ELIMINATION OF WILD RAMIFICATION

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Introduction

In 1972 Epp [E] proved a theorem asserting that in an arbitrary finite extension of discretely valued fields L/K one can eliminate wild ramification, i. e. to guarantee validity of $e(k'L/k'K)=1$ for some finite extension k'/k, where k is a constant subfield: maximal subfield of K with perfect residue field (k is canonical in the mixed characteristic case). In the papers by Ponomaryov [P1,P2] some inconsistency of the proof in [E] was removed (which took place when K is of prime characteritic). Also Ponomaryov proved a number of strong generalizations of the elimination theorem for arbitrary extensions of discretely valued fields of prime characteristic.

In [Z1] a variant of the proof of Epp's theorem in the mixed characteristic case was given. It has been also shown how one deduces from it a certain advanced version of Parshin's classification theorem for multidimensional local fields [Pa].

In the present work we consider only the case of a finite extension of complete discretely valued fields. In the case of prime characteristic we choose a constant subfield k in a given complete discretely valued field K, and we assume (outside §3) that k is fixed. We prove the following variant of Epp's theorem.

Theorem 1. *Let L/K be any finite extension of complete discretely valued fields, k the constant subfield in K. Then there exists a finite solvable extension k'/k such that $k'L/k'K$ is weakly unramified, i. e. $e(k'L/k'K)=1$.*

Let $e_L=p^h\cdot e_t$ where $(p,e_t)=1$. Then there exists desired k'/k that is a composite of a cyclic extension and an extension of degree not exceeding $(p-1)^{h+1}[L:K]e_K$.

The proof proceeds for the equal and mixed characteristic cases in parallel. It is based on the splitting of an arbitarary finite extension into a tower of an "almost constant" and "infernal" extensions. (An extension is said to be almost constant if it can be embedded into a composite of a constant extension and absolutely unramified one. An extension is said to be infernal if it possesses no non-trivial almost constant subextension.) We remark that elimination of wild ramification

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in an infernal extension yields a ferociously ramified extension of the same degree, that means an extension of ramification index 1 and purely inseparable extension of the residue fields.

The key assertion in the proof is the following theorem which is also of independent interest for the ramification theory.

Theorem 2. *Let L/K be an infernal extension, k the constant subfield of K, C/k an arbitrary deeply ramified*[1] *extension. Then there exists a finite subextension k'/k in C/k such that $k'L/k'K$ is ferociously ramified.*

Thus, in particular, k'/k can be chosen, e. g., inside any given ramified extension with the Galois group isomorphic to $\mathbb{Z}_p$.

The first 2 sections of this paper contain preliminary results. §3 is devoted to the proof of Theorems 1 and 2. In the last subsection of §3 we deduce also Theorems 3 and 4 which are new variants of structure theorems for complete fields from [Z1].

Namely, a field K is said to be standard if $e(K/k) = 1$ where k is the constant subfield. Notice that a multidimensional local field K of characteristic 0 with the first residue field of characteristic p and with perfect last residue field is standard if and only if it takes the shape $l\{\{t_1\}\}\dots\{\{t_n\}\}$.

Theorem 3. *Let $e(K/k) = p^h e_t$ where $(p, e_t) = 1$. Then there exists a finite solvable constant extension L/K such that L is a standard field. Moreover, one may assume $L = L_1 L_2$, where L_1/K is a cyclic p-extension, whereas $[L_2 : K]$ divides $(p-1)^{h+1} e(K/k)$. Furthermore, one may assume $L_1 = lK$ where l/k is a finite subextension of a given ramified constant $\mathbb{Z}_p$-extension C/k.*

We recall that in the mixed characteristic case a subfield k is determined canonically; in particular, for a multidimensional local field K with the finite last residue field, the subfield k coincides with the algebraic closure of $\mathbb{Q}_p$ in K.

A field is said to be almost standard, if some finite unramified extension of it is a standard field.

Theorem 4. *Let k be the constant subfield of K, C/k is a deeply ramified extension. Then there exists a finite subextension l/k in C/k such that lK is almost standard.*

Corollary. *Let K be a multidimensional loacal field of characteristic 0 with the first residue field of characteristic $p > 0$, k its constant subfield. Suppose that C/k is a deeply ramified extension. Then there exists a finite subextension l/k in C/k such that lK is almost standard. In particular, there exists a semi-ramified extension L/K such that $L(\zeta_{p^n})$ is a standard field for some positive integer n.*

In §4 we show that in the equal characteristic case by means of a suitable choice of the constant subfield one can force any given Galois p-extension to be infernal. This permits one to eliminate ramification by means of a cyclic k'/k (under some obvious restrictions). In particular, the following theorem takes place.

Theorem 5. *Let $\operatorname{char} K = p$ and K contain all roots of unity of degree prime to p. Then for any finite separable extension L/K there exists a constant subfield k in K and a cyclic extension k'/k such that $e(k'L/k'K) = 1$.*

[1] Deeply ramified extensions were introduced in [CG] and thoroughly studied in [F]. In the present paper the notion of a deeply ramified extension is slightly more general, see §1.

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Notation

• p always denotes a certain fixed prime number;
• ζ_m: a primitive root of unity of degree m in the separable closure of the given field;
• $\wp\colon F \to F$: the map $x \mapsto x^p - x$ in the given field F of characteristic p;
For a complete discretely valued field K with the residue field of characteristic p:
• v_K: the valuation on K, as well as its non-normalized extension onto the algebraic closure of K;
• $\mathcal{O}_K$: the valuation ring in K;
• $\mathfrak{M}_K = \{\, a \in \mathcal{O}_K : v(a) > 0 \,\}$: the maximal ideal of this ring;
• $\overline{K}$: the residue field of K;
• $\overline{a}$: the residue class in $\overline{K}$ of $a \in \mathcal{O}_K$;
• $\mathcal{R}_K$: the subgroup of Teichmüller representatives, i. e. the subgroup of K^* which consists of representatives of all non-zero elements of the maximal perfect subfield in $\overline{K}$.
• *A solvable extension* means a subextension of a finite Galois extension with solvable Galois group.

§1. Basic definitions and constructions

1.1. Almost constant extensions. Let K be a complete discretely valued field of any characteristic with the residue field $\overline{K}$ of characteristic $p > 0$.

If $\operatorname{char} K = p$, we fix *a base subfield* B in K, which is supposed to be complete with respect to v_K and to have the residue field $\mathbb{F}_p$. (The possible base subfields are exactly the subfields $\mathbb{F}_p((\tau))$ where $v_K(\tau) > 0$.) If $\operatorname{char} K = 0$, we assume $B = \mathbb{Q}_p$. Denote by $v\colon K^* \to \mathbb{Q}$ the non-normalized extension of v_B onto K; denote also $e_K = e_{K/B} = 1/v(\pi)$ where π is a prime in K. In the case $\operatorname{char} K = 0$ we have, as usual, $e_K = v_K(p)$.

Let $F = \bigcap \overline{K}^{p^i}$ be the maximal perfect subfield of $\overline{K}$. If $\operatorname{char} K = 0$, denote by k the set of all $x \in K$ which are algebraic over the fraction field k_0 of $W(F)$. In the equal characteristic case the definition of k is the same, if we take for k_0 the completion of $B(\mathcal{R}_K)$. Obviously, in both cases k is a complete subfield of K with perfect residue field, and it is maximal with respect to this property. The subfield k will be referred to as the *constant subfield* of K.

Example. Consider a standard n-dimensional local field $K = k\{\{t_1\}\} \dots k\{\{t_{n-1}\}\}$ where k is a complete discretely valued field of characteristic 0 with a perfect residue field of characteristic p. Then the constant subfield of K coincides with k.

A finite extension L/K is said to be *constant*, if $L = lK$ where l is an algebraic extension of k. Clearly, in this case l is the constant subfield of L. Notice that separable constant extensions of K are exactly the extensions of the kind $K(a)/K$, where a is algebraic and separable over k. A finite extension will be referred to as *almost constant* if it lies in a composite of a constant extension and an absolutely unfamified one. Equivalently, L/K is almost constant, if $K'L/K'$ is constant for some unramified K'/K. Notice the following properties.

1. A composite of two almost constant extensions is almost constant. Therefore, it makes sense to consider the maximal almost constant subextension in a given finite extension.

2. Any intermediate extension in an almost constant extension L/K is almost constant as well. In fact, let $K \subset K' \subset L' \subset L$. We have $L \subset CU$ where C/K is a constant extension, and U/K an unramified one. Then $L' \subset (CK')(UK')$, CK'/K' is a constant extension, UK'/K' an unramified one.

3. Any weakly ramified extension is almost constant.

4. Let $\operatorname{char} K = p$, L/K a cyclic extension of degree p. Then L/K is almost constant, if and only if $L = K(x)$, $x^p - x \in k + \mathcal{O}_K$.

5. Let $\operatorname{char} K = 0$, $\zeta_p \in K$, L/K a cyclic extension of degree p. Then L/K is almost constant, if and only if $L = K(x)$, $x^p \in k^* \cdot (1 + \mathfrak{M}_K^a)$, $a = \frac{p}{p-1} e_K$.

We shall say that K is *standard*, if $e_{K/k} = 1$. It is obvious that any unramified or constant or ferociously ramified extension of a standard field is standard.

Further, K is said to be *almost standard*, if there exists an unramified extension L/K such that L is standard. It is easy to prove that an almost constant or ferociously ramified extension of an almost standard field is almost standard.

1.2. Infernal extensions. We assign the name *infernal* to the opposite type of extensions. Namely, a finite extension L/K will be referred to as infernal if the only almost constant subextension in L/K is K itself.

1. If L/K is infernal, then any intermediate extension in L/K is infernal.

Suppose, there exist K' and L' such that $K \subset K' \subset L' \subset L$, and L'/K' is almost constant. Then $L' \subset CU$, C/K' is constant, U/K' is unramified. Obviously, $C = C_0K'$ where C_0/K is a constant extension. Next, $U = U_0K'$ where U_0/K is an unramified extension. The extension C_0U_0/K is linearly disjoint with K'/K since K'/K is infernal. We conclude that $L' = L_0K'$ where $L_0 \subset C_0U_0$. Thus, L_0/K is an almost constant subextension in L/K, whence $L_0 = K$, and $L' = K'$.

2. If L/K is infernal, then $K'L/K'$ is also infernal for any almost constant extension K'/K.

In fact, let E/K' be an almost constant subextension in $K'L/K'$. Then $(E \cap L)/K$ is almost constant and infernal at the same time, whence $E \cap L = K$. This yields $[LK' : E] = [L : K] \geqslant [LK' : K']$, and $E = K'$.

1.3. Jumps and depth of ramification. Let L/K be a finite Galois extension, $\sigma \in \operatorname{Gal}(L/K)$ a non-trivial automorphism. Then the (absolute) ramification number $s(\sigma) \in \mathbb{Q}$ is defined as $\min\{\, v(\sigma(a)a^{-1} - 1) \mid a \in \mathcal{O}_L \,\}$. All elements of $\{\, s(\sigma) \mid \sigma \in \operatorname{Gal}(L/K), \sigma \neq 1 \,\}$ are said to be the ramification jumps of L/K. If L/K is a normal inseparable extension, the ramification jumps of L/K are defined as $\{\text{jumps of } L_0/K\} \cup \{\infty\}$, where L_0/K is the maximal separable subextension. If L/K possesses an only ramification jump (this is always the case if $[L : K]$ is prime), we denote this jump by $h(L/K)$.

Let L/K be a finite separable extension. Then the depth of ramification is defined:

$$d(L/K) = \inf\{\, v(\operatorname{Tr}_{L/K}(y)/y) \mid y \in L^* \,\}.$$

It should be noted that for any intermediate N:

$$d(L/K) = d(L/N) + d(N/K)$$

[H, 2–4], and for a weakly ramified L/K we have $d(L/K) = 0$ [H, 2–12]. Finally, for a cyclic L/K of degree p we have $d(L/K) = (p-1)h(L/K)$, see [H, 2–10].

Lemma 1.3.1. *Let L_1/K and L_2/K be cyclic extensions of degree p. Then*

$$h(L_1L_2/L_1) = h(L_2/K) - h(L_1/K) + h(L_1L_2/L_2).$$

Proof. We have

$$\begin{aligned} h(L_1L_2/L_1) + h(L_1/K) &= \frac{1}{p-1}(d(L_1L_2/L_1) + d(L_1/K)) \\ &= \frac{1}{p-1}(d(L_1L_2/K)) \\ &= \frac{1}{p-1}(d(L_1L_2/L_2) + d(L_2/K)) \\ &= h(L_1L_2/L_2) + h(L_2/K). \quad \square \end{aligned}$$

Definition. An (infinite) extension l/k is said to be deeply ramified if the inertia degree $[\bar{l} : \bar{k}]$ is finite and at least one of the following two conditions is satisfied.

1) l/k possesses finite separable extensions of arbitrarily big depth of ramification, i. e. the set $\{ d(k'/k) \mid k \subset k' \subset l,\ k'/k \text{ separable} \}$ is not bounded.

2) $[l : k]_{insep} = \infty$.

It is easy to see that in the case of separable L/K we have the usual notion of deeply ramified extension, compare [F].

Lemma 1.3.2. *Let l/k be deeply ramified, k'/k a finite subextension in l/k. Then l/k' is deeply ramified.*

1.4. On the ramification in an extension of degree p. Let L/K be a cyclic extension of degree p. In the case $\operatorname{char} K = p$ we have $L = K(\gamma)$, where $\wp(\gamma) = a \in K$. If $v_K(a) > 0$, then by Hensel lemma the polynomial $\wp(x) - a$ splits completely over K, and the extension L/K is therefore trivial. This means also that $\mathfrak{M}_K \subset \wp(K)$. Obviously, in the right hand side of Artin–Schreier equation $\wp(\gamma) = a$ the element a can be replaced by any element of the class $a + \wp(K)$.

In the case $\operatorname{char} K = 0$ suppose that $\zeta_p \in K$. Then any cyclic extension L/K of degree p corresponds to an equation $x^p = a \in K$. Here a can be replaced by any element from aK^{*p}.

Proposition 1.4. *Let $\operatorname{char} K = p$, $L = K(\gamma)$ a non-trivial extension, where $\wp(\gamma) = a \in K$. In this case:*

1) if $v_K(a) = 0$, then L/K is unramified;

2) if $v_K(a) < 0$ and $(p, v_K(a)) = 1$, then L/K is totally ramified and $h(L/K) = -v(a)/p$;

3) if $v_K(a) < 0$, $p \mid v_K(a)$ and $v(a) = \max\{v(b) \mid b \in a + \wp(K)\}$, then L/K is ferociously ramified and $h(L/K) = -v(a)/p$.

Let $\operatorname{char} K = 0$, $L = K(\gamma)$ a non-trivial extension, $\gamma^p = a \in K$. In this case:

1) if $v(a-1) = \frac{p}{p-1}$, then L/K is unramified;

2) if $0 < v(a-1) < \frac{p}{p-1}$ and $(p, v_K(a-1)) = 1$, then L/K is totally ramified and $h(L/K) = \frac{1}{p-1} - \frac{1}{p}v(a-1)$;

3) if $0 < v(a-1) < \frac{p}{p-1}$, $p \mid v_K(a-1)$, *and* $v(a-1) = \max\{v(b-1) \mid b \in a(K^*)^p\}$, *then* L/K *is ferociously ramified and* $h(L/K) = \frac{1}{p-1} - \frac{1}{p}v(a-1)$;

4) if $v_K(a) = 0$, $a \notin K^p$, *then* L/K *is ferociously ramified and* $h(L/K) = \frac{1}{p-1}$;

5) if $v_K(a) > 0$, $p \mid v_K(a)$, *then* L/K *is totally ramified and* $h(L/K) = \frac{1}{p-1}$.

Proof. Let $e = e(L/K), f = f(L/K)$. Since $ef = p$, and p is prime, we have either $e = 1$, $f = p$ (the extension is weakly unramified), or $e = p$, $f = 1$ (the extension is totally ramified). Denote by π any prime element of K.

Consider first the case $\operatorname{char} K = p$. For subcases 1), 2) see [FV, Prop. III.2.5]. (In the first subcase notice also that $\overline{a} \notin \wp(\overline{K})$ in view of Hensel lemma.) Suppose now that $v_K(a) < 0, p \mid v_K(a)$ and $v(a) = \max\{v(b) \mid b \in a+\wp(K)\}$. Let $a = c\pi^{-np}$, where $v_K(c) = 0$. If $\bar{c} = (\bar{b})^p$, $b \in F$, then $v_K(a - (b\pi^{-n})^p + b\pi^{-n}) > v_K(a)$, a contradiction with the choice of a. On the other hand, denote $b = \pi^n\gamma \in \mathcal{O}_L$. Then $\overline{b}^p = \overline{c}$, whence $\overline{L}/\overline{K}$ is purely inseparable, and $\overline{L} = \overline{K}(\bar{b})$.

Now consider the case $\operatorname{char} K = 0$. We treat consequently all 5 subcases.

1) Let $\gamma = 1+\pi^{\frac{e_k}{p-1}}x$, $a = 1+\pi^{\frac{pe_k}{p-1}}b$. We obtain $v(x) = 0$, $v(b) = 0$, $x^p + \frac{p}{\pi^e}x = b$. The latter equation should not be solvable in K, whence $\overline{x}^p + \frac{p}{\pi^e}\overline{x} = \overline{b}$ has no roots in $\overline{K}$. This equation is separable, and L/K is therefore unramified.

2) Since $v(\gamma - 1) = v(a-1)/p$, L/K is totaly ramified. Let σ be any generator of $\operatorname{Gal}(L/K)$. Then $\gamma^{\sigma-1}$ is a primitive pth root of unity, whence

$$v((\gamma-1)^{\sigma-1} - 1) = \frac{1}{p-1} - \frac{1}{p}v(a-1).$$

Since $(v_L(\gamma-1), p) = 1$ one can find integral m and n such that $\pi_L = (\gamma-1)^m/\pi^n$ is a prime in L. Then

$$v(\pi_L^{\sigma-1} - 1) = \frac{1}{p-1} - \frac{1}{p}v(a-1),$$

and we are done since $\mathcal{O}_L = \mathcal{O}_K[\pi_L]$.

3) Introduce $d \in \mathcal{O}_K$ and $t \in \mathcal{O}_L$ via $a = 1 + \pi^{pl}d$, $\gamma = 1 + \pi^l t$, where $l = v_K(a-1)/p$. By the maximality of $v(a-1)$, we have $\overline{d} \notin \overline{K}^p$. On the other hand, $t^p \equiv d \mod \mathfrak{M}_L$, whence L/K is ferociously ramified, and $\mathcal{O}_L = \mathcal{O}_K[t]$. We obtain (in the above notation):

$$h(L/K) = v(t^{\sigma-1} - 1)) = v((\gamma-1)^{\sigma-1} - 1) = \frac{1}{p-1} - \frac{1}{p}v(a-1).$$

4), 5) Similarly to 3) and 2) respectively. $\square$

§2. Infernal jumps

Throughout this section K denotes a *standard* complete discretely valued field with the residue field of characteristic p, B a fixed base subfield in K, k the corresponding constant subfield in K, $v = v_B$.

2.1. Filtration on the additive group of K in the case $\operatorname{char} K = p$.

Definition. Let $\operatorname{char} K = p$. For any $\alpha \in \mathbb{Q}$, $\alpha \geqslant 0$, define Ξ_α as the subgroup in K which is generated by all $a^{p^i}c$ where $i \geqslant 0$, $a \in \mathcal{O}_K$, $c \in k$, $v(c) \geqslant -p^{i+1}\alpha$. We have obviously $\Xi_0 = \mathcal{O}_K + k$. Next, $\Xi_\alpha \subsetneqq \Xi_\beta$ for $\alpha < \beta$. For $a \in K$, put $w_K(a) = \inf\{\, \alpha \mid a \in \Xi_\alpha \,\} = \min\{\, \alpha \mid a \in K_\alpha \,\}$.

Remark. The filtration $(\Xi_\alpha + \wp(K))_\alpha$ coincides with the infernal part of the filtration on the additive group of K which is dual (in the sense of Artin-Schreier duality) to the ramification filtration on the Galois group of the maximal abelian extension of K of exponent p introduced in [Z2].

Lemma 2.1.1. *Let $a_1, a_2 \in \mathcal{O}_K$, $c_1, c_2 \in k$, $v_k(c_1) = v_k(c_2) = i$. For some integral $k_1 \geqslant k_2 \geqslant 0$ denote*
$$s = a_1^{p^{k_1}} c_1 + a_2^{p^{k_2}} c_2.$$
Then there exist $a_1', a_2' \in U_K \cup \{0\}$, $c_1', c_2' \in k$, $v_k(c_1') \geqslant i$, $v_k(c_2') > i$ such that
$$s = a_1'^{p^{k_2}} c_1' + a_2'^{p^{k_2}} c_2'.$$

Proof. We may assume that $c_2 \neq 0$. The residue class of c_1/c_2 belongs to the maximal perfect subfield in $\overline{K}$, hence there exists $r \in \mathcal{R}_K$ such that $\bar{r}^{p^{k_2}} = \bar{c}_1/\bar{c}_2$. Now take $a_1' = a_1^{p^{k_1-k_2}} + a_2/r$, $a_2' = a_2$, $c_1' = c_1$ and $c_2' = c_2 - c_1/(r^{p^{k_2}})$. If we have $0 < v_K(a_1') < \infty$, then instead of a_1' and c_1' one must take $a_1'\pi_k{}^{-v_K(a_1')}$ and $c_1'\pi_k{}^{v_K(a_1')p^{k_2}}$ respectively, where π_k is a prime element in k. If necessary, the same can be done for the pair a_2', c_2'. Now the condition $v_k(c_2') > i$ is satisfied by the choice of r, and the equality in Lemma can be easily checked. $\square$

Corollary 2.1.2. *Let c_j be certain fixed elements of k with $v_k(c_j) = j$ for all integral j.*

1. For any $\alpha > 0$ we have
$$\Xi_\alpha = \left\{ \sum_{j \gg -\infty} a_j^{p^{i(j)}} c_j \,\middle|\, a_j \in \mathcal{O}_K \right\} = \left\{ \sum_{j \gg -\infty} a_j^{p^{i(j)}} c_j \,\middle|\, a_j \in U_K \cup \{0\} \right\},$$
where $i(j) = \min\{\, i \geqslant 0 \mid j/e_k \geqslant -p^{i+1}\alpha \,\}$.

2. For any $a \in K$ the value $w_K(a)$ is equal to the minimal w such that there exists an expansion

$$a = \sum_{\substack{v(a) \leqslant j < 0 \\ j \in e_k^{-1}\mathbb{Z}}} (a_j)^{p^{\lceil \log_p(\frac{-j}{w}) - 1 \rceil}} c_j + a_0, \tag{1}$$

where $a_j \in U_K \cup \{0\}$ for $v(a) \leqslant j < 0$, $a_0 \in \mathcal{O}_K$. If there is no minimal w, then $w_K(a) = 0$. Here $\lceil x \rceil$ denotes the minimal non-negative integer which is $\geqslant x$.

Lemma 2.1.3. *Let $a \in U_K$, $\bar{a} \notin \overline{K}^p$, $c \in k$, $v_k(c) = j_0$, $u \in K$, $v_k(u - a^{p^i}c) > j_0$. Then $w_K(u) \geqslant \frac{-j_0}{p^{i+1}e_k}$.*

Proof. Let $w_K(u) < \frac{-j_0}{p^{i+1}e_k}$ which is equivalent to $u \in \Xi_\alpha$ for some

$$\alpha < \frac{-j_0}{p^{i+1}e_k}. \tag{2}$$

Corollary 2.1.2 implies

$$u = \sum_{j=j_0}^{\infty} a_j^{p^{i(j)}} c_j,$$

where $c_{j_0} = c$, and c_j with $j > j_0$ are arbitarily chosen. After division by c we obtain

$$a^{p^i} \equiv a_{j_0}^{p^{i(j_0)}} \mod \mathfrak{M}_K,$$

and $(\overline{a})^{p^i} \in \overline{K}^{p^{i(j_0)}}$. In view of $\overline{a} \notin \overline{K}^p$, we obtain $i(j_0) \leqslant i$, i. e. $j_0/e_k \geqslant -p^{i+1}\alpha$, contradicting (2). $\square$

Lemma 2.1.4. *Consider $b_0 = a^{p^n}c$ where $0 \leqslant n \in \mathbb{Z}$, $a \in U_K$, $\overline{a} \notin \overline{K}^p$, $c \in k$, $v(c) < 0$. Then*

$$w_K(b_0) = \frac{-v(c)}{p^{n+1}}.$$

Next, for any $b \in b_0 + \wp(K)$ we have

$$w_K(b) \geqslant \frac{-v(c)}{p^{n+1}}.$$

Proof. The first assertion follows immediately from Lemma 2.1.3. To prove the second assertion, we assume first that either $n = 0$ or $p \nmid v_k(c)$. Suppose there exists $b \in b_0 + \wp(K)$ with $w_K(b) < w_K(b_0)$. Among all such b we choose an element with the maximal $v(b)$; we have $b = b_0 + d^p - d$, $d \in K$. It can be seen from the Lemma 2.1.3 that $v(d) > v(b_0)/p$ implies $w_K(b_0 + d^p - d) \geqslant \frac{-v(c)}{p^{n+1}}$. Furthermore, the case $v(d) = v(b_0)/p$ is possible only if $p \mid v_k(c)$, whence $n = 0$. In this case we have

$$\overline{\left(\frac{b_0 + d^p - d}{c}\right)} = \overline{a} + \overline{\left(\frac{d^p}{c}\right)} \notin \overline{K}^p,$$

and this is also incompatible with the hypothesis $w_K(b) < w_K(b_0)$. Finally, let $v(d) < v(b_0)/p$. We can write d as $a_0^{p^{n_0}}c_0 + d'$ where $a_0 \in U_K$, $\bar{a}_0 \notin \overline{K}^p$, $c_0 \in k$ and $v(d') > v(d) = v(c_0)$. Then $b = b_0 + d^p - d = a_0^{p^{n_0+1}}c_0^p + \dots$, where dots denote terms of higher order. From Lemma 2.1.3 we have

$$w_K(b) \geqslant w_K(a_0^{p^{n_0+1}}c_0^p) = w_K(a_0^{p^{n_0}}c_0). \tag{3}$$

Put $b_1 = b - \wp(a_0^{p^{n_0}}c_0)$. Then it is seen from (3) that $w_K(b_1) \leqslant w_K(b)$ whereas $v(b_1) > v(b)$. This contradicts to the choice of b.

Now consider the remaining case when $n > 0$ and $p \mid v_k(c)$. Use induction on n. Since $p \mid v_k(c)$, we have $c = c_1^p + c'$, where $c_1, c' \in k$, and $v(c') > v(c)$. Then

$$b_0 = a^{p^n}c_1^p + a^{p^n}c' = a^{p^{n-1}}c_1 + \wp(a^{p^{n-1}}c_1) + a^{p^n}c',$$

whence $b = b_1 + a^{p^n} c'$, $b_1 \in a^{p^{n-1}} c_1 + \wp(K)$. We have

$$w_K(b_1) \geqslant \frac{-v(c_1)}{p^n} = \frac{-v(c)}{p^{n+1}}$$

by the assumption of induction, and

$$w_K(a^{p^n} c') = \frac{-v(c')}{p^{n+1}} < \frac{-v(c)}{p^{n+1}}$$

by the first assertion of Lemma. This proves the second assertion. $\square$

Lemma 2.1.5. *Let $\mathcal{B}$ be a class in $K/\wp(K)$. Then there exists $b \in \mathcal{B}$ such that $w_K(b) \leqslant w_K(b')$, and $v(b) \geqslant v(b')$ for any $b' \in \mathcal{B}$.*

Proof. Suppose first that $\mathcal{B}$ contains some $a \in \mathcal{O}_K$. Then the assertion is trivial (either a or 0 suits for b). Otherwise consider the subset $\mathcal{B}_0$ in $\mathcal{B}$ which consists of all elements $b \in \mathcal{B}$ with the maximal $v(b)$. It is clear from the definition that w_K takes finitely many values on $\mathcal{B}_0$; fix an element $b \in \mathcal{B}_0$ with the minimal $w_K(b)$.

Suppose that $w_K(b) > w_K(b')$ for some $b' \in \mathcal{B}$. Choose such a b' with the maximal $v(b')$. Then $b' = a^{p^n} \pi^{pr} + d$, where π is a prime in k, $r = v_k(b')$, $a \in U_K$, n a positive integer, $v(d) > v(b')$, either $\overline{a} \notin \overline{K}^p$ or $a = 1$. Consider

$$b'' = b' + \wp(-a^{p^{n-1}} \pi^r).$$

It follows from Lemma 2.1.3 that

$$w_K(b') \geqslant w_K(a^{p^n} \pi^{pr}) = w_K(a^{p^{n-1}} \pi^r).$$

Therefore, $w_K(b'') \leqslant w_K(b') < w_K(b)$, whereas $v(b'') > v(b')$. This contradicts to the choice of b'. $\square$

2.2. Filtration on K^* in the case $\operatorname{char} K = 0$**.**

In this subsection $\operatorname{char} K = 0$. Denote $\tilde{\wp}_K = \{\, a \in K \mid v(a) \geqslant p/(p-1) \,\}$. For a non-negative integer m and a rational number α such that $0 \leqslant \alpha \leqslant \frac{1}{p-1}$ define $\Xi(m, \alpha)$ as the subgroup in K which is generated by $\tilde{\wp}_K$ and all elements of the kind $a^{p^i} c$ where $i \geqslant 0$, $a \in U_K$, $c \in k$, $v(c) \geqslant \max(m/e_k, \frac{p}{p-1} - p^{i+1}\alpha)$.

Lemma 2.2.1. *Let $a_1, a_2 \in U_K$, $c_1, c_2 \in k$, $v(c_1) = v(c_2) = m/e_k \geqslant \frac{p}{p-1} - p^{i+1}\alpha$. For an integer $i \geqslant 0$ denote*

$$s = a_1^{p^i} c_1 + a_2^{p^i} c_2.$$

Fix $c \in k$, $v(c) = m/e_k$. Then there exists $a \in U_K \cup \{0\}$ such that

$$s \equiv a^{p^i} c \mod \Xi(m+1, \alpha).$$

Proof. As in the proof of Lemma 2.1.1, we take $r_1, r_2 \in \mathcal{R}_K$ such that $\bar{r}_1^{p^j} = \bar{c}_1/\bar{c}$, $\bar{r}_2^{p^j} = \bar{c}_2/\bar{c}$.

Denote $a = a_1 r_1 + a_2 r_2$. Then

$$a^{p^i} c \equiv {a_1}^{p^i} c {r_1}^{p^i} + a_2^{p^i} c {r_2}^{p^i} + \sum_{j=1}^{p-1} \frac{1}{p} \binom{p}{j} (a_1 r_1)^{j p^{i-1}} (a_2 r_2)^{(p-j) p^{i-1}} pc \mod p^2.$$

We have $c{r_\nu}^{p^i} \equiv c_\nu \mod \mathfrak{M}_k$, $\nu = 1, 2$, whence

$$s \equiv a_1^{p^i} c{r_1}^{p^i} + a_2^{p^i} c{r_2}^{p^i} \mod \Xi(m+1, \alpha).$$

Next, for any j we have $(a_1 r_1)^{jp^{i-1}} (a_2 r_2)^{(p-j)p^{i-1}} pc \in \Xi(m+1, \alpha)$, since $v(pc) = 1 + m/e_k \geqslant (m+1)/e_k$, and $v(pc) = 1 + m/e_k \geqslant 1 + \frac{p}{p-1} - p^{i+1}\alpha$ together with $v(pc) \geqslant 1$ implies $v(pc) \geqslant \frac{p}{p-1} - p^{(i-1)+1}\alpha$. It remains to notice that $p^2 \in \tilde{\wp}_K \subset \Xi(m+1, \alpha)$. $\square$

Corollary 2.2.2. *We have*

$$\Xi(m, \alpha) = \left\{ \sum_{j=m}^{\infty} a_j^{p^{i(j)}} c_j \,\middle|\, a_j \in \mathcal{O}_K \right\},$$

where $i(j) = \min\{ i \geqslant 0 \mid j/e_k \geqslant \frac{p}{p-1} - p^{i+1}\alpha \}$, *and* c_j *are certain fixed elements of* k *with* $v_k(c_j) = j$.

Proof. Induction on $\dfrac{pe_k}{p-1} - m$. $\square$

It is easy to see that $\Xi(0, \alpha)$ is a subring in $\mathcal{O}_K$. Lemma 2.2.1 implies that any $u \in \Xi(0, \alpha)$ with $v(u) = 0$ is invertible in $\Xi(0, \alpha)$.

Definition. Let $\operatorname{char} K = 0$. For any $\alpha \in \mathbb{Q}$, $0 \leqslant \alpha \leqslant \frac{1}{p-1}$ we define E_α as the subgroup of K^* which is equal to $\Xi(0, \alpha)^* \cdot k^*$. We have obviously $E_0 = (1 + \tilde{\wp}_K) \cdot k^*$ and $E_{\frac{1}{p-1}} = K^*$. Next, $E_\alpha \subset E_\beta$ for $\alpha < \beta$. For any $u \in K^*$, denote $w_K(u) = \inf\{ \alpha \mid u \in E_\alpha \} = \min\{ \alpha \mid u \in E_\alpha \}$.

Remark. In the case $\zeta_p \in K$ the filtration $(E_\alpha \cdot (K^*)^p)_\alpha$ coincides with the infernal part of the filtration on K^*, dual (in the sense of Kummer duality) to the ramification filtration on the Galois group of the maximal abelian extension of K of exponent p introduced in [Z2]. The requirement of such compatibilty explains the "annoying" factor p^{i+1} in the definition instead of more natural p^i.

Lemma 2.2.3. *Let* $a \in U_K$, $\overline{a} \notin \overline{K}^p$, $c \in k$, $v_k(c) = j_0 > 0$, $u \in K$, $v_k(u - 1 - a^{p^i} c) > j_0$. *Then* $w_K(u) \geqslant (\frac{p}{p-1} - \frac{j_0}{e_k})/p^{i+1}$.

Proof. Let $w_K(u) < (\frac{p}{p-1} - \frac{j_0}{e_k})/p^{i+1}$ which is equivalent to $u \in E_\alpha$ for some

$$\alpha < \frac{\frac{p}{p-1} - \frac{j_0}{e_k}}{p^{i+1}}. \tag{4}$$

We have $u - 1 \in \Xi(0, \alpha)$, $v_k(u-1) = j_0$ whence $u - 1 \in \Xi(j_0, \alpha)$. Corollary 2.2.2 implies

$$u - 1 = \sum_{j=j_0}^{\infty} a_j^{p^{i(j)}} c_j,$$

where $c_{j_0} = c$, and c_j with $j > j_0$ are arbitarily chosen. After division by c we obtain

$$a^{p^i} \equiv a_{j_0}^{p^{i(j_0)}} \mod \mathfrak{M}_K,$$

and $(\bar{a})^{p^i} \in \overline{K}^{p^{i(j_0)}}$. In view of $\bar{a} \notin \overline{K}^p$, we obtain $i(j_0) \leqslant i$, i. e. $j_0/e_k \geqslant \frac{p}{p-1} - p^{i+1}\alpha$, contradicting (4). $\square$

Lemma 2.2.4. *Consider* $b_0 = 1 + a^{p^n}c$ *where* $0 \leqslant n \in \mathbb{Z}$, $a \in U_K$, $\bar{a} \notin \overline{K}^p$, $c \in k$ *and* $0 < v(c) < \frac{p}{p-1}$. *Then*

$$w_K(b_0) = \frac{\frac{p}{p-1} - v(c)}{p^{n+1}}.$$

Next, if $b \in b_0(K^*)^p$, *then*

$$w_K(b) \geqslant \frac{\frac{p}{p-1} - v(c)}{p^{n+1}}.$$

Proof. The first assertion follows immediately from the definition of w_K and Lemma 2.2.3. To prove the second assertion, we consider first the case when either $n = 0$ or $p \nmid v_k(c)$. Suppose there exists $b \in b_0(K^*)^p$ with $w_K(b) < w_K(b_0)$. Among all such b we choose an element with $v(b) = 0$ and the maximal $v(b-1)$; we have $b = b_0 d^p$, $d \in U_K$.

It can be seen from Lemma 2.2.3 that for $v(d-1) > v(b_0-1)/p$ we have

$$w_K(b_0 d^p) \geqslant \frac{\frac{p}{p-1} - v(c)}{p^{n+1}}.$$

The case $v(d-1) = v(b_0-1)/p < \frac{1}{p-1}$ is possible only when $p \mid v_k(c)$, and it follows $n = 0$. Since in this case $\overline{(d^p-1)/c} \notin \overline{K}^p$, Lemma 2.2.3 implies the same inequality for $w_K(b_0 d^p)$. Finally, consider the case $v(d-1) < v(b_0-1)/p$. If $v(d-1) > 0$, we write d as $(1 + a_0^{p^{n_0}} c_0)d'$, $a_0 \in U_K$, $\bar{a}_0 \notin \overline{K}^p$, $c_0 \in k$ and $v(d'-1) > v(d-1) = v(c_0)$; in the case $v(d-1) = 0$ we write $d = a_0^{p^{n_0}} d'$ with the same conditions on a_0 and d'. Then for $v(d-1) > 0$ we have $b = b_0 d^p = 1 + a_0^{p^{n_0+1}} c_0^p + \dots$, where dots denote terms of higher order. Lemma 2.2.3 implies

$$w_K(b) \geqslant \frac{\frac{p}{p-1} - pv(c)}{p^{n_0+2}}. \tag{5}$$

On the other hand, it can be immediately seen that

$$w_K((1 + a_0^{p^{n_0}} c_0)^p) \leqslant \frac{\frac{p}{p-1} - pv(c)}{p^{n_0+2}}. \tag{6}$$

Denote $b_1 = b/(1 + a_0^{p^{n_0}} c_0)^p$. We observe from (5) and (6) that $w_K(b_1) \leqslant w_K(b)$, whereas $v(b_1 - 1) > v(b-1)$, a contradiction with the choice of b. In the case $v(d-1) = 0$ a similar argument works for $b_1 = b/(a_0^{p^{n_0}})^p$.

Now consider the remaining case when $n > 0$ and $p \mid v_k(c)$. Use induction on n. Consider any $c_1 \in k$ such that $v(c - c_1^p) > v(c)$; we have, of course, $v(c_1) = v(c)/p$. Denote $b' = b_0/(1 + a^{p^n} c_1^p)$. An immediate calculation using Lemma 2.2.1 shows that

$$w_K(b') > \frac{\frac{p}{p-1} - v(c)}{p^{n+1}}.$$

Similarly,

$$1 + a^{p^n} c_1^p = (1 - pa^{p^{n-1}} c_1)(1 + a^{p^{n-1}} c_1)^p b'',$$

where

$$w_K(b'') > \frac{\frac{p}{p-1} - v(c)}{p^{n+1}}$$

as well. We obtain $b = b_1 b' b''$, where $b_1 \in (1 - a^{p^{n-1}} pc_1)(K^*)^p$. By the assumption of induction,

$$w_K(b_1) \geqslant \frac{\frac{p}{p-1} - v(pc_1)}{p^n} = \frac{\frac{p}{p-1} - 1 - \frac{v(c)}{p}}{p^n} = \frac{\frac{p}{p-1} - v(c)}{p^{n+1}},$$

which proves the second assertion of Lemma. □

Lemma 2.2.5. *Let $\mathcal{B}$ be a class in $K * /(K^*)^p$. Then there exists $b \in \mathcal{B}$ such that $w_K(b) \leqslant w_K(b')$, and $v(b-1) \geqslant v(b'-1)$ for any $b' \in \mathcal{B}$.*

Proof. Similarly to Lemma 2.1.5. □

2.3. Infernal jump of a cyclic extension of degree p.

Let L/K be a cyclic extension of degree p.

Definition. In the case $\operatorname{char} K = p$ by the infernal jump of L/K, where $L = K(x)$, $x^p - x = a \in K$, we mean

$$w(L/K) = \min\{w_K(b) \mid b \in a + \wp(K)\}.$$

In the case $\operatorname{char} K = 0$ introduce $K' = K(\zeta_p)$, $L' = L(\zeta_p)$. Let $L' = K'(x)$, $x^p = a \in K'$. Then by the infernal jump of L/K we mean

$$w(L/K) = \min\{w_{K'}(b) \mid b \in a({K'}^*)^p\}.$$

If L/K is almost constant, we assume $w(L/K) = 0$.

Lemma 2.3.

1) We have $w(L/K) \leqslant h(L/K)$;

2) If K'/K is any constant totally ramified extension of degree p, then $w(L/K) = w(LK'/K')$.

Proof. Let $\operatorname{char} K = p$. It is clear from the definition of w_K that $w_K(b) \leqslant -v(b)/p$ for any $b \in K$, and we deduce the first assertion by means of Proposition 1.4. In the case $\operatorname{char} K = 0$ the argument is the same, if we start with $w_K(u) \leqslant \max(0, \frac{1}{p-1} - \frac{v(u-1)}{p})$ for $u \in U_K$.

To prove the second assertion, denote $L' = LK'$. Let first $\operatorname{char} K = p$. Assume that L/K is determined by the equation $\wp(x) = a$, where $w(L/K) = w_K(a)$. The extension L'/K' is determined by the same equation, whence $w(L'/K') \leqslant w_K(a) = w(L/K)$. It remains to prove that $w(L'/K') \geqslant w_K(a) = w(L/K)$.

Consider the expansion from Corollary 2.1.2:

$$a = \sum_{j=v(a)e_k}^{\infty} a_j^{i(j)} c_j$$

with $a_j \in U_K \cup \{0\}$. It is clear from Lemma 2.1.3 that $w_K(a_j^{p^{i(j)}} c_j) \leqslant w_K(a) = w(L/K)$ for any j, and we have equality for some j. Take the minimal j_0 such that the value of $w_K(a_{j_0}^{p^{i(j_0)}} c_{j_0})$ is equal to $w(L/K)$. Now we can assume without loss of generality that the value of j_0 is maximal possible when a is running over elements of the same class modulo $\wp(K)$ with the same value of $w_K(a)$. This assumption implies that either $i(j_0) = 0$ or $p \nmid v_k(c_{j_0})$. It follows, in particular, that the equality $w_K(a_j^{p^{i(j)}} c_j) = w_K(a_{j_0}^{p^{i(j_0)}} c_{j_0})$ is not possible for any $j > j_0$.

Thus, we have $a = a_{j_0}^{p^{i(j_0)}} c_{j_0} + b$ with $w_K(b) < w_K(a)$. Next, $w(L'/K') = w_{K'}(a')$ for some $a' \in a + \wp(d)$, $d \in K'$. By Lemma 2.1.4,

$$w_{K'}(a_{j_0}^{p^{i(j_0)}} c_{j_0} + \wp(d)) \geqslant w_{K'}(a_{j_0}^{p^{i(j_0)}} c_{j_0}) = w_K(a_{j_0}^{p^{i(j_0)}} c_{j_0}) = w(L/K),$$

whereas $w_{K'}(b) \leqslant w_K(b) < w(L/K)$. It follows $w(L'/K') \geqslant w(L/K)$.

In the case $\operatorname{char} K = 0$ the argument is similar, if one uses the assertions 2.2.2–2.2.4 instead of 2.1.2–2.1.4. $\square$

2.4. On the composite of two cyclic p-extensions.

Lemma 2.4.1. *Let L/K be a cyclic extension of degree p, $h = h(L/K) > 0$. Suppose that $w(L/K) \leqslant h/p$. Then for any $c \in k$ there exists $x \in L$ such that $v(x^p - c) \geqslant v(c) + (p-1)h$.*

Proof. First of all, notice that the assumptions $h(L/K) > 0$, $w(L/K) < h(L/K)$ imply that L/K is totally ramified.

In the case $\operatorname{char} K = p$ we assume that the extension L/K is determined by the equation $\alpha^p - \alpha = a$ where a has the maximal valuation in its class in $K/\wp(K)$, and $w_K(a) = w(L/K)$. (Such a exists by Lemma 2.1.5.) In the case $\operatorname{char} K = 0$ denote $\rho = \zeta_p - 1$. We assume that the extension L/K is determined by the equation $(1 + \rho\alpha)^p = 1 + \rho^p a$ where a is also chosen to have the maximal possible valuation and to satisfy $w_K(1 + \rho^p a) = w(L/K)$. (Such a exists by Lemma 2.2.5.)

Let $v_K(a) = -n$, where $n > 0$, we have then $n = pe_k h$. We see immediately from the condition $w(L/K) \leqslant h/p = n/(p^2 e_k)$ that

$$a = \sum_{i=-n}^{-[n/p]-1} a_i^p \pi^{-i} + a_0, \tag{1}$$

where $a_i \in \mathcal{O}_K$ for all $-n \leqslant i \leqslant -[n/p] - 1$, $a_0 \in K$, $v_K(a_0) \geqslant -[n/p]$.

Next, $v_K(\alpha) = -n/p$. Let us except for a while the case when $\operatorname{char} K = 0$ and $h = \frac{1}{p-1}$. We have then $(n, p) = 1$. There exist positive integers c and d such that $cn - dp = 1$. We see immediately that $\pi' = a^{-c}\pi^{-pd}$ is prime in K, and $\pi_1 = \alpha^{-c}\pi^{-d}$ is prime in L. We have

$$\pi_1^{-p} - \pi'^{-1} = (\alpha^{cp} - a^c){\pi_0}^{pd} = (\alpha^{cp} - (\alpha^p - \alpha)^c){\pi_0}^{pd} = (c\alpha^{p(c-1)+1} + \dots){\pi_0}^{pd},$$

where dots denote terms of higher order. Therefore,

$$v_K(\pi_1^{-p} - \pi'^{-1}) = \big(p(c-1) + 1\big)\left(-\frac{n}{p}\right) + pd = n\left(1 - \frac{1}{p}\right) - 1.$$

In the exceptional case it is well known that $L = K(\pi_1)$, where $\pi_1^p = \pi'$ is a prime in K, and π_1 is a prime in L.

For any $m \leqslant n$ we can estimate the valuation of $\pi_1^{-mp} - \pi'^{-m}$:

$$v_K(\pi_1^{-mp} - \pi'^{-m}) = \Big(-p(m-1) + \Big(n(1-\frac{1}{p}) - 1\Big)p\Big)/p = -m + n\frac{p-1}{p}.$$

In view of this observation, it remains to prove that there exists an expansion

$$c = \sum_{i=v_k(c)}^{v_k(c)+n-[n/p]-1} g_i^p \pi'^{-i} + g_0,$$

where g_0, $g_i \in K$, $v(g_i) \geqslant 0$, $v(g_0) \geqslant v(c) + (p-1)h$.

It follows from (1) that

$$\pi' = \sum_{i=1}^{n-[n/p]} d_i^p \pi^i + d_0 = f(\pi), \tag{2}$$

where all $d_i \in K$, $v_K(d_0) > n - [n/p]$ and $v_K(d_i) \geqslant 0$. It follows that

$$\pi = f^{-1}(\pi') = \sum_{i=1}^{n-[n/p]} c_i^p \pi'^i + c_0,$$

where all $c_i \in K$, $v_K(c_0) > n - [n/p]$ and $v_K(c_i) \geqslant 0$. Now it is sufficient to substitute this expression for π into any expansion of c into power series with coefficients from $\mathcal{R}_k$. In the case $\operatorname{char} K = 0$ one should also take into account that for any $x, y \in \mathcal{O}_K$ we have $v(x^p + y^p - (x+y)^p) \geqslant 1 > (n - [n/p] - 1)/e_k$. $\square$

Proposition 2.4.2. *Let K_1/K, K_2/K be cyclic extensions of degree p, $h_1 = h(K_1/K)$, $h_2 = h(K_2/K)$. Suppose that $w(K_1/K) \leqslant h_1/p$ and $w(K_2/K) \leqslant h_2/p$.*

1) If $h_1 < h_2$, then

$$h(K_1K_2/K_2) = h_1/p, \quad h(K_1K_2/K_1) = h_1/p + h_2 - h_1.$$

2) If $h_1 = h_2$, then

$$h(K_1K_2/K_2) = h(K_1K_2/K_1) \leqslant h_1/p.$$

Proof. We shall check both assertions in parallel assuming that $h_1 \leqslant h_2$. Denote $n = h_1 e_k$.

In the case $\operatorname{char} K = p$ assume that the extension K_1/K is determined by the equation $\beta^p - \beta = b$, where b has the maximal possible valuation in its class in $K/\wp(K)$, $w(K_1/K) = w_K(b)$. (Such b exists by Lemma 2.1.5.) In the case $\operatorname{char} K = 0$ denote $\rho = \zeta_p - 1$. We assume that the extension K_1/K is determined by the equation $(1+\rho\beta)^p = 1 + \rho^p b$ where b is also chosen to have the maximal possible valuation and to satisfy $w_K(1+\rho^p b) = w(K_1/K)$. (Such b exists by Lemma 2.2.5.)

By the definition of w_K, the condition $w(K_1/K) \leqslant h_1/p$ means that there exist $b_i \in U_K \cup \{0\}$ and $c_i \in k$ with $-n \leqslant i < -[n/p]$ and $b' \in K$, $v_K(b') \geqslant -[n/p]$, such that

$$b = \sum_{-n \leqslant i < -[\frac{n}{p}]} b_i^p c_i + b'.$$

Applying Lemma 2.4.1 to K_2/K, for any given i one can find $c'_i \in K_2$ such that

$$v(c_i'^p - c_i) \geqslant v(c_i) + (p-1)h_2 \geqslant (p-1)h_1.$$

In the case $\operatorname{char} K = p$ we conclude that

$$b \equiv b^* = \sum_{-n \leqslant i < -[\frac{n}{p}]} b_i c'_i + b'' \mod \wp(K_2),$$

where $v(b'') \geqslant (p-1)h_1$. Moreover, if $h_1 > h_2$, then $v(b'') > (p-1)h_1$. (Take into account that $v_K(b') \geqslant -[n/p] > -n/p$.) It follows

$$j(K_1K_2/K_2) \leqslant \frac{-v(b^*)}{p} \leqslant \frac{n}{p^2} = h_1/p.$$

If $h_1 < h_2$, we obtain $j(K_1K_2/K_2) = h_1/p$. To calculate $h(K_1K_2/K_2)$, apply Lemma 1.3.1.

The case $\operatorname{char} K = 0$ can be done similarly. □

Proposition 2.4.3. *Let K_1/K, K_2/K be cyclic extensions of degree p, $h_1 = h(K_1/K)$, $h_2 = h(K_2/K)$, $w = w(K_1/K)$. Suppose that K_2/K is constant totally ramified, maybe inseparable, and $w > h_1/p$. If $\operatorname{char} K = 0$, suppose that $\zeta_p \in k$.*

1. If $w \geqslant h_2/p + h_1 - h_2$, then K_1K_2/K_2 is ferociously ramified, and we have $h(K_1K_2/K_2) = w$.

2. If $w < h_2/p + h_1 - h_2$, then $h(K_1K_2/K_2) = h_2/p + h_1 - h_2$.

Proof. For simplicity, we treat only the case $\operatorname{char} K = p$. Then K_1/K is determined by the equation $\wp(x) = a$, where $w_K(a) = w$, and $-v(a)/p = h_1$. (We can choose a suitable a by Lemma 2.1.5.) As in the proof of Lemma 2.3, we have

$$w_K(a) = w_K(a_{j_0}^{p^{i(j_0)}} c_{j_0}) = \frac{-v(c_{j_0})}{p^{i(j_0)+1}}$$

for some j_0, where all letters have the same meaning as in the proof of Lemma 2.3. In our case we have

$$\frac{-v(c_{j_0})}{p^{i(j_0)+1}} > \frac{h_1}{p} = \frac{-v(a)}{p^2}.$$

Taking into account $-v(c_{j_0}) \leqslant -v(a)$, we obtain $i(j_0) = 0$ and $v(c_{j_0}) < v(a)/p$. Therefore, $w_K(a) = w_K(a_{j_0} c_{j_0}) = -v(c_{j_0})/p$, and $\overline{a_{j_0}} \notin \overline{K}^p$.

The case when $w = h_1$ is obvious. We may therefore assume that $w < h_1$, i. e. $j_0 < j_1 = v_K(a)$. This means $\overline{a_{j_1}^{p^{i(j_1)}}} \in \overline{K}^p$.

Suppose that $w > h_2/p + h_1 - h_2$. Let k_2 be the constant subfield in K_2. Applying Lemma 2.4.1 to k_2/k, for any $j \geqslant j_1$ we can find an element $c'_j \in k_2$ such that

$$v(c_j'^p - c_j) \geqslant je_k + (p-1)h_2 \geqslant -ph_1 + (p-1)h_2 > -pw > -h_1.$$

It follows easily that

$$a = a_{j_0} c_{j_0} + b' + \wp(b) = a' + \wp(b)$$

for some $b, b' \in K_2$, where $v(b') > -pw$. Obviously, $p | v_{K_2}(a')$, and

$$v(a') = -pw = \max\{v(a'') | a'' \in a + \wp(K_2)\}.$$

Proposition 2.1.1 implies that K_1K_2/K_2 is ferociously ramified, and $h(K_1K_2/K_2) = w$.

In the case $w = h_2/p + h_1 - h_2$ we obtain

$$a = a_{j_0}c_{j_0} + b' + a_{j_1}^{p^{i(j_1)}}(c_{j_1} - {c'_{j_1}}^p) + \wp(b),$$

and $v(c_{j_1} - {c'_{j_1}}^p) \geqslant -pw = v(a_{j_0}c_{j_0})$. Here $\overline{c_{j_0}^{-1}(a_{j_0}c_{j_0} + b' + a_{j_1}^{p^{i(j_1)}}(c_{j_1} - {c'_{j_1}}^p))} \notin \overline{K}^p$, and we continue as in the preceding case.

Finally, let $w < h_2/p + h_1 - h_2$. (In particular, $h_2 < h_1$.) Consider

$$\tilde{a} = \sum_{j=j_1}^{j_0-1} a_j^{p^{i(j)}} c_j.$$

It is clear that $w_K(\tilde{a}) \leqslant \frac{-v(c_{j_1})}{p^2} = h_1/p$. Let $\tilde{K}_1/K$ be the extension of K which corresponds to the Artin–Schreier polynomial $X^p - X - \tilde{a}$. We have $h(\tilde{K}_1/K) = h_1$ and $w(\tilde{K}_1/K) \leqslant h_1/p$. Applying Proposition 2.4.2 to $\tilde{K}_1/K$ and K_2/K, we obtain $h(\tilde{K}_1K_2/K_2) = h_2/p + h_1 - h_2$. This means that $\tilde{a} = a' + \wp(b)$, $b \in K_2$, $-v(a')/p = h_2/p + h_1 - h_2$. It remains to notice that $v(a - \tilde{a}) = -pw > -p(h_2/p + h_1 - h_2)$. □

§3. Elimination of ramification

Throughout this section K denotes a complete discretely valued field of any characteristic with the residue field of characteristic p, B a certain fixed base subfield in K, k the corresponding constant subfield in K, $v = v_B$.

3.1. On an infernal extension of degree p.

Proposition 3.1. *Let K be standard, C/k a deeply ramified extension. If $\operatorname{char} K = 0$, suppose that $\zeta_p \in k$. Then for an infernal totally ramified Galois extension L/K of degree p the extension $k'L/k'K$ is ferociously ramified for some finite subextension k'/k.*

Proof. Consider a tower $k \subset k_0 \subset k_1 \subset \ldots \subset C$, where each k_{i+1}/k_i is totally ramified of degree p, and k_0/k is finite tamely ramified. Denote $K_i = k_iK$, $L_i = k_iL$, $h_i = h(K_{i+1}/K_i) = h(k_{i+1}/k_i)$, $h'_i = h(L_i/K_i)$ for $i \geqslant 0$. The value of $w(L_i/K_i)$ does not depend on i by Lemma 2.3. This common value will be denoted by w. Notice that $w > 0$ since L/K is infernal.

The definition of a deeply ramified extension implies that at least one of the following two conditions is satisfied.

(1) There exists infinitely many i such that k_{i+1}/k_i is inseparable. (In this case $h_i = \infty$.)

(2) We have $\sum_i h_i = \infty$, where the sum is taken over all $i \geqslant 0$ such that $h_i \neq \infty$.

Suppose that for any i the extension L_i/K_i is totally ramified. Then by Propositions 2.4.2 and 2.4.3 for any i we have either $h'_{i+1} \leqslant h'_i/p$, or $h'_{i+1} = h'_i - \frac{p-1}{p}h_i$. The first variant cannot occur infinitely many times since $h'_i \geqslant w$ for all i. Therfore, there exists i_0 such that $h'_{i+1} = h'_i - \frac{p-1}{p}h_i$ for $i \geqslant i_0$. For any such i the extension k_{i+1}/k_i is separable and the series $\sum_{i=i_0}^{\infty} h_i$ does not converge. This also contradicts to the condition $h'_i \geqslant w$. □

3.2. Main theorem.

Theorem 3.2.1. *Let K be a complete discretely valued field, k its constant subfield, C/k a certain deeply ramified extension. Then for any infernal extension L/K there exists a finite subextension k'/k in C/k such that $k'L/k'K$ is ferociously ramified.*

Lemma 3.2.2. *Let L/K be a finite infernal extension, U/K a finite weakly ramified extension. If UL/U if ferociously ramified, then L/K if ferociously ramified as well.*

Proof. Obvious. □

Proof of Theorem. 1. Reduction to the case of standard K.

Suppose that the assertion of Theorem is true in the case of standard K. Choose a standard $K_0 \subset K$ such that K/K_0 is finite. (In the mixed characteristic case one can take any absolutely unramified subfield with the residue field $\overline{K}$. In the equal characterisic case we can take K_0 to be the completed composite of B an $\tilde{K}$, where $\tilde{K} \approx \overline{K}$ is any coefficient subfield in K.) Let K_1/K_0 be the maximal almost constant subextension in K/K_0. Then K_1 is almost standard, and K/K_1 is infernal. Choose unramified U/K_1 such that U is standard. Then both UK/U and UL/UK are infernal. Using our assumption, we obtain the existence of finite k'/k in C/k such that $k'UK/k'U$ is ferociously ramified. Now it is clear that $k'UK$ is standard. By Lemma 1.3.2, l/k' is deeply ramified. Using once more our assumption, we obtain the existence of a finite subextension k''/k' in l/k' such that $k''UL/k''UK$ is ferociously ramified. By Lemma 3.2.2 $k''L/k''K$ is ferociously ramified as well.

2. Reduction to the case $[L:K] = p$.

In L/K consider a chain of subextensions of degree p. Then one can apply consequently Lemma 1.3.2.

3. Reduction to the case $[L:K] = p$ with normal L/K.

Let L/K be of degree p, not necessarily normal. Denote by M/K its normal closure, and by M_0 the ramification subfield in M/K. Notice that $[M:K]$ is a divisor of $p!$. Then it is clear that M/M_0 is of degree p and normal, whereas M_0/K is weakly ramified. It follows that M_0 is almost standard. Choose an unramified U/M_0 and complete the argument similarly to part 1 of the proof.

4. Reduction to the case $\zeta_p \in K$ in the mixed characteristic case.

It is sufficient to apply Lemma 3.2.2.

5. The case $[L:K] = p$ with L/K where K is standard, and $\zeta_p \in K$ in the mixed characteristic situation.

It is sufficient to apply Proposition 3.1. □

As a particular case, we obtain the following result which has in fact already appeared in the work of Hyodo.

Corollary 3.2.3. *The assertion of Theorem 3.2.1 is true for any extension C/k with the Galois group $\mathbb{Z}_p$ assuming that C/k is not unramified.*

Proof. Let C/k be a ramified $\mathbb{Z}_p$-extension. Denote by k_n the intermediate field of degree p^n over k. Then k_{N+1}/k_N is ramified for some N, whence $d_k(k_{N+1}/k_N) = \varepsilon > 0$. Applying Hyodo inequality (see, e. g., [Z1, §1]), we can conclude that $d_k(k_{n+1}/k_n) \geqslant \varepsilon$ for all $n \geqslant N$, whence C/k is deeply ramified. □

Remark. The requirement that C/k is deeply ramified is essential for Theorem 3.2.1. We give an example of an infinite extension C/k of a complete discretely valued field with perfect residue field, for which the conclusion of the theorem is not satisfied.

Let F be an arbitrary infinite perfect field of characteristic p, $B = F((\pi))$ the base subfield. Let I be a basis of F as linear space over $\mathbb{F}_p$. For any $t \in I$ denote by x_t some root of the polynomial $X^p - X - \pi^{-1}t$, and put

$$C = B(x_t \mid t \in I).$$

Let $K = F((T))((\pi))$, $L = K(y)$, where $y^p - y = \pi^{-2}T^p$. Obviously, $k = B$ is the constant subfield in K, and L/K is infernal. We shall show that for any finite subextension k'/k in C/k the extension $k'L/k'K$ is not weakly unramified.

In fact, increasing k'/k, we may assume $k' = k(x_{t_1}, \dots, x_{t_n})$, $t_1, \dots, t_n \in I$. Put $k_i = k(x_{t_1}, \dots, x_{t_i})$, $i = 0, \dots, n$. Then Proposition 2.1.3 implies $j(k_i/k_{i-1}) \leqslant p^{-i}$, $i = 1, \dots, n$. Hence also $j(k_iK/k_{i-1}K) \leqslant p^{-i}$, $i = 1, \dots, n$.

Next, we have $w(k_iL/k_iK) = w(L/K) = 2p^{-2}$ for any i, whereas $h(L/K) = 2p^{-1}$. Applying Proposition 2.4.2, we obtain by induction on i that $h(k_iL/k_iK) \geqslant p^{-1} + p^{-(i+1)}$. It follows $h(k_nL/k_nK) > w(k_nL/k_nK)$. This means that k_nL/k_nK is not weakly unramified.

3.3. On Epp–Ponomaryov theorem. We prove the following variant of the theorem of elimination of wild ramification (compare [E, Po2, Z1]). In certain sense this result permits one to strengthen the Epp–Ponomaryov theorem [Po2, Theorem 3]. However, we consider only finite extensions of complete fields here.

Theorem 3.3.1. *Let L/K be any finite extension of complete discretely valued fields, k the constant subfield of the field K. Then there exists a finite solvable extension k'/k such that $k'L/k'K$ is weakly unramified, i. e. $e(k'L/k'K) = 1$.*

Denote by K_0 any standard subfield in K such that K/K_0 is a finite extension. Let $e(L/K_0) = p^h \cdot e_t$, where $(p, e_t) = 1$. Then there exists a desired k'/k which is a composite of a cyclic p-extension and a certain extension of degree dividing $d(p-1)^h[L : K_0]$, where $d = 1$ in the case $\operatorname{char} K = p$, *and $d = [K_0(\zeta_p) : K_0]$ in the case* $\operatorname{char} K = 0$.

Remark. Take for K_0 an absolutely unramified subfield in K such that K/K_0 is totally ramified. Then we obtain Theorem 1 of Introduction.

Lemma 3.3.2. *Let K be standard, L/K a weakly ramified extension, k the constant subfield in K. Then there exists a weakly ramified solvable extension k'/k such that $[k' : k]$ divides $[L : K]$, k' is the constant subfield in $k'L$, and $k'L/k'K$ is unramified.*

Proof. Without loss of generality we may assume that the constant subfield of L coincides with k. (If this is not so, we replace K with the maximal constant subextension in L/K.) Let L_0 be the inertia subfield in L/K. Then $L = L_0(\sqrt[e]{\pi_0})$ where $e = e(L/K)$, π_0 is a prime in L_0. Denote $k' = k(\sqrt[e]{\pi})$ where π is a prime in k. We have $\pi_0 = \pi u$, $v(u) = 0$. Then $k'L = k'L_0(\sqrt[e]{u})$ is unramified over $k'K$. Since the normal closure of k'/k is $k(\zeta_e, \sqrt[e]{\pi})/k$, k'/k is solvable. □

The following Lemma is due to F. Lorenz.

Lemma 3.3.3. *Let L/K be a totally or ferociously ramified separable extension of degree p, L'/K the normal closure of L/K, E the ramification subfield in L'/K. Then L'/E is cyclic of degree p, and E/K is cyclic of degree dividing $p - 1$.*

Proof. This is a general fact that $[L' : K] | p!$. It follows immediately that L'/E is cyclic of degree p. Let $G = \mathrm{Gal}(L'/K)$, $H = \mathrm{Gal}(L'/L)$, $G_1 = \mathrm{Gal}(L'/E)$.

Obviously, G_1 is a normal subgroup in G. Since L'/K is the normal closure of L/K, the only subgroup of H which is normal in G is the trivial subgroup.

Now consider the action of H on G_1 by conjugation. It yields a homomorphism

$$\alpha\colon H \to Aut(G_1) = \mathbb{Z}/(p-1)\mathbb{Z}.$$

The elements of $\operatorname{Ker}\alpha$ commute with the elements of G_1, and $\operatorname{Ker}\alpha$ is a normal subgroup in H. Since H and G_1 generate G, $\operatorname{Ker}\alpha$ is normal in G as well, whence $\operatorname{Ker}\alpha = id$, and H is cyclic of degree dividng $p-1$. It remains to notice that $\operatorname{Gal}(E/K)$ is isomorphic to H. $\square$

Lemma 3.3.4. *Let either* $\operatorname{char} K = 0$ *and* $\zeta_p \in K$*, or* $\operatorname{char} K = p$*; L/K a totally ramified almost constant extension of degree p, k the constant subfield. Then there exists a solvable extension k'/k such that $[k' : k] \mid p(p-1)$ and $k'L/k'K$ is unramified.*

Proof. Let L'/K be the normal closure of L/K, L_1 the ramification subfield in L'/K. Then $[L' : L_1] = p$, $[L_1 : K] \mid (p-1)$ by the previous lemma. Let l_1 be the constant subfield in L_1. It is clear from the properties 4 and 5 in **1.1** that for some cyclic k'/l_1 of degree p the extension $k'L'/k'L_1$ is unramified. Then $k'L'/k'K$ is weakly ramified, and it follows that $k'L/k'K$ is unramified since this is a p-extension lying inside $k'L'/k'K$. It remains to note that $[l_1 : k]$ divides $[L_1 : K]$, whence $[k' : k] \mid p(p-1)$. $\square$

Proof of Theorem 3.3.1. Since K can be replaced by K_0, we may assume without loss of generality that K is standard.

Denote by L_2/K the maximal almost constant subextension in L/K. We prove first of all that there exists solvable k_2/k such that k_2L/k_2K is unramified, and $[k_2 : k]$ divides $d(p-1)^h[L : K]$, where $d = 1$ in the case $\operatorname{char} K = p$ and $d = [K(\zeta_p) : K]$ in the case $\operatorname{char} K = 0$.

In this part of the proof we may increase K (if necessary) and assume that L/K has no non-trivial constant subextensions. In the case $\operatorname{char} K = 0$ we may also assume that $\zeta_p \in K$.

Let L_1 be the ramification subfield in L_2/K. By Lemma 3.3.2 there exists a weakly ramified solvable extension k_1/k of degree dividing $[L_1 : K]$ such that k_1L_1/k_1K is unramified, and k_1 is the constant subfield in k_1L_1.

The extension k_1L_2/k_1L_1 is almost constant. It is totally ramified since L_2/L_1 is a totally ramified p-extension, and k_1/k is weakly ramified. It is clear that its degree does not exceed p^h. Applying repeatedly Lemma 3.3.4, we get an extension k_2/k_1 of degree dividing $(p-1)^h[L : L_1]$ such that k_2L_2/k_2L_1 is unramified.

Now take any extension C/k with the Galois group $\mathbb{Z}_p$ which is not unramified. (It is easy to see that such an extension exists.) If $\tilde{k}$ is the constant subfield in k_2L_2, then $\tilde{k}C/\tilde{k}$ is also an extension with the Galois group $\mathbb{Z}_p$, which is not unramified. At the same time k_2L/k_2L_2 is infernal. By Corollary 3.2.3, in $\tilde{k}C/\tilde{k}$ there exists a finite subextension $\tilde{k}_3/\tilde{k}$ such that $\tilde{k}_3L/\tilde{k}_3L_2$ is ferociously ramified. Next, it is obvious that $\tilde{k}_3 = k_3\tilde{k}$ for some finite subextension k_3/k in C/k. The extension k_3/k is a cyclic p-extension, and this implies that $k' = k_2k_3$ is a desired one. $\square$

3.4. Structure theorems. Here we prove slightly advanced versions of structure theorems for complete fields from [Z1].

Put $e(K/k) = p^h \cdot e_t$, where $(p, e_t) = 1$.

Lemma 3.4.1. *The field K is a totally ramified extension of a standard field K_1 such that the constant subfield of K_1 coincides with k, i. e. with the constant subfield of K.*

Proof. If $\operatorname{char} K = 0$, then by [FV, Prop. II.5.5 and II.5.6] there exists an absolutely unramified subfield K_0 in K such that K/K_0 is totally ramified. In the case $\operatorname{char} K = p$, we assume K_0 to be the completion of $\widetilde{K}B$, where $\widetilde{K}$ is any coefficient subfield in K, B the base subfield.

Thus, in both cases we observe a subfield K_0 of K_0 such that $e_{K_0} = 1$ and K/K_0 is totally ramified. Then $K_1 = kK_0$ is the desired subfield. $\square$

Theorem 3.4.2. *There exists a constant solvable extension L/K such that L is a standard field. Moreover, $L = L_1L_2$, where L_1/K is a cyclic p-extension and $[L_2 : K]$ divides $(p-1)^{h+1}e(K/k)$. Furthermore, one may take $L_1 = lK$, where l/k is a finite subextension of a given constant $\mathbb{Z}_p$-extension C/k unless the latter is unramified.*

Proof. Let K_1 be as in Lemma 3.4.1. Then the application of Theorem 3.3.1 to K/K_1 yields the required result. $\square$

Corollary. *Let K be a multidimensional local field of characteristic 0 with the first residue field of characteristic $p > 0$, the last residue field being arbitrary perfect. Then there exists a constant solvable extension L/K such that L is isomorphic to $l\{\{t_1\}\}\dots\{\{t_n\}\}$. Moreover, one may assume $L = L_1L_2$, where L_1/K is a cyclic p-extension, and $[L_2 : K]$ divides $(p-1)^{h+1}e(K/k)$, where k is the constant subfield of K.*

Theorem 3.4.3. *Let k be the constant subfield in K, C/k a deeply ramified extension of k. Then there exists a finite subextension l/k in C/k such that lK is almost standard.*

Proof. Choose K_1 as in Lemma 3.4.1. Let K_2/K_1 be the maximal almost constant subextension in K/K_1. By Theorem 3.2.1, in C/k there exists a finite subextension l/k such that lK/lK_2 is ferociously ramified. Since K_2 is almost standard, lK_2 and lK are almost standard as well. $\square$

Corollary. *Let K be a multidimensional loacal field of characteristic 0 with the first residue field of characteristic $p > 0$, k its constant subfield. Suppose that C/k is a deeply ramified extension. Then there exists a finite subextension l/k in C/k such that lK is almost standard. In particular, there exists a semi-ramified[2] extension L/K such that $L(\zeta_{p^n})$ is a standard field for some positive integer n.*

§4. On the choice of a constant subfield

It appears that when $\operatorname{char} K = p$ and the residue field is imperfect, one can force any totally ramified p-extension of K to be infernal by means of a suitable choice of a constant subfield. Namely, the following assertion takes place.

Proposition 4.1. *Let K_0 be a field of characteristic p with imperfect residue field, K/K_0 a finite extension. Then for an arbitrary finite totally ramified Galois p-extension L/K there exists a base subfield $B \subset K_0$ such that extension L/K is infernal (with respect to B) and K_0 is standard.*

[2] i. e. unramified with respect to the rank 1 valuation

Proof. Let F be the maximal perfect subfield of $\overline{K_0}$. Choose a prime element π of K_0 and a unit $t \in K_0$ such that $\bar{t} \notin F$. A base subfield B with the required properties can be found among the subfields of type $B^{(N)} = \mathbb{F}_p((\pi t^N))$, where N is an integer. Denote by $k_0^{(N)}$ the completion of $B^{(N)}(\mathcal{R}_{K_0})$, and by $k^{(N)}$ the algebraic closure of $k_0^{(N)}$ in K. Then $k_0^{(N)}$ and $k^{(N)}$ are the constant subfields with respect to $B^{(N)}$ in the fields K_0 and K respectively, and the field K_0 is a fortiori standard.

Suppose that for any N the extension L/K is not infernal. Then for any N the extension L/K contains a non-trivial almost constant subextension L'/K. One can assume L'/K to be a Galois extension. Note that L'/K is p-extension and one can choose a normal subextension of degree p in L'/K. It appears to be an almost constant Artin-Schreier extension. We show the existence of N, for which any Artin-Schreier subextension in L/K is infernal. This yields a contradiction.

Consider the subextension K'/K in L/K, specified by the Artin-Schreier equation $x^p - x = a$. Since L/K is totally ramified, we have $v_K(a) < 0$ and we may assume $p \nmid v_K(a)$. Suppose that the extension K'/K is an almost constant one (with respect to $B^{(N)}$). If necessary, the element a can be varied by an element of $\wp(K)$, and one may assume $a \in k^{(N)} + \mathcal{O}_K$. Thus $a/c_N \in 1 + \mathfrak{M}_K$ for some $c_N \in k_N$ and $N_{K/K_0} a / N_{K/K_0} c_N \in 1 + \mathfrak{M}_{K_0}$. The element c_N is algebraic over $k_0^{(N)}$, whence $N_{K/K_0} c_N \in k_0^{(N)}$.

Let $n = v_{K_0}(N_{K/K_0} a)$. Then $\overline{N_{K/K_0} a/(\pi t^N)^n} \in F$. Taking into account that, $n \neq 0$ and $\bar{t} \notin F$, we conclude that there exist at most one N, for which the extension K'/K is almost constant.

The extension L/K contains a finite number of subextensions of degree p, because there is a finite number of subgroups in the Galois group of L/K. Therefore one can choose some $N = N_0$, for which any such subextension is infernal. This completes the proof. □

Proposition 4.2. *Let L/K be an arbitrary finite Galois extension and $[L:K] = p^r s$, where $(s,p) = 1$. Assume $\zeta_s \in K$. There exists a base subfield B in K and a cyclic constant (with respect to B) extension K'/K such that $e(K'L/K') = 1$.*

Proof. Consider the tower of subextensions $L/L_1/L_0/K$ corresponding to the filtration on the Galois group, where L_0/K is the maximal unramified subextension in L/K (inertia subfield), L_1/K the maximal tamely ramified subextension in L/K. Then L/L_1 is a totally ramified Galois p-extension.

The previous proposition enables us to choose a base subfield B in K such that L/L_1 becomes infernal and the field K standard. Let $\pi \in B$ be a prime element of K.

The extension L_1/L_0 is totally ramified Galois extension of degree that is prime to p. Let $m = [L_1 : L_0]$, $(m,p) = 1$. As m divides $[L:K]$, then ζ_m belongs to L_1. There exists a prime element $\pi_0 \in K$ such that $L_1 = L_0(\sqrt[m]{\pi_0})$ [FV, II.3.5]. Let K_1 be the decomposition field for the equation $x^m = \pi$ over K. The extension K_1/K is cyclic constant of degree m and L_1K_1/L_0K_1 is unramified, because it is obtained by adjoining a root of degree m of $\pi/\pi_0 \in U_{L_0}$.

Now we deal with the extension L/L_1. Consider an arbitrary extension C/k with the Galois group isomorphic to $\mathbb{Z}_p$, that is not unramified. By Theorem 3.2.1 it contains a subextension k_2/k such that $e(Lk_2/L_1k_2) = 1$. Denote $K_2 = k_2K$. Obviously, K_2/K is constant and cyclic, and $[K_2 : K]$ is a degree of p. Then $K' = K_1K_2$ is constant and cyclic, as a composite of two constant cyclic extensions with coprime degrees, and $e(K'L/K') = 1$. $\square$

Corollary 4.3. *Let L/K be a separable subextension of a normal extension L'/K. Let $[L' : K] = p^r s$, where $(s,p) = 1$. If $\zeta_s \in K$, then there exist a base subfield B in K and a cyclic constant (with respect to B) extension K'/K such, that $e(LK'/K') = 1$.*

Corollary 4.4. *If L/K is a Galois p-extension, there exist a base subfield B in K and a cyclic constant (with respect to B) extension K'/K such that $e(K'L/K') = 1$.*

Corollary 4.3 implies immediately

Theorem 4.5. *Let $\zeta_s \in K$ for every s prime to p. For an arbitrary separable extension L/K there exist a base subfield B in K and a cyclic constant (with respect to B) extension K'/K such that $e(K'L/K') = 1$.*

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