# Explicit formulae for the Hilbert symbol of a formal group over Witt vectors


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# EXPLICIT FORMULAE FOR THE HILBERT SYMBOL OF A FORMAL GROUP OVER WITT VECTORS

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ABSTRACT. We apply theory of $p$-adic periods, the functor field of norms and Witt explicit reciprocity law in characteristic $p > 0$, to obtain Brückner-Vostokov type explicit formulae for the Hilbert symbol of a formal group over Witt vectors.

## 0. Introduction.

0.1. Let $W(k_0)$ be Witt vectors ring with coefficients in a perfect field $k_0$ of characteristic $p > 2$. Let $G$ be a commutative formal smooth group functor over $W(k_0)$ of finite dimension $n = n(G)$. This means existence of a commutative formal group law on $\mathcal{B} = W(k_0)[[X_1, \dots, X_n]]$ with an isomorphism of group functors $G \simeq \operatorname{Spf} \mathcal{B}$.

Assume that $G$ has finite height $h = h(G)$, i.e. the isogeny $p\,\mathrm{id}_G$ induces injective morphism $p^* : \mathcal{B} \longrightarrow \mathcal{B}$ of degree $p^h$. Therefore, for any $M \in \mathbb{N}$, $G[M] = \operatorname{Ker}(p^M \mathrm{id}_G)$ is a finite flat commutative group scheme over $W(k_0)$ of order $p^{Mh}$.

Fix an algebraic closure $\bar{K}$ of the fraction field $K_0$ of the ring $W(k_0)$. If $C$ is completion of $\bar{K}$ and $m_C$ is the maximal ideal of its valuation ring, then the group homomorphism $p^M \mathrm{id}_G : G(m_C) \longrightarrow G(m_C)$ is surjective, and its kernel $G[M](m_C) \simeq (\mathbb{Z}/p^M\mathbb{Z})^h$.

Let $K$ be a finite extension of $K_0$ in $\bar{K}$. Denote its residue field by $k$ and its maximal ideal by $m_K$. Assume that all points of order $p^M$ of the group $G(m_C)$ are defined over $K$, i.e. $G[M](m_C) = G[M](m_K)$. Under this assumption for $f \in G(m_K)$, $\tau \in \operatorname{Gal}(\bar{K}/K)$ one can define the formal group symbol $(f, \tau]_G$ with values in the group $G[M](m_K)$ by the relation

$$(f, \tau]_G = \tau f_1 -_G f_1,$$

where $f_1 \in G(m_C)$ is such that $(p^M \mathrm{id}_G)(f_1) = f$. If the residue field $k$ of $K$ is finite and $\psi_K : K^* \longrightarrow \Gamma_K^{\mathrm{ab}}$ is the reciprocity map of class field theory, then we obtain the Hilbert symbol $(f, g]_G = (f, \tau]_G$, where $g \in K^*$ and $\psi_K(g)$ is the image of $\tau$ in $\Gamma_K^{\mathrm{ab}}$.

Let $o_M^1, \dots, o_M^h$ be a $\mathbb{Z}/p^M$-basis of $G[M](m_K)$, then

$$(f, g]_G = A_1 o_M^1 + \dots + A_h o_M^h,$$

where $A(f,g) = {}^t(A_1,\dots,A_h) \in (\mathbb{Z}/p^M\mathbb{Z})^h$ is vector column with coordinates in $\mathbb{Z}/p^M\mathbb{Z}$ (for any module $B$ we reserve notation $B^n$ for the module of vector columns of order $n$ with coordinates in $B$). The problem of explicit description of the symbol $(\ ,\ ]_G$ is the problem of obtaining of some analytic expression for $A(f,g)$. This expression should involve information about the structure of the formal group $G$ and of elements $f \in G(m_K)$ and $g \in K^*$.

0.2. In our setting this expression contains elements of some $W_{\mathbb{Q}_p}(k)$-algebra $\mathcal{L}_{k,\tilde{t}}$ of Laurent series in variable $\tilde{t}$ with coefficients in $W_{\mathbb{Q}_p}(k)$. To define $\mathcal{L}_{k,\tilde{t}}$, let $a \in \mathbb{Q}, a > p-1$ and let $\mathcal{L}^0_{k,\tilde{t}}(a)$ be $W(k)$-algebra of formal Laurent series $\sum_{u\in\mathbf{Z}} w_u\tilde{t}^u$, where $w_u \in W_{\mathbb{Q}_p}(k)$ and $v(w_u) \geq -u/(ae)$ for $u \geq 0$, $v(w_u) \geq -u/((p-1)e)$ for $u \leq 0$ (here $v$ is a $p$-adic valuation, such that $v(p) = 1$, and $e$ is the absolute ramification index of the field $K$). Let $\mathcal{L}^0_{k,\tilde{t}}$ be the $p$-adic closure of $\bigcup\limits_{a>p-1} \mathcal{L}^0_{k,\tilde{t}}(a)$. Then $\mathcal{L}_{k,\tilde{t}} = \mathcal{L}^0_{k,\tilde{t}} \otimes_{\mathbf{Z}_p} \mathbb{Q}_p$.

If $f = \sum\limits_{u\in\mathbf{Z}} w_u\tilde{t}^u \in \mathcal{L}_{k,\tilde{t}}$, we set $\sigma(f) = \sum\limits_{u\in\mathbf{Z}} \sigma(w_u)\tilde{t}^{up}$, if this expression has sense in $\mathcal{L}_{k,\tilde{t}}$, i.e. if $\sigma(f) \in \mathcal{L}_{k,\tilde{t}}$ (here $\sigma|_{W_{\mathbf{Q}_p}(k)}$ is usual Frobenius morphism of Witt vectors). Remark, that $\sigma$ is certainly defined on the $W_{\mathbf{Q}_p}(k)$-subalgebra $\mathcal{L}^+_{k,\tilde{t}}$ of $\mathcal{L}_{k,\tilde{t}}$, which consists of $\tilde{t}$-integral series $\sum\limits_{u\geqslant 0} w_u\tilde{t}^u$.

0.3. We can fix a structure of the formal group $G$ by taking into consideration its filtered module of $p$-adic periods $\mathcal{M}(G) = (M^0(G), M^1(G))$. If $T(G)$ is Tate module of $G$ and $\Gamma_0 = \mathrm{Gal}(\bar{K}/K_0)$, then $M^0(G) = \mathrm{Hom}^{\Gamma_0}(T(G), A_{\mathrm{cris}})$ with induced filtration and action of Frobenius $\sigma$. The structure of $\mathcal{M}(G)$ can be given in terms of fixed $W(k_0)$-basis $l_1,\dots,l_n$ of $M^1(G)$ and its complement $m_1,\dots,m_{h-n} \in M^0(G)$ to a $W(k_0)$-basis of $M^0(G)$. If $\bar{l} = {}^t(l_1,\dots,l_n)$, $\bar{m} = {}^t(m_1,\dots,m_{h-n})$ are vector columns, the structure of $\mathcal{M}(G)$ is given by matrix relation

$$\begin{pmatrix} (\sigma/p)\bar{l} \\ \sigma\bar{m} \end{pmatrix} = \mathcal{E}\begin{pmatrix} \bar{l} \\ \bar{m} \end{pmatrix},$$

where $\mathcal{E} \in \mathrm{GL}_h(W(k_0))$. We use this matrix $\mathcal{E}$ to define for all $u \in \mathbb{N}$ auxillary matrices $F_u$ and $F'_u$ (of orders $n\times n$ and $n\times(h-n)$, resp.), such that

a) If $G_{\mathcal{A}}$ is $n$-dimensional formal group over $W(k_0)$ given by the functional equation

$$\bar{l}_{\mathcal{A}}(\bar{X}) = \bar{X} + \frac{1}{p}\sum_{u\geq 1} F_u(\sigma_*\bar{l}_{\mathcal{A}})(\bar{X}^{p^u}),$$

for its logarithm vector power series $\bar{l}_{\mathcal{A}}(\bar{X}) = {}^t(l_{\mathcal{A},1}(\bar{X}),\dots,l_{\mathcal{A},n}(\bar{X}))$, then $G \simeq G_{\mathcal{A}}$.

b) If $\hat{o}^1_M,\dots,\hat{o}^h_M \in m_{k,\tilde{t}} = \tilde{t}W(k)[[\ \tilde{t}\ ]] \subset \mathcal{L}_{k,\tilde{t}}$ are such that $\hat{o}^i_M \mapsto o^i_M$ under substitution $\tilde{t} \mapsto \pi$, where $\pi$ is fixed uniformizer of the field $K$, then we set

$$\bar{m}_{\mathcal{A}}(\hat{o}^i_M) = \frac{1}{p}\sum_{u\geq 1} F'_u\sigma^u(\bar{l}_{\mathcal{A}}(\hat{o}^i_M)),$$

for $1 \le i \le h$, to create modulo $p^M$ approximation

$$\mathcal{V}_{\tilde{t}} = \begin{pmatrix} p^M \bar{l}_{\mathcal{A}}(\hat{\delta}^1_M) & \dots & p^M \bar{l}_{\mathcal{A}}(\hat{\delta}^h_M) \\ p^M \bar{m}_{\mathcal{A}}(\hat{\delta}^1_M) & \dots & p^M \bar{m}_{\mathcal{A}}(\hat{\delta}^h_M) \end{pmatrix}$$

of the matrix of values of the $p$-adic periods pairing $T(G_{\mathcal{A}}) \times M^0(G_{\mathcal{A}}) \longrightarrow A_{\text{cris}}$.

0.4. *The first explicit formula for* $(f, g]_{G_{\mathcal{A}}}$.

Let $f \in G_{\mathcal{A}}(m_K)$ and $\beta = \beta(\tilde{t}) \in m^n_{k,\tilde{t}}$ be such that $\beta(\pi) = f$. Consider $g \in K^*$, such that there exists $\delta = \delta(\tilde{t}) \in W(k)[[\ \tilde{t}\ ]][\tilde{t}^{-1}]$, such that $\sigma\delta/\delta^p \in 1 + m_{k,\tilde{t}}$, $(1/p)\log(\sigma\delta/\delta^p) = \sum_{(c,p)=1} \alpha_c \tilde{t}^c$ with all $\alpha_c \in W(k)$, and $\delta(\pi) = g$. Equivalently,

$$\delta = [\alpha_0]\tilde{t}^{a_0} \prod_{(c,p)=1} E(\alpha_c, \tilde{t}^c),$$

where $\alpha_0 \in k^*, a_0 \in \mathbb{Z}$, all $\alpha_c \in W(k)$ and

$$E(\alpha, X) = \exp(\alpha X + \dots + (\sigma^s\alpha)X^{p^s}/p^s + \dots) \in \mathbb{Z}_p[[X]].$$

Such elements $g \in K^*$ create subgroup of $K^*$ of index $p^{l_0}$, where $l_0$ is the maximal integer, such that $K$ contains a primitive root of unity of degree $p^{l_0}$.

Under above assumptions we prove the following explicit formula

$$(*_1) \qquad A(f,g) = (\mathrm{Res}_{\tilde{t}=0} \circ \mathrm{Tr}) \left\{ \mathcal{V}_{\tilde{t}}^{-1} \begin{pmatrix} \bar{l}_{\mathcal{A}}(\beta) - (\mathcal{A}^*/p)\bar{l}_{\mathcal{A}}(\beta) \\ 0 \end{pmatrix} \mathrm{d}_{\log}\delta \right\} \bmod p^M ,$$

where $\mathrm{Tr} : W(k) \longrightarrow \mathbb{Z}_p$ is trace map, $\mathrm{Res}_{\tilde{t}=0}$ is residue and $\mathcal{A}^* = \sum_{u \ge 1} F_u \sigma^u$.

This formula is obtained as a result of interpretation of the formal group symbol via Witt symbol in characteristic $p$. These symbols are related by auxillary construction of some "cristalline" symbol. In fact, this method is straight generalization of our aproach in [Ab3], where we study the case $G = \mathbb{G}_m$.

0.5. *The second explicit formula for* $(f, g]_{G_{\mathcal{A}}}$.

The above formula $(*_1)$ is not good enough, because $g$ is not arbitrary element of $K^*$ and one should relate to $g$ the special power series $\delta = \delta(\tilde{t})$. In the case $G = \mathbb{G}_m$ Brückner-Vostokov explicit formulae, c.f. [Br], [Vo1], are free from these restrictions. By purely formal arguments we transform the above formula $(*_1)$ to the formula of Brückner-Vostokov type. This result can be stated as follows.

Let $f \in G_{\mathcal{A}}(m_K)$, $\beta = \beta(\tilde{t}) \in m^n_{k,\tilde{t}}$, $\mathcal{A}^* = \sum_{u \ge 1} F_u \sigma^u$ and the matrix $\mathcal{V}_{\tilde{t}}$ be as above. For $g \in K^*$ let $\delta = \delta(\tilde{t}) \in W(k)[[\ \tilde{t}\ ]][\tilde{t}^{-1}]$ be such that $\delta(\pi) = g$ and $\delta = [\alpha_0]\tilde{t}^{a_0}(1 + \delta_1)$, where $\alpha_0 \in k^*, a_0 \in \mathbb{Z}, \delta_1 \in m_{k,\tilde{t}}$. Let

$$\bar{m}_{\mathcal{A}}(\beta) = \frac{1}{p} \sum_{u \ge 1} F'_u(\sigma^u \bar{l}_{\mathcal{A}}(\beta)) ,$$

and assume that $K$ contains a primitive root of unity $\zeta_M$ of degree $p^M$. Then

$$(*_2) \qquad A(f,g) =$$

$$(\mathrm{Res}\circ\mathrm{Tr})\left\{\mathcal{V}_{\tilde{t}}^{-1}\left[\begin{pmatrix}\bar{l}_{\mathcal{A}}(\beta)-\frac{\mathcal{A}^*}{p}\bar{l}_{\mathcal{A}}(\beta)\\ 0\end{pmatrix}\mathrm{d}_{\log}\delta-\frac{1}{p}\log\frac{\sigma\delta}{\delta^p}\,\mathrm{d}\begin{pmatrix}\frac{\mathcal{A}^*}{p}\bar{l}_{\mathcal{A}}(\beta)\\ \bar{m}_{\mathcal{A}}(\beta)\end{pmatrix}\right]\right\}\bmod p^M$$

This formula is obtained from the formula $(*_1)$ by taking into consideration $p^M$-primary elements of the group $K^*$ to avoid restriction on $g \in K^*$, by proving that values of $(*_2)$ do not depend on a choice of $\delta(\tilde{t})$ and that $(*_1)$ and $(*_2)$ have the same value under special choice of $\delta(\tilde{t})$ from n.0.4.

The formula $(*_2)$ does not contain information about $\zeta_M \in K$. Now we don't have an answer to the following question: is the formula $(*_2)$ valid without assumption $\zeta_M \in K$?

The formula $(*_2)$ can be considered as a generalization of the result from [B-V], where the case of 1-dimensional formal group symbol modulo $p$ (i.e. $M = 1$) was studied.

0.6. As it was mentioned earlier, our arguments are based on Fontaine's theory of $p$-adic periods for $p$-divisible groups (also, Fontaine-Wintenberger functor field of norms and Fontaine's interpretation of Witt reciprocity law play very important rôle). It seems, one can apply Fontaine's theory for formal groups over arbitrary local fields $K_0'$ to obtain explicit formulae at least modulo some finite defect subgroup in $G[M](m_K)$ (which is trivial, if absolute ramification index of $K_0'$ is less than $p-1$), c.f. [Fo1]. All our arguments can be directly applied in the case of $A$-modules over $A_{ur}$, one should use parallel theory of $\pi_A$-adic periods (in particular, this gives explicit formula in the Lubin-Tate case from [Vo2]), c.f. [De], [F-L]. Our method is ajusted also to study the case of "$p$-adic motives" appearing in Fontaine-Laffaille theory, but one should clarify the concept of $p$-adic points for these motives. It would be also interesting to develop this theory by methods of [Ka], where the most natural and general interpretation of explicit formulae for the group $\mathbb{G}_m$ is given.

Finally, we remark, that there is another way to obtain explicit formulae for the Hilbert symbol, which is presented by Coates-Wiles formulae in the cases of multiplicative and Lubin-Tate groups, and by Kolyvagin's ideas [Ko] for 1-dimensional groups. Recently, D. Benois (private communication) obtained in this way explicit description of Kolyvagin's normalized relations and formulae of Artin-Hasse type for the Hilbert symbol in the case of formal groups over arbitrary local field. These formulae also involve information about matrix of values of $p$-adic periods pairing.

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## 1. $p$-adic periods of a formal group over Witt vectors.

Let $K_0$ be the fraction field of Witt vectors ring $W(k_0)$, where $k_0$ is a perfect field of characteristic $p > 2$. Fix an algebraic closure $\bar{K}$ of $K_0$, and denote by $C$ a $p$-adic completion of $\bar{K}$. Let $m_C$ be the maximal ideal of the valuation ring $O_C$ of the field $C$ and $\Gamma_0 = \mathrm{Gal}(\bar{K}/K_0)$.

1.1. *Fontaine's ring* $A_{\mathrm{cris}}$, [Fo3].

Let $R = \{ (x^{(n)})_{n\geq 0} \mid x^{(n)} \in O_C, x^{(n+1)p} = x^{(n)} \}$ be a ring with operations $(x^{(n)}) + (y^{(n)}) = (z^{(n)}), (x^{(n)})(y^{(n)}) = (w^{(n)})$, where $z^{(n)} = \lim_{m\to\infty} (x^{(n+m)} + y^{(n+m)})^{p^m}$, $w^{(n)} = x^{(n)}y^{(n)}$. The ring $R$ is complete with respect to the valuation $v_R$ given by $v_R((x^{(n)})) = v(x^{(0)})$, where $v$ is a $p$-adic valuation on $C$, such that $v(p) = 1$. Residue fields of $O_C$ and of $R$ are canonically identified, in particular, there is a canonical inclusion of Witt vectors rings $W(k_0) \subset W(R)$. We use notation $m_R$ for the maximal ideal of $R$.

If $w = \sum_{n\geqslant 0} p^n[r_n] \in W(R)$, then $w \mapsto \sum_{n\geq 0} p^n r_n^{(0)}$ gives epimorphism of $W(k_0)$-algebras $\gamma : W(R) \longrightarrow O_C$, and $W^1(R) := \operatorname{Ker}\gamma$ is a principal ideal in $W(R)$. One can take as its generator any $\xi \in W^1(R)$, such that $v_R(r_0) = 1$, where $r_0 = \xi \bmod pW(R) \in R$. Remark, that $\operatorname{Ker}(\gamma : W(m_R) \longrightarrow m_C) := W^1(m_R)$ equals to $\xi W(m_R)$. $A_{\mathrm{cris}}$ is the $p$-adic completion of the divided powers envelope of $W(R)$ with respect to the ideal $\operatorname{Ker}\gamma$. $A_{\mathrm{cris}}$ has induced continuos $\Gamma_0$-action and $A_{\mathrm{cris}}^{\Gamma_0} = W(k_0)$. Absolute Frobenius $\sigma$ of $W(k_0)$ has a natural prolongation $\phi_0$ to $A_{\mathrm{cris}}$. There is a decreasing filtration $\mathrm{Fil}^i A_{\mathrm{cris}}, i \geq 0$, of divided powers of the ideal $\operatorname{Ker}\gamma$. If $\gamma_{\mathrm{cris}} : A_{\mathrm{cris}} \longrightarrow O_C$ is a natural prolongation of $\gamma$, then $\mathrm{Fil}^1 A_{\mathrm{cris}} = \operatorname{Ker}\gamma_{\mathrm{cris}}$. One has $\phi_0 \mathrm{Fil}^1 A_{\mathrm{cris}} \subset pA_{\mathrm{cris}}$, so $\phi_1 = \phi_0/p$ is well-defined $\sigma$-linear morphism from $\mathrm{Fil}^1 A_{\mathrm{cris}}$ to $A_{\mathrm{cris}}$. Sometimes we denote $\phi_0$ and $\phi_1$ simply by $\sigma$ and $\sigma/p$, respectfully.

If $M \in \mathbb{N}$, denote by $A_{\mathrm{cris},M}$ the quotient $A_{\mathrm{cris}}/p^M A_{\mathrm{cris}}$ with induced $\Gamma_0$-action, filtration and mappings $\phi_0 : A_{\mathrm{cris},M} \longrightarrow A_{\mathrm{cris},M}$, $\phi_1 : \mathrm{Fil}^1 A_{\mathrm{cris},M} \longrightarrow A_{\mathrm{cris},M}$.

1.2. *p-adic periods pairing,* [Fo1], [F-L].

Denote by $\mathrm{MF}_{W(k_0)}$ the abelian category of admissible filtered modules with filtration of length 1. Its objects are quadruples $\mathcal{M} = (M^0, M^1, \phi_0, \phi_1)$, where $M^0$ is a $W(k_0)$-module, $M^1$ is its submodule, $\phi_0 : M^0 \longrightarrow M^0$ and $\phi_1 : M^1 \longrightarrow M^0$ are $\sigma$-linear morphisms, such that for any $m \in M^1$ one has $\phi_0(m) = p\phi_1(m)$, and $M = \phi_0(M^0) + \phi_1(M^1)$. Morphisms of this category are morphisms of filtered modules, which commute with $\phi_0$ and $\phi_1$.

Let $\tilde{G}$ be a $p$-divisible group over $W(k_0)$ of finite height $h(\tilde{G})$. If $T(\tilde{G})$ is Tate module of $\tilde{G}$, let $M^0(\tilde{G}) = \mathrm{Hom}_{\Gamma_0}(T(\tilde{G}), A_{\mathrm{cris}})$ and $M^1(\tilde{G}) = \mathrm{Hom}_{\Gamma_0}(T(\tilde{G}), \mathrm{Fil}^1 A_{\mathrm{cris}})$. Then $\phi_0|_{A_{\mathrm{cris}}}$ and $\phi_1|_{\mathrm{Fil}^1 A_{\mathrm{cris}}}$ induce $\phi_0 : M^0(\tilde{G}) \longrightarrow M^0(\tilde{G})$ and $\phi_1 : M^1(\tilde{G}) \longrightarrow M^0(\tilde{G})$ and $\mathcal{M}(\tilde{G}) = (M^0(\tilde{G}), M^1(\tilde{G}), \phi_0, \phi_1)$ is the object of the category $\mathrm{MF}_{W(k_0)}$. The correspondence $\tilde{G} \mapsto \mathcal{M}(\tilde{G})$ gives fully faithfull functor from the category of $p$-divisible groups over $W(k_0)$ (of finite height) to the category $\mathrm{MF}_{W(k_0)}$. The essential image of this functor consists of $\mathcal{M}(\tilde{G})$, such that $M^0(\tilde{G})$ is free $W(k_0)$-module of finite rank. We have $\mathrm{rk}_{W(k_0)} M^0(\tilde{G}) = h(\tilde{G})$, and dimension $n(\tilde{G})$ of $\tilde{G}$ is equal to $\mathrm{rk}_{W(k_0)} M^1(\tilde{G})$.

If $G$ is a formal smooth group over $W(k_0)$ of finite height and $G[m] = \operatorname{Ker} p^m \mathrm{id}_G$, then $\tilde{G} = \{G[m]\}_{m\geq 0}$ is a $p$-divisible group over $W(k_0)$. The correspondence $G \mapsto \tilde{G}$ gives fully faithfull functor from the category of formal smooth groups of finite height to the category of $p$-divisible groups over $W(k_0)$. Corresponding objects $\mathcal{M}(G) = \mathcal{M}(\tilde{G})$ can be completely characterized by one additional property: $\phi_0$ is topologically nilpotent on $M^0(\tilde{G})$.

The above description of $p$-divisible groups $\tilde{G}$ can be interpreted as the $p$-adic periods pairing

$$T(\tilde{G}) \times M^0(\tilde{G}) \longrightarrow A_{\mathrm{cris}}.$$

This pairing is $\mathbb{Z}_p$-bilinear, nondegenerate and compatible with additional structures, i.e. with structures of $\Gamma_0$-modules on $T(\tilde{G})$ and on $A_{\mathrm{cris}}$, and with filtrations and Frobenius actions on $M^0(\tilde{G})$ and on $A_{\mathrm{cris}}$. We remark, that if $G$ is a formal group, then this pairing has values in $A^{loc}_{\mathrm{cris}} := \{a \in A_{\mathrm{cris}} \mid \phi_0^n(a) \to 0, \text{ if } n \to \infty \}$.

The above description of $p$-divisible groups exists also on the level of finite group schemes. We use it to obtain for any $M \in \mathbb{N}$ non-degenerate pairing

$$\tilde{G}[M](O_C) \times (M^0(\tilde{G}) \bmod p^M) \longrightarrow A_{\mathrm{cris},M},$$

where $\tilde{G}[M] = \mathrm{Ker}(p^M \,\mathrm{id}_{\tilde{G}})$. As earlier, this pairing is also compatible with all additional structures. In particular, $\Gamma_0$-module $\tilde{G}[M](O_C)$ can be identified with

$$\{\eta \in \mathrm{Hom}_{W(k_0)}(M^0(\tilde{G}), A_{\mathrm{cris},M}) \mid \eta(M^1(\tilde{G})) \subset \mathrm{Fil}^1 A_{\mathrm{cris},M}$$
$$\text{and } \eta\phi_0 = \phi_0\eta, \eta\phi_1 = \phi_1\eta \}.$$

1.3. *Structure of* $\mathcal{M}(\tilde{G})$.

Let $\tilde{G}$ be a $p$-divisible group over $W(k_0)$ of finite height $h$ and of dimension $n$. If $\mathcal{M}(\tilde{G}) = (M^0, M^1, \phi_0, \phi_1)$, then $W(k_0)$-module $M^1$ is a direct summand of $M^0$. So, we can choose $W(k_0)$-basis of $M^1$ and elements $m_1, \dots, m_{h-n} \in M^0$, such that $\{l_1, \dots, l_n, m_1, \dots, m_{h-n}\}$ is $W(k_0)$-basis of $M^0$.

Consider vector-columns $\bar{l} = {}^t(l_1, \dots, l_n), \bar{m} = {}^t(m_1, \dots, m_{h-n})$. Then to give on $\mathcal{M}(\tilde{G})$ the structure of an object of the category $\mathrm{MF}_{W(k_0)}$ is equivalent to giving the relation

$$(*_1) \qquad \begin{pmatrix} \phi_1(\bar{l}) \\ \phi_0(\bar{m}) \end{pmatrix} = \mathcal{E} \begin{pmatrix} \bar{l} \\ \bar{m} \end{pmatrix}$$

for some invertible matrix $\mathcal{E} \in \mathrm{GL}_h(W(k_0))$.

Let $\mathcal{E} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ be a block form of $\mathcal{E}$, such that the relation $(*_1)$ can be rewritten in a form

$$\phi_1(\bar{l}) = A_1\bar{l} + B_1\bar{m}$$
$$\phi_0(\bar{m}) = C_1\bar{l} + D_1\bar{m}$$

Now we restrict ourselves to the case of $p$-divisible groups arising from formal groups $G$. This means, that $\phi_0$ acts nilpotently on $M^0 = M^0(G)$. One can easily verify, that this additional property is equivalent in terms of the matrix $\mathcal{E}$ to the property

$$\lim_{u \to \infty} \sigma^u(D_1) \dots D_1 = 0.$$

Let $\mathcal{E}^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a block form, such that

$$(*_2) \qquad \begin{aligned} \bar{l} &= A\frac{\sigma\bar{l}}{p} + B\sigma\bar{m} \\ \bar{m} &= C\frac{\sigma\bar{l}}{p} + D\sigma\bar{m} \end{aligned}$$

(now we use notation $\sigma$ and $\sigma/p$ instead of $\phi_0$ and $\phi_1$). One can use topological nilpotency of $\sigma|_{M^0}$ to replace the above relations $(*_2)$ to equivalent relations

$$\bar{l} = \sum_{u\geq 1} F_u \frac{\sigma^u \bar{l}}{p}, \qquad \bar{m} = \sum_{u\geq 1} F'_u \frac{\sigma^u \bar{l}}{p},$$

where $F_1 = A, F_2 = B(\sigma C), \dots, F_u = B(\sigma D)\dots(\sigma^{u-1}D)(\sigma^u C)$ for $u \geq 3$, and $F'_1 = C, F'_u = D\dots(\sigma^{u-2}D)(\sigma^{u-1}C)$ for $u \geq 2$ (we use, that $\bar{m} = (\mathrm{id} - D\sigma)^{-1}C(\sigma\bar{l})/p$ and $(\mathrm{id} - D\sigma)^{-1} = \mathrm{id} + D\sigma + \dots + D(\sigma D)\dots(\sigma^{u-1}D)\sigma^u + \dots$).

1.4. *Formal group $G_{\mathcal{A}}$.*

Let $\mathcal{B} = W(k_0)[[\bar{X}]]$ be a power series ring with coefficients in $W(k_0)$ and variables $X_1, \dots, X_n$, $\mathcal{B}_{\mathbb{Q}_p} = \mathcal{B}\hat{\otimes}\mathbb{Q}_p$. Let $\Delta$ be a $\sigma$-linear operator on $\mathcal{B}_{\mathbb{Q}_p}$, such that $\Delta(wX_1^{i_1}\dots X_n^{i_n}) = \sigma(w)X_1^{pi_1}\dots X_n^{pi_n}$, where $w \in W(k_0)$, $i_1, \dots, i_n \geq 0$. Denote by $\mathcal{B}^n_{\mathbb{Q}_p}$ the space of vector-columnes of order $n$ with coordinates in $\mathcal{B}_{\mathbb{Q}_p}$. Clearly, $\Delta$ acts on $\mathcal{B}^n_{\mathbb{Q}_p}$.

Introduce $\mathbb{Z}_p$-linear operator $\mathcal{A} = \sum_{u\geq 1} F_u\Delta^u$ on $\mathcal{B}^n_{\mathbb{Q}_p}$, where $F_u, u \geq 1$, are $n \times n$- matrices from n.1.3.

Consider $\bar{l}(\bar{X}) = {}^t(l_{\mathcal{A},1}(X_1, \dots, X_n), \dots, l_{\mathcal{A},n}(X_1, \dots, X_n)) \in \mathcal{B}^n_{\mathbb{Q}_p}$, such that

$$\bar{l}_{\mathcal{A}}(\bar{X}) = \left(\mathrm{id} - \frac{\mathcal{A}}{p}\right)^{-1}(\bar{X}) = \bar{X} + \sum_{m\geq 1}\frac{\mathcal{A}^m(\bar{X})}{p^m}$$

(here $\bar{X} = {}^t(X_1, \dots, X_n) \in \mathcal{B}^n_{\mathbb{Q}_p}$). Clearly, $\bar{l}(\bar{X})$ is the unique solution in $\mathcal{B}^n_{\mathbb{Q}_p}$ of the functional equation

$$\bar{l}_{\mathcal{A}}(\bar{X}) = \bar{X} + \frac{1}{p}\sum_{u\geq 1} F_u(\sigma^u_*\bar{l}_{\mathcal{A}})(\bar{X}^{p^u}), \ \bar{l}_{\mathcal{A}}(0) = 0$$

(here $\sigma_* : \mathcal{B}^n_{\mathbb{Q}_p} \longrightarrow \mathcal{B}^n_{\mathbb{Q}_p}$ is action of $\sigma$ on coefficients of power series).

By [Ha], power series $\bar{l}_{\mathcal{A}}(\bar{X})$ can be taken as the logarithm map of some $n$-dimensional commutative formal group law $G_{\mathcal{A}}$ over $W(k_0)$. Namely, $G_{\mathcal{A}} = \mathrm{Spf}\,\mathcal{B}$ with coaddition given by the relation $\Delta_{\mathcal{A}}(\bar{X}) = \bar{l}_{\mathcal{A}}^{-1}(\bar{l}_{\mathcal{A}}(\bar{X})\hat{\otimes}1 + 1\hat{\otimes}\bar{l}_{\mathcal{A}}(\bar{X}))$.

In fact, formal groups $G$ and $G_{\mathcal{A}}$ are isomorphic. This follows from comparison of Fontaine's and of Honda's theories, c.f. [Fo1, Ch.5]. In n.1.5 below we use more precise version of this statement.

1.5. *Construction of p-adic periods pairing.*

**1.5.1. Lemma.** *$\bar{l}_{\mathcal{A}}$ induces injective continuos homomorphism of $\Gamma_0$-modules*

$$\bar{l}_{\mathcal{A}} : G_{\mathcal{A}}(W(m_R)) \longrightarrow A^n_{\mathrm{cris}} \otimes \mathbb{Q}_p.$$

*Proof.*

Let $w \in G_{\mathcal{A}}(W^1(m_R) + pW(m_R))$, then $\bar{l}_{\mathcal{A}}(w) \in A^n_{\mathrm{cris}}$. Divided powers of the ideal $W^1(m_R) + pW(m_R)$ give basis of topology on $A_{\mathrm{cris}}$. Therefore, $\bar{l}_{\mathcal{A}}$ is injective on $G_{\mathcal{A}}(W^1(m_R) + pW(m_R))$. If $w \in G_{\mathcal{A}}(W(m_R))$, then $w_1 = (p^n\,\mathrm{id}_{G_{\mathcal{A}}})(w) \in W^1(m_R) + pW(m_R)$ for some $n \geq 0$, because $p\mathrm{id}_{G_{\mathcal{A}}}$ is topologically nilpotent on $G_{\mathcal{A}}(m_R)$.

Now $\bar{l}_{\mathcal{A}}(w) = (1/p^n)\bar{l}_{\mathcal{A}}(w_1)$ and injectivity of $\bar{l}_{\mathcal{A}}|_{G_{\mathcal{A}}(W(m_R))}$ is a formal consequence of the above injectivity of $\bar{l}_{\mathcal{A}}|_{G_{\mathcal{A}}(W^1(m_R)+pW(m_R))}$.

**Corrolary.** *For any $M \in \mathbb{N}$*

$$\bar{l}_{\mathcal{A}} : G_{\mathcal{A}}(W_M(m_R)) \longrightarrow A^n_{\text{cris}} \otimes \mathbb{Q}_p \bmod p^M W(m_R)$$

*is injective continuos homomorphism of $\Gamma_0$-modules.*

*Proof.*

Any $w \in G_{\mathcal{A}}(W(m_R))$ can be written as $w = w_1 +_{G_{\mathcal{A}}} (p^M w_2)$, where $w_1, w_2 \in W(m_R)^n$ and $w_1 \bmod p^M W(m_R)$ is uniquelly determined. The statement follows, because for any $m \in \mathbb{N}$ one has $\bar{l}_{\mathcal{A}}(G_{\mathcal{A}}(p^m W(m_R))) \subset p^m W(m_R)^n$, $\bar{l}_{\mathcal{A}}$ is identical on $p^m W(m_R) \bmod p^{m+1} W(m_R)$ and, therefore, $\bar{l}_{\mathcal{A}}$ induces bijection

$$\bar{l}_{\mathcal{A}} : G_{\mathcal{A}}(p^m W(m_R)) \longrightarrow p^m W(m_R)^n.$$

1.5.2. Let $o = (o_s)_{s \geq 0} \in T(G_{\mathcal{A}})$. Here all $o_s \in G_{\mathcal{A}}(m_C)$, $(p\,\text{id}_{G_{\mathcal{A}}})(o_{s+1}) = o_s$ and $o_0 = 0$. For every $s$ choose $\hat{o}_s \in W(m_R)$, such that $\gamma(\hat{o}_s) \equiv o_s \bmod pm_C$.

In this notation one has

$$p^{s+1}\bar{l}_{\mathcal{A}}(\hat{o}_{s+1}) \equiv p^s \bar{l}_{\mathcal{A}}(\hat{o}_s) \bmod p^s \operatorname{Fil}^1 A_{\text{cris}} + p^{s+1} W(m_R).$$

Indeed, $\gamma((p\,\text{id}_{G_{\mathcal{A}}})(\hat{o}_{s+1})) \equiv (p\,\text{id}_{G_{\mathcal{A}}})(o_{s+1}) = o_s \equiv \gamma(\hat{o}_s) \bmod pm_C$, therefore, $(p\,\text{id}_{G_{\mathcal{A}}})(\hat{o}_{s+1}) \equiv \hat{o}_s \bmod W^1(m_R) + pW(m_R)$, and

$$p\bar{l}_{\mathcal{A}}(\hat{o}_{s+1}) = \bar{l}_{\mathcal{A}}((p\,\text{id}_{G_{\mathcal{A}}})(\hat{o}_{s+1})) \equiv \bar{l}_{\mathcal{A}}(\hat{o}_s) \bmod \operatorname{Fil}^1 A_{\text{cris}} + pW(m_R).$$

Let $\hat{o}'_s \in G_{\mathcal{A}}(W(m_R))$, where $\gamma(\hat{o}'_s) \equiv o_s \bmod pm_C$, $s \geq 1$, be another system of liftings. Then

$$p^s \bar{l}_{\mathcal{A}}(\hat{o}'_s) \equiv p^s \bar{l}_{\mathcal{A}}(\hat{o}_s) \bmod p^s \operatorname{Fil}^1 A_{\text{cris}} + p^{s+1} W(m_R).$$

This equivalence follows because $\hat{o}_s \equiv \hat{o}'_s \bmod W^1(m_R) + pW(m_R)$.

If we choose $\hat{o}_s \in W(m_R)$, such that $\gamma(\hat{o}_s) = o_s$, then $p^s \bar{l}_{\mathcal{A}}(\hat{o}_s) \in (\operatorname{Fil}^1 A_{\text{cris}})^n$, because $\gamma((p^s \,\text{id}_{G_{\mathcal{A}}})(\hat{o}_s)) = (p^s \,\text{id}_{G_{\mathcal{A}}})(o_s) = o_0 = 0$.

The above reasons give the following lemma, c.f. also [Co],

**Lemma.** *The correspondence $o = (o_s)_{s \geq 0} \mapsto \lim_{s \to \infty} p^s \bar{l}_{\mathcal{A}}(\hat{o}_s)$ gives well-defined element $\hat{\bar{l}} \in \operatorname{Hom}^{\Gamma_0}(T(G_{\mathcal{A}}), (\operatorname{Fil}^1 A_{\text{cris}})^n)$.*

We remark, that if $\hat{\bar{l}} = {}^t(\hat{l}_1, \dots, \hat{l}_n)$, then all $\hat{l}_i \in \operatorname{Hom}^{\Gamma_0}(T(G_{\mathcal{A}}), \operatorname{Fil}^1 A_{\text{cris}}) = M^1(G_{\mathcal{A}})$.

Let $\hat{\bar{m}} = \sum_{u \geq 1} F'_u \sigma^u \hat{\bar{l}}/p$, where $F'_u$ are the matrices from n.1.3. Then $\hat{\bar{m}} = {}^t(\hat{m}_1, \dots, \hat{m}_{h-n})$, where all $\hat{m}_i \in \operatorname{Hom}^{\Gamma_0}(T(G_{\mathcal{A}}), A_{\text{cris}})$ and $\hat{\bar{m}}(o) = \lim_{s \to \infty} p^s \bar{m}_{\mathcal{A}}(\hat{o}_s)$, where $\bar{m}_{\mathcal{A}}(\hat{o}_s) = (1/p) \sum_{u \geq 1} F'_u \sigma^u (\bar{l}_{\mathcal{A}}(\hat{o}_s))$. The functional equation for $\bar{l}_{\mathcal{A}}$ from n.1.4 gives the relation

$$\hat{\bar{l}} = \frac{\mathcal{A}^*}{p} \hat{\bar{l}},$$

where $\mathcal{A}^* = \sum_{u\geq 1} F_u\sigma^u$ is $\mathbb{Z}_p$-linear operator on $M^0(G_\mathcal{A})$. This equality can be rewritten as, c.f. n.1.3,

$$\begin{pmatrix} \phi_1(\hat{\bar{l}}) \\ \phi_0(\hat{\bar{m}}) \end{pmatrix} = \mathcal{E}\begin{pmatrix} \hat{\bar{l}} \\ \hat{\bar{m}} \end{pmatrix}.$$

Therefore, the correspondence $\bar{l} \mapsto \hat{\bar{l}}, \bar{m} \mapsto \hat{\bar{m}}$ gives morphism in the category $\mathrm{MF}_{W(k_0)}$

$$\pi_\mathcal{A} : \mathcal{M}(G) \longrightarrow \mathcal{M}(G_\mathcal{A}).$$

**Claim.** *$\pi_\mathcal{A}$ is isomorphism in $\mathrm{MF}_{W(k_0)}$ and therefore, gives rise to the isomorphism $\eta_\mathcal{A} : G \simeq G_\mathcal{A}$.*

This is more precise version of the mentioned in n.1.4 existence of an isomorphism between $G$ and $G_\mathcal{A}$. This fact follows from Fontaine's description of points of a formal group in terms of its Deudonne module, c.f. [Fo1], and from relation between covectors and $A_{\mathrm{cris}}$, c.f. [F-L].

1.5.3. From now on we use the isomorphism $\eta_\mathcal{A}$ to identify $G$ and $G_\mathcal{A}$, in particular, points of the formal group $G$ can be given by coordinates. We can use lemma of n.1.5.2 to express values $\langle\ ,\ \rangle$ of the $p$-adic periods pairing $T(G_\mathcal{A})\otimes_{\mathbb{Z}_p} M^0(G_\mathcal{A}) \longrightarrow A_{\mathrm{cris}}$:

if $o = (o_s)_{s\geq 0} \in T(G_\mathcal{A})$ and $l_1, \dots, l_n, m_1, \dots, m_{h-n}$ is the $W(k_0)$-basis of $M^0(G_\mathcal{A})$ from n.1.3, then

a) $\langle o, \bar{l}\rangle = {}^t(\langle o, l_1\rangle, \dots, \langle o, l_n\rangle) = \lim\limits_{s\to\infty} p^s \bar{l}_\mathcal{A}(\hat{o}_s)$, where $\hat{o}_s \in G(W(m_R))$ are such that $\gamma(\hat{o}_s) = o_s \bmod pm_C$;

b) $\langle o, \bar{m}\rangle = {}^t(\langle o, m_1\rangle, \dots, \langle o, m_{h-n}\rangle) = \lim\limits_{s\to\infty} p^s \bar{m}_\mathcal{A}(\hat{o}_s) = \sum\limits_{u\geq 1} F'_u\sigma^u(\langle o, \bar{l}\rangle)/p$,

where $F'_u$ are matrices from n.1.3.

Values of the modulo $p^M$ $p$-adic periods pairing

$$G_\mathcal{A}[M](m_C) \otimes_{\mathbb{Z}_p} M^0(G_\mathcal{A}) \bmod p^M \longrightarrow A_{\mathrm{cris},M}$$

can be given as follows

if $o \in G_\mathcal{A}[M](m_C)$, then

a) $\langle o, \bar{l} \bmod p^M\rangle = p^M \bar{l}_\mathcal{A}(\hat{o})$, where $\hat{o} \in G(W(m_R))$ is such that $\gamma(\hat{o}) \equiv o \bmod pm_C$;

b) $\langle o, \bar{m} \bmod p^M\rangle = \sum_{u\geq 1} F'_u\phi_1(\langle o, \bar{l} \bmod p^M\rangle)$.

1.5.4. Consider $\mathbb{Z}_p$-linear operator $\mathcal{A}^* = \sum_{u\geq 1} F_u\sigma^u$ on $A_{\mathrm{cris}}$. Claim of n.1.5.2 gives injectivity of $\hat{\bar{l}} : T(G_\mathcal{A}) \longrightarrow (\mathrm{Fil}^1 A_{\mathrm{cris}})^n$ and the equality

$$\mathrm{Im}\,\hat{\bar{l}} = \left\{\bar{x} \in (\mathrm{Fil}^1 A_{\mathrm{cris}})^n \mid \bar{x} = \frac{\mathcal{A}^*}{p}\bar{x}\right\}.$$

*Remark.* In the modulo $p^M$ situation we have induced identification of $G_\mathcal{A}[M](m_C)$ with $\{\bar{x} \in (\mathrm{Fil}^1 A_{\mathrm{cris},M})^n \mid \bar{x} = (\mathcal{A}^*/p)\bar{x}\}$.

Let $o = (o_s)_{s\geq 0} \in T(G_\mathcal{A})$ and $\hat{o}_s \in G_\mathcal{A}(W(m_R))$ are such that $\gamma(\hat{o}_s) = o_s$. Then $\lim\limits_{s\to\infty}(p^s\,\mathrm{id}_{G_\mathcal{A}})(\hat{o}_s)$ exists, doesn't depend on the above choice of liftings $\hat{o}_s$ and is an element of the group $G_{\mathcal{A}}(W^1(m_R))$. The correspondence $o \mapsto \lim_{s\to\infty}(p^s \,\mathrm{id}_{G_{\mathcal{A}}})(\hat{o}_s)$ gives rise to injective homomorphism of $\Gamma_0$-modules

$$j : T(G_{\mathcal{A}}) \longrightarrow G(W^1(m_R)),$$

such that $j \circ \bar{l}_{\mathcal{A}} = \hat{\bar{l}}$.

So, we have the following characterization of the image of $j$:

$$\mathrm{Im}\, j = \left\{ w \in G_{\mathcal{A}}(W^1(m_R)) \mid \bar{l}_{\mathcal{A}}(w) = \frac{\mathcal{A}^*}{p}\bar{l}_{\mathcal{A}}(w) \right\}.$$

1.6. *Some lemmas.*

**1.6.1. Lemma.** *$\mathcal{A}^* = \sum_{u\geq 1} F_u\sigma^u$ is invertible on $W(m_R)^n$.*

*Proof.*

If $\mathcal{E}^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, then $\mathcal{A}^* = A\sigma + B\sigma(E - D\sigma)^{-1}C\sigma$.

Let $w \in W(m_R)^n$. It is easy to see, that if $(x, w_1) \in W(m_R)^h$ is a solution of the system

$$(*_1) \qquad \begin{pmatrix} w \\ w_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \sigma x \\ \sigma w_1 \end{pmatrix},$$

where $x \in W(m_R)^n, w_1 \in W(m_R)^{h-n}$, then $\mathcal{A}^*(x) = w$. To prove solvability of the system $(*_1)$ multiply both sides of it by the matrix $\mathcal{E} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ to obtain an equivalent system

$$(*_2) \qquad \begin{aligned} \sigma x &= A_1 w + B_1 w_1 \\ \sigma w_1 &= C_1 w + D_1 w_1 \end{aligned}$$

Let

$$w_1^0 = \sigma^{-1}(C_1 w) + (\sigma^{-1}D_1)\sigma^{-2}(C_1 w) + \cdots + (\sigma^{-1}D_1)\ldots(\sigma^{-u}D_1)\sigma^{-(u+1)}(C_1 w) + \ldots .$$

This expression has sense, because

$$\lim_{u\to\infty}(\sigma^{-1}D_1)\ldots(\sigma^{-u}D_1) = \lim_{u\to\infty}\sigma^u((\sigma^{u-1}D_1)\ldots D_1) = 0,$$

c.f. n.1.3. Clearly, $\sigma w_1^0 = C_1 w + D_1 w_1^0$. Therefore, $(x^0, w_1^0)$, where $x^0 = \sigma^{-1}(A_1 w) + \sigma^{-1}(B_1 w_1^0)$ gives a solution of the system $(*_1)$.

Prove that $\mathrm{Ker}\,\mathcal{A}^* = 0$. Let $x \in W(m_R)^n$ be such that $\mathcal{A}^*(x) = 0$. Again, this is equivalent to existence of $w_1 \in W(m_R)^{h-n}$, such that $(x, w_1)$ is a solution of $(*_1)$ with $w = 0$. Then relations $(*_2)$ give

$$\sigma w_1 = D_1 w_1$$

and, therefore, $w_1 = (\sigma^{-1}D_1)(\sigma^{-2}D_1)\ldots(\sigma^{-u}D_1)w_1$ for any $n \in \mathbb{N}$. But the right hand side of this equality tends to 0, if $u \to \infty$. So, $w_1 = 0$ and, therefore, $x = 0$.

**1.6.2. Lemma.** *$G_{\mathcal{A}}(m_R)$ is uniquelly p-divisible.*

*Proof.*

Let $r = {}^t(r_1, \dots r_n) \in G_{\mathcal{A}}(m_R)$ be such that $(p\,\mathrm{id}_{G_{\mathcal{A}}})r = 0$. Then for $[r] = {}^t([r_1], \dots, [r_n]) \in G_{\mathcal{A}}(W(m_R))$, one has $p\bar{l}_{\mathcal{A}}([r]) = \bar{l}_{\mathcal{A}}(p\,\mathrm{id}_{G_{\mathcal{A}}}([r])) \in pW(m_R)^n$, and functional equation for $\bar{l}_{\mathcal{A}}$ gives

$$\mathcal{A}^*\bar{l}_{\mathcal{A}}([r]) = p\bar{l}_{\mathcal{A}}([r]) - p[r] \in pW(m_R)^n.$$

Now lemma of n.1.6.1 gives $\bar{l}_{\mathcal{A}}([r]) \in pW(m_R)^n$. But $\bar{l}_{\mathcal{A}} : G_{\mathcal{A}}(m_R) \longrightarrow A^n_{\mathrm{cris}} \otimes \mathbb{Q}_p \bmod pW(m_R)$ is injective, c.f. n.1.5.1. Therefore, $r = 0$, and $p\,\mathrm{id}_{G_{\mathcal{A}}}|_{G_{\mathcal{A}}(m_R)}$ is injective.

Let $r_0 \in G_{\mathcal{A}}(m_R)$. Take $r \in m_R^n$, such that $\mathcal{A}^*(r) = r_0$. Then $\mathcal{A}^*([r]) = [r_0] + p[r_1] + \dots + p^n[r_n] + \dots$ and

$$p\bar{l}_{\mathcal{A}}([r]) = p[r] + (\mathcal{A}^* + \dots + \mathcal{A}^{*u+1}/p^u + \dots)([r]) =$$

$$= p[r] + \bar{l}_{\mathcal{A}}([r_0]) + p\bar{l}_{\mathcal{A}}([r_1]) + \dots p^n\bar{l}_{\mathcal{A}}([r_n]) + \dots .$$

Therefore, $\bar{l}_{\mathcal{A}}([r_0]) \equiv p\bar{l}_{\mathcal{A}}([h]) \bmod pW(m_R)$, where $h = r -_{G_{\mathcal{A}}} (r_1 +_{G_{\mathcal{A}}} (p\,\mathrm{id}_{G_{\mathcal{A}}})(r_2) + \dots +_{G_{\mathcal{A}}} (p^n\,\mathrm{id}_{G_{\mathcal{A}}})(r_{n+1}) + \dots)$. By n.1.5.1 we conclude $r_0 = (p\,\mathrm{id}_{G_{\mathcal{A}}})h$. Lemma is proved.

**1.6.3. Lemma.** *For any $g \in G_{\mathcal{A}}(W(m_R))$ there exists the unique $h \in G_{\mathcal{A}}(W(m_R))$, such that $\mathcal{A}^*\bar{l}_{\mathcal{A}}(h) = \bar{l}_{\mathcal{A}}(g)$.*

*Proof.*

*Existence.* We can assume, that $g = g_1 +_{G_{\mathcal{A}}} w$, where $g_1 = [r], r \in m_R^n$, and $w \in G_{\mathcal{A}}(pW(m_R))$. Let $h_1' \in W(m_R)^n$ be such that $\mathcal{A}^*(h_1') = g_1$, c.f. lemma 1.6.1. If $h_1' = \sum_{s \geqslant 0} p^s[r_s]$, take $h_1 = \sum_{G_{\mathcal{A}}}(p^s\,\mathrm{id}_{G_{\mathcal{A}}})[r_s] \in G_{\mathcal{A}}(W(m_R))$.

Then $\mathcal{A}^*\bar{l}_{\mathcal{A}}(h_1) = \sum_{s \geq 0} p^s \mathcal{A}^*\bar{l}_{\mathcal{A}}([r_s]) =$

$$= \sum_{s \geq 0} p^s(\mathrm{id} + \mathcal{A}^*/p + \dots + \mathcal{A}^{*m}/p^m + \dots)(\mathcal{A}^*[r_s]) = (\mathrm{id} - \mathcal{A}^*/p)^{-1}\mathcal{A}^*(\sum_{s \geq 0} p^s[r_s]) =$$

$$= (\mathrm{id} - \mathcal{A}^*/p)^{-1}\mathcal{A}^*(h_1') = (\mathrm{id} - \mathcal{A}^*/p)^{-1}([r]) = \bar{l}_{\mathcal{A}}(g_1).$$

Because $\mathcal{A}^*$ and $\bar{l}_{\mathcal{A}}$ are invertible on $pW(m_R)^n$, there exists $w_1 \in G_{\mathcal{A}}(pW(m_R))$, such that $\mathcal{A}^*\bar{l}_{\mathcal{A}}(w_1) = \bar{l}_{\mathcal{A}}(w)$. Therefore, $\mathcal{A}^*\bar{l}_{\mathcal{A}}(h_1 +_{G_{\mathcal{A}}} w_1) = g$.

*Uniquiness.* It is sufficient to prove, that for $h \in G_{\mathcal{A}}(W(m_R))$ the equality $\mathcal{A}^*\bar{l}_{\mathcal{A}}(h) = 0$ implies $h = 0$. Let $h = [r] +_{G_{\mathcal{A}}} (pw)$, where $r \in m_R^n, w \in W(m_R)^n$. Then $\mathcal{A}^*\bar{l}_{\mathcal{A}}([r]) \in pW(m_R)$ and functional equation for $\bar{l}_{\mathcal{A}}$ gives $p\bar{l}_{\mathcal{A}}([r]) \in pW(m_R)$. Therefore, $(p\,\mathrm{id}_{G_{\mathcal{A}}})(r) = 0$ in $G_{\mathcal{A}}(m_R)$ (c.f. n.1.5.1) and $r = 0$ (c.f. n.1.6.2). So, $\mathcal{A}^*\bar{l}_{\mathcal{A}}(pw) = 0$, but $\mathcal{A}^*$ and $\bar{l}_{\mathcal{A}}$ are inversible on $pW(m_R)$, c.f. n.1.6.1.

1.7. *Some properties of $A_{\mathrm{cris}}$.*

1.7.1. Let $\mathbb{G}_1$ be one dimensional formal group over $W(k_0)$ with logarithm

$$l_{\mathbb{G}_1}(X) = X + \frac{X^p}{p} + \dots + \frac{X^{p^s}}{p^s} + \dots$$

$\mathbb{G}_1$ is isomorphic to the multiplicative formal group $\hat{\mathbb{G}}_m$. Therefore, Tate module $T(\mathbb{G}_1)$ has $\mathbb{Z}_p$-rank 1 and $\Gamma_0$ acts on $T(\mathbb{G}_1)$ via cyclotomic character $\chi : \Gamma_0 \longrightarrow \mathbb{Z}_p^*$. Filtered module $\mathcal{M}(\mathbb{G}_1) = (M^0(\mathbb{G}_1), M^1(\mathbb{G}_1))$, where $M^0(\mathbb{G}_1) = M^1(\mathbb{G}_1) = \mathrm{Hom}^{\Gamma_0}(T(\mathbb{G}_1), \mathrm{Fil}^1 A_{\mathrm{cris}})$ has $W(k_0)$-rank 1, and there exists $W(k_0)$-generator $y$ of $M^0(\mathbb{G}_1)$, such that $\phi_1(y) = y$.

Fix $\mathbb{Z}_p$-generator $o = (o_s)_{s\geq 0}$ of the Tate module $T(\mathbb{G}_1)$. Here all $o_s \in \mathbb{G}_1(m_C)$, $(p\,\mathrm{id}_{\mathbb{G}_1})(o_{s+1}) = o_s$, $o_0 = 0, o_1 \neq 0$. Then we can fix $W(k_0)$-generator $y$ of $M^0(\mathbb{G}_1)$ by relation

$$\langle o, y\rangle = y(o) = \lim_{s\to\infty} p^s l_{\mathbb{G}_1}(\hat{o}_s),$$

where $\hat{o}_s \in W(R)$ are such that $\gamma(\hat{o}_s) \equiv o_s \bmod pm_C$. This element $y(o)$ has the following properties: $y(o) \in \mathrm{Fil}^1 A_{\mathrm{cris}} \setminus p\,\mathrm{Fil}^1 A_{\mathrm{cris}}$, $\phi_1 y(o) = y(o)$, and for any $\tau \in \Gamma_0$ one has $\tau y(o) = \chi(\tau) y(o)$, where $\chi$ is cyclotomic character of $\Gamma_0$. So, $y(o)$ generates additive Tate submodule $\mathbb{Z}_p(1) = \{a \in \mathrm{Fil}^1 A_{\mathrm{cris}} \mid \phi_1(a) = a\}$ of $A_{\mathrm{cris}}$, and we can use standard notation $t^+ = y(o)$ from [Fo3].

1.7.2. Let $\psi = \lim_{s\to\infty} (p^s\,\mathrm{id}_{\mathbb{G}_1})(\hat{o}_s)$, where $o = (o_s) \in T(\mathbb{G}_1)$ and $\hat{o}_s$ were defined in n.1.7.1. Then $\psi \in W^1(m_R)$ and $t^+ = l_{\mathbb{G}_1}(\psi)$. Let $\psi_1 = \sigma^{-1}\psi \in W(m_R)$. Then

$$(p\,\mathrm{id}_{\mathbb{G}_1})(\psi_1) = \lim_{s\to\infty} (p^{s+1}\,\mathrm{id}_{\mathbb{G}_1})(\sigma^{-1}\hat{o}_s) = \lim_{s\to\infty} (p^s\,\mathrm{id}_{\mathbb{G}_1})(p\,\mathrm{id}_{\mathbb{G}_1})(\sigma^{-1}\hat{o}_s) = \psi,$$

because $(p\,\mathrm{id}_{\mathbb{G}_1})(\sigma^{-1}\hat{o}_s) \equiv (\sigma^{-1}\hat{o}_s)^p \equiv \hat{o}_s \bmod W^1(m_R) + pW(m_R)$.

*Remark.* We use, that if $r \in m_R$, then $(p\,\mathrm{id}_{\mathbb{G}_1})(r) = r^p$ in $\mathbb{G}_1(m_R)$. Indeed,

$$l_{\mathbb{G}_1}(p\,\mathrm{id}_{\mathbb{G}_1}([r]) = p l_{\mathbb{G}_1}([r]) = p[r] + l_{\mathbb{G}_1}([r^p]),$$

so, $p\,\mathrm{id}_{\mathbb{G}_1}([r]) \equiv [r]^p \bmod pW(m_R)$ by n.1.5.1.

Therefore,

$$(*) \qquad \psi = p\psi_1 + \psi_1^p + \sum_{i\geqslant 0} c_i \psi_1^{i+1},$$

where all $c_i \in pW(\mathbb{F}_p)$. This gives $t = \psi/\psi_1 \in W^1(m_R) = \mathrm{Ker}\,\gamma$. One can easily see, that $t$ generates the ideal $W^1(R)$ of $W(R)$. In particular, $W^1(m_R) = tW(m_R)$.

1.7.3. Remark, that $A_{\mathrm{cris}}$ is a $p$-adic completion of the ring $W(R)[t_1, \dots, t_s, \dots]$, where $t_1 = t^p/p, \dots, t_{s+1} = t_s^p/p, \dots$. Therefore, the power series ring $W(R)[[t^p/p]]$ can be identified with $p$-adic closure of $W(R)[t_1, \dots, p^{p^{s-2}+\dots+1} t_s, \dots]$ in $A_{\mathrm{cris}}$.

**Lemma.** $W(R)[[t^p/p]] = W(R)[[\psi^{p-1}/p]]$

*Proof.*

The equation $(*)$ of n.1.7.2 gives $\psi = p\psi_1 + \varepsilon\psi_1^p$, where $\varepsilon = 1 + \sum_{i\geq 0} c_i \psi_1^i \in W(m_R)^*$. Therefore, $t^p/p = \varepsilon^p \psi_1^{p(p-1)}/p + w$, where $w \in W(m_R) + pW(R)$. This gives $\psi_1^{p(p-1)}/p$ is topologically nilpotent element of $A_{\mathrm{cris}}$ and

$$W(R)[[t^p/p]] = W(R)[[\psi_1^{p(p-1)}/p]].$$

On the other hand, $\psi^{p-1}/p = \varepsilon^{p-1}\psi_1^{p(p-1)}/p + w_1$, where $w_1 \in W(m_R)$. Therefore, $\psi^{p-1}/p$ is topologically nilpotent element of $A_{\text{cris}}$ and

$$W(R)[[\psi_1^{p(p-1)}/p]] = W(R)[[\psi^{p-1}/p]].$$

1.7.4. From topological nilpotency of $\psi^{p-1}/p$ it follows, that $\eta = t^+/\psi = 1 + \psi^{p-1}/p + \dots + \psi^{p^s-1}/p^s + \dots$ is invertible element of the ring $W(R)[[\psi^{p-1}/p]]$. Therefore, $t^+$ and $\psi$ are associated elements of $W(R)[[\psi^{p-1}/p]]$.

**Lemma.** *$\sigma/p$ acts nilpotently on the ideal $\psi^{p-1}/pW(R)[[\psi^{p-1}/p]]$ of the ring $W(R)[[\psi^{p-1}/p]]$.*

*Proof.*

The above property $t^+ = \psi\eta, \eta \in W(R)[[\psi^{p-1}/p]]^*$, gives

$$\psi^{p-1}/pW(R)[[\psi^{p-1}/p]] = (t^+)^{p-1}/pW(R)[[(t^+)^{p-1}/p]].$$

But $(\sigma/p)((t^+)^{p-1}/p) = p^{p-2}((t^+)^{p-1}/p)$, q.e.d.

1.7.5. Let $\tilde{G}$ be a $p$-divisible group over $W(k_0)$ and let $\mathcal{M}(\tilde{G}) \in \mathrm{MF}_{W(k_0)}$ be given in notation of n.1.3. Then we have identification of $\Gamma_0$-modules

$$T(\tilde{G}) = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in A_{\text{cris}}^h |\ a \in (\mathrm{Fil}^1 A_{\text{cris}})^n, \begin{pmatrix} \phi_1 a \\ \phi_0 b \end{pmatrix} = \mathcal{E} \begin{pmatrix} a \\ b \end{pmatrix} \right\}.$$

Let $T_1(\tilde{G})$ be $\Gamma_0$-module cosisting of all $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \in (W(R)[[\psi^{p-1}/p]])^h$, such that $a_1 \in (W^1(R) + (\psi^{p-1}/p)W(R)[[\psi^{p-1}p]])^n$ and $\begin{pmatrix} \phi_1 a_1 \\ \phi_0 b_1 \end{pmatrix} = \mathcal{E} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$.

Clearly, a natural inclusion $W(R)[[\psi^{p-1}/p]] \longrightarrow A_{\text{cris}}$ gives $\Gamma_0$-morphism $\iota_1 : T_1(\tilde{G}) \longrightarrow T(\tilde{G})$.

The relation $(*)$ of n.1.7.2 can be rewritten as $\sigma t = p + \psi^{p-1}\varepsilon_1$, where $\varepsilon_1 = 1 + \sum_{i \geq 1} c_i \psi^i$. If $w \in W^1(R)$, let $\bar{\phi}_1(w) = (\sigma w/\sigma t) \in W(R)$.

Then

$$(*) \qquad \phi_1(w) = \frac{\sigma w}{p} = \bar{\phi}_1(w)(1 + \frac{\psi^{p-1}}{p}\varepsilon_1) \equiv \bar{\phi}_1(w) \bmod \frac{\psi^{p-1}}{p} W(R)[[\psi^{p-1}/p]].$$

Let $T_2(\tilde{G})$ be $\Gamma_0$-module, which consists of all $\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \in W(R)^h$, such that $a_2 \in W^1(R)^n$ and

$$\begin{pmatrix} \bar{\phi}_1 a_2 \\ \sigma b_2 \end{pmatrix} = \mathcal{E} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}.$$

Let $T_3(\tilde{G})$ be $\Gamma_0$-module of all $\begin{pmatrix} a_3 \\ b_3 \end{pmatrix} \in (W(R) \bmod \psi^{p-1}W(R))^h$, such that $a_3 \in (W^1(R) \bmod \psi^{p-1}W(R))^n$, and

$$\begin{pmatrix} \tilde{\phi}_1 a_3 \\ \sigma b_3 \end{pmatrix} = \mathcal{E} \begin{pmatrix} a_3 \\ b_3 \end{pmatrix}$$

(here $\tilde{\phi}_1 = \bar{\phi}_1 \bmod \psi^{p-1} : W^1(R) \bmod \psi^{p-1}W(R) \longrightarrow W(R) \bmod \psi^{p-1}W(R)$).

Clearly, projections $W(R)[[\psi^{p-1}/p]] \longrightarrow W(R) \bmod \psi^{p-1}W(R) \longleftarrow W(R)$ induce $\Gamma_0$-morphisms

$$T_1(\tilde{G}) \xrightarrow{\iota_3} T_3(\tilde{G}) \xleftarrow{\iota_2} T_2(\tilde{G}).$$

In [Ab1] lemma of n.1.7.4 and the above relation $(*)$ between $\phi_1$ and $\phi_2$ were used to prove the following

**Proposition.** *The above maps $\iota_1$, $\iota_2$ and $\iota_3$ are isomorphisms of $\Gamma_0$-modules.*

*Remark.* In particular, values of the $p$-adic periods pairing belong to the subring $W(R)[[\psi^{p-1}/p]]$.

1.7.6. Lemmas of nn. 1.7.3 and 1.7.4 give the following

**Lemma.** *Let $\bar{l}_{\mathcal{A}}$ be logarithm vector power series from n.1.3. If $w \in W^1(m_R)$, then*

$$\bar{l}_{\mathcal{A}}(w) \in \left(W^1(m_R) + \frac{\psi^{p-1}}{p} W(R)[[\psi^{p-1}/p]]\right)^n$$

*Remark.* It follows from this lemma, that values of the $p$-adic periods pairing of the formal group $G_{\mathcal{A}}$ belong to $W(m_R) + (\psi^{p-1}/p)W(R)[[\psi^{p-1}/p]]$.

1.8. *Duality.*

1.8.1. Let $\tilde{G} = \varinjlim_{s \geqslant 1} G_{\mathcal{A}}[s]$ be the $p$-divisible group associated to the formal group $G_{\mathcal{A}}$. Consider the dual $p$-divisible group $\tilde{G}^D = \varinjlim_{s \geqslant 1} G^D_{\mathcal{A}}[s]$, where $G^D_{\mathcal{A}}[s]$ are Cartier duals for the group schemes $G_{\mathcal{A}}[s]$. If $T(\tilde{G}^D)$ is Tate module of $\tilde{G}^D$, then Cartier duality gives nondegenerate pairing of $\Gamma_0$-modules

$$\langle\ ,\ \rangle_T : T(G_{\mathcal{A}}) \otimes_{\mathbb{Z}_p} T(\tilde{G}^D) \longrightarrow T(\mathbb{G}_1).$$

Fix $\mathbb{Z}_p$-basis $o^1, \dots, o^h$ of $T(G_{\mathcal{A}})$ and denote by $o^{*1}, \dots, o^{*h}$ $\mathbb{Z}_p$-basis of $T(\tilde{G}^D)$, such that $\langle o^i, o^{*j}\rangle_T = \delta_{ij} o$, where $o$ is the generator of $T(\mathbb{G}_1)$ chosen in n.1.7.

1.8.2. Let $\mathcal{M}(\tilde{G}^D) = (M^0(\tilde{G}^D), M^1(\tilde{G}^D)) \in \mathrm{MF}_{W(k_0)}$ be filtered module of the $p$-divisible group $\tilde{G}^D$. One can use functorial properties of tensor product in the category of admissible filtered modules of length of filtration 2, c.f. [F-L], [Fo2], to express Cartier duality as a morphism in this category

$$\Delta_{\mathcal{M}} : \mathcal{M}(\mathbb{G}_1) \longrightarrow \mathcal{M}(G_{\mathcal{A}}) \otimes \mathcal{M}(\tilde{G}^D).$$

If $l_1, \dots, l_n, m_1, \dots, m_{h-n}$ is a special basis of $M^0(G_{\mathcal{A}})$, chosen in n.1.3, and $y \in M^1(\mathbb{G}_1)$ is the element from n.1.7.1, then

a) $\Delta_{\mathcal{M}}(y) = l_1 \otimes m_1^* + \dots + l_n \otimes m_n^* + m_1 \otimes l_1^* + \dots + m_{h-n} \otimes l_{h-n}^* \in M^0(G_{\mathcal{A}}) \otimes M^0(\tilde{G}^D)$, where $l_1^*, \dots, l_{h-n}^*$ is $W(k_0)$-basis of $M^1(\tilde{G}^D)$ and $m_1^*, \dots, m_n^*, l_1^*, \dots, l_{h-n}^*$ is $W(k_0)$-basis of $M^0(\tilde{G}^D)$;

b) $\Delta_{\mathcal{M}}(\phi_1 y) = \phi_1(l_1) \otimes \phi_0(m_1^*) + \dots + \phi_1(l_n) \otimes \phi_0(m_n^*) + \phi_0(m_1) \otimes \phi_1(l_1^*) + \dots + \phi_0(m_{h-n}) \otimes \phi_1(l_{h-n}^*)$.

These properties of the copairing $\Delta_{\mathcal{M}}$ give the following structure of the filtered module $\mathcal{M}(\tilde{G}^D)$. If $\bar{m}^D = {}^t(m_1^*, \dots, m_n^*), \bar{l}^D = {}^t(l_1^*, \dots, l_{h-n}^*)$, then

$$(*) \qquad \begin{pmatrix} \phi_0 \bar{m}^D \\ \phi_1 \bar{l}^D \end{pmatrix} = {}^t\mathcal{E}^{-1} \begin{pmatrix} \bar{m}^D \\ \bar{l}^D \end{pmatrix}$$

where $\mathcal{E} \in \mathrm{GL}_h(W(k_0))$ gives structure of the filtered module $\mathcal{M}(G_{\mathcal{A}})$, c.f. n.1.3.

Indeed, we have $\phi_1(\bar{l}) = A_1\bar{l} + B_1\bar{m}, \phi_0\bar{m} = C_1\bar{l} + D_1\bar{m}$, where $\mathcal{E} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ (c.f. n.1.3). In evident notation we have

$$\Delta_{\mathcal{M}}(\phi_1 y) = \sum_{1 \leqslant i \leqslant n} \phi_1(l_i) \otimes \phi_0(m_i^*) + \sum_{1 \leqslant j \leqslant h-n} \phi_0(m_j) \otimes \phi_1(l_j^*) =$$

$$= \phi_1(\bar{l}) \otimes \phi_0(\bar{m}^D) + \phi_0(\bar{m}) \otimes \phi_1(\bar{l}^D) = (A_1\bar{l} + B_1\bar{m}) \otimes \phi_0(\bar{m}^D) + (C_1\bar{l} + D_1\bar{m}) \otimes \phi_1(\bar{l}^D) =$$

$$= \bar{l} \otimes ({}^tA_1\phi_0(\bar{m}^D) + {}^tC_1\phi_1(\bar{l}^D)) + \bar{m} \otimes ({}^tB_1\phi_0(\bar{m}^D) + {}^tD_1\phi_1(\bar{l}^D)).$$

Now the equality $\Delta_{\mathcal{M}}(\phi_1 y) = \Delta_{\mathcal{M}}(y) = \bar{l} \otimes \bar{m}^D + \bar{m} \otimes \bar{l}^D$ gives the relation $(*)$:

$$\begin{pmatrix} \bar{m}^D \\ \bar{l}^D \end{pmatrix} = \begin{pmatrix} {}^tA_1 & {}^tC_1 \\ {}^tB_1 & {}^tD_1 \end{pmatrix} \begin{pmatrix} \phi_0(\bar{m}^D) \\ \phi_1(\bar{l}^D) \end{pmatrix} = {}^t\mathcal{E} \begin{pmatrix} \phi_0(\bar{m}^D) \\ \phi_1(\bar{l}^D) \end{pmatrix}$$

1.8.3. Let $l_1, \dots, l_n, m_1, \dots, m_{h-n}$ and $m_1^*, \dots, m_n^*, l_1^*, \dots, l_{h-n}^*$ be above special basises of $M^0(G_{\mathcal{A}})$ and $M^0(\tilde{G}^D)$, and let $o^1, \dots, o_h$ and $o^{*1}, \dots, o^{*h}$ be special basises of $T(G_{\mathcal{A}})$ and $T(\tilde{G}^D)$ from n.1.8.1.

Consider matrices of values of the $p$-adic periods pairing in these basises

$$\mathcal{V} = \begin{pmatrix} \langle o^1, \bar{l} \rangle & \dots & \langle o^h, \bar{l} \rangle \\ \langle o^1, \bar{m} \rangle & \dots & \langle o^h, \bar{m} \rangle \end{pmatrix} \quad \mathcal{V}^D = \begin{pmatrix} \langle o^{*1}, \bar{m}^D \rangle & \dots & \langle o^{*h}, \bar{m}^D \rangle \\ \langle o^{*1}, \bar{l}^D \rangle & \dots & \langle o^{*h}, \bar{l}^D \rangle \end{pmatrix}$$

Then compatibility of the $p$-adic periods pairing with tensor product gives

**Proposition.**

$${}^t\mathcal{V}^D\mathcal{V} = t^+E_h,$$

*where $E_h$ is the unity matrix of order $h$.*

*Remark.* In evident notation matrices $\mathcal{V}$ and $\mathcal{V}^D$ satisfy the following properties

$$\begin{pmatrix} \phi_1 \\ \phi_0 \end{pmatrix} \mathcal{V} = \mathcal{E}\mathcal{V}, \ \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \mathcal{V}^D = {}^t\mathcal{E}^{-1}\mathcal{V}^D.$$

1.8.4. In notation of n.1.7.5 denote by $\iota_{\tilde{G}}$ the isomorphism of $\Gamma_0$-modules $\iota_{\tilde{G}} = \iota^{-1} \circ \iota_3 \circ \iota_2^{-1} : T(\tilde{G}) \longrightarrow T_2(\tilde{G})$. For $1 \le i \le h$ we have

$$\iota_{G_{\mathcal{A}}} \begin{pmatrix} \langle o^i, \bar{l} \rangle \\ \langle o^i, \bar{m} \rangle \end{pmatrix} = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \in W(m_R)^h,$$

where $a_i \in W^1(m_R)^n$ and $\begin{pmatrix} \bar\phi_1 a_i \\ \sigma b_i \end{pmatrix} = \mathcal{E} \begin{pmatrix} a_i \\ b_i \end{pmatrix}$.

Similarly, for $1 \le i \le h$, we have

$$\iota_{\tilde G^D} \begin{pmatrix} \langle o^{*i}, \bar m^D \rangle \\ \langle o^{*i}, \bar l^D \rangle \end{pmatrix} = \begin{pmatrix} b_i^D \\ a_i^D \end{pmatrix} \in W(m_R)^h,$$

where $a_i^D \in W^1(m_R)^{h-n}$ and $\begin{pmatrix} \sigma b_i^D \\ \bar\phi_1 a_i^D \end{pmatrix} = {}^t\mathcal{E}^{-1} \begin{pmatrix} b_i^D \\ a_i^D \end{pmatrix}$.

For the formal group $\mathbb{G}_1$ from n.1.7.1 we have $\iota_{\mathbb{G}_1}(t^+) = \psi$.

Introduce matrices of order $h$

$$\iota_{G_\mathcal{A}}(\mathcal{V}) = \hat{\mathcal{V}} = \begin{pmatrix} a_1 & \dots & a_h \\ b_1 & \dots & b_h \end{pmatrix}, \quad \iota_{\tilde G^D}(\mathcal{V}^D) = \hat{\mathcal{V}}^D = \begin{pmatrix} b_1^D & \dots & b_h^D \\ a_1^D & \dots & a_h^D \end{pmatrix}.$$

By construction we have the following properties

$$\begin{pmatrix} \bar\phi_1 \\ \sigma \end{pmatrix} \hat{\mathcal{V}} = \begin{pmatrix} \bar\phi_1 a_1 & \dots & \bar\phi_1 a_h \\ \sigma b_1 & \dots & \sigma b_h \end{pmatrix} = \mathcal{E}\hat{\mathcal{V}}$$

$$\begin{pmatrix} \sigma \\ \bar\phi_1 \end{pmatrix} \hat{\mathcal{V}}^D = \begin{pmatrix} \sigma b_1^D & \dots & \sigma b_h^D \\ \bar\phi_1 a_1^D & \dots & \bar\phi_1 a_h^D \end{pmatrix} = {}^t\mathcal{E}^{-1}\hat{\mathcal{V}}^D$$

and proposition of n.1.8.3 gives

$${}^t\hat{\mathcal{V}}^D \circ \hat{\mathcal{V}} = \psi E_h.$$

## 2. Crystalline symbol and its relation to the formal group symbol.

2.1. Recall, c.f. n. 1.5.4, that

$$j : T(G_\mathcal{A}) \longrightarrow G_\mathcal{A}(W^1(m_R))$$

is injective morphism of $\Gamma_0$-modules, and

$$\operatorname{Im} j = \left\{ w \in G_\mathcal{A}(W^1(m_R)) \mid \bar l_\mathcal{A}(w) = \frac{\mathcal{A}^*}{p} \bar l_\mathcal{A}(w) \right\}.$$

Let

$$\psi : G_\mathcal{A}(W(m_R)) \longrightarrow A_{\text{cris}}^n \otimes \mathbb{Q}_p$$

be $\Gamma_0$-morphism defined by the correspondence $g \mapsto p\bar l_\mathcal{A}(g) - \mathcal{A}^* \bar l_\mathcal{A}(g)$ for any $g \in G_\mathcal{A}(W(m_R))$.

**Proposition.** *If $\psi^1 = \psi|_{G_\mathcal{A}(W^1(m_R))}$, then $\operatorname{Im}\psi^1 = pW(m_R)^n$ and, therefore, we have exact sequence of $\Gamma_0$-modules*

$$0 \longrightarrow T(G_\mathcal{A}) \xrightarrow{j} G_\mathcal{A}(W^1(m_R)) \xrightarrow{\psi^1} pW(m_R)^n \longrightarrow 0.$$

*Proof.*

**2.2.1. Lemma.** *Let $r \in G_{\mathcal{A}}(m_R)$, and for $s \in \mathbb{N}$ let $r_s \in G_{\mathcal{A}}(m_R)$ be such that $(p^s \, \mathrm{id}_{G_{\mathcal{A}}})(r_s) = r$. Then*

a) *there exists* $\lim_{s \to \infty} (p^s \, \mathrm{id}_{G_{\mathcal{A}}})([r_s]) := \delta(r) \in G_{\mathcal{A}}(W(m_R))$;

b) $r = \delta(r) \bmod pW(m_R)$;

c) $\psi(\delta(r)) = 0$.

*Proof.*

a) and b) can be proved by arguments of n.1.5.2.

c) follows, because for any $s \in \mathbb{N}$ one has

$$p\bar{l}_{\mathcal{A}}((p^s \, \mathrm{id}_{G_{\mathcal{A}}})([r_s])) - \mathcal{A}^* \bar{l}_{\mathcal{A}}((p^s \, \mathrm{id}_{G_{\mathcal{A}}})([r_s])) =$$

$$= p^s(p\bar{l}_{\mathcal{A}}([r_s]) - \mathcal{A}^* \bar{l}_{\mathcal{A}}([r_s])) = p^{s+1}[r] \in W_{s+1}(m_R).$$

2.2.2. $\psi(G_{\mathcal{A}}(W(m_R))) = \psi(G_{\mathcal{A}}(pW(m_R)))$.

This follows from the above lemma, because if $w \in G_{\mathcal{A}}(W(m_R))$ and $r = w \bmod pW(m_R) \in G_{\mathcal{A}}(m_R)$, then $w = \delta(r) +_{G_{\mathcal{A}}} w_1$, where $w_1 \in G_{\mathcal{A}}(pW(m_R))$.

2.2.3. $\psi(G(pW(m_R))) = pW(m_R)^n$.

Indeed, let $g \in G_{\mathcal{A}}(p^i W(m_R))$ for some $i \geq 1$. Then $\bar{l}_{\mathcal{A}}(g) \equiv g \bmod p^{i+1} W(m_R)$ and

$$\psi(g) \equiv -p^i \mathcal{A}^*(r) \bmod p^{i+1} W(m_R),$$

where $r \in m_R^n$ is such that $g \equiv p^i[r] \bmod p^{i+1} W(m_R)$.

By n.1.6.1 the operator $\mathcal{A}^*|_{m_R^n}$ is invertible, therefore, for every $i \geq 1$ the map $\psi$ induces bijection of $p^i W(m_R) \bmod p^{i+1} W(m_R)$.

**2.2.4. Lemma.** *For any $a \in G_{\mathcal{A}}(m_C)$ there exists $r \in G_{\mathcal{A}}(m_R)$, such that $\gamma(\delta(r)) = a$.*

*Proof.*

Choose a sequence $\{a_s\}_{s \geq 0}$ of $a_s \in G_{\mathcal{A}}(m_C)$, such that $a_0 = a$ and $(p \, \mathrm{id}_{G_{\mathcal{A}}})(a_{s+1}) = a_s$ for all $s \geq 0$. Let $r'_s \in G_{\mathcal{A}}(m_R)$ be such that $\gamma(r'_s) = a_s$. Then one can use arguments of n.1.5.2 to verify the following properties

a) there exists $\lim_{s \to \infty} (p^s \, \mathrm{id}_{G_{\mathcal{A}}})([r'_s]) := w \in G_{\mathcal{A}}(W(m_R))$.

b) if $w \bmod pW(m_R) = r \in G_{\mathcal{A}}(m_R)$, then $w = \delta(r)$.

Now lemma is proved, because $\gamma(w) = \lim_{s \to \infty} (p^s \, \mathrm{id}_{G_{\mathcal{A}}})(a_s) = a$.

2.2.5. $\psi^1(G_{\mathcal{A}}(W^1(m_R))) = pW(m_R)^n$.

Indeed, if $w \in pW(m_R)^n$ and $g \in G_{\mathcal{A}}(W(m_R))$ is such that $\psi(g) = w$, take $r \in G_{\mathcal{A}}(m_R)$, such that $\gamma(\delta(r)) = \gamma(g) \in G_{\mathcal{A}}(m_C)$. Then $g' -_{G_{\mathcal{A}}} \delta(r) \in G_{\mathcal{A}}(W^1(m_R))$ and $\psi^1(g') = \psi(g) = w$.

Proposition is proved.

2.3. *Crystalline symbol.*

2.3.1. Fix a natural number $M$. Let $K$ be a finite extension of $K_0$ in $\bar{K}$, such that all geometric points of the group scheme $G_{\mathcal{A}}[M] = \mathrm{Ker}(p^M \, \mathrm{id}_{G_{\mathcal{A}}})$ are defined over $K$. This means, that

$$G_{\mathcal{A}}[M](m_C) = G_{\mathcal{A}}[M](m_K),$$

where $m_K$ is the maximal ideal of the valuation ring of the field $K$.

We use explicite description of the structure of the filtered module $\mathcal{M}(G_{\mathcal{A}})$ from n.1.3 and nondegenerancy of the $p$-adic periods pairing modulo $p^M$ to identify $G_{\mathcal{A}}[M](m_C)$ with the group $U_M(\mathcal{M}(G_{\mathcal{A}}))$ of vector-columns $\begin{pmatrix} y \\ z \end{pmatrix} \bmod p^M A_{\text{cris}} \in A^h_{\text{cris},M}$, such that $y \in (\text{Fil}^1 A_{\text{cris}})^n, z \in A^{h-n}_{\text{cris}}$ and $\begin{pmatrix} \phi_1(y) \\ \phi_0(z) \end{pmatrix} = \mathcal{E} \begin{pmatrix} y \\ z \end{pmatrix}$.

In these terms the equality $G_{\mathcal{A}}[M](m_C) = G_{\mathcal{A}}[M](m_K)$ means, that for any $\begin{pmatrix} y \\ z \end{pmatrix} \bmod p^M A_{\text{cris}} \in U_M(\mathcal{M}(G))$ and any $\tau \in \Gamma_K = \text{Gal}(\bar{K}/K)$ one has

$$\tau \begin{pmatrix} y \\ z \end{pmatrix} - \begin{pmatrix} y \\ z \end{pmatrix} \in p^M A^h_{\text{cris}}.$$

2.3.2. Let $t^+$ be a generator of the additive Tate module in $A_{\text{cris}}$ from n.1.7.1, and $A^{loc}_{\text{cris}}$ be the maximal subring in $A_{\text{cris}}$, where action of $\sigma$ is topologically nilpotent. Take

$$\alpha \in (A^n_{\text{cris}} \bmod t^+ A^{loc}_{\text{cris}})^{\Gamma_K} = \{\alpha \in A^n_{\text{cris}} \mid \forall \tau \in \Gamma_K \ \tau\alpha \equiv \alpha \bmod t^+ A^{loc}_{\text{cris}}\},$$

take $\tau \in \Gamma_K$ and define the value of crystalline symbol $(\alpha, \tau]_{\text{cris}} \in G_{\mathcal{A}}[M](m_K)$ as follows:

if $\begin{pmatrix} Y \\ Z \end{pmatrix} \in A^h_{\text{cris}}$ is such that $Y \in (\text{Fil}^1 A_{\text{cris}})^n$ and

$$(*) \qquad \begin{pmatrix} Y \\ Z \end{pmatrix} = \mathcal{E}^{-1} \begin{pmatrix} \phi_1(Y) \\ \phi_0(Z) \end{pmatrix} + \begin{pmatrix} \alpha \\ 0 \end{pmatrix},$$

then $(\alpha, \tau]_{\text{cris}}$ is the element $\begin{pmatrix} y \\ z \end{pmatrix} \bmod p^M A_{\text{cris}}$ of $G_{\mathcal{A}}[M](m_K) = U_M(\mathcal{M}(G_{\mathcal{A}}))$, such that

$$\tau \begin{pmatrix} Y \\ Z \end{pmatrix} - \begin{pmatrix} Y \\ Z \end{pmatrix} \equiv \begin{pmatrix} y \\ z \end{pmatrix} \bmod t^+ A^{loc}_{\text{cris}}.$$

*Remark.* This symbol is related to the filtered module $\mathcal{M}(G_{\mathcal{A}})$, or equivalently, to the formal group $G_{\mathcal{A}}$. When this dependance is important for us, we write $(\alpha, \tau]_{G_{\mathcal{A}},\text{cris}}$.

**2.3.3. Lemma.** *The above definition of $(\alpha, \tau]_{\text{cris}}$ is correct.*

*Proof.*

Solvability of the equation $(*)$ of n.2.3.2 can be deduced from n.5 of [F-L] (where it is considered a case of more general filtered modules). In fact, we apply below crystalline symbol only for $\alpha \in W(m_R)^n$, where solvability of the equation $(*)$ follows from proposition of n.2.1.

*Existence of* $\begin{pmatrix} y \\ z \end{pmatrix}$.

If $\begin{pmatrix} A \\ B \end{pmatrix} = \tau \begin{pmatrix} Y \\ Z \end{pmatrix} - \begin{pmatrix} Y \\ Z \end{pmatrix}$, then

$$\begin{pmatrix} A \\ B \end{pmatrix} = \mathcal{E}^{-1} \begin{pmatrix} \phi_1(A) \\ \phi_0(B) \end{pmatrix} + \begin{pmatrix} \alpha_\tau \\ 0 \end{pmatrix},$$

where $\alpha_\tau \in (t^+ A^{loc}_{\text{cris}})^n$. But the operator

$$\mathcal{E}^{-1} \begin{pmatrix} \phi_1 \\ \phi_0 \end{pmatrix} = \begin{pmatrix} A & pB \\ C & pD \end{pmatrix} \phi_1 : \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} A\phi_1(a) + pB\phi_1(b) \\ C\phi_1(a) + pD\phi_1(b) \end{pmatrix}$$

is nilpotent on $(t^+ A^{loc}_{\text{cris}})^h$, therefore, we can define

$$\begin{pmatrix} \beta_1 \\ \beta_0 \end{pmatrix} = \sum_{s \geq 0} \left\{ \mathcal{E}^{-1} \begin{pmatrix} \phi_1 \\ \phi_0 \end{pmatrix} \right\}^s \begin{pmatrix} \alpha_\tau \\ 0 \end{pmatrix} \in (t^+ A^{loc}_{\text{cris}})^h.$$

Clearly, $\begin{pmatrix} \beta_1 \\ \beta_0 \end{pmatrix} = \mathcal{E}^{-1} \begin{pmatrix} \phi_1(\beta_1) \\ \phi_0(\beta_0) \end{pmatrix} + \begin{pmatrix} \alpha_\tau \\ 0 \end{pmatrix}$, and we can take $\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix} - \begin{pmatrix} \beta_1 \\ \beta_0 \end{pmatrix}$.

*Uniquiness of* $\begin{pmatrix} y \\ z \end{pmatrix}$.

If $\tau \begin{pmatrix} Y \\ Z \end{pmatrix} - \begin{pmatrix} Y \\ Z \end{pmatrix} \equiv \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} \bmod t^+ A^{loc}_{\text{cris}}$, then $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} y \\ z \end{pmatrix} - \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} \in (t^+ A^{loc}_{\text{cris}})^h$ and satisfies the relation

$$\begin{pmatrix} a \\ b \end{pmatrix} = \mathcal{E}^{-1} \begin{pmatrix} \phi_1 \\ \phi_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Therefore, $\begin{pmatrix} a \\ b \end{pmatrix} = \left\{ \mathcal{E}^{-1} \begin{pmatrix} \phi_1 \\ \phi_0 \end{pmatrix} \right\}^N \begin{pmatrix} a \\ b \end{pmatrix} \underset{N \to \infty}{\longrightarrow} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, because $\mathcal{E}^{-1} \begin{pmatrix} \phi_1 \\ \phi_0 \end{pmatrix}$ is topologically nilpotent on $(t^+ A^{loc}_{\text{cris}})^h$.

*Independence of the choice of* $\begin{pmatrix} Y \\ Z \end{pmatrix}$.

If $\begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix} \in A^h_{\text{cris}}$ can be taken instead of $\begin{pmatrix} Y \\ Z \end{pmatrix}$, then $\begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$, where $\begin{pmatrix} a \\ b \end{pmatrix} \bmod p^M A_{\text{cris}} \in U_M(\mathcal{M}(G))$. But $\tau \in \Gamma_K$ and, therefore, $\tau \begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} a \\ b \end{pmatrix} \bmod p^M A^h_{\text{cris}}$, c.f. n. 2.3.1.

2.4. *Symbol* $(\alpha, \tau]_{\text{cris}}$ *in terms of operator* $\mathcal{A}^*$.

Use notation from the above definition of $(\alpha, \tau]_{\text{cris}}$. Then vector $\begin{pmatrix} Y \\ Z \end{pmatrix}$ appears as a solution of the system

$$(*) \qquad \begin{aligned} Y &= A\frac{\sigma Y}{p} + B\sigma Z + \alpha \\ Z &= C\frac{\sigma Y}{p} + D\sigma Z. \end{aligned}$$

One can easily verify, that the correspondence $\begin{pmatrix} Y \\ Z \end{pmatrix} \mapsto Y$, where $Y \in (\mathrm{Fil}^1 A_{\mathrm{cris}})^n$, $Z \in A_{\mathrm{cris}}^{h-n}$, gives one-to-one correspondence between solutions $\begin{pmatrix} Y \\ Z \end{pmatrix}$ of the above system $(*)$ and solutions $Y \in (\mathrm{Fil}^1 A_{\mathrm{cris}})^n$ of the relation

$$Y - \frac{\mathcal{A}^*}{p} Y = \alpha.$$

So, calculation of the value $o_M \in G[M](m_K)$ of $(\alpha, \tau]_{\mathrm{cris}}$ can be done as follows:

a) find $Y \in (\mathrm{Fil}^1 A_{\mathrm{cris}})^n$, such that $Y - \mathcal{A}^* Y/p = \alpha$;

b) find $y \in (\mathrm{Fil}^1 A_{\mathrm{cris}})^n$, such that $y = \mathcal{A}^* y/p$ and $\tau Y - Y \equiv y \bmod t^+ A_{\mathrm{cris}}^{loc}$;

c) find $o_M \in G[M](m_K)$, such that $y \equiv p^M \bar{l}_{\mathcal{A}}(\hat{o}_M) \bmod p^M A_{\mathrm{cris}}$, where $\hat{o}_M \in W(m_R)$ and $\gamma(\hat{o}_M) \equiv o_M \bmod p m_C$.

2.5. *Homomorphism* $\Theta_{G_{\mathcal{A}}}$.

2.5.1. Fix some uniformizer $\pi \in K$ and denote residue field of $K$ by $k$. Fix $t_0 \in m_R$, such that $t_0^{(0)} = \pi$ (it is equivalent to choosing of a sequence $\pi_s \in m_C$, such that $\pi_0 = \pi$ and $\pi_{s+1}^p = \pi_s$ for all $s \geq 0$).

Let $O_{k,\tilde{t}} = W(k)[[\tilde{t}]] \subset W(R)$, where $\tilde{t} = [t_0]$, and let $m_{k,\tilde{t}} = \tilde{t}W(k)[[\ \tilde{t}\ ]] = O_{k,\tilde{t}} \cap W(m_R)$. Remark, that $\gamma(\tilde{t}) = \pi$ and, therefore, $\gamma(O_{k,\tilde{t}}) = O_K$, $\gamma(m_{k,\tilde{t}}) = m_K$.

As usually, denote by $m_{k,\tilde{t}}^n$ the space of vector-columns of order $n$ with coordinates in $m_{k,\tilde{t}}$. We use the following abbreviated form for elements of $m_{k,\tilde{t}}^n$

$$\sum_{\bar{i} \in \mathbf{N}^n} w_{\bar{i}} \tilde{T}^{\bar{i}} = {}^t(\sum_{i_1 \in \mathbf{N}} w_{i_1} \tilde{t}^{i_1}, \ldots, \sum_{i_n \in \mathbf{N}} w_{i_n} \tilde{t}^{i_n}),$$

where $w_{\bar{i}} = {}^t(w_{i_1}, \ldots, w_{i_n}) \in W(k)^n$ and $\bar{i} = {}^t(i_1, \ldots, i_n) \in \mathbb{N}^n$.

2.5.2. Let $\sum w_{\bar{i}} \tilde{t}^{\bar{i}}$ be some element of $m_{k,\tilde{t}}^n$. If $w_{\bar{i}} = \sum_{s \geq 0} p^s [\alpha_{\bar{i},s}]$, where $\alpha_{\bar{i},s} = {}^t(\alpha_{i_1,s}, \ldots, \alpha_{i_n,s}) \in k^n$ and $[\alpha_{\bar{i},s}] = {}^t([\alpha_{i_1,s}], \ldots, [\alpha_{i_n,s}])$, set

$$\Theta_1(\sum_{\bar{i} \in \mathbf{N}^n} w_{\bar{i}} \tilde{t}^{\bar{i}}) = \sum_{\substack{(\text{in } G_{\mathcal{A}}) \\ \bar{i},s}} (p^s \operatorname{id}_{G_{\mathcal{A}}})([\alpha_{\bar{i},s}] \tilde{t}^{\bar{i}})$$

(here the right hand sum is the sum of points in the group $G_{\mathcal{A}}(m_{k,\tilde{t}})$).

So, we obtained the map $\Theta_{G_{\mathcal{A}},1} = \Theta_1 : m_{k,\tilde{t}}^n \longrightarrow G_{\mathcal{A}}(m_{k,\tilde{t}})$.

**Lemma.** *$\Theta_1$ is group isomorphism.*

*Proof.*

$$(\Theta_1 \circ \bar{l}_{\mathcal{A}})(\sum_{\bar{i} \in \mathbf{N}^n} w_{\bar{i}} \tilde{t}^{\bar{i}}) = \bar{l}_{\mathcal{A}}( \sum_{\substack{(\text{in } G_{\mathcal{A}}) \\ s,\bar{i}}} (p^s \operatorname{id}_{G_{\mathcal{A}}})([\alpha_{\bar{i},s}] \tilde{t}^{\bar{i}})) =$$

$$= \sum_{\substack{s \geqslant 0 \\ \bar{i} \in \mathbf{N}^n}} p^s \bar{l}_{\mathcal{A}}([\alpha_{\bar{i},s}] \tilde{t}^{\bar{i}}) = \sum_{\substack{s \geqslant 0 \\ \bar{i} \in \mathbf{N}^n}} p^s \left( \operatorname{id} - \frac{\mathcal{A}^*}{p} \right)^{-1} ([\alpha_{\bar{i},s}] \tilde{t}^{\bar{i}}) =$$

$$= \left(\mathrm{id} - \frac{\mathcal{A}^*}{p}\right)^{-1} \left(\sum_{\bar{\imath}\in\mathbf{N}^n} w_{\bar{\imath}} \tilde{t}^{\bar{\imath}}\right).$$

So, $\Theta_1 \circ \tilde{l}_{\mathcal{A}}$ is $\mathbb{Z}_p$-linear map $m^n_{k,\tilde{t}} \longrightarrow A^n_{\mathrm{cris}}$ and, therefore, $\Theta_1$ is group homomorphism. This formula shows also, that $\Theta_1$ is isomorphism and the correspondence

$$g \mapsto \tilde{l}_{\mathcal{A}}(g) - \frac{\mathcal{A}^*}{p} \tilde{l}_{\mathcal{A}}(g)$$

gives inverse homomorphism $\Theta_1^{-1} : G_{\mathcal{A}}(m_{k,\tilde{t}}) \longrightarrow m^n_{k,\tilde{t}}$.

*Remark.*

In the above proof we obtained the identity

$$\tilde{l}_{\mathcal{A}}(\Theta_1(\alpha)) = (\mathrm{id} + \frac{\mathcal{A}^*}{p} + \dots + \frac{\mathcal{A}^{*s}}{p^s} + \dots)(\alpha)$$

for any $\alpha \in m^n_{k,\tilde{t}}$.

2.5.3. Define the homomorphism $\Theta_{G_{\mathcal{A}}} = \Theta : m^n_{k,\tilde{t}} \longrightarrow G_{\mathcal{A}}(m_K)$ as a composition of $\Theta_1$ and of $\gamma : G_{\mathcal{A}}(m_{k,\tilde{t}}) \longrightarrow G_{\mathcal{A}}(m_K)$. Clearly, $\Theta$ is surjection.

A relation between vector power series $f(\tilde{t}) \in m^n_{k,\tilde{t}}$ and $n$-vector $\beta = \Theta(f(\tilde{t})) \in m^n_K$ can be explained in a following way.

Take any presentation of $\beta$ in a form $\beta = \sum_{\bar{\imath}\in\mathbf{N}^n} w_{\bar{\imath}} \pi^{\bar{\imath}}$, define vector power series $\beta(\tilde{t}) = \sum_{\bar{\imath}\in\mathbf{N}^n} w_{\bar{\imath}} \tilde{t}^{\bar{\imath}}$, then for

$$f(\tilde{t}) = \tilde{l}_{\mathcal{A}}(\beta(\tilde{t})) - \frac{\mathcal{A}^*}{p} \tilde{l}_{\mathcal{A}}(\beta(\tilde{t})),$$

one has $f(\tilde{t}) \in m^n_{k,\tilde{t}}$ and $\Theta(f(\tilde{t})) = \beta$.

**2.6. Lemma.** *Let $\alpha \in m^n_{k,\tilde{t}} \subset W(m_R)$. If $\tau \in \Gamma_K$, then $\tau\alpha - \alpha \in (t^+ A^{loc}_{\mathrm{cris}})^n$.*

*Proof.*

It is sufficient to check that $\tau\tilde{t} - \tilde{t} \in t^+ A^{loc}_{\mathrm{cris}}$.

One can take a generator of the additive Tate submodule in $A_{\mathrm{cris}}$ in a form $t^+ = \log[\varepsilon]$, where $\varepsilon = (\varepsilon^{(s)})_{s\geq 0} \in R$ is such that $\varepsilon^{(0)} = 1, \varepsilon^{(1)} \neq 1$. Remark, $t^+ = ([\varepsilon] - 1)\eta$, where $\eta$ invertible in $A_{\mathrm{cris}}$.

There exists $a = a_\tau \in \mathbb{Z}_p$, such that $\tau\tilde{t} = \tilde{t}[\varepsilon]^a$. This gives $\tau\tilde{t} - \tilde{t} = \tilde{t}([\varepsilon]^a - 1) = t^+ w$, where $w = \tilde{t}([\varepsilon]^a - 1)([\varepsilon] - 1)^{-1} \in W(m_R) \subset A^{loc}_{\mathrm{cris}}$.

*Remark.* Because, $t^+ A^{loc}_{\mathrm{cris}} \cap W(R) = \psi W(m_R)$, the above proposition gives $\tau\alpha - \alpha \in \psi W(m_R)^n$.

2.7. Let $f \in G_{\mathcal{A}}(m_K)$ and $\tau \in \Gamma_K$. One can consider the formal group symbol as a pairing

$$G_{\mathcal{A}}(m_K) \times \Gamma_K \longrightarrow G_{\mathcal{A}}[M](m_K).$$

Namely, $(f, \tau]_{G_{\mathcal{A}}} = \tau f_1 -_{G_{\mathcal{A}}} f_1$, where $f_1 \in G_{\mathcal{A}}(m_C)$ is such that $(p^M \mathrm{id}_{G_{\mathcal{A}}})(f_1) = f$.

**Proposition.** *If* $\alpha \in m^n_{k,\tilde{t}}$ *and* $\tau \in \Gamma_K$, *then*

$$(\alpha, \tau]_{\rm cris} = (-\,{\rm id}_{G_{\mathcal{A}}})(\Theta_{G_{\mathcal{A}}}(\alpha), \tau]_{G_{\mathcal{A}}}.$$

*Remark.* According to n.2.5.3, this statement can be reformulated in a following way. If $f \in G_{\mathcal{A}}(m_K)$ and $f(\tilde{t}) \in m^n_{k,\tilde{t}}$ is a vector power series, such that $f(\pi) = f$, then for any $\tau \in \Gamma_K$

$$(f, \tau]_{G_{\mathcal{A}}} = (-\,{\rm id}_{G_{\mathcal{A}}})\left(\bar{l}_{\mathcal{A}}(f(\tilde{t})) - \frac{\mathcal{A}^*}{p}\bar{l}_{\mathcal{A}}(f(\tilde{t})), \tau\right]_{\rm cris}.$$

2.8. *Proof of proposition* 2.7.

2.8.1. The exact sequence of n.2.1 gives a solution $Y \in (\mathrm{Fil}^1 A_{\rm cris})^n$ of the equation

$$Y - \frac{\mathcal{A}^*}{p}Y = \alpha$$

in a form $Y = \bar{l}_{\mathcal{A}}(g)$ for some $g \in G_{\mathcal{A}}(W^1(m_R))$.

By the definition of the crystalline symbol

$$\tau Y = Y + X + l_{\alpha,\tau},$$

where $l_{\alpha,\tau} \in (\mathrm{Fil}^1 A_{\rm cris})^n$, $l_{\alpha,\tau} = (\mathcal{A}^*/p)l_{\alpha,\tau}$, $l_{\alpha,\tau} \bmod p^M A_{\rm cris} = (\alpha, \tau]_{\rm cris}$ (under identification $U_M(\mathcal{M}(G_{\mathcal{A}})) = G_{\mathcal{A}}[M](m_K)$ from n.2.3.1) and $X \in (t^+ A^{loc}_{\rm cris})^n$ is such that $X - (\mathcal{A}^*/p)X = \tau\alpha - \alpha$.

We can use nilpotency of $(\mathcal{A}^*/p)|_{(t^+ A^{loc}_{\rm cris})^n}$ and the identity from remark of n.2.5.2 to express $X$ as follows

$$X = \left({\rm id} - \frac{\mathcal{A}^*}{p}\right)^{-1}(\tau\alpha - \alpha) = \tau\bar{l}_{\mathcal{A}}(\Theta_1(\alpha)) - \bar{l}_{\mathcal{A}}(\Theta_1(\alpha))$$

(here $\Theta_1 = \Theta_{G_{\mathcal{A}},1}$ is the isomorphism from n.2.5.2).

2.8.2. By lemma of n.1.6.3 take $h, y \in G_{\mathcal{A}}(W(m_R))$, such that $\mathcal{A}^{*M}\bar{l}_{\mathcal{A}}(h) = \bar{l}_{\mathcal{A}}(g)$ and $\mathcal{A}^{*M}\bar{l}_{\mathcal{A}}(y) = \bar{l}_{\mathcal{A}}(\Theta_1(\alpha))$.

Now the relation $\mathcal{A}^*Y = pY - p\alpha$ gives

$$\mathcal{A}^{*M}Y = p\mathcal{A}^{*M-1}Y - p\mathcal{A}^{*M-1}\alpha = \dots =$$

$$= p^M Y - (p^M\,{\rm id} + p^{M-1}\mathcal{A}^* + \dots + p\mathcal{A}^{*M-1})\alpha =$$

$$= \mathcal{A}^{*M}(p^M \bar{l}_{\mathcal{A}}(h) - (p^M\mathcal{A}^{*-M} + \dots + p\mathcal{A}^{*-1})\alpha).$$

Remark, that we can cancell this relation by $\mathcal{A}^{*M}$ because of the uniquiness property of lemma 1.6.3.

Applying the identity from remark of n.2.5.2, we obtain

$$(p^M\mathcal{A}^{*-M} + \dots + p\mathcal{A}^{*-1})\alpha = (p^M\mathcal{A}^{*-M} - {\rm id})({\rm id} + \frac{\mathcal{A}^*}{p} + \dots + \frac{\mathcal{A}^{*s}}{p^s} + \dots)\alpha =$$

$$= p^M \bar{l}_{\mathcal{A}}(y) - \bar{l}_{\mathcal{A}}(\Theta_1(\alpha)).$$

Therefore, for $h' = h -_{G_{\mathcal{A}}} y \in G_{\mathcal{A}}(W(m_R))$, we have the relation

$$(*_1) \qquad \bar{l}_{\mathcal{A}}(g) = p^M \bar{l}_{\mathcal{A}}(h') + \bar{l}_{\mathcal{A}}(\Theta_1(\alpha)).$$

Using this formula and the expression for $X$ from n.2.8.1, we obtain

$$(*_2) \qquad \bar{l}_{\alpha,\tau} = \bar{l}_{\mathcal{A}}(\tau g) - \bar{l}_{\mathcal{A}}(g) - X = p^M \bar{l}_{\mathcal{A}}(\tau h' -_{G_{\mathcal{A}}} h').$$

2.8.3. Apply morphism $\gamma : G_{\mathcal{A}}(m_{k,\bar{l}}) \longrightarrow G_{\mathcal{A}}(m_K)$ to both sides of the relation $(*_1)$ and use that $g \in G_{\mathcal{A}}(W^1(m_R))$. We obtain

$$(p^M \,\mathrm{id}_{G_{\mathcal{A}}})(\tilde{h}) = (-\,\mathrm{id}_{G_{\mathcal{A}}})\Theta_{G_{\mathcal{A}}}(\alpha),$$

where $\tilde{h} = \gamma(h') \in G(m_C)$.

Let $o = (o_s)_{s \geq 0} \in T(G_{\mathcal{A}})$ be such that $\bar{l}_{\alpha,\tau} = \bar{l}_{\mathcal{A}}(j(o))$, c.f. n.2.1. From 1.5.3, it follows that

$$\bar{l}_{\mathcal{A}}(j(o)) \equiv p^M \bar{l}_{\mathcal{A}}(\hat{o}_M) \,\mathrm{mod}\, p^M \bar{l}_{\mathcal{A}}(W^1(m_R)) + p^{M+1} W(m_R),$$

where $\hat{o}_M \in W(m_R)$ is such that $\gamma(\hat{o}_M) \equiv o_M \,\mathrm{mod}\, p m_C$.

Therefore, the relation $(*_2)$ gives

$$\bar{l}_{\mathcal{A}}(\hat{o}_M) \equiv \bar{l}_{\mathcal{A}}(\tau h' -_{G_{\mathcal{A}}} h') \,\mathrm{mod}\, \bar{l}_{\mathcal{A}}(W^1(m_R)) + pW(m_R).$$

So, $\hat{o}_M = \tau h' -_{G_{\mathcal{A}}} h' + \delta$, where $\delta \in G_{\mathcal{A}}(W^1(m_R) + pW(m_R))$. Applying $\gamma$, we obtain

$$o_M \equiv \tau\tilde{h} -_{G_{\mathcal{A}}} \tilde{h} \,\mathrm{mod}\, p m_C.$$

But $o_M, \tau\tilde{h} -_{G_{\mathcal{A}}} \tilde{h} \in G[M](m_C)$, $G[M](m_C) \cap p m_C^n = 0$, and, therefore, $o_M = \tau\tilde{h} -_{G_{\mathcal{A}}} \tilde{h}$.

But $o_M = (\alpha, \tau]_{\mathrm{cris}}$, and $(\Theta_{G_{\mathcal{A}}}(\alpha), \tau]_{G_{\mathcal{A}}} = (-\,\mathrm{id}_{G_{\mathcal{A}}})(\tau(\tilde{h}) -_{G_{\mathcal{A}}} \tilde{h})$.

Proposition is proved.

## 3. Relation between crystalline and Witt symbols.

3.1. Let $R_0$ be the fraction field of the ring $R$. Denote by $\mathcal{G}$ the set of formal sums

$$\left\{ \sum_{n \in \mathbb{Z}} p^n [r_n] \mid r_n \in R_0, r_n \underset{n \to -\infty}{\longrightarrow} 0 \right\}.$$

Clearly, $\mathcal{G} \supset W(R_0) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \{\sum_{n > -\infty} p^n [r_n] \mid r_n \in R_0\}$ and can be identified with completion of $W(R_0) \otimes \mathbb{Q}_p$ in $v_R$-adic topology. So, $\mathcal{G}$ has a natural structure of a $W(R)$-module, continuos action of the Galois group $\Gamma_0 = \mathrm{Gal}(\bar{K}/K_0)$ and absolute Frobenius $\sigma_{\mathcal{G}} : \sum p^n [r_n] \mapsto \sum p^n [r_n^p]$ is $\mathbb{Z}_p$-linear $\Gamma_0$-morphism. Clearly, $\mathcal{G}|_{\sigma_{\mathcal{G}} = \mathrm{id}} = W(\mathbb{F}_p) \otimes \mathbb{Q}_p = \mathbb{Q}_p$.

Let

$$\mathcal{G}(m_R) = \left\{ \sum_{n\in\mathbf{Z}} p^n[r_n] \mid r_n \in m_R, r_n \underset{n\to-\infty}{\longrightarrow} 0 \right\}.$$

Then $\mathcal{G}(m_R)$ is $\Gamma_0$-invariant $W(R)$-submodule of $\mathcal{G}$, $\sigma_{\mathcal{G}}|_{\mathcal{G}(m_R)}$ is topologically nilpotent morphism. Clearly, $\mathcal{G}(m_R)|_{\sigma_{\mathcal{G}}=\mathrm{id}} = 0$.

Let $a, b \in \mathbb{Q}$, $a > b \geq 0$. Denote by $\mathcal{G}^\circ_{a,b}$ the subset of $\mathcal{G}$, which consists of $\sum p^n[r_n]$, such that $v_R(r_n) \geq -bn$ for $n \geq 0$, and $v_R(r_n) \geq -an$ for $n \leq 0$. It is easy to see, that the $W(R)$-algebra structure on $W_{\mathbb{Q}_p}(R_0)$ induces the $W(R)$-algebra structure on $\mathcal{G}^\circ_{a,b}$. This structure is compatible with $\Gamma_0$-action and $\sigma_{\mathcal{G}}$ induces semilinear isomorphism of $\Gamma_0$-modules $\sigma_{\mathcal{G}} : \mathcal{G}^\circ_{a,b} \longrightarrow \mathcal{G}^\circ_{pa,pb}$.

If $a, b \in \mathbb{Q}, a \geq b \geq 0$, let $\mathcal{G}^\circ_{a^+,b}$ be a $p$-adic closure of $\bigcup_{c>a} \mathcal{G}^\circ_{c,b}$. Then $\mathcal{G}^\circ_{a^+,b}$ is $W(R)$-algebra with continuos $\Gamma_0$-action. We set $\mathcal{G}_{a^+,b} = \mathcal{G}^\circ_{a^+,b} \otimes \mathbb{Q}_p \subset \mathcal{G}$.

Remark, $A_{\mathrm{cris}}$ can be naturally identified with $\Gamma_0$-submodule in $\mathcal{G}$. Clearly, if $a \in A_{\mathrm{cris}}$, then $\sigma a = \sigma_{\mathcal{G}} a$.

**3.1.1. Lemma.**

a) $A_{\mathrm{cris}} \subset \mathcal{G}^\circ_{(p-1)^+,0}$;

b) *every element of* $B^+_{\mathrm{cris}} = A_{\mathrm{cris}} \otimes \mathbb{Q}_p \subset \mathcal{G}_{(p-1)^+,0}$ *is invertible in* $\mathcal{G}_{(p-1)^+,p-1}$;

c) *if* $t^+ \in A_{\mathrm{cris}}$ *is a generator of the additive Tate module* $\mathbb{Z}_p(1) \subset A_{\mathrm{cris}}$ *(c.f. n.1.7.1) and* $x_0 \in R$ *is such that* $x_0^{(0)} = p$, *then* $(t^+)^{-1} \in [x_0]^{-p/(p-1)}\mathcal{G}^0_{p,1} \subset \mathcal{G}_{(p-1)^+,1}$ *(and, therefore,* $B_{\mathrm{cris}} = B^+_{\mathrm{cris}}[1/t^+] \subset \mathcal{G}_{(p-1)^+,1}$*).*

*Proof.*

a) $A_{\mathrm{cris}}$ is a $p$-adic closure of the ring $W(R)[\{[x_0]^n/n! \mid n \geq 1\}]$, where $x_0 \in R$ is such that $x_0^{(0)} = p$. So, it is sufficient to prove, that $[x_0]^n/n! \in \mathcal{G}_{(p-1)^+,0}$. But this follows from the inequality

$$\frac{v_R(x_0^n)}{n!} = \frac{n}{[n/p] + \dots + [n/p^s] + \dots} > p - 1.$$

b) This follows from the fact, that in $p$-adic topology any element of $B^+_{\mathrm{cris}}$ is limit of finite sums $\sum p^n[r_n] \in \mathcal{G}$.

c) $\sigma t^+ = pt^+$ implies $t^+ = \sum_{n\in\mathbf{Z}} p^n[r^{p^{-n}}]$ for some $r \in m_R$. From the definition of $t^+$ it follows, that $v_R(r) = p/(p-1)$. Therefore,

$$[r^{-1}]t^+ = 1 + \sum_{n\in\mathbf{Z}\setminus\{0\}} p^n[r_n],$$

where $v_R(r_n) \geq -n$ for $n > 0$, and $v_R(r_n) \geq -pn$ for $n < 0$. Now remark, that

$$(1 + \sum_{n\in\mathbf{Z}\setminus\{0\}} p^n[r_n])^{-1} = 1 + \sum_{s\geq 1}(-1)^s(\sum_{n\in\mathbf{Z}\setminus\{0\}} p^n[r_n])^s \in \mathcal{G}^0_{p,1}.$$

This gives $(t^+)^{-1} \in [x_0]^{-p/(p-1)}\mathcal{G}^0_{p,1}$.

*Remark.* One must be carefull about compatibility of Frobenius morphisms $\sigma$ on $B_{\text{cris}}$ and $\sigma_{\mathcal{G}}$ on $\mathcal{G}$ with respect to defined in n.b) inclusion of the field of fractions $\operatorname{Frac} B^+_{\text{cris}}$ into $\mathcal{G}_{(p-1)^+,p-1}$. For example, $\sigma(1/t^+) = 1/(pt^+)$, but $\sigma_{\mathcal{G}}(1/t^+) \notin \mathcal{G}_{(p-1)^+,1}$. Compatibility of $\sigma$ and $\sigma_{\mathcal{G}}$ can be formulated as follows:

*if $b \in B_{\text{cris}}$ is such that $\sigma_{\mathcal{G}}(b) \in \mathcal{G}_{(p-1)^+,p-1}$, then $\sigma b = \sigma_{\mathcal{G}}(b)$.*

3.1.2. The above lemma gives existence of a natural inclusion of the field of fractions $\operatorname{Frac} A_{\text{cris}}$ into $\mathcal{G}_{(p-1)^+,p-1}$. Let $\psi$ be the element of $W(R)$ defined in n.1.7.2, and $x_0$ be the element of $R$ from n. c) of lemma 3.1.1.

**Lemma.**

$$\frac{1}{\psi} W(R)[[\psi^{p-1}/p]] \subset [x_0]^{-p/(p-1)} \mathcal{G}^0_{p,1}.$$

*Proof.* If $t$ is a generator of $W^1(R)$ from n.1.7.2, one can easily check, that $W(R)[[t^p/p]] = W(R)[[\,[x_0]^p/p\,]]$. Therefore, $W(R)[[\psi^{p-1}/p]] = W(R)[[\,[x_0]^p/p\,]] = \mathcal{G}^0_{p,0}$ by lemma of n.1.7.3. So, $t^+$ and $\psi$ are associated elements of the ring $\mathcal{G}^0_{p,0}$ and, therefore,

$$\frac{1}{\psi} W(R)[[\psi^{p-1}/p]] \subset [x_0]^{-p/(p-1)} \mathcal{G}^0_{p,1} \mathcal{G}^0_{p,0} = [x_0]^{-p/(p-1)} \mathcal{G}^0_{p,1}.$$

3.2. Let $\alpha = {}^t(\alpha_1, \dots, \alpha_n) \in m^n_{k,\bar{l}}$. Let $\begin{pmatrix} Y \\ Z \end{pmatrix} \in A^h_{\text{cris}}$ be such that $Y \in (\operatorname{Fil}^1 A_{\text{cris}})^n$ and

$$\begin{pmatrix} Y \\ Z \end{pmatrix} = \mathcal{E}^{-1} \begin{pmatrix} \phi_1(Y) \\ \phi_0(Z) \end{pmatrix} + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

(we use all notation from the definition of the crystalline symbol, c.f. n.2.3). Remark (c.f. n.1.7.6)

$$Y \in \left( W^1(m_R) + \frac{\psi^{p-1}}{p} W(R)[[\psi^{p-1}/p]] \right)^n,$$

$$Z \in \left( W(m_R) + \frac{\psi^{p-1}}{p} W(R)[[\psi^{p-1}/p]] \right)^{h-n}.$$

Choose some $\mathbb{Z}_p$-basis of $T(G_{\mathcal{A}})$. Let

$$\mathcal{V} = \begin{pmatrix} \langle o^1, \bar{l} \rangle & \dots & \langle o^h, \bar{l} \rangle \\ \langle o^1, \bar{m} \rangle & \dots & \langle o^h, \bar{m} \rangle \end{pmatrix}$$

be the matrix of values of the $p$-adic periods pairing with respect to the special $W(k_0)$-basis $\{l_1, \dots, l_n, m_1, \dots, m_{h-n}\}$ of $M^0(G_{\mathcal{A}})$ and the above chosen $\mathbb{Z}_p$-basis of $T(G_{\mathcal{A}})$. All elements of the matrix $\mathcal{V}$ belong to the subring $W(R)[[\psi^{p-1}/p]] \subset A_{\text{cris}}$, c.f. n.1.7.6.

By n.1.8 the matrix $\mathcal{V}$ is nondegenerate in the field of fractions $\operatorname{Frac} A_{\text{cris}}$ of $A_{\text{cris}}$. Therefore, there exist unique vector-columns $X, T \in (\operatorname{Frac} A_{\text{cris}})^h$, such that

$$\mathcal{V}X = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \quad \mathcal{V}T = \begin{pmatrix} Y \\ Z \end{pmatrix}$$

**3.3. Lemma.**

a) $X, T \in \left(\frac{1}{\psi}W(R)[[\psi^{p-1}/p]]\right)^h \subset \left([x_0]^{-p/(p-1)}\mathcal{G}^0_{p,1}\right)^h$;

b) $\sigma_{\mathcal{G}}T \in \mathcal{G}_{(p-1)+,1}$ *and, therefore,* $\sigma_{\mathcal{G}}T = \sigma T$;

c) $T - \sigma T = X$;

d) *If* $T_1 \in \left([x_0]^{-p/(p-1)}\mathcal{G}^0_{p,1}\right)^h$ *is such that* $T_1 - \sigma_{\mathcal{G}}T_1 = X$, *then* $\mathcal{V}T_1 = \begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix} \in A^h_{\mathrm{cris}}, Y_1 \in (\mathrm{Fil}^1 A_{\mathrm{cris}})^n$ *and*

$$\begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix} = \mathcal{E}^{-1}\begin{pmatrix} \phi_1(Y_1) \\ \phi_0(Z_1) \end{pmatrix} + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

*(in other words,* $\begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix}$ *can be taken instead of* $\begin{pmatrix} Y \\ Z \end{pmatrix}$ *in computation of* $(\alpha, \tau]_{\mathrm{cris}}$*).*

*Proof.*

a) This follows because ${}^t\mathcal{V}^D\mathcal{V} = t^+E_h$, $t^+$ and $\psi$ are associated elements of the ring $W(R)[[\psi^{p-1}/p]]$, and $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \begin{pmatrix} Y \\ Z \end{pmatrix} \in \left(W(R)[[\psi^{p-1}/p]]\right)^h$.

b) One can use nilpotency of $\phi_1$ on $\psi^{p-1}/pW(R)[[\psi^{p-1}/p]]$ and of $\bar{\phi}_1 = \sigma/\sigma t$ on $\psi^{p-1}W(R)$ (compare with n.1.7.5) to prove existence of the unique $\begin{pmatrix} \hat{Y} \\ \hat{Z} \end{pmatrix} \in W(m_R)^h$, such that $\hat{Y} \in W^1(m_R)^{h-n}$,

$$\begin{pmatrix} Y \\ Z \end{pmatrix} \operatorname{mod}(\psi^{p-1}/p)W(R)[[\psi^{p-1}/p]] = \begin{pmatrix} \hat{Y} \\ \hat{Z} \end{pmatrix} \operatorname{mod} \psi^{p-1}W(R)$$

and $\begin{pmatrix} \hat{Y} \\ \hat{Z} \end{pmatrix} = \mathcal{E}^{-1}\begin{pmatrix} \bar{\phi}_1(\hat{Y}) \\ \sigma(\hat{Z}) \end{pmatrix} + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$.

If $\hat{\mathcal{V}}$ is the matrix from n.1.8.4, there exist the unique $\hat{X}, \hat{T} \in (1/\psi)W(R)^h \subset \mathcal{G}^h_{(p-1)+,1}$, such that

$$(*) \qquad \hat{\mathcal{V}}\hat{X} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}, \quad \hat{\mathcal{V}}\hat{T} = \begin{pmatrix} \hat{Y} \\ \hat{Z} \end{pmatrix}.$$

It is easy to see, that $T \operatorname{mod}(\psi^{p-2}/p)W(R)[[\psi^{p-1}/p]] = \hat{T} \operatorname{mod} \psi^{p-2}W(R)$.

Consider the above equalities $(*)$ in the ring $W(R_0)$ (where $R_0$ is the field of fractions of the ring $R$). Then $\sigma_{W(R_0)} = \sigma_{\mathcal{G}}|_{W(R_0)}$ and we have

$$\begin{pmatrix} \sigma_{\mathcal{G}}/t & \\ & \sigma_{\mathcal{G}} \end{pmatrix}(\hat{\mathcal{V}}\hat{T}) = \begin{pmatrix} \bar{\phi}_1 & \\ & \sigma_{\mathcal{G}} \end{pmatrix}(\hat{\mathcal{V}})\sigma_{\mathcal{G}}\hat{T} = \mathcal{E}\hat{\mathcal{V}}\sigma_{\mathcal{G}}\hat{T}$$

$$\begin{pmatrix} \bar{\phi}_1 & \\ & \sigma_{\mathcal{G}} \end{pmatrix}\begin{pmatrix} \hat{Y} \\ \hat{Z} \end{pmatrix} = \mathcal{E}\begin{pmatrix} \hat{Y} \\ \hat{Z} \end{pmatrix} - \mathcal{E}\begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \mathcal{E}\hat{\mathcal{V}}\hat{T} - \mathcal{E}\hat{\mathcal{V}}\hat{X}.$$

Therefore, $\sigma_{\mathcal{G}}\hat{T} = \hat{T} - \hat{X} \in (1/\psi)W(R)^h \subset [x_0]^{-p/(p-1)}(\mathcal{G}^0_{p,1})^h$. If $T = \hat{T} + T_1$, where $T_1 \in (\psi^{p-2}/p)W(R)[[\psi^{p-1}/p]]$, then

$$T_1 \in \frac{1}{p}(\mathcal{G}^0_{p,0})^h \subset (\mathcal{G}_{p,0})^h$$

and $\sigma_{\mathcal{G}} T_1 \in (\mathcal{G}_{p,0})^h \subset (\mathcal{G}_{(p-1)^+,1})^h$. So, $\sigma_{\mathcal{G}} T = \sigma_{\mathcal{G}} \hat{T} + \sigma_{\mathcal{G}} T_1 \in \mathcal{G}^h_{(p-1)^+,1}$ and $\sigma_{\mathcal{G}} T = \sigma T$.

c) Because of the above n.b) we can apply operator $\begin{pmatrix} \phi_1 \\ \phi_0 \end{pmatrix}$ to both sides of the equality $\mathcal{V}T = \begin{pmatrix} Y \\ Z \end{pmatrix}$. We obtain

$$\begin{pmatrix} \langle o^1, \phi_1(\bar{l})\rangle & \dots & \langle o^h, \phi_1(\bar{l})\rangle \\ \langle o^1, \phi_0(\bar{m})\rangle & \dots & \langle o^h, \phi_0(\bar{m})\rangle \end{pmatrix} \sigma T = \begin{pmatrix} \phi_1(Y) \\ \phi_0(Z) \end{pmatrix}$$

This equality can be rewritten as

$$\mathcal{E}\mathcal{V}\sigma T = \mathcal{E}\begin{pmatrix} Y \\ Z \end{pmatrix} - \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \mathcal{E}\mathcal{V}T - \mathcal{E}\mathcal{V}X.$$

So, $\sigma T = T - X$.

d) If $t = T_1 - T$, then $t \in \mathcal{G}^h_{(p-1)^+,1}$ and $\sigma_{\mathcal{G}} t = \sigma_{\mathcal{G}} T_1 - \sigma_{\mathcal{G}} T = T_1 - T = t$. This gives

$$t \in \left(\mathbb{Q}_p \cap [x_0]^{-p/(p-1)}\mathcal{G}^0_{p,1}\right)^h = \mathbb{Z}_p^h.$$

If $t = {}^t(a_1, \dots, a_h)$, then

$$\begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix} = \mathcal{V}T_1 = \mathcal{V}T + \mathcal{V}t = \begin{pmatrix} Y \\ Z \end{pmatrix} + \sum_{1\le i\le h} a_i \begin{pmatrix} \langle o^i, \bar{l}\rangle \\ \langle o^i, \bar{m}\rangle \end{pmatrix}.$$

So, $Y_1 = Y + \sum\limits_{1\le i\le h} a_i\langle o^i, \bar{l}\rangle \in (\mathrm{Fil}^1 A_{\mathrm{cris}})^n$ and

$$\begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix} - \mathcal{E}^{-1}\begin{pmatrix} Y_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix} - \mathcal{E}^{-1}\begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}.$$

*Remark.* The correspondence $T \mapsto \mathcal{V}T$ gives one-to-one correspondence between the sets

$$\left\{T \in \left([x_0]^{-p/(p-1)}\mathcal{G}^0_{p,1}\right)^h \mid T - \sigma_{\mathcal{G}} T = X\right\}$$

and

$$\left\{\begin{pmatrix} Y \\ Z \end{pmatrix} \in A^h_{\mathrm{cris}} \mid Y \in (\mathrm{Fil}^1 A_{\mathrm{cris}})^n, \begin{pmatrix} Y \\ Z \end{pmatrix} = \mathcal{E}^{-1}\begin{pmatrix} \phi_1(Y) \\ \phi_0(Z) \end{pmatrix} + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}\right\}$$

**3.4. Lemma.** *For any $\tau \in \Gamma_K$ coordinates of the vector $\tau X - X$ belong to $W(m_R)[[\psi^{p-1}/p]] + (p^M/\psi)W(m_R)[[\psi^{p-1}/p]]$.*

*Proof.*

From $\mathcal{V}X = \begin{pmatrix}\alpha\\0\end{pmatrix}$ it follows, that $t^+X = {}^t\mathcal{V}^D\begin{pmatrix}\alpha\\0\end{pmatrix}$. Coefficients of the matrix ${}^t\mathcal{V}^D$ are elements of the ring $W(R)[[\psi^{p-1}/p]]$, $\begin{pmatrix}\alpha\\0\end{pmatrix} \in W(m_R)^h$, therefore,

$$X \in \left(\frac{1}{\psi}W(m_R)[[\psi^{p-1}/p]]\right)^h.$$

Rewrite the relation $\tau(\mathcal{V}X) - \mathcal{V}X = \begin{pmatrix}\tau\alpha - \alpha\\0\end{pmatrix}$ as

$$(*) \qquad {}^t\mathcal{V}^D(\tau\mathcal{V} - \mathcal{V})\tau X + t^+(\tau X - X) = {}^t\mathcal{V}^D\begin{pmatrix}\tau\alpha - \alpha\\0\end{pmatrix}.$$

a) All coefficients of the matrix ${}^t\mathcal{V}^D(\tau\mathcal{V} - \mathcal{V})$ belong to $p^M\mathbb{Z}_pt^+ \subset A_{\text{cris}}$. Indeed, for $1 \le i \le h$ one has

$$\tau o^i - o^i = \sum_{1\le i\le h} p^M a_{ij}o^j,$$

where all $a_{ij} \in \mathbb{Z}_p$. Therefore,

$$\tau\mathcal{V} - \mathcal{V} = \begin{pmatrix}\langle\tau o^1 - o^1, \bar{l}\rangle & \dots & \langle\tau o^h - o^h, \bar{l}\rangle\\ \langle\tau o^1 - o^1, \bar{m}\rangle & \dots & \langle\tau o^h - o^h, \bar{m}\rangle\end{pmatrix} = \mathcal{V}p^M((a_{ji}))_{1\le i,j\le h},$$

and

$${}^t\mathcal{V}^D(\tau\mathcal{V} - \mathcal{V}) = p^Mt^+((a_{ji}))_{1\le i,j\le h}.$$

b) $\tau\alpha - \alpha \in (\psi W(m_R))^h$, c.f. n.2.6, therefore,

$${}^t\mathcal{V}^D\begin{pmatrix}\tau\alpha - \alpha\\0\end{pmatrix} \in W(R)[[\psi^{p-1}/p]]\psi W(m_R) = \psi W(m_R)[[\psi^{p-1}/p]].$$

Clearly, lemma follows from the above relation $(*)$ and properties a) and b).

**Corollary.** *Let $T \in \left([x_0]^{-p/(p-1)}\mathcal{G}^0_{p,1}\right)^h$ be such that $T - \sigma_{\mathcal{G}}T = X$, then for any $\tau \in \Gamma_K$*

a) *$\tau T - T \equiv A \bmod \mathcal{G}(m_R) + p^MW(R_0)$, where $A = {}^t(A_1, \dots, A_h) \in \mathbb{Z}_p^h$;*

b) *If $\alpha \in m^n_{k,\tilde{t}}$, then*

$$(\alpha, \tau]_{\text{cris}} = A_1o^1_M + \dots + A_ho^h_M.$$

*Proof.* If $t = \tau T - T$, then $t \in [x_0]^{-p/(p-1)}(\mathcal{G}^0_{p,1})^h \subset (\mathcal{G}(m_R) + W(R_0))^h$ and all coordinates of the vector $t - \sigma_{\mathcal{G}}t = X - \tau X$ belong to $W(m_R)[[\psi^{p-1}/p]] + (p^M/\psi)W(m_R)[[\psi^{p-1}/p]] \subset \mathcal{G}(m_R) + p^MW(R_0)$.

Let $X - \tau X = X_1 + p^MX_2$, where $X_1 \in \mathcal{G}(m_R)^h$ and $X_2 \in W(R_0)^h$. If $t_1 = \sum_{s\ge0}\sigma^sX_1 \in \mathcal{G}(m_R)^h$, then $t_1 - \sigma_{\mathcal{G}}t_1 = X_1$. Take $t_2 \in W(R_0)^h$, such that $t_2 - \sigma_{\mathcal{G}}t_2 = X_2$. Then $A = t - (t_1 + p^Mt_2) \in (\mathcal{G}(m_R) + W(R_0))^h_{\sigma_{\mathcal{G}}=\text{id}} = \mathbb{Z}_p^h$.

Part a) of corollary is proved.

Let $\begin{pmatrix} Y \\ Z \end{pmatrix} = \mathcal{V}T$, then (c.f. n.3.3) $\begin{pmatrix} Y \\ Z \end{pmatrix} \in A_{\text{cris}}^h$, $Y \in (\text{Fil}^1 A_{\text{cris}})^n$ and

$$\begin{pmatrix} Y \\ Z \end{pmatrix} = \mathcal{E}^{-1} \begin{pmatrix} \phi_1(Y) \\ \phi_0(Z) \end{pmatrix} + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}.$$

By definition of the crystalline symbol, there exist $a_1, \ldots, a_h \in \mathbb{Z}_p$ and $b \in (A_{\text{cris}}^{loc})^h$, such that

$$\tau \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix} + \sum_{1 \leq i \leq h} \begin{pmatrix} \langle o^i, \tilde{l} \rangle \\ \langle o^i, \tilde{m} \rangle \end{pmatrix} a_i + t^+ b.$$

In this notation $(\alpha, \tau]_{\text{cris}} = a_1 o_M^1 + \cdots + a_h o_M^h$.
If $a = {}^t(a_1, \ldots, a_h) \in \mathbb{Z}_p^h$, then we can rewrite the above equation as

$$\tau \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix} + \mathcal{V}a + t^+ b.$$

Multiplying the both sides by the matrix ${}^t\mathcal{V}^D$ we obtain in notation from proof of the above lemma

$$p^M t^+ \tau T((a_{ji}))_{1 \leq i,j \leq h} + t^+(\tau T - T) = t^+ a + t^+({}^t\mathcal{V}^D b).$$

therefore,

$$\tau T - T = a + ({}^t\mathcal{V}^D b) + p^M c,$$

where $c \in \left([x_0]^{-p/(p-1)} \mathcal{G}_{p,1}^0\right)^h \subset (\mathcal{G}(m_R) + W(R_0))^h$. Now

$$b_1 = {}^t\mathcal{V}^D b \in \left([x_0]^{-p/(p-1)} \mathcal{G}_{p,1}^0\right)^h$$

and $\lim_{N \to \infty} \sigma^N b_1 = 0$ imply $b_1 \in \mathcal{G}(m_R)^h$.
Therefore, $a \equiv A \bmod p^M \mathbb{Z}_p$, q.e.d.

3.5. *Matrix* $\mathcal{V}_{\tilde{t}}$.

3.5.1. Recall, $K$ is a finite extension of the field $K_0$ in $\bar{K}$, such that $G_{\mathcal{A}}[M](m_C) = G_{\mathcal{A}}[M](m_K)$, where $m_K$ is the maximal ideal of the valuation ring $O_K$ of $K$.

In n.2.5.1 there was fixed uniformizer $\pi \in K$ and $t_0 \in m_R$, such that $t_0^{(0)} = \pi$. If $k$ is the residue field of $K$, then $O_{k,\tilde{t}} = W(k)[[\,\tilde{t}\,]] \subset W(R)$, where $\tilde{t} = [t_0]$. The structural morphism $\gamma : W(R) \longrightarrow O_C$ induces epimorphism of rings $\gamma : O_{k,\tilde{t}} \longrightarrow O_K$. If $m_{k,\tilde{t}} = \tilde{t}W(k)[[\,\tilde{t}\,]]$, then $\gamma(m_{k,\tilde{t}}) = m_K$. Clearly, $\text{Ker}\,\gamma|_{O_{k,\tilde{t}}} = g(\tilde{t})O_{k,\tilde{t}}$ and $\text{Ker}\,\gamma|_{m_{k,\tilde{t}}} = g(\tilde{t})m_{k,\tilde{t}}$, where $g \in W(k)[X]$ is irreducible polynom, such that $g(\pi) = 0$.

Denote by $O_{k,\tilde{t}}^{\text{DP}}$ $p$-adic closure of the divided power envelope of $O_{k,\tilde{t}}$ with respect to the ideal $(g(\tilde{t}))$. Clearly, $O_{k,\tilde{t}}^{\text{DP}} \subset A_{\text{cris}}$ and can be identified with $p$-adic closure of $O_{k,\tilde{t}}\left[\{\tilde{t}^{en}/n! \mid n \geq 1\}\right]$, where $e$ is absolute ramification index of the field $K$ (use, that $g(\tilde{t}) = \tilde{t}^e + a_1\tilde{t}^{e-1} + \cdots + a_e$, where $a_1, \ldots, a_e \in pW(k)$, $v_p(a_e) = 1$ and equality of ideals $(g(\tilde{t}), p) = (\tilde{t}^e, p)$ in the ring $O_{k,\tilde{t}}$.

3.5.2. Let $o^1 = (o^1_s)_{s\geqslant 0}, \ldots, o^h = (o^h_s)_{s\geqslant 0}$ be $\mathbb{Z}_p$-basis of $T(G_{\mathcal{A}})$. Then $o^1_M, \ldots, o^h_M$ give $\mathbb{Z}/p^M\mathbb{Z}$-basis of $G_{\mathcal{A}}[M](m_K)$. Let $\bar{l}_{\mathcal{A}}$ be logarithm vector power series from n.1.3. For $1 \leq i \leq h$ fix $\hat{o}^i_M \in m_{k,\tilde{t}}$, such that $\gamma(\hat{o}^i_M) = o^i_M$. Then

$$p^M \bar{l}_{\mathcal{A}}(\hat{o}^i_M) \in \mathrm{Fil}^1\, O^{\mathrm{DP}}_{k,\tilde{t}}.$$

Use matrices $F'_u$ from n.1.3 to set

$$\bar{m}_{\mathcal{A}}(\hat{o}^i_M) = \sum_{u\geq 1} F'_u(\sigma^u/p)\bar{l}_{\mathcal{A}}(\hat{o}^i_M).$$

Then

$$p^M \bar{m}_{\mathcal{A}}(\hat{o}^i_M) \in O^{\mathrm{DP}}_{k,\tilde{t}}.$$

In fact (compare with n.1.7.6),

$$p^M \bar{l}_{\mathcal{A}}(\hat{o}^i_M) \in m^1_{k,\tilde{t}} + \frac{\tilde{t}^{ep}}{p} O_{k,\tilde{t}}[[\tilde{t}^{ep}/p]],$$

$$p^M \bar{m}_{\mathcal{A}}(\hat{o}^i_M) \in m_{k,\tilde{t}} + \frac{\tilde{t}^{ep}}{p} O_{k,\tilde{t}}[[\tilde{t}^{ep}/p]].$$

From n.1.5 it follows

**Lemma.** *For $1 \leq i \leq h$ one has*

$$\langle o^i, \bar{l}\rangle \equiv p^M \bar{l}_{\mathcal{A}}(\hat{o}^i_M) \bmod p^M \left(W^1(m_R) + \frac{\psi^{p-1}}{p} W(R)[[\psi^{p-1}/p]]\right)$$

$$\langle o^i, \bar{m}\rangle \equiv p^M \bar{m}_{\mathcal{A}}(\hat{o}^i_M) \bmod p^M \left(W(m_R) + \frac{\psi^{p-1}}{p} W(R)[[\psi^{p-1}/p]]\right).$$

3.5.3. Consider matrix of order $h$

$$\mathcal{V}_{\tilde{t}} = \begin{pmatrix} p^M \bar{l}_{\mathcal{A}}(\hat{o}^1_M) & \ldots & p^M \bar{l}_{\mathcal{A}}(\hat{o}^h_M) \\ p^M \bar{m}_{\mathcal{A}}(\hat{o}^1_M) & \ldots & p^M \bar{m}_{\mathcal{A}}(\hat{o}^h_M) \end{pmatrix}.$$

This matrix has coefficients in $O^{\mathrm{DP}}_{k,\tilde{t}}$ (c.f. n.3.5.2) and can be considered as approximation of the matrix $\mathcal{V}$ of values of the $p$-adic periods pairing from n.1.8.3.

Lemma of the above n.3.5.2 gives the equivalence

$${}^t\mathcal{V}^D \circ \mathcal{V}_{\tilde{t}} \equiv t^+ E_h \bmod p^M \left(W^1(m_R) + \frac{\psi^{p-1}}{p} W(R)[[\psi^{p-1}/p]]\right).$$

Therefore, one can write

$${}^t\mathcal{V}^D \circ \mathcal{V}_{\tilde{t}} = t^+(E_h - p^M \Delta),$$

where $\Delta$ is some matrix of order $h$ with coefficients from

$$\frac{1}{\psi} W^1(m_R) + \frac{\psi^{p-2}}{p} W(R)[[\psi^{p-1}/p]].$$

3.5.4. Let $t$ be a generator of $W^1(R)$ from n.1.7.2.

**Lemma.**

$$\mathcal{R}=\frac{pt}{\psi}W(R)[[pt/\psi]]+\psi^{p-2}W(R)[[\psi^{p-1}/p]]$$

*is $W(R)$-subalgebra of $\mathcal{G}_{(p-1)^+,1}$.*

*Proof.*

As topological $W(R)$-module $\mathcal{R}$ is generated by elements $(pt/\psi)^{m+1}$, where $m\geq 0$, and $\psi^{p-2}(\psi^{p-1}/p)^l$, where $l\geq 0$. It is sufficient to prove for any $m,l\geq 0$, that

$$\left(\frac{pt}{\psi}\right)^{m+1}\psi^{p-2}\left(\frac{\psi^{p-1}}{p}\right)^{l}=p^{m+1-l}t^{m+1}\psi^{p-2+(p-1)l-(m+1)}\in\mathcal{R}.$$

If $m+1-l=s>0$, then this product is equal to

$$\left(\frac{pt}{\psi}\right)^{s}t^l\psi^{p-2+(p-1)l}\in\mathcal{R}.$$

If $m+1-l=-s\leq 0$, then this product can be rewritten as

$$\psi^{p-2}\left(\frac{\psi^{p-2}}{p}\right)^{s}t^{m+1}\psi^{(p-2)(m+1)}\in\mathcal{R}.$$

**Proposition.** *In notation of n.3.5.3 one has*

$$(E_h-p^M\Delta)^{-1}=E_h+p^M\Delta_1,$$

*where $\Delta_1$ is matrix of order $h$ with coefficients from*

$$\frac{1}{p}\mathcal{R}=\frac{t}{\psi}W(R)[[pt/\psi]]+\frac{\psi^{p-2}}{p}W(R)[[\psi^{p-1}/p]].$$

*Proof.* We have

$$\Delta_1=\sum_{s\geq 0}p^{Ms}\Delta^{s+1}=\frac{1}{p}\sum_{s\geq 0}p^{s(M-1)}(p\Delta)^{s+1}$$

has coefficients in $(1/p)\mathcal{R}$, because $p\Delta$ has coefficients in $\mathcal{R}$ and $\mathcal{R}$ is a ring.

**Corollary.**

*The matrix $\mathcal{V}_{\tilde{t}}$ is invertible in the field of fractions* $\operatorname{Frac}O_{k,\tilde{t}}^{\mathrm{DP}}\subset\mathcal{G}_{(p-1)^+,p-1}$.

**3.5.5. Proposition.**

$$\mathcal{V}_{\tilde{t}}^{-1}\equiv\mathcal{V}^{-1}\bmod p^M\left(\frac{1}{p\psi}\mathcal{R}\right).$$

*Proof.* We have

$$\mathcal{V}_{\tilde{t}}^{-1}=\frac{1}{t^+}{}^t\mathcal{V}^D(E_h+p^M\Delta_1)=\mathcal{V}^{-1}+p^M\left(\frac{1}{t^+}{}^t\mathcal{V}^D\right)\Delta_1.$$

Coefficients of the matrix $(1/t^+)^t\mathcal{V}^D$ belong to

$$\frac{1}{\psi}W(R)[[\psi^{p-1}/p]] = \frac{1}{\psi}W(R) + \frac{\psi^{p-2}}{p}W(R)[[\psi^{p-1}/p]].$$

By proposition of n.3.5.4 it is sufficient to prove, that

$$\frac{t}{\psi}W(R)[[pt/\psi]]\frac{\psi^{p-2}}{p}W(R)[[\psi^{p-1}/p]] \subset \frac{1}{p\psi}\mathcal{R}.$$

For $m, l \geq 0$ take the product of generators

$$\frac{t}{\psi}\left(\frac{pt}{\psi}\right)^m \frac{\psi^{p-2}}{p}\left(\frac{\psi^{p-1}}{p}\right)^l = p^{m-l-1}t^{1+m}\psi^{p-3+(p-1)l-m}.$$

If $m - l - 1 = s \geq 0$, then it can be presented as

$$\frac{t}{\psi}\left(\frac{pt}{\psi}\right)^s t^{l+1}\psi^{p-3+(p-2)l} \in \frac{t}{\psi}W(R)[[pt/\psi]].$$

If $m - l - 1 = -s \leq -1$, then $s + m = l + 1$ and the above product can be rewritten as

$$\frac{\psi^{p-3}}{p}\left(\frac{\psi^{p-1}}{p}\right)^{s-1} t^{1+m}\psi^{(p-2)m} \in \frac{\psi^{p-3}}{p}W(R)[[\psi^{p-1}/p]].$$

**Corollary.** *Coefficients of the matrix $\mathcal{V}_{\tilde{t}}^{-1}$ belong to*

$$\frac{1}{\psi}W(R)[[\psi^{p-1}/p]] + p^M\left(\frac{1}{p\psi}\mathcal{R}\right).$$

3.6. In $W(R_0)$-module $\mathcal{G}$ any expresion of a form $\sum_{s\in\mathbf{Z}} w_s p^s$, where $w_s \in W(R_0)$ and $w_s \underset{s\to-\infty}{\longrightarrow} 0$ in $v_R$-adic topology has sense, i.e. gives some element of $\mathcal{G}$.

Let

$$\mathcal{G}_{k,\tilde{t}} = \left\{ \sum_{s\in\mathbf{Z}, u\geq u_s} [\alpha_{s,u}]\tilde{t}^u p^s \mid u_s \underset{s\to-\infty}{\longrightarrow} +\infty, \text{ all } \alpha_{s,u} \in k \right\} \subset \mathcal{G}.$$

Then $O_{k,\tilde{t}} \subset \mathcal{G}_{k,\tilde{t}}$, $\mathcal{G}_{k,\tilde{t}}$ is closed $O_{k,\tilde{t}}$-submodule in $\mathcal{G}$.

One can easily see, that

a) Any element of $\mathcal{G}_{k,\tilde{t}}$ can be uniquelly presented in a form $\sum[\alpha_{s,u}]\tilde{t}^u p^s$ from its definition.

b) $O_{k,\tilde{t}}^{\mathrm{DP}} \subset \mathcal{G}_{k,\tilde{t}} \cap \mathcal{G}^0_{(p-1)+,0}$.

c) Any element of $O_{k,\tilde{t}}^{\mathrm{DP}}$ is invertible in $\mathcal{G}_{k,\tilde{t}} \cap \mathcal{G}_{(p-1)+,p-1}$.

Let $\mathcal{L}_{k,\tilde{t}} = \mathcal{G}_{k,\tilde{t}} \cap \mathcal{G}_{(p-1)+,p-1}$. Then $\mathcal{L}_{k,\tilde{t}}$ is $W_{\mathbf{Q}_p}(k)$-algebra, every element of $\mathcal{L}_{k,\tilde{t}}$ can be uniquelly expressed as $\sum_{u\in\mathbf{Z}} w_u\tilde{t}^u$, where all $w_u \in W_{\mathbf{Q}_p}(k)$. It is easy to see, that this $W_{\mathbf{Q}_p}(k)$-algebra coincides with denoted by the same symbol $W_{\mathbf{Q}_p}(k)$-algebra from introduction.

3.7. Consider the matrix $\mathcal{V}_{\tilde{t}}^{-1}$ from n.3.5. Clearly, all elements of $\mathcal{V}_{\tilde{t}}^{-1}$ belong to $\mathcal{L}_{k,\tilde{t}}$. So, they are Laurent series in variable $\tilde{t}$ with coefficients in $W_{\mathbf{Q}_p}(k)$, i.e. they can be written as $\sum_{u\in\mathbf{Z}} w_u\tilde{t}^u$, where all $w_u \in W_{\mathbf{Q}_p}(k)$.

**Proposition.** *Let* $\mathcal{V}_{\tilde{t}}^{-1} = ((v_{ij}))_{1\le i,j\le h}$ *and* $v_{ij} = \sum_{u\in\mathbf{Z}} w_{uij}\tilde{t}^u$*, where all* $w_{uij} \in W(k)\otimes\mathbb{Q}_p$*. If* $u<0$*, then* $w_{uij}\in W(k)$ *(i.e. all coefficients of the matrix* $\mathcal{V}_{\tilde{t}}^{-1}$ *have p-integral principal parts).*

*Proof.*

By corollary of n.3.5.5 for any $1\le i,j\le h$ we have

$$v_{ij}\in \mathcal{G}_{k,\tilde{t}}\cap(1/p\psi)\mathcal{R}.$$

**3.7.1. Lemma.**

$$\frac{1}{p\psi}\mathcal{R}\subset[x_0]^{-2p/(p-1)}\mathcal{G}^0_{p,1}.$$

*Proof of lemma.*

We have $t\in[x_0]W(R)+pW(R)\subset[x_0]\mathcal{G}^0_{p,1}$ and $\psi^{-1}\in[x_0]^{-p/(p-1)}\mathcal{G}^0_{p,1}$, c.f. n.3.1.2. Therefore, $pt/\psi\in p[x_0]^{-1/(p-1)}\mathcal{G}^0_{p,1}\subset\mathcal{G}^0_{p,1}$ (because $p[x_0]^{-1/(p-1)}\in\mathcal{G}^0_{p,1}$), and $(t/\psi^2)W(R)[[pt/\psi]]\subset[x_0]^{-2p/(p-1)}\mathcal{G}^0_{p,1}$.

As it was proved earlier (c.f. n.3.1.2), $W(R)[[\psi^{p-1}/p]]\subset\mathcal{G}^0_{p,1}$. We have $\psi^{p-3}\in[x_0]^{p(p-3)/(p-1)}\mathcal{G}^0_{p,1}$, because $\psi\in[x_0]^{p/(p-1)}\mathcal{G}^0_{p,1}$. Therefore,

$$\frac{\psi^{p-3}}{p}W(R)[[\psi^{p-1}/p]]\subset\frac{1}{p}[x_0]^{p-\frac{2p}{p-1}}\mathcal{G}^0_{p,1}\subset[x_0]^{-2p/(p-1)}\mathcal{G}^0_{p,1},$$

because $[x_0]^p/p\in\mathcal{G}^0_{p,1}$. So,

$$\frac{1}{p\psi}\mathcal{R}=\frac{t}{\psi^2}W(R)[[pt/\psi]]+\frac{\psi^{p-3}}{p}W(R)[[\psi^{p-1}/p]]\subset[x_0]^{-2p/(p-1)}\mathcal{G}^0_{p,1}.$$

Lemma is proved.

Clearly, our proposition is implied by the following lemma.

**3.7.2. Lemma.** *Any Laurent series from* $[x_0]^{-p}\mathcal{G}^0_{p,p-1}\cap\mathcal{L}_{k,\tilde{t}}$ *has p-integral principal part.*

*Proof.*

If $v\in[x_0]^{-p}\mathcal{G}^0_{p,p-1}\cap\mathcal{L}_{k,\tilde{t}}$, then

$$\sum_{\substack{s\leqslant-1\\ u\geqslant-ep(s+1)}}p^s[\alpha_{s,u}]\tilde{t}^u+\sum_{\substack{s\geqslant0\\ u\geqslant-e(p+s)}}p^s[\alpha_{s,u}]\tilde{t}^u,$$

where all $\alpha_{s,u}\in k$ and $e$ is absolute ramification index of the field $K$. Let $\alpha_{s_0,u_0}\ne0$ for some $s_0\le-1$. Then

$$u\ge-ep(s+1)\ge0.$$

Lemma is proved.

3.8. Let $X_{\tilde{t}}=\mathcal{V}_{\tilde{t}}^{-1}\begin{pmatrix}\alpha\\0\end{pmatrix}=\sum_{u\in\mathbf{Z}}w_u\tilde{t}^u$, where $w_u\in W_{\mathbb{Q}_p}(k)^h$.

**Proposition.**

a) $X_{\bar{t}} \equiv X \bmod p^M(1/p\psi)\mathcal{R}W(m_R)$;

b) *if* $u \leq 0$, *then* $w_u \in W(k)^h$, *i.e. "nonnegative" part* $\hat{X}_{\bar{t}} = \sum_{u\leq 0} w_u \tilde{t}^u$ *of the vector Laurent series* $X_{\bar{t}}$ *has p-integral coefficients;*

c) *if* $\mathcal{V}_{\bar{t}}^{(-1)}$ *is the matrix of principal parts of elements of the matrix* $\mathcal{V}_{\bar{t}}^{-1}$, *then "nonnegative" part of the vector* $\mathcal{V}_{\bar{t}}^{(-1)}\begin{pmatrix}\alpha\\0\end{pmatrix}$ *equals to* $\hat{X}_{\bar{t}}$;

d) $\psi\hat{X}_{\bar{t}} \in W(m_R) + p^M W(R_0)$.

*Proof.*

a) $\alpha \in m_{k,\bar{t}}^n \subset W(m_R)^n$ gives by proposition 3.5.5

$$X_{\bar{t}} = \mathcal{V}_{\bar{t}}^{-1}\begin{pmatrix}\alpha\\0\end{pmatrix} \equiv \mathcal{V}^{-1}\begin{pmatrix}\alpha\\0\end{pmatrix} = X \bmod p^M\left(\frac{1}{p\psi}\mathcal{R}\right)W(m_R).$$

b) and c) follow from proposition of n.3.7.

d) follows from corollary of n.3.5.5.

3.9. Now we can state the main result of this section.

**Proposition.** *In the above notation let* $\hat{T}_{\bar{t}} \in W(R_0)^h$ *be such that* $\hat{T}_{\bar{t}} - \sigma\hat{T}_{\bar{t}} = \hat{X}_{\bar{t}}$. *If* $\tau \in \Gamma_K$, *then* $\tau\hat{T}_{\bar{t}} - \hat{T}_{\bar{t}} \equiv a \bmod W(m_R) + p^M W(R_0)$, *where* $a = {}^t(a_1, \dots, a_h) \in \mathbb{Z}_p^h$ *and*

$$(\alpha, \tau]_{\text{cris}} = a_1 o_M^1 + \dots + a_h o_M^h.$$

*Proof.*

Let $X_{\bar{t}} = X + p^M w$, where $X$ has coordinates in $[x_0]^{-p/(p-1)}\mathcal{G}_{p,1}^0$ and $w$ has coordinates in $[x_0]^{-2p/(p-1)}\mathcal{G}_{p,1}^0 W(m_R)$, c.f. nn.3.8 a) and 3.7.1.

By corollary of n.3.4, if $T \in [x_0]^{-p/(p-1)}(\mathcal{G}_{p,1}^0)^h$ is such that $T - \sigma_{\mathcal{G}}T = X$, then

$$\tau T - T \equiv A \bmod \mathcal{G}(m_R) + p^M W(R_0),$$

where $A = {}^t(A_1, \dots, A_h) \in \mathbb{Z}_p^h$ and $(\alpha, \tau]_{\text{cris}} = A_1 o_M^1 + \dots + A_h o_M^h$.

It is easy to see, that $[x_0]^{-2p/(p-1)}\mathcal{G}_{p,1}^0 W(m_R) \subset \mathcal{G}(m_R) + W(R_0)$. Therefore, one can take $w_1 \in \mathcal{G}(m_R) + W(R_0)$, such that $w_1 - \sigma_{\mathcal{G}} w_1 = w$.

Then for $T_{\bar{t}} = T + p^M w_1$ one has $T_{\bar{t}} \in (\mathcal{G}(m_R) + W(R_0))^h$, $T_{\bar{t}} - \sigma_{\mathcal{G}}T_{\bar{t}} = X_{\bar{t}}$ and $\tau T_{\bar{t}} - T_{\bar{t}} \equiv A \bmod \mathcal{G}(m_R) + p^M W(R_0)$.

Let $X_{\bar{t}} = \hat{X}_{\bar{t}} + X_1$, where $\hat{X}_{\bar{t}} = \sum_{u\leq 0} w_u \tilde{t}^u$ is the "nonnegative" part of $X_{\bar{t}}$. Then $X_1 \in \mathcal{G}(m_R)$ and for $T'_{\bar{t}} = T_{\bar{t}} - \sum_{s\geq 0}\sigma^s X_1$ one has $T'_{\bar{t}} \in (\mathcal{G}(m_R) + W(R_0))^h$, $T'_{\bar{t}} - \sigma_{\mathcal{G}}T'_{\bar{t}} = \hat{X}_{\bar{t}}$ and $\tau T'_{\bar{t}} - T'_{\bar{t}} \equiv A \bmod(\mathcal{G}(m_R) + p^M W(R_0))$.

Therefore, $\hat{T}_{\bar{t}} - T'_{\bar{t}}$ has coordinates in $(\mathcal{G}(m_R) + W(R_0))_{\sigma_{\mathcal{G}}=\text{id}} = \mathbb{Z}_p$, $\tau\hat{T}_{\bar{t}} - \hat{T}_{\bar{t}} = \tau T'_{\bar{t}} - T'_{\bar{t}}$ has coordinates in

$$(\mathbb{Z}_p + \mathcal{G}(m_R) + p^M W(R_0)) \cap W(R_0) = \mathbb{Z}_p + W(m_R) + p^M W(R_0),$$

and $a \equiv A \bmod p^M$.

Proposition is proved.

## 4. Explicit formulae for the formal group symbol.

We use all previous notation. When reciprocity map of class field theory $\psi_K : K^* \longrightarrow \Gamma^{ab}$ is considered, we assume that the residue field $k$ of $K$ is finite.

4.1. *Preliminaries.*

4.1.1. *Functor field of norms,* [Wtb].

Let $\{\pi_s\}_{s\geq 0}$ be the sequence of elements of $\bar{K}$ chosen in n.2.5.1. This means, that $\pi_0 = \pi$ is fixed uniformizer of $K$ and $\pi_{s+1}^p = \pi_s$ for all $s \in \mathbb{Z}, s \geq 0$. Let $\tilde{K} = \bigcup_{s\geq 0} K(s)$, where $K(0) = K$ and $K(s) = K(\pi_s)$. The field $\tilde{K}$ is infinite APF-extension, and the functor field of norms $\mathcal{X}$ gives equivalence of the category of algebraic extensions of the field $\tilde{K}$ and of the category of separable extensions of the discrete valuation field $\mathcal{K} = \mathcal{X}(\tilde{K})$ of characteristic $p$. The residue field of $\mathcal{K}$ can be canonically identified with the residue field $k$ of $K$. By definition $\mathcal{K}^* = \varprojlim_s K(s)^*$ with respect to norm maps. This gives fixed uniformizer $t_0 = \varprojlim \pi_s$ in $\mathcal{K}$, so $\mathcal{K} = \operatorname{Frac} k[[t_0]] = k((t_0))$. One can fix a separable closure $\mathcal{K}_{\text{sep}}$ of $\mathcal{K}$ by $\mathcal{K}_{\text{sep}} = \mathcal{X}(\bar{K})$ and the functor $\mathcal{X}$ gives identification

$$\iota : \Gamma_{\mathcal{K}} = \operatorname{Gal}(\mathcal{K}_{\text{sep}}/\mathcal{K}) = \operatorname{Gal}(\bar{K}/\tilde{K}) \subset \Gamma_K = \operatorname{Gal}(\bar{K}/K).$$

4.1.2. *Homomorphism* $\mathcal{N} : \mathcal{K}^* \longrightarrow K^*$.

Let $\mathcal{N}$ be the projection $\mathcal{K}^* = \varprojlim K(s)^* \longrightarrow K(0)^* = K^*$. Clearly, $\mathcal{N}(t_0) = \pi$. If $\alpha \in k \subset \mathcal{K}$, then $\mathcal{N}(\alpha) = [\alpha] \in K^*$, where $[\alpha]$ is Teichmuller representative of $\alpha$ considered as element of the residue field of $K$.

Let $U_{\mathcal{K}}$ and $U_K$ be $\mathbb{Z}_p$-modules of principal units in $\mathcal{K}$ and $K$, respectfully. Then $\mathcal{N}(U_{\mathcal{K}}) \subset U_K$ and can be described explicitly as follows.

Let $\alpha \in W(k)$ and

$$E(\alpha, X) = \exp(\alpha X + \dots + (\sigma\alpha)X^p/p + \dots) \in \mathbb{Z}_p[[X]]$$

be power series from [Sh]. Any element $u \in U_{\mathcal{K}}$ can be uniquely presented in a form

$$u = \prod_{(a,p)=1} E(\alpha_a, t_0^a),$$

where all $\alpha_a \in W(k)$. With respect to this decomposition the homomorphism $\mathcal{N}$ is uniquelly defined by the property, c.f. [Ab3],

$$\mathcal{N}(E(\alpha, t_0^a)) = E(\alpha, \pi^a),$$

where $a \in \mathbb{N}, (a,p) = 1, \alpha \in W(k)$.

It can be easily shown, that $K^*/\mathcal{N}(\mathcal{K}^*)$ is cyclic group of order $p^{l_0}$, where $l_0$ is maximal integer, such that $K$ contains primitive $p^{l_0}$-root of unity (this fact is well-known modulo $K^{*p^{l_0}}$, c.f. [Sh], then one should use, that $p$-completion of $K^{*p^{l_0}}$ is generated by $\pi^{p^{l_0}}$ and all $E(p^{l_0}\alpha, \pi^a)$, where $\alpha \in W(k), a \in \mathbb{N}, (a,p) = 1$).

The group $K^*/\mathcal{N}(\mathcal{K}^*)$ is generated by the image of any $p^{l_0}$-primary element. These elements appear as principal units $E_{l_0} \in U_K$, such that $K(E_{l_0}^{p^{-l_0}})$ is unramified extension of $K$ of degree $p^{l_0}$. Equivalently, if $\psi_K : K^* \longrightarrow \Gamma_K^{\rm ab}$ is reciprocity map of class field theory, then $\psi_K(E_{l_0})(\pi^{p^{-l_0}}) = \zeta\pi^{p^{-l_0}}$ for some primitive $p^{l_0}$-root of unity $\zeta$, and $\psi_K(E_{l_0})(u^{p^{-l_0}}) = u^{p^{-l_0}}$ for all $u \in U_K$. Explicit constructions of primary elements were considered in [A-H], [Sh], [Vol]. In n.4.3 below we give Vostokov's construction of primary elements.

4.1.3. *Compatibility of class field theories,* [La].

The homomorphism $\mathcal{N}$ relates class field theories for the fields $\mathcal{K}$ and $K$. Namely, let $\psi_{\mathcal{K}} : \mathcal{K}^* \longrightarrow \Gamma_{\mathcal{K}}^{\rm ab}$ and $\psi_K : K^* \longrightarrow \Gamma_K^{\rm ab}$ be reciprocity maps of class field theory. Then for any $a \in \mathcal{K}^*$ we have $\iota^{\rm ab}(\psi_{\mathcal{K}}(\alpha)) = \psi_K(\mathcal{N}(\alpha))$, where $\iota^{\rm ab} : \Gamma_{\mathcal{K}}^{\rm ab} \longrightarrow \Gamma_K^{\rm ab}$ is induced by imbedding $\iota : \Gamma_{\mathcal{K}} \longrightarrow \Gamma_K$ from n.4.1.

4.1.4. *Witt explicit reciprocity law,* [Fo4].

The uniformizer $t_0$ of the field $\mathcal{K}$ gives $p$-basis for any separable extension $\mathcal{E}$ of the field $\mathcal{K}$. One can use $t_0$ to define functorial by $M \in \mathbb{N}$ and by $\mathcal{E} \subset \mathcal{K}_{\rm sep}$ system of liftings $O_M(\mathcal{E})$ of the field $\mathcal{E}$ modulo $p^M$. By definition $O_M(\mathcal{E})$ is flat $\mathbb{Z}/p^M\mathbb{Z}$-algebra, such that $O_M(\mathcal{E})/pO_M(\mathcal{E}) = \mathcal{E}$. These liftings can be given explicitly as

$$O_M(\mathcal{E}) = W_M(\sigma^M\mathcal{E})[\tilde{t}_0] \subset W_M(\mathcal{E}),$$

where $\tilde{t} = [t_0]$ is Teichmuller representative of $t_0$ in $W_M(\mathcal{E})$ (this version of general construction from [B-M] we use also in [Ab2]).

Multiplication by $p$ induces epimorphisms of $W(k)$-algebras $O_{M+1}(\mathcal{E}) \longrightarrow O_M(\mathcal{E})$. If $O(\mathcal{E}) = \varprojlim_M O_M(\mathcal{E})$ with respect to these epimorphisms, then $O(\mathcal{E})$ is the valuation ring of absolutely unramified field of characteristic 0 with residue field $\mathcal{E}$. Clearly, $O_M(\mathcal{K})$ is $W_M(k)$-algebra of Laurent series $W_M(k)((\tilde{t}\,)) = W_M(k)[[\tilde{t}\,]][\tilde{t}^{-1}]$ with coefficients in $W_M(k)$, and $O(\mathcal{K})$ is $p$-adic completion $W(k)((\tilde{t}\,))$ of $W_M(k)[[\tilde{t}\,]][\tilde{t}^{-1}]$.

Absolute Frobenius morphism of Witt vectors induces compatible system of Frobenius morphisms $\sigma = \sigma_{\mathcal{E}} : O(\mathcal{E}) \longrightarrow O(\mathcal{E})$. We have $O(\mathcal{K}_{\rm sep})_{\sigma={\rm id}} = W(\mathbb{F}_p)$. Action of $\Gamma_{\mathcal{K}}$ on $\mathcal{K}_{\rm sep}$ induces action of $\Gamma_{\mathcal{K}}$ on $O(\mathcal{K}_{\rm sep})$. If $\mathcal{H} \subset \Gamma_{\mathcal{K}}$ is open subgroup and $\mathcal{K}_{\rm sep}^{\mathcal{H}} = \mathcal{E}$, then $O(\mathcal{K}_{\rm sep})^{\mathcal{H}} = O(\mathcal{E})$.

Let Col : $\mathcal{K}^* \longrightarrow O(\mathcal{K})^*$ be Coleman's multiplicative section of the projection pr : $O(\mathcal{K}) \longrightarrow \mathcal{K}$, c.f. [Fo4]. In our situation, the homomorphism Col can be described explicitly in terms of generators of the group $\mathcal{K}^*$ from n.4.2. Namely, $\mathrm{Col}(t_0) = \tilde{t}$ and $\mathrm{Col}(E(\alpha, t_0^a)) = E(\alpha, \tilde{t}^a)$, where $\alpha \in W(k), a \in \mathbb{N}, (a,p) = 1$. This property gives the following simple explicit description of the homomorphism $(\mathcal{N} \circ \mathrm{pr})|_{\mathrm{Col}\,\mathcal{K}^*} : \mathrm{Col}\,\mathcal{K}^* \longrightarrow K^*$ :

*if* $g = g(\tilde{t}) \in \mathrm{Col}\,\mathcal{K}^*$, *then* $\mathcal{N}(\mathrm{pr}(g)) = g(\pi)$.

One can easily prove the following characterization of the image $\mathrm{Col}(\mathcal{K}^*)$ in $O(\mathcal{K})^*$.

**Lemma.** *Let* $g \in O(\mathcal{K})^*$. *Then* $g \in \mathrm{Col}(\mathcal{K}^*)$ *if and only if*

a) $g \in W(k)[[\tilde{t}\,]][\tilde{t}^{-1}]$;

b) $(\sigma g/g^p) \in 1 + \tilde{t}W(k)[[\tilde{t}\,]]$;

c) $\frac{1}{p}\log(\sigma g/g^p) = \sum_{(c,p)=1} \alpha_c \tilde{t}^c$, *where all* $\alpha_c \in W(k)$.

Let $f \in O(\mathcal{K})$, $g \in \mathcal{K}^*$, and let $(f,g]_W \in W(\mathbb{F}_p)$ be Witt pairing given by

$$(f,g]_W = \tau T - T,$$

where $T \in O(\mathcal{K}_{\text{sep}})$ is such that $\sigma T - T = f$ and $\tau = \psi_K(g) \in \Gamma_{\mathcal{K}}^{\text{ab}}$.

Then Witt explicit reciprocity law can be given by Fontaine formula, [Fo4]

$$(f,g]_W = (\text{Res}_{\tilde{t}=0} \circ \text{Tr}) \left( f \frac{\text{d}\,\text{Col}\, g}{\text{Col}\, g} \right),$$

where $\text{Tr} : W(k) \longrightarrow W(\mathbb{F}_p)$ is the trace map and $\text{Res}_{\tilde{t}=0}$ is residue at $\tilde{t} = 0$.

Finally, we remark, that this construction can be made in $W(R_0)$, where $R_0 = \text{Frac}\, R$. We have a natural identification of the field $\mathcal{K}$ with some subfield of $R_0$ by the correspondence: $t_0 \mapsto (\pi_s)_{s \geq 0} \in R_0$, c.f. n.2.5.1, and if $\alpha \in k \subset \mathcal{K}$, then $\alpha \mapsto ([\alpha^{p^{-s}}])_{s \geq 0} \in R_0$, where $[\ \ ]$ denotes Teichmuller representative for elements of the residue field of $K$. This embedding is a particular case of compatible system of embeddings $\mathcal{E} \subset R_0$, where $\mathcal{K} \subset \mathcal{E} \subset \mathcal{K}_{\text{sep}}$, given in [Wtb]. So, we have a natural imbedding $\mathcal{K}_{\text{sep}} \subset R_0$ compatible with Galois action (with respect to the inclusion $\iota : \Gamma_{\mathcal{K}} \longrightarrow \Gamma_K$ from n.4.1). Then by universal property of Witt vectors there exists the unique compatible with given Frobenius morphisms system of embeddings $O(\mathcal{E}) \subset W(R_0)$. So, one can compute the value $(f,g]_W$ of Witt symbol in the ring $W(R_0)$.

*Remark.* Under the above embedding $O_{\mathcal{K}} = W(k)((\ \tilde{t}\ )) \subset W(R_0)$ notation $t_0$ and $\tilde{t}$ from this n. agree with notation $t_0$ and $\tilde{t}$ from n.2.5.1.

4.2. *First explicit formula.*

Let $G = G_{\mathcal{A}}$ be the formal group with vector logarithm power series $\bar{l}_{\mathcal{A}}(\bar{X})$ from section 1. If $M$ is fixed natural number, choose $\mathbb{Z}/p^M\mathbb{Z}$-basis $o_M^1, \dots, o_M^h$ of $G_{\mathcal{A}}[M](m_K) = G_{\mathcal{A}}[M](m_C)$, take liftings of its elements $\hat{o}_M^1, \dots, \hat{o}_M^h$ in $m_{k,\tilde{t}}^n = \tilde{t}W(k)^n \subset O_{k,\pi}^n$ with respect to the epimorphism $\gamma : m_{k,\pi} \longrightarrow m_K$ given by $\tilde{t} \mapsto \pi$ and construct the matrix from n.3.5

$$\mathcal{V}_{\tilde{t}} = \begin{pmatrix} p^M \bar{l}_{\mathcal{A}}(\hat{o}_M^1) & \dots & p^M \bar{l}_{\mathcal{A}}(\hat{o}_M^h) \\ p^M \bar{m}_{\mathcal{A}}(\hat{o}_M^1) & \dots & p^M \bar{m}_{\mathcal{A}}(\hat{o}_M^h) \end{pmatrix}.$$

This matrix is invertible in the $W_{\mathbb{Q}_p}(k)$-algebra $\mathcal{L}_{k,\tilde{t}}$ and denote by $\mathcal{V}_{\tilde{t}}^{(-1)}$ the matrix obtained from $\mathcal{V}_{\tilde{t}}^{-1}$ by taking principal parts of its elements.

If $f \in G_{\mathcal{A}}(m_K)$, $g \in K^*$, then the value $(f,g]_{G_{\mathcal{A}}}$ of the formal group symbol modulo $p^M$ can be expressed as

$$(f,g]_{G_{\mathcal{A}}} = A_1(f,g)o_M^1 + \dots + A_h(f,g)o_M^h,$$

where $A(f,g) = {}^t(A_1(f,g), \dots, A_h(f,g)) \in (\mathbb{Z}/p^M\mathbb{Z})^h$.

Now propositions of nn.2.7, 3.9 and Witt explicit reciprocity law from n.4.1.4 give the following theorem

**Theorem A.** *Let* $\beta(\tilde{t}) \in m^n_{k,\tilde{t}}$, $\Theta_{G_{\mathcal{A}},1} : m^n_{k,\tilde{t}} \longrightarrow G_{\mathcal{A}}(m_{k,\tilde{t}})$ *be the isomorphism from n.2.5, and* $\delta_1 \in \mathcal{K}^*$. *Then*

$$A(\beta(\pi), \mathcal{N}(\gamma)) = (\mathrm{Res}_{\tilde{t}=0} \circ \mathrm{Tr}) \left\{ \mathcal{V}^{(-1)}_{\tilde{t}} \begin{pmatrix} \Theta^{-1}_{G_{\mathcal{A}},1}(\beta(\tilde{t})) \\ 0 \end{pmatrix} \mathrm{d}_{\log} \mathrm{Col}\, \delta_1 \right\} \bmod p^M.$$

We can use explicit description of $\Theta^{-1}_{G_{\mathcal{A}},1}$ from n.2.5.2 to give equivalent form of the above theorem

**Theorem A1.** *Let* $f \in G_{\mathcal{A}}(m_K) = m^n_K$, $g \in K^*$. *Take* $\beta(\tilde{t}) \in m^n_{k,\pi}$ *such that* $\beta(\pi) = f$ *and assume that there exists* $\delta = \delta(\tilde{t}) \in \mathrm{Col}\,\mathcal{K}^* \subset O(\mathcal{K}) = W(k)((\tilde{t}))$, *such that* $\delta(\pi) = g$ *(c.f. lemma of n.4.1.4). Then*

$$A(f, g) = (\mathrm{Res}_{\tilde{t}=0} \circ \mathrm{Tr}) \left\{ \mathcal{V}^{(-1)}_{\tilde{t}} \begin{pmatrix} l_{\mathcal{A}}(\beta(\tilde{t})) - \frac{\mathcal{A}^*}{p} l_{\mathcal{A}}(\beta(\tilde{t})) \\ 0 \end{pmatrix} \mathrm{d}_{\log} \delta \right\} \bmod p^M.$$

*Remarks.*

a) In the above theorems one can replace matrix $\mathcal{V}^{(-1)}_{\tilde{t}}$ by $\mathcal{V}^{-1}_{\tilde{t}}$, because one can compute residue also in the algebra $\mathcal{L}_{k,\tilde{t}}$ and the above replacement does not affect the value of residue.

b) Theorems A and A1 give almost all information about values of the formal group symbol. The only restriction is that the second argument can be taken only from the subgroup $\mathcal{N}(\mathcal{K}^*)$. If $K$ does not contain $p$-roots of unity, then $\mathcal{N}(\mathcal{K}^*) = K^*$, and our formula gives complete description of symbol. If $K$ contains primitive $p^M$-root of unity, one should involve in consideration $p^M$-primary elements of $K^*$, c.f. n.4.3 below.

c) Another inconvenience of the above formulae is related to the special choice of the power series $\mathrm{Col}\,\delta_1(\tilde{t}) = \delta(\tilde{t})$ to obtain $g \in K^*$ as a result of substitution $\tilde{t} \mapsto \pi$. In the case $G = \hat{\mathbb{G}}_m$ Brückner-Vostokov formula is free from this restriction. In n.4.4 below, we give similar expression for the formal group symbol.

4.3. *$p^M$-primary elements.*

Assume, that $K$ contains a primitive $p^M$-root of unity $\zeta$.

4.3.1. Let $\mathbb{G}_1$ be the formal group from n.1.7.1. If $\hat{\mathbb{G}}_m$ is the formal multiplicative group, then $\eta : X \mapsto E(1, X) = \exp(l_{\mathbb{G}_1}(X))$ gives isomorphism of formal groups $\eta : \mathbb{G}_1 \longrightarrow \hat{\mathbb{G}}_m$. In particular, $\mathbb{G}_1(m_K) \simeq \hat{\mathbb{G}}_m(m_K) = U_K$ and $o_M = \eta^{-1}(\zeta)$ is generator of $\mathbb{G}_1[M](m_C) = \mathbb{G}_1[M](m_K)$.

In notation of n.1.7.2 the matrix of values of the $p$-adic periods pairing $\mathcal{V}_{\mathbb{G}_1}$ for the formal group $\mathbb{G}_1$ equals $((t^+))$, where $t^+ = l_{\mathbb{G}_1}(\psi)$. If $\hat{o}_M \in m_{k,\tilde{t}}$ is such that $\gamma(\hat{o}_M) = o_M$ (i.e. $\hat{o}_M \mapsto o_M$ by substitution $\tilde{t} \mapsto \pi$), consider $s_M(\tilde{t}) = (p^M \mathrm{id}_{\mathbb{G}_1})(\hat{o}_M)$. Then $\mathcal{V}_{\mathbb{G}_1,\tilde{t}} = ((\ l_{\mathbb{G}_1}(s_M(\tilde{t}))\ ))$, and one can easily see, that $\mathcal{V}^{(-1)}_{\mathbb{G}_1,\tilde{t}} = ((\ s_M(\tilde{t})^{-1}\ ))$.

Let $\delta_0 \in W(k)$ be such that $\mathrm{Tr}\,\delta_0 \in \mathbb{Z}^*_p \subset \mathbb{Z}_p$, where $\mathrm{Tr} : W(k) \longrightarrow \mathbb{Z}_p$ is the trace map. Take $\hat{T}_{\tilde{t}} \in W(\bar{k}) \subset W(R_0)$, such that

$$\hat{T}_{\tilde{t}} - \sigma \hat{T}_{\tilde{t}} = \delta_0.$$

If $\tau \in \Gamma_K$, let $A_\tau = \tau \hat{T}_{\tilde{t}} - \hat{T}_{\tilde{t}} \in \mathbb{Z}_p \subset W(\bar{k})$. Clearly, the correspondence $\tau \mapsto A_\tau$ gives epimorphism $j : \Gamma_K \longrightarrow \mathbb{Z}_p$, $\operatorname{Ker} j$ is the inertia subgroup of $\Gamma_K$ and $j$ induces isomorphism $\mathrm{Gal}(K_{\mathrm{ur}}/K) \simeq \mathbb{Z}_p$, where $K_{\mathrm{ur}}$ is maximal unramified extension of $K$.

If $\Theta = \Theta_{\mathbb{G}_1}$, c.f. n.2.5.2, then by nn.2.7, 3.9 for any $\tau \in \Gamma_K$ we have

$$(*) \qquad (\ \Theta(\delta_0 s_M(\tilde{t})), \tau\ ]_{\mathbb{G}_1} = A_\tau o_M.$$

If $f_1 \in \mathbb{G}_1(m_C)$ is such that $(p^M \operatorname{id}_{\mathbb{G}_1})(f_1) = \Theta_{\mathbb{G}_1}(\delta_0 s_M(\tilde{t}))$, then the above formula $(*)$ gives $K(f_1) \subset K_{\mathrm{ur}}$ and $[K(f_1) : K] = p^M$. Therefore, we obtain $p^M$-primary element $E_M = \eta(\Theta_{\mathbb{G}_1}(\delta_0 s_M(\tilde{t})))$ by applying isomorphism $\eta : \mathbb{G}_1 \longrightarrow \hat{\mathbb{G}}_m$.

4.3.2. The above considerations give the following explicit construction of $p^M$-primary element from [Vo1].

**Proposition.** *If $s_M(\tilde{t}) = \sum_{u>0} w_u \tilde{t}^u$, where all $w_u \in W(k)$, and*

$$E_M(\tilde{t}) = \prod_{u>0} E(\delta_0 w_u, \tilde{t}^u) \in 1 + m_{k,\tilde{t}},$$

*then $\gamma(E_M(\tilde{t})) = E_M(\pi) = E_M$.*

*Proof.* If $\delta_0 w_u = \sum_{s\geq 0} [\alpha_{s,u}] p^s$, where all $\alpha_{s,u} \in k$, then

$$\Theta_{\mathbb{G}_1,1}(\delta_0 s_M(\tilde{t})) = \sum_{\substack{\text{in } \mathbb{G}_1 \\ s,u}} (p^s \operatorname{id}_{\mathbb{G}_1})([\alpha_{s,u}]\tilde{t}^u),$$

$$E_M = \eta(\Theta_{\mathbb{G}_1}(\delta_0 s_M(\tilde{t}))) = \gamma(\eta(\Theta_{\mathbb{G}_1,1}(\delta_0 s_M(\tilde{t})))) =$$

$$= \gamma(\prod_{\substack{s\geqslant 0 \\ u>0}} E(p^s[\alpha_{s,u}], \tilde{t}^u)) = \gamma(\prod_{u>0} E(\delta_0 w_u, \tilde{t}^u)) = \gamma(E_M(\tilde{t})).$$

The following corollary is also well-known.

**Corollary.** *If $E_M \in K^*$ is a $p^M$-primary element, then there exists power series $E_M(\tilde{t}) \in 1 + m_{k,\tilde{t}}$, such that $E_M(\pi) = E_M$ and $\mathrm{d}_{\log} E_M(\tilde{t}) \in p^M \Omega^1_{O_{k,\tilde{t}}}$.*

*Proof.* Indeed, take $E_M(\tilde{t})$ from the above proposition. Then

$$\mathrm{d}_{\log} E_M(\tilde{t}) = \mathrm{d}(\sum_{\substack{s\geqslant 0 \\ u>0}} \sigma^s(\delta_0 w_u) \frac{\tilde{t}^{up^s}}{p^s}) = \sum_{s\geqslant 0} \sigma^s(\tilde{t} s'_M(\tilde{t})) \frac{\mathrm{d}\,\tilde{t}}{\tilde{t}}.$$

But $s'_M(\tilde{t}) \in p^M O_{k,\tilde{t}}$, because $l_{\mathbb{G}_1}(s_M(\tilde{t})) = p^M l_{\mathbb{G}_1}(\hat{o}_M)$ and, therefore,

$$(1 + s_M(\tilde{t})^{p-1} + \cdots + s_M(\tilde{t})^{p^s - 1} + \dots)\,\mathrm{d}\, s_M(\tilde{t}) = p^M\, \mathrm{d}\, l_{\mathbb{G}_1}(\hat{o}_M) \in p^M \Omega^1_{O_{k,\tilde{t}}}.$$

4.3.3. Let $E_M \in K^*$ be a $p^M$-primary element and $G_{\mathcal{A}}$ be the formal group from section 1.

**Proposition.** *For any $f \in G_{\mathcal{A}}(m_K)$ one has*

$$(f, E_M]_{G_{\mathcal{A}}} = 0.$$

*Proof.* Choose some $\tau_M \in \Gamma_K$, such that $\psi_K(E_M)$ is the image of $\tau_M$ in $\Gamma_K^{ab}$, where $\psi_K : K^* \longrightarrow \Gamma_K^{ab}$ is reciprocity map of class field theory. We must prove, that $(f, \tau]_{G_{\mathcal{A}}} = 0$.

The statement of proposition holds for the formal group $\hat{\mathbb{G}}_m$ (c.f. n.4.1.2) and, therefore, it holds for the formal group $\mathbb{G}_1$.

Take $s_M(\tilde{t}) \in m_{k,\tilde{t}}$ from n.4.3.1. If $\alpha_0 \in m_{k,\tilde{t}}$ and $T \in W(R_0)$ is such that $T - \sigma T = \alpha_0 / s_M(\tilde{t})$, the equality $(\Theta_{\mathbf{G}_1}(\alpha_0), \tau_M]_{\mathbf{G}_1} = 0$ is equivalent to the relation

$$(*) \qquad \tau_M T - T \in W(m_R) + p^M W(R_0).$$

Let $\mathcal{V}_{\tilde{t}}$ be approximation of the matrix of values of the $p$-adic periods pairing for the formal group $G_{\mathcal{A}}$ from n.3.5, and let $\hat{X}_{\tilde{t}}$ be nonnegative part of $\mathcal{V}_{\tilde{t}}^{-1}\begin{pmatrix}\alpha \\ 0\end{pmatrix}$, c.f. n.3.8. By the part d) of proposition 3.8 vector-column $\psi \hat{X}_{\tilde{t}}$ has coordinates in $W(m_R) + p^M W(R_0)$. It is easy to see, that $\psi \equiv s_M(\tilde{t}) \bmod p^M W(R)$, therefore,

$$\hat{X}_{\tilde{t}} \in \frac{1}{s_M(\tilde{t})} m_{k,\tilde{t}}^n \bmod p^M W(R_0).$$

Now the above relation $(*)$ gives:

if $\hat{T}_{\tilde{t}} \in W(R_0)$ is such that $\hat{T}_{\tilde{t}} - \sigma \hat{T}_{\tilde{t}} = \hat{X}_{\tilde{t}}$, then $\tau_M \hat{T}_{\tilde{t}} - \hat{T}_{\tilde{t}}$ has coordinates in $W(m_R) + p^M W(R_0)$. By propositions of n.3.9 and n.2.7 this is equivalent to the statement of our proposition.

4.4. *Second explicit formula.*

4.4.1. *Agreements.*

As earlier, $G \simeq G_{\mathcal{A}}$ is the formal group over $W(k_0)$ from section 1. In particular, we use description of the structure of filtered module $\mathcal{M}(G)$ from section 1 given by the relation

$$\begin{pmatrix}\phi_1(\bar{l}) \\ \phi_0(\bar{m})\end{pmatrix} = \mathcal{E}\begin{pmatrix}\bar{l} \\ \bar{m}\end{pmatrix},$$

$\mathbb{Z}/p^M$-basis $o_M^1, \ldots, o_M^h$ of $G_{\mathcal{A}}[M](m_K)$ and the matrix $\mathcal{V}_{\tilde{t}}$, c.f. n.4.2. All appeared Laurent series $\sum_{u \in \mathbb{Z}} w_u \tilde{t}^u$, $w_u \in W_{\mathbb{Q}_p}(k)$, are elements of the $W_{\mathbb{Q}_p}(k)$-algebra $\mathcal{L}_{k,\tilde{t}} = \mathcal{G}_{k,\tilde{t}} \cap \mathcal{G}_{(p-1)^+,p-1}$, $\sigma$ denotes absolute Frobenius of $\mathcal{G}_{k,\tilde{t}}$ given by restriction of $\sigma_{\mathcal{G}}$, i.e. $\sigma\tilde{t} = \tilde{t}^p$ and $\sigma|_{W(k)}$ is usual Frobenius of Witt vectors.

If $\beta \in G_{\mathcal{A}}(m_{k,\tilde{t}})$ and $\bar{l}_{\mathcal{A}}(\bar{X})$ is vector logarithm power series of the formal group $G_{\mathcal{A}}$, then $\bar{l}_{\mathcal{A}}(\beta)$ has coordinates in $\mathcal{L}_{k,\tilde{t}} \cap \mathcal{G}_{p,0}$, so for any $u \in \mathbb{N}$ $\sigma^u \bar{l}_{\mathcal{A}}(\beta)$ has sense in $\mathcal{L}_{k,\tilde{t}}$. We use matrices $F_u, F'_u$ from n.1.3 to define $\mathbb{Z}_p$-linear operator $\mathcal{A}^* = \sum_{u \geq 1} F_u \sigma^u$ and to define for any $\beta \in G_{\mathcal{A}}(m_{k,\tilde{t}})$ vector Laurent series

$$\bar{m}_{\mathcal{A}}(\beta) = \frac{1}{p} \sum_{u \geq 1} F'_u \sigma^u(\bar{l}_{\mathcal{A}}(\beta)) \in \mathcal{G}_{k,\tilde{t}} \cap \mathcal{G}_{p,0} \subset \mathcal{L}_{k,\tilde{t}}.$$

An easy consequence of the definition of these matrices $F_u$ and $F'_u$ is the following formal identity for any $\beta \in m_{k,\tilde{t}}$

$$\mathcal{E}^{-1}\begin{pmatrix}(\sigma/p)\bar{l}_{\mathcal{A}}(\beta)\\ \sigma\bar{m}_{\mathcal{A}}(\beta)\end{pmatrix} = \begin{pmatrix}(\mathcal{A}^*/p)\bar{l}_{\mathcal{A}}(\beta)\\ \bar{m}_{\mathcal{A}}(\beta)\end{pmatrix}.$$

We denote by $\mathcal{H}_K$ multiplicative subgroup in $O_{k,\tilde{t}}[\tilde{t}^{-1}] \subset \mathcal{L}_{k,\tilde{t}}$, such that

$$\delta(\tilde{t}) \in \mathcal{H}_K \Longleftrightarrow \frac{\sigma\delta(\tilde{t})}{\delta(\tilde{t})^p} \in 1 + m_{k,\tilde{t}}.$$

So, the group $\mathcal{H}_K$ is generated by elements of $1 + m_{k,\tilde{t}}$, by $\tilde{t}$ and by $[\alpha]$, where $\alpha \in k^*$. Remark, that $\mathrm{Col}(\mathcal{K}^*) \subset \mathcal{H}_K$ and by lemma of n.4.1.4 we have

$$\delta(\tilde{t}) \in \mathrm{Col}(\mathcal{K}^*) \Longleftrightarrow \delta(\tilde{t}) \in \mathcal{H}_K \quad \text{and} \quad \frac{1}{p}\log\frac{\sigma\delta}{\delta^p} = \sum_{(c,p)=1} w_c\tilde{t}^c,$$

where all $w_c \in W(k)$.

As earlier, $\gamma : G(m_{k,\tilde{t}}) \longrightarrow G(m_K)$ and $\gamma : \mathcal{H}_K \longrightarrow K^*$ are morphisms of substitution $\tilde{t} \mapsto \pi$.

4.4.2. *Main theorem.*

Assume, that $K$ contains a primitive $p^M$-root of unity.

If $\beta = \beta(\tilde{t}) \in G_{\mathcal{A}}(m_{k,\tilde{t}})$, let

$$\Phi_1(\beta) = \mathcal{V}_{\tilde{t}}^{-1}\begin{pmatrix}\bar{l}_{\mathcal{A}}(\beta) - \frac{\mathcal{A}^*}{p}\bar{l}_{\mathcal{A}}(\beta)\\ 0\end{pmatrix} \in (\mathcal{L}_{k,\tilde{t}})^h ,$$

$$\Phi_2(\beta) = \mathcal{V}_{\tilde{t}}^{-1}\mathcal{E}^{-1}\,\mathrm{d}\begin{pmatrix}\frac{\sigma}{p}\bar{l}_{\mathcal{A}}(\beta)\\ \sigma\bar{m}_{\mathcal{A}}(\beta)\end{pmatrix} \in (\Omega^1_{\mathcal{L}_{k,\tilde{t}}})^h .$$

**Theorem B.** *Let* $\beta \in G_{\mathcal{A}}(m_{k,\tilde{t}}), \delta \in \mathcal{H}_K$, *and* $B(\beta,\delta) = {}^t(B_1,\dots,B_h) =$

$$= (\mathrm{Res}_{\tilde{t}=0}\circ\mathrm{Tr})\left(\Phi_1(\beta)\,\mathrm{d}_{\log}\delta + \frac{1}{p}\log\left(\frac{\sigma\delta}{\delta^p}\right)\Phi_2(\beta)\right) \in \mathbb{Q}_p^h ,$$

*where* $\mathrm{Tr} : W(k)\otimes\mathbb{Q}_p \longrightarrow \mathbb{Q}_p$ *is trace map, and* $\mathrm{Res}_{\tilde{t}=0}$ *is residue. Then* $B(\beta,\delta) \in \mathbb{Z}_p^h$ *and*

$$(\gamma(\beta),\gamma(\delta)]_{G_{\mathcal{A}}} = B_1 o_M^1 + \dots + B_h o_M^h.$$

*Remark.* We can use formal identity from n.4.4.1 to express the right-hand side of the above expression for $B(\beta,\delta)$ in a form, which is very close to Brückner-Vostokov formulae

$$B(\beta,\delta) = {}^t(B_1,\dots,B_h) =$$

$$(\mathrm{Res}\circ\mathrm{Tr})\left\{\mathcal{V}_{\tilde{t}}^{(-1)}\left[\left(\mathrm{id} - \begin{pmatrix}\frac{\mathcal{A}^*}{p}\\ \mathrm{id}\end{pmatrix}\right)\begin{pmatrix}\bar{l}_{\mathcal{A}}(\beta)\\ \bar{m}_{\mathcal{A}}(\beta)\end{pmatrix}\mathrm{d}_{\log}\delta - \frac{1}{p}\log\frac{\sigma\delta}{\delta^p}\begin{pmatrix}\frac{\mathcal{A}^*}{p}\\ \mathrm{id}\end{pmatrix}\mathrm{d}\begin{pmatrix}\bar{l}_{\mathcal{A}}(\beta)\\ \bar{m}_{\mathcal{A}}(\beta)\end{pmatrix}\right]\right\}$$

4.5. *Proof of theorem B.*

Consider the pairing

$$(\ ,\ ]_{k,\tilde{t}} : G_{\mathcal{A}}(m_{k,\tilde{t}}) \times \mathcal{H}_K \longrightarrow \mathbb{Q}_p^h$$

given for $\beta \in G_{\mathcal{A}}(m_{k,\tilde{t}}), \gamma \in \mathcal{H}_K$ by the expression

$$(\beta, \gamma]_{k,\tilde{t}} = (\mathrm{Res}_{\tilde{t}=0} \circ \mathrm{Tr})(\Phi_1(\beta)\, \mathrm{d}_{\log} \delta + \frac{1}{p} \log \frac{\sigma\delta}{\delta^p} \Phi_2(\beta)) \ .$$

This pairing has the following properties.

4.5.1. *For any $\beta \in G_{\mathcal{A}}(m_{k,\tilde{t}}), \delta \in \mathcal{H}_K$, one has $(\beta, \delta]_{k,\tilde{t}} \in \mathbb{Z}_p^h$.*

Indeed, all elements of the matrix $\mathcal{V}_{\tilde{t}}^{-1}$ have principal parts with coefficients from $W(k)$,

$$\bar{l}_{\mathcal{A}}(\beta) - (\mathcal{A}^*/p)\bar{l}_{\mathcal{A}}(\beta) = \Theta_{G_{\mathcal{A}},1}^{-1}(\beta) \in m_{k,\tilde{t}}^n = (\tilde{t}W(k)[[\tilde{t}\ ]])^n,$$

$$\mathrm{d}\begin{pmatrix} \frac{\sigma}{p}\bar{l}_{\mathcal{A}})(\beta) \\ \sigma\bar{m}_{\mathcal{A}}(\beta) \end{pmatrix} \in \left(\Omega^1_{W(k)[[\tilde{t}]]}\right)^h \ ,$$

$\mathrm{d}_{\log}\delta \in \tilde{t}^{-1}W(k)[[\tilde{t}]]\,\mathrm{d}\,\tilde{t}$ and $(1/p)\log(\sigma\delta/\delta^p) \in m_{k,\tilde{t}}$.

4.5.2. *The pairing $(\ ,\ ]_{k,\tilde{t}}$ is $\mathbb{Z}_p$-linear by both arguments.*

This property is obvious from the definition of pairing.

So, this pairing can be considered as homomorphism

$$(\ ,\ ]_{k,\tilde{t}} : G_{\mathcal{A}}(m_{k,\tilde{t}}) \otimes_{\mathbb{Z}_p} \mathcal{H}_K \longrightarrow \mathbb{Z}_p^h.$$

4.5.3. Proof of the following proposition will be given in n.4.6 below.

**Proposition.**

*If $\gamma(\delta) = 1$, then for any $\beta \in G_{\mathcal{A}}(m_{k,\tilde{t}})$ one has $(\beta, \delta]_{k,\tilde{t}} \equiv 0 \bmod p^M$.*

4.5.4. *Theorem B holds for any $\beta \in G_{\mathcal{A}}(m_{k,\tilde{t}})$ and $\delta \in \mathrm{Col}\,\mathcal{K}^* \subset \mathcal{H}_K$.*

By theorem A1 of n.4.2 it is sufficient to prove, that

$$\mathrm{Res}(\frac{1}{p} \log \frac{\sigma\delta}{\delta^p} \Phi_2(\beta)) \in p^M \mathbb{Z}_p^h.$$

**Lemma.** $\mathrm{d}(\mathcal{V}_{\tilde{t}}^{-1}) = p^M \mathcal{W}_{\tilde{t}}\,\mathrm{d}\,\tilde{t}$, *where the matrix $\mathcal{W}_{\tilde{t}}$ has coefficients in*

$$[x_0]^{-2p/(p-1)} \mathcal{G}^0_{p,p/(p-1)} \cap \mathcal{L}_{k,\tilde{t}}.$$

*Proof of lemma.* From definition of $\mathcal{V}_{\tilde{t}}$ it is clear, that all coefficients of $\mathrm{d}(\mathcal{V}_{\tilde{t}})$ belong to $p^M\Omega^1_{W(k)[[\tilde{t}\ ]]}$. By nn. 3.5.5, 3.1.2 and 3.7.1 the matrix $\mathcal{V}_{\tilde{t}}$ has coefficients in

$$\frac{1}{\psi}W(R)[[\psi^{p-1}/p]] + p^M\left(\frac{1}{p\psi}\mathcal{R}\right) \subset [x_0]^{-p/(p-1)}\mathcal{G}^0_{p,1} + p^M[x_0]^{-2p/(p-1)}\mathcal{G}^0_{p,1}.$$

Then the equality

$$\mathrm{d}(\mathcal{V}_{\tilde{t}}^{-1}) = -\mathcal{V}_{\tilde{t}}^{-1}\,\mathrm{d}(\mathcal{V}_{\tilde{t}})\mathcal{V}_{\tilde{t}}^{-1}$$

gives $\mathrm{d}(\mathcal{V}_{\tilde{t}}^{-1}) = p^M \mathcal{W}_{\tilde{t}}\,\mathrm{d}\,\tilde{t}$, where $\mathcal{W}_{\tilde{t}}$ has coefficients in

$$[x_0]^{-2p/(p-1)}\mathcal{G}_{p,1}^0\left[p[x_0]^{-p/(p-1)}\right] \subset [x_0]^{-2p/(p-1)}\mathcal{G}_{p,p/(p-1)}^0.$$

Lemma is proved.

Now from lemma of n.3.7.2 it follows, that principal parts of elements of $\mathcal{W}_{\tilde{t}}$ have $p$-integral coefficients. Therefore, if $\mathcal{V}_{\tilde{t}}^{-1} = ((v_{ij}(\tilde{t})\ ))_{1\le i,j\le h}$, and $v_{ij}(\tilde{t}) = \sum\limits_{u\in\mathbb{Z}} w_{uij}\tilde{t}^u$, where all $w_{uij} \in W_{\mathbb{Q}_p}(k)$, then $uw_{uij} \in p^M W(k)$ if $u < 0$. In particular, if $u < 0, (u,p) = 1$, then $w_{uij} \in p^M W(k)$.

Now remark, that there exists vector power series $F(\tilde{t}) \in (W(k)[[\tilde{t}\ ]])^h$, such that

$$\mathrm{d}\begin{pmatrix} (\sigma/p)\bar{l}_{\mathcal{A}}(\beta) \\ \sigma\bar{m}_{\mathcal{A}}(\beta)\end{pmatrix} = F(\tilde{t}^p)\frac{\mathrm{d}\,\tilde{t}}{\tilde{t}}\ ,$$

and $\delta \in \operatorname{Col}\mathcal{K} \subset \mathcal{H}_K$ implies

$$\frac{1}{p}\log\frac{\sigma\delta}{\delta^p} = \sum_{(c,p)=1}\alpha_c\tilde{t}^c\ ,$$

where $\alpha_c \in W(k)$, c.f. n.4.1.4.

Therefore, the expressions for coordinates of the residue of

$$\frac{1}{p}\log\frac{\sigma\delta}{\delta^p}\mathcal{V}_{\tilde{t}}^{-1}F(\tilde{t}^p)\frac{\mathrm{d}\,\tilde{t}}{\tilde{t}}$$

are linear combinations of $w_{uij}, (u,p) = 1$, with coefficients from $W(k)$. But all such $w_{uij} \in p^M W(k)$.

4.5.5. *Theorem B holds for* $\delta = E_M(\tilde{t}) \in 1 + m_{k,\tilde{t}}$ *from proposition* 4.3.2 *and any* $\beta \in G_{\mathcal{A}}(m_{k,\tilde{t}})$.

By proposition 4.3.3 it is sufficient to check up, that $B(\beta,\delta) \in p^M\mathbb{Z}_p^h$. By corollary of n.4.3.2 $\mathrm{d}_{\log}\delta \in p^M\Omega^1_{W(k)[[\tilde{t}\ ]]}$, principal parts of coordinates of $\Phi_1(\beta)$ have $p$-integral coefficients. So, $\operatorname{Res}(\Phi_1(\beta)\,\mathrm{d}_{\log}\delta) \in p^M\mathbb{Z}_p^h$.

From construction of $E_M(\tilde{t})$ it follows, that $(1/p)\log(\sigma\delta/\delta^p) = \delta_0 s_M(\tilde{t})$, c.f. n.4.3.1.

If $s_M(\tilde{t}) = \psi + p^M w_0$, then $w_0 = tw_1 \in W^1(m_R) = tW(m_R)$ and corollary of n.3.5.5 gives

$$s_M(\tilde{t})\mathcal{V}_{\tilde{t}}^{-1} \in (1 + p^M\frac{tw_1}{\psi})W(R)[[\psi^{p-1}/p]] + p^M(1 + p^M\frac{tw_1}{\psi})\frac{1}{p}\mathcal{R} \subset$$
$$\subset \mathcal{G}_{p,0}^0 + p^M[x_0]^{-p/(p-1)}\mathcal{G}_{p,1}^0.$$

Therefore, principal parts of coefficients of $s_M(\tilde{t})\mathcal{V}_{\tilde{t}}^{-1}$ have coefficients in $p^M W(k)$. This gives

$$\operatorname{Res}(\delta_0 s_M(\tilde{t})\Phi_2(\beta)) \equiv 0 \bmod p^M.$$

Clearly, the above properties 4.5.1-4.5.5 give the proof of theorem B.

4.6. *Proof of proposition 4.5.3.*

We can calculate in $\mathcal{G}_{(p-1)+,p-1}\otimes_{\mathcal{L}_{k,\tilde{t}}}\Omega^1_{\mathcal{L}_{k,\tilde{t}}}\supset\Omega^1_{\mathcal{L}_{k,\tilde{t}}}$.

4.6.1. $\gamma(\delta)=1$ implies existence of $\log\delta\in\mathcal{G}^0_{p,0}\cap\mathcal{L}_{k,\tilde{t}}$. Now we have

$$\mathrm{Res}(\Phi_1(\beta)\,\mathrm{d}_{\log}\delta)=-\,\mathrm{Res}(\log\delta\,\mathrm{d}\,\Phi_1(\beta)).$$

Lemma of n.4.5.4 gives

$$\mathrm{d}\,\Phi_1(\beta)\equiv\mathcal{V}_{\tilde{t}}^{-1}\,\mathrm{d}\begin{pmatrix}\bar{l}_{\mathcal{A}}(\beta)-\frac{\mathcal{A}^*}{p}\bar{l}_{\mathcal{A}}(\beta)\\ 0\end{pmatrix}\bmod p^M[x_0]^{-2p/(p-1)}\mathcal{G}^0_{p,p/(p-1)}\,\mathrm{d}\,\tilde{t}.$$

Now we can apply lemma of n.3.7.2 and the formal identity from the beginning of n.4.4 to obtain the following equivalence

$$\mathrm{Res}(\Phi_1(\beta)\,\mathrm{d}_{\log}\delta)\equiv-\,\mathrm{Res}(\log\delta\ \mathcal{V}_{\tilde{t}}^{-1}\,\mathrm{d}\begin{pmatrix}\bar{l}_{\mathcal{A}}(\beta)\\ \bar{m}_{\mathcal{A}}(\beta)\end{pmatrix}+$$

$$+\,\mathrm{Res}(\log\delta\ \mathcal{V}_{\tilde{t}}^{-1}\mathcal{E}^{-1}\,\mathrm{d}\begin{pmatrix}\frac{\sigma}{p}\bar{l}_{\mathcal{A}}(\beta)\\ \sigma\bar{m}_{\mathcal{A}}(\beta)\end{pmatrix}\bmod p^M.$$

4.6.2. $\delta\in 1+m_{k,\tilde{t}}$ implies $(1/p)\log(\sigma\delta/\delta^p)=(\sigma/p)\log\delta-\log\delta$. Now we can use the last equivalence of n.4.6.1 to write

$$\mathrm{Res}(\Phi_1(\beta)\,\mathrm{d}_{\log}\delta+\frac{1}{p}\log\frac{\sigma\delta}{\delta^p}\Phi_2(\beta))\equiv$$

$$\equiv-\,\mathrm{Res}(\log\delta\ \mathcal{V}_{\tilde{t}}^{-1}\,\mathrm{d}\begin{pmatrix}\bar{l}_{\mathcal{A}}(\beta)\\ \bar{m}_{\mathcal{A}}(\beta)\end{pmatrix})+\mathrm{Res}\left(\frac{\sigma}{p}\log\delta\ \Phi_2(\beta)\right)\bmod p^M.$$

4.6.3. Let $\omega(\beta)=\mathcal{V}_{\tilde{t}}^{-1}\,\mathrm{d}\begin{pmatrix}\bar{l}_{\mathcal{A}}(\beta)\\ \bar{m}_{\mathcal{A}}(\beta)\end{pmatrix}$.

**Lemma.**

a) $\sigma_{\mathcal{G}}(\log\delta\ \omega(\beta))\in\Omega^1_{\mathcal{L}_{k,\tilde{t}}}$ *and, therefore,*

$$\sigma_{\mathcal{G}}(\log\delta\ \omega(\beta))=\sigma(\log\delta\ \omega(\beta));$$

b) $(\sigma/p)(\log\delta\ \omega(\beta))-(\sigma/p)(\log\delta)\Phi_2(\beta)=p^MY\,\mathrm{d}\,\tilde{t}$, *where* $Y$ *has coordinates in* $\mathcal{L}_{k,\tilde{t}}\cap[x_0]^{-2p/(p-1)}\mathcal{G}_{p,p/(p-1)}$.

*Remark.* Generally, $\sigma_{\mathcal{G}}\omega(\beta)$ is not defined in $\Omega^1_{\mathcal{L}_{k,\tilde{t}}}$.

*Proof.*

a) We can write, c.f. nn.3.5.4 and 3.5.5,

$$\log\delta\ \omega(\beta)=\frac{\log\delta}{t^+}{}^t\mathcal{V}^D(E_h+p^M\Delta_1)\,\mathrm{d}\begin{pmatrix}\bar{l}_{\mathcal{A}}(\beta)\\ \bar{m}_{\mathcal{A}}(\beta)\end{pmatrix}.$$

The matrix $\Delta_1$ has coefficients in $(t/\psi)W(R)[[pt/\psi]] + \psi^{p-2}/pW(R)[[\psi^{p-1}/p]]$, c.f. n.3.5.4. By construction of $t$, c.f. n.1.7.2, $t/\psi = 1/(\sigma^{-1}\psi)$, therefore, $\sigma_{\mathcal{G}}(t/\psi) = 1/\psi \in [x_0]^{-p/(p-1)}\mathcal{G}^0_{p,1}$.

This gives

$$\sigma_{\mathcal{G}}\left( (t/\psi)W(R)[[pt/\psi]]\right) \subset [x_0]^{-p/(p-1)}\mathcal{G}^0_{p,1}\left[\, p[x_0]^{-p/(p-1)}\,\right] \subset [x_0]^{-p/(p-1)}\mathcal{G}^0_{p,p/(p-1)}.$$

Clearly,

$$\sigma\left(\psi^{p-2}/pW(R)[[\psi^{p-1}/p]]\right) \subset \psi^{p-2}/pW(R)[[\psi^{p-1}/p]] \subset [x_0]^{-p/(p-1)}\mathcal{G}^0_{p,1}.$$

So, $\sigma_{\mathcal{G}}\Delta_1$ has coefficients in $[x_0]^{-p/(p-1)}\mathcal{G}^0_{p,p/(p-1)}$ and $\sigma_{\mathcal{G}}(\Delta_1) = \sigma\Delta_1$.

$\delta \in 1 + m^1_{k,\tilde{t}}$ implies

$$\frac{\log\delta}{t^+} \in \frac{t}{\psi}W(R) + \frac{\psi^{p-2}}{p}W(R)[[\psi^{p-1}/p]].$$

By above arguments

$$\sigma_{\mathcal{G}}\left(\frac{\log\delta}{t^+}\right) \in \frac{1}{\psi}W(R) + \frac{\psi^{p-2}}{p}W(R)[[\psi^{p-1}/p]] \subset [x_0]^{-p/(p-1)}\mathcal{G}^0_{p,1}.$$

Remark, this gives

$$\sigma_{\mathcal{G}}\left(\frac{\log\delta}{t^+}\right) = \sigma\left(\frac{\log\delta}{t^+}\right) = \frac{\sigma}{p}(\log\delta)\frac{1}{t^+},$$

because $\sigma(\log\delta) = \sigma\left(\frac{\log\delta}{t^+}\right)\sigma t^+$ and $\sigma t^+ = pt^+$.

Clearly, $\sigma_{\mathcal{G}}({}^t\mathcal{V}^D), \sigma_{\mathcal{G}}(\bar{l}_{\mathcal{A}}(\beta)), \sigma_{\mathcal{G}}(\bar{m}_{\mathcal{A}}(\beta))$ have coordinates in $\mathcal{G}^0_{p,0}$. So, $\sigma_{\mathcal{G}}(\log\delta\,\omega(\beta))$ has coordinates in

$$\left([x_0]^{-2p/(p-1)}\mathcal{G}^0_{p,p/(p-1)} \cap \mathcal{G}_{k,\tilde{t}}\right)\mathrm{d}\,\tilde{t} \subset \Omega^1_{\mathcal{L}_{k,\tilde{t}}}\ .$$

b) Calculations of n.a) give

$$\sigma(\log\delta\ \omega(\beta)) = (\frac{\sigma}{p}\log\delta)\frac{1}{t^+}{}^t(\sigma\mathcal{V}^D)(E_h + p^M\sigma\Delta_1)\,\mathrm{d}\begin{pmatrix}\sigma\bar{l}_{\mathcal{A}}(\beta)\\ \sigma\bar{m}_{\mathcal{A}}(\beta)\end{pmatrix}.$$

Consider vector differential form

$$\omega_1 = \left(\frac{\sigma}{p}\log\delta\right)\frac{1}{t^+}{}^t(\sigma\mathcal{V}^D)(\sigma\Delta_1)\,\mathrm{d}\begin{pmatrix}\sigma\bar{l}_{\mathcal{A}}(\beta)\\ \sigma\bar{m}_{\mathcal{A}}(\beta)\end{pmatrix}.$$

Then estimates of n.a) give $\omega_1 = pY_1\,\mathrm{d}\,\tilde{t}$, where $Y_1 \in [x_0]^{-2p/(p-1)}\left(\mathcal{G}^0_{p,p/(p-1)}\right)^h$, because $\mathrm{d}\begin{pmatrix}\sigma\bar{l}_{\mathcal{A}}(\beta)\\ \sigma\bar{m}_{\mathcal{A}}(\beta)\end{pmatrix} \in \left(p\Omega^1_{W(k)[[\tilde{t}]]}\right)^n$.

If $\Delta$ is the matrix from n.3.5.3, then by the same reasons

$$\omega_2 = \left(\frac{\sigma}{p}\log\delta\right)\Delta\frac{1}{t^+}{}^t(\sigma\mathcal{V}^D)\,\mathrm{d}\begin{pmatrix}\sigma\bar{l}_{\mathcal{A}}(\beta)\\ \sigma\bar{m}_{\mathcal{A}}(\beta)\end{pmatrix} = pY_2\,\mathrm{d}\,\tilde{t},$$

where $Y_2$ has coordinates in $[x_0]^{-2p/(p-1)}\mathcal{G}^0_{p,1}$.

Therefore, $\sigma(\log\delta\ \omega(\beta)) =$

$$= (\frac{\sigma}{p}\log\delta)(E_h - p^M\Delta)\frac{1}{t^+}{}^t(\sigma\mathcal{V}^D)\,\mathrm{d}\begin{pmatrix}\sigma\bar{l}_{\mathcal{A}}(\beta)\\ \sigma\bar{m}_{\mathcal{A}}(\beta)\end{pmatrix} + p^{M+1}(Y_1 - Y_2)\,\mathrm{d}\,\tilde{t}.$$

Now we use properties of the matrix $\mathcal{V}^D$ from n.1.8.

$$\frac{1}{p}{}^t(\sigma\mathcal{V}^D)\,\mathrm{d}\begin{pmatrix}\sigma\bar{l}_{\mathcal{A}}(\beta)\\ \sigma\bar{m}_{\mathcal{A}}(\beta)\end{pmatrix} = {}^t\{\begin{pmatrix}\sigma & \\ & \sigma/p\end{pmatrix}\mathcal{V}^D\}\,\mathrm{d}\begin{pmatrix}\sigma/p\bar{l}_{\mathcal{A}}(\beta)\\ \sigma\bar{m}_{\mathcal{A}}(\beta)\end{pmatrix} =$$

$$= {}^t\mathcal{V}^D\mathcal{E}^{-1}\,\mathrm{d}\begin{pmatrix}\sigma/p\bar{l}_{\mathcal{A}}(\beta)\\ \sigma\bar{m}_{\mathcal{A}}(\beta)\end{pmatrix} = t^+\mathcal{V}^{-1}\mathcal{E}^{-1}\,\mathrm{d}\begin{pmatrix}\sigma/p\bar{l}_{\mathcal{A}}(\beta)\\ \sigma\bar{m}_{\mathcal{A}}(\beta)\end{pmatrix}.$$

From definition of the matrix $\Delta$ it follows, that $(E_h - p^M\Delta)\mathcal{V}^{-1} = \mathcal{V}_{\tilde{t}}^{-1}$, so, we obtained

$$\sigma(\log\delta\ \omega(\beta)) = \sigma(\log\delta)\mathcal{V}_{\tilde{t}}^{-1}\mathcal{E}^{-1}\,\mathrm{d}\begin{pmatrix}\sigma/p\bar{l}_{\mathcal{A}}(\beta)\\ \sigma\bar{m}_{\mathcal{A}}(\beta)\end{pmatrix} + p^{M+1}Y\,\mathrm{d}\,\tilde{t},$$

where $Y \in [x_0]^{-2p/(p-1)}\mathcal{G}^0_{p,p/(p-1)} \cap \mathcal{L}_{k,\tilde{t}}$.

Lemma is proved.

4.6.4. The vector $Y$ from the above lemma has $p$-integral principal part, therefore,

$$\mathrm{Res}(\frac{\sigma}{p}\log\delta\ \Phi_2(\beta)) \equiv \mathrm{Res}\left(\frac{\sigma}{p}(\log\delta\ \omega(\beta))\right) = \sigma\,\mathrm{Res}(\log\delta\ \omega(\beta))\bmod p^M.$$

So, the equivalence of n.4.6.2 gives

$$(\beta,\delta]_{k,\tilde{t}} \equiv -\mathrm{Tr}(\mathrm{Res}(\log\delta\ \omega(\beta))) + \mathrm{Tr}(\sigma\,\mathrm{Res}(\log\delta\ \omega(\beta))) = 0\bmod p^M.$$

Proposition is proved.

## References


[Ab1] V.A. Abrashkin, *Ramification in etale cohomology*, Invent. Math. **101** (1990), 631-640.

[Ab2] V.A. Abrashkin, *Ramification filtration of the Galois group of a local field II*, Proceeding of Steklov Math. Inst. **208** (1995).

[Ab3] V.A. Abrashkin, *A remark about Brückner-Vostokov explicit reciprocity law*, Preprint MPI/ 94-61, Bonn.

[A-H] E. Artin, H. Hasse, *Die beiden Ergänzungssätze zum Reziprozitätsgesetz der $l^n$-ten Einheitzwurzeln*, Abh. Math. Sem. Univ. Hamburg **6** (1928), 46-162.

[B-V] D.G. Benua (Benois), S.V. Vostokov, *The norm pairing in formal groups, and Galois representations*, Algebra i analiz **2** (1991); Engl. transl. in Leningrad Math. J. **2** (1991), 1221-1249.

[Br] H. Brückner, *Eine explizite Formel zum Reziprozitätsgesetz für Primzahlexponenten p*, In: Hasse, Roquette, Algebraische Zahlentheorie, Bericht einer Tagung des Math. Inst. Obervolfach 1964. Bibliographisches Institut, Manheim 1966.

[Co] P. Colmez, *Periodes p-adiques des varietes abeliennes*, Math. Ann. **292** (1992), 629-644.

[De] J.-M. Decauwert, *Classification des A-modules formels*, C.R. Acad. Sci. Paris, Serie A **282**, 1413-1416.

[Fo1] J.-M. Fontaine, *Groupes p-divisibles sur les corps locaux*, Asterisque **47-48** (1977), Paris.

[Fo2] J.-M. Fontaine, *Modules galoisiens, modules filtres et anneaux de Barsotti-Tate*, in Journee de Geometrie Algebrique de Rennes, Soc. Math. de France, vol. 65, Paris, 1979, p. 3-80.

[Fo3] J.-M. Fontaine, *Sur certains types de representations p-adiques du group de Galois d'un corps local; construction d'un anneau de Barsotti-Tate*, Ann. Math. **115** (1982), 529-577.

[Fo4] J.-M. Fontaine, *Representations p-adiques des corps locaux (1-ere partie)*, in The Grothendieck Festschrift, A Collection of Articles Written in Honor of the 60th Birthday of Alexander Grothendieck, vol. II, p. 249-309.

[F-L] J.-M. Fontaine, G. Laffaille, *Construction de representations p-adiques*, Ann. Sci. E.N.S. 4 serie **15** (1982), 547-608.

[Ha] M.Hazewinkel, *Formal groups and applications*, New York: Academic Press, 1978.

[Ka] K. Kato, *The explicit reciprocity law and the cohomology of Fontaine-Messing*, Bull. Soc. math. France **119** (1991), 397-441.

[Ko] V.A. Kolyvagin, *Formal groups and the norm residue symbol*, Izv. Akad. Nauk SSSR. Ser. Mat. **43** (1979), 1054-1120; Engl. transl. in Math. USSR Izv. **15** (1980), 79-138.

[La] F. Laubie, *Extensions de Lie et groupes d'automorphismes de corps locaux*, Compos. Math. **67** (1988), 165-189.

[Sh] I.R. Shafarevich, *A general reciprocity law*, Mat. Sb. **26(68)** (1950), 113-146.

[Vo1] S.V. Vostokov, *Explicit form on the law of reciprocity*, Izv. Akad. Nauk SSSR. Ser. Mat. **42** (1978), 1288-1321; Engl. transl. in Math. USSR Izvestija **13** (1979), 557-588.

[Vo2] S.V. Vostokov, *A norm pairing in formal modules*, Izv. Akad. Nauk SSSR. Ser. Mat. **42** (1979), 765-794; Engl. transl. in Math. USSR Izv. **15** (1980), 25-52.

[Wtb] J.-P. Wintenberger, *Le corps des normes de certaines extensions infinies des corps locaux; application*, Ann. Sci. Ec. Norm. Super., IV. Ser **16** (1983), 59-89.