REMARKS ON FORMAL DEFORMATIONS

AND BATALIN-VILKOVISKY ALGEBRAS

Vadim Schechtman

Introduction

This note consists of two parts. Part I is an exposition of (a part of) the V.Drinfeld's letter, [D]. The responsibility for the style lies entirely on the author of the present note.

The sheaf of algebras of polyvector fields $\Lambda^\bullet \mathcal{T}_X$ on a Calabi-Yau manifold X, equipped with the Schouten bracket, admits a structure of a Batalin-Vilkovisky algebra. This fact was probably first noticed by Z.Ran, [R]. Part II is devoted to some generalizations of this remark.

To get the above BV structure, one uses a trivialization of the canonical bundle $\mathcal{K}_X$. We note that, in fact, it is sufficient to have an integrable connection on $\mathcal{K}_X$. Moreover, one has a canonical bijection between the set of integrable connections on $\mathcal{K}_X$, or, what is the same, the set of *right* $\mathcal{D}_X$-module structures on $\mathcal{O}_X$, and the set of BV structures on $\Lambda^\bullet \mathcal{T}_X$, cf. **Theorem 4.3**. We give some generalizations of this correspondence, cf. §4.

An interesting example of these data is a pair (a non compact CY manifold X, a function $f \in \Gamma(X, \mathcal{O}_X)$). If we fix a trivialization $\mathcal{K}_X \cong \mathcal{O}_X$, we get the connection on $\mathcal{K}_X$ defined by the exact form df, see §2.

This note is an extended version of the talks given at Max-Planck-Institut für Mathematik in January, 1998. I am very grateful to MPI for the hospitality and the excellent working conditions. I am grateful to Vladimir Hinich for the enlightening correspondence. I am especially obliged to Yu.I. Manin, whose question triggered this work. The contents of Sections 1 and 2 of Part II was influenced by the M. Kontsevich's talk at MPI in January, 1997.

TABLE OF CONTENTS

PART I. MAURER-CARTAN AND FORMAL DEFORMATIONS

§1. Standard functors and Maurer-Cartan equations

1.1. In this Part, we fix once and for all a ground field k of characteristic 0. A "vector space" will mean a vector space over k; $\otimes$ will mean $\otimes_k$.

$Vect$ will denote the category of vector spaces; it is a symmetric monoidal category, with respect to the tensor product. Ass will denote the category of associative algebras in $Vect$ (without unit). The forgetful functor $Ass \longrightarrow Vect$ admits a left adjoint functor of a free associative algebra, to be denoted $\mathcal{F}_{Ass}$. For $V \in Vect$, $\mathcal{F}_{Ass}(V)$ is equal to the tensor algebra

$$T^{\geq 1}(V) = \oplus_{n\geq 1} V^{\otimes n}, \tag{1.1}$$

with the obvious multiplication.

Let Com denote the category of commutative algebras in $Vect$ (assotiative, without unit). The forgetful functor $Com \longrightarrow Vect$ admits a left adjoint, $\mathcal{F}_{Com}$. We have $\mathcal{F}_{Com}(V) = S^{\geq 1}(V)$ where

$$S^{\geq 1}(V) = \oplus_{n\geq 1} S^n(V) \tag{1.2}$$

Here $S^n(V)$ is the n-th symmetric power of V, i.e. the quotient of $V^{\otimes n}$ over the obvious action of the symmetric group Σ_n.

The canonical projection $V^{\otimes n} \longrightarrow S^n(V)$ admits a canonical splitting $i:\ S^n(V) \hookrightarrow V^{\otimes n}$, given by

$$i(x_1 \cdot \ldots \cdot x_n) = \frac{1}{n!} \sum_{\sigma\in\Sigma_n} x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)} \tag{1.3}$$

We denote by Lie the category of Lie algebras in $Vect$, and by $\mathcal{F}_{Lie}$ the functor of a free Lie algebra, the left adjoint to the forgetful functor $Lie \longrightarrow Vect$.

1.2. Let C be a coassociative coalgebra in $Vect$, with comultiplication $\Delta:\ C \longrightarrow C\otimes C$. For $n \geq 2$, define a map

$$\Delta^{(n)}:\ C \longrightarrow C^{\otimes n} \tag{1.4}$$

by induction. We set $\Delta^{(2)} = \Delta$, and

$$\Delta^{(n+1)} = \left(id_{A^{\otimes(n-1)}} \otimes \Delta\right) \circ \Delta^{(n)} \tag{1.5}$$

Let us consider the following condition

(F) *For each $x \in C$, there exists n such that $\Delta^{(n)}(x) = 0$.*

We denote by $Coass$ the category of coassociative coalgebras in $Vect$ (without counit) satisfying **(F)**. The forgetful functor $Coass \longrightarrow Vect$ admits a *right* adjoint, the functor of a *cofree* coalgebra, to be denoted $\mathcal{F}_{Coass}$. For $V \in Vect$, $\mathcal{F}_{Coass}(V) = T^{\geq 1}(V)$ as a vector space. The comultiplication is defined by

$$\Delta(v_1 \otimes \ldots \otimes v_n) = \sum_{i=1}^{n-1} (v_1 \otimes \ldots \otimes v_i) \otimes (v_{i+1} \otimes \ldots \otimes v_n) \tag{1.6}$$

We denote by $Cocom$ the category of cocommutative coalgebras in $Vect$ (coassociative, without counit). The forgetful functor $Cocom \longrightarrow Vect$ admits a right adjoint, the functor of a cofree cocommutative coalgebra, to be denoted by $\mathcal{F}_{Cocom}$. For $V \in Vect$, $\mathcal{F}_{Cocom}(V) = S^{\geq 1}(V)$ as a vector space. The comultiplication is defined as the composition

$$S^{\geq 1}(V) \hookrightarrow T^{\geq 1}(V) \longrightarrow T^{\geq 1}(V) \otimes T^{\geq 1}(V) \longrightarrow S^{\geq 1}(V) \otimes S^{\geq 1}(V) \tag{1.7}$$

Here the first arrow is the canonical injection (1.3), the second one is the comultiplication in $T^{\geq 1}(V)$, and the third one is the canonical projection.

1.3. By a *graded vector space*, we mean a collection of vector spaces $V^\bullet = \{V^i\}_{i\in\mathbb{Z}}$, indexed by integers. Graded vector spaces form the category $GVect$, with obvious morphisms. It is a monoidal tensor category. For $V^\bullet,\ W^\bullet \in GVect$, their tensor product $V^\bullet \otimes W^\bullet$ is defined by

$$(V^\bullet \otimes W^\bullet)^i = \oplus_{p+q=i}\ V^p \otimes W^q \tag{1.8}$$

The associativity constraint is the obvious one. The commutativity isomorphisms $s_{V^\bullet,W^\bullet}:\ V^\bullet \otimes W^\bullet \xrightarrow{\sim} W^\bullet \otimes V^\bullet$ are given by

$$s_{V^\bullet,W^\bullet}(a \otimes b) = (-1)^{pq} b \otimes a, \tag{1.9}$$

for $a \in V^p,\ b \in W^q$. We often use the notation $|a| = p$.

We have the *shift functors* $[i]:\ GVect \longrightarrow GVect$ $(i \in \mathbb{Z})$, given by $(V^\bullet[i])^j = V^{i+j}$. By a *morphism of degree* i between $V^\bullet$ and $W^\bullet$ we mean a morphism $V^\bullet \longrightarrow W^\bullet[i]$ in $GVect$. We define $\mathcal{H}om(V^\bullet, W^\bullet) \in GVect$ by

$$\mathcal{H}om(V^\bullet, W^\bullet)^i = Hom_{GVect}(V^\bullet, W^\bullet) \tag{1.10}$$

We define the categories $GAss, GCoass, GCom, GCocom$ and $GLie$ as the categories of associative algebras, coassociative coalgebras, commutative algebras, cocommutative coalgebras (all without unit or counit) and Lie algebras in $GVect$. *The coalgebras are assumed to satisfy the condition* **(F)**. We regard $Vect$, etc. as a full subcategory of $GVect$, etc., consisting of objects living in degree 0.

We have the functors of (co)free (co)algebras $\mathcal{F}_{GAss}, \mathcal{F}_{GCoass}, \mathcal{F}_{GCom}, \mathcal{F}_{GCocom}$ and $\mathcal{F}_{GLie}$, defined as in the case of non-graded algebras.

If $A^\bullet \in GAss$ and $B^\bullet \in GCoass$ then $\mathcal{H}om(B^\bullet, A^\bullet)$ admits a canonnical structure of a coassociative coalgebra. For two maps $f \in \mathcal{H}om(B^\bullet, A^\bullet)^i,\ g \in \mathcal{H}om(B^\bullet, A^\bullet)^j$, their product $f \cdot g$ is defined as a composition

$$f \cdot g:\ B^\bullet \longrightarrow B^\bullet \otimes B^\bullet \xrightarrow{f\otimes g} A^\bullet[i] \otimes A^\bullet[j] \longrightarrow A^\bullet[i+j] \tag{1.11}$$

Here the first arrow is the comultiplication in $B^\bullet$, and the third one is the multiplication in $A^\bullet$. The second one is defined by

$$(f \otimes g)(b \otimes c) = (-1)^{j|b|} f(b) \otimes g(c) \tag{1.12}$$

Similarly, if $\mathfrak{g}^\bullet \in GLie$ and $B^\bullet \in GCocom$, $\mathcal{H}om(B^\bullet, \mathfrak{g}^\bullet)$ admits a canonical structure of a graded Lie algebra.

If $A^\bullet \in GCom$ then $A^\bullet \otimes \mathfrak{g}^\bullet$ is canonically a graded Lie algebra, with the bracket defined by

$$[a \otimes x, b \otimes y] = (-1)^{|x||a|} ab \otimes [x, y] \tag{1.13}$$

A *differential graded (dg)* vector space, or a *complex*, is a graded vector space $V^\bullet$ equipped with a differential of degree 1, $d:\ V^\bullet \longrightarrow V^\bullet[1],\ d^2 = 0$. We can repeat all the discussion of this Subsection, adding the word "differential" to "graded". The corresponding categories are denoted by $DGVect, DGAss$, etc.

1.4. Let A be an associative algebra. Consider the cofree graded coalgebra $\mathcal{F}_{GCoass}(A[1])$. There exists a unique differential d on this graded space, making it a dg coassociative coalgebra, whose degree -2 component

$$d^{-2}:\ \mathcal{F}_{GCoass}(A[1])^{-2} = A \otimes A \longrightarrow \mathcal{F}_{GCoass}(A[1])^{-1} = A$$

coincides with the multiplication μ in A. Note that the condition $d^2 = 0$ is equivalent to the associativity of μ.

Let us denote this dg coalgebra by $\mathcal{C}_{Coass}(A)$. This way we get a functor $\mathcal{C}_{Coass}:$ $Ass \longrightarrow DGCoass$ which extends naturally to a functor

$$\mathcal{C}_{Coass}:\ DGAss \longrightarrow DGCoass \tag{1.14}$$

Conversely, let B be a coassociative coalgebra. Consider the free graded algebra $\mathcal{F}_{GAss}(B[-1])$. There exists a unique differential d on this space, making it a dg associative algebra, whose degree 1 component

$$d^1:\ \mathcal{F}_{GAss}(B[-1])^1 = B \longrightarrow \mathcal{F}_{GAss}(B[-1])^2 = B \otimes B \tag{1.15}$$

coincides with the comultiplication ν in B. The condition $d^2 = 0$ is equivalent to the coassociativity of ν.

Let us denote this dg algebra by $\mathcal{C}_{Ass}(B)$. This way we get a functor $\mathcal{C}_{Ass}:$ $Coass \longrightarrow DGAss$, which extends naturally to a functor

$$\mathcal{C}_{Ass}:\ DGCoass \longrightarrow DGAss \tag{1.16}$$

1.5. Let $A^\bullet$ be a dg associative algebra. The **Maurer-Cartan equation** in $A^\bullet$ is the equation

$$da + a \cdot a = 0, \tag{1.17}$$

$a \in A^1$. The set of all $a \in A^1$ satisfying (1.7) will be denoted $MC(A^\bullet)$.

1.7. Let $\widehat{Vect}$ denote the category whose objects are topological vector spaces V, complete in a linear topology. Thus, the topology admits a base of neighbourhoods of 0 consisting of subspaces $V^{(i)}$, and V is the inverse limit of the spaces with discrete topology,

$$V = \lim_{\longleftarrow}\ V/V^{(i)} \tag{1.19}$$

Morphisms are continuous linear maps. This is a monoidal tensor category, with the tensor product $\widehat{\otimes}$ given by

$$V\widehat{\otimes}W = \lim_{\longleftarrow} V/V^{(i)} \otimes W/W^{(j)} \tag{1.20}$$

We regard the tensor category $Vect$ as embedded in $\widehat{Vect}$, as a full subcategory of spaces with the discrete topology.

Let $\widehat{Coass}, \widehat{Cocom}$, etc. denote the category of coassociative coalgebras, cocommutative coalgebras, etc., in $\widehat{Vect}$. *We do not impose on the coalgebras the finiteness condition* **(F)**.

The forgetful functor $\widehat{Coass} \longrightarrow \widehat{Vect}$, resp., $\widehat{Cocom} \longrightarrow \widehat{Vect}$, admits a right adjoint

$$\mathcal{F}_{\widehat{Coass}}(V) = \widehat{T}^{\geq 1}(V) := \prod_{n\geq 1} V^{\otimes n}, \tag{1.21}$$

resp.,

$$\mathcal{F}_{\widehat{Cocom}}(V) = \widehat{S}^{\geq 1}(V) := \prod_{n\geq 1} S^n(V) \tag{1.22}$$

We define the category $\widehat{GCoass}, \widehat{DGCoass}$, etc., in the obvious way.

1.9. We define the functor

$$\widehat{\mathcal{C}}_{Coass}:\ \widehat{DGAss} \longrightarrow \widehat{DGCoass} \tag{1.23}$$

in the same way as (1.14).

Given $A^\bullet \in \widehat{DGAss}$, a solution a of (1.17) gives rise to a 0-cocycle of the dg coalgebra $\widehat{\mathcal{C}}_{Coass}(A^\bullet)$, given by

$$\sum_{n=1}^{\infty} a^{\otimes n} \in \widehat{T}^{\geq 1}(A^1) \subset \widehat{\mathcal{C}}_{Coass}(A^\bullet)^0 \tag{1.24}$$

1.10. Theorem. *Let $A^\bullet \in DGAss$, $B^\bullet \in DGCoass$. We have canonically*

$$Hom_{DGAss}(\mathcal{C}_{Ass}(B^\bullet), A^\bullet) = MC(\mathcal{H}om(B^\bullet, A^\bullet)) =$$

$$= Hom_{DGCoass}(B^\bullet, \mathcal{C}_{Coass}(A^\bullet)) \tag{1.25}$$

In particular, the functor $\mathcal{C}_{Ass}$ is left adjoint to $\mathcal{C}_{Coass}$. $\triangle$

1.11. Let $\mathfrak{g}$ be a Lie algebra. Consider the cofree coalgebra $\mathcal{F}_{GCocom}(\mathfrak{g}[1])$ on the graded space $\mathfrak{g}[1]$. There exists a unique differential

$$d:\ \mathcal{F}_{GCocom}(\mathfrak{g}[1]) \longrightarrow \mathcal{F}_{GCocom}(\mathfrak{g}[1])[1]$$

making $\mathcal{F}_{GCocom}\mathfrak{g}[1]$ a dg cocommutative coalgebra, and whose degree -2 component

$$d^{-2}:\ \mathcal{F}_{Cocom}(\mathfrak{g}[1])^2 = \Lambda^2(\mathfrak{g}) \longrightarrow \mathcal{F}_{Cocom}(\mathfrak{g}[1])^1 = \mathfrak{g}$$

coincides with the bracket in $\mathfrak{g}$. Let us denote this dg coalgebra $\mathcal{C}_{Cocom}(\mathfrak{g})$. This way we get a functor $\mathcal{C}_{Cocom} : Lie \longrightarrow DGCocom$, which extends naturally to a functor

$$\mathcal{C}_{Cocom} : DGLie \longrightarrow DGCocom \tag{1.26}$$

Conversely, let B be a cocommutative coalgebra. Consider the free graded Lie algebra $\mathcal{F}_{Lie}(B[-1])$. There exists a unique differential on this space, making it a dg Lie algebra, whose degree 1 component

$$d^1 : \mathcal{F}_{Lie}(B[-1])^1 = B \longrightarrow \mathcal{F}_{Lie}(B[-1])^2 = S^2(B)$$

coincides with the comultiplication in B. Let us denote this dg Lie algebra by $\mathcal{C}_{Lie}(B)$. This way we get a functor $\mathcal{C}_{Lie} : Cocom \longrightarrow DGLie$ which extends naturally to a functor

$$\mathcal{C}_{Lie} : DGCocom \longrightarrow DGLie \tag{1.27}$$

In a similar manner, one defines the functor

$$\widehat{\mathcal{C}}_{Cocom} : DG\widehat{Lie} \longrightarrow DG\widehat{Cocom} \tag{1.28}$$

1.12. Let $\mathfrak{g}^\bullet$ be a dg Lie algebra. The **Maurer-Cartan equation** in $\mathfrak{g}^\bullet$ is the equation

$$da + \frac{1}{2}[a, a] = 0, \tag{1.30}$$

$a \in \mathfrak{g}^1$. The set of all $a \in \mathfrak{g}^1$ satisfying (1.3) will be denoted by $MC(\mathfrak{g}^\bullet)$.

A solution a of (1.30) gives rise to a 0-cocycle of the dg coalgebra $\widehat{\mathcal{C}}_{Cocom}(\mathfrak{g}^\bullet)$, given by

$$\sum_{n=1}^{\infty} \frac{a^n}{n!} \in \widehat{S}^{\geq 1}(\mathfrak{g}^1) \subset \widehat{\mathcal{C}}_{Cocom}(\mathfrak{g}^\bullet) \tag{1.31}$$

1.13. Theorem. *Let $\mathfrak{g}^\bullet \in DGLie$, $B^\bullet \in DGCocom$. We have canonically*

$$Hom_{DGLie}(\mathcal{C}_{Lie}(B^\bullet), \mathfrak{g}^\bullet) = MC(\mathcal{H}om(B^\bullet, \mathfrak{g}^\bullet)) =$$

$$= Hom_{DGCocom}(B^\bullet, \mathcal{C}_{Cocom}(\mathfrak{g}^\bullet)) \tag{1.32}$$

In particular, the functor $\mathcal{C}_{Lie}$ is left adjoint to $\mathcal{C}_{Cocom}$.

Assume for simplicity that B is concentrated in degree 0. By definition, a map of graded Lie algebras $\alpha : \mathcal{C}_{Lie}(B) \longrightarrow \mathfrak{g}^\bullet$ is the same as a map of vector spaces $\alpha^1 : B \longrightarrow \mathfrak{g}^1$; the map α is compatible with the differentials iff the map α^1 satisfies the Maurer-Cartan equation. This proves the first equality in this case. The rest is proved similarly. $\triangle$

§2. Koszul complex and Maurer-Cartan scheme

A. KOSZUL

2.0. In this section we will use (dg) algebras and coalgebras with unit (resp., counit). We will use the notation $(\cdot)^+$ for the objects obtained by formally adjoining a unit or a counit. We denote by Com^+, $Cocom^+$ the categories of unital algebras, resp., coalgebras.

For a (dg) vector space V, we denote

$$S^\bullet(V) = \oplus_{n\geq 0}\, S^n(V) \tag{2.0}$$

It is the underlying vector space for $\mathcal{F}_{Com}(V)^+$ and $\mathcal{F}_{Cocom}(V)^+$.

2.1. Let $\mathfrak{g}^\bullet$ be a graded Lie algebra. Assume that $\mathfrak{g}^1$ is finite dimensional. Then the Maurer-Cartan equation (1.30) defines a closed subscheme of $\mathfrak{g}^1$, considered as an affine space, to be called the **Maurer-Cartan scheme** of $\mathfrak{g}^\bullet$, and denoted also by $MC(\mathfrak{g}^\bullet)$. It represents the functor $Com^+ \longrightarrow Sets$ which assigns to a k-algebra B the set $MC(\mathfrak{g}^\bullet \otimes B)$, cf. (1.3).

More explicitely, set

$$A = S^\bullet(\mathfrak{g}^{1*}) \tag{2.1}$$

$\mathfrak{g}^{1*}$ denotes the dual space. We have $\mathfrak{g}^1 = Spec(A)$.

Let us choose a base $\{e_i\}$ of $\mathfrak{g}^{2*}$. Define the elements

$$f_i = (e_i \circ d, e_i \circ [\ ,\ ]) \in \mathfrak{g}^{1*} \oplus S^2(\mathfrak{g}^{1*}) \subset A \tag{2.2}$$

Here $d:\ \mathfrak{g}^1 \longrightarrow \mathfrak{g}^2$ is the component of the differential, and $[\ ,\ ]:\ S^2\mathfrak{g}^1 \longrightarrow \mathfrak{g}^2$ is the component of the bracket.

We have

$$MC(\mathfrak{g}^\bullet) = Spec\big(A/(f_i)\big) \tag{2.3}$$

Here we have denoted by (f_i) the ideal generated by all f_i.

2.2. Let us assume that $\mathfrak{g}^i = 0$ for $i \neq 1, 2$. Such a dg Lie algebra is the same as the following set of data:

two vector spaces $V = \mathfrak{g}^1$, $W = \mathfrak{g}^2$, a linear map $d:\ V \longrightarrow W$, and a symmetric map $b:\ S^2(V) \longrightarrow W$.

We have

$$\mathcal{C}_{Cocom}(\mathfrak{g}^\bullet)^+ = S^\bullet(V) \otimes \Lambda^\bullet(W) \tag{2.4}$$

as a graded coalgebra. Here $\Lambda^\bullet(W) = \oplus_{n\geq 0}\, \Lambda^n(W)$ is the exterior algebra. In (2.4), V has grading 0, and W has grading 1.

The differential acts as

$$0 \longrightarrow S^\bullet(V) \longrightarrow S^\bullet(V) \otimes W \longrightarrow S^\bullet(V) \otimes \Lambda^2(W) \longrightarrow \ldots \tag{2.5}$$

This complex is dual to a Koszul complex. To make a precise statement, assume that V, W are finite dimensional. Set $A = S^\bullet(V^*)$, as in (2.1).

Then the (restricted) dual of (2.5) is

$$\ldots \longrightarrow \Lambda^2(W) \otimes A \longrightarrow W \otimes A \longrightarrow A \longrightarrow 0 \tag{2.6}$$

If we choose a basis $\{e_i\}$ of the linear space W^*, we get the elements $f_i \in \mathfrak{m}_A$, as in (2.2). The complex (2.5) is nothing but the Koszul complex $K(A; (f_i))$, cf. II, 1.1.

2.3. Corollary. *In the assumptions of* 2.2, *one has canonical isomorphism of commutative algebras*

$$\widehat{\mathcal{O}}_{MC(\mathfrak{g}^\bullet);0} = \left(H^0(\mathcal{C}_{Cocom}(\mathfrak{g}^\bullet)^+)\right)^* \tag{2.7}$$

Here in the left hand side stays the completion of the local ring of the origin of the scheme $MC(\mathfrak{g}^\bullet)$.

2.4. More generally,

$$\mathcal{C}_{Cocom}(\mathfrak{g}^\bullet)^+ = S^\bullet(\mathfrak{g}^{odd}) \otimes \Lambda^\bullet(\mathfrak{g}^{ev}) \tag{2.8}$$

for an arbitrary dg Lie algebra $\mathfrak{g}^\bullet$. Here $\mathfrak{g}^{odd} = \oplus_{i\in\mathbb{Z}}\ \mathfrak{g}^{2i+1},\ \mathfrak{g}^{ev} = \oplus_{i\in\mathbb{Z}}\ \mathfrak{g}^{2i}$.

Assume that $\mathfrak{g}^i = 0$ for $i \leq 0$. Then

$$\mathcal{C}_{Cocom}(\mathfrak{g}^\bullet)^+ = \oplus_{p=1}^{\infty}\ \oplus_{i_1,\ldots,i_p\geq 0}\ \left(S^{i_1}(\mathfrak{g}^1) \otimes \Lambda^{i_2}(\mathfrak{g}^2) \otimes \ldots \otimes F^{i_p}(\mathfrak{g}^p)\right) \tag{2.9}$$

where $F = S$ if p is odd, and $F = \Lambda$ if p is even. In this space the component $\mathfrak{g}^p$ has degree $p-1$. With the differential, this complex looks as

$$0 \longrightarrow S^\bullet(\mathfrak{g}^1) \longrightarrow S^\bullet(\mathfrak{g}^1) \otimes \mathfrak{g}^2 \longrightarrow S^\bullet(\mathfrak{g}^1) \otimes \left(\Lambda^2(\mathfrak{g}^2) \oplus \mathfrak{g}^3\right) \longrightarrow \ldots \tag{2.10}$$

In other words, the very beginning depends only on $\mathfrak{g}^{\leq 2}$. Therefore, we get

2.5. Theorem. *Let $\mathfrak{g}^\bullet$ be a dg Lie algebra sitting in degrees ≥ 1. Assume that $\mathfrak{g}^1$ is finite dimensional. We have a canonical isomorphism of commutative algebras*

$$\widehat{\mathcal{O}}_{MC(\mathfrak{g}^\bullet);0} = \left(H^0(\mathcal{C}_{Cocom}(\mathfrak{g}^\bullet)^+)\right)^* \tag{2.11}$$

B. CHEVALLEY-EILENBERG

2.6. A dg Lie algebra $\mathfrak{g}^\bullet$ such that $\mathfrak{g}^i = 0$ for $i \neq 0, 1$, is the same as the following set of data:

a Lie algebra $U = \mathfrak{g}^0$, a U-module $V = \mathfrak{g}^1$, and a linear map $d:\ U \longrightarrow V$ such that $d([u, u']) = u \cdot du' - u' \cdot du$, for all $u, u' \in U$.

Here we have denoted by $u \cdot v$ the result of the action of an element $u \in U$ on $v \in V$.

We want to describe the dg coalgebra $\mathcal{C}_{Cocom}(\mathfrak{g}^\bullet)^+$. We have

$$\mathcal{C}_{Cocom}(\mathfrak{g}^\bullet)^+ = \Lambda^\bullet(U) \otimes S^\bullet(V) \tag{2.12}$$

as a graded coalgebra, where we assign the degree -1 to U, and the degree 0 to V.

Consider the cocommutative counital coalgebra $B = S^\bullet(V)$. The action of the Lie algebra U on V induces the U-action on all symmetric powers

$$U \otimes S^n(V) \longrightarrow S^n(V)$$

On the other hand, the differential $d:\ U \longrightarrow V$ induces the maps

$$U \otimes S^n(V) \stackrel{d\otimes id}{\longrightarrow} V \otimes S^n(V) \longrightarrow S^{n+1}(V), \tag{2.13}$$

the last arrow being the multiplication. Adding up, we get the maps

$$U \otimes S^n(V) \longrightarrow S^n(V) \oplus S^{n+1}(V)$$

Adding over n, we get a map

$$U \otimes B \longrightarrow B \tag{2.14}$$

which defines the action of the Lie algebra U by coderivations on the coalgebra B.

Geometrically, this action can be interpreted as follows. Assume that the space V is finite dimensional. Consider the commutative algebra $A = S^\bullet(V^*)$, and regard V as an affine scheme $Spec(A)$. The Lie algebra U acts on V by affine vector fields; namely, to an element $u \in U$, there corresponds the vector field τ_u, whose value at a point $v \in V$ is equal to $du + u \cdot v$. Here we have identified the tangent space $T_{V;v}$ with V. Therefore, we get the action of U on the functions

$$U \otimes A \longrightarrow A \tag{2.15}$$

Now, the action (2.10) is nothing but the dual to (2.11).

Consider the homological Chevalley-Eilenberg complex of the Lie algebra U with coefficients in the B, where the action is (2.10),

$$C_{homol}(U; B):\ \ldots \longrightarrow \Lambda^2(U) \otimes B \longrightarrow U \otimes B \longrightarrow B \longrightarrow 0 \tag{2.16}$$

Let us regard it as living in degrees ≤ 0.

2.7. Lemma. *The complex* (2.16) *coincides with* $\mathcal{C}_{Cocom}(\mathfrak{g}^\bullet)^+$*, if we use the identification* (2.12).

2.8. Corollary. *Assume that* V *is finite dimensional. Then we have a canonical isomorphism of pro-artinian algebras,*

$$\widehat{\mathcal{O}}^U_{\mathcal{O}_V;0} = \left(H^0(\mathcal{C}_{Cocom}(\mathfrak{g}^\bullet)^+)\right)^* \tag{2.17}$$

Here the superscript $(\cdot)^U$ means the subspace of U-invariants.

2.9. Let us introduce a shorthand notation

$$C(\mathfrak{g}^\bullet)=\mathcal{C}_{Cocom}(\mathfrak{g}^\bullet)^+ \tag{2.18}$$

Assume that $\mathfrak{g}^i=0$ for $i<0$. The dg Lie algebra $\mathfrak{g}^{\ge 1}$ is a dg module over the Lie algebra $\mathfrak{g}^0$, therefore its symmetric powers are also dg $\mathfrak{g}^0$-modules, i.e. we have the action maps

$$\mathfrak{g}^0\otimes S^n(\mathfrak{g}^\bullet)\longrightarrow S^n(\mathfrak{g}^\bullet) \tag{2.19}$$

On the other hand, we have the maps

$$\mathfrak{g}^0\otimes S^n(\mathfrak{g}^1)\longrightarrow S^{n+1}(\mathfrak{g}^1) \tag{2.20}$$

defined as in (2.13). Adding the maps (2.19) and (2.20) and summing up over $n\ge 0$ and $p\ge 1$, we get a map

$$\mathfrak{g}^0\otimes S^\bullet(\mathfrak{g}^{\ge 1}[1])\longrightarrow S^\bullet(\mathfrak{g}^{\ge 1}[1]) \tag{2.21}$$

which defines a structure of a $\mathfrak{g}^0$-module on $S^\bullet(\mathfrak{g}^{\ge 1}[1])=C(\mathfrak{g}^{\ge 1})$.

2.10. Theorem. *The cocommutative dg coalgebra $C(\mathfrak{g}^\bullet)$ is canonically isomorphic to the Chevalley-Eilenberg complex $C_{homol}(\mathfrak{g}^0;C(\mathfrak{g}^{\ge 1}))$, where the action of $\mathfrak{g}^0$ on $C(\mathfrak{g}^{\ge 1})$ is defined in* (2.21).

The coalgebra structure on the Chevalley-Eilenberg complex comes from the cocommutative coalgebra structure on $C(\mathfrak{g}^{\ge 1})$.

2.11. For a more general discussion, I refer the reader to the very nice paper [H].

§3. Deformations

3.1. Let $C^\bullet$ be a graded coalgebra. Recall that a *coderivation* of degree d of $C^\bullet$ is a linear mapping $D:\ C^\bullet\longrightarrow C^\bullet[d]$ satisfying the *co-Leibnitz rule*

$$\Delta(D(x))=\sum\ (D(a)\otimes b+(-1)^{d\cdot|a|}a\otimes D(b)) \tag{3.1}$$

if $\Delta(x)=\sum\ a\otimes b$. Here Δ is the comultiplication in $B^\bullet$. The coderivations form a fraded Lie algebra, to be denoted $Coder(C^\bullet)$. The bracket is given by

$$[D_1,D_2]=D_1D_2-(-1)^{|D_1||D_2|}D_2D_1 \tag{3.2}$$

Let V be a vector space. Consider the cofree graded coalgebra $\mathcal{F}_{GCoass}(V[-1])$. Set

$$\mathfrak{g}_V^\bullet=Coder(\mathcal{F}_{GCoass}(V[-1])) \tag{3.3}$$

As a graded vector space,

$$\mathfrak{g}_V^\bullet=\oplus_{i\ge 0}\ Hom(V^{\otimes i},V), \tag{3.4}$$

where $\mathfrak{g}^i = Hom(V^{\otimes(i+1)}, V)$.

3.2. Lemma. *Suppose we are given a map*

$$f:\ V\otimes V \longrightarrow V$$

It is associative if and only if, when considered as an element of $\mathfrak{g}^1_V$, *it satisfies the equation*

$$\frac{1}{2}[f,f]=0$$

Given such f, if we extend it to the coderivation of degree 1 of

$$\mathcal{F}_{Coass}\ (V[-1]) = \oplus_{i\geq 1}\ V^{\otimes i}$$

its component

$$f_3:\ V^{\otimes 3} \longrightarrow V^{\otimes 2}$$

is given by

$$f_3(a\otimes b\otimes c) = f(a\otimes b)\otimes c - a\otimes f(b\otimes c),$$

by the co-Leibnitz rule. Therefore the component of $\frac{1}{2}[f,f]=f\cdot f$ acting from $V^{\otimes 3}$ to V, sends $a\otimes b\otimes c$ to $f(f(a\otimes b)\otimes c) - f(a\otimes f(b\otimes c))$. The Lemma follows. $\triangle$.

3.3. Let $(A, f:A\otimes A\longrightarrow A)$ be an associative algebra. Let us define the dg Lie algebra $\mathfrak{g}^\bullet_f$, which is equal to

$$Coder(\mathcal{F}_{GCoass}(A[1])) = \oplus_{i\geq 0}\ Hom(A^{\otimes i}, A) \tag{3.5}$$

as a graded Lie algebra, and with the differential $d_f = ad(f)$.

Here we consider f as an element of $\mathfrak{g}^1_f$, and by the previous Lemma, $[f,f]=0$.

Now, let $f' = f+h:\ A\otimes A\longrightarrow A$ be another candidate for the multiplication. We have

$$\frac{1}{2}[f+h,f+h] = [f,h]+\frac{1}{2}[h,h] = d_f h + \frac{1}{2}[h,h]$$

Let M_A denote the set of associative multiplications on A.

More generally, define M_A as a functor $Com^+\longrightarrow Sets$ which assigns to a commutative unital k-algebra B the set of associative multiplications on $A\otimes B$.

We have proven

3.4. Lemma. *Assume that* A *is finite dimensional. Then* M_A *is represented by an affine scheme. We have a canonical isomorphism of schemes*

$$M_A = MC(\mathfrak{g}^\bullet_f)$$

which takes $f\in M_A(k)$ *to* $0\in MC(\mathfrak{g}^\bullet_f)(k)$.

$\triangle$

Applying Theorem 2.5, we get

3.5. Theorem. *We have a canonical isomorphism of formal commutative algebras*

$$\widehat{\mathcal{O}}_{M_A;f} = \left(H^0(\mathcal{C}_{Cocom}(\mathfrak{g}^{\geq 1}_f)^+)\right)^* \tag{3.6}$$

PART II. MILNOR RING AND BATALIN-VILKOVISKY ALGEBRAS

§1. Koszul complex and Milnor ring

1.1. Koszul complex. Let A be a commutative ring, and $f_1, \ldots , f_r$ a sequence of its elements.

We denote by $K^\bullet(A; f_1, \ldots , f_r)$ the complex

$$0 \longrightarrow \Lambda^r(A^r) \longrightarrow \Lambda^{r-1}(A^r) \longrightarrow \ldots \longrightarrow A^r \longrightarrow A \longrightarrow 0 \tag{1.1}$$

The exterior powers are over A. The differential is given by

$$d(a_{i_1} \wedge \ldots \wedge a_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} f_{i_j} a_{i_1} \wedge \ldots \wedge \hat{a}_{i_j} \wedge \ldots \wedge a_{i_p} \tag{1.2}$$

Here for $a \in A$ and $1 \leq i \leq r$, we denote by $a_i \in A^r$ the element $(0, \ldots , a, \ldots , 0)$, with a on i-th place. We consider this complex as living in degrees from $-r$ to 0.

We have

$$K^\bullet(A; f_1, \ldots , f_r) = K^\bullet(A; f_1) \otimes_A \ldots \otimes_A K^\bullet(A; f_r) \tag{1.3}$$

Equipped with the obvious wedge product, $K^\bullet(A; f_1, \ldots , f_r)$ becomes a commutative dg A-algebra.

In fact, the differential (1.2) is the unique differential on the exterior algebra $\Lambda^\bullet(A^r)$, whose component $A^r \longrightarrow A$ is given by

$$d(a_1, \ldots , a_r) = \sum_{i=1}^{r} f_i a_i, \tag{1.4}$$

and making it a commutative dg algebra.

1.2. Assume that $(f_1, \ldots , f_r)$ is a regular sequence, i.e. the operator of multiplication by f_i on $A/(f_1, \ldots , f_{i-1})$ is injective, for all i. Then the Koszul complex $K^\bullet(A; f_1, \ldots , f_r)$ is acyclic in negative degrees.

This is proved easily by induction on r, using (1.3), cf. [EGA III], Prop. 1.1.4.

Obviously,

$$H^0(K^\bullet(A; f_1, \ldots , f_r)) = A/(f_1, \ldots , f_r) \tag{1.5}$$

1.3. From now on, until mentioned otherwise, the ring A will be equal to a polynomial algebra $k[x_1, \ldots , x_n]$, k being a fixed ground commutative ring. We denote by

$$\mathcal{T}(A) = Der_k(A)$$

the Lie algebra of k-linear derivations of A. We identify $\mathcal{T}(A)$ with A^n using the basis $\partial_1, \ldots , \partial_n$, where $\partial_i := \partial/\partial x_i$.

We fix a function $f \in A$ such that $f(0) = 0$. We denote by $M^\bullet(A; df)$ the Koszul complex

$$M^\bullet(A; df) = K^\bullet(A; \partial_1 f, \dots, \partial_n f) \tag{1.6}$$

Thus,

$$M^\bullet(A; df) = \Lambda^\bullet \mathcal{T}(A) \tag{1.7}$$

as a graded commutative algebra, with $\mathcal{T}(A)$ having the degree -1. The $(-1)-$component of the differential $\mathcal{T}(A) \longrightarrow A$ sends $\tau \in \mathcal{T}(A)$ to $\tau(f)$, cf. the remark at the end of 1.1.

1.4. Lemma. *The following conditions are equivalent*:

(i) $\dim(A/(\partial_1 f, \dots, \partial_n f)) < \infty$;

(ii) $M^\bullet(A; df)$ *is acyclic in negative degrees;*

(iii) $(\partial_1 f, \dots, \partial_n f)$ *is a regular sequence.*

This is a standard fact of Commutative Algebra, and follows from [S], Ch. IV, B.2, Théorème 2, Ch. III, B.3.

If these equivalent conditions are fulfilled, we say that f *has an isolated singularity* at 0. The only non-zero cohomology space,

$$A/(\partial_1 f, \dots, \partial_n f) = H^0(M^\bullet(A; df)) \tag{1.8}$$

is called the **Milnor ring** of f at 0, cf. [M], §7.

Thus, the commutative dg algebra $M^\bullet(A; df)$ may be considered as a resolution of the Milnor ring.

1.5. Definition. *A* **Gerstenhaber algebra** *is a commutative $\mathbb{Z}$-graded algebra $G^\bullet$, equipped with a bracket*

$$[\ ,\ ]:\ G^\bullet \otimes G^\bullet \longrightarrow G^\bullet[1]$$

of degree 1*, which makes $G^\bullet[-1]$ a graded Lie algebra.*

The multiplication and the bracket must satisfy the following compatibility,

$$[a, b \cdot c] = [a, b] \cdot c + (-1)^{(|a|-1)|b|} b \cdot [a, c] \tag{1.9}$$

for all homogeneous a, b, c.

Note the following equivalent form of (1.9),

$$[a \cdot b, c] = a \cdot [b, c] + (-1)^{(|c|-1)|b|} [a, c] \cdot b \tag{1.10}$$

The equivalence of (1.9) and (1.10) is proved using the symmetry relations

$$a \cdot b = (-1)^{|a||b|} b \cdot a \tag{1.11}$$

and

$$[a, b] = -(-1)^{(|a|-1)(|b|-1)} [b, a] \tag{1.12}$$

1.6. Theorem. *There exists a unique bracket of degree* 1 *on the algebra of polyvector fields* $\Lambda^\bullet\mathcal{T}(A)$*, where to* $\mathcal{T}(A)$ *the degree* -1 *and to* $\mathcal{A}$ *the degree* 0 *is assigned, such that*

(a) *for* $a \in A,\ \tau \in \mathcal{T}(A),\ \ [\tau, a] = \tau(a)$*;*

(b) *the bracket on* $\mathcal{T}(A)$ *coincides with the usual bracket of vector fields*;

(c) *together with the wedge product, this algebra becomes a Gerstenhaber algebra.*

It is called the **Schouten bracket**, [Sch]. The Gerstenhaber algebra defined in this Theorem, will be called the **Schouten algebra**, and denoted $\mathcal{T}^\bullet(A)$.

Now, let us identify $\Lambda^\bullet\mathcal{T}(A)$ with $M^\bullet(A; df)$.

1.7. Theorem. *Equipped with the Schouten bracket, the complex* $M^\bullet(A; df)[-1]$ *is a dg Lie algebra.*

We have to prove that

$$d([a, b]) = [da, b] + (-1)^{|a|-1}[a, db] \tag{1.13}$$

for $a \in \Lambda^{|a|}$. Using (1.12), it is easy to see that (1.13) is true for (a, b) if and only if it is for (b, a).

1.8. Lemma. *If* (1.13) *holds true for* (a, b) *and* (a, c) *then it holds true for* (a, bc)*.*

This Lemma is proved by a straightforward computation, which we omit. One uses (1.10) and

$$d(a \cdot b) = da \cdot b + (-1)^{|a|} a \cdot db \tag{1.14}$$

△

Returning to the Theorem, every element in $\Lambda^\bullet\mathcal{T}(A)$ is a sum of elements, which are either functions, or products of vector fields. Therefore, due to the Lemma, we have to prove (1.13) only in the case when a, b are either functions or vector fields. The only non-trivial identity comes out when both are vector fields. We have

$$d([\tau_1, \tau_2]) = [\tau_1, \tau_2](f),$$

$\tau_i \in \mathcal{T}(A)$. On the other hand,

$$[d\tau_1, \tau_2] + (-1)^{-1-1}[\tau_1, d\tau_2] = [\tau_1(f), \tau_2] + [\tau_1, \tau_2(f)] =$$

$$= -\tau_2\tau_1(f) + \tau_1\tau_2(f) = [\tau_1, \tau_2](f)$$

The Theorem is proved. △

1.9. Let us call a **degenerate Batalin-Vilkovisky algebra** a commutative dg algebra $G^\bullet$, together with a bracket $G^\bullet \otimes G^\bullet \longrightarrow G^\bullet[1]$ of degree 1 making $G^\bullet[-1]$ a dg Lie algebra, such that with the differential forgotten, $G^\bullet$ is a Gerstenhaber algebra.

The previous considerations show that the complex $M^\bullet(A; df)$, together with the wedge product of polyvector fields and the Schouten bracket, is a degenerate BV algebra.

§2. Batalin-Vilkovisky algebras on Calabi-Yau manifolds

2.1. Definition. *A* **Batalin-Vilkovisky algebra** *is a Gerstenhaber algebra $B^\bullet$, together with a differential d of degree 1 on it, such that*

$$(-1)^{|a|}[a,b] = d(a\cdot b) - da\cdot b - (-1)^{|a|}a\cdot db \tag{2.1}$$

for all homogeneous a, b.

It follows from (2.1) that

$$d([a,b]) = [da,b] + (-1)^{|a|-1}[a,db], \tag{2.2}$$

i.e. $B^\bullet[-1]$ is a dg Lie algebra.

2.2. Let us return to the situation of the last Section. Consider the de Rham complex

$$\Omega^\bullet(A):\ 0\longrightarrow A\overset{d_{DR}}{\longrightarrow}\Omega^1(A)\overset{d_{DR}}{\longrightarrow}\ldots\overset{d_{DR}}{\longrightarrow}\Omega^n(A)\longrightarrow 0 \tag{2.3}$$

Let us fix the top differential form

$$\omega = dx_1\wedge\ldots\wedge dx_n \tag{2.4}$$

Using ω, one gets the isomorphisms

$$\omega_*:\ \Lambda^p\mathcal{T}(A)\overset{\sim}{\longrightarrow}\Omega^{n-p}(A) \tag{2.5}$$

defined by

$$\omega_*(a) = a\omega \tag{2.6}$$

for $a\in A=\Lambda^0\mathcal{T}(A)$, and

$$\omega_*(\tau_1\wedge\ldots\wedge\tau_p) = i_{\tau_1}\cdot\ldots\cdot i_{\tau_p}(\omega) \tag{2.7}$$

for $p\geq 1$. Explicitely,

$$\omega_*(a\partial_{i_1}\wedge\ldots\wedge\partial_{i_p}) = (-1)^{(\Sigma_{j=1}^p\ i_j)-p}a dx_1\wedge\ldots\wedge\hat{dx}_{i_1}\wedge\ldots\wedge\hat{dx}_{i_p}\wedge\ldots\wedge dx_n \tag{2.8}$$

for $i_1<\ldots<i_p$.

Let us identify the shifted de Rham complex $\Omega^\bullet(A)[n]$ with $\Lambda^\bullet\mathcal{T}(A)$, using the isomorphisms (2.5). We can transfer the de Rham differential d_{DR} to $\Lambda^\bullet\mathcal{T}(A)$. Here $\mathcal{T}(A)$ will have the degree -1.

2.3. Theorem. *Equipped with the differential coming from $-d_{DR}$, the wedge product and the Schouten bracket, the algebra of polyvector fields $\Lambda^\bullet\mathcal{T}(A)$ becomes a BV algebra.*

We have to check the relation (2.1). First note that if (2.1) holds true for a pair (a,b) then it does for (b,a).

2.4. Lemma. *Suppose that* (2.1) *is true for pairs* (a,b)*,* (b,c) *and* (ab,c)*. Then it holds true the pair* (a,bc)*.*

Indeed, we have

$$(-1)^{|a|}[a,bc]=(-1)^{|a|}[a,b]\cdot c+(-1)^{|a||b|+|a|+|b|}b\cdot[a,c], \tag{2.9}$$

by (1.9). On the other hand,

$$d(a(bc))-da\cdot bc-(-1)^{|a|}a\cdot d(bc)=$$

$$=(-1)^{|a|+|b|}[ab,c]+d(ab)\cdot c+(-1)^{|a|+|b|}ab\cdot d(c)-$$
$$-da\cdot bc-(-1)^{|a|}a\cdot\big((-1)^{|b|}[b,c]+db\cdot c+(-1)^{|b|}b\cdot dc\big)$$

(we have applied (2.1) for (ab,c) and (b,c))

$$=(-1)^{|a|+|b|}[ab,c]+\big((-1)^{|a|}[a,b]+da\cdot b+(-1)^{|a|}a\cdot db\big)\cdot c-$$

$$-da\cdot bc-(-1)^{|a|+|b|}a\cdot[b,c]-(-1)^{|a|}a\cdot db\cdot c$$

(we have applied (2.1) for (a,b))

$$=(-1)^{|a|+|b|}[ab,c]+(-1)^{|a|}[a,b]\cdot c-(-1)^{|a|+|b|}a\cdot[b,c] \tag{2.10}$$

The right hand sides of (2.9) and (2.10) are equal, by the relation (1.10). This proves the Lemma. $\triangle$

In view of this Lemma, to prove the Theorem, it is enough to check (2.1) when a is either a function, or a vector field. Let us do that.

If both a and b are functions, (2.1) is trivial.

Let us check that

$$[a,b_1\partial_{i_1}\wedge\ldots\wedge b_p\partial_{i_p}]=$$
$$=d(ab_1\partial_{i_1}\wedge b_2\partial_{i_2}\wedge\ldots\wedge\partial_{i_p})-a\cdot d(b_1\partial_{i_1}\wedge\ldots\wedge b_p\partial_{i_p}), \tag{2.11}$$

where $a\in A$, $i_1<\ldots<i_j$. Using the definition, one checks that

$$d(b_1\partial_{i_1}\wedge\ldots\wedge b_p\partial_{i_p})=\sum_{j=1}^{p}(-1)^j\partial_{i_j}(b_1\cdot\ldots\cdot b_p)\partial_{i_1}\wedge\ldots\wedge\hat{\partial}_{i_j}\wedge\ldots\wedge\partial_{i_p} \tag{2.12}$$

On the other hand, the repeated application of (1.9) gives

$$[a,b_1\partial_{i_1}\wedge\ldots\wedge b_p\partial_{i_p}]=\sum_{j=1}^{p}(-1)^j b_j\partial_{i_j}(a)b_1\partial_{i_1}\wedge\ldots\wedge\widehat{b_j\partial_{i_j}}\wedge\ldots\wedge b_p\partial_{i_p} \tag{2.13}$$

These two formulas easily imply (2.11). This completes the check of (2.1) for $a\in A$.

The check of (2.1) for $a\in\mathcal{T}(A)$ is similar; we leave it to the reader. The Theorem is proven. $\triangle$

2.5. Now let us turn the function f on. First of all, note that when we identify $\Omega^\bullet(A)$ with $\Lambda^\bullet\mathcal{T}(A)$ using ω_* as in (2.2), the multiplication by df becomes the Koszul differential in $M^\bullet(A;df)$, (1.6).

2.6. Theorem. *The Schouten algebra $\mathcal{T}^\bullet(A)$* (cf. 1.6) *equipped with the differential coming from $-d_{DR} + df$, is a Batalin-Vilkovisky algebra.*

2.7. Lemma. *Let $S^\bullet$ be a Gerstenhaber algebra. Let d_0, d_1 be two anticommuting differentials on $S^\bullet$, such that $(S^\bullet, d_0)$ is a degenerate BV-algebra* (cf. 1.9), *and $(S^\bullet, d_1)$ is a BV-algebra. Then $(S^\bullet, d_0 + d_1)$ is a BV-algebra.*

The proof is obvious. The anticommutation of d_0 and d_1 is needed for the identity $(d_0 + d_1)^2 = 0$. $\triangle$

Applying the Lemma to the Schouten algebra $\mathcal{T}^\bullet(A)$, to d_0 induced from df, and d_1 induced from $-d_{DR}$, we get the Theorem, in view of 2.5 and Theorems 1.7 and 2.3. $\triangle$

The BV-algebra defined in Theorem 2.6 will be denoted $\mathcal{M}(A, \omega; df)$.

2.8. Let us make a base change $k \longrightarrow k[\lambda]$ where λ is an independent variable. Using 2.6, we get a BV-algebra over $k[\lambda]$, $\mathcal{M}(A[\lambda], \omega; \lambda \cdot df)$. The degenerate BV-algebra $M^\bullet(A; df)$ may be regarded as a "limit" of $\mathcal{M}(A, \omega; \lambda \cdot df)$ at $\lambda \longrightarrow \infty$.

2.9. We can apply the same construction by taking as an algebra A a localization of the polynomial algebra $k[x_1, \dots, x_n]$, and as a form ω an arbitrary generator

$$\omega = c(x) dx_1 \wedge \ldots \wedge dx_n, \tag{2.14}$$

of the A-module $\Omega^n(A)$. Here $c(x)$ is a unit in A.

Theorem 2.3 remains true, with the same proof. We should apply the following modifications of the formulas (2.11) and (2.13).

$$d(a\partial_i) = -\partial_i(a) - a \cdot \partial_i \log c, \tag{2.15}$$

and

$$d(a\partial_i \wedge b\partial_j) = -\big(\partial_i(ab) + ab \cdot \partial_i \log c\big)\partial_j + \big(\partial_j(ab) + ab \cdot \partial_j \log c\big)\partial_i \tag{2.16}$$

Here

$$\partial_i \log c = \partial_i c \cdot c^{-1} \tag{2.17}$$

Theorem 2.6 remains true, with the same proof.

2.10. Let A be an arbitrary commutative k-algebra, smooth over k. Assume that A is étale over a polynomial algebra $B = k[x_1, \dots, x_n]$. By definition, cf. [SGA 1], Exposé II, Déf. 1.1, this holds true for any smooth k-algebra, locally over $Spec(A)$.

The derivations $\partial/\partial_{x_i} \in \mathcal{T}(B)$ extend uniquely to derivations $\partial_i \in \mathcal{T}(A)$, which form a basis of $\mathcal{T}(A)$ as an A-module, and anticommute. Their duals dx_i form a basis of $\Omega^n(A)$.

Choose a generator

$$\omega = c dx_1 \wedge \ldots \wedge dx_n \tag{2.17}$$

of $\Omega^n(A)$. It induces the isomorphisms

$$\omega_* : \ \Lambda^\bullet \mathcal{T}(A) \xrightarrow{\sim} \Omega^{n-\bullet}(A) \tag{2.18}$$

Theorems 2.3 and 2.6 remain true, with the same proof. This follows from 2.9.

2.11. Now let us pass to the global situation. Suppose we are given a scheme X, smooth of relative dimension n over k. Assume that the canonical bundle Ω^n_X is trivial.

Let us choose a trivialization

$$\omega : \mathcal{O}_X \xrightarrow{\sim} \Omega^n_X \tag{2.19}$$

It induces the isomorphisms of graded $\mathcal{O}_X$-modules

$$\omega_* : \Lambda^\bullet \mathcal{T}_X \xrightarrow{\sim} \Omega^{n-\bullet}_X \tag{2.20}$$

2.12. Theorem. *Given a closed one-form $\phi \in \Gamma(X; \Omega^1_X)$, consider the Schouten algebra $\Lambda^\bullet \mathcal{T}_X$, equipped with the differential obtained, using the identification* (2.18), *from the differential*

$$-d_{DR} + \phi \tag{2.21}$$

on $\Omega^\bullet_X$. This way we get a sheaf of Batalin-Vilkovisky algebras on X.

Indeed, the statement is local, and we can assume we are in the situation 2.10. Now, consider the situation of Theorem 1.7, with *df* replaced by an *arbitrary* one-form $\phi \in \Omega^1(A)$. In the corresponding complex $M^\bullet(A;\phi)$, the differential is equal to the multiplication by ϕ, if we use the identification (2.18).

In particular the differential $\mathcal{T}(A) \longrightarrow A$ is given by

$$\tau \mapsto \langle \tau, \phi \rangle \tag{2.22}$$

The conclusion of Theorem 1.7 remains true in these more general assumptions. Indeed, we can use the same Lemma 1.8, and replace the last part of the proof of 1.7 by the Lemma below.

2.13. Lemma. *For all $\tau_1, \tau_2 \in \mathcal{T}(A)$, and $\phi \in \Omega^1(A)$, we have*

$$\langle [\tau_1, \tau_2], \phi \rangle = \tau_1(\langle \tau_2, \phi \rangle) - \tau_2(\langle \tau_1, \phi \rangle) \tag{2.23}$$

In fact, we can assume that $\tau_1 = a\partial_i, \tau_2 = b\partial_j$ and $\phi = dx_k$. Then the left hand side of (2.23) is equal to

$$\langle a\partial_i(b)\partial_j - b\partial_j(a)\partial_i, dx_k \rangle = a\partial_i(b)\delta_{jk} - b\partial_j(a)\delta_{ik},$$

which is equal to the right hand side of (2.23). $\triangle$

Using this generalization of Theorem 1.7, we get a generalization of Theorem 2.6, with *df* replaced by an arbitrary *closed* one form ϕ, and A as in 2.10. (The closedness of ϕ is necessary and sufficient condition for the equality $(-d_{DR} + \phi)^2 = 0$.) This generalization of 2.6 implies, in turn, Theorem 2.12. $\triangle$

The sheaf of BV algebras defined in this Theorem, will be denoted $\mathcal{M}^\bullet(X, \omega; \phi)$.

§3. Recollections on D-modules

For more details about the generalities below, cf. [BD1], Chapter 2, §1, and [B].

3.1. From now on, we assume that our ground ring k is a field of characteristic 0. Let X be a scheme, smooth over k, of dimension n. We will consider the categories $Mod(\mathcal{D}_X)^\ell$ and $Mod(\mathcal{D}_X)^r$ of left (resp., right) $\mathcal{D}_X$-modules (:= sheaves of left (resp., right) $\mathcal{D}_X$-modules, quasicoherent over $\mathcal{O}_X$).

The sheaf of rings $\mathcal{D}_X$ is generated by two $\mathcal{O}_X$-modules, $\mathcal{O}_X$ and $\mathcal{T}_X$, and relations

$$\tau \cdot a - a \cdot \tau = \tau(a) \tag{3.1}$$

and

$$\tau \cdot \tau' - \tau' \cdot \tau = [\tau, \tau'], \tag{3.2}$$

for $\tau, \tau' \in \mathcal{T}_X,\ a \in \mathcal{O}_X$.

Therefore, a left $\mathcal{D}_X$ module is a quasicoherent $\mathcal{O}_X$-module M, equipped with the left multiplication by $\mathcal{T}_X$, such that

$$a(\tau m) = (a\tau) \cdot m \tag{3.3a}$$

$$\tau(am) = a(\tau m) + \tau(a) \cdot m \tag{3.3b}$$

$$\tau(\tau' m) - \tau'(\tau m) = [\tau, \tau'] \cdot m, \tag{3.3c}$$

for $m \in M$.

Similarly, a right $\mathcal{D}_X$-module is a quasicoherent $\mathcal{O}_X$-module N, equipped with the right multiplication by $\mathcal{T}_X$, such that

$$(na)\tau = n \cdot (a\tau) \tag{3.4a}$$

$$(n\tau)a = n(a\tau) + n \cdot \tau(a) \tag{3.4b}$$

$$(n\tau)\tau' - (n\tau')\tau = n \cdot [\tau, \tau'], \tag{3.4c}$$

for $n \in N$. Here we agree that

$$n \cdot a = a \cdot n \tag{3.5}$$

3.2. For $M, M' \in Mod(\mathcal{D}_X)^\ell$, the tensor product $M \otimes_{\mathcal{O}_X} N$ admits a canonical structure of a left $\mathcal{D}_X$-module, defined by the Leibnitz rule

$$\tau \cdot (m \otimes m') = \tau m \otimes m' + m \otimes \tau m', \tag{3.6}$$

for $\tau \in \mathcal{T}_X$. This way we get a tensor structure on $Mod(\mathcal{D}_X)^\ell$ (it is denoted by $\otimes^!$ in [BB], cf. *op. cit.*, 2.2.1).

The sheaf $Hom_{\mathcal{O}_X}(M, M')$ is also canonically a left $\mathcal{D}_X$-module, with the $\mathcal{T}_X$-action given by

$$(\tau f)(m) = \tau(f(m)) - f(\tau m), \tag{3.7}$$

for $f \in Hom_{\mathcal{O}_X}(M, M')$. The rule (3.7) is invented in such a way that the canonical morphism,

$$Hom_{\mathcal{O}_X}(M, M') \otimes M \longrightarrow M' \tag{3.8}$$

would be a morphism of $\mathcal{D}_X$-modules.

3.3. For $M \in Mod(\mathcal{D}_X)^\ell$, $N \in Mod(\mathcal{D}_X)^r$, the tensor product $M \otimes N$ is canonically a right $\mathcal{D}_X$-module, with the multiplication by $\mathcal{T}_X$ defined by

$$(m \otimes n) \cdot \tau = m \otimes n\tau - \tau m \otimes n \tag{3.9}$$

For $N' \in Mod(\mathcal{D}_X)^r$, the $\mathcal{O}_X$-module $Hom_{\mathcal{O}_X}(N, N')$ is canonically a left $\mathcal{D}_X$-module, with the multiplication by $\mathcal{T}_X$ given by

$$(\tau f)(n) = f(n\tau) - f(n) \cdot \tau \tag{3.10}$$

Again, this rule is written in such a way that the canonical adjunction morphism is a morphism of right $\mathcal{D}_X$-modules.

3.4. The structure sheaf $\mathcal{O}_X$ has an obvious structure of a left $\mathcal{D}_X$-module.

The canonical bundle Ω^n_X is canonically a right $\mathcal{D}_X$-module. The multiplication by $\mathcal{T}_X$ is given by

$$\omega \cdot \tau = -Lie_\tau(\omega), \tag{3.11}$$

for $\omega \in \Omega^n_X$.

We have the **Cartan formula**

$$i_\tau \circ d + d \circ i_\tau = Lie_\tau \tag{3.12}$$

Consequently,

$$\omega \cdot \tau = -d(i_\tau \omega) \tag{3.13}$$

Let us check the relations (3.4a-c). We have

$$\omega \cdot (a\tau) = -di_{a\tau}(\omega) = -di_\tau(a\omega) = (\omega a) \cdot \tau,$$

which proves (3.4a). We have

$$\omega \cdot (a\tau) = -Lie_\tau(a\omega) = -\tau(a)\omega - aLie_{\tau(\omega)},$$

which proves (3.4b). The relation (3.4c) is obvious.

Using 3.3, we get the functors

$$Mod(\mathcal{D}_X)^\ell \longrightarrow Mod(\mathcal{D}_X)^r,\ M \mapsto M \otimes \Omega^n_X, \tag{3.14}$$

and

$$Mod(\mathcal{D}_X)^r \longrightarrow Mod(\mathcal{D}_X)^\ell,\ N \mapsto Hom(\Omega^n_X, N) \tag{3.15}$$

which are mutually inverse equivalences.

3.5. De Rham complex. We have the De Rham complex of our scheme X,

$$\Omega^\bullet_X:\ 0 \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow \ldots \longrightarrow \Omega^n_X \longrightarrow 0 \tag{3.16}$$

We can regard $\Omega^\bullet_X$ as a twisted version of the standard cochain complex of the Lie algebra of vector fields $\mathcal{T}_X$ with coefficients in the module of functions $\mathcal{O}_X$. Namely, let us identify

$$\Omega^i_X = Hom_{\mathcal{O}_X}(\Lambda^i\mathcal{T}_X, \mathcal{O}_X) \tag{3.17}$$

Thus, we get the identification

$$\Omega^\bullet_X = Hom_{\mathcal{O}_X}(\Lambda^\bullet\mathcal{T}_X, \mathcal{O}_X), \tag{3.18}$$

with the differential defined by

$$d\omega(\tau_1 \wedge \ldots \wedge \tau_p) = \sum_{i=1}^{p} (-1)^{i+1}\tau_i(\omega(\tau_1 \wedge \ldots \wedge \hat{\tau}_i \wedge \ldots \wedge \tau_p))+$$

$$+ \sum_{1\le i<j\le p} (-1)^{i+j+1}\ \omega([\tau_i, \tau_j] \wedge \tau_1 \wedge \ldots \wedge \hat{\tau}_i \wedge \ldots \wedge \hat{\tau}_j \wedge \ldots \wedge \tau_p), \tag{3.19}$$

for $\omega \in Hom(\Lambda^p\mathcal{T}_X, \mathcal{O}_X)$. cf. [KN], Chapter I, Prop. 3.11.

Twisted, since the bracket in $\mathcal{T}_X$ is not $\mathcal{O}_X$-linear. The differentials are not $\mathcal{O}_X$-linear, but are differential operators of the first order.

In the same sense, $\mathcal{D}_X$ is the twisted enveloping algebra of $\mathcal{T}_X$,

$$\mathcal{D}_X = U_{\mathcal{O}_X}\mathcal{T}_X \tag{3.20}$$

More generally, for a left $\mathcal{D}_X$-module $M \in Mod(\mathcal{D}_X)^\ell$, we have the De Rham complex

$$DR^\ell(M):\ 0 \longrightarrow M \longrightarrow \Omega^1_X \otimes M \longrightarrow \ldots \longrightarrow \Omega^n_X \otimes M \longrightarrow 0 \tag{3.21}$$

Using the identifications (3.17), we can identify this complex with the twisted cochain complex of the Lie algebra $\mathcal{T}_X$ with coefficients in M,

$$DR^\ell(M) = Hom_{\mathcal{O}_X}(\Lambda^\bullet\mathcal{T}_X, M), \tag{3.22}$$

the differential being defined by the same formula (3.19).

For a right $\mathcal{D}_X$-module $N \in Mod(\mathcal{D}_X)^r$, we have the De Rham complex

$$DR(N)^r:\ 0 \longrightarrow N \otimes \Lambda^n\mathcal{T}_X \longrightarrow \ldots \longrightarrow N \otimes \mathcal{T}_X \longrightarrow N \longrightarrow 0 \tag{3.23}$$

We consider it as living in degrees from $-n$ to 0. It may be regarded as the twisted version of the standard chain complex of the Lie algebra $\mathcal{T}_X$ with coefficients in N,

$$DR(N)^r = N \otimes_{\mathcal{O}_X} \Lambda^\bullet\mathcal{T}_X \tag{3.24}$$

The differential acts as follows

$$d(n \otimes \tau_1 \wedge \ldots \wedge \tau_p) = \sum_{i=1}^{p} (-1)^{i-1} n\tau_i \otimes \tau_1 \wedge \ldots \wedge \hat{\tau}_i \wedge \ldots \wedge \tau_p+$$

$$+\sum_{1\le i<j\le p}(-1)^{i+j}n\otimes[\tau_i,\tau_j]\wedge\ldots\wedge\hat{\tau}_i\wedge\ldots\wedge\hat{\tau}_j\wedge\ldots\wedge\tau_p \tag{3.25}$$

Let us check that (3.25) is a well defined map. For example, we have to check that

$$d(n\otimes a\tau_1\wedge\tau_2)=d(n\otimes\tau_1\wedge a\tau_2), \tag{3.26}$$

for $a\in\mathcal{O}_X$. We have

$$d(n\otimes a\tau_1\wedge\tau_2)=na\tau_1\otimes\tau_2-n\tau_2\otimes a\tau_1-n\otimes[a\tau_1,\tau_2]=$$

$$=n\tau_1a\otimes\tau_2-n\tau_1(a)\otimes\tau_2-n\tau_2\otimes a\tau_1-n\otimes a[\tau_1,\tau_2]+n\otimes\tau_2(a)\tau_1$$

We have used the formula

$$[a\tau_1,\tau_2]=a[\tau_1,\tau_2]-\tau_2(a)\tau_1 \tag{3.27}$$

which is a particular case of (1.10). On the other hand,

$$d(n\otimes\tau_1\wedge a\tau_2)=n\tau_1\otimes a\tau_2-na\tau_2\otimes\tau_1-n\otimes[\tau_1,a\tau_2]=$$

$$=n\tau_1\otimes a\tau_2-n\tau_2a\otimes\tau_1+n\tau_2(a)\otimes\tau_1-n\otimes a[\tau_1,\tau_2]-n\otimes\tau_1(a)\tau_2$$

which is equal to the left hand side of (3.26). In the same manner, the analog of (3.26) with an arbitrary number of τ's is checked. This proves that the map (3.25) is well defined.

The complexes DR^ℓ and DR^r are mapped to each other under the canonical equivalence between left and right $\mathcal{D}_X$-modules, (3.14), (3.15).

§4. Connections and Batalin-Vilkovisky structures

4.1. We keep the assumptions of the last Section. Let us call a **Calabi-Yau data** on X an integrable connection

$$\nabla:\ \Omega^n_X\longrightarrow\Omega^1_X\otimes\Omega^n_X \tag{4.1}$$

on the canonical bundle Ω^n_X.

In other words, a CY data is a structure of a left $\mathcal{D}_X$-module on Ω^n_X. By 3.4, this is the same as a structure of a *right* $\mathcal{D}_X$-module on the structure sheaf $\mathcal{O}_X=Hom_{\mathcal{O}_X}(\Omega^n_X,\Omega^n_X)$, i.e. a map

$$\nabla^r:\ \mathcal{T}_X=\mathcal{O}_X\otimes_{\mathcal{O}_X}\mathcal{T}_X\longrightarrow\mathcal{O}_X \tag{4.2}$$

such that

$$\nabla^r(a\tau)=a\nabla^r(\tau)-\tau(a), \tag{4.3}$$

for $a\in\mathcal{O}_X,\ \tau\in\mathcal{T}_X$.

4.2. Let us call a **Batalin-Vilkovisky data** on X a differential on the Schouten algebra $\Lambda^\bullet\mathcal{T}_X$, making it a BV algebra.

4.3. Theorem. *Given a CY data on X, the De Rham complex $DR^r(\mathcal{O}_X)$ of the corresponding right $\mathcal{D}_X$-module $\mathcal{O}_X$ is a BV data on X.*

Conversely, given a BV data on X, the component $\mathcal{T}_X \longrightarrow \mathcal{O}_X$ of the corresponding differential on $\Lambda^\bullet \mathcal{T}_X$ gives a map (4.2) which defines a CY data on X.

These two procedures establish the mutually inverse bijections between the sets of CY data and BV data on X.

Suppose we are given a CY data. We have to show the $DR^r(\mathcal{O}_X)$ is a BV algebra. This is a generalization of Theorem 2.12. We have to check the identity (2.1). We will use the formula

$$d(\tau_1 \wedge \ldots \wedge \tau_p) = \sum_{j=1}^{p} (-1)^{j-1} \nabla^r(\tau_j) \tau_1 \wedge \ldots \wedge \hat{\tau}_j \wedge \ldots \wedge \tau_p +$$

$$+ \sum_{i<j} (-1)^{i+j} [\tau_i, \tau_j] \wedge \tau_1 \wedge \ldots \wedge \hat{\tau}_i \wedge \ldots \wedge \hat{\tau}_j \wedge \ldots \wedge \tau_p \tag{4.4}$$

which is a version of (3.25).

Application of Lemma 2.4 shows that it is enough to check (2.1) when a is either a function or a vector field. If both a, b are functions, (2.1) is trivial.

Let $a \in \mathcal{O}_X$. We have

$$[a, \tau_1 \wedge \ldots \wedge \tau_p] = \sum_{j=1}^{p} (-1)^p \tau_j(a) \tau_1 \wedge \ldots \wedge \hat{\tau}_j \wedge \ldots \wedge \tau_p$$

On the other hand, it follows from (4.4) that

$$d(a\tau_1 \wedge \ldots \wedge \tau_p) = \nabla^r(a\tau_1) \wedge \tau_2 \wedge \ldots \wedge \tau_p + \sum_{j=2}^{p} (-1)^{j-1} \nabla^r(\tau_j) a\tau_1 \wedge \ldots \wedge \hat{\tau}_j \wedge \ldots \wedge \tau_p +$$

$$+ \sum_{1<j} (-1)^{1+j} [a\tau_1, \tau_j] \wedge \tau_2 \wedge \ldots \wedge \hat{\tau}_j \wedge \ldots \wedge \tau_p + \sum_{1<i<j} (-1)^{i+j} a[\tau_i, \tau_j] \wedge \tau_1 \wedge \ldots \wedge \hat{\tau}_i \wedge \ldots \wedge \hat{\tau}_j \wedge \ldots \wedge \tau_p =$$

$$= \big(a\nabla^r(\tau_1) - \tau_1(a)\big) \wedge \tau_2 \wedge \ldots \wedge \tau_p + \sum_{j=2}^{p} (-1)^{j-1} a\nabla^r(\tau_j) \tau_1 \wedge \ldots \wedge \hat{\tau}_j \wedge \ldots \wedge \tau_p +$$

$$+ \sum_{1<j} (-1)^{1+j} \big(a[\tau_1, \tau_j] - \tau_j(a)\big) \wedge \tau_2 \wedge \ldots \wedge \hat{\tau}_j \wedge \ldots \wedge \tau_p +$$

$$+ \sum_{1<i<j} (-1)^{i+j} a[\tau_i, \tau_j] \wedge \tau_1 \wedge \ldots \wedge \hat{\tau}_i \wedge \ldots \wedge \hat{\tau}_j \wedge \ldots \wedge \tau_p$$

Subtracting from this $a \cdot d(\tau_1 \wedge \ldots \wedge \tau_p)$, we get (2.1) in this case.

It remains to check the following particular case of (2.1),

$$-[\tau_0, \tau_1 \wedge \ldots \wedge \tau_p] = d(\tau_0 \wedge \ldots \wedge \tau_p) - d\tau_0 \wedge \tau_1 \wedge \ldots \wedge \tau_p + \tau_0 \wedge d(\tau_1 \wedge \ldots \wedge \tau_p) \tag{4.5}$$

The left hand side of (4.5) is equal to

$$\sum_{j=1}^{p}(-1)^j[\tau_0,\tau_j]\wedge\tau_1\wedge\ldots\wedge\hat{\tau}_j\wedge\ldots\wedge\tau_p$$

The right hand side is equal to

$$\sum_{j=0}^{p}(-1)^j\nabla^r(\tau_j)\wedge\tau_0\wedge\ldots\wedge\hat{\tau}_j\wedge\ldots\wedge\tau_p+\sum_{1\le j}(-1)^j[\tau_0,\tau_j]\wedge\tau_0\wedge\ldots\wedge\hat{\tau}_j\wedge\ldots\wedge\tau_p+$$

$$+\sum_{1\le i<j}(-1)^{i+j}[\tau_i,\tau_j]\wedge\tau_0\wedge\ldots\wedge\hat{\tau}_i\wedge\ldots\wedge\hat{\tau}_j\wedge\ldots\wedge\tau_p-\nabla^r(\tau_0)\wedge\tau_1\wedge\ldots\wedge\tau_p+$$

$$+\sum_{j=1}^{p}(-1)^{j-1}\nabla^r(\tau_j)\tau_0\wedge\ldots\wedge\hat{\tau}_j\wedge\ldots\wedge\tau_p+\sum_{1\le i<j}(-1)^{i+j}\tau_0\wedge[\tau_i,\tau_j]\wedge\tau_1\wedge\ldots\wedge\hat{\tau}_i\wedge\ldots\wedge\hat{\tau}_j\wedge\ldots\wedge\tau_p$$

which is equal to the left hand side. This proves (4.5) and completes the construction of the arrow α : (CY data) $\longrightarrow$ (BV data).

Conversely, given a BV data, a particular case of (2.1) is

$$[a,\tau]=\nabla^r(a\tau)-a\nabla^r(\tau)$$

which is nothing but (4.3). This gives the arrow β : (BV data) $\longrightarrow$ (CY data). It is clear that the composition $\beta\alpha$ is the identity. On the other hand, given a BV data, the differential in $\Lambda^\bullet\mathcal{T}_X$ is uniquely determined by its last component, and (2.1). This implies that $\alpha\beta$ is the identity.

The Theorem is proven. $\triangle$

4.4. Let $\mathcal{A}$ be a commutative $\mathcal{D}_X$-algebra, i.e. a commutative (implied: associative, unital) algebra in the category of $Mod(\mathcal{D}_X)^\ell$, with the $\otimes^!$ tensor structure, cf. 3.2. On the sheaf of $\mathcal{A}$-modules

$$\mathcal{T}_\mathcal{A}:=\mathcal{A}\otimes_\mathcal{O}\mathcal{T}_X, \tag{4.6}$$

one defines a Lie algebra structure by

$$[a\tau,b\tau']=a\tau(b)\tau'-b\tau'(a)\tau+ab[\tau,\tau'], \tag{4.7}$$

Here by $a\tau$ we denoted $a\otimes\tau$, for $a\in\mathcal{A},\ \tau\in\mathcal{T}_X$, and by $\tau(a)$ we denoted $\tau\cdot a\in\mathcal{A}$, defined by the left $\mathcal{D}_X$-module structure on $\mathcal{A}$.

The given $\mathcal{O}_X$-linear map $\mathcal{T}_X\longrightarrow Der(\mathcal{A})$ is extended by linearity to the $\mathcal{A}$-linear map

$$\mathcal{T}_\mathcal{A}\longrightarrow Der(\mathcal{A})$$

which defines on $\mathcal{A}$ a structure of a module over the Lie algebra $\mathcal{T}_\mathcal{A}$. The action of $\mathcal{T}_\mathcal{A}$ on $\mathcal{A}$ will also be denoted by $\tau(a),\ a\in\mathcal{A},\ \tau\in\mathcal{A}$. We have

$$[\tau,a\tau']=a[\tau,\tau']+\tau(a)\tau' \tag{4.8}$$

for all $\tau, \tau' \in \mathcal{T}_{\mathcal{A}},\ a \in \mathcal{A}$. Thus, $\mathcal{T}_{\mathcal{A}}$ is a *Lie algebroid over* $\mathcal{A}$, in the terminology of [BD2], 3.5.10.

Let $\Lambda^\bullet(\mathcal{T}_{\mathcal{A}})$ denote the exterior algebra of $\mathcal{T}_{\mathcal{A}}$ *over* $\mathcal{A}$. We regard it as a graded commutative algebra, with $\mathcal{A}$ (resp. $\mathcal{T}_{\mathcal{A}}$) having degree 0 (resp. -1).

We have the following generalization of Theorem 1.6.

4.5. Theorem. *There exists a unique Lie bracket of degree* 1 *on* $\Lambda^\bullet(\mathcal{T}_{\mathcal{A}})$ *such that*

(a) *for* $a \in \mathcal{A},\ \tau \in \mathcal{T}_{\mathcal{A}},\ [\tau, a] = \tau(a)$;

(b) *the bracket on* $\mathcal{T}_{\mathcal{A}}$ *is given by* (4.7);

(c) *equipped with this bracket,* $\Lambda^\bullet(\mathcal{T}_{\mathcal{A}})$ *becomes a Gerstenhaber algebra.*

4.6. Let $\mathcal{D}_{\mathcal{A}}$ denote the sheaf of associative algebras generated by the algebra $\mathcal{A}$ and the sheaf $\mathcal{T}_X$, subject to relations

(a) $\tau \cdot a - a \cdot \tau = \tau(a)$;

(b) $\tau \cdot \tau' - \tau' \cdot \tau = [\tau, \tau']$.

We have a canonical filtration $\{\mathcal{D}_{\mathcal{A}}^{\leq i}\}$ on $\mathcal{D}_{\mathcal{A}}$, with the graded algebra

$$gr^\bullet \mathcal{D}_{\mathcal{A}} = S^\bullet(\mathcal{T}_{\mathcal{A}}), \tag{4.9}$$

the symmetric algebra over $\mathcal{A}$.

One has a canonical isomorphism of left $\mathcal{A}$-modules

$$\mathcal{D}_{\mathcal{A}} = \mathcal{A} \otimes_{\mathcal{O}} \mathcal{D}_X \tag{4.10}$$

We have canonical embeddings of $\mathcal{O}_X$-algebras

$$\mathcal{D}_X \hookrightarrow \mathcal{D}_{\mathcal{A}};\ \ \mathcal{A} \hookrightarrow \mathcal{D}_{\mathcal{A}}$$

(We have just described the *enveloping algebra* construction from [BB], 1.2.5.)

4.7. The algebra $\mathcal{A}$ has an obvious structure of a left $\mathcal{D}_{\mathcal{A}}$-module. Assume we are given a structure of a right $\mathcal{D}_{\mathcal{A}}$-module on $\mathcal{A}$ which induces an obvious $\mathcal{A}$-module structure. After restriction, $\mathcal{A}$ becomes a right $\mathcal{D}_X$-module, such that

$$(a \cdot b) \cdot \tau = a \cdot (b \cdot \tau) - b \cdot \tau(a), \tag{4.11}$$

for all $a, b \in \mathcal{A},\ \tau \in \mathcal{T}_X$. Define a map $\nabla^r :\ \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{A}$ by

$$\nabla^r(a\tau) = a \cdot \tau \tag{4.12}$$

We have

$$\nabla^r(a\nu) = a\nabla^r(\nu) - \nu(a) \tag{4.13}$$

for all $a \in \mathcal{A},\ \nu \in \mathcal{T}_{\mathcal{A}}$.

4.8. Lemma. *The previous definitions establish bijections between the data* (i) — (iv) *below.*

(i) *The structures of a right $\mathcal{D}_{\mathcal{A}}$-module on $\mathcal{A}$, inducing the obvious $\mathcal{A}$-module structure.*

(ii) *The structures of a right $\mathcal{D}_X$-module on $\mathcal{A}$ inducing the given $\mathcal{O}_X$-module structure, such that* (4.11) *is satisfied.*

(iii) *The maps*

$$\nabla^r : \mathcal{T}_X \longrightarrow \mathcal{A}$$

such that (4.13) *is satisfied, for all* $a \in \mathcal{O}_X,\ \nu \in \mathcal{T}_X$.

(iv) *The maps*

$$\nabla^r : \mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{A}$$

such that (4.13) *is satisfied.*

A structure described in this Lemma will be called a $\mathbf{CY}_{\mathcal{A}}$**-structure**.

4.9. Theorem. *Given a $CY_{\mathcal{A}}$-structure, there exists a unique differential of degree* 1 *on the Gerstenhaber algebra $\Lambda^\bullet\mathcal{T}_{\mathcal{A}}$, whose first component coincides with* (4.13) *and making $\Lambda^\bullet\mathcal{T}_{\mathcal{A}}$ a Batalin-Vilkovisky algebra.*

Let us call a differential on $\Lambda^\bullet\mathcal{T}_{\mathcal{A}}$ making it a BV algebra a $\mathbf{BV}_{\mathcal{A}}$**-structure**. Given a $\mathrm{BV}_{\mathcal{A}}$-structure, the first component of the differential is a map $\mathcal{T}_{\mathcal{A}} \longrightarrow \mathcal{A}$ satisfying (4.13).

4.10. Theorem. *The above constructions establish the two inverse bijections between the set of $CY_{\mathcal{A}}$-data and the set of $BV_{\mathcal{A}}$-data.*

This is a generalization of Theorem 4.3. The proof is the same. On just notes that in the proof of 4.3, we have used only the identity (4.13).

4.11. What is a Gerstenhaber algebra (over X) living in degrees $-1, 0$? It is the same as a sheaf of commutative algebras $\mathcal{A}^0$, a sheaf of Lie algebras $\mathcal{A}^{-1}$, which is also an $\mathcal{A}^0$-module, and acts on $\mathcal{A}^0$ by derivations. The corresponding map $\mathcal{A}^{-1} \longrightarrow Der(\mathcal{A}^0)$ should be $\mathcal{A}^0$-linear, and the identity (4.8) should hold for all $a \in \mathcal{A}^0,\ \tau \in \mathcal{A}^{-1}$. In other words, $\mathcal{A}^{-1}$ is a Lie algebroid over $\mathcal{A}^0$.

Let $Gerst$ denote the category of Gerstenhaber algebras over X, and let

$$Gerst^{[-1,0]} \subset Gerst \tag{4.14}$$

be the full subcategory consisting of Gerstenhaber algebras living in degrees $-1, 0$. We claim that the obvious trucation functor

$$t : \ Gerst \longrightarrow Gerst^{[-1,0]} \tag{4.15}$$

admits a left adjoint

$$S : \ Gerst^{[-1,0]} \longrightarrow Gerst \tag{4.16}$$

In fact, given $\mathcal{A} \in Gerst^{[-1,0]}$, set

$$S(\mathcal{A}) = S^\bullet_{\mathcal{A}^0}(\mathcal{A}^{-1}[1]), \tag{4.17}$$

the symmetric algebra over $\mathcal{A}^0$. In other words, as a graded algebra

$$S(A) = \Lambda^\bullet_{\mathcal{A}^0}(\mathcal{A}^{-1}), \tag{4.18}$$

with $\mathcal{A}^{-1}$ having degree -1.

4.12. Theorem. *There exists a unique Lie bracket of degree* 1 *on* $S(\mathcal{A})$, *making it a Gerstenhaber algebra, such that the obvious embedding* $\mathcal{A} \hookrightarrow S(\mathcal{A})$ *is a morphism of graded Lie algebras.*

This is a version of Theorems 4.5 and 1.6. This bracket is called the *Schouten bracket.* This way we get a structure of a Gerstenhaber algebra on $S(\mathcal{A})$, called the **Schouten algebra of** $\mathcal{A}$.

The functor S is the left adjoint to (4.15). Obviously, the composition $t \circ S$ is the identity.

4.13. Given $\mathcal{A} \in Gerst^{[-1,0]}$, assume we have a BV structure on $S(\mathcal{A})$, i.e. a differential of degree 1 on this algebra, making it a BV algebra. Its first component is a mapping

$$\nabla^r : \mathcal{A}^{-1} \longrightarrow \mathcal{A}^0 \tag{4.19}$$

such that

$$\nabla^r(a\tau) = a\nabla^r(\tau) - \tau(a) \tag{4.20}$$

for all $a \in \mathcal{A}^0,\ \tau \in \mathcal{A}^{-1}$.

Conversely, given a mapping (4.19) satisfying (4.20), there exists a unique extension of this mapping to a differential d on $S(\mathcal{A})$, making it a BV algebra. In fact, the uniqueness of d is obvious. To prove the existence, one defines d by the formula (4.4); the computations after (4.4) show that we get indeed a BV structure.

Thus, we have proven

4.14. Theorem. *The above constructions establish two inverse bijections between the set of mappings* (4.19) *satisfying* (4.20) *and the set of BV structures on* $S(\mathcal{A})$. $\triangle$

This Theorem generalizes Theorems 4.10 and 4.3.

References

[BB] A. Beilinson, J. Bernstein, A proof of Jantzen conjectures, in: I.M. Gelfand Seminar, S. Gelfand, S. Gindikin (Eds.), *Adv. in Sov. Math.* **16**, Part I, American Math. Society, Providence, Rhode Island, 1993, 1-50.

[BD1] A. Beilinson, V. Drinfeld, Chiral algebras I, Preprint.

[BD2] A. Beilinson, V. Drinfeld, Quantization of Hitchin's integrable system and Hecke eigensheaves, Preprint, 1997.

[B] A. Borel et al., Algebraic D-modules, Academic Press, Boston et al., 1987.

[D] V. Drinfeld, Letter to the author, September 1988.

[EGA III] A. Grothendieck, J. Dieudonné, Éléments de Géometrie Algébrique, III. Étude cohomologique des faisceaux cohérents (Première Partie), *Publ. Math. IHES*, **11**, 1961.

[SGA 1] A. Grothendieck, Revêtements étales et groupe fondamental, *Lect. Notes in Math.* **224**, Springer-Verlag, Berlin-Heidelberg- New York, 1971.

[H] V. Hinich, Descent of Deligne groupoids, *IMRN*, 1997, No. 5, 223-239.

[KN] S. Kobayashi, K. Nomizu, Foundations of differential geometry, Volume I, Interscience Publishers, New York-London, 1963.

[M] J. Milnor, Singular points of complex hypersurfaces, Princeton University Press, Princeton, New Jersey, 1968.

[R] Z. Ran, Thickening Calabi-Yau spaces, in: Mirror Symmetry II, *AMS/IP Stud. Adv. Math.,* **1**, Amer. Math. Soc., Providence, Rhode Island, 1997, 393-400.

[Sch] J.A. Schouten, Über Differentialkomitanten zweier kontravarianter Grossen, *Nederl. Acad. Wetensch. Proc.*, Ser. A, **43** (1940), 449-452.

[S] J.-P. Serre, Algèbre Locale Multiplicités, *Lect. Notes in Math.*, **11**, Springer-Verlag, Berlin-Heidelberg-New York, 1975.

Max-Planck-Institut für Mathematik,

Gottfried-Claren-Straße 26, 53225 Bonn

e-mail: vadik@mpim-bonn.mpg.de