual in this first case.

In the second case we change notation and we can suppose that

$$\lambda = S_1, S_2, ..., S_\nu, [a, a+1], S_{\nu+1}, ..., S_2^*, S_1^*$$

where we also allow the the sector $[a, a+1]$ to be at the left or right. Then it is immediately clear that there does not exist a weakly selfdual derivative λ_j different from λ_i except if $[a, a+1]$ is the rightmost or leftmost sector of λ. Without restriction of generality we may assume it is the leftmost sector of λ, i.e. $\lambda = S_0, S_1, S_2, ..., S_2^*, S_1^*$ for $S_0 = [a, a+1]$ holds, i.e. $S_\nu^* = S_{k+1-\nu}$. If λ_j is obtained by $S_j \mapsto \partial S_j$, we distinguish two cases: The first is where $j = \min(j, k+1-j)$ and the second is where $j = \max(j, k+1-j)$. In the first case we obtain $S_0^* = S_k = S_1^*, S_1^* = S_{k-1} = S_2^*, ..., S_{j-1}^* = S_j^*$ and it is immediately clear that $\partial S_j = \emptyset$. It is then clear, that there is a conflict with the symmetry of distances at j unless $j = k$. In this case $j = \max(j, k+1-j)$. So let us turn to this case now, where it follows in a similar way that the only possible case is $j = k$. Then we can easily show that all distances of λ are the same and all sectors have length two. Hence in the second case again λ is weakly selfdual.

In the third case λ_i is the unique weakly selfdual derivative of λ. □

Corollary A.7. *Suppose a maximal atypical weight λ that is not weakly selfdual admits a weakly selfdual derivative λ_i for some $i = 1, ..., k$. Then λ_i is unique with this property and we are in case 3 above.*

Appendix B. Pairings

Selfdual objects $L(\lambda)$ will give rise to groups of type B, C, D according to section 6. In order to distinguish between the orthogonal and the symplectic case we check whether these representations are even or odd in the sense defined below.

B.1. **Strong selfduality.** We say that an object M is strongly selfdual, if there exists an isomorphism $\rho : M \to M^\vee$ such that $\rho^\vee = \pm\rho$ holds and call it even or odd depending on the sign. Here $\rho^\vee : M \to M^\vee$ is the dual morphism of ρ. Here we use the canonical identification $M = (M^\vee)^\vee$, since a priori we only have $\rho^\vee : (M^\vee)^\vee \to M^\vee$. Note that any selfdual irreducible object is strongly selfdual in this sense. Slightly more general: If

L is an invertible object in a tannakian category and $\rho : M \cong M^\vee \otimes L$, then $(\rho^\vee \otimes id_L) \circ (id_M \otimes coeval_L) = \pm\rho$. Furthermore any multiplicity one retract of a strongly selfdual object is strongly selfdual. Finally, if F is a tensor functor between rigid symmetric tensor categories, then $F(M)$ is strongly selfdual if M is strongly selfdual. We remark that we can define the similar notion of strong selfduality for $*$-duality.

By [Sch79, (4.30)] a supersymmetric invariant bilinear form on a representation (V, ρ) in T defines a skew-supersymmetric invariant bilinear form on the representation $\Pi(V, \rho)$.

Suppose $L \cong L^\vee$ in $\mathcal{R}$ is a maximal atypical self dual representation. We consider now irreducible representations of the form $[\lambda] = [\lambda_1, \ldots, \lambda_{n-1}, 0]$. We call these positive. For general λ we can twist with an appropriate Berezin power to get this form. We will induct on the degree $\sum \lambda_i$, hence we start with the case S^1.

Lemma B.1. *S^1 is an even selfdual representation.*

Proof. Obviously $S^1 \cong (S^1)^\vee$, and therefore there exists a nondegenerate super bilinear form

$$B : S^1 \otimes S^1 \to \mathbf{1} = k \ .$$

Note that the adjoint representation of G_n on $\mathbb{A} := \mathfrak{g}_n$ carries the nondegenerate invariant Killing form

$$K : \mathfrak{g}_n \otimes \mathfrak{g}_n \to \mathbf{1} = k \ .$$

This bilinear form is supersymmetric: $K(S(x \otimes y)) = K(x \otimes y)$ for the symmetry constraint $S : \mathfrak{g}_n \otimes \mathfrak{g}_n \cong \mathfrak{g}_n \otimes \mathfrak{g}_n$, or $K(x, y) = (-1)^{|x||y|} K(y, x)$. Let $\mathfrak{g}_n^0$ denote the kernel of the supertrace $\mathfrak{g}_n \to \mathbf{1}$. Then $S^1 = \mathfrak{g}_n^0/z$, where z is the center of G_n. The Killing form K restricts to a supersymmetric form on $\mathfrak{g}_n^0$ which becomes nondegenerate on $S^1 = \mathfrak{g}_n^0/z$. Hence S^1 carries a nondegenerate supersymmetric bilinear form. □

We now treat the general $[\lambda] = [\lambda_1, \ldots, \lambda_{n-1}, 0]$-case. Recall that the direct summands of $V^{\otimes r} \otimes (V^\vee)^{\otimes s}$ are called mixed tensors. The maximal atypical mixed tensors are parametrized by partitions λ satisfying $k(\lambda) \leq n$ for an integer $k(\lambda)$ defined in [BS12b, 6.17] [Hei14, Section 4]. We furthermore recall from [Hei14, Theorem 12.3]: For every such $[\lambda]$ the mixed tensor $R(\lambda)$ contains $[\lambda]$ with multiplicity 1 in the middle Loewy layer. $[\lambda]$ is the constituent of highest weight of $R(\lambda)$. If we define $deg\ [\lambda] = \sum_{i=1}^n \lambda_i$, then $[\lambda]$ has larger degree then all other constituents. We denote the degree of a partition by $|\lambda|$. We recall further: If λ and μ are two partitions of length $\leq n$, the tensor product $R(\lambda) \otimes R(\mu)$ splits in $\mathcal{R}_n$ as

$$R(\lambda) \otimes R(\mu) = \bigoplus_{|\nu|=|\lambda|+|\mu|, k(\nu) \leq n} (c^\nu_{\lambda\mu})^2 R(\nu) \oplus \bigoplus_{|\nu|<|\lambda|+|\mu|, k(\nu) \leq n} d^\nu_{\lambda\mu} R(\nu)$$

for some coefficients $d^{\nu}_{\lambda\mu} \in \mathbb{N}$, the Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$ and the invariant $k(\lambda)$ [Hei14, Lemma 14.4].

Proposition B.2. *Let $[\lambda]$ be positive of degree r. Then $R(\lambda)$ occurs as a direct summand with multiplicity 1 in a tensor product $\mathbb{A} \otimes R(\lambda_i)$ where $l(\lambda_i) \leq n$, $|\lambda_i| = r - 1$ and $R(\lambda_i)$ is a direct summand in $\mathbb{A}^{\otimes r-1}$. The constituent $[\lambda]$ occurs with multiplicity 1 as a composition factor in the tensor product $\mathbb{A} \otimes R(\lambda_i)$.*

Proof. For λ, μ of length $\leq n$ we know that

$$R(\lambda) \otimes R(\mu) = \bigoplus_{|\nu|=|\lambda|+|\mu|, k(\nu)\leq n} (c^{\nu}_{\lambda\mu})^2 R(\nu) \oplus \tilde{R}$$

where $\tilde{R}$ are the terms of lower degree. We apply this for $\lambda = \mu = (1)$ (i.e. $\mathbb{A} \otimes \mathbb{A}$) and then to tensor products of the form $R(\lambda) \otimes \mathbb{A}$. Since every summand in a tensor product of the standard representation of $SL(n)$ with any other irreducible module has multiplicity 1, $\mathbb{A}^{\otimes r}$ decomposes as the standard representation of $SL(n)$ modulo contributions of lower degree and contributions of length $l(\nu) > n$. Since every irreducible $SL(n)$-representation with highest weight λ of degree $deg(\lambda) = \sum \lambda_i = r$ occurs as a summand in $st^{\otimes r}$, every mixed tensor $R(\lambda)$ with $l(\lambda) \leq n$ and $deg(\lambda) = r$ occurs as a direct summand in $\mathbb{A}^{\otimes r}$. Hence there exists in $\mathbb{A}^{\otimes r-1}$ a mixed tensor $R(\lambda_i)$ of length $\leq n$ and degree $deg(\lambda_i) = r - 1$ with

$$\mathbb{A} \otimes R(\lambda_i) = R(\lambda) \oplus \bigoplus R(\nu_i)$$

and $\nu_i \neq \lambda$ for all i. $R(\lambda)$ contains the composition factor $[\lambda]$ with multiplicity 1 and no other mixed tensor in this decomposition contains $[\lambda]$. Indeed if $deg(\nu_i) < r$, its constituent of highest weight has degree $< r$. If $R(\nu_i)$ has degree r and $l(\nu_i) \leq n$, its constituent of highest weight is $[\nu_i] \neq [\lambda]$ and if $R(\nu_i)$ has degree r and $l(\nu_i) > n$, its constituent of highest weight has degree $< r$ by [Hei14, Section 14]. $\square$

This applies in particular to positive $[\lambda]$ which are (Tannaka) self-dual. Every such $[\lambda]$ occurs as a multiplicity 1 constituent in a multiplicity 1 summand in a tensor product $\mathbb{A} \otimes R(\lambda_i)$ for $|\lambda_i| = r - 1$ which in turn appears as a multiplicity 1 summand in a tensor product $\mathbb{A} \otimes R(\lambda_{i_2})$ with $|\lambda_{i_2}| = r - 2$ etc.

Corollary B.3. *The selfdual representation $[\lambda] = [\lambda_1, \ldots, \lambda_{n_1}, 0]$ is even. Its parity shift $\Pi[\lambda]$ is odd.*

Proof: The parity is inherited to super tensor products (look at the even parts) and to multiplicity 1 summands. $\square$

If the representation $L(\lambda)$ is only selfdual up to a Berezin twist, we can simply restrict to $SL(n|n)$.

Appendix C. Technical lemmas on derivatives and superdimensions

C.1. Derivatives.

Lemma C.1. *Suppose L is a simple module and suppose the trivial module $\mathbf{1}$ is a constituent in $H^0_D(L)$, then $L \cong \mathbf{1}$.*

Proof. Suppose $H^0_D(L)$ contains $\mathbf{1}$ and suppose $L \ncong \mathbf{1}$. Then theorem 4.1 implies that L has two sectors with sector structure $[-n+2, ..., 0, 1, ..., n-1]S_1$ and $r(S_1) = 1$, hence

$$L \cong Ber \otimes S^i$$

for some $i \geq n-1$, or has sector structure $S_2[-n+2, ..., 0, 1, ..., n-1]$ with $r(S_2) = 1$ and hence

$$L \cong (Ber \otimes S^i)^\vee$$

for some $i \geq n-1$. However $H^\nu_D(Ber \otimes S^i) \cong Ber \otimes H^{\nu-1}_D(S^i)$ vanishes unless $\nu - 1 = 0$ with $H^0_D(S^i) = S^i$ or $\nu - 1 = i - (n-1) \geq 0$, as follows from the next lemma C.2. Hence this implies $H^0_D(Ber \otimes S^i) = 0$. Similarly then also $H^0_D((Ber \otimes S^i)^\vee) = 0$ holds by duality. This contradiction proves our claim. □

Lemma C.2. *Suppose $i \geq 1$. Then for S^i in $\mathcal{R}_n$ the cohomology is $H^\nu(S^i) = S^i$ for $\nu = 0$ and $H^\nu(S^i) = Ber^{-1}$ for $\nu = max(0, i-n+1)$, and $H^\nu(S^i)$ is zero otherwise.*

Proof. An easy consequence of theorem 4.1 and [HW14, Proposition 22.1]. □

The following lemma is an immediate consequence of theorem 4.1 or lemma C.1.

Lemma C.3. *$DS(L(\lambda))$ has a summand of superdimension 1 only if $L(\lambda) \cong Ber^r \otimes S^i$ for some r, i.*

Recall that an irreduble representation is weakly selfdual (or of type (SD)) if $L(\lambda)^\vee \cong Ber^r \otimes L(\lambda)$ for some $r \in \mathbf{Z}$.

Lemma C.4. *A (weakly) selfdual irreducible object $L = L(\lambda)$ with odd superdimension* $\mathrm{sdim}(L)$ *is a power of the Berezin determinant.*

Proof. For (weakly) selfdual maximal atypical irreducible objects $L = L(\lambda)$ of odd dimension their plot has sectors $S_1, \cdots S_k$ from left to right of lengths say $2r_1, ..., 2r_k$ that must satisfy

$$r_{k+1-i} = r_i$$

and hence in particular $r_1 = r_k$. By [Wei10][HW14] the superdimension is divisible by the multinomial coefficient $n!/(\prod_i r_i!)$ for $n = \sum_i r_i$. Hence, in case $k \geq 2$, the superdimension is divisible by the integer $(r_1 + r_k)!/(r_1! r_k!)$, which is $(2r_1)!/(r_1)!(r_1)!$ and hence *even*. Therefore $\mathrm{sdim}(L) \notin 2\mathbb{Z}$ implies $k = 1$, i.e. the associated plot only has a single sector. For this sector, we may continue with the same argument using the recursion formula for the superdimension given in [Wei10][HW14]. □

Lemma C.5. *Let $L(\lambda)$ be a maximal atypical weight with k sectors of rank $r_1, \ldots, r_k$ and derivatives $L(\lambda_j)$, $j = 1, \ldots, k$. Then for all $j = 1, .., k$*

$$\mathrm{sdim}(L(\lambda)) = \mathrm{sdim}(L(\lambda_j)) \cdot \frac{n}{r_j} \ .$$

Proof. By the superdimension formula [HW14]

$$\mathrm{sdim}(L(\lambda)) = \binom{n}{r_1, ..., r_k} \cdot T(S_1, ..., S_k)$$

for a term $T(S_1, ..., S_k)$ that only depends on the sektors S_j such that

$$T(S_1, ..., S_k) = T(S_1, ..., \partial S_j, ..., S_k) \ .$$

Since

$$\mathrm{sdim}(V_j) = \binom{n-1}{r_1, ..., r_j - 1, ..., r_k} T(S_1, ..., \partial S_i, ..., S_k) \ ,$$

this implies for all $j = 1, .., k$

$$\mathrm{sdim}(L(\lambda)) = \mathrm{sdim}(L(\lambda_j)) \cdot \frac{n}{r_j} \ .$$

□

C.2. **Small superdimensions.** According to lemma 8.1 a small representation belongs to one of four infinite families of regular cases or to a finite list of exceptional cases. The largest dimension occuring in the exceptional cases is 64 (the spin representations of D_7). Assume that V_λ restricted to G_λ splits as $V_\lambda = W_1 \oplus \ldots \oplus W_s$. We may assume $\dim(W_1) \leq \frac{1}{s}\dim(V_\lambda)$. The rank estimates in section 10.3 show that W_1 belongs to the regular cases of lemma 8.1 if $s \geq 3$. We therefore consider here the case where V_λ restricted to G_λ splits into at most two representations $V_\lambda = W \oplus W^\vee$. We want to rule out that W or $W^\vee$ is one of the exceptional cases. The dimension of W is $\dim(V_\lambda)/2$. Therefore we compute all superdimensions of irreducible weakly selfdual representations up to superdimension 128. Except for the numbers 20 and 56 none of the exceptional dimensions is equal to either the superdimension or half the superdimension of an irreducible weakly selfdual representation in $\mathcal{T}_n^+$.

Lemma C.6. *If $[\lambda]$ is a basic representation of $\mathcal{T}_n^+$, then $[\lambda, 0]$ is a basic representation of $\mathcal{T}_{n+1}^+$ of the same superdimension. Every basic representation of $\mathcal{T}_{n+1}^+$ with one sector is of this form.*

Therefore we can always assume that the irreducible representations have at least two sectors. Note also that a weakly selfdual representation cannot have an even number of sectors if n is odd. For a list of the basic representations in the case $n = 3$ and $n = 4$ we refer to the examples in section 13.

C.2.1. *Basic selfdual weights for* $n = 5$.

$[4, 3, 2, 1, 0]$,	sdim 120;	$[3, 3, 2, 0, 0]$,	sdim 30
$[4, 1, 1, 1, 0]$,	sdim 20;	$[1, 0, 0, 0, 0]$,	sdim 2
$[2, 1, 0, 0, 0]$,	sdim 6;	$[3, 2, 1, 0, 0]$,	sdim 24
$[2, 2, 0, 0, 0]$,	sdim 6;	$[3, 1, 1, 0, 0]$,	sdim 12

C.2.2. *Basic selfdual weights for* $n = 6$. By lemma C.6 we can focus on the case of two or more sectors. These basic weights are listed below.

$[5, 4, 3, 2, 1, 0]$,	sdim 720;	$[3, 3, 3, 0, 0, 0]$,	sdim 20
$[4, 3, 3, 1, 0, 0]$,	sdim 80;	$[5, 1, 1, 1, 1, 0]$,	sdim 30
$[4, 4, 2, 2, 0, 0]$,	sdim 120;	$[3, 3, 2, 2, 0, 0]$,	sdim 180
$[5, 3, 3, 1, 1, 0]$,	sdim 180;	$[0, 1, 2, 2, 3, 4]$,	sdim 360

C.2.3. *Basic selfdual weights for* $n = 7$. By lemma C.6 we can focus on the case of two or more sectors. Since n is odd, a weakly selfdual weight cannot have an even number of sectors. If the weight has ≥ 5 sectors, its superdimension exceeds 128. Therefore we list the basic SD weights with 3 sectors.

$[4, 4, 4, 3, 0, 0, 0]$,	sdim 140;	$[4, 4, 2, 2, 2, 0, 0]$,	sdim 210
$[6, 1, 1, 1, 1, 1, 0]$,	sdim 30;	$[6, 3, 3, 1, 1, 1, 0]$,	sdim 252
$[6, 4, 3, 2, 1, 1, 0]$,	sdim 1008		

C.2.4. *Basic selfdual weights for* $n \geq 8$. If the weight has 2 sectors for $n \geq 8$, then the biggest possible superdimension is $\geq n!/((n/2)!(n/2)!)$. This equals the case $[\lambda] = [n/2, n/2, \ldots, n/2, 0, 0, \ldots, 0]$ (each $n/2$ times). For $n = 8$ the superdimension is then 70, for $n = 9$ it is already 252. All other weights with 2 sectors have superdimension > 128.

If the weight has 3 sectors for $n \geq 9$, the smallest superdimension is given by the hook weight $[n-1, 1, \ldots, 1, 0]$ of superdimension $n(n-1)$. The next smallest superdimension is given by the irreducible representation $[n-1, 2, 1, \ldots, 1, 0]$ of superdimension $2 \cdot n(n-1)$. For $n = 8$ these superdimensions are 56 and 112. For $n \geq 9$ the second case has superdimension larger than 128. In the first case the superdimensions are 72 ($n = 9$), 90 ($n = 10$), 110 ($n = 11$) and exceed 128 otherwise.

If $n \geq 8$ and the weight has ≥ 4 sectors, its superdimension exceeds 128.

C.2.5. *Comparison with the exceptional cases.* We compare the superdimensions above with the dimensions of the exceptional cases in lemma 8.1. Except for the cases where the superdimension is 20 or 56 the dimensions are different. If the dimension is 20, then the irreducible representation is $\Lambda^3(st)$ for $SL(6)$. If the dimension is 56, then the irreducible representation is either $\Lambda^3(st)$ for $SL(8)$ or the irreducible representation of minimal dimension of E_7.

If V_λ or W is of the form $\Lambda^3(st)$, then so is its restriction $Res(W)$ to $G_{\lambda'}$ since Λ^3 commutes with Res, in contradiction to the induction assumption.

In the dim = 56-case with V_λ irreducible upon restriction to G_λ, the corresponding $L(\lambda)$ is the hook weight $[n-1, 2, 1, \ldots, 1, 0, \ldots, 0]$ for $n \geq 8$. For $n = 8$

$$DS(L(\lambda)) \cong Ber \otimes S^6 \oplus (Ber \otimes S^6)^\vee \oplus [7, 1, 1, 1, 1, 1, 0].$$

The connected derived Tannaka group of $[7, 1, 1, 1, 1, 1, 0]$ is either $SO(42)$, $Sp(42)$ or $SL(24)$ and doesn't embed into E_7. If $n \geq 9$, the hook weight $[n-1, 2, 1, \ldots, 1, 0, \ldots, 0]$ has one sector and therefore one derivative, hence the corresponding Tannaka group contains either $SO(42)$, $Sp(42)$ or $SL(24)$.

If V_λ decomposes as $W \oplus W^\vee$ and $\dim(W) = 56$, then $\dim(V_\lambda) = 112$. This happens for $L(\lambda) \cong [n-1, 2, 1, \ldots, 1, 0]$ for $n = 8$. In this case

$$DS(L(\lambda)) \cong [7, 2, 1, \ldots, 1] \oplus [7, 2, 1, \ldots, 1]^\vee.$$

Since this weight is NSD, its connected derived Tannaka group is $SL(56)$ which doesn't embed into E_7.

C.2.6. *The regular cases.* We can now assume that we are in one of the regular cases of lemma 8.1. If V is either $S^2(st)$, $S^2(st^\vee)$, $\Lambda^2(st)$, $\Lambda^2(st^\vee)$ or the nontrivial irreducible representation of $\Lambda^2(st)$ in the C_r-case, we get a contradiction to the induction assumption since restriction commutes with Schur functors. Therefore the representation is a standard representation or its dual for type A, B, C, D.

Corollary C.7. *If the selfdual irreducible representation V_λ is irreducible upon restriction to G_λ or splits in the form $W \oplus W^\vee$, the group G_λ is a simple group of type ABCD and V_λ and W are its standard representation (or its dual.)*

Appendix D. Clean decomposition

The ambiguity in the determination of G_λ is only due to the fact that we cannot exclude special elements with 2-torsion in $\pi_0(H_n)$. We show that $I \cong \mathbf{1}$ if $I \otimes I^\vee \simeq \mathbf{1} \oplus Proj$ holds. We then discuss the occurence of projective summands in tensor products of irreducible modules and show that $I \cong \mathbf{1}$ in some cases for $n = 4$.

D.1. **Endotrivial modules.** Our condition $I^{\otimes 2} \simeq \mathbf{1} \oplus N$ resembles the definition of an endotrivial representation.

Lemma D.1. *The following conditions are equivalent:*

(1) $I^{\otimes 2} \simeq \mathbf{1} \oplus Proj$.
(2) $DS(I) = \mathbf{1}$.

Proof. If $DS(I) = \mathbf{1}$, then $\mathrm{sdim}(I) = 1$, hence $I^{\otimes 2} \simeq \mathbf{1} \oplus negl$. But $ker(DS)$ (restricted to $\mathcal{T}_n^+$) is $Proj$. □

Modules M with the property $M \otimes M^\vee \simeq \mathbf{1} \oplus Proj$ are called endotrivial. If I satisfies $I^{\otimes 2} \simeq \mathbf{1} \oplus Proj$ or equivalently $DS(I) \simeq \mathbf{1}$, I is endotrivial (since $I^\vee \simeq I$).

Theorem D.2. [Tal15] *All endotrivial modules for $\mathcal{T}_n$ are of the form $Ber^j \otimes \Omega^i(\mathbf{1}) \oplus Proj$ or $\Pi(Ber^j \otimes \Omega^i(\mathbf{1})) \oplus Proj$ for some $i, j \in \mathbf{Z}$ where $\Omega^i(\mathbf{1})$ denotes the i-th syzygy of $\mathbf{1}$.*

We remark that we can split the projective resolution defining the $\Omega^i(M)$ into exact sequences

$$1 \to \Omega^i(M) \to P \to \Omega^{i-1}(M) \to 1$$

with some projective object P. It follows $\mathrm{sdim}(\Omega^i(M) = -\mathrm{sdim}(\Omega^{i-1}(M)$ since $\mathrm{sdim}(P) = 0$.

Lemma D.3. *If $I^{\otimes 2} \simeq \mathbf{1} \oplus Proj$ with I as above, then $I \simeq \mathbf{1}$.*

Proof. By restricting to $SL(n|n)$ we can ignore Berezin twists. By the classification of endotrivial modules $I \simeq \Pi^j \Omega^i(\mathbf{1}) \oplus Proj$ for some $i, j \geq 0$. Hence according to our list of properties of I

$$L \otimes \Pi^j \Omega^i(\mathbf{1}) \oplus Proj \cong L \oplus Proj.$$

On the other hand

$$\Omega^i(M) \otimes N \simeq \Omega^i(M \otimes N) \oplus Proj$$

holds for all N and i. Hence for $M \simeq \mathbf{1}$ we would have (using $\Pi\Omega^i(M)) = \Omega^i(\Pi M))$

$$L \otimes \Pi^i \Omega^i(\mathbf{1}) \simeq \Omega^i(\Pi^j L) \oplus Proj \simeq L \oplus Proj$$

which is absurd since $\Omega^i(\Pi^j L) \not\simeq L \oplus Proj$ for $i > 0$. In fact using the short exact sequences

$$1 \to \Omega^i(M) \to P \to \Omega^{i-1}(M) \to 1$$

and $DS(Proj) = 0$ we obtain

$$H^l(\Omega^i(L)) \simeq H^{l+i}(L).$$

Hence $i = 0$ and so $I \simeq \Omega^0(\mathbf{1}) \simeq \mathbf{1}$. □

D.2. **Clean decomposition.** We say a direct sum is *clean* if none of the summands is negligible. We say a negligible module N in $\mathcal{T}_n$ is potentially projective of degree r if $DS^{n-r}(N) \in \mathcal{T}_r$ is projective and $DS^i(N)$ is not for $i \leq n-r$.

Now consider the special representations S^i. Then we proved in [HW15] the surprising fact that the projection of $S^i \otimes S^j$ or $S^i \otimes (S^j)^\vee$ on the maximal atypical block is clean. To prove the result we establish the $n = 2$-case by a brute force calculation. The theory of mixed tensors [Hei14] then shows that the Loewy length of any summand in $S^i \otimes S^j$ is less or equal to 5. This implies the result since the Loewy length of a projective cover is $2n + 1$.

Lemma D.4. *Every maximal atypical negligible summand in a tensor product $L(\lambda) \otimes L(\mu)$ is potentially projective of degree at least 3.*

Proof. The decomposition of $S^i \otimes S^j$ in $\mathcal{R}_2$ is clean. Further DS sends negligible modules in $\mathcal{T}_n^+$ to negligible modules in $\mathcal{T}_{n-1}^+$ and the kernel of DS on $\mathcal{T}_n^+$ consists of the projective elements. Since $DS^{n-2}(L(\lambda) \otimes L(\mu)) \in \mathcal{T}_2$ splits into a direct sum of irreducible representations of the form $Ber^j S^i$ for some $i, j \in Z$ by the main theorem of [HW14], $DS^{n-2}(N) = 0$. □

We show below that the decomposition of the tensor product $L(\lambda) \otimes L(\mu)$ is also clean in the case $n = 3$ unless $\lambda_{basic} = \mu_{basic} = (2, 1, 0)$ and that projective summands can occur only under strong restrictions in the case $n = 4$.

Question. Let $L(\lambda),\ L(\mu)$ be maximal atypical. Is the projection of the decomposition of $L(\lambda) \otimes L(\mu)$ on the maximal atypical block always clean?

An affirmative answer would immediately imply $I \simeq \mathbf{1}$. To prove that decompositons are always clean, it would be enough to prove that the tensor product of two irreducible maximally atypical representations never contains a maximally atypical projective summand since repeated applications of DS to a negligible representation results in a direct sum of projective representations. A positive answer to this question would also imply that the tensor product decomposition of two maximal atypical irreducible representations behaves classically after projection to the maximal atypical block (and not just modulo vanishing superdimension).

Example D.5. If I is a direct summand in $[2, 2, 0, 0]^{\otimes 2}$ as above, then $I \cong \mathbf{1}$. This follows from the S^i-computations. Consider $L = [2, 2, 0, 0] \in \mathcal{R}_n$. Then $DS(I) \simeq \mathbf{1}$. In fact $DS(L) = [2, 2, 0] + B^{-1}S^3$. Hence $DS(L) \otimes DS(L)$ is a tensor product involving only S^i's and their duals (or their Berezin twists). Their decomposition is clean according to [HW15] (or use appendix E). Hence any negligible module in $[2, 2, 0, 0] \otimes [2, 2, 0, 0]$ maps to zero under DS. In particular $DS(I) = \mathbf{1}$ and hence $I \simeq \mathbf{1}$.

Appendix E. The depth of a representation

Due to the results of appendix D it is important to know when a maximal atypical projective module P occurs in a tensor product of two irreducible modules $L(\lambda), L(\mu)$. Some conditions can be obtained from a restriction of $L(\lambda)$ and $L(\mu)$ to $\mathfrak{g}_0$ as we show in this appendix.

E.1. **Depths.** The restriction of a maximally atypical module L of $\mathfrak{g} = \mathfrak{gl}(n|n)$ to the classical subalgebra $\mathfrak{g}_0$ decomposes completely into a direct sum of irreducible $\mathfrak{g}_0$-modules. We write $\rho \boxtimes \tilde{\rho}$ for these. An irreducible representation ρ of $\mathfrak{gl}(n)$ is described by a highest weight vector $(\lambda_1, ..., \lambda_n)$ with $\lambda_1 \geq ... \geq \lambda_n$, and we define the degree $deg(\rho)$ of ρ to be the sum $\sum_{i=1}^n \lambda_i$.

- Let a be the maximal degree $deg(\rho)$ for all $\rho \boxtimes \tilde{\rho}$ in the restriction of $L(\lambda)$ to $\mathfrak{g}_0$, and
- let b be the minimal degree $deg(\rho)$ for all $\rho \boxtimes \tilde{\rho}$ in the restriction of $L(\lambda)$ to $\mathfrak{g}_0$.

Define the **depth** to be $depth(L) = a - b$ or

$$depth(L) = deg(\text{highest } \mathfrak{g}_0\text{-weight of } L) - deg(\text{lowest } \mathfrak{g}_0\text{-weight of } L)$$

We often write $depth(\lambda)$ for $depth(L(\lambda))$. Rather obviously we have

$$depth(A \otimes B) = depth(A) + depth(B) \ .$$

and $A \hookrightarrow B$ implies $depth(A) \leq depth(B)$. Two remarks are in order:

- For every weight μ of the Cartan algebra in the representation space of the irreducible representation space of $\mathfrak{gl}(n)$ defined by the highest weight ρ the degrees $deg(\mu) = deg(\rho)$ (defined as above) coincide.
- For any irreducible representation $\rho \boxtimes \tilde{\rho}$ of $\mathfrak{g}_0$ in the restriction of an irreducible max. atyp. representation L of $\mathfrak{g}$ one has $deg(\tilde{\rho}) = -deg(\rho)$.

If we consider the restriction of $L = L(\lambda)$, the maximal degree $deg(\rho)$ for all $\rho \boxtimes \tilde{\rho}$ in the restriction is $a = \sum_{i=1}^n \lambda_n = deg(L)$. One easily shows

$$depth(L) = a - b = deg(L) - (-deg(L^\vee)) = deg(L) + deg(L^\vee) \ .$$

For any highest weight submodule $W = W(\tau) \hookrightarrow L \otimes L^\vee$ we therefore get

$$deg(\tau) \leq depth(L) \ .$$

Indeed $deg(\tau) \leq deg(L) + deg(L^\vee)$; here $deg(L)$ denotes the degree of the highest weight of L. We also conclude

$$depth(L(\tau)) \leq 2 \cdot depth(L) \ .$$

If we consider $W = L \otimes L^\vee$ for $L = L(\lambda)$, then the highest weight in W has degree $deg(\lambda) + deg(\lambda^\vee) = depth(\lambda) = depth(L)$. Since $depth(W) =$

$depth(L) + depth(L^\vee) = 2depth(L)$, therefore all weights in W have degrees within

$$[-depth(L), depth(L)] \ .$$

The weights λ^0 and λ^c. We recall the definition of the weight λ^0 attached to λ from [BS10a]. In the weight diagram of λ add n cups to the diagram by repeatedly connecting $\wedge\vee$-pairs of vertices that are neighbours in the sense that there are no vertices in between not yet connected to cups. Then λ^0 is the weight whose associated cup diagram is the cup diagram just constructed.

Example E.1. If $depth(L) = n(n-1)$, then $L(\lambda^0) = L(\lambda) \otimes Ber^{-1}$.

Example E.2. If $depth(L) = 0$, then $L(\lambda^0) = L(\lambda) \otimes Ber^{-n}$.

The assignment of weights

$$\lambda \mapsto \lambda^0$$

has a unique inverse

$$\mu \mapsto \mu^c \ ,$$

where μ^c is obtained from μ by a total left move (in the language of cup diagrams). Hence μ^c is the weight attached to the complementary plot of the plot corresponding to μ (for the notion complementary plot we refer to [HW14]; i.e. one passes from the plot to the complements in each sector of the plot).

Lemma E.3. *$depth(L) = deg(L) + deg(L^\vee) = depth(L_{basic}) = 2deg(L_{basic})$ holds for irreducible $\mathfrak{g}$-modules L.*

Proof. Consider the Kac module $V(\rho_\lambda)$ or $V(\lambda)$ for short. Its irreducible socle (as a $\mathfrak{g}$-module) is $L(\lambda^0)$; see [BS10a, Theorem 6.6]. The restriction of $L(\lambda)$ to $\mathfrak{g}_0$ contains the weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ of maximal degree, whereas the restriction of λ^0 to $\mathfrak{g}_0$ contains the weight

$$\tau = \lambda \otimes det^{-n}$$

where τ is the minimal highest weight of $\mathfrak{g}_0$ in the restriction of $L(\lambda^0)$. We also write $(\lambda^0)_{min}$ instead of $\tau = \lambda \otimes det^{-n}$. Indeed the lowest $\mathfrak{g}_0$-representation τ in $V(\lambda)|_{\mathfrak{g}_0} \cong \rho_\lambda \otimes \Lambda^\bullet(g/p)$ is $\rho_\lambda \otimes \Lambda^{n^2}(g/p)$, that corresponds in our notation to the representation $\rho_\lambda \otimes det^{-n}$. We conclude $deg(\lambda) - deg(\lambda^0) + depth(\lambda^0) = deg(det^n) = n^2$. In terms of $\mu = \lambda^0$, we conclude from the above arguments

Lemma E.4. *The maximal degree resp. minimal degree for the highest $\mathfrak{g}_0$-weights of the restriction of the irreducible $\mathfrak{g}$-module $L(\mu)$ are the degrees of the extremal highest weights μ resp. $\mu_{min} = \mu^c - (n, ..., n)$ and*

$$depth(\lambda) = deg(\lambda) - deg(\lambda^c - (n, ..., n)) = deg(\lambda) - deg(\lambda^c) + n^2 \ .$$

Furthermore μ_{min} is the unique irreducible $\mathfrak{g}_0$-constituent in $L(\mu)$ of minimal degree.

Note that $\deg(\lambda)$ is the same as $S(\lambda)+n(n-1)/2$ where $S(\lambda)$ is the sum of the points x in the support of the plot $\lambda(x)$. Since $S(\lambda)-S(\lambda^c)$ only depends on the associated basic weight λ_{basic}, we find that

$$depth(\lambda) = depth(\lambda_{basic}) \ .$$

Since $\lambda^\vee = \lambda^*$, we have $depth(\lambda) = deg(\lambda) + deg(\lambda^*)$ for basic weights $\lambda = \lambda_{basic}$. Since $deg(\lambda^*) = deg(\lambda)$, we obtain for basis weights $\lambda = \lambda_{basic}$ the formula

$$depth(\lambda_{basic}) = 2 \cdot deg(\lambda_{basic}) \ .$$

This proves lemma E.3. □

Corollary E.5. *For all weights $\lambda = (\lambda_1, ..., \lambda_n)$ we have*

$$0 \leq depth(\lambda) \leq n(n-1) \ .$$

Proof. Obvious from $deg(\lambda_{basic}) \leq deg((n-1,...,1,0)) = n(n-1)/2$. □

Note that $V(\lambda)^\vee$ again is a Kac-module whose cosocle now is the dual $L(\lambda^0)^\vee$ of $L(\lambda^0)$. Hence $V(\lambda)^\vee = V((\lambda^0)^\vee)$. Its highest weight is the highest weight $(\lambda^0)^\vee$ of $L((\lambda^0)^\vee)$.

We also remark that $Ber^{-n}V(\lambda^\vee)^\vee = V(\lambda_{min})$ for $L(\lambda^\vee) := L(\lambda)^\vee$ and $\lambda_{min} = \lambda^c det^{-n}$. Indeed we can replace λ by λ^0. Then $V(\lambda^\vee)^\vee$ becomes $V(\lambda)$ and λ_{min} becomes $\tau = \lambda det^{-n}$.

E.2. **Projectives in tensor products.** Let L' and L'' be irreducible $\mathfrak{g} = \mathfrak{gl}(n|n)$-modules that are maximal atypical. Let us assume that P is an irreducible projective maximal atypical module with the property

$$P \subseteq L' \otimes L'' \ .$$

We assume $L' = L(\rho')$ resp. $L'' = L(\rho'')$. In P there exists a Kac module $V = V(\rho) \subset P$ whose socle $L(\tau) = L(\rho^0)$ is the socle of $P = P(\tau)$. Its precise structure will be not important at the moment, except that the anti-Kac module $V(\tau)^*$ is also a g-submodule of P with cosocle $L(\tau^0)$. Indeed, $V(\tau)$ is a quotient module of $P(\tau)$ and hence $V(\tau)^*$ is a submodule of $P(\tau)^* \cong P(\tau)$. Consider the inclusion

$$i : V = V(\rho) \hookrightarrow L(\rho') \otimes L(\rho'')$$

and its restriction to the subalgebra $\mathfrak{g}_0 = \mathfrak{gl}(n) \times \mathfrak{gl}(n)$ and similarly for $V(\tau)^*$.

As a representation of $\mathfrak{g}_0$ the module $V' = V(\rho')$ becomes

$$V'|_{\mathfrak{g}_0} \cong \rho' \otimes W^\bullet$$

where $W^\bullet = \bigoplus_\mu \rho_{-\mu} \boxtimes \rho_{-\mu^*}$ holds for the irreducible representations ρ_μ of $\mathfrak{gl}(n)$ with highest weights $\mu = (\mu_1, .., \mu_n)$ running over all μ such that $n \geq \mu_1 \geq ... \geq \mu_n \geq 0$; here μ^* denotes the weight with the transposed Young diagram of the weight μ. In particular, the degree $deg(\mu) = \sum_{i=1}^n \mu_i$

varies between 0 and n^2. In particular, in the degree grading of $W^\bullet$ we have $deg(W^i) = -i$ and

$$L'|_{\mathfrak{g}_0} \subseteq \rho' \otimes \bigotimes_{i=0}^{depth(L')} W^i .$$

Similarly for L'' instead off L'.

The projective module $P = P(\tau)$ contains the irreducible $\mathfrak{g}$-modules $L(\rho)$, $L(\tau) = L(\rho^0)$ and $L(\tau^0) = L((\rho^0)^0)$.

$$\begin{tikzcd}
& & P(\tau) \arrow[dl, dash] \arrow[dr, dash] & & \\
L(\tau^0) \arrow[d, dotted, dash] & V(\tau)^* \arrow[l, two heads] \arrow[dr, dash] & & V(\rho) \arrow[r, two heads] \arrow[dl, dash] & L(\rho) \\
\sigma & & L(\tau) = L(\rho^0) \arrow[d, dotted, dash] & & \\
& & \rho \otimes det^{-n} & &
\end{tikzcd}$$

The restriction of P to $\mathfrak{g}_0$ contains then the irreducible $\mathfrak{g}_0$-modules

$$\rho, \tau, \rho det^{-n}, \tau^0, \sigma$$

for the lowest $\mathfrak{g}_0$-weight $\sigma = \tau \otimes det^{-n}$ of τ^0. Note that $\sigma = (\tau^0)^c \otimes det^{-n} \cong \tau \otimes det^{-n}$. Furthermore

$$deg(\sigma) = deg(\tau^0) - depth(\tau^0)$$

$$deg(\tau) - depth(\tau) = deg(\rho) - n^2$$

and so

$$deg(\tau) - deg(\tau^0) = n^2 - depth(\tau^0) \ .$$

These equations imply

$$depth(P) = deg(\rho) - deg(\sigma) = 2n^2 - depth(\tau) \geq n(n+1) \ .$$

Note that $P \hookrightarrow L \otimes L^\vee$ implies $depth(P) \leq depth(L) + depth(L^\vee) = 2depth(L)$, and hence

$$2n^2 \leq depth(L(\tau)) + 2depth(L) \ .$$

In particular, we then obtain

$$2n \leq depth(L(\tau)) \quad , \quad n(n+1)/2 \leq depth(L) \ .$$

From the above we get

Proposition E.6. *If the tensor product of maximally atypical irreducible $\mathfrak{g}$-modules $L(\rho')$ and $L(\rho'')$ contains a maximal atypical projective module $P = P(\tau)$, then the irreducible $\mathfrak{gl}(n)$- representations defined by τ, τ^0 and $\tau^c = \rho$ and $\tau \otimes det^{-n} = \sigma$ and $\tau^c \otimes det^{-n} = \rho \otimes det^{-n}$ are constituents of*

$$\rho' \otimes \rho'' \otimes \bigoplus_{i=0}^{depth(\rho')} W^i \otimes \bigoplus_{j=0}^{depth(\rho'')} W^j$$

and

$$(\rho')^c \otimes (\rho'')^c \otimes det^{-2n} \otimes \bigoplus_{i=0}^{depth(\rho')} (W^i)^{dual} \otimes \bigoplus_{j=0}^{depth(\rho'')} (W^j)^{dual} \ .$$

Since the degrees in this tensor product are between $deg(\rho') + deg(\rho'')$ and $deg(\rho') + deg(\rho'') - depth(\rho') - depth(\rho'')$, in the situation of the last proposition the following holds

$$deg(\rho) - deg(\sigma) \leq depth(\rho') + depth(\rho'') \ .$$

Hence we get

Corollary E.7. *$L(\sigma') \otimes L(\sigma'')$ can not contain a projective maximal atypical g-module unless*

$$deg(\rho'_{basic}) + deg(\rho''_{basic}) \geq n(n+1)/2 \ .$$

E.3. **The case** $n = 3$**.** . Let us assume first $n = 3$. Here the condition $deg(\rho'_{basic}) + deg(\rho''_{basic}) \geq 6$ may be only satisfied for $\rho'_{basic} = \rho''_{basic} = (2, 1, 0)$. Consider $P(\tau) \subseteq L' \otimes L''$. From depth comparison with projective modules we get $depth(P(\tau)) = 2n^2 - depth(\tau) \leq depth(L') + depth(L'')$. Hence $depth(\tau) \geq 6$ and therefore $depth(\tau) = 6$ resp. $\tau_{basic} = (2, 1, 0)$. Hence

$$depth(P(\tau)) = 2n^2 - depth(\tau) = 18 - 6 = 6 + 6 = depth(L') + depth(L'') \ .$$

This implies that the highest weight of $L' \otimes L''$ must have the same degree as the highest weight of $P(\tau)$, i.e.

$$deg(\lambda' + \lambda'') = deg(\rho),$$

where $\rho^0 = \tau$. Note that ρ cannot be the highest weight $\lambda' + \lambda''$ because of the next lemma.

Lemma E.8. *The highest weight constituent in a tensor product of two irreducible representations $L(\lambda') \otimes L(\lambda'')$ is never contained in a projective module.*

Proof. We can assume that $L(\lambda') = [\lambda'_1, \ldots, \lambda'_{n-1}, 0]$ and likewise for $L(\lambda'')$. Then the result follows from [Hei14, corollary 13.11]. $\square$

Corollary E.9. *For $n = 3$ a projective maximal atypical module P can not be contained in the tensor product $L' \otimes L''$ unless $L'_{basic} = L''_{basic} = [2, 1, 0]$.*

Remark E.10. We note that this implies that $I \cong \mathbf{1}$ for I as above if I is a direct summand in $L(\lambda) \otimes L(\mu)$ with $\lambda_{basic} = \mu_{basic} = (2,2,0,0)$. Indeed none of the direct summands of $DS([2,2,0,0])$ is of basic type $(2,1,0)$.

Example E.11. A brute force computations shows that $R(2,1)^{\otimes 2}$ only contains $R(3,2,1) = P[2,1,0]$ as a maximal atypical projective summand. Since $[2,1,0]^{\otimes 2}$ is a subquotient of $R(2,1)^{\otimes 2}$ this means that the only possible maximal atypical projective summand in $[2,1,0]^{\otimes 2}$ is $P[2,1,0]$.

Appendix F. Determinants

F.1. **Determinants in Deligne categories.** Following Deligne [Del07] we define for $\delta \in \mathbf{Z}$ the following triples (G, ϵ, X) where G is a supergroup, ϵ an element of order 2 such that $int(\epsilon)$ induces on $\mathcal{O}(G)$ its grading modulo 2 and $X \in \mathrm{Rep}(G, \epsilon)$:

- $\delta = m \geq 0 : \ (O(m) = OSp(m|0), id, V)$, ($V$ will always denote the standard representation)[2]
- $\delta = -2n < 0 : \ (Sp(2n) = OSp(0|2n), -id, V$ seen as odd),
- $\delta = 1 - 2n < 0 : \ (OSp(1|2n), diag(1, -1, \ldots, -1), V)$.

By the universal property [Del07, Proposition 9.4], the assignment $R(1) \mapsto X$ (see below for notation) defines a tensor functor $F_\delta : \underline{\mathrm{Rep}}(O_\delta) \to \mathrm{Rep}(G, \epsilon)$.

Theorem F.1. [Del07, Théorème 9.6]. *The tensor functor F_δ induces an equivalence of categories*

$$\underline{Rep}(O_\delta)/\mathcal{N} \to \mathrm{Rep}(G, \epsilon)$$

where $\mathcal{N}$ denotes the tensor ideal of negligible morphisms.

Recall that the indecomposable objects in $\underline{\mathrm{Rep}}(O_\delta)$ are classified by partitions and we denote by $R(\lambda)$ the corresponding indecomposable object. Then $R(0) = \mathbf{1}$ and $R(1)$ is the distinguished element in Deligne's category (the analogue of the standard representation). For an object X of dimension $d \neq 0$ we write $det(X)$ for $\Lambda^d(X)$.

Theorem F.2. *The following hold in $\underline{Rep}(O_\delta)$ for $\delta \in 2\mathbf{Z}$:*

$$Sym^{|\delta|}(R(1)) = \mathbf{1} \oplus \ \textit{negligible if } \delta < 0,$$
$$\Lambda^\delta(R(1)) = J \oplus \ \textit{negligible if } \delta > 0$$

for indecomposable $J \neq \mathbf{1}$ of dimension 1 such that $J^{\otimes 2} \cong \mathbf{1}$.

Proof. For $\delta < 0$ $\underline{\mathrm{Rep}}(O_\delta)/\mathcal{N} \cong Rep(Sp(\delta))$. In the latter we have $\Lambda^{|\delta|}(V) \cong \mathbf{1}$. Since $\Lambda^{|\delta|}(V) \cong Sym^{|\delta|}(\Pi V)$ and $F_\delta(R(1)) \cong \Pi V$, we obtain

$$F_\delta(Sym^{|\delta|}(R(1)) = \mathbf{1}.$$

[2]For the case $\delta = 0$, $O(0)$ is the trivial group, $V = 0$, and $\mathrm{Rep}(G, \epsilon)$ is equivalent to the category of finite dimensional k-vector spaces.

Since F_δ induces a bijection between the isomorphism classes of non-negligible indecomposable elements in $\underline{\mathrm{Rep}}(O_\delta)$ and the isomorphism classes of irreducible elements in the quotient, this implies

$$Sym^{|\delta|}(R(1)) = R(0) \oplus \text{ negligible}.$$

For $\delta > 0$ $\underline{\mathrm{Rep}}(O_\delta)/\mathcal{N} \cong Rep(O(\delta))$ and $R(1)$ maps to the (even) standard representation V. The classical result

$$\Lambda^\delta(V) \cong J \quad (\text{ sign representation})$$

implies now as in the $\delta < 0$-case

$$\Lambda^\delta(V) \cong J \oplus \text{ negligible}$$

where $J = R(.)$ is indecomposable, $J \neq \mathbf{1}$, $J^{\otimes 2} \cong \mathbf{1} \oplus$ negligible. □

Theorem F.3. *In $\underline{Rep}(Gl_\delta)$, $\delta \in \mathbf{Z}_+$*

$$\Lambda^\delta(R(1)) \cong R \oplus \textit{ negligible}$$

for some indecomposable $R \neq \mathbf{1}$ of dimension 1. If we denote its dual $R^\vee$ by R^{-1} and the i-fold tensor product $R \otimes R \otimes \ldots \otimes R$ by R^i and likewise for R^{-1}, we obtain

$$R^i \otimes R^j \cong R^{i+j} \oplus \textit{ negligible}$$

for all $i, j \in \mathbf{Z}$ (using the convention $R^0 := \mathbf{1}$).

Proof. Use the functor $F_\delta : \underline{\mathrm{Rep}}(Gl_\delta) \to Rep(Gl(\delta))$. Now

$$F_\delta(\Lambda^\delta(R(1)) = \Lambda^\delta(V) = det = L(1, \ldots, 1)$$

and the result follows from the usual properties of the determinant as in the $\underline{\mathrm{Rep}}(O_\delta)$-case. □

F.2. **Determinants in $\mathcal{T}_n$.** We apply this now to compute the top exterior power $det(L) := \Lambda^\delta(L)$ where L is irreducible in $\mathcal{T}_n^+$ and $\mathrm{sdim}(L) = \delta$. We use the following notation: With $\mathcal{S}_n$ and $\mathcal{S}_n^+$ we denote the analogous categories to $\mathcal{T}_n$ and $\mathcal{T}_n^+$ in the $SL(n|n)$-case.

If L is irreducible in $\mathcal{T}_n$ and F_L is the corresponding functor from the Deligne category to $\mathcal{T}_n$, we can typically say very little about $F_L(X)$ for some indecomposable element X, even if $\dim(X) = 1$. The situation changes if we know that the image of F_L is in $\mathcal{T}_n^+$! The *positivity* of the latter implies then in case $\dim(X) = 1$ that

$$F_L(X) = R \oplus \text{ negligible}$$

where R is indecomposable with $sdim(X) = 1$.

F.2.1. *Case 1: odd SD-case.* Consider $L = L(\lambda)$ irreducible and maximal atypical such that $L^\vee \cong L \otimes Ber^r$ in $\mathcal{T}_n$ for some $r \in \mathbf{Z}$ and therefore $L^\vee \cong L$ in $\mathcal{S}_n$. The symmetric pairing

$$e : L^\vee \otimes L \to \mathbf{1}$$

commutes with the symmetry-constraint of S_n. This defines a functor

$$F = F_L : \underline{\mathrm{Rep}}(O_\delta) \to \mathcal{S}_n$$

such that $F(R(1)) = L$ for $\delta = \mathrm{sdim}(L)$. Now apply F to

$$Sym^{|\delta|}(R(1)) = R(0) \oplus \text{ negligible if } \delta < 0.$$

Then $F_L(Sym^{|\delta|}(R(1))) = Sym^{|\delta|}(L)$. Since $\mathrm{sdim}(L) < 0$ we replace it by its parity shift ΠL to be in S_n^+. Using $Sym^{|\delta|}(L) = \Lambda^{|\delta|}(\Pi L)$, $R(1) \mapsto L$ and $Sym^{|\delta|}(R(1)) = \mathbf{1} \oplus$ negligible we obtain

$$\Lambda^\delta(\Pi L) \cong \mathbf{1} \oplus \text{ negligible in } \mathcal{S}_n$$

and therefore

$$\Lambda^\delta(\Pi L) \cong Ber^r \oplus \text{ negligible in } \mathcal{T}_n.$$

Remark F.4. If L is proper selfdual $L \cong L^\vee$, then we can actually consider the functor $F_L : \underline{\mathrm{Rep}}(O_\delta) \to \mathcal{T}_n^+$ instead to $\mathcal{S}_n^+$. Then we obtain at once $\Lambda^{|\delta|}(L) \cong \mathbf{1} \oplus$ negligible (and not just some Berezin power).

Remark F.5. Note that the specific Berezin power was computed in section 12.

F.2.2. *Case 2: even SD-case.* If L has positive superdimension it is even. As before we compute

$$\begin{aligned} \Lambda^\delta(L) &\cong \Lambda^\delta(F_L(R(1))) \\ &\cong F_L(\Lambda^\delta(R(1))) \\ &\cong F_L(J \oplus \text{ negligible}) \\ &\cong J \oplus \text{ negligible} \end{aligned}$$

where J in $\mathcal{T}_n^+$ is indecomposable of superdimension 1 and $J^{\otimes 2} \cong \mathbf{1}$ ($J \neq \mathbf{1}$ might not hold in S_n). To explain the notation: $F_L(J)$ is the direct sum of an indecomposable module of superdimension 1 (called again J by abuse of notation) and a bunch of negligible ones. In the last step we used again the positivity of superdimensions in $\mathcal{S}_n^+$.

F.2.3. *Case 3: NSD-case.* Let L be of type NSD. Without loss of generality we consider the case $\mathrm{sdim}(L) > 0$. We denote by F_L the functor

$$F_L : \underline{\mathrm{Rep}} Gl_\delta \to \mathcal{T}_n \text{ or } \mathcal{T}_n^+.$$

Then we compute

$$\begin{aligned}\Lambda^{\delta}(L) &\cong \Lambda^{\delta}(F_L(R(1)))\\ &\cong F_L(\Lambda^{\delta}(R(1)))\\ &\cong F_L(R\oplus \text{ negligible})\\ &\cong R\oplus \text{ negligible }.\end{aligned}$$

To explain the notation: $F_L(R) \cong R\oplus$ negligible where R denotes again (by abuse of notation) an indecomposable module of superdimension 1. The properties of R (as an element in $\underline{\mathrm{Rep}}(Gl_{\delta})$) from theorem F.3 carry over. Here we use the following notation: $F_L(R^i)$ is a direct sum of an indecomposable module of superdimension 1 and a direct sum of negligible modules. The summand of superdimension 1 will again be called R^i. Then

(1) R^i is indecomposable in $\mathcal{T}_n^+$ of superdimension 1 for any i.
(2) $(R^i)^{\vee} = R^{-i}$.
(3) R^i is *-dual.
(4) $R^i \otimes R^j \cong R^{i+j} \oplus$ negligible.
(5) If we assume by induction that determinants are given by Berezin powers, we obtain also $DS(R^i) = \Pi^s Ber^i \oplus$ negligible for some $s \in \mathbf{Z}$.

Remark F.6. A priori R might be trivial (even though it is not in the Deligne category). If we know already that the determinant is nontrivial in $\mathcal{T}_{n-1}$, then R has to be nontrivial as well.

Remark F.7. We remark that $DS(R) = \Pi^s Ber^i$ would imply that $R \cong \Pi^s Ber$ using the classification of endotrivial modules. For this we would restrict to $\mathcal{S}_n$ and obtain $DS(R) \cong \mathbf{1}$. Then use $R\otimes R^{\vee} \cong \mathbf{1}\oplus negligible$. Since $ker(DS) = Proj$ the negligible part must therefore be projective, hence R is endotrivial. The endotrivial modules in $\mathcal{S}_n$ are of the form $\Pi^s\Omega^i(\mathbf{1})$ for some $i, s \in \mathbf{Z}$. These modules are not $*$-invariant unless $i = 0$.

References

[AP95] Henning Haahr Andersen and Jan Paradowski, *Fusion categories arising from semisimple Lie algebras.*, Commun. Math. Phys., 169, 1995.

[AK02] Yves André and Bruno Kahn, *Nilpotence, radicaux et structures monoïdales.* Rend. Semin. Mat. Univ. Padova, 108, 2002.

[AK05] Yves André and Bruno Kahn, *Erratum: nilpotence, radicaux et structures monoïdales.*, Rend. Semin. Mat. Univ. Padova, 113, 2005.

[AK02] Yves André and Bruno Kahn, *Construction inconditionnelle de groupes de Galois motiviques.*, C. R., Math., Acad. Sci. Paris, 334, 2002.

[AVE67] E.M. Andreev and È.B. Vinberg and A.G. Èlashvili, *Orbits of greatest dimension in semisimple linear Lie groups.*, Funkts. Anal. Prilozh., 1, 4, 1967.

[BK01] B. Bakalov and A.jun. Kirillov, *Lectures on tensor categories and modular functors.*, 2001.

[BKN09a] Boe, B. D. and Kujawa, J. R. and Nakano, D. K., *Complexity and module varieties for classical Lie superalgebras*, Int. Math. Res. Not., 2011 3, 2011.

[BKN09b] Boe, Brian D. and Kujawa, Jonathan R. and Nakano, Daniel K., *Cohomology and support varieties for Lie superalgebras. II.*, Proc. Lond. Math. Soc. (3), 98, 1, 2009.

[BKN10] Boe, Brian D. and Kujawa, Jonathan R. and Nakano, Daniel K., *Cohomology and support varieties for Lie superalgebras.*, Trans. Am. Math. Soc., 362, 12, 2010.

[Bru03] Brundan, Jonathan, *Kazhdan-Lusztig polynomials and character formulae for the Lie superalgebra* $GL(m|n)$., J. Am. Math. Soc., 16, 1, 2003.

[BS10a] Brundan, Jonathan and Stroppel, Catharina, *Highest weight categories arising from Khovanov's diagram algebra. II: Koszulity.*, Transform. Groups, 15, 1, 2010.

[BS11] Brundan, Jonathan and Stroppel, Catharina, *Highest weight categories arising from Khovanov's diagram algebra. I: cellularity.*, Mosc. Math. J., 11, 4, 2011.

[BS12a] Brundan, Jonathan and Stroppel, Catharina, *Highest weight categories arising from Khovanov's diagram algebra. IV: the general linear supergroup.*, J. Eur. Math. Soc. (JEMS), 14, 2, 2012.

[BS12b] Brundan, J. and Stroppel, C., *Gradings on walled Brauer algebras and Khovanov's arc algebra*, Adv. Math., 231, 2, 2012.

[Cl37] Clifford, A.H., *Representations induced in an invariant subgroup.*, Ann. Math. (2), 38, 1937.

[CW11] Comes, J. and Wilson, B., *Deligne's category* $Rep(GL_\delta)$ *and representations of general linear supergroups*, Represent. Theory, 16, (2012).

[Co18] Coulembier, K., *Tensor ideals, Deligne categories and invariant theory*, Sel. Math., New Ser., 24, 5, 2018.

[Del02] Deligne, P., *Catégories tensorielles. (Tensor categories).*, Mosc. Math. J. , 2, 2, 2002.

[Del90] Deligne, P., *Catégories tannakiennes. (Tannaka categories).*, The Grothendieck Festschrift, Collect. Artic. in Honor of the 60th Birthday of A. Grothendieck. Vol. II, Prog. Math. 87, 111-195 (1990)..

[DM82] Deligne, Pierre and Milne, J.S., *Tannakian categories.*, Hodge cycles, motives, and Shimura varieties, Lect. Notes Math. 900, 101-228 (1982)..

[Del07] Deligne, P., *La catégorie des représentations du groupe symétrique* S_t*, lorsque* t *n'est pas un entier naturel.*, Mehta, V. B. (ed.), Algebraic groups and homogeneous spaces. Proceedings of the international colloquium, Mumbai, India, January 6–14, 2004. 209-273 (2007)., 2007.

[Dro09] Drouot, Francois, *Quelques proprietes des representations de la super-algebre der Lie* $GL(m,n)$, PhD thesis, 2009.

[DS05] Duflo, M. and Serganova. V., *On associated variety for Lie superalgebras.*, arXiv:math/0507198v1, 2005.

[EO18] Etingof, P. and Ostrik, V., On semisimplification of tensor categories, ArXiv e-prints, 1801.04409, 2018.

[Ger98] Germoni, Jérôme, *Indecomposable representations of special linear Lie superalgebras.*, J. Algebra , 209, 2, 1998.

[GKPM11] Geer, Nathan and Kujawa, Jonathan and Patureau-Mirand, Bertrand, *Generalized trace and modified dimension functions on ribbon categories.*, Sel. Math., New Ser., 17, 2, 2011.

[GQS07] Goetz, Gerhard and Quella, Thomas and Schomerus, Volker, *Representation theory of* $\mathfrak{sl}(2|1)$., J. Algebra, 312, 2, 2007.

[Hei14] Heidersdorf, Thorsten, *On supergroups and their semisimplified representation categories*, to appear in Algebr. Represent. Theor. ArXiv e-prints, 1512.03420, 2015.

[Hei14] Heidersdorf, Thorsten, *Mixed tensors of the General Linear Supergroup*, J.Algebra, 491, 2017.

[HW14] Heidersdorf, Th and Weissauer, R., *Cohomological tensor functors on representations of the General Linear Supergroup* to appear in Mem. Am. Math. Soc., ArXiv e-prints, 1406.0321, 2014.

[HW15] Heidersdorf, T. and Weissauer, R., *Pieri type rules and $GL(2|2)$ tensor products* ArXiv e-prints, 1509.07443, 2015

[Ja92] Jannsen, U., *Motives, numerical equivalence, and semi-simplicity.*, Invent. Math., 107, 1992

[Kac78] Kac, V., *Representations of classical Lie superalgebras.*, Differ. geom. Meth. math. Phys. II, Proc., Bonn 1977, Lect. Notes Math. 676, 597-626 (1978)..

[KrW15] Krämer, Th. and Weissauer, R., *Semisimple super tannakian categories with a small tensor generator*, Pacific J. Math., 276, 2015

[Kuj11] Kujawa, J., *The generalized Kac-Wakimoto conjecture and support varieties for the Lie superalgebra $osp(m|2n)$*, Recent developments in Lie algebras, groups and representation theory. 2009-2011 Southeastern Lie theory workshop series (2012)

[Li13] Linckelmann, M. *Representation theory of finite-dimensional algebras*, `http://www.ltcc.ac.uk/courses/representation_theory/Notes/reptheory2013.pdf`, 2013

[MS11] Ian M. Musson and Vera Serganova, *Combinatorics of character formulas for the Lie superalgebra $\mathfrak{gl}(m,n)$.*, Transform. Groups, 16, no.2, 2011.

[Sch79] M. Scheunert, *The theory of Lie superalgebras. An introduction.*, Lecture Notes in Mathematics. 716. Berlin-Heidelberg-New York: Springer-Verlag. X, 271 p. (1979).

[Ser10] Serganova, V., *On the superdimension of an irreducible representation of a basic classical Lie super algebra*, Supersymmetry in mathematics and physics. UCLA Los Angeles, USA 2010. Papers based on the presentations at the workshop, Februar 2010., 2011.

[Ser06] Serganova, V. *Blocks in the category of finite-dimensional representations of* $\mathfrak{gl}(m,n)$, preprint, 2006.

[Tal15] Talian, A. J., *Endotrivial modules for the general linear Lie superalgebra*, J. Algebra, 433, 2015.

[Wei10] Weissauer, Rainer, *Monoidal model structures, categorial quotients and representations of super commutative Hopf algebras II: The case $GL(m,n)$*, ArXiv e-prints, 2010.

T. H.: Max-Planck Institut für Mathematik, Bonn
E-mail address: `heidersdorf.thorsten@gmail.com`

R. W.: Mathematisches Institut, Universität Heidelberg
E-mail address: `weissaue@mathi.uni-heidelberg.de`