THE FANO SURFACE OF THE KLEIN CUBIC THREEFOLD

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Abstract. We prove that the Klein cubic threefold F is the only one cubic threefold which has an order 11 automorphism. We calculate the period lattice of the intermediate Jacobian of F and study its Fano surface S. We compute the set of fibrations of S on a curve of positive genus and the intersection between the fibres of these fibrations. These fibres generate an index 2 sub-group of the Néron-Severi group and we obtain the generators of this group. The Néron-Severi group of S has rank $25 = h^{1,1}$ and discriminant 11^{10}.

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0.1. **Introduction.** Let $F \hookrightarrow \mathbb{P}^4$ be a smooth cubic threefold. Its intermediate Jacobian

$$J(F) := H^{2,1}(F, \mathbb{C})^*/H_3(F, \mathbb{Z})$$

is a principally polarised Abelian variety $(J(F), \Theta)$ of dimension 5 that has the role in the analysis of curves on F similar to the role of the Jacobian variety in the study of divisors on a curve.
The Hilbert scheme of lines on F is a smooth surface S called the Fano surface of F; the Abel-Jacobi map $\vartheta : S \to J(F)$ is an embedding that induce an isomorphism Alb(S) $\to$ J(F) where

$$\mathrm{Alb(S)} := \mathrm{H^o(\Omega_S)^*/H_1(S, \mathbb{Z})}$$

is the Albanese variety of S and $H^o(\Omega_S) := H^o(S, \Omega_S)$ ([5] 0.6, 0.8).

The cotangent bundle theorem ([5] Lemma 12.5) ables us to recover the cubic F if we know only its Fano surface and moreover it gives a natural isomorphism beetwen the spaces $H^o(\Omega_S)$ and $H^o(F, \mathcal{O}_F(1)) = H^o(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(1))$. As we mainly work with the Fano surface, we will identify the homogenous coordinates $x_1, .., x_5$ of $\mathbb{P}^4$ with elements of $H^o(\Omega_S)$. We will also identify the Abelian variety $J(F)$ with Alb(S).

In [10], we give the classification of elliptic curve configurations on a Fano surface. It is proved that this classification is equivalent to the classification of the automorphism sub-groups of S that are generated by certain involutions. Moreover, it is also proved that the automorphism groups Aut(F) and Aut(S) of a cubic and its Fano surface are isomorphic. In the present paper, we pursue the study of these groups. By [10] Corollary 1.19, the order of Aut(S) divides

$$11.7.5^2 3^6 2^{23}.$$

This legitimates the study of the Fano surfaces which have an order 11 automorphism. A. Adler [1] has proved that automorphism group of the Klein cubic:

$$F:\, x_1x_5^2+x_5x_3^2+x_3x_4^2+x_4x_2^2+x_2x_1^2=0$$

is isomorphic to $PSL_2(\mathbb{F}_{11})$. We prove in the present paper that:

Proposition 0.1. *The Klein cubic is the only one smooth cubic threefold which has an order* 11 *automorphism.*

If a curve admits a sufficiently large group of automorphisms, Bolza has found a method to compute a period matrix of its Jacobian (see [4], 11.7). As for the case of curves, we will use the fact that the Klein cubic F admits a large group of automorphisms to compute the period lattice of its intermediate Jacobian $J(F)$ or, what is the same thing, the period lattice $H_1(S,\mathbb{Z})\subset H^o(\Omega_S)^*$ of the two dimensional variety S.

Let $\nu=\frac{-1+\sqrt{-11}}{2}$ and let $\mathbb{E}$ be the elliptic curve $\mathbb{E}=\mathbb{C}/\mathbb{Z}[\nu]$. Let us denote by $e_1,..,e_5\in H^o(\Omega_S)^*$ the dual basis of $x_1,..,x_5$. Let ξ be a primitive 11-th root of unity and let $v_i\in H^o(\Omega_S)^*$ be :

$$v_i=\xi^i e_1+\xi^{9i}e_2+\xi^{3i}e_3+\xi^{4i}e_4+\xi^{5i}e_5.$$

For s a point of the Hilbert sheme S, we denote by C_s the divisor on S that parametrizes the lines on F that cut the line corresponding to the point s.

Theorem 0.2. *The period lattice $H_1(S,\mathbb{Z})\subset H^o(\Omega_S)^*$ is equal to:*

$$\frac{\mathbb{Z}[\nu]}{1+2\nu}(v_0-3v_1+3v_2-v_3)+\frac{\mathbb{Z}[\nu]}{1+2\nu}(v_1-3v_2+3v_3-v_4)+\bigoplus_{k=0}^{2}\mathbb{Z}[\nu]v_k.$$

The Néron-Severi group NS(S) *of S has rank $25=h^{1,1}(S)$ and discriminant 11^{10}. The image of the morphism*

$$\vartheta^*:\mathrm{NS(Alb(S))}\to\mathrm{NS(S)}$$

is sub-lattice of index 2 *and* NS(S) *is generated by this lattice and the class of the incidence divisor C_s.*
The set of numerical classes of fibres of fibrations of S in a curve of positive genus is in natural bijection with $\mathbb{P}^4_{\mathbb{Z}}(\mathbb{Z}[\nu])$ and generate ϑ^*NS(Alb(S)).

We remark that $J(F)\simeq$ Alb(S) is isomorphic to $\mathbb{E}^5$ but by [5] (0.12), this isomorphism is not an isomorphism of principally polarised Abelian varieties. The fact that $J(F)$ is isomorphic to $\mathbb{E}^5$ is mentioned in [2].

The main properties used to prove Theorem 0.2 are the fact that the class of $S\hookrightarrow$ Alb(S) is equal to $\frac{1}{3!}\Theta^3$ and the fact that the action of the group Aut(S) on Alb(S) preserves the polarisation Θ.

To close this introduction, let us mention two known facts: (1) as the plane Klein quartic, the Klein cubic threefold has a modular interpretation see [7], (2) the cotangent sheaf of its Fano surface is ample [10].

0.2. **Properties of the Fano surfaces.** Let us recall some facts proved in [10] and fix the notations:
An automorphism f of F preserves the lines on F and induces an automorphism $\rho(f)$ of the Fano surface S.
An automorphism σ of S induces an automorphism σ' of the Albanese variety of S such that the following diagram:

$$\begin{array}{ccc} S & \stackrel{\vartheta}{\to} & \mathrm{Alb(S)} \\ \downarrow\sigma & & \downarrow\sigma' \\ S & \stackrel{\vartheta}{\to} & \mathrm{Alb(S)} \end{array}$$

is commutative (where $\vartheta : S \to \mathrm{Alb(S)}$ is a fixed Albanese morphism). We denote by $M_\sigma \in GL(H^o(\Omega_S)^*)$ the linear part of the differential of σ' and we denote by

$$q : GL(H^o(\Omega_S)^*) \to PGL(H^o(\Omega_S)^*)$$

the natural quotient map. We have ([10], Theorem 1.15):

Theorem 0.3. *A) The morphism $q(M_\sigma)$ is an automorphism of $F \hookrightarrow \mathbb{P}(H^o(\Omega_S)^*)$.*
B) The morphisms $\rho : \mathrm{Aut(F)} \to \mathrm{Aut(S)}$ and $\sigma \to q(M_\sigma)$ are reciprocal isomorphisms.
C) For all $\sigma \in \mathrm{Aut(S)}$, the automorphism σ' is an automorphism of the principally polarised Abelian variety $(\mathrm{Alb(S)}, \Theta)$.

0.3. **The unique cubic with an order 11 automorphism.** Let us prove that the Klein cubic is the only one that possesses an order 11 automorphism.

Suppose that a cubic threefold F has an order 11 automorphism f. Let $\tau = \rho(f)$ be the induced automorphism of the Fano surface S. The Proposition 13.2.5 and the Theorem 13.2.8 of [4] imply that the eigenvalues of M_τ are 5 pairewise non-complex conjugate 11-th primitive root of unity.
We denote by $\mathcal{O}$ the set of sets of 5 pairwise non-complex conjugate 11-th primitive root of unity : $\mathcal{O}$ contains 2^5 elements. The group $\mathrm{Aut}(\mathbb{C})$ of automorphisms of $\mathbb{C}$ acts on $\mathcal{O}$.
Let ξ be a 11-th primitive root of unity. For $i \in \{0, 1, ..10\}$, we denote by χ_i the 1 dimensional representation:

$$x \to \xi^i x \in \mathbb{C}.$$

Let us suppose that $\{\xi, \xi^9, \xi^3, \xi^4, \xi^5\} \in \mathcal{O}$ is the set of eigenvalues of M_τ. The third symmetric power of the dimension 5 representation:

$$(x_1, x_2, x_3, x_4, x_5) \to (\xi x_1, \xi^9 x_2, \xi^3 x_3, \xi^4 x_4, \xi^5 x_5)$$

is decomposed in the following direct sum:

$$H^o(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) = 5\chi_0+2\chi_1+3\chi_2+3\chi_3+4\chi_4+3\chi_5+3\chi_6+3\chi_7+3\chi_8+3\chi_9+3\chi_{10}.$$

The space $5\chi_0$ is generated by:

$$x_1x_5^2,\ x_5x_3^2,\ x_3x_4^2,\ x_4x_2^2,\ x_2x_1^2$$

By an appropriate variable change, we see that any smooth cubic of this space is isomorphic to the Klein cubic:

$$x_1x_5^2 + x_5x_3^2 + x_3x_4^2 + x_4x_2^2 + x_2x_1^2 = 0.$$

The other stable spaces define singular cubic threefolds.
The orbit $\mathcal{O}_0 \subset \mathcal{O}$ of $\{\xi, \xi^9, \xi^3, \xi^4, \xi^5\} \in \mathcal{O}$ by $\mathrm{Aut}(\mathbb{C})$ is:

$$\mathcal{O}_0 = \{\{\xi, \xi^9, \xi^3, \xi^4, \xi^5\}, \{\xi^{10}, \xi^2, \xi^8, \xi^7, \xi^6\}\}.$$

Let us study the representation $\chi_1+\chi_2+\chi_3+\chi_4+\chi_5$. Its third symmetric power is:

$$H^o(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) = 4\chi_0+3\chi_1+2\chi_2+3\chi_3+2\chi_4+\chi_5+3\chi_6+4\chi_7+4\chi_8+5\chi_9+4\chi_{10}.$$

A basis of $5\chi_9$ is :

$$x_2x_3x_4,\ x_1x_3x_5,\ x_5x_2^2,\ x_1x_4^2,\ x_3^3$$

and any cubic of this space is singular at the point: $(1:0:0:0:0)$. As we can verify, the other stable spaces give also singular threefolds.
Hence there is no Fano surface with an order 11 automorphism τ such that the eigenvalues of M_τ is the set $\{\xi, \xi^2, \xi^3, \xi^4, \xi^5\}$.
The orbit $\mathcal{O}_1 \subset \mathcal{O}$ of the element $\{\xi, \xi^2, \xi^3, \xi^4, \xi^5\} \in \mathcal{O}$ by $\mathrm{Aut}(\mathbb{C})$ has order 10. Hence we have studied 10 representations and no one gives a smooth cubic threefold.

The set $\{\xi, \xi^2, \xi^3, \xi^4, \xi^6\} \in \mathcal{O}$ is not an element of the orbits $\mathcal{O}_0$ and $\mathcal{O}_1$. The third symmmetic power of the representation:

$$(x_1, x_2, x_3, x_4, x_5) \to (\xi x_1, \xi^2 x_2, \xi^3 x_3, \xi^4 x_4, \xi^6 x_5)$$

is decomposed in:

$$H^o(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) = 3\chi_0+3\chi_1+2\chi_2+3\chi_3+2\chi_4+3\chi_5+3\chi_6+4\chi_7+4\chi_8+4\chi_9+4\chi_{10}.$$

As we can verify, no element of these 11 spaces gives a smooth cubic threefold. The orbit $\mathcal{O}_2$ of the set $\{\xi, \xi^2, \xi^3, \xi^4, \xi^6\}$ by the action of $\mathrm{Aut}(\mathbb{C})$ has order 10.

The set $\{\xi, \xi^2, \xi^3, \xi^5, \xi^7\} \in \mathcal{O}$ is not an element of the orbits $\mathcal{O}_0$, $\mathcal{O}_1$ and $\mathcal{O}_2$. The third symmmetic power of the representation:

$$(x_1, x_2, x_3, x_4, x_5) \to (\xi x_1, \xi^2 x_2, \xi^3 x_3, \xi^5 x_4, \xi^7 x_5)$$

is decomposed in:

$$H^o(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3)) = 4\chi_0+2\chi_1+3\chi_2+2\chi_3+4\chi_4+3\chi_5+4\chi_6+3\chi_7+3\chi_8+4\chi_9+3\chi_{10}.$$

No one elements of these 11 spaces gives a smooth cubic threefold. The orbit $\mathcal{O}_3$ of the set $\{\xi, \xi^2, \xi^3, \xi^5, \xi^7\}$ by the action of $\mathrm{Aut}(\mathbb{C})$ has order 10.

The union of the orbits $\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ is equal to $\mathcal{O}$. This shows that the Klein cubic

$$x_1x_5^2 + x_5x_3^2 + x_3x_4^2 + x_4x_2^2 + x_2x_1^2 = 0$$

is (up to isomorphism) the only one smooth cubic on which $\mathbb{Z}/11\mathbb{Z}$ acts. □

Remark 0.4. By the same method, we can prove that there is no smooth cubic threefold with an order 7 automorphism.

0.4. **The period lattice of the intermediate Jacobian of the Klein cubic.** Let F be the Klein cubic and let S be its Fano surface. Let $\vartheta : S \to$ Alb(S) be a fixed Albanese morphism; it is an embedding. We compute here the period lattice $H_1(S,\mathbb{Z}) \subset H^o(\Omega_S)^*$ of the Albanese variety of S.

The order 5 automorphism:

$$g : (x_1 : x_2 : x_3 : x_4 : x_5) \to (x_5 : x_1 : x_4 : x_2 : x_3)$$

acts on F. Let be $\sigma = \rho(g)$. By Theorem 0.3, there exists a 5-th root of unity θ such that $M_\sigma \in GL(H^o(\Omega_S)^*)$ is equal to:

$$M_\sigma : x \to \theta(x_5, x_1, x_4, x_2, x_3).$$

Since the Klein cubic and g are defined over $\mathbb{Q}$, we have $\theta = 1$.
Moreover, we know that M_τ verifies:

$$M_\tau : x \to (\xi x_1, \xi^9 x_2, \xi^3 x_3, \xi^4 x_4, \xi^5 x_5)$$

where $\tau = \rho(f)$ is defined in paragraph 0.2.

Let q_1 be the endomorphism of Alb(S) defined by the linear part of:

$$\sum_{k=0}^{k=4} (\sigma')^k$$

(where $\sigma' \circ \vartheta = \vartheta \circ \sigma$. Its differential is:

$$dq_1 : x \to (x_1 + x_2 + x_3 + x_4 + x_5)(e_1 + e_2 + e_3 + e_4 + e_5)$$

and its image is an elliptic curve which we denote by $\mathbb{E}$. Let us take $\xi = e^{\frac{2i\pi}{11}}$ where $i^2 = -1$. The restriction of the linear part of $q_1 \circ \tau' :$ Alb(S) $\to \mathbb{E}$ to $\mathbb{E}$ is the multiplication by:

$$\nu = \xi + \xi^9 + \xi^3 + \xi^4 + \xi^5 = \frac{-1 + i\sqrt{11}}{2}.$$

The curve $\mathbb{E}$ has complex multiplication by the principal ideal domain $\mathbb{Z}[\nu]$. Up to a normalization of the basis $e_1, .., e_5$ by a multiplication by a scalar, we can suppose that:

$$H_1(S,\mathbb{Z}) \cap \mathbb{C}v_0 = \mathbb{Z}[\nu]v_0$$

(such normalization preserves the equation of F).
For $i \in \mathbb{Z}/11\mathbb{Z}$, let v_i be the vector:

$$\begin{aligned} v_i &= (M_\tau)^i v_0 \in H^o(\Omega_S)^* \\ &= \xi^i e_1 + \xi^{9i} e_2 + \xi^{3i} e_3 + \xi^{4i} e_4 + \xi^{5i} e_5. \end{aligned}$$

Let $\Lambda_0 \subset H^o(\Omega_S)^*$ be the $\mathbb{Z}$-module generated by the v_i, $i \in \mathbb{Z}/11\mathbb{Z}$. The $\mathbb{Z}$-module Λ_0 is leaved stable by M_τ and $\Lambda_0 \subset H_1(S,\mathbb{Z})$.

Lemma 0.5. *The $\mathbb{Z}$-module Λ_0 is equal to the lattice:*

$$R_0 = \mathbb{Z}[\nu]v_0 + \mathbb{Z}[\nu]v_1 + \mathbb{Z}[\nu]v_2 + \mathbb{Z}[\nu]v_3 + \mathbb{Z}[\nu]v_4.$$

Proof. We have:

$$\nu v_0 = v_1 + v_3 + v_4 + v_5 + v_9,$$

hence νv_0 is an element of Λ_0. This implies that the vectors $\nu v_i = (M_\tau)^i \nu v_0$ are elements of Λ_0 for all i, hence: $R_0 \subset \Lambda_0$.
Reciprocally, we have:

$$v_5 = v_0 + (1+\nu)v_1 - v_2 + v_3 + \nu v_4.$$

This proves that R_0 is leaved stable by M_τ and that the lattice R_0 contains the vectors $v_i = (M_\tau)^i v_0$. Hence we have: $R_0 = \Lambda_0$. □

We need to know the first Chern class $c_1(\Theta)$ of the Θ divisor of Alb(S).

Lemma 0.6. *Let H be the matrix of the Hermitian form of the polarisation Θ in the basis $e_1, .., e_5$. There exists a positive integer such that:*

$$H = a\frac{2}{\sqrt{11}} I_5$$

where I_5 is the size 5 identity matrix.

Proof. The automorphism τ' preserves the polarisation Θ (see [10], Lemma 1.18). By [4] Lemma 2.17, this implies that:

$${}^t M_\tau H \bar{M_\tau} = H$$

where $\bar{M_\tau}$ is the matrix in the basis $e_1, .., e_5$ whose coefficients are conjugated of those of M_τ and where ^{t} is the transposition. The only Hermitian matrices that verify this equality are the diagonal matrices. By the same reasonning with σ instead of τ, we obtain that these diagonal coefficients are equal and:

$$H = a\frac{2}{\sqrt{11}} I_5$$

where a is a positive real (H is a positive definite Hermitian form). As H is a polarisation, the alternating form $c_1(\Theta) = \Im m(H)$ take integer values on $H_1(S,\mathbb{Z})$. But v_1 and v_2 are elements of $H_1(S,\mathbb{Z})$ and:

$$\Im m(H(v_1, v_2)) = -a$$

hence a is an integer. □

Let be $k \in \mathbb{Z}/11\mathbb{Z}$. The differential of the linear part of the morphism $q_1 \circ (\tau')^k$ is:

$$x \to \ell_k(x)(e_1 + .. + e_5)$$

where ℓ_k is the linear form:

$$\ell_k = \xi^k x_1 + \xi^{9k} x_2 + \xi^{3k} x_3 + \xi^{4k} x_4 + \xi^{5k} x_5 \in H^o(\Omega_S).$$

Let be $\lambda \in H_1(S,\mathbb{Z})$. As

$$H_1(S,\mathbb{Z}) \cap \mathbb{C}v_0 = \mathbb{Z}[\nu]v_0,$$

the scalar $\ell_k(\lambda)$ is an element of $\mathbb{Z}[\nu]$. Put:

$$\Lambda_4 = \{u \in \mathbb{C}^5 / \forall k,\, 0 \leq k \leq 4,\, \ell_k(u) \in \mathbb{Z}[\nu]\}.$$

The set Λ_4 contains $H_1(S,\mathbb{Z})$.

Lemma 0.7. *The $\mathbb{Z}$-module Λ_4 is equal to the lattice:*

$$R_1 = \sum_{i=0}^{i=3} \frac{\mathbb{Z}[\nu]}{1+2\nu}(v_i - v_{i+1}) + \mathbb{Z}[\nu]v_0.$$

Moreover M_τ leaves stable Λ_4.

Proof. The element $u = \sum u_i e_i \in H^o(\Omega_S)^*$ is in Λ_4 if and only if

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ \xi & \xi^9 & \xi^3 & \xi^4 & \xi^5 \\ \xi^2 & \xi^7 & \xi^6 & \xi^8 & \xi^{10} \\ \xi^3 & \xi^5 & \xi^9 & \xi & \xi^4 \\ \xi^4 & \xi^3 & \xi & \xi^5 & \xi^9 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix} \in \begin{pmatrix} \mathbb{Z}[\nu] \\ \mathbb{Z}[\nu] \\ \mathbb{Z}[\nu] \\ \mathbb{Z}[\nu] \\ \mathbb{Z}[\nu] \end{pmatrix}.$$

The group Λ_4 is writted in the basis $b = v_0, .., v_4$:

$$\frac{1}{1+2\nu} \begin{pmatrix} -1 & -\nu & 0 & -1 & 1-\nu \\ -\nu & 2 & 0 & -\nu & 3+\nu \\ 0 & 0 & 0 & 1 & -1 \\ -1 & -\nu & 1 & -2 & 1-\nu \\ 1-\nu & 3+\nu & -1 & 1-\nu & 2+2\nu \end{pmatrix} \begin{pmatrix} \mathbb{Z}[\nu] \\ \mathbb{Z}[\nu] \\ \mathbb{Z}[\nu] \\ \mathbb{Z}[\nu] \\ \mathbb{Z}[\nu] \end{pmatrix}$$

and by elementary operations with coefficients in $\mathbb{Z}[\nu]$ on the rows, we obtain that Λ_4 is equal to R_1.
Now, we use the fact that:

$$v_5 = v_0 + (1+\nu)v_1 - v_2 + v_3 + \nu v_4$$

to prove that $\frac{1}{1+2\nu}(v_4 - v_5)$ is an element of Λ_4. As $M_\tau(v_j) = v_{j+1}$, this ables us to conclude that Λ_4 is stable by M_τ. □

We denote by $\phi : \Lambda_4 \to \Lambda_4/\Lambda_0$ the quotient map. The ring $\mathbb{Z}[\nu]/(1+2\nu)$ is the finite field with 11 elements.

Lemma 0.8. *The quotient Λ_4/Λ_0 is a $\mathbb{Z}[\nu]/(1+2\nu)$-vector space with basis:*

$$\begin{array}{ll} t_1 = \frac{1}{1+2\nu}(v_0 - v_1) + \Lambda_0, & t_2 = \frac{1}{1+2\nu}(v_1 - v_2) + \Lambda_0 \\ t_3 = \frac{1}{1+2\nu}(v_2 - v_3) + \Lambda_0, & t_4 = \frac{1}{1+2\nu}(v_3 - v_4) + \Lambda_0. \end{array}$$

Proof. The quotient Λ_4/Λ_0 is an hyperplane of the 5 dimensional $\mathbb{Z}[\nu]/(1+2\nu)$-vector space $\frac{1}{1+2\nu}\Lambda_0/\Lambda_0$. □

Let R be a lattice such that : $\Lambda_0 \subset R \subset \Lambda_4$. The group $\phi(R)$ is a sub-vector space of Λ_4/Λ_0 and:

$$\phi^{-1}\phi(R) = R + \Lambda_0 = R.$$

Because M_τ preserves Λ_0, the morphism M_τ induces a morphism $\widehat{M_\tau}$ on the quotient Λ_4/Λ_0 such that $\phi \circ M_\tau = \widehat{M_\tau} \circ \phi$. As M_τ leaves stable $H_1(S,\mathbb{Z})$, the vector space $\phi(H_1(S,\mathbb{Z}))$ is stable by $\widehat{M_\tau}$. We denote:

$$\begin{array}{rcl} w_1 & = & -t_1 + 3t_2 - 3t_3 + t_4 \\ w_2 & = & t_1 - 2t_2 + t_3 \\ w_3 & = & -t_1 + t_2 \\ w_4 & = & t_1. \end{array}$$

The matrix of $\widehat{M_\tau}$ in the basis $w_1, .., w_4$ of Λ_4/Λ is:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The sub-spaces leaved stable by $\widehat{M_\tau}$ are the space $W_0 = \{0\}$ and the spaces W_i, $1 \leq i \leq 4$ generated by $w_1, .., w_i$. Let Λ_i be the lattice $\phi^{-1}W_i$, then:

Theorem 0.9. *The lattice $H_1(S,\mathbb{Z})$ is equal to Λ_2, and Λ_2 is equal to*

$$R_2 = \frac{\mathbb{Z}[\nu]}{1+2\nu}(v_0 - 3v_1 + 3v_2 - v_3) + \frac{\mathbb{Z}[\nu]}{1+2\nu}(v_1 - 3v_2 + 3v_3 - v_4) + \bigoplus_{k=0}^{2} \mathbb{Z}[\nu]v_k.$$

Moreover, the Hermitian matrix associated to Θ is equal to $\frac{2}{\sqrt{11}}I_5$ in the basis $e_1, .., e_5$.

Proof. Let $c_1(\Theta) = \Im m(H) = i\frac{a}{\sqrt{11}}\sum dx_k \wedge d\bar{x}_k$ be the alternating form of the principal polarisation Θ. Let $\lambda_1, .., \lambda_{10}$ be a basis of a lattice Λ. The Pfaffian $Pf_\Theta(\Lambda)$ of Λ is the determinant of the matrix

$$(c_1(\Theta)(\lambda_j, \lambda_k))_{1 \leq j,k \leq 10}\,.$$

Since Θ is a principal polarisation, we have $Pf_\Theta(H_1(S,\mathbb{Z})) = 1$.
It is easy to find a basis of Λ_j ($j \in \{0, .., 4\}$). For example, the space W_2 is generated by $w_2 = t_1 - 2t_2 + t_3$ and $w_1 + w_2 = t_2 - 2t_3 + t_4$ and as

$$\begin{array}{c} \phi(\frac{1}{1+2\nu}(v_0 - 3v_1 + 3v_2 - v_3)) = w_2 \\ \phi(\frac{1}{1+2\nu}(v_1 - 3v_2 + 3v_3 - v_4)) = w_1 + w_2, \end{array}$$

the lattice R_2 (that contains Λ_0) is equal to Λ_2.
Then, with the help of a computer, we can calculate the Pfaffian P_j of the lattice Λ_j and verify that it is equal to:

$$a^{10}11^{4-2j}$$

where a is the integer of Lemma 0.6. As a is a positive, the only possibility that P_j equals 1 is $j=2$ and $a=1$. $\square$

0.5. **The Néron-Severi Group of the Fano surface of the Klein cubic.** Let us define:

$$u_1 = \tfrac{1}{1+2\nu}(v_0 - 3v_1 + 3v_2 - v_3),\ u_2 = \tfrac{1}{1+2\nu}(v_1 - 3v_2 + 3v_3 - v_4)$$
$$u_3 = v_0,\ u_4 = v_1,\ u_5 = v_2.$$

Let $y_1, ..., y_5 \in H^o(\Omega_S)$ be the linear forms such that:

$$\sum_{k=1}^{k=4} x_k e_k = \sum_{k=1}^{k=4} y_k u_k.$$

Let be k, $1 \leq k \leq 5$. The image of $H_1(S,\mathbb{Z})$ by $y_k \in H^o(\Omega_S)$ is $\mathbb{Z}[\nu]$, and this form is the differential of an Abelian varieties morphism

$$r_k : \mathrm{Alb(S)} \to \mathbb{E} = \mathbb{C}/\mathbb{Z}[\nu]$$

The morphisms $r_1, .., r_5$ form a basis of the $\mathbb{Z}[\nu]$-module $\mathrm{Hom}_{\mathrm{ab\ var}}(\mathrm{Alb(S)}, \mathbb{E})$.

We denote by Λ_A^* the free $\mathbb{Z}[\nu]$-module of rank 5 generated by $y_1, .., y_5$ and for $\ell \in \Lambda_A^*$, we denote by $\Gamma_\ell : \mathrm{Alb(S)} \to \mathbb{E}$ the morphism whose differential is $\ell : H^o(\Omega_S)^* \to \mathbb{C}$.
Let $\vartheta : S \to \mathrm{Alb(S)}$ be a fixed Albanese morphism. We denote by $\gamma_\ell : S \to \mathbb{E}$ the morphism $\gamma_\ell = \Gamma_\ell \circ \vartheta$ and we denote by F_ℓ the numerical equivalence class of a fibre of γ_ℓ (this class is independant of the choice of ϑ).
We define the scalar product of two forms $\ell, \ell' \in \Lambda_A^*$ by:

$$\langle \ell, \ell' \rangle = \sum_{k=1}^{k=5} \ell(e_k)\overline{\ell'(e_k)}$$

and the norm of ℓ by:

$$\|\ell\| = \sqrt{\langle \ell, \ell \rangle}.$$

We denote by NS(X) the Néron-Severi group of a variety X. The aim of this paragraph is to prove the following result:

Theorem 0.10. *Let ℓ, ℓ' be elements of Λ_A^*. The fibre F_ℓ has genus:*

$$g(F_\ell) = 1 + 3\,\|\ell\|^2\,,$$

verifies $F_\ell C_s = 2\,\|\ell\|^2$ and :

$$F_\ell F_{\ell'} = \|\ell\|^2\,\|\ell'\|^2 - \langle \ell, \ell' \rangle \langle \ell', \ell \rangle\,.$$

The image of the morphism $\vartheta^* : \mathrm{NS(Alb(S))} \to \mathrm{NS(S)}$ is a rank 25 sub-lattice of discriminant $2^2 11^{10}$.
The following 25 fibres

$$F_{y_k},\ k \in \{1, .., 5\} \quad F_{y_k + y_l},\ 1 \leq k < l \leq 5$$
$$F_{y_k + \nu y_l},\ 1 \leq k < l \leq 5$$

are a $\mathbb{Z}$-basis of $\vartheta^*\mathrm{NS(Alb(S))}$ and together with the class of the incident divisor C_s ($s \in S$) they generate the Néron-Severi group of S.

We begin by the following lemma:

Lemma 0.11. *The Néron-Severi group of* Alb(S) *is generated by the* 25 *forms:*

$$\frac{i}{\sqrt{11}}dy_k \wedge d\bar{y}_k,\ k \in \{1,..,5\} \qquad \frac{i}{\sqrt{11}}(dy_k \wedge d\bar{y}_l + dy_l \wedge d\bar{y}_k),\ 1 \leq k < l \leq 5$$
$$\frac{i}{\sqrt{11}}(\nu dy_k \wedge d\bar{y}_l + \bar{\nu} dy_l \wedge d\bar{y}_k),\ 1 \leq k < l \leq 5.$$

Proof. The Hermitian form $H' = \frac{2}{\sqrt{11}} I_5$ in the basis $u_1, .., u_k$ defines a principal polarisation of Alb(S). Let $\mathrm{End}^s(\mathrm{Alb(S)})$ be the group of symmetric morphisms for the Rosati involution associated to H'. There exists an isomorphism

$$\phi_{H'} : \mathrm{NS(Alb(S))} \to \mathrm{End}^s(\mathrm{Alb(S)}).$$

The group $\mathrm{End}^s(\mathrm{Alb(S)})$ is easily calculated and we obtain the lemma when we take the inverse morphism of $\phi_{H'}$ (see [4] Proposition 5.2.1). □

The Néron-Severi group of the curve $\mathbb{E} = \mathbb{C}/\mathbb{Z}[\nu]$ is the $\mathbb{Z}$-module generated by the form:

$$\eta = \frac{i}{\sqrt{11}} dz \wedge d\bar{z}.$$

Let $\ell \in \Lambda_A^*$, we have:

$$\Gamma_\ell^* \eta = \frac{i}{\sqrt{11}} d\ell \wedge d\bar{\ell}$$

and this form is the Chern class of the divisor $\Gamma_\ell^* 0$.

Lemma 0.12. *The* 25 *forms:*

$$\eta_k = \Gamma_{y_k}^* \eta,\ k \in \{1,..,5\} \quad \eta_{k,l}^1 = \Gamma_{y_k+y_l}^* \eta,\ 1 \leq k < l \leq 5$$
$$\eta_{k,l}^\nu = \Gamma_{y_k+\nu y_l}^* \eta,\ 1 \leq k < l \leq 5$$

are a basis of the Néron-Severi group of Alb(S).

Proof. Let $1 \leq k \leq 5$ be an integer. The element $\Gamma_{y_k}^* \eta = \frac{i}{\sqrt{11}} dy_k \wedge d\bar{y}_k$ is in the basis of Lemma 0.11. Let $1 \leq l < k \leq 5$ be integers, let be $a \in \{1, \nu\}$, and $\ell = y_k + a y_l$. We have:

$$\Gamma_\ell^* \eta = \frac{i}{\sqrt{11}} (dy_k \wedge d\bar{y}_k + \bar{a} dy_k \wedge d\bar{y}_l + a dy_l \wedge d\bar{y}_k + a\bar{a} dy_l \wedge d\bar{y}_l),$$

this proves, when we take $a = 1$ and next $a = \nu$, that the forms of the basis of Lemma 0.11 are $\mathbb{Z}$-linear combinaisons of the forms $\eta_k, \eta_{k,l}^1, \eta_{k,l}^\nu$, $1 \leq k, l \leq 5$. □

Let us prove the Theorem 0.10.

Proof. As the homology class of S in Alb(S) is equal to $\frac{\Theta^3}{3!}$, the intersection of the fibres F_ℓ and $F_{\ell'}$ is equal to:

$$\int_A \frac{1}{3!} \wedge^3 c_1(\Theta) \wedge \Gamma_\ell^* \eta \wedge \Gamma_{\ell'}^* \eta.$$

Write ℓ in the basis $x_1,..,x_5$: $\ell = a_1x_1+..+a_5x_5$ and $\ell' = b_1x_1+..+b_5x_5$, then:

$$\frac{1}{3!}(\frac{i}{\sqrt{11}})^2 d\ell\wedge d\bar{\ell}\wedge d\ell'\wedge d\overline{\ell'}\wedge(\wedge^3 c_1(\Theta))$$

is equal to:

$$\begin{array}{c}(\frac{i}{\sqrt{11}})^5(\sum a_jx_j)\wedge(\sum \bar{a}_j\bar{x}_j)\wedge(\sum b_jx_j)\wedge(\sum \bar{b}_j\bar{x}_j)\\ \wedge\sum_{h<j<k} dx_h\wedge d\bar{x}_h\wedge dx_j\wedge d\bar{x}_j\wedge dx_k\wedge d\bar{x}_k\end{array}$$

that is equal to:

$$(\sum_{k\neq j} a_k\bar{a}_kb_j\bar{b}_j - a_k\bar{a}_jb_j\bar{b}_k)\frac{1}{5!}\wedge^5 c_1(\Theta).$$

But : $\int_A \frac{1}{5!}\wedge^5 c_1(\Theta) = 1$ because Θ is a principal polarisation of Alb(S), hence:

$$\begin{aligned}F_\ell F_{\ell'} = \int_A \tfrac{1}{3!}\wedge^3 c_1(\Theta)\wedge\Gamma_\ell^*\eta\wedge\Gamma_{\ell'}^*\eta \quad &= \textstyle\sum_{k\neq j} a_k\bar{a}_kb_j\bar{b}_j - a_k\bar{a}_jb_j\bar{b}_k\\ &= \|\ell\|^2\|\ell'\|^2 - \langle\ell,\ell'\rangle\langle\ell',\ell\rangle\,.\end{aligned}$$

By [5] (10.9) and Lemma 11.27, $\frac{3}{2}\vartheta^*c_1(\Theta)$ is the Poincaré dual of a canonical divisor K of S, hence:

$$KF_\ell = \frac{3}{2}\vartheta^*c_1(\Theta)\vartheta^*\Gamma_\ell^*\eta = \frac{3}{2}\int_A \frac{1}{3!}\wedge^4 c_1(\Theta)\wedge\Gamma_\ell^*\eta$$

and:

$$KF_\ell = \int_A 6(\frac{i}{\sqrt{11}})^5(\sum a_jdx_j)\wedge(\sum \bar{a}_jd\bar{x}_j)\wedge\sum_{1\leq k\leq 5}(\wedge_{j\neq k}(dx_j\wedge d\bar{x}_j))$$

so $KF = 6\sum_{k=1}^{k=5} a_k\bar{a}_k = 6\|\ell\|^2$. Hence we have $g(F_\ell) = (KF_\ell+0)/2+1 = 3\|\ell\|^2+1$.

Lemma 0.12 give us a basis $\eta_1,...,\eta_{25}$ of NS(Alb(S)) and we know the intersections $\vartheta^*\eta_k\vartheta^*\eta_l$ in the Fano surface. With the help of a computer, we can verify that the determinant of the intersection matrix:

$$(\vartheta^*\eta_k\vartheta^*\eta_l)_{1\leq k,l\leq 25}$$

is equal to $2^2 11^{10}$. By general results of [10], Proposition 1.17, the index of ϑ^*NS(Alb(S)) $\subset$ NS(S) is 2 and NS(S) is generated by ϑ^*NS(Alb(S)) and the class of an incidence divisor C_s. $\square$

We obtain also the following corollary:

Corollary 0.13. *Let C be a smooth curve of genus >0 and let $\gamma: S\to C$ be a fibration with connected fibres. Then there exists an isomorphism $j:\mathbb{E}\to C$ and an $\ell\in\Lambda_A^*$ such that $\gamma = j\circ\gamma_\ell$.*
The connected fibrations (in a curve of genus >0) up to isomorphism are in bijection with $\mathbb{P}^4_{\mathbb{Z}}(\mathbb{Z}[\nu])$.

Proof. The natural morphism $\wedge^2 H^o(\Omega_S) \to H^o(S, \wedge^2\Omega_S)$ is an isomorphism, hence if $\gamma : S \to C$ is fibration on a curve of genus > 0, the curve C has genus 1. This implies that there is a morphism Γ : Alb(S) $\to$ C such that $\gamma = \Gamma \circ \vartheta$. Moreover Γ has connected fibres hence C is isomorphic to $\mathbb{E}$ (here we use the fact that $\mathbb{Z}[\nu]$ is principal).
Let $\ell \in \Lambda_A^*$, $\ell = t_1y_1 + .. + t_5y_5$, the fibration Γ_ℓ has connected fibres if and only if $t_1, .., t_5$ generates $\mathbb{Z}[\nu]$. □

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