VON ZAHLEN UND FIGUREN[1]

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§0. Introduction

Geometry is a large subfield of mathematics, but also a label for a certain mindset of a practising mathematician. The same can be told about *Algebra* (understood here broadly, as the language of mathematics, as opposed to its content, and so including *Logic*.) A natural or acquired predilection towards geometric or algebraic thinking and respective mental objects is often expressed in strong pronouncements, like Hermann Weyl's exorcising "the devil of abstract algebra" who allegedly struggles with "the angel of geometry" for the soul of each mathematical theory. (One is reminded of an even more sweeping truth: *"L'enfer – c'est les autres"*.)

Actually, the most fascinating thing about algebra and geometry is the way they struggle to help each other to emerge from the chaos of non–being, from those dark depths of subconscious where all roots of intellectual creativity reside. What one "sees" geometrically must be conveyed to others in words and symbols. If the resulting text can never be a perfect vehicle for the private and personal vision, the vision itself can never achieve maturity without being subject to the test of written speech. The latter is, after all, the basis of the social existence of mathematics.

A skillful use of the interpretative algebraic language possesses also a definite therapeutic quality. It allows one to fight the obsession which often accompanies contemplation of enigmatic Rorschach's blots of one's imagination.

When a significant new unit of meaning (technically, a mathematical definition or a mathematical fact) emerges from such a struggle, the mathematical community spends some time elaborating all conceivable implications of this discovery. (As an example, imagine the development of the idea of a continuous function, or a Riemannian metric, or a structure sheaf.) Interiorized, these implications prepare new firm ground for further flights of imagination, and more often than not reveal the limitations of the initial formalization of the geometric intuition. Gradually the discrepancy between the limited scope of this unit of meaning and our newly educated and enhanced geometric vision becomes glaring, and the cycle repeats itself.

A very special role in this cyclic process is played by problems. The importance of a new theoretical development is generally judged by its success (or otherwise) in

[1]Talk at the International Conference "Géométrie au vingtième ciècle: 1930–2000", Paris, Institut Henri Poincaré, Sept. 2001. The title is a homage to Hans Rademacher and Otto Toeplitz whose book fascinated the author many years ago.

throwing light on an old problem or two. Conversely, a problem can stimulate the emergence of a new geometric vision expressing a sudden perception of a hidden analogy, as in André Weil's famous recognition of Lefschetz's formula in the context of algebraic equations over finite fields.

A problem/conjecture is usually represented by a short mathematical statement allowing a yes or no answer. "Yes" and "no" do not play symmetric roles: a positive answer to a good question usually validates a certain intuitive picture, whereas a "no" answer often shows only the limitations of this picture rather than its total lack of value. A "counterexample" disposing of a resistant conjecture can have a certain sportive value, but becomes really important only if pursued so far as to reveal some positive truth which has been escaping our understanding for some time.

The geometry of XX century is a huge patchwork of ideas, visions, problems and its solutions. A brief list of platitudes I sketched above could be used to organize a narrative dedicated to the contemporary history of this discipline or its separate episodes. I have chosen instead to present at this conference a narrative based on my current work which at its key point remains purely conjectural, if not outright speculative.

I hope that the lack of a real mathematical breakthrough to report can be compensated in the context of this conference by certain freshness of perception accompanying such early stages of research. Besides, a discussion of partial successes and failures of this enterprise can serve as an illustration of some general issues of geometry and algebra.

Briefly, the research in question concerns explicit construction of numbers generating abelian extensions of algebraic number fields.

The archetypal problem is that of understanding abelian extensions of $\mathbf{Q}$. As we know after Kronecker and Weber, the maximal abelian extension of $\mathbf{Q}$ is generated by roots of unity. Roots of unity of degree n form in the complex plane vertices of a regular n–gone. Which of these n–gons can be constructed using only ruler and compass, was a famous problem solved by Gauss. His first publication dated April 18, 1796 (and *Tagebuch* entry of March 30, cf. [Ga]) is an announcement that a regular 17–gon has this property. Gauss was not quite 19 then; apparently, this discovery prompted him to dedicate his life to mathematics.

One remarkable feature of Gauss' result is the appearance of a hidden symmetry group. In fact, the definitions of a regular n–gon and ruler and compass constructions are given in terms of Euclidean plane geometry and make practically "evident" that the relevant symmetry group is that of rigid rotations $SO\,(2)$ (perhaps, extended by reflections and shifts). This conclusion turns out to be totally misleading: instead, one should rely upon $\mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})$. The action of the latter group upon roots of unity of degree n factors through the maximal abelian quotient and is given by

$\zeta \mapsto \zeta^k$, with k running over all $k \bmod n$ with $(k,n) = 1$, whereas the action of the rotation group is given by $\zeta \mapsto \zeta_0\zeta$ with ζ_0 running over all n–th roots. Thus, $\mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})$ does not conserve angles between vertices which seem to be basic for the initial problem. Instead, it is compatible with addition and multiplication of complex numbers, and this property proves to be crucial.

This gem of classical mathematics contains in a nutshell some basic counterpoints of my presentation: geometry and spatial imagination vs refined language of algebra; physics (kinematics of solid bodies) vs number theory; relativity of continuous and discrete. I look at them from the modern perspective of non–commutative geometry, inspired by several deep insights of Alain Connes:

(a) Connes' bold attack on the Riemann Hypothesis ([Co3], [Co4]);

(b) the discovery (joint with J. Bost, [BoCo]) of a statistical system with the symmetry group $\mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})$ and spontaneous symmetry breaking at the pole $s = 1$ of the Riemann zeta;

(c) Connes' idea that approximately finite dimensional simple central C^*–algebras will furnish the missing nontrivial Brauer theory for archimedean arithmetic infinity ([Co3], p. 72 and [Co4], p. 38).

I discuss some of these and related themes and add to them the fourth suggestion:

(d) Real Multiplication project, in which elliptic curves with complex multiplication are replaced by two–dimensional quantum tori T_θ whose K_0–group (or rather, its image wrt the normalized trace map $\mathbf{Z} + \mathbf{Z}\theta$) is a subgroup of a real quadratic field. More details will be given in a paper in preparation [Ma4].

§1. Real multiplication: an introduction

Let K be a local or global field in the sense of number theory, that is, a field of algebraic numbers, a field of functions on a curve over $\mathbf{F}_q$, or a completion of one of these fields. Denote by K^{ab} its maximal abelian (separable) extension. Class field theory provides a description of the Galois group $\mathrm{Gal}\,(K^{ab}/K)$ and some partial information on its action upon K^{ab}. However, specific generators of K^{ab}, together with exact action of $\mathrm{Gal}\,(K^{ab}/K)$ upon them, are not generally known. Hilbert's twelfth problem addresses this question. Stark's conjectures, still unproved in general, provide a partial answer: see [St1], [St2], [Ta]. Below we will define Stark's numbers directly for $\mathbf{Q}$ (see 2.1) and real quadratic fields (see 4.1).

The most elementary, complete and satisfactory description is furnished by the Kronecker–Weber theorem which we already mentioned.

The Kronecker–Weber machine (KW) can be successfully imitated for at least three more classes of ground fields K: complex quadratic extensions of $\mathbf{Q}$ (referred to as CM, for complex multiplication theory), finite extensions of $\mathbf{Q}_p$ (LT, for Lubin–Tate formal groups), and global fields of finite characteristics (D, for Drinfeld's modules).

The CM case was elaborated first, and all variations on the CM theme, including KW, can be described according to the following scheme. (Only the Lubin–Tate case slightly diverges from the general pattern at this point, see below).

Let K be a global field as above, $O \subset K$ its appropriate subring with quotient field K. We denote by $\mathbf{A}$ the analytic (or formal) additive group defined over K and construct the quotient group $\mathbf{A}/O$ where O is embedded as a subgroup of K–points of $\mathbf{A}$. By construction, multiplication by O induces on $\mathbf{A}/O$ endomorphisms in an appropriate category of K–groups: this is the *geometric* multiplication.

We construct a function t on $\mathbf{A}/O$ taking algebraic (over K) values at the O–torsion points $\xi \in K/O \subset \mathbf{A}/O$. The action of a Galois group upon these torsion points will have an abelian image, if it is shown to commute with the geometric action of O. This presupposes a careful study of the fields of definition of various objects involved.

Class field theory plus special properties of the function t allows us then to establish that the values of t on K/O generate (almost) all of K^{ab}/K. Here are some more details.

1.1. Case (KW). Here $K := \mathbf{Q}$, $O := \mathbf{Z}$. For $\mathbf{A}$ we will take the additive group considered as a complex–analytic space supplied by an additive coordinate z. The map $a:\ z \mapsto e^{2\pi i z}$ identifies $\mathbf{A}/O = \mathbf{G}_a^{an}/\mathbf{Z}$ with the (analytic) multiplicative group $\mathbf{G}_m^{an}$. The latter, of course, has a canonical algebraic structure, with a multiplicative coordinate which we will denote $1+t$, $a^*(1+t) = e^{2\pi i z}$, and this t will be the function mentioned in the general description. The lattice $\mathbf{Z}$ acts upon $\mathbf{G}_m$ by $1+t \mapsto (1+t)^n$: this is the geometric action. The values of $1+t$ at the torsion points are all roots of unity. The Galois group commutes with geometric semigroup and therefore can be identified with its profinite completion $\widehat{\mathbf{Z}}^*$.

1.2. Case (CM). Here we denote the ground field (former K) by L and assume that it is an imaginary quadratic extension of $\mathbf{Q}$. For O we take the ring O_L of all integers in L, and for $\mathbf{A}$ the complex analytic additive group as above. The homomorphism of analytic groups $a:\ \mathbf{G}_a \to X_L$ with the kernel O_L is the universal covering of the elliptic curve X_L. Over $\mathbf{C}$, its field of algebraic functions is generated by the Weierstrass function $\wp(z, O_L)$ and its derivative. The action of O_L upon X_L, in particular, produces rational expressions for $\wp(\alpha z, O_L)$, $\alpha \in O_L$, in terms of $\wp(z, O_L)$, $\wp'(z, O_L)$.

(i) First description of L^{ab}. An important difference from the (KW)–case is that elliptic curves, unlike $\mathbf{G}_M$, are not rigid but occur in families. In particular, the isomorphism class of X_L over $\mathbf{C}$ is determined by the invariant $j(O_L) \in \mathbf{C}$. It turns

out that $L(j(O_L))$ is exactly the maximal unramified abelian extension of L, i.e. its Hilbert class field.

Over $L(j(O_L))$, the maximal abelian extension of L can be generated by the values of the function t such that $a^*(t) := \wp^{-u}(z, O_L)$, where $2u$ is the order of the group O_L^* of units in O_L. Generally, $u = 1$; but $u = 2$ (resp. $u = 3$) for $L = \mathbf{Q}(e^{\pi i/2})$ (resp. $L = \mathbf{Q}(e^{\pi i/3})$).

A more geometric description of t is this: X_L/O_L^* is a projective line, and t is a coordinate on it vanishing at the image of the zero point of X_L.

The appearance of X_L/O_L^* rather than X_L itself in this description might seem a minor matter. However, in the conjectural picture of real multiplication it will acquire a great importance, because O_L^* for *real* quadratic fields is always infinite, and the study of the respective quotient will again require tools of noncommutative geometry.

(ii) Second description of L^{ab}. Instead of using only the curve X_L with complex multiplication by the whole O_L, one can consider the family of all elliptic curves Y_L whose endomorphism ring is an order in O_L. It turns out that their absolute invariants $j(Y_L)$ generate a big subfield of L^{ab}: to obtain all of L^{ab}, it remains however to make an additional 2–extension (of infinite degree). The curves Y_L are all isogenous to X_L, and since the points of finite order of X_L "essentially" generate L^{ab}, this gives an intuitive explanation of why this might be so.

However, the two description differ not only by the details of the calculations. What is important in the second picture, is the implicit appearance of the tower or stack of modular curves, parametrizing all elliptic curves, rather than of only one elliptic curve X_L.

This tower consists of the compactified algebraic models of analytic spaces H/Γ, Γ, where H is the upper complex half–plane, and Γ runs over congruence subgroups of $PSL(2, \mathbf{Z})$.

For a readable review of two approaches, see [Se] and [Ste].

1.3. Case (D). Here the ground field is the field of functions $k = \mathbf{F}_q(C)$ on a smooth algebraic curve defined over the finite base field with q elements. We choose a point $\infty \in C(\mathbf{F}_q)$ as a part of the structure and denote by O_k the ring of regular functions on the affine curve $C \setminus \{\infty\}$. For $\mathbf{A}$ we take the additive group considered as an analytic group over the quotient field k_∞ of the completion of $\mathcal{O}_{C,\infty}$.

In this situation we have an analogue of exponential function on $\mathbf{A}$:

$$e(z) := z \prod_{\alpha \in O_L \setminus \{0\}} (1 - \frac{z}{\alpha})$$

This function is entire. It is however additive rather than multiplicative: $e(z_1 + z_2) = e(z_1) + e(z_2)$. A large abelian extension of k, as in the KW–case, is generated by the values of $e(\lambda z)$ at the "points of finite O_k–order", where λ is an appropriate analogue of $2\pi i$. The action of O_k on $\mathbf{G}_a$ can be alternatively described by the embedding O_k into the (algebraic) crossed product of k_∞ with the semigroup of non–negative powers of the Frobenius endomorphism $x \mapsto x^p$ acting upon k_∞.

This construction can be considerably generalized by replacing λO_k in the construction of the exponential function with any O_k lattice of rank $d \geq 1$. We get in this way the notion of the Drinfeld module of rank d. The case $d = 1$ produces abelian extensions and is similar both to (KW) and (CM) cases. The rank 2 case looks like a version of the theory of elliptic curves, and an appropriate modification of the general d case produces the Langlands type description of the algebraic closure of K.

A thorough historical and mathematical discussion of the theories (KW), (CM) and (D) can be found in the book [V].

1.4. Case (LT). Here K is a finite extension of $\mathbf{Q}_p$, $O = O_K$ is the ring of integers in K, and the group G_{LT} denoted formerly by $\mathbf{A}/O$ above, is an one–dimensional formal group over O_K equipped with a homomorphism $O_K \to \mathrm{End}\, G_{LT}$ sending any $a \in O_K$ to a formal map with the linear term $x \mapsto ax$.

I do not know whether G_{LT} can be interpreted as a quotient $\mathbf{A}/O$ in an appropriate category, although the theory of logarithmic functions developed in the context of p–adic Hodge–Tate theory indicates that such an interpretation might exist. Lubin and Tate construct G_{LT} directly using an ingenious calculation with formal series providing simultaneously a description of the category of such formal groups.

1.5. Real multiplication (RM) and noncommutative geometry. The simplest class of fields M for which no direct description of M^{ab} is known consists of real quadratic extensions of $\mathbf{Q}$.

Below we sketch a possible approach to this problem via non–commutative geometry and suggest its relation with the Stark conjectures. The idea is straightforward enough. An elliptic curve with complex multiplication by, say, the maximal order O_L in a complex quadratic field L has a complex analytic model $\mathbf{C}/O_L$, where O_L is considered as lattice in $\mathbf{C}$. Similarly, one could try to imagine a space $\mathbf{R}/O_M$ where O_M is the ring of integers of a real quadratic field M. Of course, O_M is not discrete anymore, but the first principle of the non–commutative philosophy says that one should not shy away from such situations: $\mathbf{R}/O_M$ exists as a non–commutative space represented e.g. by a two–dimensional torus T_θ where $O_M = \mathbf{Z} + \mathbf{Z}\,\theta$. Recall that the C^*–algebra of T_θ is the universal algebra generated by two unitaries U, V with the commutation rule $UV = e^{\pi i\theta} VU$. This algebra is the crossed product of $C(\mathbf{R}/\mathbf{Z})$ and the irrational rotation automorphism of $\mathbf{R}/\mathbf{Z}$ induces by the shift $t \mapsto t + \theta$. Crossed products generally serve to represent quotients with respect to "bad" (and occasionally good) equivalence relations: cf, [Co1], pp. 85–91 for a very livrly account of this principle. (With a hindsight, we noted that in the construction of Drinfeld modules the central role was played by a universal crossed product of an algebra of power series and the semigroup generated by the Frobenius endomorphism.)

As the next step, we want to make sense of the statement that T_θ admits real multiplication by O_M. This is obvious enough for $\mathbf{R}/O_M$ understood set–theoretically: the action of O_M is simply induced by the multiplication $O_M \times \mathbf{R} \to \mathbf{R}$. But crossed product C^*–algebras are not functorial in any naive sense. One way to deal with this difficulty is to replace the usual homomorphisms of associative rings by Morita functors between the categories of their (say, right) modules. Morita functors are given by bimodules, and composition of functors corresponds to the tensor product of bimodules.

Similarly, a coarse moduli space of two–dimensional quantum tori up to Morita equivalence can be seen as a quotient $PGL(2, \mathbf{Z}) \setminus \mathbf{P}^1(\mathbf{R})$; various rigidities lead to its modular covers. Basics of topology and function theory of these spaces were studied in [MaMar].

Finally, a key issue for arithmetics is the question of fields of definition of various objects: of the "non–commutative elliptic curves with real multiplication T_θ, " of the Galois–Hopf automorphism algebras of such objects (replacing their points of finite order) etc. An implication is that we need algebraic geometric, finitely or countably generated objects, preferably over $\mathbf{Z}$, from which the functional analytic structures like C^*–algebras can be obtained by extension of the base field and an appropriate completion.

In the next section I will address some of these challenges. I will start it with a discussion of several constructions in which algebraic numbers, C^*–algebras and related noncommutative rings appear in combinations that are likely to be susceptible to generalizations.

§2. Noncommutative geometry and arithmetic

Oversimplifying, one can say that in commutative geometry algebraic numbers appear as values of algebraic functions, whereas in noncommutative geometry they appear as values of traces of projections, or more generally values of appropriate states on observables. In both cases, a control of the action of the Galois group is gained, if this action commutes with an action of certain "geometric" endomorphisms, or correspondences, whenever the latter are defined over the ground field.

We will deal with three basic situations:

(A) V. Jones theory of indices of subfactors (cf. [Jo1], [Jo2], [GoHaJo]) and its further developments.

(B) Bost–Connes "spontaneous symmetry breaking", with the symmetry group $\mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})$ (cf. [BoCo]).

(C) V. Drinfeld's embedding of $\mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})$ into the Grothendieck–Teichmüller group: cf. [Dr], [DE], [LoS].

2.1. Jones indices. Consider first the Temperley–Lieb algebra $TL\,(n+1,\tau)$ which arose in statistical physics. Over the central subring $\mathbf{Z}[\tau]$, it is generated by the idempotents $e_1,\dots,e_n$ satisfying the relations $e_ie_j=e_je_i$ for $|i-j|\ge 2$ and $e_ie_{i\pm1}e_i=\tau e_i$. It has a finite dimension, and is semisimple for generic τ. For a critical value of the form $\tau^{-1}=4\cos^2\frac{m\pi}{n+1}$, $TL\,(n+1,\tau)$ fails to be semisimple.

It is interesting that Stark's numbers for the ground field $\mathbf{Q}$ ([Ta], p. 79) differ from these critical numbers only by a sign change and shift by 4:

$$\exp\,(-2\zeta'_{(m,n)}(0))=4\sin^2\frac{m\pi}{n}$$

where

$$\zeta_{(m,n)}(s):=\sum_{k\in m+n\mathbf{Z}}|k|^{-s}.$$

In particular, TL–critical values and Stark numbers generate the maximal real subfield of $\mathbf{Q}^{ab}$ and are stable with respect to the Galois group of $\overline{\mathbf{Q}}$.

One can extend the base ring of TL–algebras to $\mathbf{C}$, define the $*$–involution on them by $e_i^*=e_i$ (so that e_i become projections), and then pass to the inductive limit with respect to the obvious injections $TL\,(n+1,\tau)\mapsto TL\,(n+2,\tau)$. It turns out that the resulting algebra admits an involutive representation in a complex Hilbert space iff either $\tau^{-1}=4\cos^2\frac{\pi}{n}$ for some $n\ge 3$, or $\tau^{-1}\ge 4$.

This statement constitutes an essential part of the famous result due to V. Jones who proved that for any pair of II_1–factors $N\subset M$ the index $[M:N]$ lies in the set $\{4\cos^2\frac{\pi}{n},\ n\ge 3\}\cup[4,\infty]$, and that for the hyperfinite M this is exactly the set of values of indices. Index itself can be defined as a value of the Hattori–Stallings rank, or as a measure of the growth rate of the minimal number of generators of the left N–module $M^{\otimes n}:=M\otimes_N\cdots\otimes_N M$ as $n\to\infty$.

Notice that although the discrete part of the values of the index constitutes a part of the TL–spectrum, it is not $\mathrm{Gal}\,(\mathbf{Q}^{ab}/\mathbf{Q})$–invariant. The reason is that by passing to the C^*–limit, we have implicitly chosen a non–archimedean valuation of $\mathbf{Q}^{ab}$.

2.2. Bost–Connes spontaneous symmetry breaking. In the remarkable paper [BoCo] the action of $\mathrm{Gal}\,(\mathbf{Q}^{ab}/\mathbf{Q})$ on roots of unity appears in yet another setting. Instead of (the union of) the Temperley–Lieb algebras, consider the Hecke algebra $\mathcal{H}$ with involution over $\mathbf{Q}$ given by the following presentation. The generators are denoted μ_n, $n \in \mathbf{Z}_+$, and $e(\gamma)$, $\gamma \in \mathbf{Q}/\mathbf{Z}$. The relations are

$$\mu_n^* \mu_n = 1, \quad \mu_{mn} = \mu_m \mu_n, \quad \mu_m^* \mu_n = \mu_n \mu_m^* \text{ for } (m,n) = 1;$$

$$e(\gamma)^* = e(-\gamma), \quad e(\gamma_1 + \gamma_2) = e(\gamma_1)\, e(\gamma_2);$$

$$e(\gamma)\mu_n = \mu_n e(n\gamma), \quad \mu_n e(\gamma) \mu_n^* = \frac{1}{n} \sum_{n\delta = \gamma} e(\delta).$$

The idèle class group $\widehat{\mathbf{Z}}^*$ of $\mathbf{Q}$ acts upon $\mathcal{H}$ in a very explicit and simple way: on $e(\gamma)$'s the action is induced by the multiplication $\widehat{\mathbf{Z}}^* \times \mathbf{Q}/\mathbf{Z} \to \mathbf{Q}/\mathbf{Z}$, whereas on μ_n's it is identical.

The algebra $\mathcal{H}$ admits an involutive representation ρ in $l^2(\mathbf{Z}_+)$: denoting by $\{\epsilon_k\}$ the standard basis of this space, we have

$$\rho(\mu_n)\, \epsilon_k = \epsilon_{nk}, \quad \rho(e(\gamma))\epsilon_k = e^{2\pi i k \gamma} \epsilon_k.$$

From this, one can produce the whole $\mathrm{Gal}\,(\mathbf{Q}^{ab}/\mathbf{Q})$–orbit $\{\rho_g\}$ of such representations, applying $g \in \mathrm{Gal}\,(\mathbf{Q}^{ab}/\mathbf{Q})$ to all roots of unity occuring at the right hand sides of the expressions for $\rho(e(\gamma))\epsilon_k$. All these representations can be canonically extended to the C^*–algebra completion C of $\mathcal{H}$ constructed from the regular representation of $\mathcal{H}$. Let us denote them by the same symbol ρ_g.

To formulate the main theorem of [BoCo], we need some more explanations. The algebra C admits a canonical action of $\mathbf{R}$, which can be interpreted as *time evolution* represented on the algebra of observables. This is a general (and deep) fact in the theory of C^*–algebras, but for C the action of $\mathbf{R}$ can be quite explicitly described on the generators. Let us denote by σ_t the action of $t \in \mathbf{R}$. *A KMS_β state at inverse temperature* β on (C, σ_t) is defined as a state φ on C such that for any $x, y \in C$ there exists a bounded holomorphic function $F_{x,y}(z)$ defined in the strip $0 \le \operatorname{Im} z \le \beta$ and continuous on the boundary, satisfying

$$\varphi(x\sigma_t(y)) = F_{x,y}(t), \;\; \varphi(\sigma_t(y)x) = F_{x,y}(t + i\beta).$$

Now denote by H the positive operator on $l^2(\mathbf{Z}_+)$: $H\epsilon_k = (\log k)\, \epsilon_k$. Then for any $\beta > 1$, $g \in \mathrm{Gal}\,(\mathbf{Q}^{ab}/\mathbf{Q})$ one can define a KMS_β state $\varphi_{\beta,g}$ on (C, σ_t) by the following formula:

$$\varphi_{\beta,g}(x) := \zeta(\beta)^{-1}\, \mathrm{Trace}\,(\rho_g(x)\, e^{-\beta H}), \quad x \in C$$

where ζ is the Riemann zeta–function. The map $g \mapsto \varphi_{\beta,g}(x)$ is a homeomorphism of $\mathrm{Gal}\,(\mathbf{Q}^{ab}/\mathbf{Q})$ with the space of extreme points of the Choquet simplex of all KMS_β states.

To the contrary, for $\beta < 1$ there is a unique KMS_β state. This is a remarkable "arithmetical symmetry breaking" phenomenon.

2.3. $\mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})$ **and the Grothendieck–Teichmüller group.** Here we briefly describe the setting studied in [Dr]. Consider the following tower of fields

$$\mathbf{Q}(t) \subset \overline{\mathbf{Q}}(t) \subset F$$

where F is the maximal algebraic extension of $\overline{\mathbf{Q}}(t)$ ramified only at $t = 0, 1, \infty$. The Galois group $\mathrm{Gal}\,(F/\overline{\mathbf{Q}}(t))$ is an extension of $\mathcal{G} := \mathrm{Gal}\,(\overline{\mathbf{Q}}/\mathbf{Q})$ by $\Pi := \mathrm{Gal}\,(F/\overline{\mathbf{Q}}(t))$. Hence $\mathcal{G}$ acts upon Π by outer automorphisms. Now, Π is a profinite completion of the fundamental group $\pi_1(\mathbf{P}^1(\mathbf{C}) \setminus \{0, 1, \infty\})$. Drinfeld prefers to work with the "formal" monodromy group of the differential equation

$$G'(z) = \frac{1}{2\pi i}\left(\frac{A}{x} + \frac{B}{z-1}\right) G(z)$$

where A, B are non–commuting symbols, and with Knizhnik–Zamolodchikov (KZ) scattering data for such an equation treated in a purely algebraic way. This allows him to connect the Galois action with his study of quasi–Hopf algebras.

2.4. Summary. The three constructions briefly summarized above at the moment do not form a part of a coherent picture. However, they shed some light upon each other.

The Temperley–Lieb algebras were recently understood as a kind of Galois symmetry objects (quantum groupoids) responsible for the classification of pairs of factors $N \subset M$ with finite width and index: see the review [NiVa] and other references quoted therein. Apparently, the action of the cyclotomic Galois group on them comes from a setting similar to that described in [Dr], which however has to be made explicit yet. Although in the Grothendieck–Drinfeld picture the stress was usually made on the dessins d'enfant rather than unwieldy non–commutative objects, this might be only one of several possibilities.

Approximately finite dimensional algebras are completed inductive limits of split central semisimple algebras, i.e. direct sums of matrix algebras. It would be interesting to develop the theory of inductive limits of general central semisimple algebras over non–closed, in particular, number fields.

No explicit connection was made between Bost–Connes theory and Jones theory. Are there any? Since zeta function appears explicitly in [BoCo] as a partition function, cyclotomic Stark numbers might surface naturally in this model.

Finally, all three theories are absolute in the sense that their ground field is $\mathbf{Q}$. It is striking that their relative analogs (over arbitrary number fields K) are not known.

To be more precise, some rather straightforward extensions of [BoCo] were studied in [HaL], [ArLR], [Coh1], [Coh2] but none of them has the π_0 of the idèle class group of K as a symmetry group when $K \neq \mathbf{Q}$. In fact, the archimedean part of the idèles is not properly accommodated, which returns us to the problem of the right Brauer theory for archimedean primes.

One can expect that a real understanding of the classical abelian class field theory will depend on such a generalization.

§3. Real multiplication of quantum tori: geometry

3.1. Lattices and pseudolattices. The category of elliptic curves over $\mathbf{C}$ is equivalent to the category of period lattices. Formally, a lattice is a discrete and cocompact embedding $j : \Lambda \to V$ where Λ is isomorphic to $\mathbf{Z}^2$ and V is an one-dimensional complex vector space; morphisms are linear maps $V \to V'$ sending $j(\Lambda)$ into $j'(\Lambda')$. The functor establishing equivalence sends (Λ, V, j) to the complex torus $V/j(\Lambda)$.

Let us denote by Λ_τ the lattice $\mathbf{Z}+\mathbf{Z}\tau$, $\tau \in \mathbf{C}\setminus\mathbf{R}$. Then any lattice is isomorphic to some Λ_τ, and each non–zero morphism $\Lambda_{\tau'} \to \Lambda_\tau$ is represented by a non–degenerate matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M\,(2, \mathbf{Z})$$

such that

$$\tau' = \frac{a\tau + b}{c\tau + d},$$

It is an isomorphism, iff $g \in GL(2, \mathbf{Z})$. Thus the coarse moduli space classifying lattices up to an isomorphism is

$$PGL(2, \mathbf{Z}) \setminus (\mathbf{P}^1(\mathbf{C}) \setminus \mathbf{P}^1(\mathbf{R}))$$

which is usually written as $PSL(2, \mathbf{Z}) \setminus H$ where H is the upper half–plane.

The map j can degenerate so that j acquires a kernel of rank one; the image remains discrete but is not cocompact. The coarse moduli space admitting such degenerations is $PSL(2, \mathbf{Z}) \setminus (H \cup \mathbf{P}^1(\mathbf{Q}))$: a cusp is added. This cusp (or several cusps if a rigidity is imposed) can be seen on algebraic geometric models of the modular curves and canonical elliptic fibrations over them: they essentially correspond to the degenerations of elliptic curves with (stably) multiplicative connected component of the closed fiber.

Classes of the irrational real points constitute another part of the boundary $PGL(2, \mathbf{Z}) \setminus \mathbf{P}^1(\mathbf{R})$. This part classifies degenerations of lattices which I will call *pseudolattices*: a pseudolattice is an embedding $j : L \to V$ where L is isomorphic to

$\mathbf{Z}^2$, V is an one–dimensional $\mathbf{C}$–space as above, and the closure of $j(V)$ is a real line. Morphisms are defined exactly as for lattices. Denoting by L_θ the pseudolattice $\mathbf{Z}+\mathbf{Z}\theta$, we see as above that any pseudolattice is isomorphic to an L_θ, and morphisms correspond to the fractional linear transformations of θ's with integral coefficients.

These degenerations are invisible in algebraic geometry because $V/j(L)$ makes no sense as an algebraic or analytic curve. But the machinery of noncommutative geometry of Connes is designed to deal with such spaces. Choosing L_θ as a representative of the respective isomorphism class, we can naively replace $\mathbf{C}/(\mathbf{Z}+\mathbf{Z}\theta)$ by $\mathbf{C}^*/(e^{2\pi i\theta})$ ("Jacobi uniformization"), and then interpret the last quotient as an "irrational rotation algebra", or two–dimensional quantum torus T_θ. We recall that this torus is (represented by) the universal C^*–algebra A_θ generated by two unitaries U, V with the commutation rule $UV = e^{2\pi i\theta}VU$.

The next task is to define morphisms between these quantum tori, so that we could compare their category with that of pseudolattices. Already isomorphisms present a problem: we want invertible fractional linear transforms to produce isomorphic quantum tori. M. Rieffel's seminal discovery (cf. [Ri3]–[Ri5], [RiSch]) was that to this end we should consider Morita equivalences between appropriate categories of modules as isomorphisms between the tori themselves. Taking this lead, we will formally introduce the general Morita morphisms of associative rings, stressing those traits of the formalism that play a central role in the structure theory of quantum tori (of arbitrary dimension).

For a considerably more sophisticated version of such a theory for von Neumann algebras, see [Co1], VB, and a report [Jo3]. Physical motivations can be found in [Sch1], [Wi], [CoDSch].

3.2. Morita category and projective modules. Let A, B be two associative rings. A *Morita morphism* $A \to B$ by definition, is the isomorphism class of a bimodule ${}_AM_B$, which is projective and finitely generated separately as module over A and B.

The composition of morphisms is given by the tensor product ${}_AM_B \otimes {}_BM'_C$, or ${}_AM \otimes {}_BM'_C$ for short.

If we associate to ${}_AM_B$ the functor

$$\mathrm{Mod}_A \to \mathrm{Mod}_B : \ N_A \mapsto N \otimes_A M_B,$$

the composition of functors will be given by the tensor product, and isomorphisms of functors will correspond to the isomorphisms of bimodules.

We imagine an object A of the (opposite) Morita category as a noncommutative space, right A–modules as sheaves on this space, and the tensor multiplication by ${}_AM_B$ as the pull–back functor.

Two bimodules ${}_AM_B$ and ${}_BN_A$ supplied with two bimodule isomorphisms ${}_AM\otimes_B N_A \to {}_AA_A$ and ${}_BN\otimes_A M_B \to {}_BB_B$ define mutually inverse Morita isomorphisms (equivalences) between A and B. The basic example of this kind is furnished by $B = \mathrm{Mat}\,(n,A)$, $M = {}_AA^n{}_B$ and $N = {}_BA^n{}_A$.

Projective right A–modules up to isomorphism are exactly ranges of idempotents in various matrix rings $\mathrm{Mat}\,(n,A)$ acting from the left upon (column) vector modules A^n.

We prefer to work with all n simultaneously. So we will denote by $\mathcal{M}A$ the ring of infinite matrices $(a_{ij})\,i,j\geq 1$, $a_{ij}\in A$, $a_{ij}=0$ for $i+j$ big enough (depending on the matrix in question). Similarly, denote by A^∞ the left $\mathcal{M}A$–module of infinite columns (a_i), $i\geq 1$, with coordinates in A, almost all zero.

Denote by Pr_A the category of finitely generated right A–modules. Denote by pr_A the category whose objects are projectors $p\in\mathcal{M}A$, $p^2=p$, whereas morphisms are defined by

$$\mathrm{Hom}\,(p,q) := q\,\mathcal{M}A\,p$$

with the composition induced by the multiplication in $\mathcal{M}A$.

There is a natural functor $pr_A \to Pr_A$ defined on objects by $p\mapsto pA^\infty$. In order to define it on morphisms, we remark that morphisms $pA^\infty \to qA^\infty$ can be naturally described by matrices in the following way. Clearly, pA^∞ contains the columns p_k of p which generate pA^∞ as a right A–module. We can then apply any $\varphi : pA^\infty \to qA^\infty$ to all p_k and arrange the resulting vectors into a matrix $\Phi\in\mathcal{M}A$ with k–th column $\varphi(p_k)$. One checks that $\Phi p = \Phi$, and since also $q\Phi = \Phi$, we have $\Phi\in q\mathcal{M}Ap$. Conversely, any such matrix determines a unique morphism $pA^\infty \to qA^\infty$.

3.2.1. Claim. *(a) The functor $pr_A \to Pr_A$ described above is an equivalence of categories.*

(b) If pA^∞ is isomorphic to qA^∞, then $p-q\in[\mathcal{M}A,\mathcal{M}A]$.

A trace of A is any homomorphism of additive groups $t : A\to G$ vanishing on commutators; by definition, it factors through the universal trace $A\to A/[A,A]$. Combining it with the matrix trace, we get its canonical extension to $\mathcal{M}A$. From the Claim above it follows that $t(p)$ depends only on the isomorphism class of pA^∞. The class $p \bmod [A,A]$ is called *the Hattori–Stallings rank* of pA^∞.

We define $K_0(A)$ as the Grothendieck group of Pr_A. If $N_A\in Pr_A$, $[N_A]$ denotes its class in $K_0(A)$. If N_A is the range of an idempotent p and t is a trace, $t(p)$ depends only on $[N_A]$ and is additive on exact triples, hence t becomes a homomorphism of $K_0(A)$ (it is called dimension in the theory of von Neumann algebras).

Assume now that A is endowed with an additive (linear or antilinear in the case of algebras) involution $a\mapsto a^*$, $(ab)^* = b^*a^*$, $a^{**}=a$. It extends to matrix algebras:

$(B^*)_{ij} := B^*_{ji}$. similarly, it extends to $A^\infty \to A_\infty$ and $A_\infty \mapsto A^\infty$, compatibly with the module structures.

In such a context, it makes sense to consider only those projective modules that are ranges of *projections*, that is, $*$–invariant idempotents.

3.3. Morita category of two–dimensional quantum tori. By definition, two–dimensional quantum tori are objects of the category $\mathcal{QT}$ whose morphisms are isomorphism classes of bimodules ${}_AM_B$ corresponding to $*$–invariant projections. The algebra A_θ has a unique trace t_A which is normalized by the condition $t_A(1) = 1$. We can now define a functor K from $\mathcal{QT}$ to the category of pseudolattices $\mathcal{PL}$. On objects, we put:

$$K(T) = (L_A, V_A, j_A, s_A).$$

Here $L_A := K_0(A)$, the K_0–group of the category of right projective A–modules (as above, given by projections in finite matrix algebras over A); V_A is the target group of the universal trace on A, that is, the quotient space of A modulo the completed commutator subspace $[A, A]$. Furthermore, $j_A = t_A : K_0(A) \to V_A$ is this universal trace extended to matrix algebras; its value on the class of a module, as we already explained, is its value at the respective projection. The pseudolattice $K(A_\theta)$ comes in fact with an additional structure which we will call *orientation*: namely, the cone of effective elements in $K_0(A_\theta)$. The information it carries is exactly the choice of an orientation of the real closure of $j_A(L_A)$.

On morphisms, we put

$$K({}_AM_B)([N_A]) := [N \otimes_A M_B].$$

It takes some work to show that this functor is well defined. Clearly, it is compatible with orientations.

Unlike the case of elliptic curves, K is not quite an equivalence of categories. It is essentially surjective on objects and morphisms of oriented pseudolattices conserving orientations. However, it glues together some Morita morphisms. The most clear cut case of this is furnished by Morita equivalences: if $\otimes_A M_B$ and $\otimes_A M'_B$ produce Morita equivalences, and induce the same isomorphism of pseudolattices, these functors differ by an automorphism of the category Mod_A which is induced by an automorphism of the ring A.

Thus, in order to achieve an equivalence of categories, one should count as equivalent many morphisms of noncommutative tori induced by ring homomorphisms, contrary to the intuition educated on affine schemes. In fact, when the trace is unique, all the standard automorphisms of A_θ act trivially on $K_0(A_\theta)$. But in any case, all endomorphisms of pseudolattices keeping orientation are induced by bimoduli, which is a basis of real multiplication to which we finally turn.

3.4. Complex Multiplication and Real Multiplication. Endomorphisms of a lattice Λ and of the respective elliptic curve form a ring which always contains $\mathbf{Z}$. The situations when it is strictly larger than $\mathbf{Z}$ can be described as follows.

(a) $\mathrm{End}\,\Lambda \neq \mathbf{Z}$ iff there exists a complex quadratic subfield K of $\mathbf{C}$ such that Λ is isomorphic to a lattice contained in K.

(b) If this is the case, denote by O_K the ring of integers of K. There exists a unique integer $f \geq 1$ (conductor) such that $\mathrm{End}\,\Lambda = \mathbf{Z} + fO_K =: R_f$, and Λ is a projective module of rank 1 over R_f. Every K, f and a projective module over R_f come from a lattice.

(c) If lattices Λ and Λ' have the same K and f, they are isomorphic if and only if their classes in the Picard group $\mathrm{Pic}\,R_f$ coincide.

Automorphisms of a lattice generally form a group $\mathbf{Z_2}$ (ψ is multiplication by ± 1.) However, integers of two imaginary quadratic fields obtained by adjoining to $\mathbf{Q}$ a primitive root of unity of degree 4 (resp. 6) furnish examples of lattices with automorphism group of order 4 (resp. 6). Only these two fields produce lattices with such extra symmetries.

Pseudolattices with real multiplication admit a completely similar description.

Endomorphisms of a pseudolattice L (we omit other structures in notation if there is no danger of confusion) form a ring $\mathrm{End}\,L$ with composition as multiplication. It contains $\mathbf{Z}$ and comes together with its embedding in $\mathbf{R}$ as $\{a \in \mathbf{R} \,|\, aj(L) \subset j(L)\}$.

(a′) $\mathrm{End}\,L \neq \mathbf{Z}$ iff there exists a real quadratic subfield K of $\mathbf{R}$ such that L is isomorphic to a pseudolattice contained in K.

(b′) If this is the case, we will say that L is an RM pseudolattice. Denote by O_K the ring of integers of K. There exists a unique integer $f \geq 1$ (conductor) such that $\mathrm{End}\,L = \mathbf{Z} + fO_K =: R_f$, and L is a projective module of rank 1 over R_f.

The module L is endowed with a total ordering.

Every K, f and a ordered projective module over R_f come from a lattice.

(c′) If pseudolattices L and L' have the same K and f, they are isomorphic if and only if their classes in the Picard group $\mathrm{Pic}\,R_f$ coincide.

Unlike the case of lattices, the automorphism group of a pseudolattice is always infinite, it is isomorphic to $\mathbf{Z} \times \mathbf{Z}_2$.

3.5. Quantum tori as "limits" of elliptic curves. Comparison of the relevant geometric categories suggests that two–dimensional quantum tori can be thus considered as limits of elliptic curves. More specifically, take a family of Jacobi parametrized curves $E_\tau = \mathbf{C}/(e^{2\pi i\tau})$ with $\mathrm{Im}\,\tau > 0$ and $\tau \to \theta \in \mathbf{R}$. It is then natural to imagine T_θ as a limit of E_τ.

Fixing a Jacobi uniformization of an elliptic curve (or abelian variety of any dimension) as a part of its structure is necessary, for example, in problems connected with mirror symmetry. In such contexts our intuition seemingly provides a sound picture (cf. a similar discussion in [So], pp. 100, 113–114).

However, limitations of this viewpoint become quite apparent if one has no reason to keep a Jacobi uniformization as a part of the structure, and is interested only in the isomorphim classes of elliptic curves, perhaps somewhat rigidified by a choice of a level structure.

In this case one must contemplate the dynamics of the limiting process not on the closed upper half–plane but on a relevant modular curve X. Letting τ tend to θ along a geodesic, we get a parametrized real curve on X which, when θ is irrational, does not tend to any limiting point. This is what can happen.

(a) Let θ be a real quadratic irrationality, θ' its conjugate. Consider the oriented geodesic in H joining θ' to θ. The image of this geodesic on any modular curve X is supported by a closed loop, which we denote $(\theta', \theta)_X$.

(b) Let θ be as above, and let τ tend to θ along an arbitrary geodesic. Then the image of this geodesic on X has $(\theta', \theta)_X$ as a limit cycle (in positive time).

(c) Each closed geodesic on X is the support of a closed loop $(\theta', \theta)_X$. The union of them is dense in X. It is a strange attractor for the geodesic flow in the following sense. Having chosen a sequence of loops $(\theta_i', \theta_i)_X$, a sequence of integers $n_i \geq 1$, and a sequence of real numbers $\epsilon_i > 0$, $i = 1, 2, \dots$, one can find an oriented geodesic winding $\geq n_i$ times in the ϵ_i–neighborhood of $(\theta_i', \theta_i)_X$ for each i, before jumping to the next loop.

Now let us imagine that we have constructed a certain object $R(E_\tau)$ depending on the isomorphism class of E_τ (perhaps, with rigidity). This object can be a number, a function of the lattice, a linear space, a category ... Suppose also that we have constructed a similar object $\mathcal{R}(T_\theta)$ depending on the isomorphism class of T_θ, and that we want to make sense of the intuitive notion that $\mathcal{R}(T_\theta)$ is "a limit of $R(E_\tau)$." Since in the most interesting for us case (a) E_τ keeps rotating around the same loop, there are two natural possibilities:

(i) *The object $R(E_\tau)$ actually "does not depend on τ", and $\mathcal{R}(T_\theta)$ is its constant value. Here independence generally means a canonical identification of different $R(E_\tau)$, e.g. via a version of flat connection defined along the loop.*

(ii) *The object $R(E_\tau)$ does depend on τ, and $\mathcal{R}(T_\theta)$ is obtained by a kind of integrating or averaging various $R(E_\tau)$ along the loop.*

The second case looks more interesting, however, it is not immediately obvious that such objects occur in nature. Remarkably, they do, and precisely in the context of real multiplication and Stark's conjecture. In fact, this is how we will interpret the beautiful old calculational tricks due to Hecke: see [He1], [He2], [Her], [Za1]. See also [Dar] for a similar observation related to what Darmon calls Stark–Heegner points of elliptic curves.

In this section we will only explain the geometric meaning of Hecke's substitution, whereas the (slightly generalized) calculation itself will be treated in the next section.

Let $K \subset \mathbf{R}$ be a real quadratic subfield of $\mathbf{R}$ and $L \subset K$ an RM pseudolattice. From now on, we denote by $l \mapsto l'$ the nontrivial element of the Galois group of $K/\mathbf{Q}$.

For any real t, consider the following subset of $\mathbf{C}$:

$$\Lambda_t = \Lambda_t(L) := \{\lambda_t = \lambda_t(l) := le^{t/2} + il'e^{-t/2} \,|\, l \in L\}$$

3.5.1. Lemma. *(a) $\Lambda_t(L)$ is a lattice.*

(b) Any isomorphism $a : L_1 \to L$ in the narrow sense induces isomorphisms $\Lambda_t(L_1) \to \Lambda_{t+c}(L)$ where c is a constant depending only on a and t is arbitrary.

(c) The image of the curve $\{\Lambda_t \,|\, t \in \mathbf{R}\}$ on any modular curve is a closed geodesic. The affine coordinate t along this curve is the geodesic length.

3.5.2. Remark. One can try to relate elliptic curves to quantum tori by treating these curves themselves as objects on noncommutative geometry represented by some version of the relevant crossed product algebra. In the most direct approach, the latter is a completion of the non–unitary toric algebra generated by U, V with $UV = e^{2\pi i\tau}VU$. Representation of such an algebra are in fact closely related to vector bundles on E_τ as was shown in [BEG] (following [BG]). Developing further this approach, one can hope to see better what happens when one passes to the unitary limit $\tau \to \theta$.

§4. Stark's numbers for real quadratic fields

In this section we will explain Stark's conjecture for real quadratic fields and slightly generalize Hecke's method of calculation of these numbers. Since it involves integration along the geodesic loops introduced above, we conclude that Stark's numbers have something to do with the respective quantum tori. However, key parts of the picture are still missing.

4.1. Stark's numbers at $s = 0$. In this section we fix a real quadratic subfield $K \subset \mathbf{R}$. Denote by $l \mapsto l'$ the action of the nontrivial element of the Galois group of K, and by O_K the ring of integers of K, and put $N(l) = ll'$.

Let L be an arbitrary integral ideal of K which, together with its embedding in $\mathbf{R}$ and the induced ordering, will be considered as a pseudolattice.

Choose also an $l_0 \in O_K$ so that the pair (L, l_0) satisfies the following restrictions:

(i) *The ideals* $\mathfrak{b} := (L, l_0)$ *and* $\mathfrak{a}_0 := (l_0)\mathfrak{b}^{-1}$ *are coprime with* $\mathfrak{f} := L\mathfrak{b}^{-1}$.

(ii) *Let* ε *be a unit of* K *such that* $\varepsilon \equiv 1 \bmod \mathfrak{f}$. *Then* $\varepsilon' > 0$.

Put now

$$\zeta(L, l_0, s) := \operatorname{sgn} l_0' \, N(\mathfrak{b})^s \sum_{l \in L}^{(u)} \frac{\operatorname{sgn}(l_0 + l)'}{|N(l_0 + l)|^s} \tag{4.1}$$

where (u) at the summation sign means that one should take one representative from each coset $(l_0+l)\varepsilon$ where ε runs over all units $\equiv 1 \bmod \mathfrak{f}$. Notice that $(l_0+L)\varepsilon = l_0 + L$ precisely for such units.

With this conventions, our $\zeta(L, l_0, s)$ is exactly Stark's function denoted $\zeta(s, \mathfrak{c})$ on the page 65 of [St1]: our $\mathfrak{b}, \mathfrak{f}$ have the same meaning in [St1], and our l_0 is Stark's γ. The meaning of Stark's $\mathfrak{c}$ is explained below.

The Stark number of (L, l_0) is defined as

$$S_0(L, l_0) := e^{\zeta'(L, l_0, 0)} . \tag{4.2}$$

The simplest examples correspond to the cases when $(L, l_0) = (1)$, $\mathfrak{f} = L$, in particular, $l_0 = 1$.

Notice that pseudolattices which are integral ideals have conductor $f = 1$.

4.2. Stark's conjecture for real quadratic fields. In [St1], Stark conjectures that $S_0(L, l_0)$ are algebraic units generating abelian extensions of K. To be more precise, let us first describe an abelian extension M/K associated with (L, l_0) using the classical language of class field theory. (Our M is Stark's K, whereas our K corresponds to Stark's k.)

In 4.1 above we constructed, starting with (L, l_0), the ideals $\mathfrak{f}$ and $\mathfrak{b}$ in O_K. Let $I(\mathfrak{f})$ be the group of fractional ideals of K generated by the prime ideals of K not dividing $\mathfrak{f}$, and $S(\mathfrak{f})$ be its subgroup called the principal ray class modulo $\mathfrak{f}$. Then Artin's reciprocity map identifies $G(\mathfrak{f}) := I(\mathfrak{f})/S(\mathfrak{f})$ with the Galois group of M/K.

Consider all pairs (L, l_0) as above with fixed $\mathfrak{f}$. It is not difficult to establish that on this set, $S_0(L, l_0)$ in fact depends only on the class $\mathfrak{c}$ of $(l_0)\mathfrak{b}^{-1}$ in $G(\mathfrak{f})$. Denote the respective number $E(\mathfrak{c})$.

4.2.1. Conjecture. *The numbers* $E(\mathfrak{c})$ *are units belonging to* M *and generating* M *over* K. *If the Artin isomorphism associates with* $\mathfrak{c}$ *an automorphism* σ, *we have* $E(1)^\sigma = E(\mathfrak{c})$.

(We reproduced here the most optimistic form of the Conjecture 1 on page 65 of [St1] involving $m = 1$ and Artin's reciprocity map).

4.3. Hecke's formulas. In this section we will work out Hecke's approach to the computation of sums of the type (4.1), cf. [He2]. It starts with a Mellin transform so that instead of Dirichlet series (4.1) we will be dealing with a version of theta–functions for real quadratic fields. We start with introducing a class of such theta functions more general than strictly needed for dealing with (4.1) (and more general than Hecke's one).

4.4. Theta functions of pseudolattices. Let $K\subset \mathbf{R}$ be as in 4.1. We choose and fix the following data: a pseudolattice $L\subset K$, two numbers $l_0, m_0\in K$ and a number $\eta=\eta_0+i\eta_1\in\mathbf{C}$. A complex variable v will take values in the upper half plane; $\sqrt{-iv}$ is the branch which is positive on the upper part of the imaginary axis.

Finally, choose an infinite cyclic group U of totally positive units in K such that the following conditions hold:

(a) $u(l_0+L)=l_0+L$ for all $u\in U$.

(b) $\mathrm{tr}\, ulm_0\equiv \mathrm{tr}\, lm_0 \bmod \mathbf{Z}$, $\mathrm{tr}\, ul_0m_0\equiv \mathrm{tr}\, l_0m_0 \bmod \mathbf{Z}$ for all $l\in L$, $u\in U$, where $\mathrm{tr}:=\mathrm{tr}_{K/\mathbf{Q}}$.

Let $\varepsilon>1$ be a generator of U.

Put now

$$\Theta^U_{L,\eta}\begin{bmatrix} l_0\\ m_0\end{bmatrix}(v):=$$

$$\sum_{l_0+l \bmod U}(\eta_0\,\mathrm{sgn}\,(l_0'+l')+\eta_1\,\mathrm{sgn}\,(l_0+l))\,e^{2\pi i\,v|(l_0+l)(l_0'+l')|}e^{-2\pi i\,\mathrm{tr}\,lm_0}e^{-\pi i\,\mathrm{tr}\,l_0m_0}. \tag{4.3}$$

Notation $l_0+l \bmod U$ means that we sum over a system of representatives of orbits of U acting upon l_0+L.

Notice that such U always exists, and that if we choose a smaller subgroup $V\subset U$, then

$$\Theta^V_{L,\eta}\begin{bmatrix} l_0\\ m_0\end{bmatrix}(v)=[U:V]\,\Theta^U_{L,\eta}\begin{bmatrix} l_0\\ m_0\end{bmatrix}(v).$$

In order to relate these thetas to Stark's numbers, consider the function

$$\Theta^U_{L,1}\begin{bmatrix} l_0\\ 0\end{bmatrix}(v)=\sum_{l_0+l \bmod U}\mathrm{sgn}\,(l_0'+l')\,e^{2\pi i\,v|(l_0+l)(l_0'+l')|}. \tag{4.4}$$

Then we have

$$\sum_{l_0+l \bmod U}\frac{\mathrm{sgn}\,(l_0'+l')}{|N(l_0+l)|^s}=\frac{(2\pi)^s}{\Gamma(s)}\int_0^{i\infty}(-iv)^s\Theta^U_{L,1}\begin{bmatrix} l_0\\ 0\end{bmatrix}(v)\,\frac{dv}{v}. \tag{4.5}$$

We will now show that these RM thetas can be obtained by averaging some theta constants (related to the complex lattices) along the closed geodesics described above.

4.5. Theta constants along geodesics. Starting with the same data as in 4.4, we consider first of all the Hecke family of lattices $\Lambda_t = \Lambda_t(L)$ (see 3.5 above). From l_0 which was used to shift L, we will produce a shift of Λ_t:

$$\lambda_{0,t} := l_0\, e^{t/2} + il_0'\, e^{-t/2}.$$

The number m_0 determines a character of L appearing in (3.3): $l \mapsto e^{-2\pi i\, \mathrm{tr}\, lm_0}$. Similarly, we will produce a character of Λ_t from

$$\mu_{0,t} := m_0\, e^{t/2} + im_0'\, e^{-t/2}$$

by using the scalar product on $\mathbf{C}$

$$(x \cdot y) = \mathrm{Im}\, xy = x_0y_1 + x_1y_0 \tag{4.6}$$

where $x = x_0 + ix_1$, $y = y_0 + iy_1$. Since $l_0, m_0 \in L \otimes \mathbf{Q}$, we have similarly $\lambda_{0,t}, \mu_{0,t} \in \Lambda_t \otimes \mathbf{Q}$. Omitting t for brevity, we put:

$$\theta_{\Lambda,\eta}\begin{bmatrix}\lambda_0\\ \mu_0\end{bmatrix}(v) := \sum_{\lambda\in\Lambda}((\lambda_0+\lambda)\cdot\eta)\, e^{\pi iv|\lambda_0+\lambda|^2} e^{-2\pi i(\lambda\cdot\mu_0)-\pi i(\lambda_0\cdot\mu_0)}. \tag{4.7}$$

The two types of thetas are related by Hecke's averaging formula:

4.6. Proposition. *We have*

$$\Theta^U_{L,\eta}\begin{bmatrix}l_0\\ m_0\end{bmatrix}(v) = \sqrt{-iv}\int_{-\log\varepsilon}^{\log\varepsilon}\theta_{\Lambda_t,\eta}\begin{bmatrix}\lambda_{0,t}\\ \mu_{0,t}\end{bmatrix}(v)\, dt. \tag{4.8}$$

Proof. The following formulas are valid for $\mathrm{Im}\, v > 0$:

$$e^{2\pi i\, v|mm'|} = \sqrt{-iv}\,|m'|\int_{-\infty}^{\infty} e^{-t/2}e^{\pi iv(m^2e^t+m'^2e^{-t})}\, dt =$$

$$\sqrt{-iv}\,|m|\int_{-\infty}^{\infty} e^{t/2}e^{\pi iv(m^2e^t+m'^2e^{-t})}\, dt \tag{4.9}$$

(see e. g. [La], pp. 270–271). In the rhs of (4.3), replace the first exponent by the integral expressions (4.9), using the first version at η_0 and the second at η_1. We get:

$$\Theta^U_{L,\eta}\begin{bmatrix}l_0\\ m_0\end{bmatrix}(v) =$$

$$\sqrt{-iv}\int_{-\infty}^{\infty}\sum_{l_0+l \bmod U}(\eta_0\,(l_0'+l')\,e^{-t/2}+\eta_1\,(l_0+l)\,e^{t/2})\times$$
$$e^{\pi iv((l_0+l)^2e^t+(l_0'+l')^2e^{-t})}e^{-2\pi i\mathrm{tr}\,lm_0}e^{-\pi i\mathrm{tr}\,l_0m_0}\,dt\,. \tag{4.10}$$

In view of (4.6) we have

$$\eta_0\,(l_0'+l')\,e^{-t/2}+\eta_1\,(l_0+l)\,e^{t/2}=((\lambda_{0,t}+\lambda_t)\cdot\eta),$$
$$(l_0+l)^2e^t+(l_0'+l')^2e^{-t}=|\lambda_{0,t}+\lambda_t|^2,$$

and similarly

$$\mathrm{tr}\,lm_0=(\lambda_t\cdot\mu_{0,t}),\quad \mathrm{tr}\,l_0m_0=(\lambda_{0,t}\cdot\mu_{0,t}).$$

Inserting this into (4.10), we obtain

$$\sqrt{-iv}\int_{-\infty}^{\infty}dt\sum_{l_0+l \bmod U}((\lambda_{0,t}+\lambda_t)\cdot\eta)e^{\pi i\,v|\lambda_{0,t}+\lambda_t|^2}\,e^{-2\pi i\,(\lambda_t\cdot\mu_{0,t})}e^{-\pi i\,(\lambda_{0,t}\cdot\mu_{0,t})}. \tag{4.11}$$

Replacing l_0+l by $\varepsilon(l_0+l)$ is equivalent to replacing t by $t+2\log\varepsilon$. Hence finally the right hand side of (4.11) can be rewritten as

$$\sqrt{-iv}\int_{-\log\varepsilon}^{\log\varepsilon}dt\sum_{\lambda_t\in\Lambda_t}((\lambda_{0,t}+\lambda_t)\cdot\eta)\,e^{\pi i\,v|\lambda_{0,t}+\lambda_t|^2}e^{-2\pi i\,(\lambda_t\cdot\mu_{0,t})-\pi i\,(\lambda_{0,t}\cdot\mu_{0,t})} \tag{4.12}$$

which is the same as (4.8).

We will now apply Poisson formula in order to derive functional equations for Hecke's thetas.

4.7. Poisson formula. Let V be a real vector space, $\widehat{V}$ its dual. We will denote by $(x\cdot y)\in\mathbf{R}$ the scalar product of $x\in V$ and $y\in\widehat{V}$. Choose a lattice (discrete subgroup of finite covolume) $\Lambda\subset V$ and put

$$\Lambda^! := \{\mu\in\widehat{V}\,|\,\forall\lambda\in\Lambda,\,(\lambda\cdot\mu)\in\mathbf{Z}\}. \tag{4.13}$$

Choose also a Haar measure dx on V and define the Fourier transform of a Schwarz function f on V by

$$\widehat{f}(y) := \int_V f(x)\,e^{-2\pi i(x\cdot y)}dx. \tag{4.14}$$

If $f(x)$ in this formula is replaced by $f(x+x_0)\,e^{-2\pi i(x\cdot y_0)-\pi i(x_0\cdot y_0)}$ for some $x_0\in V$, $y_0\in\widehat{V}$, its Fourier transform $\widehat{f}(y)$ gets replaced by $\widehat{f}(y+y_0)\,e^{2\pi i(x_0\cdot y)+\pi i(x_0\cdot y_0)}$.

The Poisson formula reads

$$\sum_{\lambda\in\Lambda} f(\lambda) = \frac{1}{\int_{V/\Lambda} dx} \sum_{\mu\in\Lambda^!} \widehat{f}(\mu), \tag{4.15}$$

and for shifted functions as above

$$\sum_{\lambda\in\Lambda} f(\lambda_0+\lambda)\, e^{-2\pi i(\lambda\cdot\mu_0)-\pi i(\lambda_0\cdot\mu_0)} = \frac{1}{\int_{V/\Lambda} dx} \sum_{\mu\in\Lambda^!} \widehat{f}(\mu_0+\mu)\, e^{2\pi i(\lambda_0\cdot\mu)+\pi i(\lambda_0\cdot\mu_0)}. \tag{4.16}$$

4.8. Functional equations for θ and Θ. In order to transform (4.12) using the Poisson formula, we put

$$V = \mathbf{C} = \{x_0 + ix_1\},\ \widehat{V} = \mathbf{C} = \{y_0 + iy_1\}, \tag{4.17}$$

and take (4.6) for the scalar product.

4.8.1. Lemma. *Let the lattice $\Lambda_t \subset \mathbf{C}$ be the Hecke lattice. Then the dual lattice $\Lambda_t^!$ with respect to the pairing (4.6) has the similar structure*

$$\Lambda_t^! = \Lambda_t(M) := \{me^{t/2} + im'e^{-t/2}\,|\, m \in M\} \tag{4.18}$$

where we denoted by $M = L^?$ the pseudolattice

$$M := \{m \in K\,|\,\forall l \in L,\ \mathrm{tr}_{K/\mathbf{Q}}(l'm) \in \mathbf{Z}.\}.$$

Proof. Denote by Γ the lattice (4.18). For any $\lambda = le^{t/2} + il'e^{-t/2} \in \Lambda_t$ and $\mu = me^{t/2} + im'e^{-t/2} \in \Gamma$ we have

$$(\lambda\cdot\mu) = \mathrm{Im}\,\lambda\mu = lm' + l'm = \mathrm{tr}_{K/\mathbf{Q}}(lm'). \tag{4.19}$$

Therefore this scalar product lies in $\mathbf{Z}$ if $m \in M$ so that $\Gamma \subset \Lambda_t^!$. Clearly, then, Γ must be commensurable with $\Lambda_t^!$, so that the right hand side of (4.19) can be used for computing $(\lambda\cdot\mu)$ on the whole $\Lambda_t^!$. This finishes the proof.

For example, $O_K^? = \mathfrak{d}^{-1}$ where $\mathfrak{d}$ is the different. In fact, this is the standard definition of the different.

Now let l_1, l_2 be two generators of the pseudolattice L. Put

$$\Delta(L) := |l_1l_2' - l_1'l_2|. \tag{4.20}$$

Clearly, this number does not depend on the choice of generators.

4.8.2. Lemma. *Let the Haar measure on* V *be* $dx = dx_0\, dx_1$. *Choose generators* l_1, l_2 *of* L. *Then*

$$\int_{V/\Lambda_t} dx = \Delta(L). \tag{4.21}$$

Proof. If Λ_t is generated by ω_1, ω_2, then the volume (4.21) equals

$$|\mathrm{Re}\,\omega_1\,\mathrm{Im}\,\omega_2 - \mathrm{Re}\,\omega_2\,\mathrm{Im}\,\omega_1|.$$

Taking

$$\omega_1 = l_1 e^{t/2} + il_1' e^{-t/2}, \quad \omega_2 = l_2 e^{t/2} + il_2' e^{-t/2},$$

we get (4.20).

4.8.3. Lemma. *The Fourier transform of*

$$f_{v,\eta}(x) := (x\cdot\eta)\, e^{\pi i v |x|^2},\ \eta = \eta_0 + i\eta_1 \tag{4.22}$$

equals

$$g_{v,\eta}(y) := \frac{i}{v^2}\,(y\cdot i\bar\eta)\, e^{-\frac{\pi i}{v}|y|^2} \tag{4.23}$$

Proof. Putting $w = -iv$ we have

$$f_{v,\eta}(x) = (x_0\eta_1 + x_1\eta_0)\, e^{-\pi w(x_0^2+x_1^2)},$$

so that its Fourier transform by (4.13) and (4.14) is

$$\eta_1 \int_{-\infty}^{\infty} e^{-\pi w x_0^2}\, e^{-2\pi i x_0 y_1}\, x_0\, dx_0 \cdot \int_{-\infty}^{\infty} e^{-\pi w x_1^2}\, e^{-2\pi i x_1 y_0}\, dx_1 +$$
$$\eta_0 \int_{-\infty}^{\infty} e^{-\pi w x_0^2}\, e^{-2\pi i x_0 y_1}\, dx_0 \cdot \int_{-\infty}^{\infty} e^{-\pi w x_1^2}\, e^{-2\pi i x_1 y_0}\, x_1\, dx_1 =$$
$$(\eta_0 y_0 + \eta_1 y_1)\,\frac{1}{iw^2}\, e^{-\pi\frac{y_0^2+y_1^2}{w}}.$$

This is (4.23).

4.8.4. A functional equation for θ. Let us now write (4.16) for $f = f_{v,\eta}$ and Λ_t:

$$\sum_{\lambda\in\Lambda_t} ((\lambda_{0,t}+\lambda)\cdot\eta)\, e^{\pi i v|\lambda_{0,t}+\lambda|^2} e^{-2\pi i(\lambda\cdot\mu_{0,t}) - \pi i(\lambda_{0,t}\cdot\mu_{0,t})} =$$
$$\frac{i}{\Delta(L)\,v^2} \sum_{\mu\in\Lambda_t^!} ((\mu_{0,t}+\mu)\cdot i\bar\eta)\, e^{-\frac{\pi i}{v}|\mu_0+\mu|^2} e^{2\pi i(\lambda_{0,t}\cdot\mu) + \pi i(\lambda_{0,t}\cdot\mu_{0,t})}.$$

In the notation (4.7) this means:

$$\theta_{\Lambda_t,\eta}\begin{bmatrix}\lambda_{0,t}\\ \mu_{0,t}\end{bmatrix}(v) = \frac{i}{\Delta(L)\,v^2}\,\theta_{\Lambda_t^!,i\bar\eta}\begin{bmatrix}\mu_{0,t}\\ -\lambda_{0,t}\end{bmatrix}\left(-\frac{1}{v}\right). \tag{4.24}$$

We now can establish a functional equation for Θ^U as well:

4.9. Proposition. *We have*

$$\Theta^U_{L,\eta}\begin{bmatrix} l_0 \\ m_0 \end{bmatrix}(v) = \frac{1}{\Delta(L)\,v}\,\Theta^U_{L^?,i\bar\eta}\begin{bmatrix} m_0 \\ -l_0 \end{bmatrix}\left(-\frac{1}{v}\right). \tag{4.25}$$

Proof. This is a straightforward consequence of (4.8) and (4.24).

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