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RELATIVISTIC TODA CHAIN AT ROOT OF UNITY II. MODIFIED Q-OPERATOR

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ABSTRACT. Matrix elements of quantum intertwiner as well as the modified Q-operator for the quantum relativistic Toda chain at root of unity are constructed explicitly. Modified Q-operators make isospectrality transformations of quantum transfer matrices so that the classical counterparts of Q-operators correspond to the Bäcklund isospectrality transformations of the classical transfer matrices. Separated vectors for the Functional Bethe Ansatz are constructed with the help of modified Q-operators.

CONTENTS

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Introduction

This paper is the continuation of Ref. [1] where the quantum relativistic Toda chain at a root of unity was investigated. In the previous paper we have established several features of the chain, concerning the relationship between its finite dimensional (spin) counterpart and its nontrivial classical counterpart.

For several integrable models, based on the simple Weyl algebra at N-th root of unity, the $\mathbb{C}$-numerical N-th powers of Weyl generators form a classical discrete integrable system. So the parameters of the unitary representation of Weyl algebra form the classical counterpart of the finite dimensional integrable system (i.e. spin integrable system), see [2, 3, 4, 5]. Most important finite dimensional operators, arising in the spin systems, have functional counterparts, defined as rational mappings in the space of Nth powers of the parameters. So the finite dimensional operators are the secondary objects: one has to define first the mapping of the parameters, and then the finite dimensional operators are to be constructed in the terms of initial and final values of the parameters. Usual finite dimensional integrable models correspond to the case when for all the operators involved the initial and final parameters coincide. Algebraically these conditions are the origin e.g. of Baxter's curve in the case of the chiral Potts model [6, 7], or of the spherical triangle parameterization for the Zamolodchikov-Bazhanov-Baxter model.

The simplest case of such combined system is the so-called quantum Toda chain at root of unity. It differs significantly from the usual Toda chain [8] as well as from the relativistic Toda chain with arbitrary q [9]. For RTC at root of unity L-operators, quantum as well as classical, and the quantum intertwiners have rather simple form [9]. In [1] it was shown that the trace of monodromy of quantum intertwiners, giving usually Baxter's Q-operator, has the nontrivial classical counterpart. Under several special additional conditions, the classical counterpart of Q operator is the Bäcklund transformation [10, 11] for the classical counterpart of relativistic Toda chain. Spin counterpart of the corresponding Bäcklund transformation makes an isospectral transformation of the spin system. Parameters of Weyl algebra are described in the terms of τ-function[1], and the transformation mentioned maps $(n-1)$-solitonic state into n-solitonic state. Initial homogeneous state is supposed to be 0-solitonic, while the maximal number of solitons is $M-1$ for the chain of the length M. The isospectral transformation of the spin system depends on the $(n-1)$-solitonic origin and n-solitonic image of the classical states. Finite dimensional similarity operators, called the modified Q-operators, of course, do not form a commutative family. Nevertheless these isospectrality deformations play the important rôle in the Functional Bethe Ansatz [1] (see [12, 8, 13] for the details concerning the method of FBA). Namely, for a special set of the solitonic amplitudes the similarity operator becomes a projector into an eigenstate of non-diagonal element of the quantum Toda chain monodromy matrix. It was shown in [1], but we did not give explicit formulas for matrix elements of all finite dimensional operators (quantum intertwiner, similarity operators as well as the separating operator). All these is the subject of the current paper.

This paper is organized as follows. First, we recall the formulation of the quantum relativistic Toda chain at root of unity and its classical counterpart. Next, we recall the form of the auxiliary L-operators (which are the particular case of Bazhanov-Stroganov's L-operator [7]), its classical counterpart, the intertwining relations and the isospectrality transformations of the Toda transfer matrix. In the third section the matrix elements of

[1]in this paper we use the term "τ-function" in the sense of bilinear discrete equation with constant coefficients, i.e. as a trigonometric limit of the θ-function on a high genus algebraic curve

quantum R-matrix and modified Q-operator are given in an appropriate basis. In the fourth section we recall the parameterization of τ-function in application to the relativistic Toda chain. In the fifth section we parameterize arguments of the modified Q-operators for the whole set of Bäcklund transformations and so construct explicitly the similarity operator transforming the homogeneous chain into the general $(M-1)$-parametric inhomogeneous chain. In the sixth section, we describe an appropriate limiting procedure for the degenerative final state and construct the quantum separating operator. Finally, in conclusion, we discuss several further applications of the method described.

1. Formulation of the model

In this section we recall briefly the subject of the model called the quantum relativistic Toda chain at root of unity [1, 9].

1.1. *L*-operators.

Let the chain is formed by M sites with the periodical boundary conditions. m-th site of the Toda chain is described by the following local Lax matrix:

$$\ell_m(\lambda) \;=\; \begin{pmatrix} 1 + \dfrac{\kappa}{\lambda}\,\mathbf{u}_m\,\mathbf{w}_m & , & -\dfrac{\omega^{1/2}}{\lambda}\,\mathbf{u}_m \\ & & \\ \mathbf{w}_m & , & 0 \end{pmatrix} . \tag{1}$$

Here λ is the spectral parameter, κ is an extra complex parameter, common for all sites, i.e. the modulus. Elements $\mathbf{u}_m$ and $\mathbf{w}_m$ form the ultra-local Weyl algebra,

$$\mathbf{u}_m \cdot \mathbf{w}_m \;=\; \omega\; \mathbf{w}_m \cdot \mathbf{u}_m \;, \tag{2}$$

and $\mathbf{u}_m$, $\mathbf{w}_m$ for different sites commute. Weyl's parameter ω is the primitive root of unity,

$$\omega \;=\; \mathbf{e}^{2\,\pi\, i/N}\;, \quad \omega^{1/2} \;=\; \mathbf{e}^{i\,\pi/N}\;. \tag{3}$$

N-th powers of the Weyl elements are centers of the algebra. We will deal with the finite dimensional unitary representation of the Weyl algebra, i.e.

$$\mathbf{u}_m \;=\; u_m\,\mathbf{x}_m\;, \quad \mathbf{w}_m \;=\; w_m\,\mathbf{z}_m\;, \tag{4}$$

where u_m and w_m are $\mathbb{C}$-numbers, and

$$\mathbf{x}_m = 1\otimes 1\otimes \ldots \otimes \underbrace{\mathbf{x}}_{\substack{m-\text{th}\\ \text{place}}} \otimes \ldots\,, \quad \mathbf{z}_m = 1\otimes 1\otimes \ldots \otimes \underbrace{\mathbf{z}}_{\substack{m-\text{th}\\ \text{place}}} \otimes \ldots \tag{5}$$

Convenient representation of $\mathbf{x}$ and $\mathbf{z}$ on the N-dimensional vector space $|\alpha\rangle = |\alpha \mod N\rangle$ is

$$\mathbf{x}\,|\alpha\rangle \;=\; |\alpha\rangle\,\omega^{\alpha}\,,\quad \mathbf{z}\,|\alpha\rangle \;=\; |\alpha+1\,\rangle\,,\quad \langle\alpha|\beta\rangle \;=\; \delta_{\alpha,\beta}\,. \tag{6}$$

Thus $\mathbf{x}$ and $\mathbf{z}$ are normalized to the unity ($\mathbf{x}^N = \mathbf{z}^N = 1$) $N\times N$ dimensional matrices, and the Nth powers of the local Weyl elements are $\mathbb{C}$-numbers

$$\mathbf{u}_m^N \;=\; u_m^N\,,\quad \mathbf{w}_m^N \;=\; w_m^N\,. \tag{7}$$

In general, all u_m and w_m are different, so we deal with the inhomogeneous chain.

Variables u_m^N and w_m^N form the classical counterpart of the quantum relativistic Toda chain, and the classical Lax matrix is

$$L_m(\lambda^N) \;\stackrel{def}{=}\; \begin{pmatrix} 1+\dfrac{\kappa^N}{\lambda^N}\,u_m^N w_m^N & , & \dfrac{u_m^N}{\lambda^N} \\ & & \\ w_m^N & , & 0 \end{pmatrix}. \tag{8}$$

1.2. **Transfer matrix and integrability.** Ordered product of the quantum L-operators (1)

$$\widehat{\mathbf{t}}(\lambda) \;\stackrel{def}{=}\; \ell_1(\lambda)\,\ell_2(\lambda)\,\cdots\,\ell_M(\lambda) \;=\; \begin{pmatrix} \mathbf{a}(\lambda) & \mathbf{b}(\lambda) \\ & \\ \mathbf{c}(\lambda) & \mathbf{d}(\lambda) \end{pmatrix}, \tag{9}$$

and its trace

$$\mathbf{t}(\lambda) \;=\; \mathbf{a}(\lambda)+\mathbf{d}(\lambda) \;=\; \sum_{k=0}^{M} \lambda^{-k}\,\mathbf{t}_k\,. \tag{10}$$

are the monodromy and the transfer matrices of the model.

Transfer matrices, defined with the given κ and with the given set of u_m, w_m, $m = 1...M$,

$$\mathbf{t}(\lambda) \;=\; \mathbf{t}(\lambda,\kappa\,;\{u_m,w_m\}_{m=1}^M)\,. \tag{11}$$

form the commutative set: $\forall\;\lambda,\mu$

$$\Big[\,\mathbf{t}(\lambda,\kappa\,;\{u_m,w_m\})\,,\;\mathbf{t}(\mu,\kappa\,;\{u_m,w_m\})\,\Big] \;=\; 0\,. \tag{12}$$

Eq. (12) is provided by the intertwining relation for L-operators (1) in the auxiliary two-dimensional vector spaces with the help of well known six-vertex trigonometric R-matrix, see [1] for the details.

In the spectral decomposition of the transfer matrix (10) the utmost operators are

$$\mathbf{t}_0 = 1\ , \quad \mathbf{t}_M = \kappa^M \prod_{m=1}^{M} (-\omega^{1/2} u_m w_m)\, \mathbf{Y}\ , \tag{13}$$

where

$$\mathbf{Y} = \prod_{m=1}^{M} (-\omega^{-1/2} \mathbf{x}_m \mathbf{z}_m)\ , \quad \mathbf{Y}^N = 1\ . \tag{14}$$

Further we will need also the monodromy of the classical Lax matrices (8),

$$\widehat{T} \;=\; L_1\, L_2\, \cdots\, L_M \;=\; \begin{pmatrix} A(\lambda^N) & B(\lambda^N) \\ & \\ C(\lambda^N) & D(\lambda^N) \end{pmatrix} \tag{15}$$

and its trace

$$T(\lambda^N) = A(\lambda^N) + D(\lambda^N)\ . \tag{16}$$

M nontrivial coefficients in the decomposition of $T(\lambda^N)$ are involutive with respect to the Poisson brackets, associated with the following symplectic form:

$$\Omega = \sum_{m=1}^{M} \frac{d\, u_m}{u_m} \wedge \frac{d\, w_m}{w_m}\ . \tag{17}$$

It is the consequence of (2) and (12).

The utmost coefficient in the decomposition of the classical transfer matrix $T(\lambda^N)$, analogous to (14), generates the simple gauge transformation of L-operators (1). Thus we may fix without lost of generality

$$\prod_{m=1}^{M} (-\omega^{1/2} u_m) \;=\; \prod_{m=1}^{M} (-w_m) \;=\; 1\ . \tag{18}$$

1.3. **Quantum separation of variables.** Eigenstates of off-diagonal element of the monodromy matrix (9) $\mathbf{b}(\lambda)$ play the important rôle in the constructing of the spectrum of the quantum transfer matrix. This method is known as the method of the Functional Bethe Ansatz, or the method of the quantum separation of variables [12, 8, 13]. It is useful to parameterize the spectrum of $\mathbf{b}(\lambda)$ by its zeros λ_k, $k = 1...M-1$. To deal with the eigenstates of $\mathbf{b}(\lambda)$ explicitly is not useful. We should better define the set of vectors $|\{\lambda_k\}_{k=1}^{M-1}, \gamma\rangle$, such that

$$\mathbf{b}(\lambda_j)\,|\{\lambda_k\}_{k=1}^{M-1}, \gamma\rangle \;=\; 0\;, \quad \lambda_j \in \{\lambda_k\}_{k=1}^{M-1}\;, \tag{19}$$

and

$$\mathbf{Y}\,|\{\lambda_k\}_{k=1}^{M-1}, \gamma\rangle \;=\; |\{\lambda_k\}_{k=1}^{M-1}, \gamma\rangle\,\omega^{\gamma}\;, \tag{20}$$

where the integral of motion $\mathbf{Y}$ is given by (14).

The matrix element between an eigenstate of $\mathbf{t}_k$ and $|\{\lambda\}, \gamma\rangle$ is a product of $(M-1)$ Baxter's functions $Q(\lambda_k)$, taken for the same λ_k, $k = 1...M-1$, as in (19), see e.g. [12, 13]. For the details concerning the Baxter equation for the quantum relativistic Toda chain at root of unity see [1], and later we will recall it. Note here, this Q is a meromorphic function on Baxter's curve. In our parameterization it is given by

$$\lambda^N \;=\; \delta^N \;+\; \frac{\lambda^N}{\delta^N} \;+\; \kappa^N\;. \tag{21}$$

Convenient Baxter's form $x^N + y^N \;=\; 1 + k^2 x^N y^N$ may be obtained by the substitution $x = 1/\delta$, $y = \delta/\lambda$, $k^2 = -\kappa^N$.

Spectra of $\mathbf{a}$, ..., $\mathbf{d}$ are described by A, ..., D, see [14]. For example,

$$\prod_{n\in\mathbb{Z}_N} \mathbf{b}(\omega^n\lambda) \;=\; B(\lambda^N)\;. \tag{22}$$

Eq. (22) allows one to find the set of λ_k. We will do it later for the homogeneous model.

The real reason why we are considering the classical counterpart is that the quantum separation of variables is associated with the Bäcklund transformation of the classical model.

2. Intertwining relations

In this section we recall the auxiliary L-operators, quantum R-matrix and the Darboux relation for the classical counterpart, providing the solution of the isospectrality problem, construction of modified Q-operators and Bäcklund transformation.

2.1. **Auxiliary L-operator.** Toda L-operators (1) are intertwined in the quantum spaces with the Bazhanov-Stroganov L-operators [7]. This fact is the origin of the relationship between the relativistic Toda chain at the Nth root of unity and the N-state chiral Potts model. In general the intertwiner is rather complicated, but there exists a special limit of Bazhanov-Stroganov's L-operator such that the intertwiner simplifies. This limit is the analogue of the so-called dimer self-trapping L-operators for usual Toda chain, see [10, 8, 9] for the details. We define the auxiliary L-operator as follows:

$$\widetilde{\ell}_\phi(\lambda,\lambda_\phi) = \tag{23}$$
$$= \begin{pmatrix} 1 - \omega^{1/2}\kappa_\phi\dfrac{\lambda_\phi}{\lambda}\mathbf{w}_\phi & , & -\dfrac{\omega^{1/2}}{\lambda}(1-\omega^{1/2}\kappa_\phi\mathbf{w}_\phi)\mathbf{u}_\phi \\ \\ -\omega^{1/2}\lambda_\phi\mathbf{u}_\phi^{-1}\mathbf{w}_\phi & , & \mathbf{w}_\phi \end{pmatrix} .$$

Here λ and λ_ϕ are two spectral parameters (actually, up to a gauge transformation, $\widetilde{\ell}$ depends on their ratio). κ_ϕ is the module as well as κ. $\mathbf{u}_\phi$ and $\mathbf{w}_\phi$ form the same Weyl algebra at the root of unity (4),

$$\mathbf{u}_\phi = u_\phi\,\mathbf{x}_\phi\,, \quad \mathbf{w}_\phi = w_\phi\,\mathbf{z}_\phi\,. \tag{24}$$

In all these notations the subscript ϕ stands as the "number" of additional Weyl algebra.

Classical counterpart of $\widetilde{\ell}$ is

$$\widetilde{L}_\phi = \begin{pmatrix} 1+\kappa_\phi^N\dfrac{\lambda_\phi^N}{\lambda^N}w_\phi^N & , & \dfrac{u_\phi^N}{\lambda^N}(1+\kappa_\phi^N w_\phi^N) \\ \\ \lambda_\phi^N\dfrac{w_\phi^N}{u_\phi^N} & , & w_\phi^N \end{pmatrix} . \tag{25}$$

L-operators (23) are intertwined in their two dimensional auxiliary vector spaces by the same six-vertex trigonometric R-matrix as (1). Also there exists the fundamental quantum intertwiner for (23): it is the R-matrix for the chiral Potts model such that two rapidities are fixed to a special singular value. The method describing in this paper may be applied directly to the model, defined by (23), this is the subject of the forthcoming paper.

2.2. **Intertwining.** We are going to write out some quantum intertwining relation for L-operators (1) as well as for the whole monodromy matrix (9). So we use notations, applicable for the recursion in m. Also in this section we will point out parameters u_m, w_m, (4), as the arguments of L-operators.

Proposition 1. *There exists unique (up to a constant multiplier) $N^2 \times N^2$ matrix $\mathbf{R}_{m,\phi}(\lambda_\phi)$, such that $\mathbf{R}_{m,\phi}$, ℓ_m and $\widetilde{\ell}$ obey the modified intertwining relation*

$$\widetilde{\ell}_\phi(\lambda,\lambda_\phi;u_{\phi,m},w_{\phi,m})\cdot\ell_m(\lambda;u_m,w_m)\,\mathbf{R}_{m,\phi}(\lambda_\phi) \;= \tag{26}$$
$$=\;\mathbf{R}_{m,\phi}(\lambda_\phi)\,\ell_m(\lambda;u'_m,w'_m)\cdot\widetilde{\ell}_\phi(\lambda,\lambda_\phi;u_{\phi,m+1},w_{\phi,m+1})\;,$$

if and only if their classical counterparts of L_m and $\widetilde{L}$ obey the following Darboux relation:

$$\widetilde{L}(u^N_{\phi,m},w^N_{\phi,m})L(u^N_m,w^N_m)\;=\;L(u'^N_m,w'^N_m)\widetilde{L}(u^N_{\phi,m+1},w^N_{\phi,m+1})\;, \tag{27}$$

Proof: the direct verification. Eq. (26) may be rewritten as

$$\widetilde{\ell}_\phi(\lambda;\mathbf{u}_\phi,\mathbf{w}_\phi)\cdot\ell_m(\lambda;\mathbf{u}_m,\mathbf{w}_m)\;=\;\ell_m(\lambda;\mathbf{u}'_m,\mathbf{w}'_m)\cdot\widetilde{\ell}_\phi(\lambda;\mathbf{u}'_\phi,\mathbf{w}'_\phi)\;, \tag{28}$$

where

$$\mathbf{u}'_m = u'_m\,\mathbf{R}_{m,\phi}\mathbf{x}_m\mathbf{R}^{-1}_{m,\phi}\;,\quad \mathbf{w}'_m = w'_m\,\mathbf{R}_{m,\phi}\mathbf{z}_m\mathbf{R}^{-1}_{m,\phi}\;, \tag{29}$$

and

$$\mathbf{u}'_\phi = u_{\phi,m+1}\,\mathbf{R}_{m,\phi}\mathbf{x}_\phi\mathbf{R}^{-1}_{m,\phi}\;,\quad \mathbf{w}'_\phi = w_{\phi,m+1}\,\mathbf{R}_{m,\phi}\mathbf{z}_\phi\mathbf{R}^{-1}_{m,\phi}\;, \tag{30}$$

Solution of (28) with respect to the primed *operators* is unique and given by

$$
\left\{
\begin{aligned}
\mathbf{u}'_m \;&=\; \frac{\kappa_\phi}{\kappa}\,\mathbf{u}_\phi\;,\\[2ex]
\mathbf{w}'_m \;&=\; \mathbf{w}_m\mathbf{w}_\phi\;-\;\omega^{1/2}\lambda_\phi\mathbf{u}_\phi^{-1}\mathbf{w}_\phi\;,\\[2ex]
\mathbf{u}'_\phi \;&=\; \left(\lambda_\phi\;+\;\kappa\mathbf{u}_m\mathbf{w}_m\;-\;\omega^{1/2}\mathbf{w}_m\mathbf{u}_\phi\right)^{-1}\lambda_\phi\mathbf{u}_m\;,\\[2ex]
\mathbf{w}'_\phi \;&=\; \frac{\kappa}{\kappa_\phi}\,\mathbf{u}_m\mathbf{w}_m\,\left(\mathbf{w}_m\mathbf{u}_\phi\;-\;\omega^{1/2}\lambda_\phi\right)^{-1}
\end{aligned}
\right.
\tag{31}
$$

Nth powers of all primed operators in (31) give

$$
u_{\phi,m+1}^N \;=\; \frac{\lambda_\phi^N\,u_m^N}{\lambda_\phi^N+\kappa^N u_m^N w_m^N+w_m^N u_{\phi,m}^N}\;,\quad w_{\phi,m+1}^N\;=\;\frac{\kappa^N}{\kappa_\phi^N}\,\frac{u_m^N w_m^N}{\lambda_\phi^N+w_m^N u_{\phi,m}^N}\;,
\tag{32}
$$

and

$$
u_m'^N\;=\;\frac{\kappa_\phi^N}{\kappa^N}\,u_{\phi,m}^N\;,\qquad w_m'^N\;=\;\frac{\kappa^N}{\kappa_\phi^N}\,\frac{u_m^N w_m^N}{u_{\phi,m}^N}\,\frac{w_{\phi,m}^N}{w_{\phi,m+1}^N}\;.
\tag{33}
$$

Eqs. (32) and (33) are the exact and unique solution of (27). Therefore eq. (27) is the consistency condition for (26). Next, eq. (26) is a set of linear equations for the matrix elements of $\mathbf{R}_{m,\phi}$, this set is defined appropriately, and this provides the uniqueness of $\mathbf{R}_{m,\phi}$.

The local transformation

$$
u_m, w_m, u_{\phi,m}, w_{\phi,m}\;\;\mapsto\;\; u'_m, w'_m, u_{\phi,m+1}, w_{\phi,m+1}\;,
\tag{34}
$$

given by eqs. (32) and (33), is called usually as the Darboux transformation for the classical relativistic Toda chain. Eqs. (32) and (33) define the mapping (34) up to N-th roots of unity. These roots are the additional discrete parameters of the transformation (34). Note, matrix $\mathbf{R}_{m,\phi}$ is unique if all these roots are fixed.

2.3. **$\mathcal{Q}$-transformation.** Relations (26) and (27) may be iterated for the whole chain. The functional counterpart gives

$$
\widetilde{L}(u_{\phi,1}^N,w_{\phi,1}^N)\widehat{T}(\{u_m^N,w_m^N\}_{m=1}^M)\;=\;\widehat{T}(\{u_m'^N,w_m'^N\}_{m=1}^M)\widetilde{L}(u_{\phi,M+1}^N,w_{\phi,M+1}^N)\;,
\tag{35}
$$

where the mapping $u_{\phi,1}, w_{\phi,1}\mapsto u_{\phi,M+1},\,w_{\phi,M+1}$ is M-th iteration of (34).

For the periodic chain the cyclic boundary conditions for the recursion (32) are to be imposed,

$$u_{\phi,M+1} = u_{\phi,1}\ , \quad w_{\phi,M+1} = w_{\phi,1}\ . \tag{36}$$

Now suppose (32) and (36) are solved, i.e. $u_{\phi,m}$, $w_{\phi,m}$ are parameterized in the terms of u_m, w_m, $m = 1...M$, and some extra parameters, possible degrees of freedom of (32) and (36) (e.g. λ_ϕ). Then (33) defines in general the transformation $\mathcal{Q}_\phi$

$$\mathcal{Q}_\phi\ :\ \{u_m, w_m\}_{m=1}^M \ \mapsto\ \{u'_m, w'_m\}_{m=1}^M \tag{37}$$

The transfer matrices of two sets, $\{u_m, w_m\}$ and $\{u'_m, w'_m\}$, have the same spectrum, because of there exists $N^M \times N^M$ matrix $\mathbf{Q}_\phi$, nondegenerative in general,

$$\mathbf{Q}_\phi \stackrel{def}{=} \mathrm{tr}_\phi\ \left(\mathbf{R}_{1,\phi}\, \mathbf{R}_{2,\phi}\ \ldots\ \mathbf{R}_{M,\phi}\right)\ , \tag{38}$$

such that

$$\mathbf{t}(\lambda\,; \{u_m, w_m\})\ \cdot\ \mathbf{Q}_\phi\ =\ \mathbf{Q}_\phi\ \cdot\ \mathbf{t}(\lambda\,; \{u'_m, w'_m\})\ . \tag{39}$$

Subscript ϕ of $\mathbf{Q}_\phi$ stands as the reminder for the parameters, arising in the solution of recursion, including at least the spectral parameter λ_ϕ.

Consider now the repeated application of the transformations $\mathcal{Q}_\phi$, eq. (37),

$$\{u_m, w_m\} \stackrel{\mathcal{Q}_{\phi_1}}{\mapsto} \{u'_m, w'_m\} \stackrel{\mathcal{Q}_{\phi_2}}{\mapsto} ... \stackrel{\mathcal{Q}_{\phi_n}}{\mapsto} \{u_m^{(n)}, w_m^{(n)}\}\ , \tag{40}$$

such that the set of isospectral quantum transfer matrices

$$\mathbf{t}^{(n)}(\lambda)\ =\ \mathbf{t}(\lambda, \kappa; \{u_m^{(n)}, w_m^{(n)}\}_{m=1}^M) \tag{41}$$

has arisen. Sequence (40) defines the transformation $\mathcal{K}$,

$$\mathcal{K}^{(n)}\ :\ \{u_m \equiv u_m^{(0)}, w_m \equiv w_m^{(0)}\}_{m=1}^M\ \mapsto\ \{u_m^{(n)}, w_m^{(n)}\}_{m=1}^M\ , \tag{42}$$

with the finite dimensional counterpart

$$\mathbf{K}^{(n)}\ =\ \mathbf{Q}_{\phi_1}^{(1)}\, \mathbf{Q}_{\phi_2}^{(2)} ... \mathbf{Q}_{\phi_n}^{(n)}\ , \tag{43}$$

where

$$\mathbf{Q}_{\phi_n}^{(n)}\ =\ \mathrm{tr}_{\phi_n}\ \left(\mathbf{R}_{1,\phi_n}^{(n)}\, \mathbf{R}_{2,\phi_n}^{(n)}\ \ldots\ \mathbf{R}_{M,\phi_n}^{(n)}\right) \tag{44}$$

makes

$$\mathbf{t}^{(n-1)}(\lambda)\ \mathbf{Q}^{(n)}_{\phi_n}\ =\ \mathbf{Q}^{(n)}_{\phi_n}\ \mathbf{t}^{(n)}(\lambda)\ , \tag{45}$$

and

$$\mathbf{t}^{(0)}(\lambda)\ \mathbf{K}^{(n)}\ =\ \mathbf{K}^{(n)}\ \mathbf{t}^{(n)}(\lambda)\ . \tag{46}$$

3. Matrix $\mathbf{R}$

In this section we construct explicitly the finite dimensional matrix $\mathbf{R}_{m,\phi}$, obeying (26), in the basis (6). But first we have to introduce several notations.

3.1. w-function.

Let p be a point on the Fermat curve $\mathcal{F}$

$$p\ \stackrel{def}{=}\ (x,y)\ \in\mathcal{F}\quad\Leftrightarrow\quad x^N+y^N=1\ . \tag{47}$$

Very useful function on the Fermat curve is $w_p(n)$, $p\in\mathcal{F}$, $n\in\mathbb{Z}_N$, defined as follows:

$$\frac{w_p(n)}{w_p(n-1)}\ =\ \frac{y}{1\ -\ x\,\omega^n}\ ,\quad w_p(0)\ =\ 1\ . \tag{48}$$

Function $w_p(n)$ has a lot of remarkable properties, see the appendix of ref. [15] for an introduction into ω-hypergeometry. In this paper it is necessary to mention just a couple of properties of w-function. Let O be the following automorphism of the Fermat curve:

$$p\ =\ (x,y)\quad\Leftrightarrow\quad Op\ =\ (\omega^{-1}x^{-1},\omega^{-1/2}x^{-1}y)\ . \tag{49}$$

Then

$$w_{Op}(n)\ =\ \frac{1}{\Phi(n)w_p(-n)}\ , \tag{50}$$

where

$$\Phi(n)\ =\ (-)^n\,\omega^{n^2/2}\ . \tag{51}$$

In the subsequent sections we will use implicitly two simple automorphisms else:

$$w_{(x,\omega y)}(n)=\omega^n w_{(x,y)}(n)\ ,\quad w_{(x,y)}(n+1)=\frac{y}{1-\omega x}w_{(\omega x,y)}(n)\ . \tag{52}$$

Define also three special points on the Fermat curve:

$$q_0=(0,1)\ ,\quad q_\infty=Oq_0\ ,\quad q_1=(\omega^{-1},0)\ . \tag{53}$$

Then

$$w_{q_0}(n) = 1 \;, \quad w_{q_\infty}(n) = \frac{1}{\Phi(n)} \;, \quad \frac{1}{w_{q_1}(n)} \;=\; \delta_{n,0} \;. \tag{54}$$

3.2. **Matrix elements of $\mathbf{R}_{m,\phi}$.** Consider the $N^2 \times N^2$ matrix $\mathbf{R}_{m,\phi}(p_1, p_2, p_3)$ with the following matrix elements:

$$\begin{aligned} &\langle \alpha_m, \alpha_\phi | \mathbf{R}_{m,\phi} | \beta_m, \beta_\phi \rangle \;= \\ &=\; \omega^{(\alpha_m - \beta_m)\beta_\phi} \, \frac{w_{p_1}(\alpha_\phi - \alpha_m)\, w_{p_2}(\beta_\phi - \beta_m)}{w_{p_3}(\beta_\phi - \alpha_m)} \, \delta_{\alpha_\phi, \beta_m} \;. \end{aligned} \tag{55}$$

Here p_1, p_2, p_3 are three points on the Fermat curve, such that

$$x_1\, x_2 \;=\; x_3 \;. \tag{56}$$

Eq. (56) and the spin structure of (55) provide the dependence of (55) on two continuous parameters, say x_1 and x_3, and on two discrete parameters, say the phase of y_1 and the phase of y_3.

Proposition 2. *Matrix $\mathbf{R}_{m,\phi}(p_1, p_2, p_3)$, whose matrix elements (55) are given in the basis (6), makes the following mapping:*

$$\left\{ \begin{aligned} \mathbf{R}_{m,\phi} \mathbf{x}_m \mathbf{R}_{m,\phi}^{-1} \;&=\; \mathbf{x}_\phi \;, \\ \mathbf{R}_{m,\phi} \mathbf{z}_m \mathbf{R}_{m,\phi}^{-1} \;&=\; \frac{y_3}{y_2} \mathbf{z}_m \mathbf{z}_\phi - \omega \frac{x_3 y_1}{x_1 y_2} \mathbf{x}_\phi^{-1} \mathbf{z}_\phi \;, \\ \mathbf{R}_{m,\phi} \mathbf{x}_\phi^{-1} \mathbf{R}_{m,\phi}^{-1} \;&=\; \omega x_3 \mathbf{x}_m^{-1} - \omega \frac{x_1 y_3}{y_1} \mathbf{x}_m^{-1} \mathbf{z}_m \mathbf{x}_\phi + \frac{y_3}{y_1} \mathbf{z}_m \;, \\ \mathbf{R}_{m,\phi} \mathbf{z}_\phi^{-1} \mathbf{R}_{m,\phi}^{-1} \;&=\; \frac{y_3}{y_2} \mathbf{x}_\phi \mathbf{x}_m^{-1} - \omega \frac{x_3 y_1}{x_1 y_2} \mathbf{z}_m^{-1} \mathbf{x}_m^{-1} \;. \end{aligned} \right. \tag{57}$$

Proof: direct verification with the help of (52).

Compare (57) with (31). Obviously, $\mathbf{R}_{m,\phi}$ solves (26) if

$$x_{1,m} \;=\; \omega^{-1/2} \, \frac{u_{\phi,m}}{\kappa\, u_m} \;, \quad \frac{x_{3,m}\, y_{1,m}}{x_{1,m}\, y_{3,m}} \;=\; \omega^{-1/2} \, \frac{\lambda_\phi}{u_{\phi,m} w_m} \;, \tag{58}$$

and

$$
(59)\qquad \begin{array}{ll}
u'_m \;=\; \dfrac{\kappa_\phi}{\kappa}\, u_{\phi,m}\,, & w'_m \;=\; w_m w_{\phi,m}\, \dfrac{y_{2,m}}{y_{3,m}}\,,\\[3ex]
u_{\phi,m+1} \;=\; \omega\, x_{3,m}\, u_m\,, & w_{\phi,m+1} \;=\; \dfrac{\kappa}{\kappa_\phi}\, \dfrac{u_m}{u_{\phi,m}}\, \dfrac{y_{3,m}}{y_{2,m}}\,.
\end{array}
$$

Eqs. (58) and (59) allow one to parameterize $\mathbf{R}^{(n)}_{m,\phi_n}$ in (44) in terms of $u_m, w_m, \ldots$. So for any appropriate set of $\{u_m^{(k)}, w_m^{(k)}\}$, $m = 1 \ldots M$ and $k = 0 \ldots n$, see formula (40), we may construct the corresponding $\mathbf{K}^{(n)}$ explicitly via the parameterization (58,59), form of the matrix elements (55), and the formulas (44,43).

Further we will consider the sequences like (40) for the homogeneous initial state $\{u_m = u_0, w_m = w_0\}_{m=1}^M$, and parameterize $\{u_m^{(n)}, w_m^{(n)}\}$ in the terms of solitonic Hirota-type expressions. To do it, several notations are to be introduced.

4. Rational Θ-function

In this section we introduce several useful functions and notations. The main purpose is to introduce the form of solitonic τ-function for the classical relativistic Toda chain. The reader may find the details concerning the corresponding classical integrable model, reduction from 2DToda lattice hierarchy and so on in Ref. [17].

Fix any sequence of ϕ_k from the set $\mathfrak{F}_M$:

$$
(60)\qquad \phi_k \;\in\; \mathfrak{F}_M \;\stackrel{def}{=}\; \{\frac{\pi}{M}, \frac{2\pi}{M}, \ldots, \frac{(M-1)\pi}{M}\}\,,
$$

Introduce the functions

$$
(61)\qquad \begin{cases}
\Delta_\phi \;=\; \mathbf{e}^{i\phi}\left(\sqrt{\cos^2\phi + \kappa^N} \;+\; \cos\phi\right)\,,\\[2ex]
\Delta^*_\phi \;=\; \mathbf{e}^{-i\phi}\left(\sqrt{\cos^2\phi + \kappa^N} \;+\; \cos\phi\right)\,,\\[2ex]
\Lambda_\phi \;=\; \Delta_\phi\, \Delta^*_\phi\,.
\end{cases}
$$

Expressions for Δ and Δ^* parameterize the curve

$$
(62)\qquad \Delta\Delta^* \;=\; \Delta + \Delta^* + \kappa^N \;\text{ in the terms of }\; \mathbf{e}^{2i\phi} \;=\; \frac{\Delta}{\Delta^*}\,.
$$

Rational Θ-functions are to be defined recursively as

$$\Theta_m^{(0)} \;=\; 1\;,$$

(63) $$\Theta_m^{(1)} \;=\; 1 \;-\; f_1\,\mathbf{e}^{2im\phi_1}\;,$$

$$\Theta_m^{(2)} \;=\; 1 - f_1\mathbf{e}^{2im\phi_1} - f_2\mathbf{e}^{2im\phi_2} + d_{1,2}f_1f_2\mathbf{e}^{2im(\phi_1+\phi_2)}\;,$$

and so on,

(64) $$\Theta_m^{(n)} \;=\; \Theta_m^{(n-1)}(\{f_k,\phi_k\}_{k=1}^{n-1}) - f_n\mathbf{e}^{2im\phi_n}\Theta_m^{(n-1)}(\{d_{k,n}f_k,\phi_k\}_{k=1}^{n-1})\;,$$

where all ϕ_k are different and belong to the set (60). The phase shift $d_{j,k} = d_{\phi_j,\phi_k}$ is given by (see (61))

(65) $$d_{\phi,\phi'} \;=\; \frac{\left(\Delta_\phi - \Delta_{\phi'}\right)\;\left(\Delta_\phi^* - \Delta_{\phi'}^*\right)}{\left(\Delta_\phi - \Delta_{\phi'}^*\right)\;\left(\Delta_\phi^* - \Delta_{\phi'}\right)}\;.$$

Arguments of $\Theta_m^{(n)}$ are the set of $\phi_k \in \mathfrak{F}_M$ and corresponding amplitudes f_k, $k = 1...n$. Maximal Θ-function is

(66) $$\Theta_m \;\stackrel{def}{=}\; \Theta_m^{(M-1)}(\{f_k\}_{k=1}^{M-1})\;.$$

$\mathfrak{F}_M$-set, (60), provides the M-periodicity of Θ-functions, $\Theta_{m+M} \equiv \Theta_m$.

Introduce next a couple of useful functions

(67) $$s_\phi(\xi) \;=\; \frac{\Delta_\phi^*}{\Delta_\phi}\,\frac{\Delta_\phi \;-\; \xi}{\Delta_\phi^* \;-\; \xi}\;,$$

and

(68) $$s_{\phi,\phi'} \;=\; s_\phi(\Delta_{\phi'}^*) \;=\; \frac{\Delta_\phi^*}{\Delta_\phi}\,\frac{\Delta_{\phi'}^* \;-\; \Delta_\phi}{\Delta_{\phi'}^* \;-\; \Delta_\phi^*}\;.$$

Necessary is to consider the re-parameterization of the amplitudes f_k in the special form: for any n,k, $0 \le n < M$, $1 \le k \le n$ let

(69) $$f_k^{(n)}(\{g\}) \;=\; g_k \prod_{\substack{j=1\\ j\neq k}}^{n} s_{k,j}\;,$$

where $s_{k,j}$ is given by (68). Denote the corresponding Θ-functions as follows:

(70) $$\Theta_m^{(n)}(\{g\}) \;=\; \Theta_m^{(n)}(\{f_k^{(n)}(\{g\}),\phi_k\}_{k=1}^{n})$$

In advance, we will always imply the free amplitudes g_k, $k = 1,\ldots$, as the arguments of Θ-function, (69,70).

Particular case when all $g_k = s_k(\xi)$ is also important: let

$$\Theta_m^{(n)}(\xi) \;=\; \Theta_m^{(n)}(\{g_k = s_k(\xi)\}_{k=1}^n) \tag{71}$$

This particular case has the following property:

Proposition 3. *For any n*

$$\Theta_n^{(n)}(\xi) \;=\; \prod_{k=1}^n \frac{\Delta_k^* \,-\, \Delta_k}{\Delta_k^* \,-\, \xi} \prod_{1\leq j<k\leq n} \frac{\Lambda_j \,-\, \Lambda_k}{\Delta_j^* \,-\, \Delta_k^*}\;, \tag{72}$$

and for $m = 0\ldots n$

$$\Theta_m^{(n)}(\xi) \;=\; \left(-\,\kappa^N\right)^{(n-m)(n-m-1)/2} \left(\frac{\xi}{\Delta_1\ldots\Delta_n}\right)^{n-m} \Theta_n^{(n)}(\xi) \tag{73}$$

If $\xi = 1$, eq. (73) is valid also for $m = -1$.

5. Parameterization of $\mathbf{K}$

5.1. **Sequence of the Bäcklund transformations.** Turn now to the original quantum chain. Let the initial parameters u_m, w_m are homogeneous:

$$u_m \;=\; -\omega^{-1/2}\;,\;\; w_m = -1\;. \tag{74}$$

It corresponds to (18).

For the sequence of $\phi_k \in \mathfrak{F}_M$ fixed, and for generic set of corresponding g_k, $k = 1\ldots M-1$, let (see eqs. (69) and (70) and the note right after (70))

$$\left(\tau_m^{(n)}\right)^N \;=\; \Theta_m^{(n)}(\{g_k\}_{k=1}^n)\;,\quad \left(\theta_m^{(n)}\right)^N \;=\; \Theta_m^{(n)}(\{g_k s_k(1)\}_{k=1}^n)\;. \tag{75}$$

The phases of $\tau_m^{(n)}$ and $\theta_m^{(n)}$ are arbitrary.

Proposition 4. *Consider the set of $(M-1)$ transformations (37), such that the initial state is the homogeneous one (74), and the mapping $\mathcal{Q}_{\phi_n}$ is derived for*

$$\lambda_{\phi_n}^N \;=\; \Lambda_{\phi_n}\;, \tag{76}$$

where sequence of $\phi_n \in \mathfrak{F}_M$ is fixed, and function Λ_ϕ is given by the last formula of (61). Let besides

$$\kappa_{\phi_n}^N \;=\; \frac{\kappa^N}{\Delta_{\phi_n}}\,. \tag{77}$$

Then the sequence (40) of $\{u_m^{(n)}, w_m^{(n)}\}$ may be parameterized as follows:

$$u_m^{(n)} \;=\; -\omega^{-1/2}\,\frac{\tau_{m-1}^{(n)}}{\tau_m^{(n)}}\;,\quad w_m^{(n)} \;=\; -\,\frac{\theta_m^{(n)}}{\theta_{m-1}^{(n)}}\;,\;;\;\; m\,\in\,\mathbb{Z}_M\;, \tag{78}$$

where $\tau_m^{(n)}$, $\theta_m^{(n)}$ are given by (75) with arbitrary $g_1 ... g_{M-1}$.

See [1] for the sketch of the proof of this proposition. Eq. (77) needs a comment. Arbitrary value of κ_{ϕ_n} corresponds to a gauge transformation of

$$\{u_m^{(n)}, w_m^{(n)}\}_{m=1}^M \;\mapsto\; \{c u_m^{(n)}, c^{-1} w_m^{(n)}\}_{m=1}^M \tag{79}$$

with some c proportional to κ_{ϕ_n}, while the structure of τ and θ is not changed. So we impose (77) and obtain (78) without lost of generality.

5.2. **Parameterization of modified Q-operators.** Let now for the shortness $\lambda_{\phi_n} = \lambda_n$ are given by (76), and let us fix further the roots for δ_n and c_n, $1 \le n < M$:

$$\delta_n^N = \Delta_{\phi_n}\;,\quad \kappa_{\phi_n} = \frac{\kappa}{\delta_n}\;,\quad c_n^N = \frac{\Delta_{\phi_n}^* - 1}{\Delta_{\phi_n}^*}\;. \tag{80}$$

Auxiliary values in (58,59) are given by

$$u_{\phi_n,m} \;=\; -\,\omega^{-1/2}\,\delta_n\,\frac{\tau_{m-1}^{(n)}}{\tau_m^{(n)}}\;,\quad w_{\phi_n,m+1} \;=\; \omega^{-1/2}\,\frac{\delta_n}{\lambda_n c_n}\,\frac{\theta_m^{(n-1)}\tau_m^{(n)}}{\tau_m^{(n-1)}\theta_m^{(n)}}\;. \tag{81}$$

$\mathbf{R}_{m,\phi_n}^{(n)}$-matrices, entering to (44),have the arguments

$$\mathbf{R}_{m,\phi_n}^{(n)} \;:\; p_{1,m}^{(n)},\; p_{2,m}^{(n)},\; p_{3,m}^{(n)}, \tag{82}$$

given by

$$\begin{aligned} x_{1,m}^{(n)} \;&=\; \omega^{-1/2}\,\frac{\delta_n}{\kappa}\,\frac{\tau_m^{(n-1)}\tau_{m-1}^{(n)}}{\tau_{m-1}^{(n-1)}\tau_m^{(n)}}\;, \qquad x_{3,m}^{(n)} \;=\; \omega^{-1}\delta_n\,\frac{\tau_m^{(n-1)}\tau_m^{(n)}}{\tau_{m-1}^{(n-1)}\tau_{m+1}^{(n)}}\;,\\ \frac{y_{3,m}^{(n)}}{y_{2,m}^{(n)}} \;&=\; \omega^{-1/2}\,\frac{\delta_n}{\lambda_n c_n}\,\frac{\theta_m^{(n-1)}\tau_{m-1}^{(n)}}{\tau_{m-1}^{(n-1)}\theta_m^{(n)}}\;,\quad \frac{y_{1,m}^{(n)}}{y_{3,m}^{(n)}} \;=\; \omega^{1/2}\,\frac{\lambda_n}{\kappa\delta_n}\,\frac{\theta_{m-1}^{(n-1)}\tau_{m+1}^{(n)}}{\theta_m^{(n-1)}\tau_m^{(n)}}\;, \end{aligned} \tag{83}$$

and with (56) implied,

$$x_{2,m}^{(n)} \;=\; \omega^{-1/2}\kappa\,\frac{(\tau_m^{(n)})^2}{\tau_{m-1}^{(n)}\tau_{m+1}^{(n)}}\;. \tag{84}$$

The trace of quantum monodromy (44) may be calculated explicitly in the basis (6), and the answer is

$$\langle\alpha|\mathbf{Q}_{\phi_n}^{(n)}|\beta\rangle = \prod_{m\in\mathbb{Z}_M}\;\omega^{(\alpha_m-\beta_m)\beta_{m+1}}\,\frac{w_{p_{1,m}^{(n)}}(\beta_m-\alpha_m)\,w_{p_{2,m}^{(n)}}(\beta_{m+1}-\beta_m)}{w_{p_{3,m}^{(n)}}(\beta_{m+1}-\alpha_m)}\;. \tag{85}$$

Operator $\mathbf{K}$, (43), calculated as the product of all $(M-1)$ $Q^{(n)}$-operators, is thus defined explicitly.

Proposition 5. *Operator $\mathbf{K}(\{g\})$, given by (43), does not depend on the ordering of (g_k,ϕ_k) (up to a multiplier, arising in general when one takes arbitrary phases for $\tau_m^{(n)}$ and $\theta_m^{(n)}$). Thus without lost of generality one may regard $\phi_n = \dfrac{\pi n}{M}$.*

Thus for the generic set of amplitudes g_k constructed is the similarity operator $\mathbf{K}(\{g\})$, transforming the homogeneous initial transfer matrix into most generic isospectral one.

5.3. **Arbitrary value of λ_ϕ.** Previously we considered the set of $\phi_k\in\mathfrak{F}_M$. Actually, all the calculations in the classical relativistic Toda chain are based on (62). Therefore, in general, we may consider a generic sequence of complex numbers ϕ_k, $k\in\mathbb{Z}_+$. For any generic complex ϕ one has to restore λ_ϕ and δ_ϕ via (61), and take into account (80). It corresponds to the Q operator for the generic value of λ_ϕ. Then all the formulas for the parameterization of transfer matrix in the terms of τ-functions (78,75,70) as well as the parameterization of Q-operator (83,85) are valid if the amplitudes g_ϕ, corresponding to those $\phi\notin\mathfrak{F}_M$ and $-\phi\notin\mathfrak{F}_M$ are zeros, and in the case when $\phi\in\mathfrak{F}_M$ or $-\phi\in\mathfrak{F}_M$ the amplitudes are arbitrary. Important is that in the calculation of the amplitudes (69) all ϕs are to be taken into account. Also, if in the generic sequence of ϕ_k it happens $\phi_k=\phi_m$ or $\phi_k=-\phi_m$, then Θ-function is to be understood as the corresponding residue of formal singular expression (70).

For example, usual Q-operators for the homogeneous initial state corresponds to $\phi_1 = \phi\notin\mathfrak{F}_M$, $g_1=0$. The cases when $\phi_k=\pm\phi_m$ correspond to the annihilation of the solitons.

Nevertheless, considering operator $\mathbf{K}$, (43), we will always imply $\phi_k\in\mathfrak{F}_M$.

5.4. **Baxter equation.** Consider initial homogeneous transfer matrix $\mathbf{t}(\lambda)$ and corresponding Q operators with arbitrary value of λ_ϕ. As it was mentioned in the previous subsection, corresponding functional counterpart of Q-operator is trivial, i.e. such Q-operators commute with the transfer matrices and form the commutative family. Such Q-operators are usual Baxter's Q-operators, obeying Baxter's equation.

Explicit form of simple Q-operator is given by (85) with the homogeneous parameterization

$$p_{1,m} = p_1 \ , \quad p_{2,m} = p_2 \ , \quad p_{3,m} = p_3 \tag{86}$$

with

$$\begin{gathered} x_1 = \omega^{-1/2}\frac{\delta_\phi}{\kappa} \ , \quad x_2 = \omega^{-1/2}\kappa \ , \quad x_3 = \omega^{-1}\delta_\phi \ , \\ \frac{y_3}{y_2} = \frac{\omega^{-1/2}\delta_\phi}{\lambda_\phi c_\phi} \ , \quad \frac{y_1}{y_3} = \frac{\omega^{1/2}\lambda_\phi}{\kappa\delta_\phi} \ , \end{gathered} \tag{87}$$

where δ_ϕ, λ_ϕ and c_ϕ are given by (80,76) and (61), but with generic value of ϕ. Using (87), we may regard

$$\mathbf{Q} \ = \ \mathbf{Q}(\lambda_\phi, \delta_\phi) \ . \tag{88}$$

Several simple properties of simple Q-operator may be derived with the help of the matrix elements of (85) and (87), and with the help of (52):

$$\begin{aligned} \mathbf{X}\mathbf{Q}(\lambda_\phi, \delta_\phi)\mathbf{X}^{-1} \ &= \ \mathbf{Q}(\omega^{-1}\lambda_\phi, \delta_\phi) \ , \\ \mathbf{Z}\mathbf{Q}(\lambda_\phi, \delta_\phi)\mathbf{X} \ &= \ \left(\omega^{1/2}\frac{\lambda_\phi}{\delta_\phi}\,\frac{1-\delta_\phi}{\kappa - \omega^{1/2}\delta_\phi}\right)^M \mathbf{Q}(\omega\lambda_\phi, \omega\delta_\phi) \ , \\ \mathbf{Y}\mathbf{Q}(\lambda_\phi, \delta_\phi) \ = \ \mathbf{Q}(\lambda_\phi, \delta_\phi)\mathbf{Y} \ &= \ \left(-\frac{\lambda_\phi}{\delta_\phi}\,\frac{1-\delta_\phi}{\kappa - \omega^{1/2}\delta_\phi}\right)^M \mathbf{Q}(\lambda_\phi, \omega\delta_\phi) \ , \end{aligned} \tag{89}$$

where

$$\mathbf{X} = \prod_{m=1}^{M} \mathbf{x}_m \ , \quad \mathbf{Z} = \prod_{m=1}^{M} \mathbf{z}_m \ , \tag{90}$$

and $\mathbf{Y}$ is given by (14).

Let us derive now Baxter's equation for general inhomogeneous chain. To do it, one has to consider the degeneration point of $\widetilde{\ell}$, (23):

$$\widetilde{\ell}_\phi(\lambda_\phi, \lambda_\phi) \;=\; \begin{pmatrix} 1-\omega^{1/2}\kappa_\phi \mathbf{w}_\phi \\ -\omega^{1/2}\lambda_\phi \mathbf{u}_\phi^{-1}\mathbf{w}_\phi \end{pmatrix} \cdot \begin{pmatrix} 1\;, & -\omega^{1/2}\lambda_\phi^{-1}\mathbf{u}_\phi \end{pmatrix} . \tag{91}$$

Therefore eq. (26) in the degeneration point may be rewritten in the following forms:

$$\begin{aligned} &\begin{pmatrix} 1\;, & -\omega^{1/2}\dfrac{u_{\phi,m}}{\lambda_\phi}\,\mathbf{x}_\phi \end{pmatrix} \cdot \ell_m(\lambda_\phi\,;u_m,w_m) \,\cdot\, \mathbf{R}_{m,\phi} \;= \\ &= \;\mathbf{R}^{(1)}_{m,\phi}\cdot \begin{pmatrix} 1\;, & -\omega^{1/2}\dfrac{u_{\phi,m+1}}{\lambda_\phi}\,\mathbf{x}_\phi \end{pmatrix} , \end{aligned} \tag{92}$$

and

$$\begin{aligned} &\ell_m(\lambda_\phi\,;u'_m,w'_m)\,\cdot\,\mathbf{R}_{m,\phi}\,\cdot \begin{pmatrix} \omega^{1/2}\dfrac{u_{\phi,m+1}}{\lambda_\phi}\mathbf{x}_\phi \\ \\ 1 \end{pmatrix} \;= \\ &= \begin{pmatrix} \omega^{1/2}\dfrac{u_{\phi,m}}{\lambda_\phi}\mathbf{x}_\phi \\ \\ 1 \end{pmatrix} \cdot\, \mathbf{R}^{(2)}_{m,\phi}\;, \end{aligned} \tag{93}$$

where

$$\begin{aligned} \mathbf{R}^{(1)}_{m,\phi} \;&=\; \frac{u_m}{u_{\phi,m+1}}\;\; \mathbf{x}_m\,\mathbf{R}_{m,\phi}\,\mathbf{x}_\phi^{-1}\;, \\ \mathbf{R}^{(2)}_{m,\phi} \;&=\; \omega^{1/2}\,\frac{u_{\phi,m+1}\,w_m}{\lambda_\phi}\;\; \mathbf{z}_m\,\mathbf{R}_{m,\phi}\,\mathbf{x}_\phi\;. \end{aligned} \tag{94}$$

Let further $\mathbf{Q}^{(1)}_\phi$ and $\mathbf{Q}^{(2)}_\phi$ are the traces of the monodromies of $\mathbf{R}^{(1)}_{m,\phi}$ and $\mathbf{R}^{(2)}_{m,\phi}$ ($\mathbb{Z}_M$ boundary condition for $u_{\phi,m}$ is taken into account again – the subscript ϕ reminds this). Then from (92) and (93) it follows the Baxter equation in its operator form:

$$\mathbf{t}(\lambda_\phi)\,\mathbf{Q}_\phi \;=\; \mathbf{Q}_\phi\,\mathbf{t}'(\lambda_\phi) \;=\; \mathbf{Q}^{(1)}_\phi \;+\; \mathbf{Q}^{(2)}_\phi\;. \tag{95}$$

Using the first relation of (57), one may obtain

$$\mathbf{Q}^{(1)}_\phi = \prod_m \frac{u_m}{u_{\phi,m}}\;\mathbf{X}\mathbf{Q}\mathbf{X}^{-1}\;,\quad \mathbf{Q}^{(2)} = \prod_m \omega^{1/2}\frac{u_{\phi,m}w_m}{\lambda_\phi}\;\mathbf{Z}\mathbf{Q}\mathbf{X}\;, \tag{96}$$

where $\mathbf{X}$ and $\mathbf{Z}$ are given by (90).

Turn now to the homogeneous chain. Using (89), we obtain

$$\mathbf{t}(\lambda_\phi)\mathbf{Q}(\lambda_\phi) = \delta_\phi^{-M}\mathbf{Q}(\omega^{-1}\lambda_\phi) + \left(-\omega^{-1/2}\frac{\delta_\phi}{\lambda_\phi}\right)^M \mathbf{Y}\mathbf{Q}(\omega\lambda_\phi)\ , \tag{97}$$

where δ_ϕ-argument of all $\mathbf{Q}$-s remains unchanged, and $\mathbf{t}$ and $\mathbf{Q}$ may be diagonalized simultaneously. Let $t(\lambda)$ and $q_t(\lambda)$ be the eigenvalues of $\mathbf{t}$ and $\mathbf{Q}$ for the same eigenvector, then (97) provides the functional equation

$$t(\lambda)q_t(\lambda) = \delta^{-M}q_t(\omega^{-1}\lambda) + \left(-\omega^{-1/2}\frac{\delta}{\lambda}\right)^M \omega^\gamma q_t(\omega\lambda)\ , \tag{98}$$

where ω^γ is the eigenvalue of $\mathbf{Y}$, and q_t is the meromorphic function on the curve (21). Detailed investigations of (98) show that for generic λ any solution of (98) such that $t(\lambda)$ is a polynomial and $q_t(\lambda,\delta)$ is a meromorphic function on the curve (21), gives the eigenvalue of $\mathbf{t}(\lambda)$ and $\mathbf{Q}(\lambda)$.

6. Separating operator

6.1. **Spectrum of b.** For the homogeneous chain (74) the value of $B(\lambda^N)$, (15), may be calculated immediately:

$$B(\lambda^N) \;=\; (-)^{N-1}\frac{1}{\lambda^N}\prod_{k=1}^{M-1}\left(1 - \frac{\Lambda_{\phi_k}}{\lambda^N}\right)\ , \tag{99}$$

where Λ_ϕ is given by the last equation of (61) and ϕ_k is exactly the set $\mathfrak{F}_M$, (60). This defines the parameterization of the vectors $|\{\lambda_k\}_{k=1}^{M-1},\gamma\rangle$, see (19).

In [1] it was established the following fact:

Proposition 6. *Let $\lambda_k = \lambda_{\phi_k}$ are defined by (76) and the phases of λ_k are fixed. Operator $\mathbf{K}_{\{\lambda\}}$, corresponding to the case when all $g_k = 1$, $k = 1...M-1$ in eqs. (75,70), makes the quantum separation of variables:*

$$\mathbf{b}(\lambda_k)\,\mathbf{K}_{\{\lambda\}} \;=\; 0\ ,\quad k = 1...M-1\ . \tag{100}$$

This means

$$\mathbf{K}_{\{\lambda\}} \;=\; \sum_\gamma\ |\{\lambda_k\}_{k=1}^{M-1},\gamma\rangle\,\langle\chi_\gamma|\ , \tag{101}$$

where χ_γ are some vectors, such that $\langle\chi_\gamma|\,\mathbf{Y} = \omega^\gamma\,\langle\chi_\gamma|$, where $\mathbf{Y}$ is given by (14). Parameterization

$$(102)\qquad \left(\tau_m^{(n)}\right)^N \;=\; \Theta_m^{(n)}(0)\;,\quad \left(\theta_m^{(n)}\right)^N \;=\; \Theta_m^{(n)}(1)$$

is not appropriate for (83), because of $\tau_m^{(n)} = 0$ for $m = 0...n-1$, and $u_m^{(n)}$ are ambiguous. Thus one needs to define in addition a limiting procedure for (102). In general, the ambiguity in (78) corresponds to the arbitrariness of the right eigenstates χ_γ of $\mathbf{K}_{\{\lambda\}}$.

6.2. **Limiting procedure.** In this section we describe an appropriate limiting procedure. Most simple $\mathbf{K}_{\{\lambda\}}$ appears when we choose $u_m^{(M-1)} = 0$ for $m = 1...M-1$. Namely, consider the set of infinitely small numbers,

$$(103)\qquad \varepsilon_k \;\mapsto\; 0\;,\quad k = 1...M-1\;,$$

such that any ratio $\varepsilon_n/\varepsilon_m \neq 1$ is finite, and

$$(104)\qquad u_m^{(M-1)} \;=\; -\omega^{-1/2}\,\frac{\varepsilon_{M-m}}{\delta_1....\delta_{M-1}}\;,\quad m = 1...M-1\;,$$

so that

$$(105)\qquad u_M^{(M-1)} \;=\; -\omega^{-1/2}\,\frac{(\delta_1...\delta_{M-1})^{M-1}}{\varepsilon_1...\varepsilon_{M-1}}\;.$$

Recursion (32,33) with

$$(106)\qquad \kappa_{\phi_n} \;=\; \frac{\kappa}{\delta_n}$$

implies

$$(107)\qquad u_m^{(n)} \;=\; -\omega^{-1/2}\,\frac{\varepsilon_{n-m+1}}{\delta_1...\delta_n} \;+\; o(\varepsilon_1)\;,\quad m = 1...n$$

while for $m = n+1...M-1$ all $u_m^{(n)}$ are regular. This means

$$(108)\qquad \tau_m^{(n)} \;=\; \tau_n^{(n)}\,\frac{\varepsilon_1...\varepsilon_{n-m}}{(\delta_1...\delta_n)^{n-m}} \;+\; o(\varepsilon_1^{n-m})\;,\quad m = 0...n\;.$$

$w_m^{(n)}$ as explicit functions of ε_k are rather complicated, and speaking truly, we may say nothing about $\mathbf{t}^{(M-1)}$ in the meantime. But in the limit $\varepsilon_k = 0$ the following formulas may be chosen:

$$(109)\qquad \frac{\tau_n^{(n)}}{\theta_n^{(n)}} \;=\; \prod_{k=1}^{n} c_k\;,$$

where c_k are given by (80) and

$$\theta_m^{(n)} = (\omega^{1/2}\kappa)^{(n-m)(n-m-1)/2} \frac{\theta_n^{(n)}}{(\delta_1 ... \delta_n)^{n-m}} . \tag{110}$$

On N-th powers these formulas follow from proposition (3). Eq. (110) implies in part

$$\frac{w_{m+1}^{(n)}}{w_m^{(n)}} = \omega^{1/2}\kappa , \quad m = 0...n-1 . \tag{111}$$

Now in the limit $\varepsilon_k \mapsto 0$ the following parameterization of the ambiguous and singular points arises (notation for the singular points are described by (53,54)):

- $\underline{m = 1...n-1}$:

$$p_{3,m}^{(n)} = q_1 , \quad p_{2,m}^{(n)} = O(p_{1,m}^{(n)}) . \tag{112}$$

  where $x_{1,m}^{(n)} = \dfrac{1}{\omega^{1/2}\kappa} \dfrac{\varepsilon_{n-m+1}}{\varepsilon_{n-m}}$.
- $\underline{m = n}$:

$$p_{1,n}^{(n)} = q_0 , \quad p_{2,n}^{(n)} = q_\infty \quad \forall\, n \tag{113}$$

  and $p_{3,n}^{(n)}$ are regular, except

$$p_{3,M-1}^{(M-1)} = q_\infty . \tag{114}$$

- $\underline{m = n+1...M-2}$: All $p_{j,m}^{(n)}$ in this region are regular.
- $\underline{m = M-1}$: $p_{1,M-1}^{(n)}$ are regular except

$$p_{1,M-1}^{(M-1)} = q_0 , \tag{115}$$

  and

$$p_{2,M-1}^{(n)} = p_{3,M-1}^{(n)} = q_\infty \quad \forall\, n . \tag{116}$$

- $\underline{m = M}$:

$$p_{1,M}^{(n)} = q_\infty , \quad p_{2,M}^{(n)} = p_{3,M}^{(n)} = q_0 \quad \forall\, n. \tag{117}$$

Substituting these expressions into (85), one obtains the following form of n-th modified Q-matrix, $n \neq M-1$:

$$\begin{aligned}
\langle \alpha | \mathbf{Q}^{(n)} | \beta \rangle \;=\; & \frac{1}{\Phi(\beta_1+\beta_M-\alpha_M)} \prod_{m=1}^{n-1} \delta_{\alpha_m,\beta_{m+1}} \times \\
& \Phi(\alpha_n)\, w_{Op^{(n)}_{3,n}}(\alpha_n-\beta_{n+1}) \times \\
& \prod_{m=n+1}^{M-2} \omega^{(\alpha_m-\beta_m)\beta_{m+1}} \frac{w_{p^{(n)}_{1,m}}(\beta_m-\alpha_m)\, w_{p^{(n)}_{2,m}}(\beta_{m+1}-\beta_m)}{w_{p^{(n)}_{3,m}}(\beta_{m+1}-\alpha_m)} \times \\
& \frac{\Phi(\alpha_{M-1})}{\Phi(\beta_{M-1})}\, w_{p^{(n)}_{1,M-1}}(\beta_{M-1}-\alpha_{M-1})\;,
\end{aligned} \tag{118}$$

and for the last $(M-1)$th one

$$\langle \alpha | \mathbf{Q}^{(M-1)} | \beta \rangle \;=\; \frac{1}{\Phi(\beta_1+\beta_M-\alpha_M)} \prod_{m=1}^{M-2} \delta_{\alpha_m,\beta_{m+1}}\, \Phi(\alpha_{M-1})\;. \tag{119}$$

Here Φ is given by (51). Explicit form of the modified Q-operators (118) and (119) allows one to prove the following

Proposition 7.

$$\mathbf{K}_{\{\lambda\}}\,(\mathbf{z}_m-\mathbf{z}_{m+1}) \;=\; 0\;, \quad m \in \mathbb{Z}_M\;. \tag{120}$$

So, in our limiting procedure χ is simple:

$$\langle \chi_\gamma | \alpha \rangle \;=\; \chi_\gamma(\overline{\alpha}) \;\equiv\; \chi_\gamma(\overline{\alpha} \mod N), \tag{121}$$

where

$$\overline{\alpha} \;\stackrel{def}{=}\; \sum_{m\in\mathbb{Z}_M} \alpha_m \tag{122}$$

and

$$\frac{\chi_\gamma(\overline{\alpha}+M)}{\chi_\gamma(\overline{\alpha})} \;=\; (-\omega^{-1/2})^M\, \omega^{\gamma-\overline{\alpha}}\;. \tag{123}$$

7. Discussion

In this paper we investigated the simplest integrable model, associated with the local Weyl algebra at the root of unity. All such models always contain the classical discrete dynamic of parameters. Nontrivial solution of the classical counterpart provides the solution of the isospectrality problem of the finite dimensional counterpart. Well known result by Sklyanin, Kuznetsov at al was that in the classical limit of the usual Toda chain (and many other models) Baxter's quantum Q-operator corresponds to the Bäcklund transformation of the classical system, see e.g. [8, 10, 11, 12, 16] etc. In our case we have the Bäcklund transformation of the classical counterpart and modified Q-operator in the quantum space simultaneously. Unusual is that solving the quantum isospectrality problem, we miss the commutativity of the modified Q-operators.

Nevertheless, our main result, we think, is the explicit construction of $(M-1)$-parametric family of quantum inhomogeneous transfer matrices with the same spectrum as the initial homogeneous one, and the explicit construction of the corresponding similarity operator (43). We hope, the solution of the isospectrality problem will help to solve the model with arbitrary N explicitly.

As one particular application of it we have obtained the quantum separation of variables. Previously there was a hypothesis, formulated for the usual quantum Toda chain, that the product of *operators* Q, taken in the special points, is related to the quantum Functional Bethe Ansatz. In this paper we have established it explicitly, but for the product of *modified operators* Q. Below some explanations are given.

Let the solitonic amplitudes g_k are in general position. Let $|\Psi_t^{(n)}\rangle$ be the complete set of eigenvectors of $\mathbf{t}^{(n)}(\lambda)$, and $\langle\Psi_t^{(n)}|$ be the corresponding set of co-vectors,

$$\mathbf{t}^{(n)}(\lambda) \;=\; \sum_t \,|\Psi_t^{(n)}\rangle\, t(\lambda)\, \langle\Psi_t^{(n)}|\;. \tag{124}$$

Then

$$\mathbf{Q}_{\phi_n}^{(n)} \;=\; \sum_t \,|\Psi_t^{(n-1)}\rangle\, q_t(\lambda_n)\, \langle\Psi_t^{(n)}|\;, \tag{125}$$

where $q_t(\lambda)$ is evidently the eigenvalues of simple Q-operators, as in (98). Note, the functional Baxter equation exists also and for arbitrary inhomogeneous chain, but to obtain it from (95) one has to use the decomposition like (125).

With (125) taken into account, $\mathbf{K}$-operator for general set of g_k is

$$\mathbf{K} \;=\; \sum_t \,|\Psi_t^{(0)}\rangle\, q_t(\lambda_1)...q_t(\lambda_{M-1})\,\langle\Psi_t^{(M-1)}|\;, \tag{126}$$

In the limit when all $g_k = 1$, the sets $\langle\Psi_f^{(n)}|$ become degenerative, e.g.

$$\langle\Psi_t^{(n)}|\,(\mathbf{z}_1 - \mathbf{z}_M) \;=\; 0 \quad \forall\, t\;. \tag{127}$$

For us it is still mysterious what happens with the set $|\Psi_t^{(n)}\rangle$ and co-set $\langle\Psi_t^{(n)}|$ when $g_k = 1$ – this is the subject of separate investigation. But nevertheless, in [1] and in this paper it is proven that $\mathbf{K}$ is given by (126) and simultaneously by (101). Thus Sklyanin's formula appears explicitly:

$$\langle\Psi_t^{(0)}|\{\lambda_k\}_{k=1}^{M-1},\gamma\rangle \;=\; \text{const}\,\prod_{k=1}^{M-1} q_t(\lambda_k)\;. \tag{128}$$

Well known is that to solve Baxter's equation for $N > 2$ is quite hopeless (pure fermionic case $N = 2$ is trivial, (98) may be solved in one string). Only way is to avoid Baxter's equation for high genus curve. So, we think, the attention is to be paid to the special cases of the isospectral transfer matrices like $g_k = 1$, when several simplifications for the eigenstates are expected.

Note in conclusion, this method may be applied to any model, based on the local Weyl algebra. Mentioned are to be the chiral Potts model [6, 7] and the Zamolodchikov-Bazhanov-Baxter model in the vertex formulation [15]. To be exact, all two dimensional integrable models with the local Weyl algebra are particular cases of the general three dimensional model, and their classical counterparts are known [5].

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