$K3$ PROJECTIVE MODELS IN SCROLLS

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Abstract. We study the projective models of complex $K3$ surfaces polarized by a line bundle L such that all smooth curves in $|L|$ have non-general Clifford index. Such models are in a natural way contained in rational normal scrolls.

We use this study to classify and describe all projective models of $K3$ surfaces of genus $g \leq 10$.

1. Introduction

It is well known that all $K3$ surfaces are diffeomorphic, and that there is a 20-dimensional family of analytical isomorphism classes of $K3$ surfaces. Moreover there is a countable union of 19-dimensional families of $K3$ surfaces with base point free line bundles on them.

A pair (S, L) of a $K3$ surface S and a base point free line bundle L with $L^2 = 2g - 2$ will be called a polarized $K3$ surface of genus g. The sections of L give a map ϕ_L of S into $\mathbf{P}^g$, and the image is called a projective model of S. If ϕ_L is birational, the image is a surface of degree $2g - 2$ in $\mathbf{P}^g$.

The following is well known: For $g = 3$ the birational projective model is a quartic surface and for $g = 4$ a complete intersection of a quadric and a cubic hypersurface. For $g = 5$ the general model is a complete intersection of three hyperquadrics. For $6 \leq g \leq 10$ and $g = 12$ it is shown by Mukai in [Mu1] and [Mu2] that the general projective model is a complete intersection in certain homogeneous varieties contained in projective spaces of larger dimension than g.

We will study the projective models of a particular kind of polarized $K3$ surfaces. Recall the result of Green and Lazarsfeld [G-L2], which states that if L is a base point free line bundle on a $K3$ surface S, then Cliff C is constant for all smooth irreducible $C \in |L|$, and if Cliff $C < \lfloor \frac{g-1}{2} \rfloor$, then there exists a line bundle M on S such that $M_C := M \otimes \mathcal{O}_C$ computes the Clifford index of C for all smooth irreducible $C \in |L|$.

It therefore makes sense to define the Clifford index Cliff L of a base point free line bundle, or the Clifford index $\mathrm{Cliff}_L(S)$ of a polarized $K3$ surface (S, L) as the Clifford index of the smooth curves in $|L|$.

By the *Existence Theorem* 3.8 below, for any pair of integers (g, c) such that $g \geq 2$ and $0 \leq c \leq \lfloor \frac{g-1}{2} \rfloor$, there exists an 18-dimensional family of polarized $K3$ surfaces of genus g and of Clifford index c.

In this paper we study and classify projective models of non-Clifford general polarized $K3$ surfaces of any genus larger than two.

The central point is that by the result of Green and Lazarsfeld, there exists in these cases a divisor class D on S computing the Clifford index of L. We will show that $0 \leq D^2 \leq c+2$. We can choose such a divisor that is base point free, so that in particular the general member of $|D|$ is a smooth curve. Such a divisor (class) will be called a *free Clifford divisor* for L. (The definition only depends on the class of D.)

The images of the members of $|D|$ by ϕ_L span sublinear spaces inside $\mathbf{P}^g$. Each subpencil $\{D_\lambda\}$ within the complete linear system $|D|$ then gives rise to a pencil of sublinear spaces.

For each fixed pencil the union of these spaces will be a rational normal scroll $\mathcal{T}$. We will investigate under which conditions this scroll is smooth. These scrolls are the natural ambient spaces for non-Clifford general $K3$ surfaces.

In the cases $c = 1$ and 2 with $D^2 = 0$, the description of the projective models is particularly nice, since they are then complete intersections in their corresponding scrolls.

If $\mathcal{T}$ is smooth, we will be able to find a resolution (up to certain invariants) of $\phi_L(S)$ inside $\mathcal{T}$ for arbitrary c.

If $\mathcal{T}$ is singular, we take the blow up $f : \tilde{S} \rightarrow S$ at the D^2 base points of the pencil $\{D_\lambda\}$ and show that the projective model $\phi_H(\tilde{S})$ of $\tilde{S}$ by the base point free line bundle $H := f^*L + f^*D - E$, where E is the exceptional divisor, is contained in a smooth rational normal scroll $\mathcal{T}_0$ which is a smoothing of $\mathcal{T}$. We find a resolution of $\phi_H(\tilde{S})$ inside $\mathcal{T}_0$ and investigate the possibility of pushing down this resolution to a resolution of $\phi_L(S)$ inside $\mathcal{T}$.

We also give a description of those projetive models for $g \leq 10$ that are Clifford general, but still not general in the sense of Mukai (i.e. they are not complete intersections in homogeneous spaces). These models are also contained in scrolls, and can be analysed in a similar manner. This can also be done for $g = 12$, which we leave to the reader.

Together with Mukai's results this will then give a complete picture of the birational projective models for $g \leq 10$ (and $g = 12$). For $g = 11$ and $g \geq 13$ our description of non-Clifford general projective models is not supplemented by any description of general projective models at all. We hope, however, that our description of the non-general models may have some interest in themselves.

The paper is organized as follows.

In Section 2 we recall some basic facts about rational normal scrolls, and how to obtain such scrolls from surfaces with pencils on them. In Section 3 we define the essential concept of a free Clifford divisor. We end the section by proving the above mentioned existence theorem.

In Section 4 we study in detail the singular loci of the projective model $\phi_L(S)$ and the scroll $\mathcal{T}$ in which we choose to view this model as contained. We show that we can always find a free Clifford divisor D such that the singular locus of $\mathcal{T}$ is "spanned" by the images of the base points of the pencil $\{D_\lambda\}$ and the contractions of smooth rational curves across the members of the pencil. We also include a study of the projective model if $c = 0$ (the hyperelliptic case), which is due to Saint-Donat [SD]. Some of the longer proofs of the results in this section are postponed until Section 5.

In Section 6 we study the resolution of $\phi_L(S)$ inside its scroll $\mathcal{T}$ when $\mathcal{T}$ is smooth. In this case a general hyperplane section of $\mathcal{T}$ is a scroll formed in a similar way from a pencil computing the gonality on a canonical curve C of genus g (the gonality is $c+2$). Such scrolls were studied in [Sc], and our results for $K3$ surfaces in smooth scrolls are quite parallel to those of [Sc].

In Section 7 we treat the case when the scroll $\mathcal{T}$ is singular. The approach is to study the blow up $f : \tilde{S} \rightarrow S$ at the D^2 base points of the pencil $\{D_\lambda\}$ and the projective model $S'' := \phi_H(\tilde{S})$ of $\tilde{S}$ by the base point free line bundle $H := f^*L + f^*D - E$, where E is the exceptional divisor. The pencil $|f^*D - E|$ defines a smooth rational normal scroll $\mathcal{T}_0$ that contains S'' and is a smoothing of $\mathcal{T}$. We use results from [Gr] to show how resolutions of $\phi_L(D_\lambda)$ in its linear span in $\mathbf{P}^g$, for each D_λ (from the chosen subpencil of $|D|$), can be used to obtain a resolution of S'' in $\mathcal{T}_0$. We prove that S'' is normal, and use this to give more details about the resolution. We give conditions under which we can push down the resolution to one of $\phi_L(S)$ in $\mathcal{T}$. Here we use results from [Sc]. We end the section by investigating some examples for low genera.

In Section 8 we study the set of projective models in smooth scrolls for $c = 1$, 2 and 3 ($< \lfloor \frac{g-1}{2} \rfloor$). We study the sets of projective models in $(c+2)$-dimensional scrolls of given types. Since the scroll type is dependent on which rational curves that exist on S, and therefore on the Picard lattice, it is natural that the dimension of the set of models in question in a scroll as described, is dependent on the scroll type. We study this interplay, and obtain a fairly clear picture for $c = 1$ and 2. For $c = 3$ we study a Pfaffian map of the resolution of $\phi_L(S)$ in the scroll. We give a conjecture for the dimension of the set of projective $K3$ models inside a smooth scroll of a given type, for arbitrary $c < \lfloor \frac{g-1}{2} \rfloor$).

In Section 9 we study the projective models for $g \leq 10$ that are Clifford general but not general in the sense of Mukai (i.e. they are not complete intersections in homogeneous spaces). By the concrete description in [Mu2] of such surfaces, it follows that their projective models are also contained in scrolls. We analyse them in a similar manner.

We conclude by giving a complete list and descripton of *all* birational projective models of $K3$ surfaces for $g \leq 10$.

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1.2. **Notation and conventions.** We use standard notation from Algebraic Geometry.

The ground field is the field of complex numbers. All surfaces are *algebraic surfaces.*

By a *curve* on a surface S is always meant an *irreducible curve* (possibly singular), i.e. a prime divisor. Line bundles and divisors are used with little or no distinction, as well as the multiplicative and additive notation. Linear equivalence of divisors is denoted by $\sim$, and numerical equivalence by $\equiv$. Note that on a $K3$ surface linear and numerical equivalence is the same.

If L is any line bundle on a surface, L is said to be *numerically effective*, or simply *nef*, if $L.C \geq 0$ for all curves C on S. In this case L is said to be *big* if $L^2 > 0$.

If $\mathcal{F}$ is any coherent sheaf on a variety V, we shall denote by $h^i(\mathcal{F})$ the complex dimension of $H^i(V, \mathcal{F})$, and by $\chi(\mathcal{F})$ the Euler characteristic $\sum(-1)^i h^i(\mathcal{F})$.

If D is any divisor on a normal surface S, Riemann-Roch for D is $\chi(\mathcal{O}_S(D)) = \frac{1}{2}D.(D + K_S) + \chi(\mathcal{O}_S)$.

By a $K3$ surface is meant a smooth surface S with trivial canonical bundle and such that $H^1(\mathcal{O}_S) = 0$. In particular $h^2(\mathcal{O}_S) = 1$ and $\chi(\mathcal{O}_S) = 2$.

We will make use of the following results of Saint-Donat on line bundles on $K3$ surfaces. The first result will be used repeatedly, without further mention.

Proposition 1.1. [SD, Cor. 3.2] *A complete linear system on a $K3$ surface has no base points outside of its fixed components.*

Proposition 1.2. [SD, Prop. 2.6] *Let L be an invertible sheaf on a $K3$ surface S such that $|L| \neq \emptyset$ and such that $|L|$ has no fixed components. Then either*

(i) *$L^2 > 0$ and the general member of $|L|$ is an irreducible curve of arithmetic genus $L^2/2 + 1$. In this case $h^1(L) = 0$, or*
(ii) *$L^2 = 0$, then $L \simeq \mathcal{O}_S(kE)$, where k is an integer ≥ 1 and E is an irreducible curve of arithmetic genus 1. In this case $h^1(L) = k - 1$ and every member of $|L|$ can be written as a sum $E_1 + \cdots + E_k$, where $E_i \in |E|$ for $i = 1, \ldots, k$.*

Note that by Bertini the general members in $|L|$ and $|E|$ are smooth.

Lemma 1.3. [SD, 2.7] *Let L be a nef line bundle on a $K3$ surface S. Then $|L|$ is not base point free if and only if there exist smooth irreducible curves E, Γ and an integer $k \geq 2$ such that*

$$L \sim kE + \Gamma, \quad E^2 = 0, \quad \Gamma^2 = -2, \quad E.\Gamma = 1.$$

In this case, every member of $|L|$ is of the form $E_1 + \cdots + E_k + \Gamma$, where $E_i \in |E|$ for all i.

To show the existence of $K3$ surfaces with certain divisors on it, a very useful result is the following by Morrison:

Proposition 1.4. [Mo, Cor. 2.9(i)] *Let $\rho \leq 10$ be an integer. Then every even lattice of signature $(1, \rho - 1)$ occurs as the Picard group of some algebraic $K3$ surface.*

Consider now the group generated by the Picard-Lefschetz reflections

$$\begin{array}{rcl} \phi_\Gamma : \operatorname{Pic} S & \longrightarrow & \operatorname{Pic} S \\ D & \mapsto & D + (D.\Gamma)\Gamma \end{array}$$

where $\Gamma \in \operatorname{Pic} S$ satisfies $\Gamma^2 = -2$. Note that a reflection leaves the intersections between divisors invariant.

The following result will also be useful for our purposes:

Proposition 1.5. [B-P-V, VIII, Prop. 3.9] *A fundamental domain for this action is the big-and-nef cone of S.*

This means that given a certain Picard lattice, we can perform Picard-Lefscetz reflections on it, and thus assume that some particular line bundle chosen (with positive self-intersection) is nef.

2. Surfaces in scrolls

In the beginning of this section we briefly review some basic facts that can be found in [Sc].

Definition 2.1. *Let $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_d)$, with $e_1 \geq \ldots \geq e_d \geq 0$ and $f = e_1 + \cdots + e_d \geq 2$. Consider the linear system $\mathcal{L} = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ on the corresponding $\mathbf{P}^{d-1}$-bundle $\mathbf{P}(\mathcal{E})$ over $\mathbf{P}^1$. We map $\mathbf{P}(\mathcal{E})$ into $\mathbf{P}^r$ with the complete linear system $H^0(\mathcal{L})$. Then $r = f + d - 1$. The image T is by definition a rational normal scroll of type* $\mathbf{e}= (e_1, \ldots, e_d)$. *The image is smooth, and isomorphic to $\mathbf{P}(\mathcal{E})$, if and only if $e_d \geq 1$.*

Remark 2.2. Some authors, like in [P-S], use the term *rational normal scroll* only if $e_d \geq 1$ (so that T is smooth), but for our purposes it will be more convenient to use the more liberal definition above.

Let L be a spanned line bundle on a smooth surface S. We denote by ϕ_L the natural morphism

$$\phi_L : S \longrightarrow \mathbf{P}^{h^0(L)-1} := \mathbf{P}^g$$

defined by the complete linear system $|L|$.

Assume that L can be decomposed as

$$L \sim M + N,$$

where $M \simeq \mathcal{O}_S(D)$ and $N \simeq \mathcal{O}_S(F)$ are line bundles on S (with D and F their corresponding divisors) satisfying $h^0(M)$, $h^0(N) \geq 2$. Choose a 2-dimensional subspace $W \subseteq H^0(S,D)$, which then defines a pencil

$$\{D_\lambda\}_{\lambda \in \mathbf{P}^1} \subseteq |D|.$$

Each $\phi_L(D_\lambda)$ will span a $(h^0(L) - h^0(L-D) - 1)$-dimensional subspace of $\mathbf{P}^g$, which is called the linear span of $\phi_L(D_\lambda)$ and denoted by $\overline{D_\lambda}$. It turns out that the variety swept out by these linear spaces,

$$T = \cup_{\lambda \in \mathbf{P}^1} \overline{D_\lambda} \subseteq \mathbf{P}^g,$$

is a rational normal scroll:

Proposition 2.3. [Sc] *The multiplication map*

$$W \otimes H^0(S,F) \longrightarrow H^0(S,L)$$

yields a $2 \times h^0(F)$ matrix with linear entries whose 2×2 minors vanish on $\phi_L(S)$. The variety T defined by these minors contains $\phi_L(S)$ and is a rational normal scroll of degree $f := h^0(F)$ and dimension $d := h^0(L) - h^0(L-D)$. In particular $d + f = g + 1$.

Decomposing the pencil $\{D_\lambda\}$ into its moving part $\{D'_\lambda\}$ and fixed part Δ,

$$D_\lambda \sim D'_\lambda + \Delta,$$

the type $(e_1, \dots, e_d)$ of the scroll T is given by

$$e_i = \#\{j \mid d_j \geq i\} - 1, \tag{1}$$

where

$$\begin{aligned} d = d_0 \quad &:= \quad h^0(L) - h^0(L-D), \\ d_1 \quad &:= \quad h^0(L-D) - h^0(L - 2D' - \Delta), \\ &\vdots \\ d_i \quad &:= \quad h^0(L - iD' - \Delta) - h^0(L - (i+1)D' - \Delta), \\ &\vdots \end{aligned}$$

Conversely, if $\phi_L(S)$ is contained in a scroll T of degree f, the ruling of T will cut out on S a pencil of divisors (possibly with base points) $\{D_\lambda\} \subseteq |D|$ with $h^0(L-D) = f \geq 2$, whence inducing a decomposition

$$L \sim M + N,$$

as above.

We will study these scrolls more thoroughly in the case where S is a $K3$ surface. We will see that in the $K3$ case natural decompositions of L as above will occur when the smooth curves in $|L|$ do not have the general Clifford index. We will turn back to this in the next section. First we get some general results.

For any decomposition $L \sim D + F$, with $h^0(D)$, $h^0(F) \geq 2$, write $D.F = c + 2$, for some integer c. Assuming $D.L \leq F.L$, or equivalently $D^2 \leq F^2$, we have by the Hodge index theorem that D satisfies the numerical conditions below:

$$2D^2 \overset{(i)}{\leq} L.D = D^2 + c + 2 \overset{(ii)}{\leq} 2c + 4$$

with equality in (i) or (ii) if and only if $L \equiv 2D$ and $L^2 = 4c + 8$.

Indeed, the condition $D.L \leq F.L$ can be rephrased as $2D.L \leq L^2$, and by the Hodge index theorem $2D^2(D.L) \leq D^2L^2 \leq (D.L)^2$, with equalities only if $L \equiv 2D$.

If the set

$$\mathcal{A}(L) := \{D \in \operatorname{Pic} S \mid h^0(D), h^0(L-D) \geq 2\}$$

is nonempty, define the integer $\mu(L)$ as

$$\begin{aligned} \mu(L) &:= \min\{D.F - 2 \mid L \sim D + F \text{ and } D, F \in \mathcal{A}(L)\} \\ &= \min\{D.L - D^2 - 2 \mid D \in \mathcal{A}(L)\} \end{aligned}$$

and set

$$\mathcal{A}^0(L) := \{D \in \mathcal{A}(L) \mid D.(L-D) = \mu(L) + 2\}$$

For $K3$ surfaces we have the following result:

Proposition 2.4. *Let L be a spanned and big line bundle on a $K3$ surface S such that $\mathcal{A}(L) \neq \emptyset$. Then $\mu(L) \geq 0$ and any divisor D in $\mathcal{A}^0(L)$ will have the following properties:*

(i) *the (possibly empty) base locus Δ of D satisfies $L.\Delta = 0$,*

(ii) $h^1(D) = 0$.

Proof. The first statement follows from the fact that any member of the complete linear system of a spanned and big line bundle on a $K3$ surface is numerically 2-connected (see [SD, (3.9.6)], or [Kn1, Thm. 1.1] for a more general statement).

We first show (i).

If D is nef but not base point free, then by Lemma 1.3, $D \sim kE + \Gamma$, for an integer $k \geq 2$ and divisors E and Γ satisfying $E^2 = 0$, $\Gamma^2 = -2$ and $E.\Gamma = 1$. Since L is spanned, we must have $E.L \geq 2$ [Kn1, Thm. 1.1], so $D.L - D^2 = (kE + \Gamma).L - (2k-2) \geq kE.L - 2k - 2 \geq E.L + 2(k-1) - 2(k-1) = E.L$, which implies $E.L = 2$, $\mu(L) = 0$, and as asserted $\Gamma.L = 0$.

If D is not nef, there exists a smooth rational curve Γ such that $\Gamma.D < 0$. Letting $D' := D - \Gamma$ and we have $D' \in \mathcal{A}(L)$ and $L.D' - D'^2 - 2 = D.(L-D) - L.\Gamma + 2\Gamma.D + 2 \geq D.(L-D)$, whence $L.\Gamma = 0$, $\Gamma.D = -1$, $D'^2 = D^2$, $L.D' = L.D$ and $D'(L-D') = D.(L-D) = c + 2$. Continuing inductively, we get that $\Delta.L = 0$, as desired.

Since $\Delta.L = 0$ and $(D-\Delta).L - (D-\Delta)^2 \geq D.L - D^2$, we must have $D^2 \geq (D-\Delta)^2 \geq 0$.

We now prove (ii).

If $h^1(D) \neq 0$, there exists by Ramanujam's lemma an effective decomposition $D \sim D_1 + D_2$ such that $D_1.D_2 \leq 0$. By the Hodge index theorem (and the fact that $D^2 \geq 0$) we can assume $D_1^2 \geq 0$ and $D_2^2 \leq 0$, with equalities occurring simultaneously. The divisor D_1 is in $\mathcal{A}(L)$, and writing $F := L - D$ we get $D_1.(F + D_2) = D.F + D_1.D_2 - D_2.F \geq F.D$, whence $D_2.F \leq D_1.D_2 \leq 0$. But L is nef, so $D_2.L = D_2.D + D_2.F \geq 0$, which implies $D_2.F = D_1.D_2 = D_1^2 = D_2^2 = 0$. Now the same argument works for D_1, so $D_1.F = 0$ and we get the contradiction $D.F = (D_1 + D_2).F = 0$.

□

Writing $L \sim D + F$, the above result is of course symmetric in D and F. It turns out that we can choose one of them to have an additional property. More precisely, we have :

Proposition 2.5. *Let L be a spanned line bundle on a $K3$ surface S such that $\mathcal{A}(L) \neq \emptyset$. We can find a divisor D in $\mathcal{A}^0(L)$ such that either $|D|$ or $|L-D|$ (but not necessarily both at the same time) is base point free and its general member is smooth and irreducible. If L is ample, then for any divisor D in $\mathcal{A}^0(L)$ the above conditions will be satisfied for both $|D|$ and $|L-D|$.*

Proof. Let $D \in \mathcal{A}^0(L)$. Denote its base locus by Δ and assume it is not zero. Then $L.\Delta = 0$ by the previous proposition.

If D is nef but not base point free, then $D \sim kE + \Gamma$ as above and the smooth curve E will satisfy the desired conditions.

If D is not nef, there exists a smooth rational curve Γ such that $\Gamma.D < 0$. Letting $D' := D - \Gamma$, we can argue inductively as above until we reach a divisor which is base point free or of the form $kE + \Gamma$.

This procedure can of course not be performed on both D and $L-D$ simultaneously, but if L is ample, they are both automatically base point free.

The fact that the general member of $|D|$ (or $|L-D|$) is a smooth curve now follows from Proposition 1.2, since $h^1(D) = 0$. $\square$

Remark 2.6. Note that by the proofs of the two previous propositions, if D is not nef, then $D'^2 = D^2$. This means that given a divisor $D \in \mathcal{A}(L)$, we can find a divisor $D_0 \in \mathcal{A}(L)$ satisfying the additional conditions in Proposition 2.5 and such that $D_0^2 \leq D^2$.

In the next section we will study the perhaps most important example of such decompositions of the line bundle L, namely the case where all smooth curves in $|L|$ have a non-general Clifford index.

3. The Clifford index of smooth curves in $|L|$ and the definition of the scrolls $\mathcal{T}(c, D, \{D_\lambda\})$

We briefly recall the definition and some properties of gonality and Clifford index of curves.

Let C be a smooth irreducible curve of genus $g \geq 2$. We denote by g^r_d a linear system of dimension r and degree d and say that C is *k-gonal* (and that k is its *gonality*) if C posesses a g^1_k but no g^1_{k-1}. In particular, we call a 2-gonal curve *hyperelliptic* and a 3-gonal curve *trigonal*. We denote by $\operatorname{gon} C$ the gonality of C. Note that if C is k-gonal, all g^1_k's must necessarily be base point free and complete.

If A is a line bundle on C, then the *Clifford index* of A is the integer

$$\operatorname{Cliff} A = \deg A - 2(h^0(A) - 1).$$

The *Clifford index of C* itself is defined as

$$\operatorname{Cliff} C = \min\{\operatorname{Cliff} A \mid h^0(A) \geq 2, h^1(A) \geq 2\}.$$

Clifford's theorem then states that $\operatorname{Cliff} C \geq 0$ with equality if and only if C is hyperelliptic and $\operatorname{Cliff} C = 1$ if and only if C is trigonal or a smooth plane quintic.

At the other extreme, we get from Brill-Noether theory (cf. [A-C-G-H, V]) that the gonality of C satisfies $\operatorname{gon} C \leq \lfloor \frac{g+3}{2} \rfloor$, whence $\operatorname{Cliff} C \leq \lfloor \frac{g-1}{2} \rfloor$. For the general curve of genus g, we have $\operatorname{Cliff} C = \lfloor \frac{g-1}{2} \rfloor$.

We say that a line bundle A on C *contributes to the Clifford index of C* if $h^0(A), h^1(A) \geq 2$ and that it *computes the Clifford index of C* if in addition $\operatorname{Cliff} C = \operatorname{Cliff} A$.

Note that $\operatorname{Cliff} A = \operatorname{Cliff} \omega_C \otimes A^{-1}$.

The *Clifford dimension* of C is defined as

$$\min\{h^0(A) - 1 \mid A \text{ computes the Clifford index of } C\}.$$

A line bundle A which achieves the minimum and computes the Clifford index, is said to *compute* the Clifford dimension. A curve of Clifford index c is $(c+2)$-gonal if and only if it has Clifford dimension 1. For a general curve C, we have $\operatorname{gon} C = c + 2$.

Lemma 3.1. [C-M, Theorem 2.3] *The gonality k of a smooth irreducible projective curve C satisfies*

$$\operatorname{Cliff} C + 2 \leq k \leq \operatorname{Cliff} C + 3.$$

The curves satisfying $\operatorname{gon} C = \operatorname{Cliff} C + 3$ are conjectured to be very rare and called *exceptional* (cf. [Ma, (4.1)]).

Recall the following result of Green and Lazarsfeld:

Theorem 3.2. [G-L2] *Let L be a base point free line bundle on a $K3$ surface S with $L^2 > 0$. Then* Cliff C *is constant for all smooth irreducible $C \in |L|$, and if* Cliff $C < \lfloor \frac{g-1}{2} \rfloor$*, then there exists a line bundle M on S such that $M_C := M \otimes \mathcal{O}_C$ computes the Clifford index of C for all smooth irreducible $C \in |L|$.*

Note that since $(L - M) \otimes \mathcal{O}_C \simeq \omega_C \otimes {M_C}^{-1}$, the result is symmetric in M and $L - M$.

Based on this result, we make the following definition:

Definition 3.3. *Let L be a base point free and big line bundle on a $K3$ surface. We define the* Clifford index of L *to be the Clifford index of all the smooth curves in $|L|$ and denote it by* Cliff L.

Similarly, if (S, L) is a polarized $K3$ surface we will often call Cliff L *the* Clifford index of S *and denote it by* $\operatorname{Cliff}_L(S)$.

It turns out that we can choose the line bundle M appearing in Theorem 3.2 above so that it satisfies certain properties. We will need the following result in the sequel.

Lemma 3.4. [Kn1, Lemma 8.3] *Let L be a base point free line bundle on a $K3$ surface S with $L^2 = 2g - 2 \geq 2$ and* Cliff $L = c$.

If $c < \lfloor \frac{g-1}{2} \rfloor$, then there exists a smooth curve D on S satisfying $0 \leq D^2 \leq c + 2$, $2D^2 \leq D.L$ (either of the latter two inequalities being an equality if and only if $L \sim 2D$) and

$$\operatorname{Cliff} C = \operatorname{Cliff}(\mathcal{O}_S(D) \otimes \mathcal{O}_C) = D.L - D^2 - 2$$

for any smooth curve $C \in |L|$.

It is also known (see e.g. [Ma]) that D satisfies $h^0(D \otimes \mathcal{O}_C) = h^0(D)$ and $h^0((L - D) \otimes \mathcal{O}_C) = h^0(L - D) = h^1(D \otimes \mathcal{O}_C)$ for any smooth curve $C \in |L|$.

From the results in the previous section, it is also clear that

$$\operatorname{Cliff} C = \min\{\mu(L), \lfloor \frac{g-1}{2} \rfloor\}$$

for any smooth $C \in |L|$.

Summarizing, and using Propositions 2.4 and 2.5, we have that if L is a spanned line bundle on a $K3$ surface S, of *sectional genus*

$$g = g(L) = \frac{1}{2}L^2 + 1,$$

and the smooth curves in $|L|$ have Clifford index

$$c < \lfloor \frac{g-1}{2} \rfloor = \lfloor \frac{L^2}{4} \rfloor,$$

(which in particular implies $L^2 \geq 4c + 4$), then there exists a divisor (class) D on S with the following properties (with $F := L - D$):

(C1) $c = D.L - D^2 - 2 = D.F - 2$,

(C2) $D.L \leq F.L$ (eqv. $D^2 \leq F^2$) and if equality occurs, then either $L \sim 2D$ or $h^0(2D-L) = 0$,
(C3) $h^1(D) = h^1(F) = 0$.

A divisor (class) D with the properties (C1) and (C2) above will be called a *Clifford divisor* for L. This means in other words that D and $L-D$ compute the Clifford index of all smooth curves in $|L|$. The property (C2) can be considered an ordering of D and $L-D$. Any Clifford divisor will automatically fulfill property (C3) and the (possibly empty) base loci Δ of $|D|$ and Δ' of $|L-D|$ will satisfy $L.\Delta = L.\Delta' = 0$ by Proposition 2.4.

By Proposition 2.5 we can find a Clifford divisor D satisfying the properties

(C4) the (possibly empty) base locus Δ of F satisfies $L.\Delta = 0$,
(C5) $|D|$ is base point free and its general member is a smooth curve,

A divisor D satisfying all properties (C1)-(C5) will be called a *free Clifford divisor* for L. Since this definition only depends on the class of D, we will by abuse of notation never distinguish between D and its divisor class. Hopefully, this will not cause any confusion.

We will now take a closer look at two particular kinds of free Clifford divisors, namely:

(a) $D^2 = c+1$, or
(b) $D^2 = c$, $L \sim 2D + \Delta$, with $\Delta > 0$.

It turns out that these free Clifford divisors are of a particular form.

Proposition 3.5. *Let L be a spanned and big line bundle of non-general Clifford index c on a $K3$ surface, and let D be a free Clifford divisor.*

If D is as in (a) above, then $L^2 = 4c+6$ and

(E0) $L \sim 2D + \Gamma$, *where Γ is a smooth rational curve satisfying* $\Gamma.D = 1$.

If D is as in (b) above, then $L^2 = 4c+4$ and (letting all Γ_i be smooth rational curves) either

(E1) $L \sim 2D + \Gamma_1 + \Gamma_2$, $D^2 = c$, $D.\Gamma_1 = D.\Gamma_2 = 1$, $\Gamma_1.\Gamma_2 = 0$, *or*
(E2) $L \sim 2D + 2\Gamma_0 + 2\Gamma_1 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$, $D^2 = c$, *and the following configuration:*

$$\begin{tikzcd} D \arrow[r, no head] & \Gamma_0 \arrow[r, dashed, no head] & \Gamma_N \arrow[r, no head] \arrow[d, no head] & \Gamma_{N+1} \\ & & \Gamma_{N+2} & \end{tikzcd}$$

Furthermore, there can only exist Clifford divisors of one of the three types (E0)-(E2) (on the same surface), and all such are linearly equivalent.

Proof. If D is as in case (a), the proof of the first statement is similar to the proof of [Kn1, Lemma 7.4] and is left to the reader.

Now assume D is as in case (b). Then we have $L^2 = 2D.L + \Delta.L = 2(D^2+c+2) = 4c+4$. This gives $\Delta^2 = (L-2D)^2 = -4$ and $D.\Delta = \frac{1}{2}(L.\Delta - \Delta^2) = 2$. By Proposition 4.3 below, any smooth rational curve Γ in Δ such that $\Gamma.D > 0$, satisfies $\Gamma.D = 1$, and any two such curves are disjoint. So we have to distinguish between two cases.

If there exist two distinct rational curves Γ_1 and Γ_2 in Δ such that $\Gamma_1.D = \Gamma_2.D = 1$ and $\Gamma_1.\Gamma_2 = 0$, write $L \sim 2D + \Gamma_1 + \Gamma_2 + \Delta'$, for some $\Delta' \geq 0$. Then $0 = \Gamma_i.L = 2 - 2 + \Gamma_i.\Delta'$ gives $\Delta'.\Gamma_i = 0$, for $i = 1, 2$. Clearly $D.\Delta' = 0$, so

$$(2D + \Gamma_1 + \Gamma_2).\Delta' = 0,$$

whence $\Delta' = 0$, since L is numerically 2-connected, and we are in case (E1).

If there exists a rational curve Γ occurring with multiplicity 2 in Δ such that $\Gamma.D = 1$, write $L \sim 2D + 2\Gamma + \Delta'$, for some $\Delta' \geq 0$. Since $0 = \Gamma.L = 2 - 4 + \Delta'.\Gamma$, we get $\Delta'.\Gamma = 2$. Iterating the process, we get case (E2).

Assume now D is given and B is any free Clifford divisor as in (E0)-(E2). We want to show that $B \sim D$.

If D is of type (E0), we must have $L^2 = 4c + 6$, so B must also be of type (E0), which means that $L \sim 2B + \Gamma_0$, where Γ is a smooth rational curve such that $B.\Gamma_0 = 1$.

Since

$$0 = \Gamma_0.L = 2D.\Gamma_0 + \Gamma.\Gamma_0,$$

and D is nef, we get the two possibilities

$$\text{(i) } D.\Gamma_0 = \Gamma.\Gamma_0 = 0 \quad \text{or} \quad \text{(ii) } D.\Gamma_0 = 1, \Gamma = \Gamma_0.$$

If (i) were to happen, we would get

$$4D.B = (L - \Gamma)(L - \Gamma_0) = L^2 = 4c + 6,$$

which is clearly impossible.

Hence $\Gamma = \Gamma_0$ and $D \sim B$.

If D is of type (E1) or (E2), we have $L^2 = 4c+4$, so B must also be of type (E1) or (E2) and will therefore satisfy either

1) $L \sim 2B + \Gamma'_1 + \Gamma'_2$, or
2) $L \sim 2B + 2\Gamma'_0 + \cdots + 2\Gamma'_N + \Gamma'_{N+1} + \Gamma'_{N+2}$,

where the Γ'_i are smooth rational curves with configurations as in the cases (E1) and (E2).

Assume now that D is of type (E1). The proof if D is of type (E2) is similar.

If 2) holds, we get from

$$0 = \Gamma'_i.L = 2D.\Gamma'_i + \Gamma_1.\Gamma'_i + \Gamma_2.\Gamma'_i,$$

and the fact that D is nef, that

$$\Gamma_j.\Gamma'_i = 0 \quad \text{or} \quad \Gamma_j = \Gamma'_i, \quad i = 0, \ldots, N+2, \quad j = 1, 2.$$

This gives

$$4D.B = (L - \Gamma_1 - \Gamma_2)(L - 2\Gamma'_0 - \cdots - 2\Gamma'_N + \Gamma'_{N+1} - \Gamma'_{N+2}) \leq L^2 = 4(c+1),$$

whence $D.B \leq c + 1$ and $(D - B)^2 \geq -2$, and by Riemann-Roch, if $D \not\sim B$, either $D - B$ or $B - D$ is effective. The argument below is symmetric in those two cases, so assume $D \sim B + \Sigma$, for Σ effective and $\Sigma^2 \geq -2$. Then $\Sigma.L = 0$ and $\Sigma^2 = -2$ by the Hodge index theorem. Furthermore,

$$L \sim 2D + \Gamma_1 + \Gamma_2 \sim 2B + \Gamma_1 + \Gamma_2 + 2\Sigma \sim 2B + 2\Gamma'_0 + \cdots + 2\Gamma'_N + \Gamma'_{N+1} + \Gamma'_{N+2},$$

whence

$$2\Sigma \sim 2\Gamma'_0 + \cdots + 2\Gamma'_N + \Gamma'_{N+1} + \Gamma'_{N+2} - \Gamma_1 - \Gamma_2,$$

and $2\Sigma.B = 2 - (\Gamma_1 + \Gamma_2).B \leq 2$ (since B is nef). By

$$0 = \Sigma.L = 2B.\Sigma + (\Gamma_1 + \Gamma_2).\Sigma + 2\Sigma^2,$$

we get $(\Gamma_1 + \Gamma_2).\Sigma \geq 2$, and

$$2\Sigma.D = (L - \Gamma_1 - \Gamma_2).\Sigma \leq -2,$$

contradicting the nefness of D. So we are in case 1) above and again from

$$0 = \Gamma'_i.L = 2D.\Gamma'_i + \Gamma_1.\Gamma'_i + \Gamma_2.\Gamma'_i,$$

and the fact that D is nef, we get the three possibilities:

(i) $D.\Gamma'_1 = 1$, $\Gamma'_1 = \Gamma_1$, $D.\Gamma'_2 = \Gamma_2.\Gamma'_2 = 0$,

(ii) $D.\Gamma'_i = \Gamma_1.\Gamma'_i = \Gamma_2.\Gamma'_i = 0$, i=1,2,

(iii) $D.\Gamma'_i = 1$, $\Gamma'_i = \Gamma_i$, $i = 1, 2$.

In case (i) we get the absurdity $4D.B = (L - \Gamma_1 - \Gamma_2).(L - \Gamma'_1 - \Gamma'_2) = 4(c + 1) - 2$.

In case (ii) we get $4D.B = 4(c+1)$, whence $D.B = c+1$. We calculate $(D-B)^2 \geq -2$, and by Riemann-Roch, if $D \not\sim B$, either $D - B$ or $B - D$ is effective. Writing $D \sim B + \Sigma$, for Σ effective and $\Sigma^2 = -2$, we get the same contradiction as above.

So we are in case (iii) and $B \sim D$. □

Note that any Clifford divisor D will satisfy the numerical conditions:

$$2D^2 \overset{(i)}{\leq} L.D = D^2 + c + 2 \overset{(ii)}{\leq} 2c + 4$$

$(*)$ with equality in (i) or (ii) if and only if $L \sim 2D$ and $L^2 = 4c + 8$.

In particular,

$$D^2 \leq c + 2, \text{with equality if and only if } L \sim 2D \text{ and } L^2 = 4c + 8, \tag{2}$$

and by the Hodge index theorem

$$D^2 L^2 \leq (L.D)^2 = (D^2 + c + 2)^2. \tag{3}$$

It turns out that free Clifford divisors of self-intersection $c+1$ will play a particular role. The following proposition describes this case.

Proposition 3.6. *Let L be a spanned and big line bundle on a $K3$ surface and let c be the Clifford index of all smooth curves in $|L|$. Then the following conditions are equivalent:*

(i) *all smooth curves in $|L|$ are exceptional (i.e. have gonality $c + 3$),*

(ii) *there is a free Clifford divisor of type (E0),*

(iii) *all free Clifford divisors are linearly equivalent and of type (E0).*

Proof. The equivalence between (i) and (iii) follows from the proof of [Kn1, Prop. 8.6]. We now show that if D is a free Clifford divisor of type (E0) and D' is any other free Clifford divisor, then $D' \sim D$.

So let $B := D - D'$. Define $R' := L - 2D'$ and note that $R' \sim 2B + \Gamma$. We have

$$\begin{aligned} c + 2 = D.(L - D) &= (D' + B).(D' + B + \Gamma) \\ &= {D'}^2 + (2B + \Gamma).D' + B.(B + \Gamma) = c + 2 + B.(B + \Gamma), \end{aligned}$$

whence $B^2 + B.\Gamma = 0$. Combined with $\Gamma.D = \Gamma.(D' + B) = 1$, and since $\Gamma.D' = 0$ or 1 by Lemma 5.3(c), we get

$$\Gamma.D' = 1, B^2 = 0, B.\Gamma = 0,$$

whence

$$B.R' = B.(2B + \Gamma) = 0 \quad \text{and} \quad R'.D' = 2B.D' + 1.$$

This gives

$$L.B = 2D'.B + R'.B = 2D'.B = R'.D' - 1.$$

But this implies

$$B.L - B^2 - 2 = R'.D' - 3 < {D'}^2 + R'.D' - 2 = c,$$

whence we must have $B \sim 0$, as desired. □

Remark 3.7. This result can be seen as a generalization of [SD, Rem. 7.13] and [E-L-M-S, Thm. 4.3].

Now we return to the theory of scrolls. Let D be a free Clifford divisor.

If $D^2 = 0$, then $|D| = \{D_\lambda\}_{\lambda \in \mathbf{P}^1}$ is a pencil, which defines in a natural way a scroll containing $\phi_L(S)$.

If $D^2 > 0$, then $\dim |D| = \frac{1}{2}D^2 + 1 > 1$, and we choose a subpencil $\{D_\lambda\}_{\lambda \in \mathbf{P}^1} \subseteq |D|$ as follows: Pick any two smooth members D_1 and $D_2 \in |D|$ intersecting in D^2 distinct points and such that none of these points belong to the union of the finite set of curves

$$\{\Gamma \ | \ \Gamma \text{ is a smooth rational curve}, \Gamma.L \leq c+2\}. \tag{4}$$

Then

$$\{D_\lambda\}_{\lambda \in \mathbf{P}^1} := \text{the pencil generated by } D_1 \text{ and } D_2.$$

The pencil $\{D_\lambda\}$ will be without fixed components (but with D^2 base points) and define in a natural way a scroll containing $\phi_L(S)$ and of type determined as in equation (1) by the integers

$$d_i = h^0(L - iD) - h^0(L - (i+1)D), \quad i \geq 0. \tag{5}$$

Since all choices of subpencils of $|D|$ will give scrolls of the same type, and scrolls of the same type are isomorphic, the scrolls arising are up to isomorphism only dependent on D. We denote these scrolls by $\mathcal{T} = \mathcal{T}(c, D, \{D_\lambda\})$.

Since $h^1(L) = 0$, we get by the conditions (C1) and (C5) that $\mathcal{T}$ has dimension

$$\dim \mathcal{T} = d_0 = h^0(L) - h^0(F) = c + 2 + \frac{1}{2}D^2, \tag{6}$$

and degree

$$\deg \mathcal{T} = h^0(F) = g - c - 1 - \frac{1}{2}D^2. \tag{7}$$

Furthermore, each $D_\lambda \in \{D_\lambda\}$ has linear span

$$\overline{D_\lambda} = \mathbf{P}^{c+1+\frac{1}{2}D^2}. \tag{8}$$

Given integers $g \geq 2$ and $0 \leq c \leq \lfloor \frac{g-1}{2} \rfloor$, one may ask whether there actually exists a pair (S, L), where S is a $K3$ surface, $L^2 = 2g - 2$ and all smooth curves in $|L|$ have Clifford index c.

The following result gives a positive answer to this question. We will conclude this section by giving its proof.

Theorem 3.8. *Let g and c be integers such that $g \geq 3$ and $0 \leq c \leq \lfloor \frac{g-1}{2} \rfloor$. Then there exists a polarized $K3$ surface of genus g and Clifford index c.*

The theorem is an immediate consequence of the following

Proposition 3.9. *Let d and g be integers such that $g \geq 3$ and $2 \leq d \leq \lfloor \frac{g-1}{2} \rfloor + 2$, and let S be a $K3$ surface with $\operatorname{Pic} S = \mathbf{Z}L \oplus \mathbf{Z}E$, where $L^2 = 2(g-1)$, $E.L = d$ and $E^2 = 0$. Then L is base point free and*

$$c := \operatorname{Cliff} L = d - 2 < \lfloor \frac{g-1}{2} \rfloor$$

Furthermore, E is the only Clifford divisor for L (modulo equivalence class).

To prove this proposition, we first need the following basic existence result:

Lemma 3.10. *Let $g \geq 3$ and $d \geq 2$ be integers. Then there exists a $K3$ surface S with* $\operatorname{Pic} S = \mathbf{Z}L \oplus \mathbf{Z}E$*, such that L is base point free and E is a smooth curve, $L^2 = 2(g-1)$, $E.L = d$ and $E^2 = 0$.*

Proof. By Propositions 1.4 and 1.5, we can find a $K3$ surface S with $\operatorname{Pic} S = \mathbf{Z}L \oplus \mathbf{Z}E$, with intersection matrix

$$\begin{bmatrix} L^2 & L.E \\ E.L & E^2 \end{bmatrix} = \begin{bmatrix} 2(g-1) & d \\ d & 0 \end{bmatrix}$$

and such that L is nef. If L is not base point free, there exists by Proposition 1.3 a curve B such that $B^2 = 0$ and $B.L = 1$. An easy calculation shows that this is impossible. By [Kn2, Proposition 4.4], we have that $|E|$ contains a smooth curve. □

Proof of Proposition 3.9. Let S, L and E be as in Lemma 3.10, with $d \leq \lfloor \frac{g-1}{2} \rfloor + 2$. Note that since E is irreducible, we have $h^1(E) = 0$. By the cohomology of the short exact sequence

$$0 \longrightarrow \mathcal{O}_S(E-L) \longrightarrow \mathcal{O}_S(E) \longrightarrow \mathcal{O}_C(E) \longrightarrow 0,$$

where C is any smooth curve in $|L|$, we find that $h^0(\mathcal{O}_C(E)) \geq h^0(E) = 2$ and $h^1(\mathcal{O}_C(E)) = h^0(L-E) \geq 2$, so $\mathcal{O}_C(E)$ contributes to the Clifford index of C and

$$c \leq \operatorname{Cliff} \mathcal{O}_C(E) \leq E.L - E^2 - 2 = d - 2 < \lfloor \frac{g-1}{2} \rfloor.$$

Now since $c < \lfloor \frac{g-1}{2} \rfloor$, then there has to exist an effective divisor D on S satisfying

$$c = \operatorname{Cliff} \mathcal{O}_C(D) = D.L - D^2 - 2.$$

Since both D and $L - D$ must be effective and E is nef, we must have

$$D.E \geq 0 \text{ and } (L-D).E \geq 0.$$

Writing $D \sim xL + yE$ this is equivalent to

$$dx \geq 0 \text{ and } d(1-x) \geq 0,$$

which gives $x = 0$ or 1. These two cases give, respectively, $D = yE$ and $L - D = yE$. This shows that $c = E.L - E^2 - 2 = d - 2$. This argument also shows that there are no other Clifford divisors but E. □

This concludes the proof of Theorem 3.8.

Also note that the proof of Proposition 3.9 gives the following result, which is of its own interest:

Theorem 3.11. *Let g and k be integers such that $g \geq 2$ and $2 \leq k \leq \lfloor \frac{g+3}{2} \rfloor$. Then there exists a $K3$ surface containing a smooth curve of genus g and gonality k.*

The surfaces constructed in Proposition 3.9 all have the property that the only free Clifford divisor (modulo equivalence class) is a smooth elliptic curve E. One could also perform the same construction with lattices of the form

$$\begin{bmatrix} L^2 & L.D \\ D.L & D^2 \end{bmatrix}$$

with $D^2 > 0$ (and satisfying the constraints given by equations (2) and (3)), but for each pair (g, c) there might be values of D^2 that cannot occur. See Proposition 10.1 for a result concerning low values of c. We will in Sections 9 and 10 perform more such constructions, also with lattices of higher ranks.

4. The singular locus of the surface S' and the scroll $\mathcal{T}$

We start this section by describing the image $S' := \phi_L(S)$ by the complete linear system $|L|$ on the $K3$ surface S.

Proposition 4.1. *Let L be a spanned and big line bundle on a $K3$ surface S, and denote by ϕ_L the corresponding morphism and by c the Clifford index of the smooth curves in $|L|$.*

(i) *If $c = 0$, ϕ_L is $2:1$ onto a surface of degree $\frac{1}{2}L^2$,*

(ii) *If $c > 0$, then ϕ_L is birational onto a surface of degree L^2 (in fact it is an isomorphism outside of finitely many contracted smooth rational curves), and $S' := \phi_L(S)$ is normal and has only double rational singularities. In particular $K_{S'} \simeq \mathcal{O}_{S'}$, and $p_a(S') = 1$.*

Proof. These are well-known results due to Saint-Donat [SD] (see also [Kn1] for further discussions). □

Let D be a free Clifford divisor and $\{D_\lambda\}$ a subpencil of $|D|$ chosen as described in the previous section.

Define the subset $\mathcal{D}$ of the pencil $\{D_\lambda\}$ by

$$\mathcal{D} := \{D_\lambda \in \{D_\lambda\} \mid \phi_L \text{ does not contract any component of } D_\lambda\}.$$

We then have

Lemma 4.2. *If $c > 0$, then L_{D_λ} is very ample for all $D_\lambda \in \mathcal{D}$.*

Proof. By [C-F, Thm. 3.1] it is sufficient to show that for any subdivisor A of D_λ we have $L.A \geq A^2 + 3$.

If $A^2 \geq 0$, then we have $L.A \geq A^2 + c + 2 \geq A^2 + 3$ (which actually holds for any divisor A on S). If $A^2 \leq -2$, then $L.A \geq 1 \geq A^2 + 3$, unless $L.A = 0$, which proves the lemma. □

In the rest of this section we focus on the singular locus of the rational normal scroll $\mathcal{T} = \mathcal{T}(c, D, \{D_\lambda\})$ constructed as in the previous section.

It is well-known that the singular locus of a rational normal scroll of type $(e_1, \ldots, e_d)$ has dimension $r - 1$, where

$$r := \#\{e_i \mid e_i = 0\}. \tag{9}$$

From equation (1) we have

$$r = d_0 - d_1.$$

By property (C2), when $L \not\sim 2D$,we have for $R := L - 2D$ by Riemann-Roch

$$h^0(R) = \frac{1}{2}R^2 + 2 + h^1(R).$$

Note that we have $h^0(R) > 0$ if $L^2 \geq 4c + 6$, and $h^0(R) = 0$ if and only if $L^2 = 4c + 4$ and $h^1(R) = 0$.

If $L \sim 2D$, we have $D^2 = c + 2$ and $h^0(L - 2D) = h^0(\mathcal{O}_S) = 1$. Combining this with the lemma above, we get the following expression for r:

$$r = \begin{cases} D^2 + h^1(L - 2D) & \text{if } L \not\sim 2D \text{ (equiv. } D^2 \neq c + 2), \\ D^2 - 1 & \text{if } L \sim 2D \text{ (equiv. } D^2 = c + 2) \end{cases} \tag{10}$$

The next results will show that the term D^2 (or $D^2 - 1$) can be interpreted geometrically as follows: The pencil $\{D_\lambda\}$ has $n = D^2$ distinct base points, denote their images by ϕ_L by $x_1, \ldots, x_n$. The linear spaces $\overline{D_\lambda}$ that sweep out the scroll $\mathcal{T}$ will intersect in the linear space spanned by these points, which we denote by $< x_1, \dots, x_n >$. This is a $\mathbf{P}^{n-1}$ when $L \not\sim 2D$ and a $\mathbf{P}^{n-2}$ when $L \sim 2D$.

Define the set

$$\mathcal{R}_{L,D} := \{\Gamma \mid \Gamma \text{ is a smooth rational curve, } \Gamma.L = 0 \text{ and } \Gamma.D > 0\}. \tag{11}$$

The members of $\phi_L(\{D_\lambda\})$ will intersect in the points $\{\phi_L(\Gamma)\}_{\Gamma \in \mathcal{R}_{L,D}}$ in addition to the images of the D^2 base points of $\{D_\lambda\}$. If these extra points pose new independent conditions, they will contribute to the singular locus of $\mathcal{T}$. We will show below that among all free Clifford divisors, we can choose one such that the term $h^1(L-2D)$ will correspond exactly to the singularities of the scroll arising from the contractions of the curves in $\mathcal{R}_{L,D}$.

The contraction of smooth rational curves Γ which are not in $\mathcal{R}_{L,D}$, will occur in some fiber. Indeed, since $D.\Gamma = 0$ one calculates $h^0(D-\Gamma) = h^0(D) - 1$, whence Γ will be a component of a unique reducible member of $\{D_\lambda\}$. Clearly, such contractions which occur in some fiber, and not transversally to the fibers, will not influence the singularities of $\mathcal{T}$.

The proofs of the next three propositions are rather long and tedious, and will therefore be postponed until the next section.

Proposition 4.3. *Let D be a free Clifford divisor for L and Γ a curve in $\mathcal{R}_{L,D}$.*

Then $D.\Gamma = 1$, $F.\Gamma = -1$ and Γ is contained in the base locus Δ of F. As a consequence, $\Delta.D = \#\mathcal{R}_{L,D}$, where the elements are counted with the multiplicity they have in Δ.

Furthermore, if γ is any reduced and connected effective divisor such that $\gamma.L = 0$ and $\gamma.D > 0$, then $D.\gamma = 1$.

In particular, the curves in $\mathcal{R}_{L,D}$ are disjoint.

We defined the cases (E0)-(E2) in the previous section. We also need to define the following two cases for $c = 0$:

(E3) $L \sim 3D + 2\Gamma_0 + \Gamma_1$, Γ_0 and Γ_1 are smooth rational curves, $c = D^2 = 0$, $L^2 = 6$, $D.\Gamma_0 = 1$, $D.\Gamma_1 = 0$, $\Gamma_0.\Gamma_1 = 1$.

(E4) $L \sim 4D + 2\Gamma$, Γ is a smooth rational curve, $c = D^2 = 0$, $L^2 = 8$, $D.\Gamma = 1$.

Note that in all cases (E0)-(E4) we have $h^1(L-2D) = \Delta.D - 1$. More precisely we have:

(E0) $\Delta = \Gamma$, $\Delta.D = 1$, $h^1(L-2D) = 0$,
(E1) $\Delta = \Gamma_1 + \Gamma_2$, $\Delta.D = 2$, $h^1(L-2D) = 1$,
(E2) $\Delta = 2\Gamma_0 + 2\Gamma_1 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$, $\Delta.D = 2$, $h^1(L-2D) = 1$,
(E3) $\Delta = 2\Gamma_0 + \Gamma_1$, $\Delta.D = 2$, $h^1(L-2D) = 1$,
(E4) $\Delta = 2\Gamma$, $\Delta.D = 2$, $h^1(L-2D) = 1$.

Remark 4.4. If D is any free Clifford divisor not of type (E0)-(E4), then it will follow from equation (28) in Section 5 that $h^1(L-2D) \geq \Delta.D$.

Proposition 4.5. *Among all free Clifford divisors for L there is one, call it D, with the following property (denoting by Δ the base locus of $F := L - D$):*

If D is not of type (E0)-(E4), then

$$h^1(L-2D) = \Delta.D.$$

We will also need the following bound on $\Delta.D$:

Proposition 4.6. *We have for D a free Clifford divisor*

$$D.\Delta \leq \frac{1}{2}c + 1 - D^2,$$

except for the case $L^2 \leq 4c + 6$ and $\Delta = 0$, the cases (E0)-(E4) *above, and the case*

$$L^2 = 4c + 4, D.\Delta = 1, \Delta^2 = -2. \tag{12}$$

In this latter case, $D^2 < c$.

We now study the singular locus V of the scroll $\mathcal{T}$. By equation (10) we know its dimension $r-1$, and in the following results we will see which points in $\phi_L(S)$ that span V and how $\phi_L(S)$ intersects V. We will divide the treatment into the two cases $c=0$ and $c>0$. We recall from Proposition 4.1 that these two cases are naturally different.

We will now treat the case $c>0$. Since we choose the base points of the pencil $\{D_\lambda\}$ to be distinct and to lie outside of the finitely many curves in $\mathcal{R}_{L,D}$, the images by ϕ_L of these points will be $n=D^2$ distinct points in $\phi_L(S)$, denote them by $x_1,\dots,x_n$, and their preimages by $p_1,\dots,p_n$. Let $m=D.\Delta$ and let

$$\mathcal{R}_{L,D} = \{\Gamma_1,\dots,\Gamma_r\}, \tag{13}$$

and define

$$m_i := \text{ multiplicity of } \Gamma_i \text{ in } \Delta. \tag{14}$$

Then $m=\sum_{i=1}^r m_i$. Denote by $y_1,\dots,y_r$ the images (distinct from $x_1,\dots,x_n$) of the contractions of the curves in $\mathcal{R}_{L,D}$, and by $q_{1,\lambda},\dots,q_{r,\lambda}$ their corresponding preimages in each fiber. So $q_{i,\lambda}=\Gamma_i\cap D_\lambda$.

In the cases (E0)-(E2) of Proposition 4.5, we use the following notation:

(E0) $y=\phi_L(\Gamma)$,
(E1) $y_1=\phi_L(\Gamma_1)$, $y_2=\phi_L(\Gamma_2)$,
(E2) $y_0=\phi_L(\Gamma_0)$.

We will denote by q_λ, $q_{1,\lambda}$, $q_{2,\lambda}$ and $q_{0,\lambda}$ their respective preimages in the fiber D_λ.

Also, denote the special case $L\sim 2D$ by (Q).

For each $D_\lambda\in\mathcal{D}$, we can identify D_λ with its image $D'_\lambda := \phi_L(D_\lambda)$ on S' by Lemma 4.2. Moreover, we clearly have that the multiplicities of the points $p_1,\dots,p_n,q_{1,\lambda},\dots,q_{r,\lambda}$ on each D_λ is one, hence these points are all smooth points of D_λ, and consequently all $x_1,\dots,x_n,y_1,\dots,y_r$ are smooth points of D'_λ.

For any $D_\lambda\in\mathcal{D}$, we define Z_λ to be the zero-dimensional subscheme of length $n+m$ of D_λ defined by

$$Z_\lambda := p_1+\cdots+p_n+m_1q_{1,\lambda}+\cdots+m_rq_{r,\lambda}. \tag{15}$$

In particular

$$\mathcal{O}_{D_\lambda}(Z_\lambda)\simeq\mathcal{O}_{D_\lambda}(D+\sum_{i=1}^r m_i\Gamma_i). \tag{16}$$

This zero-dimensional scheme can, by the isomorphism between D_λ and D'_λ, be identified with the following zero-dimensional subscheme of D'_λ, which we by abuse of notation denote by the same name:

$$Z_\lambda = x_1+\cdots+x_n+m_1y_{1,\lambda}+\cdots+m_ry_{r,\lambda}.$$

Note that in the case (Q) all the Z_λ are equal to $p_1+\cdots+p_n$ and will be denoted by Z.

In the special cases (Q), (E0)-(E2) we will also define the following zero-dimensional subschemes of Z_λ (which we again will identify to their corresponding subschemes of D'_λ):

(Q) $Z^i := p_1+\cdots+\hat{p}_i+\cdots+p_n$,
(E0) $Z_{0,\lambda} := p_1+\cdots+p_n$,
(E1) $Z_{i,\lambda} := p_1+\cdots+p_n+q_{i,\lambda}$, $i=1,2$,
(E2) $Z_{0,\lambda} := p_1+\cdots+p_n+q_{0,\lambda}$.

By $< Z >$ we will mean the linear span of the zero-dimensional scheme Z on S'.
The following is the main result of this section:

Theorem 4.7. *Assume $c > 0$. Among all free Clifford divisors for L there is one, call it D, satisfying the property in Proposition 4.5 and with the following three additional properties:*

(a) *If D is not of type (Q), (E0), (E1) or (E2), then for all $D_\lambda \in \mathcal{D}$ we have*

$$V =< Z_\lambda >\simeq \mathbf{P}^{n+m-1},$$

and if D is of one of the particular types above, then:

(Q) $V =< Z^i >=< Z >\simeq \mathbf{P}^{n-2}$,
(E0) $V =< Z_{0,\lambda} >=< Z_\lambda >\simeq \mathbf{P}^{n-1}$,
(E1) $V =< Z_{1,\lambda} >=< Z_{2,\lambda} >=< Z_\lambda >\simeq \mathbf{P}^{n}$,
(E2) $V =< Z_{0,\lambda} >=< Z_\lambda >\simeq \mathbf{P}^{n}$.

(b) *V does not intersect S' (set-theoretically) outside the points in the support of Z_λ.*

(c) *For any irreducible D_λ, we have*

$$V \cap D_\lambda = Z_\lambda.$$

In the theorem above, the following convention is used: $\mathbf{P}^{-1} = \emptyset$ (which happens if and only if $n = m = 0$ and implies that the scroll is smooth).

Remark 4.8. *If D is any free Clifford divisor, we have $V \supseteq< Z_\lambda >\simeq \mathbf{P}^{n+m-1}$, except in the cases (Q), (E0)-(E2), where the property (a) is automatically fulfilled.*

If D is not of type (E1) or (E2), the properties (b) and (c) automatically hold. If D is of type (E1) or (E2), then it might be that V intersects S' outside of the support of $< Z_\lambda >$.

The proof of Theorem 4.7 will be divided in the general case and in the special cases (Q), (E0)-(E2). We will only prove the two first properties. The last one will be left to the reader.

In this section, we give the proofs for the general case and the case (Q). The proofs for the other cases are postponed until the next section.

We will write $\lambda \in \mathcal{D}$ for a λ such that $D_\lambda \in \mathcal{D}$.

Proof of Theorem 4.7 in the general case. Let $s := n+m$. To prove that $< Z_\lambda >\simeq \mathbf{P}^{n+m-1}$ it suffices to prove that the natural map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{Z_\lambda})$$

is surjective for all $\lambda \in \mathcal{D}$.

So assume this map is not surjective for some λ. Then there exists a subscheme $Z' \subseteq Z_\lambda$ of length $s' \leq s$, for some integer $s' \geq 2$ (since L is base point free), such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{Z'})$ is not surjective, but such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{Z''})$ is surjective for all proper subschemes $Z'' \subsetneqq Z'$. We now use Proposition 4.6.

If $\Delta = 0$ and $L^2 \leq 4c + 6$, we have $n = D^2 \leq c$ and $m = 0$, so

$$L^2 \geq 4(c+1) \geq 4(n+1) = 4(s+1).$$

If we are in the case given by (12), we have

$$L^2 \geq 4(c+1) \geq 4(n+2) = 4(s+1).$$

In all other cases, we have $s = m + n \leq \lfloor \frac{1}{2}c \rfloor + 1$ by Proposition 4.6, so

$$L^2 \geq 4(c+1) \geq 4(\lfloor \frac{1}{2}c \rfloor + 2) \geq 4(s+1).$$

Therefore, by [Kn1], there exists an effective divisor B passing through Z' and satisfying $B^2 \geq -2$, $h^1(B) = 0$ and the numerical conditions

$$2B^2 < B.L \leq B^2 + s' < 2s'.$$

If $B^2 \geq 0$, then B would induce a Clifford index $c_B \leq s' - 2 \leq n + m - 2$ on the smooth curves in $|L|$. If $\Delta = 0$ and $L^2 \leq 4c + 6$, we get the contradiction $c_B \leq n - 2 \leq c - 2$. If we are in the case given by (12), we get the contradiction $c_B \leq n - 1 \leq c - 2$. Finally, in all other cases, we have $c_B \leq n + m - 2 < \lfloor \frac{1}{2} c \rfloor$, again a contradiction.

Hence $B^2 = -2$ and B is supported on a union of smooth rational curves. Furthermore, $B.L \leq s' - 2$ and $B.D \geq s'$ (the last inequality follows since D_λ passes through Z_λ).

We now consider the effective decomposition

$$L \sim (D + B) + (F - B).$$

Firstly note that $L.(D + B) \leq n + s' + c$ and $(D + B)^2 \geq n + 2s' - 2$, whence $(F - B)^2 = (L - D - B)^2 \geq 2c - n + 2 \geq c + 2 > 0$, so that $h^0(F - B) \geq 2$.

Secondly, $L.(D + B) - (D + B)^2 - 2 \leq c - s' < c$, a contradiction.

For the second statement, it suffices to show that there is no point $x_0 \in S' - \{x_1, \dots, x_n, y_1, \dots, y_r\}$ such that S' has an $(s + 1)$-secant $(s - 1)$-plane through Z_λ and x_0 for all λ.

Assume, to get a contradiction, that there is such a point x_0. Choose any preimage p_0 of x_0, and denote by X_λ the zero-dimensional scheme defined as the union of Z_λ and p_0. Fix any λ such that D_λ is irreducible.

In these terms we have that the natural map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{X_\lambda})$$

is not surjective.

Then there exists a subscheme $X' \subseteq X_\lambda$ of length $s' + 1 \leq s + 1$, for some integer $s' \geq 1$, such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{X'})$ is not surjective, but such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{X''})$ is surjective for all proper subschemes $X'' \subsetneqq X'$.

Since $L^2 \geq 4(s + 1)$ by the above, there exists by [Kn1] again an effective divisor B passing through X' and satisfying $B^2 \geq -2$, $h^1(B) = 0$ and the numerical conditions

$$2B^2 \leq B.L \leq B^2 + s' + 1 \leq 2s' + 2.$$

As above, if $B^2 \geq 0$, we would get a contradiction on the Clifford index c. Hence $B^2 = -2$ and B is supported on a union of smooth rational curves. Furthermore, $B.L \leq s' - 1$ and $B.D \geq s'$ (the last inequality follows since D_λ is irreducible).

As above, the effective decomposition

$$L \sim (D + B) + (F - B)$$

induces a Clifford index $< c$ on the smooth curves in $|L|$, unless $s' = 1$, $B.L = 0$ and $B.D = 1$. This means that p_0 lies in some divisor which is contracted to one of the points $y_1, \dots, y_r$. Hence x_0 is one of these points, a contradiction. □

Proof of Theorem 4.7 in the case $L \sim 2D$. It suffices to prove that if there is a point $x_0 \in S' - \{x_1, \dots, \hat{x_i}, \dots, x_n\}$ for some i, such that S' has an n-secant $(n - 2)$-plane through x_0 and Z^i, then $x_0 = x_i$.

Choose any preimage p_0 of x_0, and denote by X_i the zero-dimensional scheme defined by p_0 and Z^i. We will show that if the natural map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{X_i})$$

is not surjective, then $p_0 = p_i$.

Let $X' \subseteq X_i$ be a subscheme of length $n' \leq n$, for some integer $n' \geq 2$, such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{X'})$ is not surjective, but such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{X''})$ is surjective for all proper subschemes $X'' \subsetneqq X'$.

By assumption, we have $n = D^2 = c + 2$ and $L^2 = 4c + 8 = 4n$. Hence, by [Kn1], there exists an effective divisor B passing through X' and satisfying $B^2 \geq -2$ and the numerical conditions

$$2B^2 \overset{(a)}{\leq} L.B \leq B^2 + n' \overset{(b)}{\leq} 2n',$$

with equality in (a) or (b) implying $L \sim 2B$.

Since B passes through X', we have $B.D \geq n' - 1$, whence $B.L \geq 2n' - 2$. From the inequalities above, we get $B^2 \geq n' - 1 \geq 0$, so we have $n' = n$ and $B.L = B^2 + n$, since otherwise B would induce a Clifford index $< n - 2 = c$ on the smooth members of $|L|$. This leaves us with the two possibilities:

$$(i) \quad B^2 = n \text{ and } L \sim 2B, \quad \text{or} \quad (ii) \quad B^2 = n - 1.$$

In the second case, one easily finds $L \sim 2B + \Gamma$, for Γ a smooth rational curve, which is impossible, since $L \sim 2D$. So we are in case (i), and $B \in |D|$.

Now recall by [Kn1] how the divisor B is obtained: It sits in an exact sequence

$$0 \longrightarrow \mathcal{O}_S(B) \longrightarrow E \longrightarrow \mathcal{O}_S(B) \longrightarrow 0,$$

where E is a rank two bundle given as an extension

$$0 \longrightarrow \mathcal{O}_S \longrightarrow E \longrightarrow L \otimes \mathcal{J}_{X'} \longrightarrow 0.$$

Tensoring both sequences with $\mathcal{O}_S(-B)$ and taking cohomology, one gets

$$h^0(B \otimes \mathcal{J}_{X'}) = h^0(E - B) = 2,$$

so there is a pencil P of divisors in $|D|$ passing through X'.

We claim that any divisor $D_0 \in |D|$ passing through $n - 1$ of the points $p_1, \ldots, p_n$, will also pass through the last one. Indeed, by the surjectivity of the map $H^0(D) \to H^0(\mathcal{O}_{D_0}(D))$, we reduce to the same statement for $\mathcal{O}_{D_0}(D)$. By Riemann-Roch, this is equivalent to $h^0(\mathcal{O}_{D_0}(Z)) \geq 2$ and $\mathcal{O}_{D_0}(Z)$ base point free, which are both satisfied since $\mathcal{O}_{D_0}(Z) \simeq \mathcal{O}_{D_0}(D)$, and $\mathcal{O}_S(D)$ is base point free.

Since Z contains the points $p_0, \ldots, \hat{p_i}, \ldots, p_n$, we have that all the members in P contain all the points $p_0, \ldots, p_n$. Therefore, P is the pencil $\{D_\lambda\}$, whose general member is smooth and irreducible. Since all the members intersect in n points, we have $p_0 = p_i$, as asserted. $\square$

It will be convenient to make the following definition:

Definition 4.9. *A free Clifford divisor satisfying the properties described in Proposition 4.5 and Theorem 4.7 will be called a* perfect *Clifford divisor.*

In the next section we will prove Proposition 4.5 and Theorem 4.7, thus proving that we can find a perfect Clifford divisor.

The main advantage of choosing a perfect Clifford divisor is that we then get a nice description of the singular locus of $\mathcal{T}$ and how it intersects S' as in Theorem 4.7 above. This theorem states that $\operatorname{Sing} \mathcal{T}$ is "spanned" by the images of the base points of the chosen subpencil of $|D|$ and the contracted curves, and moreover that it intersects S' in only these points. If D is not perfect, then $\operatorname{Sing} \mathcal{T} \supsetneqq Z_\lambda$, as seen in Remarks 4.4 and 4.8. In Proposition 7.31 below we will see an example where this occurs.

It will also be practical, for classification purposes, to restrict the attention to perfect Clifford divisors, as we will do in Section 10.

Apart from this, any free Clifford divisor will be equally fit for our purposes.

We include an additional description of the case (Q):

Proposition 4.10. *Assume D is a free Clifford divisor of type (Q) and $c \geq 2$. Then $\phi_L(S)$ is the 2-uple embedding of $\phi_D(S)$, unless $c = 2$ and there exists a smooth elliptic curve E such that $E.D = 2$ (in which case D is hyperelliptic).*

Proof. By [SD, Thm. 6.1] ϕ_L is the 2-uple embedding of $\phi_D(S)$, when D is not hyperelliptic.

Conversely, if D is hyperelliptic, then ϕ_D is not birational, so ϕ_L cannot be the 2-uple embedding of $\phi_D(S)$.

If D is hyperelliptic, then there either exists a smooth curve E satisfying $E^2 = 0$ and $E.D = 2$, or a smooth curve B such that $B^2 = 2$ and $D \sim 2B$. In the first case, we compute

$$E.L - E^2 - 2 = 2E.D - 2 = 2,$$

whence $c = 2$, and in the second we get the contradiction

$$D.L - D^2 - 2 = 6 > 4 = B.L - B^2 - 2.$$

□

The special case appearing in the proposition will be thouroughly described in Proposition 7.31 below.

If $c = 0$, there exist two kinds of (free) Clifford divisors for L, namely:

1. $D^2 = 0$, $D.L = 2$ and
2. $D^2 = 2$, $L \sim 2D$.

In both these cases $\phi_L(S)$ is $2:1$ on each fiber.

In the case $c = 0$ we have the following result:

Proposition 4.11. *Assume $c = 0$. Let D be a free Clifford divisor for L. Then $D^2 = 0$ and $V = \emptyset$ except for the following cases:*

- (Q) *$L \sim 2D$, $D^2 = 2$, $V = \{x\}$, where x is the common image of the two base points of the chosen pencil $\{D_\lambda\}$,*
- (E1) *$D^2 = 0$, $L \sim 2D + \Gamma_1 + \Gamma_2$, $V = \{\phi_L(\Gamma_1)\} = \{\phi_L(\Gamma_2)\}$,*
- (E2) *$D^2 = 0$, $L \sim 2D + 2\Gamma_0 + 2\Gamma_1 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$, $V = \{\phi_L(\Gamma_0)\}$,*
- (E3) *$D^2 = 0$, $L \sim 3D + 2\Gamma_0 + \Gamma_1$, $V = \{\phi_L(\Gamma_0)\}$,*
- (E4) *$D^2 = 0$, $L \sim 4D + 2\Gamma$, $V = \{\phi_L(\Gamma)\}$.*

Proof. For $D^2 = 0$, this follows from the fact that except for the cases (E1)-(E4), the base locus Δ of $L - D$ is zero, which is shown in the proof of [SD, Prop. 5.7]. In the other case, it follows from the equation (9). □

All these cases have been completely described in [SD, Prop. 5.6 and 5.7].

When $V = \emptyset$, then $\phi_L(S)$ is a rational ruled surface.

The cases where there are contractions across the fibers, are the cases (E1)-(E4). In these cases $\phi_L(S)$ is a cone.

In the case (Q), $\phi_L(S)$ is the Veronese surface in $\mathbf{P}^5$.

5. Postponed proofs

In this section we will give the proofs omitted in the previous section.

Throughout this section L will be a spanned and big line bundle of non-general Clifford index c. In particular, this implies $L^2 \geq 4c + 4$.

Also we write $F := L - D$ and $R := L - 2D = F - D$, and denote the (possibly zero) base divisor of $|F|$ by Δ. Recall that $L.\Delta = 0$ and that we have $h^0(R) = 0$ if and only if $L^2 = 4c + 4$ and $h^1(R) = 0$. In particular, $h^1(R) > 0$ implies that $R > 0$.

Furthermore, we have

Lemma 5.1. *If $h^0(R) = 0$, then $\Delta = 0$.*

Proof. We have $L^2 = 4c + 4$, so we cannot be in the cases (Q) or (E1), whence $D^2 \leq c$. Choose any smooth curve $D_0 \in |D|$ and let $F_{D_0} := F \otimes \mathcal{O}_{D_0}$. Then $\deg F_{D_0} = c + 2 \geq D^2 + 2 = 2g(D_0)$, whence F_{D_0} is base point free.

We first will show that this implies that F is nef.

Taking cohomology of the short exact sequence

$$0 \longrightarrow R \longrightarrow F \longrightarrow F_{D_0} \longrightarrow 0,$$

and of the same sequence tensored with $-\Delta$, we get the following two exact sequences (using $h^0(R) = h^0(R - \Delta) = h^1(R) = 0$)

$$\begin{tikzcd}
0 \arrow[r] & H^0(F) \arrow[r] \arrow[d, equal] & H^0(F_{D_0}) \arrow[r] & 0 \\
0 \arrow[r] & H^0(F - \Delta) \arrow[r] & H^0((F - \Delta)_{D_0}).
\end{tikzcd}$$

This gives $h^0((F - \Delta)_{D_0}) \geq h^0(F_{D_0})$, whence $\Delta.D = 0$, since F_{D_0} is base point free. This means that for any smooth rational curve Γ in the support of Δ, we have $\Gamma.D = \Gamma.F = 0$. Hence F is nef.

By Lemma 1.3 it now suffices to show that F is not of the type $F \sim kE + \Gamma$, for E a smooth elliptic curve and Γ a smooth rational curve satisfying $E.\Gamma = 1$ and an integer $k \geq 2$. But if this were the case, we would have $E.L = 2 + c/k$. If $c \neq 0$, this would mean that E induces a lower Clifford index than c on the smooth curves in $|L|$, a contradiction. If $c = 0$, we get $D.F = 2$ and $D^2 = 0$. But this would give $R^2 = (F - D)^2 \geq -2$ and by Riemann-Roch, we would then get the contradiction $h^0(F - D) \geq 1$. □

By this lemma, if $h^0(R) = 0$, the Propositions 4.3, 4.5 and 4.6 will automatically be satisfied. So for the rest of this section, we will assume $R > 0$.

Let F_0 be the moving component of $|F|$. Since $R > 0$, we can write $F_0 \sim D + A$ for some divisor $A \geq 0$. Thus we have

$$F \sim D + R \sim F_0 + \Delta \sim D + A + \Delta, \tag{17}$$

and

$$L \sim 2D + A + \Delta. \tag{18}$$

We will first study the divisors above more closely.

Lemma 5.2. *Except for the cases (E3) and (E4), the general member of $|F_0|$ is smooth and irreducible.*

Proof. Since $|F_0|$ is base point free, by Proposition 1.2 we only need to show that $F_0 \not\sim kE$, for E a smooth elliptic curve and an integer $k \geq 2$.

Assume, to get a contradiction, that $F_0 \sim kE$, then by (17), we have $D \sim E$ and $A \sim (k-1)E$. Let $d := c + 2 = E.L \geq 2$.

Since $L \sim (k+1)E + \Delta$, we get $L^2 = d(k+1)$, so $h^0(L) = d(k+1)/2 + 2$, and $h^0(F) = h^0(kE) = k + 1$. On the other hand, by equation (7), we have $h^0(F) = h^0(L) - d$.

Combining the last three equations, we get

$$k+1 = d(k-1)/2 + 2,$$

which is only possible if $d = 2$, i.e. $c = 0$. A case by case study as in the proof of [SD, Prop. 5.7] establishes the lemma in this latter case. □

We gather some basic properties of R.

Lemma 5.3. (a) *If $R = R_1 + R_2$ is an effective decomposition, then $R_1.R_2 \geq 0$*
(b) *If γ is an effective divisor satisfying $\gamma^2 = -2$ and $\gamma.R < 0$, then $\gamma.R = -1$ or -2.*
(c) *If γ is an effective divisor satisfying $\gamma^2 = -2$ and $\gamma.L = 0$, then either $\gamma.D = \gamma.F = \gamma.R = 0$ or $\gamma.D = 1$, $\gamma.F = -1$ and $\gamma.R = -2$.*
(d) *If Γ is a smooth rational curve, then $\Gamma \in \mathcal{R}_{L,D}$ if and only if $\Gamma.R = -2$ and $\Gamma.L = 0$.*

Proof. To prove (a), one immediately sees that if $R_1.R_2 < 0$, then the effective decomposition $L \sim (D + R_1) + (D + R_2)$ would induce a Clifford index $< c$.

The other assertions are immediate consequences of (a). □

This concludes the proof of Proposition 4.3.

Lemma 5.4. *Except for the cases (E3) and (E4), the following holds:*
$\Delta^2 = -2D.\Delta$ *and* $\Delta.A = 0$.

Proof. By Lemma 5.2, we have $h^1(F_0) = 0$. From $0 = \Delta.L = 2\Delta.D + \Delta.A + \Delta^2$, we get

$$\Delta^2 = -2\Delta.D - \Delta.A. \tag{19}$$

Furthermore, we also have

$$h^0(F_0) = h^0(F) = \frac{1}{2}F_0^2 + F_0.\Delta + \frac{1}{2}\Delta^2 + 2 = h^0(F_0) + (D + A).\Delta + \frac{1}{2}\Delta^2,$$

which implies

$$\Delta^2 = -2\Delta.D - 2\Delta.A. \tag{20}$$

Combining equations (19) and (20), we get $\Delta.A = 0$ and $\Delta^2 = -2D.\Delta$. □

We have seen in Proposition 3.6, that if there exists a free Clifford divisor of type (E0), then all free Clifford divisors are linearly equivalent and of type (E0).

We now take a closer look at the types (E1) and (E2).

Proposition 5.5. *Let L be a spanned and big line bundle of non-general Clifford index c on a $K3$ surface and let D be a free Clifford divisor of type (E1) or (E2).*

If $D' \not\sim D$ is any other free Clifford divisor, then $B := D - D' > 0$ and

$$\Delta.D' = 0, \Delta.B = 2, B^2 = -2. \tag{21}$$

Proof. Let $R' := L - 2D'$ as usual, and note that $R' \sim 2B + \Delta$.

Since $R'^2 = L^2 - 4(c+2) = \Delta^2 = -4$, we get $B^2 + B.\Delta = 0$. Combined with $\Delta.D = \Delta.(D' + B) = 2$, we get the two possibilities
(a) $\Delta.D' \geq 2$, $\Delta.B \leq 0$, $B^2 \geq 0$,
(b) $\Delta.D' = 0$, $\Delta.B = 2$, $B^2 = -2$.

Using $D'^2 \leq c$, we calculate

$$\begin{aligned} B.L &= \frac{1}{2}(R' - \Delta).L = \frac{1}{2}R'.L = \frac{1}{2}(L - 2D').L \\ &= \frac{1}{2}(L^2 - 2(D'^2 + c + 2)) \geq 2c + 2 - c - c - 2 = 0. \end{aligned}$$

In case (a) we then must have $B.L > 0$ by the Hodge index theorem, so $B \geq 0$ by Riemann-Roch. We get

$$B.R' = B.(2B + \Delta) = B^2 \geq 0 \quad \text{and} \quad R'.D' = 2B.D' + \Delta.D' \geq 2B.D' + 2,$$

which gives

$$L.B = 2D'.B + R'.B = 2D'.B + B^2 \leq R'.D' + B^2 - 2.$$

But this implies

$$B.L - B^2 - 2 \leq R'.D' - 4 < D'^2 + R'.D' - 2 = c,$$

whence we must have $B \sim 0$.

So we must be in case (b), and by Riemann-Roch we have either $B > 0$ or $-B > 0$. We see that $B.L > 0$ unless $D'^2 = D^2 = c$. But if the latter holds, since both D' and D are assumed to be smooth Clifford divisors (so that $h^1(D) = h^1(D') = 0$), we have $h^0(D) = h^0(D')$, whence $D \sim D'$ and $B \sim 0$, a contradiction. Hence $D.L > 0$, so $B > 0$ and we are done. □

As seen below, we will distinguish between inclusions $D' < D$ as in Proposition 5.5 with $\Delta' = 0$ and $\Delta' \neq 0$ (where Δ' is the base divisor of $|L - D'|$).

By Propositions 3.5 and 3.6 it is clear that we can choose a free Clifford divisor D with the two additional properties (recall that Δ as usual denotes the base divisor of $|L - D|$):

(C6) If D' is any other free Clifford divisor such that $D' > D$, then $\Delta \neq 0$ and D' is of type (E1) or (E2).

(C7) If D is of type (E1) or (E2) above, and D' is any other free Clifford divisor satisfying (C6), then $D' \sim D$.

Property (C6) is a *maximality condition*: it means that we choose a free Clifford divisor which is not contained in any other free Clifford divisor, unless possibly when $\Delta \neq 0$ and it is contained in some free Clifford divisor of type (E1) or (E2).

Property (C7) means that if we can, we will choose among all free Clifford divisors satisfying (C6), one that is not of type (E1) or (E2).

It turns out that free Clifford divisors satisfying the additional properties (C6) and (C7) will be perfect, i.e. they will satisfy Propositions 4.5 and 4.7.

Now assume $R = R_1 + R_2$ is an effective decomposition such that $R_1.R_2 = 0$. Then $L \sim (D + R_1) + (D + R_2)$ is an effective decomposition satisfying

$$(D + R_1).(D + R_2) = D^2 + D.(R_1 + R_2) = D.F = c + 2,$$

so this decomposition induces the same Clifford index c. This means that either $D + R_1$ or $D + R_2$ is a Clifford divisor. This enables us to prove the following:

Proposition 5.6. *Assume D is not of type (E3) or (E4) and satisfies (C6) and (C7). Assume furthermore that there exists an effective decomposition $R = R_1 + R_2$ such that $R_1.R_2 = 0$ and such that $D + R_1$ is a Clifford divisor.*

Then either $\Delta \neq 0$ and $D + R_1$ is of type (E1) or (E2), or there exists a smooth rational curve Γ satisfying either

(I) $\Gamma.D = \Gamma.F = \Gamma.L = 0$, $\Gamma.R_1 = -1$, $\Gamma.R_2 = 1$, *or*

(II) $\Gamma.D = 1$, $\Gamma.F = -1$, $\Gamma.L = 0$, $\Gamma.R_1 = -2$, $\Gamma.R_2 = 0$.

Proof. Let $D_1 := D + R_1$ and $D_2 := D + R_2$. Since D_1 is a Clifford divisor containing D, we have by condition (C6) that either D_1 is not a free Clifford divisor, or $\Delta \neq 0$ and D_1 is of type (E1) or (E2).

So we can assume D_1 is not a free Clifford divisor, which means that D_1 is not base point free.

If D_1 is nef, then by Lemma 1.3 it is of the form

$$D_1 \sim lE + \Gamma_0,$$

for some smooth elliptic curve E and smooth rational curve Γ_0 satisfying $E.\Gamma_0 = 1$, and some integer $l \geq 2$. This gives

$$R_1 \sim (l-1)E + \Gamma_0 \quad \text{and} \quad D \sim E.$$

Write

$$D_2 = D + R_2 \sim D + M + B,$$

where $B \geq 0$ is the base divisor of D_2 and $M \geq 0$. Note that $M + B \sim R_2$.

We have

$$0 = R_1.R_2 = ((l-1)D + \Gamma_0).(M+B) = (l-1)D.M + (l-1)D.B + \Gamma_0.M + \Gamma_0.B.$$

Also, we have an effective decomposition

$$R \sim ((l-1)D + M + B) + \Gamma_0,$$

such that

$$((l-1)D + M + B).\Gamma_0 = l - 1 + \Gamma_0.M + \Gamma_0.B = (l-1)(1 - D.M - D.B).$$

By [SD, Lemma 3.7], if $M \neq 0$, either $M \sim kD$, for some integer $k \geq 1$, or $D.M \geq 2$. In this latter case, the latter product would be negative, contradicting Lemma 5.3. So we must have $M \sim kD$, for some integer $k \geq 0$ and $D.B = 0$ or 1.

So $R \sim R_1 + R_2 \sim (l-1)D + \Gamma_0 + M + B \sim (k+l-1)D + \Gamma_0 + B$ and

$$c + 2 = D.F = D.(D+R) = ((k+l)D + \Gamma_0 + B).D \leq 2,$$

which gives $c = 0$ and $B.D = 1$. A short analysis as in part (b) of the proof of [SD, Lemma 5.7.2] shows that D is then of type (E3) or (E4).

So D_1 is not nef, which means that there exists a smooth rational curve Γ such that $\Gamma.D_1 < 0$, whence Γ is fixed in $|D_1|$ and $\Gamma.L = 0$, by Proposition 2.4. Combining $\Gamma.D_1 = \Gamma.D + \Gamma.R_1 \leq -1$ and $0 = \Gamma.L = 2\Gamma.D + \Gamma.R_1 + \Gamma.R_2$, we get

$$1 - \Gamma.R_2 \leq \Gamma.D \leq -1 - \Gamma.R_1. \tag{22}$$

Furthermore, by Lemma 5.3(b), we have

$$\Gamma.R = \Gamma.R_1 + \Gamma.R_2 \geq -2. \tag{23}$$

If $R_1 = \Gamma$, we are done by Lemma 5.3(c), so we can assume that $R_1 - \Gamma > 0$. Then by Lemma 5.3(a) we have $(R_1 - \Gamma).(R_2 + \Gamma) = R_1.R_2 + \Gamma.R_1 - \Gamma.R_2 + 2 \geq 0$, which implies

$$\Gamma.R_1 - \Gamma.R_2 \geq -2. \tag{24}$$

Combining (23) and (24), we get

$$-2 - \Gamma.R_1 \leq \Gamma.R_2 \leq 2 + \Gamma.R_1 \quad \text{and} \quad \Gamma.R_1 \geq -2. \tag{25}$$

Combining (25) with (22) and Lemma 5.3(c), we end up with the two possibilities given by (I) and (II) above. □

We now need a basic lemma about A.

Lemma 5.7. *If $A = 0$, then D is of one of the types (E0)-(E2).*

If $A^2 \leq -2$, then one of the following holds:

(a) $A^2 = -4$, $\Delta = 0$, $L^2 = 4c + 4$,

(b) $A^2 = -2$, $\Delta = 0$, $L^2 = 4c + 6$,
(c) $A^2 = -2$, $\Delta^2 = -2$, $D.\Delta = 1$, $L^2 = 4c + 4$.

Moreover, in case (c) we have $D^2 < c$.

Proof. If $A = 0$, we must have $-4 \leq \Delta^2 = R^2 \leq -2$, whence $\Delta^2 = -4$ or -2, $D.\Delta = 2$ or 1 respectively (by Lemma 5.4), and $L^2 = 4c + 4$ or $4c + 6$ respectively. An analysis as in Proposition 3.5 now gives that D is as in one of the cases (E0)-(E2).

If $A^2 \leq -2$, we have by $R^2 = A^2 + \Delta^2 = L^2 - 4(c + 2)$ (where we have used Lemma 5.4) that either $\Delta = 0$ and we are in case (a) or (b) above, or that $A^2 = -2$, $\Delta^2 = -2$, $D.\Delta = 1$ (by Lemma 5.4) and $L^2 = 4c + 4$, i.e. case (c).

In this latter case, we have

$$c + 2 = D.F = D^2 + D.A + D.\Delta = D^2 + D.A + 1,$$

whence $D^2 = c + 1 - D.A$. Since $D + A \sim F_0$ is base point free, we have $D.A \geq 2$ by [SD, (3.9.6)], whence $D^2 < c$. $\square$

We can now prove Proposition 4.6.

Indeed, when we are not in the exceptional cases of the proposition (which are the cases (E0)-(E4) and the cases (a)-(c) of the last lemma), we have $A \neq 0$ and $A^2 \geq 0$.

We calculate

$$c + 2 = F.D = (D + A + \Delta).D = D^2 + A.D + \Delta.D \tag{26}$$

and

$$A.(2D + \Delta) = 2A.D \tag{27}$$

(by Lemma 5.4). Since D is a Clifford divisor, we must have $A.D \geq D^2 + \Delta.D$.

We also have from (26) that $A.D = c + 2 - D^2 - \Delta.D$. Combining this with the last inequality, we get

$$c/2 + 1 - D^2 \geq \Delta.D.$$

This concludes the proof of Proposition 4.6.

Before proving the next result, we will need the following easy lemma.

Lemma 5.8. *Assume D is not as in (E3) or (E4). If $A \sim A_1 + A_2$ is an effective decomposition such that $A_1.A_2 \leq 0$, then*

$$A_1.A_2 = A_1.\Delta = A_2.\Delta = 0,$$

and either $D + A_1$ or $D + A_2$ is a Clifford divisor.

Proof. By Lemma 5.4 we have $\Delta.A = 0$, so we can assume (possibly after interchanging A_1 and A_2) that $A_1.\Delta \leq 0$ and $A_2.\Delta \geq 0$. Then $R \sim A_1 + (A_2 + \Delta)$ is an effective decomposition of R such that

$$A_1.(A_2 + \Delta) = A_1.A_2 + A_1.\Delta \leq 0.$$

By Lemma 5.3(a) we must have equality, whence $A_1.A_2 = A_1.\Delta = A_2.\Delta = 0$.

If $A_1.L > A_2.L$ (resp. $A_2.L > A_1.L$), then clearly $D + A_1$ (resp. $D + A_2$) is a Clifford divisor by condition (C2).

If $A_1.L = A_2.L$, then $D + A_i$ is not a Clifford divisor if and only if $h^0((D + A_i) - (D + A_{3-i} + \Delta)) = h^0(A_i - A_{3-i} - \Delta) > 0$. Clearly this condition cannot hold for both $i = 1$ and 2. So we are done. $\square$

The next result is the crucial one to prove Proposition 4.5.

Proposition 5.9. *If D satisfies (C6) and (C7), then $H^1(A) = 0$ except for the case (E4).*

Proof. The result is trivial if $A = 0$. So we will assume $A > 0$. Also, the result is fulfilled in the case (E3), so we can assume D is not as in (E3) or (E4). In particular, we can use the Lemmas 5.4 and 5.8.

If $h^1(-A) = h^1(A) > 0$, then A cannot be numerically 1-connected, whence there exists an effective decomposition $A \sim A_1 + A_2$ such that $A_1.A_2 \leq 0$. By the previous lemma, we have $A_1.A_2 = A_1.\Delta = A_2.\Delta = 0$, and (possibly after interchanging A_1 and A_2) we can assume that $D + A_1$ is a Clifford divisor.

Assume first that D and $D+A_1$ are as in the special case where $\Delta \neq 0$ and $D' := D+A_1$ is a free Clifford divisor of type (E1) or (E2), so the base divisor Δ' of

$$F' := L - D' \sim D + A_2 + \Delta \sim D + A_1 + \Delta'$$

satisfies ${\Delta'}^2 = -4$. Furthermore, by Proposition 5.5, $\Delta'.D = 0$, $A_1^2 = -2$ and $A_1.\Delta' = 2$. Also, since $\Delta'.L = 0$ and $D'L = F'.L$, we must have $A_1.L = A_2.L$. Also note that $A_1 \not\sim A_2$, since $A_1.A_2 = 0$.

Since $(A_1 - A_2).L = 0$, we must by the Hodge index theorem have $A^2 = (A_1 - A_2)^2 \leq -2$. By Lemma 5.7 and the fact that $\Delta \neq 0$, this gives us

$$A_1^2 = -2, \quad A_2^2 = 0, \quad \Delta^2 = -2,$$

We then get from $L^2 = 4c+4 = (2D + A + \Delta).L = 2D.L + A.L = 2D^2 + 2c + 4 + A.L$, that

$$A.L = 2(c - D^2).$$

Since $A_1.L = A_2.L$, we have

$$A_1.L = A_2.L = c - D^2.$$

So A_2 would induce a Clifford index $\leq L.A_2 - A_2^2 - 2 = c - D^2 - 2 < c$ on the smooth curves in $|L|$, a contradiction.

So we can now use Proposition 5.6 and find a smooth rational curve satisfying one of the two conditions:

(I) $\Gamma.D = 0$, $\Gamma.A_1 = -1$, $\Gamma.(A_2 + \Delta) = 1$,
(II) $\Gamma.D = 1$, $\Gamma.A_1 = -2$, $\Gamma.(A_2 + \Delta) = 0$.

In case (I) we get $\Gamma.A = \Gamma.F_0 - \Gamma.D \geq 0$, whence $\Gamma.A_2 \geq 1$. Since $\Gamma.A_1 = -1$, we have $A_1 - \Gamma > 0$, and we get an effective decomposition $A \sim (A_1 - \Gamma) + (A_2 + \Gamma)$ such that

$$(A_1 - \Gamma).(A_2 + \Gamma) = A_1.A_2 - \Gamma.A_2 + \Gamma.A_1 - \Gamma^2 \leq 0,$$

so by Lemma 5.8, we must have $\Gamma.A_2 = 1$ and $(A_1 - \Gamma).(A_2 + \Gamma) = 0$. Obviously, $D + A_1 - \Gamma$ is a Clifford divisor, and we can now repeat the process with A_1 and A_2 replaced by $A_1 - \Gamma$ and $A_2 + \Gamma$. This will eventually bring us in case (II) after a finite number of steps.

So we can assume that A_1 and A_2 are as in case (II). Again, by $\Gamma.A = \Gamma.F_0 - \Gamma.D \geq -1$, we have $\Gamma.A_2 \geq 1$, whence $\Gamma \neq A_1$ and $A_1 - \Gamma > 0$. Since

$$(A_1 - \Gamma).(A_2 + \Gamma) = A_1.A_2 - \Gamma.A_2 + \Gamma.A_1 - \Gamma^2 \leq -1,$$

we have a contradiction by Lemma 5.8.

This concludes the proof of the proposition. □

We can now prove Proposition 4.5.

In the cases (E3) and (E4) the proposition is clear, so by Lemma 5.4, we can assume $A.\Delta = 0$ and $\Delta^2 = -2D.\Delta$.

One easily sees that the base divisor of R must contain Δ, so $h^0(A) = h^0(R) = h^0(A+\Delta)$.

If $A > 0$, we have

$$h^0(A) = h^0(A+\Delta) = \frac{1}{2}A^2 + 2 + \frac{1}{2}\Delta^2 + h^1(R) = h^0(A) - h^1(A) + D.\Delta + h^1(R), \tag{28}$$

whence $h^1(R) = D.\Delta + h^1(A)$. If we choose D such that it satisfies (C6) and (C7), then $h^1(A) = 0$ by Proposition 5.9.

If $A = 0$, then $R = \Delta$ and D is of one of the types (E0)-(E2) by Lemma 5.7. This gives

$$1 = h^0(R) = \frac{1}{2}R^2 + 2 + h^1(R) = -D.\Delta + 2 + h^1(R), \tag{29}$$

whence $h^1(R) = D.\Delta - 1$.

This concludes the proof of Proposition 4.5.

We now give the proofs of Theorem 4.7 in the cases (E0)-(E2)

Proof of Theorem 4.7 in the case (E0). We first prove that $< Z_{0,\lambda} > \simeq \mathbf{P}^{n-1}$ for all λ. If this were not true, the natural map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{Z_{0,\lambda}})$$

would not be surjective for some λ.

As usual let $Z' \subseteq Z_{0,\lambda}$ be a subscheme of length $n' \leq n$, for some integer $n' \geq 2$, such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{Z'})$ is not surjective, but such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{Z''})$ is surjective for all proper subschemes $Z'' \subsetneqq Z'$.

Since $L^2 = 4n + 2 = 4(n-1) + 6$, we get by [Kn1] that there exists an effective divisor B passing through Z' such that $B^2 \geq -2$, $h^1(B) = 0$ and

$$2B^2 < B.L \leq B^2 + n' < 2n'.$$

If $B^2 \geq 0$, we would get that B induces a Clifford index $c_B \leq n' - 2 \leq c - 1$ on the smooth curves in $|L|$, a contradiction.

So $B^2 = -2$, and B is necessarily supported on a union of smooth rational curves, since $h^1(B) = 0$. But $B.L \leq n' - 2 \leq n - 2 = c - 1$ and Z' has length ≥ 2, so it necessarily contains some of the base points of $\{D_\lambda\}$. This means that B passes through some of these base points, which contadicts the fact that we have chosen these base points to lie outside of smooth rational curves of degree $\leq c + 2$ with respect to L.

So $< Z_{0,\lambda} > \simeq \mathbf{P}^{n-1}$, and by equation (10) and Proposition 4.5 we know that $V \simeq \mathbf{P}^{n-1}$, so the point y does not pose any additional conditions.

To prove the last assertion, assume to get a contradiction that there exists a point $x_0 \in S' - \{x_1, \ldots, x_n, y\}$ such that S' has an $(n+1)$-secant $(n-1)$-plane through x_0 and $Z_{0,\lambda}$. Choose any preimage p_0 of x_0 and denote by X_λ the zero-dimensional scheme defined by p_0 and $Z_{0,\lambda}$. We then have that the natural map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{X_\lambda})$$

is not surjective. Fix a λ.

Again let $X' \subseteq X_\lambda$ be a subscheme of length $n'+1 \leq n+1$, for some integer $n' \geq 1$, such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{X'})$ is not surjective, but such that the map $H^0(L) \to H^0(L \otimes \mathcal{O}_{X''})$ is surjective for all subschemes $X'' \subseteq X'$.

Since $L^2 = 4n+2$ and L is not divisible by assumption, we get by [Kn1] again that there exists an effective divisor B passing through X' satisfying $B^2 \geq -2$, $h^1(B) = 0$ and the numerical conditions

$$2B^2 < B.L \leq B^2 + n' + 1 < 2n' + 2.$$

If $B^2 = -2$ we argue as above, using $B.L \leq n' - 1 \leq n - 1 = c$, and get a contradiction to our choice of the base points of $\{D_\lambda\}$ to lie outside of smooth rational curves of degree $\leq c + 2$.

So $B^2 \geq 0$, $n = n'$ and $X' = X_\lambda$, so B is a Clifford divisor (the condition $L - B \geq B$ will be fullfilled by [Kn1]) and by Proposition 3.6, we have $D \sim B$. By the proof of Proposition 7.5 in [Kn1], we then have $\Gamma \cap X_\lambda \neq \emptyset$, whence we conclude that $p_0 \in \Gamma$. This gives the desired contradiction $x_0 = y$. $\square$

Proof of Theorem 4.7 in the case (E1). We first show that $< Z_{i,\lambda} > \simeq \mathbf{P}^n$ for $i = 1, 2$ and any λ. If this were not true, the natural map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{Z_{i,\lambda}})$$

would not be surjective.

Let $Z' \subseteq Z_{i,\lambda}$ be a subscheme of length $n'+1 \leq n+1$, for some integer $n' \geq 1$, such that the map $H^0(L) \rightarrow H^0(L \otimes \mathcal{O}_{Z'})$ is not surjective, but such that the map $H^0(L) \rightarrow H^0(L \otimes \mathcal{O}_{Z''})$ is surjective for all proper subschemes $Z'' \subsetneqq Z'$.

Since $L^2 = 4c+4 = 4(n+1)$, we have by [Kn1] again that there exists an effective divisor B passing through Z' such that $B^2 \geq -2$, $h^1(B) = 0$ and

$$2B^2 \leq B.L \leq B^2 + n' + 1 \leq 2n' + 2.$$

If $B^2 \geq 0$, we would get that B induces a Clifford index $c_B \leq n' - 1 \leq c - 1$ on the smooth curves in $|L|$, a contradiction.

So $B^2 = -2$, and B is necessarily supported on a union of smooth rational curves, since $h^1(B) = 0$. But $B.L \leq n' - 1 \leq n - 1 = c - 1$ and Z' has length ≥ 2, so B passes through some of the base points of $\{D_\lambda\}$. This contadicts the fact that we have chosen these base points to lie outside of smooth rational curves of degree $\leq c + 2$ with respect to L.

To prove the second assertion, we will show that if there is a point $x_0 \in S' - \{x_1, \ldots, x_n, y_1\}$ such that S' has an $(n+2)$-secant n-plane through x_0 and $Z_{1,\lambda}$, then $x_0 = y_2$. By symmetry, this will suffice.

As usual choose any preimage p_0 of x_0 and denote by $X_{1,\lambda}$ the zero-dimensional scheme defined by p_0 and $Z_{1,\lambda}$. We then have that the natural map

$$H^0(L) \longrightarrow H^0(L \otimes \mathcal{O}_{X_{1,\lambda}})$$

is not surjective for any λ.

As usual let Let $X'_{1,\lambda} \subseteq X_{1,\lambda}$ be a subscheme of length $n'_{1,\lambda} + 2 \leq n + 2$, for some integer $n'_{1,\lambda} \geq 0$, such that the map $H^0(L) \rightarrow H^0(L \otimes \mathcal{O}_{X'_{1,\lambda}})$ is not surjective, but such that the map $H^0(L) \rightarrow H^0(L \otimes \mathcal{O}_{X''_{1,\lambda}})$ is surjective for all proper subschemes $X''_{1,\lambda} \subsetneqq X'_{1,\lambda}$.

Since $n = D^2 = c \geq 2$ and $L^2 = 4c + 4 = 4(n + 1)$, we again have by [Kn1] that there for each λ exists an effective divisor $B_{1,\lambda}$ passing through $X_{1,\lambda}$ and satisfying $B^2_{1,\lambda} \geq -2$, $h^1(B_{1,\lambda}) = 0$ and the numerical conditions

$$2B^2_{1,\lambda} \overset{(a)}{\leq} L.B_{1,\lambda} \leq B^2_{1,\lambda} + n'_{1,\lambda} + 2 \overset{(b)}{\leq} 2n'_{1,\lambda} + 4,$$

with equality in (a) or (b) implying $L \sim 2B_{1,\lambda}$.

Assume first that $B^2_{1,\lambda} = -2$ for some λ. We then get the same contradiction on the choice of the base points of $\{D_\lambda\}$, since $B_{1,\lambda}.L \leq n'_{1,\lambda} \leq n = c$.

So we must have $B^2_{1,\lambda} \geq 0$ for all λ. Then $n'_{1,\lambda} = n$, $X'_{1,\lambda} = X_{1,\lambda}$, $L.B_{1,\lambda} = B^2_{1,\lambda} + n + 2$, and $B_{1,\lambda}$ is a Clifford divisor. The moving part $B'_{1,\lambda}$ of $|B_{1,\lambda}|$ is then a free Clifford divisor, so by condition (C7) we have that either $B'_{1,\lambda} \sim D$ or there exists a free Clifford divisor $P_{1,\lambda}$

such that $B'_{1,\lambda} \leq P_{1,\lambda} < D$ with the last inclusion as described in Proposition 5.5, with the additional property that $|L - P_{1,\lambda}|$ has no fixed divisor, by the conditions (C6) and (C7).

We will show that this latter case cannot occur.

We have that $B_{1,\lambda}$ passes through $X_{1,\lambda}$. Now a (possible) base divisor in $|B_{i,\lambda}|$ cannot pass through any of the points $p_0, \dots, p_n$, since these points lie outside all the rational curves contracted by L. So we must have $B'_{i,\lambda}.D \geq n$.

In addition, by Proposition 5.5 we must have

$$D \sim B'_{1,\lambda} + \gamma_{1,\lambda},$$

for some $\gamma_{1,\lambda} > 0$ satisfying $\gamma_{1,\lambda}^2 = -2$. Moreover, $B'_{1,\lambda}.\Delta = 0$. Hence

$$B'_{1,\lambda}.L = B'_{1,\lambda}.(L - \Delta) = 2B'_{1,\lambda}.D \geq 2n,$$

so that

$${B'_{1,\lambda}}^2 \geq n - 2 = D^2 - 2.$$

Since $h^1(B'_{1,\lambda}) = h^1(P_{1,\lambda}) = h^1(D) = 0$, we clearly must have $B'_{1,\lambda} \sim P_{1,\lambda}$.

This means that $B'_{1,\lambda} \sim B_{1,\lambda}$, so $B_{1,\lambda} < D$ and $B_{1,\lambda}$ is a free Clifford divisor. Since $h^0(B_{1,\lambda} \otimes \mathcal{J}_{X_{1,\lambda}}) > 0$, there must exist an element of $|D|$ of the form $B_{1,\lambda} + A_{1,\lambda}$ passing through $Z_{1,\lambda}$, for $A_{i,\lambda} > 0$. But since there is only one element of $|D|$ passing through $p_1, \dots, p_n, q_{1,\lambda}$, which we called D_λ and which is smooth and irreducible, we have $B_{1,\lambda} = D_\lambda$, a contradiction.

So we must have $B'_{1,\lambda} \sim D$. Recall by [Kn1] that either $L - B_{1,\lambda} \geq B_{1,\lambda}$, or we are in a case where a certain exact sequence splits (namely the sequence (3.2)), in which case we have both $h^0(B_{1i,\lambda} \otimes \mathcal{J}_{X_{1,\lambda}})$ and $h^0((L - B_{1,\lambda}) \otimes \mathcal{J}_{X_{1,\lambda}}) \neq 0$. This gives us the two possibilities:

1. $B_{1,\lambda} \in |D|$,
2. $B_{1,\lambda} \in |D| + \Gamma_{j(\lambda)}$, for $j(\lambda) = 1$ or 2, and there exists an $F_{1,\lambda} \in |D| + \Gamma_{3-j(\lambda)}$ passing through $X_{1,\lambda}$.

In case 1., since there is only one member of $|D|$ containing $p_1, \dots, p_n, q_{1,\lambda}$, which we called D_λ, we have $B_{1,\lambda} = D_\lambda$. But this would mean that $p_0 \in D_\lambda$ for all λ, a contradiction.

In case 2. one easily sees that the only option is $p_0 \in \Gamma_2$, which means that $x_0 = y_2$, as desired. □

The proof of Theorem 4.7 in the case (E2) is similar, and therefore left to the reader.

Since we have seen that the crucial point in proving Propositon 4.5 is to prove that $h^1(A) = 0$, we get the following result (by checking that the proof of Theorem 4.7 goes through):

Lemma 5.10. *Let D be a free Clifford divisor, not of type (E1) or (E2). If $h^1(A) = 0$, then D is perfect.*

6. Projective models in smooth scrolls

In this section we will study the case when the scroll $\mathcal{T}$ associated to S and L as above is smooth. When D is perfect, this happens if and only if $D^2 = 0$ and $\mathcal{R}_{L,D} = \emptyset$. We recall that $\phi_L(S)$ is denoted by S'.

Since $\mathcal{T}$ is smooth, it can be identified with the $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbf{P}^1}(e_1) \oplus \mathcal{O}_{\mathbf{P}^1}(e_2) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(e_{c+2})$, and $(e_1, e_2, \dots, e_{c+2})$ is the type of the scroll.

We will construct a resolution of the structure sheaf $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathcal{T}}$-module.

The contents in this section will be very similar to that in [Sc], where canonical curves of genus g are treated. This is quite natural, since a general hyperplane section of S' is indeed such a canonical curve.

The following are well-known facts about $\mathcal{T}$ in $\mathbf{P}^g$ (see [Ha] and [E-H]):

(1) $\deg \mathcal{T} = g - c - 1$.
(2) $\dim \mathcal{T} = c + 2$.
(3) The Chow ring of $\mathcal{T}$ is $\mathbf{Z}[\mathcal{H}, \mathcal{F}]/(\mathcal{F}^2, \mathcal{H}^{c+3}, \mathcal{H}^{c+2}\mathcal{F}, \mathcal{H}^{c+2} - (g-c-1)\mathcal{H}^{c+1}\mathcal{F})$, where $\mathcal{H}$ is the hyperplane section, and $\mathcal{F}$ is the class of the ruling.
(4) The canonical class of $\mathcal{T}$ is $-(c+2)\mathcal{H} + (g-c-3)\mathcal{F}$.
(5) The class of S' in the Chow ring of $\mathcal{T}$ is $(c+2)\mathcal{H}^c + (c^2+3c-cg)\mathcal{H}^{c-1}\mathcal{F}$.

We will need the betti-numbers of the $\phi_L(D_\lambda)$ in $\mathbf{P}^{c+1}$. These can be found also when $\mathcal{T}$ is singular, and will be needed in this case later on.

Lemma 6.1. *Let (S, L) be a polarized $K3$ surface of genus g and of non-general Clifford index $c > 0$. Let D be a free Clifford divisor satisfying $D^2 = 0$. For $c \geq 2$, all the $\phi_L(D_\lambda)$ in $\mathbf{P}^{c+1}$ have minimal resolutions*

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^{c+1}}(-(c+2)) \longrightarrow \mathcal{O}_{\mathbf{P}^{c+1}}(-c)^{\beta_{c-1}} \longrightarrow \mathcal{O}_{\mathbf{P}^{c+1}}(-(c-1))^{\beta_{n-3}} \longrightarrow \cdots$$
$$\longrightarrow \mathcal{O}_{\mathbf{P}^{c+1}}(-3)^{\beta_2} \longrightarrow \mathcal{O}_{\mathbf{P}^{c+1}}(-2)^{\beta_1} \to \mathcal{O}_{\mathbf{P}^{c+1}} \to \mathcal{O}_{\phi_L(D_\lambda)} \longrightarrow 0,$$

where

$$\beta_i = i\binom{c+1}{i+1} - \binom{c}{i-1}.$$

For $c = 1$ all the $\phi_L(D_\lambda)$ in $\mathbf{P}^2$ have the resolution

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^2}(-3) \longrightarrow \mathcal{O}_{\mathbf{P}^2} \longrightarrow \mathcal{O}_{\phi_L(D_\lambda)} \longrightarrow 0.$$

Proof. In Proposition 7.6 below we will show that in general (not only for $D^2 = 0$) the betti-numbers of all $\phi_L(D_\lambda)$ are constant. Granted this, and the fact that L is very ample on D_λ for general λ by Lemma 4.2, the result follows since any smooth $\phi_L(D_\lambda)$ is an elliptic normal curve in $\mathbf{P}^{c+1}$, with a resolution as above. Even so, we will now give a special and different proof for the present case $D^2 = 0$.

Pick any $D'_\lambda := \phi_L(D_\lambda)$. Its betti-numbers are equal to those of a general hyperplane section of it. It is sufficient that the linear term defining the hyperplane is not a zero divisor in its coordinate ring R_λ. This is essentially [Na, Theorem 27.1].

Now choose a sufficiently general hyperplane H_λ in $\mathbf{P}^g$ so that $C_\lambda := H_\lambda \cap S'$ is a smooth canonical curve, H_λ does not contain any of the linear spaces $\overline{D_\lambda}$, and the hyperplane section $A_\lambda := H_\lambda \cap D_\lambda$ is not a zero divisor of R_λ.

We can identify C_λ with an element in $|L|$, and by abuse of notation write $\mathcal{O}_{C_\lambda}(A_\lambda) = \mathcal{O}_{C_\lambda}(D_\lambda) = \mathcal{O}_{C_\lambda}(D)$. This linear system is complete and base point free (in fact it is a pencil computing the gonality) of degree $c+2$ on C_λ. By [Sc, Lemma p.119] (where there is a misprint) and [Sc, Proposition 4.3] the zero-dimensional scheme A_λ then has the betti-numbers $\beta_{i,i+1} = \beta_i = i\binom{c+1}{i+1} - \binom{c}{i-1}$.

In particular, these numbers are independent of λ. □

The following result is analogous to [Sc, Corollary (4.4)].

Proposition 6.2. *Let S be a polarized $K3$ surface of non-general Clifford-index $c > 0$, whose associated scroll $\mathcal{T}$ as above is smooth.*

(a) *$\mathcal{O}_{S'}$ has a unique $\mathcal{O}_{\mathcal{T}}$-resolution F_* (up to isomorphism). If $c = 1$, the resolution is:*

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + (g-4)\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0,$$

If $c \geq 2$, the resolution is of the following type:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-(c+2)\mathcal{H}+(g-c-3)\mathcal{F}) \longrightarrow \oplus_{k=1}^{\beta_{c-1}}\mathcal{O}_{\mathcal{T}}(-c\mathcal{H}+b_{c-1}^k\mathcal{F}) \longrightarrow \cdots$$
$$\longrightarrow \oplus_{k=1}^{\beta_1}\mathcal{O}_{\mathcal{T}}(-2\mathcal{H}+b_1^k\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0,$$

where $\beta_i = i\binom{c+1}{i+1} - \binom{c}{i-1}$.

(b) *F_* is self-dual: $\mathcal{H}om(F_*, \mathcal{O}_{\mathcal{T}}(-(c+2)\mathcal{H}+(g-c-3)\mathcal{F})) \simeq F_*$.*

(c) *If all $b_i^k \geq -1$, then an iterated mapping cone*

$$[[\mathcal{C}^{g-c-3}(-(c+2)) \longrightarrow \oplus_{k=1}^{\beta_{c-1}}\mathcal{C}^{b_c^k}(-c)]\dots] \longrightarrow \mathcal{C}^0$$

is a (not necessarily minimal) resolution of $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathbf{P}^g}$-module.

(d) *The b_i^k satisfy the following polynomial equation in n if $c \geq 2$:*

$$\binom{n+c+1}{c+1}\Big(\frac{n(g-c-1)}{c+2}+1\Big) - n^2(g-1) - 2 \quad =$$
$$\sum_{i=1}^{c-1}\Big((-1)^{i+1}\binom{n-i+c}{c+1}\Big(\frac{((n-i-1)(g-c-1)+(c+2))\beta_i}{c+2} + \sum_{k=1}^{\beta_i} b_i^k\Big) \quad +$$
$$(-1)^{c+1}\binom{n-1}{c+1}\Big(\frac{(n-c-2)(g-c-1)}{c+2} + g - c - 2\Big).$$

Proof. We start by proving (a). We have $\overline{D_\lambda} \simeq \mathbf{P}^{c+1}$ by (8). The $\phi_L(D_\lambda)$ have betti-numbers $\beta_{i,j,\lambda} = \dim(Tor_i^{R_\lambda}(R,k)_j)$, where R is the homogeneous coordinate ring of $\mathbf{P}^{c+1}$, and R_λ the coordinate ring R/I_λ of $\phi_L(D_\lambda)$. Following [Sc], for $c \geq 2$ it is enough to prove:

(1) For fixed i, j the $\beta_{i,j,\lambda}$ are the same for all λ.
(2) If $c \geq 2$, then $\beta_{i,j,\lambda} = 0$, unless $j = i+1$ and $i \leq c-1$, or $(i,j) = (c, c+2)$.
(3) The common value $\beta_{i,i+1} = \beta_{i,i+1,\lambda}$ is $\beta_i = i\binom{c+1}{i+1} - \binom{c}{i-1}$ for $i \leq c-1$, and $\beta_{c,c+2} = 1$.

This follows immediately from the lemma above.

The easier case $c = 1$ is dealt with in an analogous manner.

The proof of (b) is almost identical to that of [Sc, Corollary 4.4(b)]. In our case we have $\mathcal{E}xt_{\mathcal{T}}^i(\mathcal{O}_{S'}, \omega_{\mathcal{T}}) = \omega_{S'}$ if $i = c$, and zero otherwise, $\omega_{S'} = \mathcal{O}_{S'}$, and $\omega_{\mathcal{T}} = \mathcal{O}_{\mathcal{T}}(-(c+1)\mathcal{H} + (g-c-3)\mathcal{F})$.

The proof of (c) is identical to that of [Sc, Prop. 4.4(c)].

Denote the term i places to the left of $\mathcal{O}_{\mathcal{T}}$ in the resolution F_* by F_i. The proof of (d) then follows from the identity

$$\chi(\mathcal{O}_{\mathcal{T}}(n\mathcal{H})) - \chi(\mathcal{O}_{S'}(n\mathcal{H})) = \sum_i (-1)^i \chi(F_i(n\mathcal{H})).$$

The contribution from the F_c-term is written out separately. Moreover it is clear that for all large n, we have $\chi(F_i(n\mathcal{H})) = h^0(F_i(n\mathcal{H})$, for all i, and $\chi(\mathcal{O}_{\mathcal{T}}(n\mathcal{H})) = h^0(\mathcal{O}_{\mathcal{T}}(n\mathcal{H}))$ since $\mathcal{H}$ is (very) ample on $\mathcal{T}$. Then one uses the following well-known fact for $a \geq 0$:

$$h^0(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(a\mathcal{H}+b\mathcal{F})) = h^0(\mathbf{P}^1, Sym^a(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^1}(b)). \tag{30}$$

□

Remark 6.3. Part (d) of the proposition only gives us the sums of the b_i^k for each fixed i. The values $n = 2, 3, \dots, c$ give enough equations to determine these sums. The duality of part (c) gives $\beta_i = \beta_{c-i}$, for $i = 1, \dots, c-1$, and after a possible renumeration of the b_i^k, for $k = 1, \dots, \beta_i$, we also have $b_{c-i}^k = g - c - 3 - b_i^k$ for these k. In particular this enables us to identify the sums of the b_i^k with those of the $g-c-3-b_{c-i}^k$. To obtain more information about the individual b_i^k a more refined study is necessary.

7. Projective models in singular scrolls

In this section we will study more thouroughly the case where the scroll $\mathcal{T}$ is singular. As seen above, this necessarily happens if $D^2>0$ or the set $\mathcal{R}_{L,D}$ is non-empty.

We will always assume $c>0$, so that $\phi_L : S \rightarrow S'$ is birational.

The type $(e_1,\ldots,e_d)$ of the scroll, where $d=\frac{1}{2}D^2+c+2$, is such that the last r of the e_i are zero, where r is defined as in equation (9) and can be computed as in equation (10).

As we have seen, when D is perfect we have

$$r = \begin{cases} D^2-1 & \text{if } D \text{ is of type (Q)}, \\ D^2+D.\Delta-1 & \text{if } D \text{ is of one of the types (E0)-(E2)}, \\ D^2+D.\Delta & \text{otherwise.} \end{cases} \tag{31}$$

We will however not assume that D is perfect, unless explicitly stated.

Let $n:=D^2$ and denote by $p_1,\ldots,p_n$ the n base points of the pencil $\{D_\lambda\}$. Let

$$\tilde{S} \xrightarrow{f} S$$

be the blow up of S at $p_1,\ldots,p_n$. Denote by E_i the exceptional line over p_i and let

$$E := \sum_{i=1}^{n} E_i$$

denote the exceptional divisor. Define

$$H := f^*L + f^*D - E.$$

The first observation is:

Lemma 7.1. *H is generated by its global sections, $h^1(H)=0$ and ϕ_H is birational; in fact ϕ_H is an isomorphism outside of finitely many smooth rational (-2)-curves.*

Moreover, a smooth rational curve γ is contracted by H if and only if $\gamma = f^*\Gamma$, for some smooth rational curve Γ on S such that $\Gamma.L=\Gamma.D=0$.

Proof. Since $H-E \sim (f^*L-E)+(f^*D-E)$ is clearly nef, we have $h^1(H)=0$. Since furthermore $(H-E)^2 \geq 10$, it follows from the well-known results of Reider [R, Thm. 1(i) and Cor. 2] that if $|H|$ is not base-point free or ϕ_H is not birational, then there exists an effective divisor B on $\tilde{S}$ such that $B^2=-1$ and $B.(H-E)=0$, or $B^2=0$ and $B.(H-E)\leq 2$.

Assume there exists such a divisor B.

If $B^2=-1$, then $B=E_i$ for some i, say $i=1$, and $B.(H-E)=E_1(H-E)=-2E_1^2=2$, a contradiction.

If $B^2=0$, write $B=f^*A-rE$ for some divisor $A>0$ on S and integer $r\geq 0$. Then $A^2=r^2$ and by the Hodge index theorem we get $A.L>2r$. Furthermore, $A.D\geq 2$, since D is base point free. This yields the contradiction

$$B.(H-E)=(f^*A-rE)(f^*L+f^*D-2E)=A.(L+D)-2r>2r+2-2r\geq 2.$$

The last statement is easily checked. □

We have $h^0(H)=\frac{1}{2}H.(H-E)+2=\frac{1}{2}L^2+\frac{1}{2}D^2+c+4=g+\frac{1}{2}D^2+c+2=g+d+1$. Denote by S'' the surface $\phi_H(\tilde{S})$ in $\mathbf{P}^{g+d}$.

One easily obtains $\deg S''=2g+2c+2+2D^2$.

Proposition 7.2. *The surface S'' is normal, $p_a(S'') = 1$, and $K_{S''} \simeq \mathcal{O}_{S''}(E')$, where E' is the sum of D^2 lines that are (-1)-curves on S''.*

Proof. The two last assertions are immediate consequences of S'' being normal, by [Ar].

Consider the surface $\tilde{S}$ and the commutative diagram in the previous lemma.

Denote by $\mathcal{E}_H$ the set of irreducible curves $\tilde{\Gamma}$ on $\tilde{S}$ such that $\tilde{\Gamma}.H = 0$. From the Hodge Index theorem it follows that such a curve has negative self-intersection. Moreover, by Lemma 7.1

$$\tilde{\Gamma} = f^*\Gamma,$$

for some smooth rational curve Γ on S such that $\Gamma.L = \Gamma.D = 0$. Thus we can write

$$\mathcal{E}_H = f^*(\mathcal{E}_L - \mathcal{R}_{L,D}).$$

Now let $\tilde{\delta}$ be the fundamental cycle of a connected component of $\mathcal{E}_H$, p the image of $\tilde{\delta}$ on S'' and U the inverse image of an affine open neighborhood of p. To prove the normality of p it will be sufficient to prove the surjectivity of

$$H^0(U, \mathcal{O}_U(H - \tilde{\delta})) \longrightarrow H^0(\tilde{\delta}, \mathcal{O}_{\tilde{\delta}}(H - \tilde{\delta})),$$

hence of

$$H^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(H - \tilde{\delta})) \longrightarrow H^0(\tilde{\delta}, \mathcal{O}_{\tilde{\delta}}(H - \tilde{\delta})).$$

To show the latter, it will suffice to show

$$H^1(\tilde{S}, \mathcal{O}_{\tilde{S}}(H - 2\tilde{\delta})) = 0.$$

By the degeneration of the Leray spectral sequence

$$0 \longrightarrow H^1(S, f_*(H - 2\tilde{\delta})) \longrightarrow H^1(\tilde{S}, H - 2\tilde{\delta}) \longrightarrow H^0(S, R^1 f_*(H - 2\tilde{\delta}))$$

it will suffice to show that

$$h^1(S, f_*(H - 2\tilde{\delta})) = h^0(S, R^1 f_*(H - 2\tilde{\delta})) = 0.$$

Denote by δ the divisor on S such that $f^*\delta = \tilde{\delta}$. Then δ is connected and $\delta^2 = -2$ (δ is in fact a fundamental cycle for a connected component of $\mathcal{E}_L$ minus a curve Γ that is a tail of δ and is such that $\Gamma.D = 1$. The fact that $\delta^2 = -2$ can be checked by inspection for each of the five platonic configurations [Ar]). We then have

$$f_*(H - 2\tilde{\delta}) = (L + D - 2\delta) \otimes \mathcal{J}_Z,$$

where Z is the zero-dimensional scheme corresponding to the n blown up points, and

$$R^1 f_*(H - 2\tilde{\delta}) = R^1 f_*(-E) \otimes (L + D - 2\delta).$$

Since $f_*\mathcal{O}_E \simeq \mathcal{O}_Z$, we have $R^1 f_*(-E) = 0$, whence we are reduced to proving the vanishing of $H^1((L + D - 2\delta) \otimes \mathcal{J}_Z)$. This will be proved in Lemma 7.3 below. □

Lemma 7.3. *With the notation as above, $H^1((L + D - 2\delta) \otimes \mathcal{J}_Z) = 0$.*

Proof. We will first need the following fact:

$$h^1(L + D - 2\delta) = 0.$$

The proof for this is rather long and tedious, but does not involve any new ideas and is similar in principle to the proof of [Co, Lemma 5.3.5]. We therefore leave it to the reader.

Note that if $D^2 = 0$, then $Z = \emptyset$, and we are done. So we will from now on assume that $n = D^2 > 0$.

Because of the vanishing of $H^1(L+D-2\delta)$, the vanishing of $H^1((L+D-2\delta)\otimes \mathcal{J}_Z)$ is equivalent to the surjectivity of the map

$$H^0(L+D-2\delta) \longrightarrow H^0((L+D-2\delta)\otimes \mathcal{O}_Z).$$

Assume, to get a contradiction, that this map is not surjective. Let $Z' \subseteq Z$ be a subscheme of length $l+1 \leq n = D^2$, for some integer $l \geq 1$, such that $H^0(L+D-2\delta) \longrightarrow H^0((L+D-2\delta)\otimes \mathcal{O}_{Z'})$ is not surjective, but $H^0(L+D-2\delta) \longrightarrow H^0((L+D-2\delta)\otimes \mathcal{O}_{Z''})$ is for all proper subschemes Z''.

Since $(L+D-2\delta)^2 > 4l+4$, we can argue as in [Kn1, Lemma 3.5], with the assumption that $(L+D-2\delta)$ be big and nef (which it is not) replaced by the weaker condition $h^1(L+D-2\delta)=0$, and get the following:

There is an effective decomposition $L+D-2\delta \sim A+B$ such that $A \geq B$, $A.B \leq l+1$, $h^1(B)=0$, $B^2 \geq -2$ and B passes through Z'.

If $B^2=-2$ (so that B is necessarily supported on a union of smooth rational curves), then we use the fact that we have chosen Z to lie outside of any rational curve Γ such that $\Gamma.L \leq c+2$ by (4) and

$$L.B \leq (L+D).B \leq l-1+2\delta.B \leq D^2-2+2\delta.B \leq c+2\delta.B,$$

to conclude that we must have $\delta.B \geq 2$. Hence $(\delta+B)^2 \geq 0$.

This yields that we in all cases have

$$h^0(\delta+B) \geq 2.$$

We now want to show that also

$$h^0(\delta+A-D) \geq 2.$$

We can write

$$F \sim A+B+2\delta-2D \sim (A+\delta-D)+(B+\delta-D) := F_1+F_2.$$

This is not necessarily an effective decomposition, but we have $F_1 \geq F_2$, since $A \geq B$.

We can easily calculate

$$F_1.F_2 = A.B-D^2-c \leq -c < 0,$$

and since $F^2 = {F_1}^2+{F_2}^2+2F_1.F_2 \geq D^2 \geq 2$, we must have ${F_1}^2 \geq 2$ or ${F_2}^2 \geq 2$.

If ${F_1}^2 \geq 2$, then either $h^0(F_1) \geq 2$ or $h^0(-F_1) \geq 2$ by Riemann-Roch. Since $L.F_1 = L.A-L.D \geq \frac{1}{2}(L^2+L.D)-L.D = \frac{1}{2}(c+2+F^2) > 0$, we must have $h^0(F_1) \geq 2$, and we are done.

If ${F_2}^2 \geq 2$, then either $h^0(F_2) \geq 2$ or $h^0(-F_2) \geq 2$. In the first case, we get $h^0(F_1) \geq h^0(F_2) \geq 2$. In the second, we get $F_1 \sim F-F_2 > F$, since $-F_2$ is effective, whence $h^0(F_1) \geq h^0(F) \geq 2$ again.

So we have an effective decomposition of L as

$$L \sim (B+\delta)+(A+\delta-D),$$

such that both $h^0(B+\delta)$ and $h^0(A+\delta-D) \geq 2$ and such that

$$(B+\delta).(A+\delta-D) = A.B-D.B+2 \leq l-D.B+3.$$

Since $l+1 \leq D^2 \leq c+2$, and $D.B \geq 2$, since D is base point free and $h^0(B+\delta) \geq 2$, we must have $D.B=2$ and $l+1=n=D^2=c+2$. But since B passes through Z, we must have $D.B \geq n$, whence the contradiction $c=0$.

This concludes the proof of the lemma and hence of Proposition 7.2. □

Define the following line bundle on $\tilde{S}$:

$$\tilde{D} := f^*D - E.$$

The members of $|\tilde{D}|$ are in one-to-one correspondence with the members of the pencil $\{D_\lambda\}$. One computes $\tilde{D}^2 = 0$, so $|\tilde{D}|$ is a pencil of disjoint members. Furthermore

$$h^0(H - \tilde{D}) = h^0(f^*L) > 2,$$

so $|\tilde{D}|$ defines a rational normal scroll $\mathcal{T}_0$ containing S''.

Proposition 7.4. *$\mathcal{T}_0$ has dimension d and degree $g+1$ and is of type $(e_1+1, \dots, e_d+1)$.*

Proof. The two firts assertions are easily checked.

We have to calculate the numbers $h^0(\tilde{S}, H - i\tilde{D}) = h^0(\tilde{S}, f^*(L-(i-1)D + (i-1)E)$ for all $i \geq 0$.

One easily sees that $(i-1)E$ is a fixed divisor in $|f^*(L-(i-1)D+(i-1)E|$ for all $i \geq 1$, so we get for all $i \geq 1$:

$$h^0(\tilde{S}, f^*(L-(i-1)D+(i-1)E) = h^0(\tilde{S}, f^*(L-(i-1)D) = h^0(S, L-(i-1)D). \tag{32}$$

We also have

$$h^0(H) - h^0(H - \tilde{D}) = d. \tag{33}$$

Defining $d'_i := h^0(\tilde{S}, H - i\tilde{D}) - h^0(\tilde{S}, H-(i+1)\tilde{D})$, we get by combining (32) and (33) that

$$d'_0 = d_0 \quad \text{and} \quad d'_i = d_{i-1} \quad \text{for} \quad i \geq 1.$$

It follows immediately that the type of $\mathcal{T}_0$ is as claimed. □

This means that $\mathcal{T}_0$ is smooth, so that $\mathcal{T}_0 \simeq \mathbf{P}(\mathcal{E})$, where $\mathcal{E} = \oplus_{i=1}^{d} \mathcal{O}_{\mathbf{P}^1}(e_i+1)$. Also, we have the maps

$$\begin{tikzcd} \mathbf{P}(\mathcal{E}) \arrow[r, "j"] \arrow[d, "\pi"] & \mathcal{T}_0 \subseteq \mathbf{P}^{g+d} \\ \mathbf{P}^1 & \end{tikzcd}$$

where j is an isomorphism. Then the Picard group of $\mathbf{P}(\mathcal{E})$ satisfies

$$\operatorname{Pic} \mathbf{P}(\mathcal{E}) \simeq \mathbf{Z}\mathcal{H}_0 \oplus \mathbf{Z}\mathcal{F},$$

where $\mathcal{H}_0 := j^*\mathcal{O}_{\mathbf{P}^{g+d}}(1)$ and $\mathcal{F} := \pi^*\mathcal{O}_{\mathbf{P}^1}(1)$.

Furthermore, the Chow ring of $\mathbf{P}(\mathcal{E})$ is

$$\mathbf{Z}[\mathcal{H}_0, \mathcal{F}]/(\mathcal{F}^2, \mathcal{H}_0^{c+3+\frac{1}{2}D^2}, \mathcal{H}_0^{c+2+\frac{1}{2}D^2}\mathcal{F}, \mathcal{H}_0^{c+2+\frac{1}{2}D^2} - (g+1)\mathcal{H}_0^{c+1+\frac{1}{2}D^2}\mathcal{F}). \tag{34}$$

Consider now the morphism i given by the base point free line bundle $\mathcal{H} := \mathcal{H}_0 - \mathcal{F}$, where $\mathcal{H}_0 = \mathcal{H} + \mathcal{F}$:

$$i : \mathbf{P}(\mathcal{E}) \longrightarrow \mathbf{P}^g.$$

One easily sees that i maps $\mathbf{P}(\mathcal{E})$ onto a rational normal scroll of dimension d and type $(e_1, \dots, e_d)$, whence isomorphic to $\mathcal{T}$. So we can assume that i maps $\mathbf{P}(\mathcal{E})$ onto $\mathcal{T}$. By abuse of notation we write

$$i : \mathcal{T}_0 \longrightarrow \mathcal{T},$$

and this is a rational resolution of singularities of $\mathcal{T}$. Furthermore one easily sees that by construction i restricts to a map

$$g : S'' \longrightarrow S'$$

which is a resolution of some singularities of S' (precisely the singularities of S' arising from the contractions of rational curves across the fibers in S, i.e. the curves in $\mathcal{R}_{L,D}$) and a blow up at the images of the base points of $\{D_\lambda\}$.

We get the following commutative diagram:

$$\begin{tikzcd}
\mathbf{P}(\mathcal{E}) \arrow[r, equal, "j"] & \mathcal{T}_0 \arrow[r, "i"] & \mathcal{T} \arrow[r] & \mathbf{P}^g \\
 & S'' \arrow[u] \arrow[r, "g"] & S' \arrow[u] & \\
 & \tilde{S} \arrow[u, "\phi_H"] \arrow[r, "f"] & S. \arrow[u, "\phi_L"'] &
\end{tikzcd}$$

By construction, one has $g \circ \phi_H = \phi_{f^*L}$.

We would like to study the resolution of S'' in $\mathbf{P}(\mathcal{E}) \simeq \mathcal{T}_0$. By [Sc], if S'' has constant betti-numbers $\beta_{ij} = \beta_{ij}(\lambda)$ over $\mathbf{P}^1$, we can (at least in principle) find a resolution of $\mathcal{O}_{S''}$ by free $\mathcal{O}_{\mathbf{P}(\mathcal{E})}$-modules which restricts to the minimal resolution of $\mathcal{O}_{S''_\lambda}$ on each fiber $\mathbf{P}(\mathcal{E})_\lambda \simeq \mathbf{P}^{c+1+\frac{1}{2}D^2}$.

Clearly, since the map i is the identity on each fiber, the betti-numbers of S''_λ are the same as the betti-numbers of $\phi_L(D_\lambda)$.

Recall that a projective variety V is called arithmetically normal if the natural map

$$S_k H^0(V, \mathcal{O}_V(1)) \longrightarrow H^0(V, \mathcal{O}_V(k))$$

is surjective for all $k \geq 0$.

We start by showing that the $\phi_L(D_\lambda)$ are all arithmetically normal.

Proposition 7.5. *All the $\phi_L(D_\lambda)$ are arithmetically normal in $\overline{D_\lambda} = \mathbf{P}^{c+1+\frac{1}{2}D^2}$.*

Proof. We can easily show that

$$h^1(\mathcal{O}_S(qL - D)) = 0 \text{ for all } q. \tag{35}$$

Furthermore, by [SD, Thm. 6.1], we have that

$$S_k H^0(S, L) \longrightarrow H^0(S, kL) \text{ is surjective for all } k \geq 0. \tag{36}$$

We have a commutative diagram

$$\begin{tikzcd}
H^0(\mathcal{O}_{\mathbf{P}^g}(q)) \arrow[r] \arrow[d, "\alpha_1"] & H^0(\mathcal{O}_{\overline{D_\lambda}}(qL)) \arrow[d, "\alpha_3"] \\
H^0(\mathcal{O}_S(qL)) \arrow[r, "\alpha_2"] & H^0(\mathcal{O}_{D_\lambda}(qL)).
\end{tikzcd}$$

Now α_2 is surjective by (35) and α_1 is surjective by (36). Hence α_3 is surjective and $\phi_L(D_\lambda)$ is arithmetically normal. □

For each $\lambda \in \mathbf{P}^1$ define

$$B^\lambda := \oplus_{q \in \mathbf{Z}} H^0(D_\lambda, qL) \quad \text{and} \quad V^\lambda := H^0(D_\lambda, L).$$

Then B^λ is a V^λ-module in a natural way.

The symmetric algebra $S(V^\lambda)$ of V^λ satisfies

$$S(V^\lambda) \simeq R,$$

where R is the homogeneous coordinate ring of $\mathbf{P}^{c+1+\frac{1}{2}D^2}$, and B^λ is a graded R-module.

We have the *Koszul complex*

$$\cdots \longrightarrow \wedge^{i+1}V^\lambda \otimes B^\lambda_{j-1} \xrightarrow{d^\lambda_{i+1,j-1}} \wedge^i V^\lambda \otimes B^\lambda_j \xrightarrow{d^\lambda_{i,j}} \cdots$$

with the *Koszul cohomology groups* defined by

$$\mathcal{K}_{i,j}(B^\lambda, V^\lambda) := \frac{\ker d^\lambda_{i,j}}{\operatorname{im} d^\lambda_{i+1,j-1}}.$$

For each λ we have a minimal free resolution of B^λ as an R-module:

$$\cdots \longrightarrow \oplus_{i,j}R(-j)^{\beta_{i,j}} \longrightarrow \cdots \longrightarrow \oplus_{i,j}R(-j)^{\beta_{1,j}} \longrightarrow \oplus_{i,j}R(-j)^{\beta_{0,j}} \longrightarrow B^\lambda \longrightarrow 0,$$

and the $\beta_{i,j}$ are the *(graded) betti-numbers* for $\phi_L(D_\lambda)$ (since $\phi_L(D_\lambda)$ is arithmetically normal).

By the well-known *Syzygy Theorem* [Gr, Thm. (1.b.4)], we have

$$\beta_{i,j} = \dim \mathcal{K}_{i,j}(B^\lambda, V^\lambda)$$

(where the dimension is as vector space over $\mathbf{C}$).

So to show that the $\phi_L(D_\lambda)$ all have the same betti-numbers, it is necessary and sufficient to show that the $\mathcal{K}_{i,j}(B^\lambda, V^\lambda)$ are isomorphic (as vector spaces) for all λ. This is the contents of the following

Proposition 7.6. *The groups $\mathcal{K}_{i,j}(B^\lambda, V^\lambda)$ are isomorphic as vector spaces over $\mathbf{C}$ for all λ.*

Proof. Let D_{λ_1} and D_{λ_2} be two distinct members of the pencil and define B^{λ_1}, B^{λ_2}, V^{λ_1} and V^{λ_2} as above. Since $H^1(qL-D) = 0$ for all $q \in \mathbf{Z}$, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \oplus H^0(S, L^q \otimes \mathcal{J}_{D_{\lambda_1}}) \arrow[r] \arrow[d] & \oplus H^0(S, L^q) \arrow[r] \arrow[d, equal] & B^{\lambda_1} \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \oplus H^0(S, L^q \otimes \mathcal{J}_{D_{\lambda_2}}) \arrow[r] & \oplus H^0(S, L^q) \arrow[r] & B^{\lambda_2} \arrow[r] & 0,
\end{tikzcd}$$

where the isomorphism ψ is induced by the isomorphism on the left and the equality in the middle. By restriction, ψ also induces an isomorphism $V^{\lambda_1} \to V^{\lambda_2}$.

We want to show that the map

$$\wedge^i V^{\lambda_1} \otimes B^{\lambda_1}_j \xrightarrow{\wedge^i\psi\otimes\psi_j} \wedge^i V^{\lambda_2} \otimes B^{\lambda_2}_j$$

induced by ψ descends to Koszul cohomology and gives an isomorphism

$$\mathcal{K}_{i,j}(B^{\lambda_1}, V^{\lambda_1}) \longrightarrow \mathcal{K}_{i,j}(B^{\lambda_2}, V^{\lambda_2}).$$

For this it suffices to show that

$$\psi(v)\cdot\psi(b) = \psi(v\cdot b) \quad \text{for all} \quad v \in V^{\lambda_1} \quad \text{and} \quad b \in B^{\lambda_1}.$$

This follows from the commutative diagram above. □

Corollary 7.7. *The betti-numbers of the $\phi_L(D_\lambda)$ are the same for all λ.*

Example 7.8. As an example we look at the case where $D^2 = 0$ and $\mathcal{T}$ is singular (i.e. D is not perfect or $\mathcal{R}_{L,D}$ is non-empty). In this case the scroll $\mathcal{T}_0$ can be analyzed with the techniques of Section 6. Proposition 7.2 gives that the canonical sheaf on S'' is trivial. Lemma 6.1 gives us the betti-numbers of all the D_λ. Hence the analogue of Proposition 6.2 goes through completely (we need the triviality of the canonical sheaf to prove the analogue of part (b)) to give a resolution of $\mathcal{O}_{S''}$ as an $\mathcal{O}_{\mathcal{T}_0}$-module. Set $g_0 = g + d = g + c + 2$. Since $\mathcal{T}_0$ has degree $g_0 - c - 1$, dimension $c + 2$, and spans $\mathbf{P}^{g_0}$, we only need to replace g by g_0 in Theorem 6.2.

Unfortunately, finding the betti-numbers β_{ij} for the $\phi_L(D_\lambda)$ when $D^2 > 0$ is not as easy as in the case $D^2 = 0$. In fact, we are not able to compute all of them.

Since the betti-numbers are the same for all $\lambda \in \mathbf{P}^1$ we can choose to work with any smooth irreducible D_λ. We recall from Lemma 4.1 that $\phi_L(D_\lambda)$ is isomorphic to D_λ and also smooth.

Recall that the line bundle $L_{D_\lambda} := L \otimes \mathcal{O}_{D_\lambda}$ on D_λ is said to satisfy property N_p if the betti-numbers satisfy the following:

$$\beta_{0,j} = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0 \end{cases} \quad \text{and} \quad \beta_{i,j} \neq 0 \text{ if and only if } j = i + 1, \text{ for } 0 < i \leq p. \tag{37}$$

This means that B^λ has a resolution of the form

$$\cdots \longrightarrow R(-p-1)^{\beta_{p,p+1}} \longrightarrow \cdots \longrightarrow R(-3)^{\beta_{2,3}} \longrightarrow R(-2)^{\beta_{1,2}} \longrightarrow R \longrightarrow B^\lambda \longrightarrow 0.$$

In our case, we have

Proposition 7.9. *Assume $c > 0$. Then L_{D_λ} satisfies property N_{c-1} but not N_c.*

Proof. The first statement is a direct consequence of [Gr, Thm. (4.a.1)], since $\deg L_{D_\lambda} = 2g(D_\lambda) + c$ and $h^1(L_{D_\lambda}) = 0$.

For the second statement, we have by [Kn1] and the conditions $(*)$ that L_{D_λ} fails to be $(c+1)$-very ample, and the result follows from [G-L1, Thm. 2]. □

From [Gr, Thm. (4.a.1)] it also follows that

$$\mathcal{K}_{i,j}(D_\lambda, L) = 0 \text{ for } j \geq 3,$$

whence

$$\beta_{ij} = 0 \quad \text{for} \quad j \geq i + 3.$$

Of course we also have

$$\beta_{ij} = 0 \quad \text{for} \quad i \geq h^0(L_{D_\lambda}) - 1 = c + 1 + \frac{1}{2}D^2.$$

Also, by the Theorem in [G-L3], we have that for $D^2 > 0$:

$$\beta_{c,c+1} \neq 0.$$

Indeed, $L_{D_\lambda} \simeq F_{D_\lambda} + \omega_{D_\lambda}$, and $D^2 \leq 2c$ for $c > 0$, and we calculate

$$h^0(F_{D_\lambda}) = h^0(F) - \chi(F - D) = c + 2 - \frac{1}{2}D^2 \geq 2,$$

$$h^0(\omega_{D_\lambda}) = \frac{1}{2}D^2 + 1 \geq 1$$

and

$$h^0(F_{D_\lambda}) + h^0(\omega_{D_\lambda}) - 3 = c.$$

Summing up, we have

Proposition 7.10. *For $c > 0$, the betti-numbers of the $\phi_L(D_\lambda)$ satisfy:*

(a) $\beta_{0,j} = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0 \end{cases}$.

(b) *For $0 < i \leq c-1$, $\beta_{i,j} \neq 0$ if and only if $j = i+1$,*

(c) *$\beta_{ij} = 0$ for $i \geq c + 1 + \frac{1}{2}D^2$.*

(d) *$\beta_{ij} = 0$ for $j \geq i + 3$.*

(e) *$\beta_{c,c+1} \neq 0$ for $D^2 > 0$.*

(f) *$\beta_{c,c+2} \neq 0$.*

So for $D^2 > 0$, the $\phi_L(D_\lambda)$ all have a resolution of the form:

$$\begin{aligned} 0 \longrightarrow R(-\frac{1}{2}D^2 - c - 1)^{\beta_{\frac{1}{2}D^2+c,\frac{1}{2}D^2+c+1}} \oplus R(-\frac{1}{2}D^2 - c - 2)^{\beta_{\frac{1}{2}D^2+c,\frac{1}{2}D^2+c+2}} &\longrightarrow \cdots \\ \cdots \longrightarrow R(-c-2)^{\beta_{c+1,c+2}} \oplus R(-c-3)^{\beta_{c+1,c+3}} & \\ \longrightarrow R(-c-1)^{\beta_{c,c+1}} \oplus R(-c-2)^{\beta_{c,c+2}} \longrightarrow R(-c)^{\beta_{c-1,c}} &\longrightarrow \cdots \\ \cdots \longrightarrow R(-3)^{\beta_{2,3}} \longrightarrow R(-2)^{\beta_{1,2}} \longrightarrow R \longrightarrow B^\lambda &\longrightarrow 0, \end{aligned}$$

where all the betti-numbers in the last two lines are nonzero, whereas the betti-numbers in the upper two lines might be zero.

Example 7.11. As an example we study the case (E0) with $c = 1$, $D^2 = 2$ and $g = 6$. Then the smooth curves $\phi_L(D_\lambda)$ are (hyperelliptic) of genus 2 in $\mathbf{P}^3$. The minimal resolutions of such curves are well known (see [Si]) to be

$$0 \longrightarrow R(-4)^2 \longrightarrow R(-2) \oplus R(-3)^2 \longrightarrow R \longrightarrow B_\lambda \longrightarrow 0.$$

Remark 7.12. We are not able to find the $\beta_{i,j}$ directly in all cases, but if we twist the resolution above with n and use the additivity of the Euler characteristic, we obtain the following polynomial identity in the variable n:

$$\begin{aligned} (c + 2 + D^2)n - \frac{1}{2}D^2 \quad &= \\ \binom{n + c + 1 + \frac{1}{2}D^2}{c + 1 + \frac{1}{2}D^2} \quad &+ \quad \sum_{j=2}^{\frac{1}{2}D^2+c+2} (-1)^j \binom{n + c + \frac{1}{2}D^2 + 1 - j}{c + 1 + \frac{1}{2}D^2} (\beta_{j-2,j} - \beta_{j-1,j}). \end{aligned}$$

It is easy to see that this identity determines the $\beta_{j-2,j} - \beta_{j-1,j}$ uniquely. Since $\beta_{j-2,j} = 0$, for $j \leq c+1$, the $\beta_{j-1,j}$, for $j = 2, \ldots, c+1$ are determined uniquely.

Setting $n = 2$, we obtain

$$\beta_{1,2} = \frac{1}{8}(4c^2 + 4c - 2D^2 + 4cD^2 - 8 + (D^2)^2).$$

Setting $n = 0$, we obtain

$$\beta_{\frac{1}{2}D^2+c,\frac{1}{2}D^2+c+2} = \frac{1}{2}D^2 + 1.$$

Example 7.13. As another example we study the case when $D^2 = 2$ and $c \geq 2$. Then any smooth D' in $\{\phi_L(D_\lambda)\}$ has genus 2 and degree $c+4$ in $\mathbf{P}^{c+2}$. In this $\mathbf{P}^{c+2}$, we have $h^1(\mathcal{J}_{D'}(2)) = 0$ by [G-L-P, Theorem p.492]. This gives $\beta_{1,2} = h^0(\mathcal{J}_{D'}(2)) = \frac{c^2}{2} + \frac{3}{2}c - 1$, using the exact sequence

$$0 \longrightarrow H^0(\mathcal{J}_{D'}(2)) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^{c+2}}(2)) \longrightarrow H^0(\mathcal{O}_{D'}(2)) \longrightarrow 0.$$

We see that this is the same as we get by using the last part of Remark 7.12. From Remark 7.12 we also get $\beta_{c+1,c+3} = 2$.

To be more explicit, we will assume $c = 2$ in the rest of the example. Then the resolution above reduces to:

$$0 \longrightarrow R(-4)^{\beta_{3,4}} \oplus R(-5)^2 \longrightarrow R(-3)^{\beta_{2,3}} \oplus R(-4)^{\beta_{2,4}} \longrightarrow R(-2)^4 \to R \longrightarrow B^\lambda \longrightarrow 0.$$

The identity in Remark 7.12 reduces to:

$$\binom{n+4}{4} - 6n + 1 = \sum_{j=2}^{5} (-1)^j \binom{n+4-j}{4} (\beta_{j-1,j} - \beta_{j-2,j}),$$

where $\beta_{1,3} = 0$. We have already used $\beta_{3,5} = 2$ and $\beta_{1,2} = 4$ from inserting $n = 0$ and $n = 2$, respectively. Inserting $n = 1$ gives nothing, while $n = 3$ gives $\beta_{2,3} = 2$. Inserting $n = 4$ we get $\beta_{2,4} - \beta_{3,4} = 3$. Inserting other values of n or looking at the coefficients in the polynomial identity gives nothing more than this.

Hence the resolution looks like:

$$0 \longrightarrow R(-4)^a \oplus R(-5)^2 \to R(-3)^2 \oplus R(-4)^{a+3} \longrightarrow R(-2)^4 \longrightarrow R \longrightarrow B^\lambda \longrightarrow 0,$$

for a non-negative integer a.

We now return to the general resolution, following Proposition 7.10. From Corollary 7.7, Example 7.8 for the special case $D^2 = 0$ and [Sc, Thm. (3.2)], we obtain the following:

Proposition 7.14. *If $D^2 = 0$ and $c = 1$, then the $\mathcal{O}_{\mathcal{T}_0}$-resolution F_* of $\mathcal{O}_{S''}$ is*

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + (g-1)\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0.$$

If $D^2 = 0$ and $c \geq 2$, the resolution is of the following type:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-(c+2)\mathcal{H}_0 + (g-1)\mathcal{F}) \longrightarrow \oplus_{k=1}^{\beta_{c-1}} \mathcal{O}_{\mathcal{T}_0}(-c\mathcal{H}_0 + b_{c-1}^k \mathcal{F}) \longrightarrow \cdots \\ \longrightarrow \oplus_{k=1}^{\beta_1} \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + b_1^k \mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0, \end{aligned}$$

where $\beta_i = i\binom{c+1}{i+1} - \binom{c}{i-1}$.

If $D^2 \geq 2$, then $\mathcal{O}_{S''}$ has a $\mathcal{O}_{\mathcal{T}_0}$-resolution F_* of the following type:

$$\begin{aligned} 0 &\longrightarrow F_{\frac{1}{2}D^2+c} \cdots \longrightarrow F_{c+1} \longrightarrow F_c \\ &\longrightarrow \oplus_{k=1}^{\beta_{c-1}} \mathcal{O}_{\mathcal{T}_0}(-c\mathcal{H}_0 + b_{c-1}^k \mathcal{F}) \longrightarrow \cdots \longrightarrow \oplus_{k=1}^{\beta_2} \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + b_2^k \mathcal{F}) \\ &\longrightarrow \oplus_{k=1}^{\beta_1} \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + b_1^k \mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0. \end{aligned}$$

Here $\beta_i = \beta_{i,i+1}$, for $i = 1, \ldots, c$, and F_c is an extension of the non-zero term

$$\oplus_{k=1}^{\beta_{c,c+2}} \mathcal{O}_{\mathcal{T}_0}(-(c+2)\mathcal{H}_0 + b_{c,c+2}^k \mathcal{F})$$

by the non-zero term

$$\oplus_{k=1}^{\beta_{c,c+1}} \mathcal{O}_{\mathcal{T}_0}(-(c+1)\mathcal{H}_0 + b_{c,c+1}^k \mathcal{F}).$$

Moreover F_i is an extension of the term

$$\oplus_{k=1}^{\beta_{i,i+2}} \mathcal{O}_{\mathcal{T}_0}(-(i+2)\mathcal{H}_0 + b_{i,i+2}^k \mathcal{F})$$

by the term

$$\oplus_{k=1}^{\beta_{i,i+1}} \mathcal{O}_{\mathcal{T}_0}(-(i+1)\mathcal{H}_0 + b_{i,i+1}^k \mathcal{F})$$

for $i = c+1, \ldots, \frac{1}{2}D^2 + c$.

Proof. Since the betti-numbers are the same for all λ, this is a direct application of [Sc, Thm. (3.2)]. □

Proposition 7.15. *Let $\mathcal{J}_{S''/\mathcal{T}_0}$ denote the ideal sheaf of S'' in $\mathcal{T}_0$ and $\mathcal{J}_{S'/\mathcal{T}}$ the ideal sheaf of S' in $\mathcal{T}$.*

We have $\mathcal{J}_{S'/\mathcal{T}} = i_*\mathcal{J}_{S''/\mathcal{T}_0}$.

Proof. This follows since $i_*\mathcal{O}_{\mathcal{T}_0} = \mathcal{O}_{\mathcal{T}}$ and $i_*\mathcal{O}_{S''} = \mathcal{O}_S$. The latter fact is a consequence of g being a birational map of normal surfaces. □

We recall that the Chow ring of $\mathcal{T}_0$ is given by (34). Define $\mathcal{H}_{\mathcal{T}}$ and $\mathcal{F}_{\mathcal{T}}$ to be the push-down of cycles by i of $\mathcal{H}$ and $\mathcal{F}$ respectively.

We have the following description:

Proposition 7.16. (a) *The class of S'' in the Chow ring of $\mathcal{T}_0$ is*

$$(D^2+c+2)\mathcal{H}_0^{d-2} + (c - cg - D^2(g-1))\mathcal{H}_0^{d-3}\mathcal{F}.$$

(b) *The class of S' in the Chow group of $\mathcal{T}$ is*

$$(D^2+c+2)(\mathcal{H}_{\mathcal{T}})^{d-2} + (D^2(d-1-g) - 4 - cg - c + cd + 2d)(\mathcal{H}_{\mathcal{T}})^{d-3}\mathcal{F}_{\mathcal{T}}.$$

Proof. The class of S'' is of the type $m\mathcal{H}_0^{d-2} + n\mathcal{H}_0^{d-3}\mathcal{F}$, for two integers m and n. To determine m and n one has the equations $S''\mathcal{H}_0^2 = \deg S'' = 2g + 2c + 2 + 2D^2$ and $S''\mathcal{H}_0\mathcal{F} = \deg(\phi_L(D)) = c + 2 + D^2$.

Statement (b) is an immediate consequence of i being birational by using the cap product map $A^*(\mathbf{P}^g) \otimes A_*(\mathcal{T}) \to A_*(\mathcal{T})$. □

We recall the definition $\beta_i = \beta_{i,i+1}$, for $i = 1, \dots, c-1$, and $d = c + 2 + \frac{1}{2}D^2 = \dim \mathcal{T}$.

Proposition 7.17. *The $b_{i,j}^k$ and b_i^k in Proposition 7.14 satisfy the following polynomial equation in n (set $b_{i,i+2}^k = \beta_{i,i+2}^k = 0$ for all i and k if $D^2 = 0$, and set $b_{i,i+1}^k = b_i^k$, for $i = 1, \dots, c-1$ for all values of D^2):*

$$\binom{n+d-1}{d-1}\Big(\frac{n(g+1)}{d}+1\Big) - n^2(g+1+c+D^2) + \frac{1}{2}nD^2 - 2 \quad =$$

$$\sum_{i=1}^{c+\frac{1}{2}D^2} (-1)^{i+1}\binom{n+d-i-2}{d-1}\Big(\frac{((n-i-1)(g+1)+d)\beta_{i,i+1}}{d} + \sum_{k=1}^{\beta_{i,i+1}} b_{i,i+1}^k\Big) \quad +$$

$$\sum_{i=c}^{c+\frac{1}{2}D^2} (-1)^{i+1}\binom{n+d-i-3}{d-1}\Big(\frac{((n-i-2)(g+1)+d)\beta_{i,i+2}}{d} + \sum_{k=1}^{\beta_{i,i+2}} b_{i,i+2}^k\Big).$$

Proof. Denote the term i places to the left of $\mathcal{O}_{\mathcal{T}_0}$ in the resolution F_* by F_i. The result follows, similarly as in the proof of [Sc, Prop. 4.4(c)] from the identity

$$\chi(\mathcal{O}_{\mathcal{T}_0}(n\mathcal{H}_0)) - \chi(\mathcal{O}_{S''}(n\mathcal{H}_0)) = \sum_i (-1)^{i+1}\chi(F_i(n\mathcal{H}_0)).$$

To calculate $\chi(\mathcal{O}_{S''}(n\mathcal{H}_0))$ one uses Riemann-Roch on S'' and $\deg S'' = 2g + 2c + 2 + 2D^2$.

Moreover it is clear that for all large n, we have $\chi(F_i(n\mathcal{H})) = h^0(F_i(n\mathcal{H})$, for all i, and $\chi(\mathcal{O}_{\mathcal{T}_0}(n\mathcal{H}_0)) = h^0(\mathcal{O}_{\mathcal{T}_0}(n\mathcal{H}_0))$ since $\mathcal{H}_0$ is (very) ample on $\mathcal{T}_0$. Then one uses (30) again. □

Remark 7.18. From the last result it is clear that the sums $\sum_{k=1}^{\beta_{i,j}} b_{i,j}^k$ are uniquely determined, but this does not necessarily apply to the $b_{i,j}^k$ individually. If $D^2 > 0$, it is not even a priori clear that the $b_{i,j}^k$ are independent of the choice of pencil inside $|D|$ giving rise to $\mathcal{T}(c, D, \{D_\lambda\})$.

Corollary 7.19. (a) *We have*

$$\sum_{k=1}^{\beta_{1,2}} b^k_{1,2} = (\frac{1}{2}D^2 + c - 1)g + (1 - c - D^2).$$

(b) *If $D^2 > 0$, then*

$$\sum_{k=1}^{\beta_{c+\frac{1}{2}D^2,c+\frac{1}{2}D^2+2}} b^k_{c+\frac{1}{2}D^2,c+\frac{1}{2}D^2+2} = g(\frac{1}{2}D^2 + 1) + 1.$$

Proof. We insert $n = 2$ in Proposition 7.17. That gives part (a) directly. Then we insert $n = 0$ in Proposition 7.17. That gives

$$-1 = g\beta_{c+\frac{1}{2}D^2,c+\frac{1}{2}D^2+2} - \sum_{k=1}^{\beta_{c+\frac{1}{2}D^2,c+\frac{1}{2}D^2+2}} b^k_{c+\frac{1}{2}D^2,c+\frac{1}{2}D^2+2}.$$

This immediately gives the statement of part (b), since we have shown in Remark 7.12 that

$$\beta_{c+\frac{1}{2}D^2,c+\frac{1}{2}D^2+2} = \frac{1}{2}D^2 + 1.$$

□

Example 7.20. We return to the situation studied in Example 7.13, with $D^2 = c = 2$, and we will apply Proposition 7.17. From that example and Corollary 7.19 we see that $\beta_{3,5} = 2$ and

$$\sum_{k=1}^{\beta_{3,5}} b^k_{3,5} = 2g - 1.$$

Setting $n = 1$ in Proposition 7.17, we get nothing, but setting $n = 2$ we obtain

$$\sum_{k=1}^{\beta_1} b^k_1 = 2g + 1 - \beta_{1,2} = 2g - 3.$$

Setting $n = 3$ we obtain

$$\sum_{k=1}^{\beta_{2,3}} b^k_{2,3} = 2g - 3 - \beta_{2,3} = 2g - 5.$$

Continuing this way, we find the difference of the $b^k_{3,4}$ and the $b^k_{2,4}$ in terms of $\beta_1, \beta_{2,3}, \beta_{2,4}, \beta_{3,4}$, by setting $n = 4$. This gives

$$\sum_{k=1}^{\beta_{3,4}} b^k_{3,4} - \sum_{k=1}^{\beta_{2,4}} b^k_{2,4} = (\beta_{2,3} - 4)g + (\beta_{2,3} + \beta_{3,4} - \beta_{2,4} - 6) = -2g - 7.$$

This is also the general picture. One uses the values $n = 2, 3, \ldots, c$ to obtain the sums of the corresponding $b^k_{n-1,n}$ in terms of the corresponding and previous $\beta_{i,i+1}$, and one uses the values $n = c + 1, \ldots, d - 1 = c + 1 + \frac{1}{2}D^2$ to obtain the differences of the sums of the $b_{n-2,n}$ and $b_{n-1,n}$ (and $n = 0$ as in Corollary 7.19) in terms of their corresponding and previous $\beta_{i,i+1}$ and $\beta_{i,i+2}$.

Setting $n = 5$ in our concrete example above, we get

$$\sum_{k=1}^{\beta_{3,5}} b_{3,5}^k = (\beta_{2,3} + \beta_{2,4} - \beta_{3,4} - 3)g + (\beta_{2,3} + \beta_{2,4} - \beta_{3,4} - 4) - \beta_{3,5} = 2g - 1.$$

But this gives nothing new.

Definition 7.21. *We define, for integers a and b,*

$$\mathcal{O}_{\mathcal{T}}(a\mathcal{H} + b\mathcal{F}) := i_*\mathcal{O}_{\mathcal{T}_0}(a\mathcal{H} + b\mathcal{F}).$$

In particular, by the projection formula,

$$i_*\mathcal{O}_{\mathcal{T}_0}(a\mathcal{H}_0 + b\mathcal{F}) = i_*\mathcal{O}_{\mathcal{T}_0}(a\mathcal{H} + (a+b)\mathcal{F}) = \mathcal{O}_{\mathcal{T}}(a) \otimes i_*((a+b)\mathcal{F}).$$

We now return to the general situation. As a consequence of Proposition 7.14 we have the following two results:

Proposition 7.22. *Assume $b_i^k \geq i$, for $i = 1, \dots, c-1$ and all k and $b_{i,j}^k \geq j-1$ for $j = i+1, i+2$, $i = c, \dots, \frac{1}{2}D^2 + c$, and all k.*

If $D^2 = 0$ and $c \geq 2$, then

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-(c+2)\mathcal{H} + (g-1)\mathcal{F}) \longrightarrow \oplus_{k=1}^{\beta_{c-1}} \mathcal{O}_{\mathcal{T}}(-c\mathcal{H} + (b_{c-1}^k - c)\mathcal{F}) \longrightarrow \cdots \\ \longrightarrow \oplus_{k=1}^{\beta_1} \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + (b_1^k - 2)\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0, \end{aligned}$$

is an $\mathcal{O}_{\mathcal{T}}$-resolution of $\mathcal{O}_{S'}$.

If $D^2 \geq 2$, then $\mathcal{O}_{S'}$ has a $\mathcal{O}_{\mathcal{T}}$-resolution F'_* of the following type:

$$\begin{array}{rcccl} 0 \longrightarrow F'_{\frac{1}{2}D^2+c} \longrightarrow & \cdots & \longrightarrow F'_{c+1} \longrightarrow F'_c \\ \longrightarrow \oplus_{k=1}^{\beta_{c-1}} \mathcal{O}_{\mathcal{T}}(-c\mathcal{H} + (b_{c-1}^k - c)\mathcal{F}) \longrightarrow & \cdots & \longrightarrow \oplus_{k=1}^{\beta_2} \mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + (b_2^k - 3)\mathcal{F}) \\ \longrightarrow \oplus_{k=1}^{\beta_1} \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + (b_1^k - 2)\mathcal{F}) \longrightarrow & \mathcal{O}_{\mathcal{T}} & \longrightarrow \mathcal{O}_{S'} \longrightarrow 0. \end{array}$$

Here $F'_i = i_*(F_i)$, for all i.

Proof. See [Sc, p.117]. The essential fact is that the map $i : \mathcal{T}_0 \simeq \mathbf{P}(\mathcal{E}) \to \mathcal{T}$ is a rational resolution of singularities, and that we therefore have $R^1 i_*\mathcal{O}_{\mathcal{T}_0} = 0$. Moreover $i_*\mathcal{O}_{S''} = \mathcal{O}_{S'}$, and $i_*\mathcal{O}_{\mathcal{T}_0} = \mathcal{O}_{\mathcal{T}}$. The condition on the b_i and the $b_{i,j}$ gives that each term (except $\mathcal{O}_{S''}$) in the resolution of $\mathcal{O}_{S''}$ in Proposition 7.14 is an extension of terms of the form $\mathcal{O}_{\mathcal{T}_0}(a\mathcal{H} + b\mathcal{F})$, with $b \geq -1$. As in [Sc, (3.5)] we then conclude that the resolution therefore remains exact after pushing down. □

Remark 7.23. By Proposition 7.15 we already know that the ideal of S' in $\mathcal{T}$ is the push-down by i of the ideal of S'' in $\mathcal{T}_0$.

If $b_2^k \geq 2$ for all k when $c \geq 3$ (resp. $b_{2,3}^k \geq 2$ and $b_{2,4}^k \geq 3$ for all k when $c \leq 2$), then it automatically follows that $R^1 i_* F_2 = R^1 i_* F_1 = 0$, so that we get an exact pushed-down right end

$$i_* F_1 \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

This means that the ideal of S' in $\mathcal{T}$ is generated by the push-down by i of the generators of the ideal of S'' in $\mathcal{T}_0$.

The next two results give the first examples of applications of the proposition.

Corollary 7.24. *Assume $D^2 = 0$.*

(a) If $c = 1$ then $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + (g-4)\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

(b) If $c = 2$, then $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H} + (g-1)\mathcal{F}) &\longrightarrow \\ \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + a_1\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + a_2\mathcal{F}) &\longrightarrow \quad \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0, \end{aligned}$$

for two integers a_1 and a_2 such that $a_1 \geq a_2 \geq 0$ and $a_1 + a_2 = g - 5$.

Proof. Set $g_0 = g + c + 2$.

If $c = 1$, then by Example 7.8 a resolution of $\mathcal{O}_{S''}$ as an $\mathcal{O}_{\mathcal{T}_0}$-module is:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + (g_0 - 4)\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0.$$

Here $g_0 - 4 \geq 5$, whence (a) follows.

If $c = 2$, we have $g_0 \geq 10$ and Proposition 6.2(a), gives that a resolution of $\mathcal{O}_{S''}$ as an $\mathcal{O}_{\mathcal{T}_0}$-module is:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0 + (g_0-5)\mathcal{F}) &\longrightarrow \\ \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + b_1\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + b_2\mathcal{F}) &\longrightarrow \quad \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0, \end{aligned}$$

for two integers b_1 and b_2 such that $b_1 \geq b_2 \geq 0$ and $b_1 + b_2 = g_0 - 5$. From [Br, Thm. 5.1] we have that S'' is singular along a curve if $b_1 > \frac{2(e_1+e_2+e_3)}{3}$, where (e_1, e_2, e_3) denotes the type of $\mathcal{T}_0$. This is equivalent to $b_2 < \frac{g_0 - 9 + 2e_4}{3}$. Hence $b_2 \geq 1$ and (b) follows. $\square$

Corollary 7.25. *Let $c = 1$, $D^2 = 2$ and $g = 6$ as in Example 7.11. Then $\mathcal{O}_{S''}$ has the following $\mathcal{O}_{\mathcal{T}_0}$-resolution:*

$$\begin{aligned} 0 \quad &\longrightarrow \quad \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0 + 6\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0 + 5\mathcal{F}) \\ &\longrightarrow \quad \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + 4\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + 4\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + 3\mathcal{F}) \\ &\longrightarrow \quad \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0. \end{aligned}$$

In particular, $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$\begin{aligned} 0 \quad &\longrightarrow \quad \mathcal{O}_{\mathcal{T}}(-4\mathcal{H} + 2\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-4\mathcal{H} + \mathcal{F}) \\ &\longrightarrow \quad \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + 2\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + \mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}) \\ &\longrightarrow \quad \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0. \end{aligned}$$

Proof. Corollary 7.19 gives $b^1_{1,2} = 4$ and $b^1_{2,4} + b^2_{2,4} = 11$, while inserting $n = 3$ in Proposition 7.17 gives $b^1_{1,3} + b^2_{1,3} = 7$. Other values of n gives no more information. Hence we have a resolution

(38)
$$0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0 + b^1_{2,4}\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0 + (11 - b^1_{2,4})\mathcal{F}) \longrightarrow F_1 \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0,$$

where F_1 is an extension

(39)
$$0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + 4\mathcal{F}) \longrightarrow F_1 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + b^1_{1,3}\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + (7 - b^1_{1,3})\mathcal{F}) \longrightarrow 0.$$

Without loss of generality we assume $b := b^1_{1,3} \geq 4$, and $a := b^1_{2,4} \geq 6$. The type of $\mathcal{T}_0$ is $(3, 2, 1, 1)$.

Look at the composite morphism given by (38) and (39)

$$\alpha : \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0 + a\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0 + (11 - a)\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + b\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + (7 - b)\mathcal{F}).$$

Now α can be expressed by a matrix

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix},$$

with

$$\begin{array}{rcl} \alpha_1 & \in & H^0(\mathcal{H}_0 + (b-a)\mathcal{F}) \\ \alpha_2 & \in & H^0(\mathcal{H}_0 + (a+b-11)\mathcal{F}) \\ \alpha_3 & \in & H^0(\mathcal{H}_0 + (7-a-b)\mathcal{F}) \\ \alpha_4 & \in & H^0(\mathcal{H}_0 + (a-b-4)\mathcal{F}), \end{array}$$

whose determinant gives a section $g \in H^0(2\mathcal{H}_0 - 4\mathcal{F})$ whose zero scheme contains S''.

If $(a,b) \neq (6,4)$, we have

$$H^0(\mathcal{H}_0 + (7-a-b)\mathcal{F}) = H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(10-a-b) \oplus \mathcal{O}_{\mathbf{P}^1}(9-a-b) \oplus \mathcal{O}_{\mathbf{P}^1}(8-a-b)^2) = 0,$$

whence $\alpha_3 = 0$ and g is a product of two sections of $\mathcal{H}_0 + (b-a)\mathcal{F}$ and $\mathcal{H}_0 + (a-b-4)\mathcal{F}$ respectively. But then S'' would have degenerate fibers S''_λ, contradicting Proposition 7.10(b).

So $(a,b) = (6,4)$ and we compute

$$\begin{array}{rcl} \mathrm{Ext}^1(\mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + 4\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + 3\mathcal{F}), \mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + 4\mathcal{F}) & = & \\ H^1(\mathcal{H}_0) \oplus H^1(\mathcal{H}_0 + \mathcal{F}) & = & \\ H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1}(1)^2) \oplus H^1(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(4) \oplus \mathcal{O}_{\mathbf{P}^1}(3) \oplus \mathcal{O}_{\mathbf{P}^1}(2)^2) & = & 0, \end{array}$$

whence the sequence (39) splits and the first assertion follows.

The second is then an immediate consequence of Proposition 7.22. □

Note that by this result, S'' is cut out in $\mathcal{T}_0$ by three sections q, c_1 and c_2 of $\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}_0 - 4\mathcal{F})$, $\mathcal{O}_{\mathcal{T}_0}(3\mathcal{H}_0 - 4\mathcal{F})$ and $\mathcal{O}_{\mathcal{T}_0}(3\mathcal{H}_0 - 3\mathcal{F})$ respectively.

Now look at the three dimensional subvariety V of $\mathcal{T}_0$ defined by $q \in \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}_0 - 4\mathcal{F})$. Arguing as in the proof of Proposition 7.16 we find that the class of $i(V)$ in the Chow group of $\mathcal{T}$ is $2\mathcal{H}_{\mathcal{T}} - 2\mathcal{F}_{\mathcal{T}}$, whence $i(V)$ has degree 4 and dimension 3 in $\mathbf{P}^6$. As in [SD, (7.12)] we have that $i(V)$ is a cone over the Veronese surface (whose vertex is the image of Γ) and that this variety is the (reduced) intersection of all quadrics containing S'.

A very useful result is the following, involving so called "rolling factors" coordinates (see for example [Ha, p.59], [St, p.3] or [Re]):

Lemma 7.26. *The sections of $a\mathcal{H} - b\mathcal{F}$ on a smooth rational normal scroll of type $(e_1, \dots, e_d)$ can be identified with weighted-homogeneous polynomials of the form*

$$P = \sum P_{i_1,\dots,i_d}(t,u) Z_1^{i_1} \dots Z_d^{i_d},$$

where $i_1 + \cdots + i_d = a$, and $P_{i_1,\dots,i_d}(t,u)$ is homogeneous of degree $-b + (i_1 e_1 + \cdots + i_d e_d)$.

If we multiply P by a homogeneous polynomial of degree b in t,u, then we get a defining equation of the zero scheme of the section, in term of homogeneous coordinates of the projective space, within which the scroll is embedded. Here $X_{k,j} = t^j u^{d_k - j} Z_k$, for $k = 1, \dots, d$, and $j = 0, \dots, d_k$, are coordinates for this space. The equation is uniquely determined modulo the homogeneous ideal of the scroll.

As a first application, we prove the analogous of Corollary 7.24 for $c = 3$.

Corollary 7.27. *Let $D^2 = 0$ and $c = 3$. Then a resolution of $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathcal{T}}$-module is:*

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-5\mathcal{H}_0 + (g-6)\mathcal{F}) \longrightarrow \oplus_{i=1}^{5}\mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + a_i\mathcal{F}) &\longrightarrow \\ \oplus_{i=1}^{5}\mathcal{O}_{\mathcal{T}}(-2\mathcal{H}_0 + b_i\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} &\longrightarrow 0, \end{aligned}$$

where all the a_i and $b_i \geq -1$ and satisfy $\sum_{i=1}^{5} b_i = 2g - 12$ and $a_i = g - 6 - b_i$.

Proof. First recall that $g \geq 9$.

As a special case of Proposition 7.14 we obtain that the resolution of $\mathcal{O}_{S''}$ as an $\mathcal{O}_{\mathcal{T}_0}$-module is:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-5\mathcal{H}_0 + (g-1)\mathcal{F}) \longrightarrow \oplus_{k=1}^{5}\mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + b_2^k\mathcal{F}) \longrightarrow \\ \oplus_{k=1}^{5}\mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + b_1^k\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0. \end{aligned}$$

From Corollary 7.19 we get $\sum_{i=1}^{5} b_1^k = 2g - 2$. The self-duality of the resolution in this particular case gives $b_2^k = g - 1 - b_1^k$, if we for example order the b_1^k in a non-increasing way, and the b_2^k in a non-decreasing way.

We will show that for all $g \geq 9$ all the $b_1^k \geq 1$ and all the $b_2^k \geq 2$, so that we can push down the resolution to one of $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathcal{T}}$-module.

Look at the map

$$\Phi : \oplus_{k=1}^{5}\mathcal{O}_{\mathcal{T}_0}(-3\mathcal{H}_0 + b_2^k\mathcal{F}) \longrightarrow \oplus_{k=1}^{5}\mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0 + b_1^k\mathcal{F}).$$

Just like in the analysis of pentagonal curves in [Sc], it follows from [B-E] that the map Φ is skew-symmetrical and that its Pfaffians generate the ideal of S'' in $\mathcal{T}_0$. See also [Wa].

Let the type of $\mathcal{T}_0$ be $(e_1, \ldots, e_5)$, where $e_5 = 1$ and $\sum_{i=1}^{4} e_i = g$.

A key observation is the following: $b_1^1 \leq e_1 + e_3$ and $b_1^1 \leq 2e_2$. The first inequality holds, since otherwise we would have a quadric relation of the form $f(t, u, Z_1, Z_2) = 0$ in each fiber. Hence the general fiber would be reducible, a contradiction. The second inequality follows since its negation implies that Z_1 is factor in one quadric relation satisfied by the points of S'', a contradiction. This gives $b_1^1 \leq \frac{2(e_1+e_2+e_3)}{3} = \frac{2(g-e_4)}{3}$. Hence

$$b_2^1 = g - 1 - b_1^1 \geq g - 1 - \frac{2g}{3} + \frac{2e_4}{3} = \frac{g - 3 + 2e_4}{3} \geq 2,$$

since $g \geq 9$. Hence $b_2^k \geq b_2^1 \geq 3$ for all k.

Another key observation is the following: $b_1^2 \leq e_1 + e_4$ and $b_1^2 \leq e_2 + e_3$. The first inequality holds, since otherwise the two-step projection of the general fiber D'' of S'' into the Z_1, Z_2, Z_3-plane from $P = (0,0,0,0,1)$ and $Q = (0,0,0,1,0)$ would be contained in 2 quadrics. This in only possible if the projected image is a line, and in that case the general fiber would be degenerate (contained in the $\mathbf{P}^3$ spanned by this line and P and Q). This is impossible. The second inequality holds, since otherwise there would be two independent relations of the form

$$Z_1 f(t, u, Z_1, \ldots, Z_5) + aZ_2^2 = 0$$

for each fiber. In that case we could eliminate the Z_2^2-term and obtain one relation with Z_1 as a factor, a contradiction.

These two inequalities for b_1^2 imply

$$b_1^2 \leq \frac{e_1 + e_2 + e_3 + e_4}{2} = \frac{g}{2}.$$

Now we assume for contradiction that $b_1^5 \leq 0$. Then we get

$$b_2^5 - b_1^k \geq b_2^5 - b_1^2 = (g - 1 - b_1^5) - b_1^2 \geq g - 1 - \frac{g}{2} = \frac{g}{2} - 1,$$

for $k = 2, 3, 4$. In the matrix description of the map Φ there is one submaximal minor with one column consisting of zero and sections of $\mathcal{H}_0 - (b_2^5 - b_1^k)\mathcal{F}$, for $k = 2, 3, 4$. If all entries of this column have Z_1 as a factor, that would lead to a contradiction, since the minor is the square of one of the generators of the (Pfaffian) ideal of S'' on $\mathcal{T}_0$. To avoid that Z_1 is a factor in each such entry, we must have $e_2 \geq \frac{g}{2} - 1$. This gives $e_1 + e_2 \geq g - 2$, and $e_3 + e_4 \leq g - (e_1 + e_2) \leq g - (g - 2) = 2$. Hence $e_3 = e_4 = e_5 = 1$. But this implies that $\mathcal{R}_{L,D}$ has at least 3 members, whence $D.\Delta \geq 3$. But this is impossible by Proposition 4.6.

Hence the assumption $b_1^5 \leq 0$ leads to a contradiction, and $b_1^k \geq 1$, for all k. Hence the entire resolution can be pushed down to one of $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathcal{T}}$-module and the result follows. □

Remark 7.28. We know by Proposition 7.14 that a finite set of sections of line bundles of type $a\mathcal{H}_0 - b\mathcal{F}$ generate the ideal of the surface S'' on the smooth rational normal scroll $\mathcal{T}_0$ of type $(e_1, \dots, e_{d-r-1}, e_{d-r}, \dots, e_d)$, where $e_{d-r} = \cdots = e_d = 1$, $e_{d-r-1} \geq 2$, and $V = \operatorname{Sing}\mathcal{T} \simeq \mathbf{P}^r$ for some $r \geq 0$. Let $W = i^{-1}(V)$. This is a subscroll of $\mathcal{T}_0$ of type $(1, \dots, 1)$, that is $\mathbf{P}^r \times \mathbf{P}^1$. The ideal generators can be classified into 3 types:

(a) Those that are sections of $a\mathcal{H}_0 - b\mathcal{F} = a\mathcal{H} - (b - a)\mathcal{F}$, with $b > a$.
(b) Those that are sections of $a\mathcal{H}_0 - b\mathcal{F} = a\mathcal{H} - (b - a)\mathcal{F}$, with $b = a$.
(c) Those that are sections of $a\mathcal{H}_0 - b\mathcal{F} = a\mathcal{H} - (b - a)\mathcal{F}$, with $b < a$.

For those of type (a) it is clear from Lemma 7.26 that their zero scheme contains W. Likewise one sees that those of type (b) can be written as

$$f(t, u, Z_1, \dots, Z_d) + g(Z_{s+1}, \dots, Z_d),$$

where Z_1 or Z_2 or ... or Z_s is a factor in every monomial of $f(t, u, Z_1, \dots, Z_d)$, while g is homogeneous of degree $a = b$. Those of type (c) can be written as

$$f(t, u, Z_1, \dots, Z_d) + h(t, u, Z_{s+1}, \dots, Z_d),$$

where f is as described for type (b), while h is bihomogeneous, of degree $a - b > 0$ in t, u and degree a in $Z_{s+1}, \dots, Z_d$.

There is one fundamental difference between the sections of types (a) and (b) on one hand, and those of type (c) on the other. Those of types (a) and (b) are "constant" on the fibers of i, their zero scheme contains either the whole fiber, or no point on the fiber, for each $\mathbf{P}^1$, which is a fiber of i. For the sections of type (c) this is only true if h is the zero polynomial, and then its zero scheme contains all of W.

We therefore see that if all $b_{1,j}^k$ of the resolution of $\mathcal{O}_{S''}$ as an $\mathcal{O}_{\mathcal{T}_0}$-module are at least j, then the ideal of S'' is generated by "fiber constant" equations, and if Q is a point on $\mathcal{T}_0$ not on S'', then there is a fiber constant section of the type described, which does not contain Q in its zero scheme. In short, fiber constant equations cutting out S'' in $\mathcal{T}_0$ are also equations of S' in $\mathcal{T}$.

In Section 10 we will classify the possible projective models for $g \leq 10$, and in particular the singular scrolls $\mathcal{T}$ appearing as $\mathcal{T}(c, D, \{D_\lambda\})$ in the various cases, and we will also show that projective models givings scrolls of all these types exist. In Corollaries 7.24, 7.25 and 7.27 we showed that in some particular cases we can push down the entire resolution of $\mathcal{O}_{S''}$ to one of $\mathcal{O}_{S'}$. In the rest of this section, through a series of additional examples, we take a closer look at the rest of the singular scrolls appearing for $g \leq 10$, and using Lemma 7.26 we will find restrictions on the b_1^k. (See Remark 7.18.)

Example 7.29. We return to the situation studied in Example 7.13 and Example 7.20. We recall $c = 2$, $D^2 = 2$ and $g \geq 7$. From Example 7.20 we see that the ideal of S'' in $\mathcal{O}_{\mathcal{T}_0}$ is generated by four sections of the type $2\mathcal{H}_0 - b_k\mathcal{F}$, where $\sum_{k=1}^4 b_k = 2g - 3$.

The type of $\mathcal{T}_0$ is $(e_1+1, e_2+1, e_3+1, 1, 1)$, where all $e_i \geq 0$ and $e_1+e_2+e_3 = g-4$. Let Q be the subscroll of $\mathcal{T}_0$ formed by the two last directrices, so Q is the inverse image by i of the line in $\mathbf{P}^g$ spanned by the images by ϕ_L of the basepoints of D. We see that Q is a quadric in $\mathbf{P}^3$. All the four sections of type $2\mathcal{H}_0 - b_k\mathcal{F}$ must intersect Q in the two lines that form the inverse image by i in $\mathcal{T}_0$ of the images by ϕ_L of the basepoints of D . But this is simply the two last directrices. The intersection with Q for one such section is obtained by using Lemma 7.26 to express each of the sections, and set $Z_1 = Z_2 = Z_3 = 0$. What remains must be a term of the type $P_{2-b_k}(t,u)Z_4Z_5$, where P_{2-b_k} is zero if $b_k \geq 3$, and a polynomial of degree $2-b_k$ otherwise.

We order the b_k as $b_1 \geq b_2 \geq b_3 \geq b_4$. We see that if $b_4 \leq 1$, in particular if $b_4 \leq 0$, then $b_3 \leq 2$, since otherwise the total intersection of K with the four sections will consist of $2-b_4$ lines transversal to the two directrices in addition to the two directrices.

If $g = 7$, it is clear that $\mathcal{T}_0$ has type $(2,2,2,1,1)$ or $(3,2,1,1,1)$. Then Z_1 is a factor in all sections of $2\mathcal{H}_0 - b\mathcal{F}$ for $b \geq 5$ for both scroll types. Hence $b_1 \leq 4$. If $b_1 = 3$, then the only possible combination is $(b_1, \ldots, b_4) = (3,3,3,2)$, since $\sum b_k = 11$. If $b_1 = 4$, then the only a priori possibilities are $(4,4,2,1)$ and $(4,3,2,2)$. But $(4,4,2,1)$ is impossible for type $(2,2,2,1,1)$, since we then have two quadric relations between Z_1, Z_2, Z_3 only. To see this, let D'' be any smooth curve in $|f^*D - E|$, which can be identified with its image under ϕ_H. The variables Z_1, Z_2, Z_3 restricted to D'' correspond to sections of $(H-E)_{D_0}$. Since this line bundle has degree $2g(D'') = 4$, it is base point free and its sections map D'' into $\mathbf{P}^2$ by a one-to-one or two-to-one map. This means that there is at most one quadric relation between Z_1, Z_2, Z_3, whence $b_2 \leq 3$. So for type $(2,2,2,1,1)$ the only possibilities for $(b_1, \ldots, b_4)$ are

$$(3,3,3,2) \quad \text{and} \quad (4,3,2,2).$$

For the type $(3,2,1,1,1)$, for each fiber of $\mathcal{T}_0$, the equations with $b_k \leq 2$, restricted to the subscroll $Z_1 = Z_2 = 0$ with plane fibers, must cut out a subscheme of length 4, such that each subscheme of length 3 spans a $\mathbf{P}^2$ (these are the cases (E1) and (E2)). It takes 2 equations to do this. Hence $b_3 \leq 2$. Moreover $b_2 \leq 3$. Assume $b_2 \geq 4$. Then we would have two independent equations of type

$$Z_1 f(t,u,Z_1,Z_2,Z_3,Z_4,Z_5) + Z_2^2$$

for general (t,u). From these equations we can eliminate the Z_2^2-term, and derive one equation with Z_1 as a factor. This is a contradiction.

Hence the only possibility for $(b_1, \ldots, b_4)$ for the type $(3,2,1,1,1)$ is $(4,3,2,2)$.

If $g = 8$, then $\mathcal{T}_0$ has type $(3,2,2,1,1)$. A similar argument as for $g = 7$, gives that the only possible combinations for $(b_1, \ldots, b_4)$ are

$$(5,4,2,2), \quad (5,3,3,2) \quad \text{and} \quad (4,4,3,2).$$

.

For $g = 9$ the type of $\mathcal{T}_0$ is a priori either $(3,3,2,1,1)$ or $(4,2,2,1,1)$. In Section 10 the type $(4,2,2,1,1)$ is ruled out when D is perfect. For the type $(3,3,2,1,1)$ we see that $b_1 \geq 6$ is impossible, since $2\mathcal{H}_0 - 6\mathcal{F}$ only has sections of the form $f(Z_1, Z_2)$. Moreover $b_4 \leq 2$, since we need to cut out the exceptional fibers. Hence $b_1 = 5$, otherwise the sum of the b_i would be at most 14, and it is 15. Any section of $2\mathcal{H}_0 - 5\mathcal{F}$ can be written in terms of t, u, Z_1, Z_2, Z_3 only. As for one case with $g = 7$ we see that we cannot have two quadric relations between Z_1, Z_2, Z_3 for general fixed (t,u), so $b_2 \leq 4$. We then see that the only possible combination is $(b_1, b_2, b_3, b_4) = (5,4,4,2)$.

For $g = 10$ the type of $\mathcal{T}_0$ is a priori $(3,3,3,1,1)$, $(4,3,2,1,1)$ or $(5,2,2,1,1)$. In Section 10 it is shown that only the type $(4,3,2,1,1)$ occurs when D is perfect. In this case a more detailed analysis gives that the only possible combinations for $(b_1, \dots, b_4)$ are

$$(6,5,4,2) \quad \text{and} \quad (5,5,5,2).$$

Example 7.30. Let us study the case $c = 2$, $D^2 = 4$ and $g = 9$. This gives a non-primitive projective model, with $L \simeq 2D$ (it is the case (Q) described in the text above Theorem 4.7). The scroll $\mathcal{T}$ necessarily has type $(2,1,1,0,0,0)$ and $\mathcal{T}_0$ has type $(3,2,2,1,1,1)$. By Proposition 7.10 we have $\beta_{1,3} = 0$, and by Remark 7.12 we have $\beta_{1,2} = 7$ and $\sum_{k=1}^{7} b^k = 3g - 5 = 22$.

We will now find the b_k.

Lemma 7.26 gives $b_k \leq 4$, for all k, since Z_1 is factor in every section of $2\mathcal{H}_0 - b\mathcal{F}$, for $b \geq 5$. The complete intersection $Z_1 = Z_2 = Z_3 = 0$ in $\mathcal{T}_0$ is a subscroll N of type $(1,1,1)$ with a plane in each fiber. The 7 equations cutting out S'' in $\mathcal{T}_0$ must together cut out four points in each plane fiber, such that no three of these points are collinear, by Theorem 4.7. It is clear that such a configuration of points is contained in exactly two quadrics in each plane. All sections of $2\mathcal{H}_0 - b\mathcal{F}$ with $b \geq 3$ vanish on N, so we must have at least two of the b_k less than 3. Moreover, every section of $2\mathcal{H}_0 - 4\mathcal{F}$ can be written

$$Z_1 f(t, u, Z_1, \dots, Z_5) + aZ_2^2 + bZ_2Z_3 + cZ_3^2.$$

If $b_4 \geq 4$, there are four independent equations of this kind, so we could eliminate the three last terms and obtain one relation with Z_1 as a factor. This is a contradiction, so $b_4 \leq 3$.

This leaves the unique possibility $(b_1, \dots, b_7) = (4,4,4,3,3,2,2)$.

The following proposition describes the particular case in Proposition 4.10 above, that is we have $L \sim 2D$, $D^2 = 4$ and $c = 2$ and D is hyperelliptic, which is also a particular case of the last example. In this case there is a smooth curve E (which is a perfect Clifford divisor for D) satisfying $E^2 = 0$ and $E.D = 2$. Since $(D - E)^2 = 0$ and $(D - E).L = 4$, we have $D > E$, so E does not satisfy the conditions (C6) and (C7). We will also see that E is not a perfect Clifford divisor for L.

Proposition 7.31. *Let L and D be as in the particular case of Proposition 4.10 (where S' is not the 2-uple embedding of $\phi_D(S)$).*

Then we are in one of the following three cases:

(i) *$\mathcal{R}_{L,E} = \emptyset$ and $D \sim E + E'$, where E' is a smooth elliptic curve such that $E.E' = 2$.*
(ii) *$\mathcal{R}_{L,E} = \{\Gamma_1, \Gamma_2\}$ and $D \sim E + \Gamma_1 + \Gamma_2$.*
(iii) *$\mathcal{R}_{L,E} = \{\Gamma_0\}$ and $D \sim E + \Delta_0$, where Δ_0 has a configuration with respect to E as in (E2).*

Moreover, in case (i), $\mathcal{O}_{S'}$ has one of the two following $\mathcal{O}_{\mathcal{T}}$-resolutions for a scroll $\mathcal{T}$ of type $(2,2,2,0)$ in $\mathbf{P}^9$:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H} + 4\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + 2\mathcal{F})^2 \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0,$$

or

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H} + 4\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + 4\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

The latter case occurs only if $\operatorname{Sing} \mathcal{T} \cap S' = \emptyset$, so $S' \simeq S''$ where S'' sits in the smooth scroll $\mathcal{T}_0$ of type $(3,3,3,1)$. In the cases (ii) and (iii), $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolutions for a scroll $\mathcal{T}$ of type $(4,2,0,0)$ in $\mathbf{P}^9$, whose singular locus is spanned by $< Z_\lambda >$ (using the same notation as in Theorem 4.7):

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H} + 4\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + 4\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

Proof. The three cases follow from Proposition 4.11, by noting that we clearly have $\mathcal{R}_{L,E} = \mathcal{R}_{D,E}$.

We have $h^0(L) = 10$ and $h^0(L-E) = 6$. We leave it to the reader to verify that in case (i) we have

$$h^0(L-2E) = 3 \quad \text{and} \quad h^0(L-3E) = 0,$$

and that we in case (ii) have

$$h^0(L-2E) = 3, \;\; h^0(L-3E) = 2, \;\; h^0(L-4E) = 1 \quad \text{and} \quad h^0(L-5E) = 0.$$

This yields the two scroll types $(2,2,2,0)$ and $(4,2,0,0)$ respectively.

In the cases (ii) and (iii), one can show as in the proof of Theorem 4.7 that $\operatorname{Sing}\mathcal{T} =< Z_\lambda >$.

In Subsection 8.4 we will study projective models with $c = 2$ and the statements about the resolutions will follow from Remarks 8.13 and 8.14. □

Example 7.32. Let us study the case $c = 3$, $D^2 = 2$ and $g = 9$. In Section 10 we will show that projective models with such invariants occur, and that the scroll type of $\mathcal{T}$ is either $(1,1,1,1,0,0)$ or $(2,1,1,0,0,0)$ when D is perfect. By Proposition 7.14 we have $\beta_{1,3} = 0$, and by Remark 7.12 we have $\beta_{1,2} = 8$ and $\sum_{k=1}^{8} b^k = 3g - 4 = 23$.

Assume first that the type is $(1,1,1,1,0,0)$, which implies that $\mathcal{T}_0$ has type $(2,2,2,2,1,1)$. Lemma 7.26 gives $b_k \leq 4$, for all k, since $h^0(2\mathcal{H}_0 - b\mathcal{F}) = 0$ for $b \geq 5$. The complete intersection $Z_1 = Z_2 = Z_3 = Z_4$ in $\mathcal{T}_0$ is a subscroll Q of type $(1,1)$ with a line in each fiber. The 8 equations cutting out S'' in $\mathcal{T}_0$ must together cut out two points in each fiber of Q (the inverse image in S'' of $\operatorname{Sing}\mathcal{T} \cap S'$). For a general fiber, call these points P_1 and P_2. Order the b_k as in the previous example. To cut out the two points we must have $b_8 \leq 2$, since all sections of $2\mathcal{H}_0 - b\mathcal{F}$ vanish on Q for $b \geq 3$.

For general (t,u), where the fiber D'' of S'' is smooth, D'' is a smooth curve of degree 7 and genus 2, which can be identified with a smooth curve in $|f^*D - E|$, mapped by the complete linear system $|H_{D''}|$, which is of degree $2g(D'') + 1 = 5$ and in particular very ample. Now Z_1, Z_2, Z_3, Z_4 (restricted to D'') span $H^0(H_{D''})$, which embeds D'' as a curve of degree 5 and genus 2 in $\mathbf{P}^3$. As in Example 7.11 we conclude from [Si] that this curve is contained in only one quadric surface. On the other hand all sections of $2\mathcal{H}_0 - 4\mathcal{F}$ can be expressed in terms of Z_1, Z_2, Z_3, Z_4 only. Hence no more than one of the b_k can be 4. This leaves us with only two possible cases:

$$(b_1, \ldots, b_8) = (4,3,3,3,3,3,2,2) \quad \text{or} \quad (3,3,3,3,3,3,3,2).$$

Assume now that the type is $(2,1,1,0,0,0)$, which implies that $\mathcal{T}_0$ has type $(3,2,2,1,1,1)$. Lemma 7.26 gives $b_k \leq 4$, for all k, since all sections of $2\mathcal{H}_0 - b\mathcal{F}$ have Z_1 as factor if $b \geq 5$. The complete intersection $Z_1 = Z_2 = Z_3$ in $\mathcal{T}_0$ is a subscroll N of type $(1,1,1)$ with a plane in each fiber. The 8 equations cutting out S'' in $\mathcal{T}_0$ must together cut out three independent points in each fiber of N (the inverse image in S'' of $\operatorname{Sing}\mathcal{T} \cap S'$).

Order the b_k as in the previous example. To cut out the three points we must have $b_6 \leq 2$, since all sections of $2\mathcal{H}_0 - b\mathcal{F}$ vanish on N for $b \geq 3$, and a net of three quadrics is needed to cut out three independent points in a plane. On the other hand every section of $2\mathcal{H}_0 - 4\mathcal{F}$ can be written

$$Z_1 f(t,u,Z_1,\ldots,Z_5) + aZ_2^2 + bZ_2Z_3 + cZ_3^2.$$

This gives $b_4 \leq 3$ as in Example 7.30. This leaves

$$(b_1, \ldots, b_8) = (4,4,4,3,2,2,2,2) \quad \text{or} \quad (4,4,3,3,3,2,2,2),$$

as the only possibilities.

Example 7.33. The case $c = 3$, $D^2 = 2$ and $g = 10$ is very similar to the analogous one for $g = 9$, treated in Example 7.32 and one can show in a similar way that

$$(b_1, \dots, b_8) = (4, 4, 4, 3, 3, 3, 3, 2).$$

We show in Section 10 that the only possible scroll type for $\mathcal{T}$ is $(2, 1, 1, 1, 0, 0)$ when D is perfect, corresponding to the type $(3, 2, 2, 2, 1, 1)$ for $\mathcal{T}_0$.

Example 7.34. Similarly, in the case $c = 3$, $D^2 = 4$ and $g = 10$, which is of type (E0) (with $\beta_{1,2} = 12$ and $\sum b_k = 4g - 6 = 34$), one can show that

$$(b_1, \dots, b_{12}) = (4, 4, 4, 3, 3, 3, 3, 2, 2, 2, 2, 2).$$

In Section 10 it is shown that the only scroll type occurring for $\mathcal{T}$ is $(2, 1, 1, 0, 0, 0, 0)$, which means $(3, 2, 2, 1, 1, 1, 1)$ for $\mathcal{T}_0$.

Example 7.35. Let $D^2 = 0$ and $c = 3$, as in Corollary 7.27.

We will show in Section 10 that for $g = 9$ the only smooth scroll occurring as $\mathcal{T} = \mathcal{T}(3, D)$ is of type $(1, 1, 1, 1, 1)$, and the singular scrolls occurring are of types $(2, 1, 1, 1, 0)$, $(2, 2, 1, 0, 0)$ and $(3, 1, 1, 0, 0)$, corresponding to the smooth types $(3, 2, 2, 2, 1)$, $(3, 3, 2, 1, 1)$ and $(4, 2, 2, 1, 1)$ for $\mathcal{T}_0$. We also show that all these occur. By using similar techniques as in the previous examples, one can show that $(b_1, b_2, b_3, b_4) = (2, 1, 1, 1)$ for the type $(1, 1, 1, 1, 1)$, $(b_1, b_2, b_3, b_4) = (4, 3, 3, 3, 3)$, $(4, 4, 3, 3, 2)$ or $(4, 4, 4, 2, 2)$ for the type $(3, 2, 2, 2, 1)$, and $(b_1, b_2, b_3, b_4) = (4, 4, 3, 3, 2)$ or $(4, 4, 4, 2, 2)$ for the types $(3, 3, 2, 1, 1)$ and $(4, 2, 2, 1, 1)$.

For $g = 10$ we will show in Section 10 that the only smooth scroll occurring as $\mathcal{T} = \mathcal{T}(3, D)$ is of type $(2, 1, 1, 1, 1)$, and the singular scrolls occurring are of types $(2, 2, 1, 1, 0)$, $(2, 2, 2, 0, 0)$, $(3, 2, 1, 0, 0)$, corresponding to the types $(3, 3, 2, 2, 1)$, $(3, 3, 3, 1, 1)$ and $(4, 3, 2, 1, 1)$ for $\mathcal{T}_0$. We also show that all these occur. Again one can show that $(b_1, b_2, b_3, b_4) = (2, 2, 2, 1, 1)$ for the type $(2, 1, 1, 1, 1)$, $(b_1, b_2, b_3, b_4) = (4, 4, 4, 3, 3)$ or $(4, 4, 4, 4, 2)$ for the scroll type $(3, 3, 3, 1, 1)$, and $(b_1, b_2, b_3, b_4) = (5, 5, 4, 2, 2)$, $(5, 5, 3, 3, 2)$, $(5, 4, 4, 3, 2)$, $(4, 4, 4, 3, 3)$ or $(4, 4, 4, 4, 2)$ for the scroll types $(3, 3, 2, 2, 1)$ and $(4, 3, 2, 1, 1)$.

8. More on projective models in smooth scrolls of $K3$ surfaces of low Clifford-index

In this section we will have a closer look at the situation described in Section 6 for $c = 1$, 2 and 3. We recall that $\mathcal{T}(c, D, \{D_\lambda\})$ is smooth, which is equivalent to the conditions $D^2 = 0$ and $\mathcal{R}_{L,D} = \emptyset$ when D is perfect. The resolution of $\mathcal{O}_{S'}$ as an $\mathcal{O}_{\mathcal{T}}$-module is given in Proposition 6.2.

From the proof of Theorem 3.8 it is clear that for each of the possible combinations of c and g there is an 18-dimensional family of isomorphism classes of polarized $K3$ surfaces with smooth scroll $\mathcal{T}(c, D, \{D_\lambda\})$. Moreover it follows that there will be an $18 + \dim(\mathrm{Aut}(\mathbf{P}^g)) = 17 + (g + 1)^2$-dimensional family of such projective models of $K3$ surfaces. This is true, simply because there is only a finite number of linear automorphisms that leaves a smooth polarized $K3$ surface invariant.

For each value of c (and g) one can pose several questions about the set (or subscheme of the Hilbert scheme) of projective models S' of $K3$ surfaces S with elliptic free Clifford divisor D and such that $\mathcal{T}$ is smooth.

All scrolls of the same type are projectively equivalent, and hence the configuration of projective models of $K3$ surfaces in one such scroll is a projectively equivalent copy of that in another. Some questions one can pose, are: How many scrolls are there of a given type?

How many projective models S' are there within each scroll? In how many scrolls of a given type is a given S' included?

The answer to the first question is well-known, the remaining ones we will study more closely.

We start with the following well-known result from [Ha]:

Proposition 8.1. *The dimension of the set of scrolls of type* $\mathbf{e}$ *and dimension* d *in* $\mathbf{P}^g$ *is*

$$\dim(\mathrm{Aut}(\mathbf{P}^g)) - \dim(\mathrm{Aut}(\mathbf{P}(\mathcal{E}))) = (g+1)^2 - 3 - d^2 - \delta_1,$$

where $\delta_1 = \sum_{i,j} \max(0, e_i - e_j - 1)$.

If $D^2 = 0$, we recall that $d = c + 2$ for the scroll $\mathcal{T}(c, D, \{D_\lambda\})$.

8.1. **Projective models with $c = 1$.** We have $g \geq 5$. Let the projective model S' and the smooth scroll $\mathcal{T}$ be given. As is seen from Proposition 6.2 the surface S' corresponds to the divisor class $3\mathcal{H} - (g-4)\mathcal{F}$ on $\mathcal{T}$. Moreover, part (c) of the proposition can be applied so we can obtain a resolution in $\mathbf{P}^g$. This is even minimal, by the comment in [Sc, Example 3.6].

Assume $\mathcal{T}$ has scroll type (e_1, e_2, e_3), with $e_1 \geq e_2 \geq e_3$

Proposition 8.2. *The dimension of the set of sections of divisor type* $3\mathcal{H} - (g-4)\mathcal{F}$ *in* $\mathcal{T}$ *is equal to* $29 + \delta_2$*, where* $\delta_2 = \sum \max(0, g - 5 - \sum_{i=1}^{3} a_i e_i)$*. Here the first summation is taken over those triples* (a_1, a_2, a_3) *such that* $a_i \geq 0$*, for* $i = 1, 2, 3$*, and* $\sum_{i=1}^{3} a_i = 3$*. If* S' *is smooth, a general section is a smooth projective model of a* $K3$ *surface.*

Proof. We use the formula $h^0(\mathbf{P}(\mathcal{E}), (a\mathcal{H} + b\mathcal{F})) = h^0(\mathbf{P}^1, \mathrm{Sym}^a(\mathcal{E}) \otimes \mathcal{O}_{\mathbf{P}^1}(b))$, with $a = 3$ and $b = g - 4$. This gives $29 + \delta_2$. Being a smooth model of a $K3$ surface is an open condition on the set of sections of $3\mathcal{H} - (g-4)\mathcal{F}$, and since one section, the one giving S', is smooth, a general section of the linear system is so, too. □

We also have:

Proposition 8.3. *Each projective model* S' *of a* $K3$ *surface* S *of Clifford index 1 in* $\mathbf{P}^g$ *for* $g \geq 5$*, with* $\mathcal{T}(c, D, \{D_\lambda\})$ *smooth, is contained in only one smooth rational normal scroll of dimension 3.*

Proof. In [SD, Part (7.12)] it is shown that the scroll $\mathcal{T}$ is the intersection of all quadric hypersurfaces containing S'. Moreover, any other smooth scroll $\mathcal{T}_2$ containing S' is an intersection of quadric hypersurfaces ([SD, Prop. 1.5(ii)]), each of course containing S'. Hence $\mathcal{T}_2$ contains $\mathcal{T}$. Since the two scrolls have the same dimension and degree, they must be equal. □

From this we conclude:

Corollary 8.4. *There is a set of dimension*

$$(g+1)^2 + 17 + \delta_2 - \delta_1 = \dim(\mathrm{Aut}(\mathbf{P}^g)) + 18 + \delta_2 - \delta_1,$$

where $\delta_1 \geq \delta_2$*, parametrizing projective models of* $K3$ *surfaces in smooth scrolls of the given type.*

Proof. From Proposition 8.1 we see that there is a $((g+1)^2 - 12 - \delta_1)$-dimensional set of scrolls of the same type as $\mathcal{T}$ in $\mathbf{P}^g$. We know that each S' in each such scroll is contained in only one scroll. In each scroll there is a $(29 + \delta_2)$-dimensional set of projective models of $K3$ surfaces as described. We have $\delta_1 \geq \delta_2$, since otherwise there would be too many models with Clifford index 1. □

Remark 8.5. For $c = 1$ we have

$$\delta_1 = \max(0, e_1 - e_2 - 1) + \max(0, e_1 - e_3 - 1) + \max(0, e_2 - e_3 - 1),$$

and

$$\begin{aligned}\delta_2 &= \max(0, e_1 - e_2 - 3) + \max(0, e_1 - e_3 - 3) + \max(0, e_2 - e_3 - 3) + \\ &\quad \max(0, e_1 + e_2 - 2e_3 - 3) + \max(0, e_1 - 2e_2 + e_3 - 3).\end{aligned}$$

Moreover $\delta_1 = 0$ if and only if the scroll type is maximally balanced, and $\delta_2 = 0$ if the scroll type is "reasonably well balanced". It is clear that $\delta_1 = 0$ implies $\delta_2 = 0$. We also see that if $5 \leq g \leq 8$, then $\delta_2 = 0$. Hence the cases $(e_1, e_2, e_3) = (3, 1, 1)$ or $(3, 2, 1)$ are cases where the number $(g+1)^2+17+\delta_2-\delta_1$ is strictly less than $(g+1)^2+17 = \dim(\mathrm{Aut}(\mathbf{P}^g))+18$, and it is clear that scrolls of these types cannot represent the general projective models with $c = 1$ and fixed g since by the construction as in Proposition 3.8 we get an 18-dimensional family of such models.

The inequality $\delta_1 \geq \delta_2$ does not follow directly from the formulas in Remark 8.5, for example since $\delta_1 < \delta_2$, for scroll types $(g - 4, 1, 1)$, when $g \geq 11$. This enables us to conclude that these and other scroll types with $\delta_1 < \delta_2$ do not occur for the scrolls formed from projective models of $K3$ surfaces as described. This statement will be strengthened to apply for $g \geq 8$ below. On the other hand the mentioned type $(3, 1, 1)$ does occur for $g = 7$. This can be seen by using Lemma 7.26.

For $g = 7$ and type $(3, 1, 1)$ one then gets a polynomial P of the form:

$$\begin{gathered}P_{6,1}(t,u)Z_1^3 + P_{4,1}(t,u)Z_1^2Z_2 + P_{4,2}Z_1^2Z_3 + P_{2,1}(t,u)Z_1Z_2^2 + \\ P_{2,2}(t,u)Z_1Z_2Z_3 + P_{2,3}(t,u)Z_1Z_3^2 + c_1Z_2^3 + c_2Z_2^2Z_3 + c_3Z_2Z_3^2 + c_4Z_3^3,\end{gathered}$$

where the $P_{i,j}$ are homogeneous of degree i, and the c_k are constants. For any fixed (t, u) and any fixed point in the $\mathbf{P}^2$ thus obtained, we see that we can avoid that point lying on the zero scheme of P by choosing the $P_{i,j}$ and c_k properly, so we conclude that the general section of $3\mathcal{H} - (g - 4)\mathcal{F} = 3\mathcal{H} - 3\mathcal{F}$ is base point free, and hence smooth, by Bertini. Irreducibility also follows by a similar argument.

Using Lemma 7.26, we see that for $g = 8$ any section of $3\mathcal{H} - (g - 4)\mathcal{F} = 3\mathcal{H} - 4\mathcal{F}$ on a scroll of type $(3, 2, 1)$ can be identified with a P of the form

$$\begin{gathered}P_{5,1}(t,u)Z_1^3 + P_{4,1}(t,u)Z_1^2Z_2 + P_{3,1}Z_1^2Z_3 + P_{3,2}(t,u)Z_1Z_2^2 + \\ P_{2,1}(t,u)Z_1Z_2Z_3 + P_{1,1}(t,u)Z_1Z_3^2 + P_{2,2}Z_2^3 + P_{1,2}Z_2^2Z_3 + c_1Z_2Z_3^2.\end{gathered}$$

So here there is no Z_3^3-term, and from that we conclude that any section of $3\mathcal{H} - 4\mathcal{F}$ on a scroll of type $(3, 2, 1)$ must have the directrix line, say Γ, corresponding to $e_3 = 1$ as a part of its zero scheme. The fact that $\delta_2 < \delta_1$ indicates that if we can form smooth projective models of $K3$ surfaces this way, the surface must have a Picard lattice of higher rank than two. We may check this. If L, E and Γ sit inside a lattice of rank 2, then we can write $\Gamma = aL + bE$, for rational numbers a, b. In addition we must have $\Gamma^2 = -2$ and $L.\Gamma = E.\Gamma = 1$. It is easy to check that this is impossible.

It is clear that the set of base points of the linear system $3\mathcal{H} - 4\mathcal{F}$ is just the directrix line. This is true since for a fixed value of (t, u) (each fixed fiber) and a point Q outside $(0, 0, 1)$, we can just change the $P_{6,1}(t, u)Z_1^3$-term or the $P_{2,2}(t, u)Z_2^3$-term, if we want to avoid Q. If we choose $c_1 \neq 0$, then we obtain that the zero scheme of the corresponding section intersects all fibers in curves, smooth at $(0, 0, 1)$. (Here we set $Z_3 = 1$ in order to write the equation of the curve in affine coordinates around $(0, 0, 1)$. The existence of the non-zero linear term c_1Z_2 gives smoothness at this point.) Hence the zero scheme of a general section is smooth on all of $\mathcal{T}$. We have basically used the identities $3e_2 \geq g - 4$ and

$e_2 + 2e_3 = g - 4$, to conclude as we do. See Remark 8.8 for references to other authors who have already used this kind of reasoning.

Using Lemma 7.26 again we see that if $g \geq 8$, then any section of $3\mathcal{H} - (g-4)\mathcal{F}$ on a scroll of type $(g-4,1,1)$ corresponds to a polynomial P with Z_1 as a factor, which means that its zero scheme must be reducible as a sum of sections $\mathcal{H} - (g-4)\mathcal{F}$ (a subscroll of type $(1,1)$) and $2\mathcal{H}$. Hence these scroll types cannot occur for $\mathcal{T}(c, D, \{D_\lambda\})$. In a similar way one can draw conclusions about sections on other scroll types. The observation above also has an interesting consequence for the types of singular scrolls $\mathcal{T}(c, D, \{D_\lambda\})$ arising for the case $c = 1$, $D^2 = 0$, $g \geq 5$.

Corollary 8.6. *The type of $\mathcal{T}(c, D, \{D_\lambda\})$ is never $(g-2,0,0)$ for $c = 1$ and $D^2 = 0$.*

Proof. Assume the type of $\mathcal{T}(c, D, \{D_\lambda\})$ is $(g-2,0,0)$. Then the type of the associated scroll $\mathcal{T}_0$ is $(g-1,1,1) = (g_0-4,1,1)$, and the divisor type of S'' in $\mathcal{T}_0$ is $3\mathcal{H}_0 - (g_0-4)\mathcal{F}$ (see Example 7.8 and the proof of Corollary 7.24). But we just observed that this is impossible. □

In general we conclude in the same way:

Proposition 8.7. *If a type (a,b,c) is impossible for a smooth scroll $\mathcal{T}(c, D, \{D_\lambda\})$ in $\mathbf{P}^g$ with $c = 1$ and $D^2 = 0$, then the type $(a-1, b-1, c-1)$ is impossible for any scroll $\mathcal{T}(c, D, \{D_\lambda\})$ in $\mathbf{P}^{g-3}$ with $c = 1$ and $D^2 = 0$.*

We will make a list including all possible scroll types for smooth $\mathcal{T}(c, D, \{D_\lambda\})$, for $g \leq 13$, with $c = 1$ and $D^2 = 0$. By the previous lemma, this will give a list including all scroll types of $\mathcal{T}(c, D, \{D_\lambda\})$, smooth or singular, for $g \leq 10$ and $c = 1$ and $D^2 = 0$. In the column with headline "# mod." we give the value of $18 - \delta_1 + \delta_2$.

The information in the list is essentially contained in [Re] and [St, p.8-10]. We include it for completeness, and for the benefit of the reader we also include, in Remark 8.8 below, a few words about how the information can be obtained.

g	scroll type	# mod.	g	scroll type	# mod.	g	scroll type	# mod.
5	$(1,1,1)$	18	9	$(3,2,2)$	18	12	$(5,3,2)$	16
6	$(2,1,1)$	18	10	$(5,2,1)$	16	12	$(4,4,2)$	17
7	$(3,1,1)$	16	10	$(4,3,1)$	17	12	$(4,3,3)$	18
7	$(2,2,1)$	18	10	$(4,2,2)$	16	13	$(7,3,1)$	18
8	$(3,2,1)$	17	10	$(3,3,2)$	18	13	$(6,3,2)$	16
8	$(2,2,2)$	18	11	$(5,3,1)$	17	13	$(5,4,2)$	17
9	$(4,2,1)$	16	11	$(4,3,2)$	17	13	$(5,3,3)$	16
9	$(3,3,1)$	17	11	$(3,3,3)$	18	13	$(4,4,3)$	18
			12	$(6,3,1)$	17			

This gives the following possibilities for singular types:

g	singular scroll types
5	$(2,1,0)$
6	$(3,1,0)$, $(2,2,0)$
7	$(4,1,0)$, $(3,2,0)$
8	$(4,2,0)$
9	$(5,2,0)$
10	$(6,2,0)$

The dimensions of the families on the singular scrolls of type (e_1, e_2, e_3) in $\mathbf{P}^g$ are equal to those of type (e_1+1, e_2+1, e_3+1) on the corresponding smooth scrolls in $\mathbf{P}^{g+3}$.

Remark 8.8. Among the smooth scroll types listed above, we may immediately conclude that a general section of $3\mathcal{H}-(g-4)\mathcal{F}$ is smooth, and hence a smooth projective model of a $K3$ surface, for the types

$$(1,1,1),(2,1,1),(3,1,1),(2,2,1),(2,2,2),(3,2,2),$$
$$(4,2,2),(3,3,2),(3,3,3),(4,3,3),(5,3,3),(4,4,3).$$

These are the ones with $3e_3 \geq g-4$. The last inequality implies that the complete linear system $3\mathcal{H}-(g-4)\mathcal{F}$ has no base points, and hence a Bertini argument gives smoothness of the general section. The remaining types have the third directrix (of degree e_3) as base locus. We have seen above that for the type $(3,2,1)$ the zero scheme of the general section of $3\mathcal{H}-(g-4)F$ is smooth, since $3e_2 \geq g-4$ and $e_2+2e_3 = g-4$. The same identities hold for the types $(4,3,2)$, $(4,4,2)$ and $(5,4,2)$ also, so the zero scheme of a general section of $3\mathcal{H}-(g-4)\mathcal{F}$ is smooth for these types too.

A similar argument can be made for the types $(3,3,1)$, $(4,3,1)$, $(5,3,1)$, $(6,3,1)$ and $(7,3,1)$. Here the identity $3e_3 < g-4$ gives that the third directrix curve (a line) consists of base points for the linear system. The identity $3e_2 \geq g-4$ gives that there are no other base points. The identity $e_1+2e_3 = g-4$ gives that in each fiber the curve that arises as the intersection of that fiber and the zero scheme of a section of $3\mathcal{H}-(g-4)\mathcal{F}$ is smooth at $(0,0,1)$, provided we choose the section with a non-zero $cZ_1Z_3^2$-term. The total zero scheme is also smooth then.

The remaining possible smooth scroll types for $g \leq 13$ on the list above are different, in the sense that for a general section of $3\mathcal{H}-(g-4)\mathcal{F}$, the zero scheme of the section is singular at finitely many points. It turns out that for these types, which are $(4,2,1)$, $(5,2,1)$, $(5,3,2)$ and $(6,3,2)$, the general zero schemes are singular at exactly one point each.

The reason is the following: Since $3e_3 < g-4$, the third directrix curve consists of base points for the linear system. Since $3e_2 \geq g-4$ for these types, there are no other base points. Since $e_2+2e_3 < g-4$, there is no $Z_2Z_3^2$-term for any section. Since $e_1+2e_3 > g-4$, in fact $e_1+2e_3 = (g-4)+1$ for all these types, there is no $cZ_1Z_3^2$-term with c a constant, but there is an $L(t,u)Z_1Z_3^2$-term with $L(t,u)$ a linear expression in t and u. If the section is chosen general, $L(t,u)$ is not identically equal to zero. For all fixed (t,u) where $L(t,u) \neq 0$, the zero scheme of the section of $3\mathcal{H}-(g-4)\mathcal{F}$ is then smooth. For the single zero of $L(t,u)$, the zero scheme is however singular.

A comment about the types not appearing on the list above: The smooth scroll types $(4,4,1),(5,4,1),(6,4,1),(5,5,1)$ are eliminated the following way: A section of $3\mathcal{H}-(g-4)\mathcal{F}$ can have no term containing a $Z_3^2Z_i$-term for these scroll types. Hence all plane cubics in the pencil are singular where they meet the linear directrix. But this is a contradiction, since the general element in the pencil $|D|$ is smooth. Here we have used the identity $e_1+2e_3 < g-4$ for these types. The other types are eliminated because Z_1 must be a factor in each relevant section, a contradiction. These are the types with $3e_2 < g-4$.

The necessary and sufficient condition "$e_1+2e_3 < g-4$ or $3e_2 < g-4$" for eliminating scroll types is given in [Re], as quoted in [St, Lemma 1.8.]. In [St, p.9] one also describes on which scroll types a general section of $3\mathcal{H}-(g-4)\mathcal{F}$ is singular in a finite number of points.

In Section 10 we will show that all the types listed above for $g \leq 10$ actually occur.

Remark 8.9. One does not have to use the resolution from Proposition 6.2 to see that a projective model S' of a $K3$ surface of Clifford index one in a smooth scroll $\mathcal{T}$ as above must be of divisor type $3\mathcal{H}-(g-4)\mathcal{F}$ in $\mathcal{T}$. Define the vector space $W = H^0(\mathcal{J}_{S'}(3))/H^0(\mathcal{J}_{\mathcal{T}}(3))$. In a natural way W represents the space of cubic functions on $\mathcal{T}$ that vanish on S'.

Study the exact sequences:

$$0 \longrightarrow H^0(\mathcal{J}_{S'}(3)) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^g}(3)) \longrightarrow H^0(\mathcal{O}_{S'}(3)) \longrightarrow H^1(\mathcal{J}_{S'}(3)) \longrightarrow 0$$

and

$$0 \longrightarrow H^0(\mathcal{J}_{\mathcal{T}}(3)) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}^g}(3)) \longrightarrow H^0(\mathcal{O}_{\mathcal{T}}(3)) \longrightarrow H^1(\mathcal{J}_{\mathcal{T}}(3)) \longrightarrow 0.$$

One obtains $\dim W = h^0(\mathcal{J}_{S'}(3)) - h^0(\mathcal{J}_{\mathcal{T}}(3)) = g-3$, since $h^0(\mathcal{O}_{\mathcal{T}}(3)) - h^0(\mathcal{O}_{S'}(3)) = g-3$ and $h^1(\mathcal{J}_{S'}(3)) = h^1(\mathcal{J}_{\mathcal{T}}(3)) = 0$ (see [SD, Prop 1.5(i)]). Take $g-3$ arbitrary fibers F of the ruling on $\mathcal{T}$, that is $g-3$ planes. For each plane it is one linear condition on the elements in W to contain it (since this is equivalent to contain an point in the plane outside S'). Hence containing all the $g-3$ planes imposes $g-3$ conditions. These conditions must be independent, since otherwise there would be a cubic in $\mathbf{P}^g$, not containing $\mathcal{T}$, and containing the union of S' and $g-3$ planes. This union has degree $(2g-2)+(g-3) = 3g-5$. But by Bezout's theorem the cubic and $\mathcal{T}$ intersect in a surface of degree $3g-6$. Hence, in particular, any choice of $g-4$ planes gives independent conditions, and there is one, and only one, hypercubic (modulo the ideal of $\mathcal{T}$), which contains S' and $g-4$ planes in the pencil. By Bezout's theorem, it does not contain more. Hence S' in a natural way is a section of $3\mathcal{H}-(g-4)\mathcal{F}$.

8.2. **Projective models with** $c = 2$**.** We have $g \geq 7$. Let the scroll $\mathcal{T}$ formed from S' as usual have type (e_1, e_2, e_3, e_4). Proposition 6.2(a), gives that S' is a complete intersection in $\mathcal{T}$ of two divisors of type $2\mathcal{H} - b_1\mathcal{F}$ and $2\mathcal{H} - b_2\mathcal{F}$. By convention, we set $b_1 \geq b_2$. Part (d) of the same proposition gives the well-known fact that $b_1 + b_2 = g-5$. Such a situation has been thoroughly investigated in [Br]. As already mentioned in the proof of Corollary 7.27, it follows from [Br, Thm. 5.1] that S' is singular along a curve if $b_1 > \frac{2(e_1+e_2+e_3)}{3}$, or equivalently $b_2 < \frac{g-9+2e_4}{3}$. Hence $b_2 \geq 0$ for $g \geq 7$, for complete intersections with only finitely many singularities. Since in particular $b_2 \geq -1$, it is clear that part (b) and (c) of Proposition 6.2 can be used to give a resolution of S' in $\mathbf{P}^g$. This resolution is minimal because of the comment in [Sc, Example 3.6]. The fact that $b_2 \geq 0$ means that S' can be viewed geometrically as a complete intersection of one hyperquadric containing b_1 three-planes in the pencil, and another containing b_2 three-planes (throwing away the three-planes and taking the closure of what remains).

Let us study projective models in smooth scrolls for $c = 2$ and $D^2 = 0$ in general. We see from Proposition 8.1 that the set parametrizing the scrolls having the same type as $\mathcal{T}$ has dimension $(g+1)^2 - 19 - \delta_1 = \dim(\mathrm{Aut}(\mathbf{P}^g)) - 18 - \delta_1$, where $\delta_1 = \sum_{i,j} \max(0, e_i - e_j - 1)$. Therefore one expects the set of projective models of smooth $K3$ surfaces in each scroll to have dimension 36 if the scroll type is reasonably well balanced, to get a set of total dimension $\dim(\mathrm{Aut}(\mathbf{P}^g)) + 18$. This "expectation" is based on the natural assumption that a set of total dimension $\dim(\mathrm{Aut}(\mathbf{P}^g)) + 18$ arises from S' that sit inside maximally balanced scrolls. In this case there are two different sources of imbalance; that of the e_i, and that of the b_k. We will look more closely at this.

Set $\delta_2 = \max(0, b_1 - b_2 - 1)$. Assume first $b_1 > b_2$. We calculate

$$\dim |\mathcal{O}_{\mathcal{T}}(2\mathcal{H} - b_1\mathcal{F})| = 5g - 6 - 10b_1 + \delta_3,$$

where $\delta_3 = h^1(\mathrm{Sym}^2 \mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^1}(-b_1)) = 0$ if and only if $e_4 \geq \frac{b_1-1}{2}$.

By the same sort of calculation we of course get

$$\dim|\mathcal{O}_{\mathcal{T}}(2\mathcal{H}-b_2\mathcal{F})| = 5g-6-10b_2+\delta_4,$$

where $\delta_4 = h^1(\mathrm{Sym}^2\,\mathcal{E}\otimes\mathcal{O}_{\mathbf{P}^1}(-b_2)) = 0$ if and only if $e_4 \geq \frac{b_2-1}{2}$.

Now fix a zero scheme Y of a section s of $2\mathcal{H}-b_1\mathcal{F}$, and study the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}((b_1-b_2)\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}}(2\mathcal{H}-b_2\mathcal{F}) \longrightarrow \mathcal{O}_Y(2\mathcal{H}-b_2\mathcal{F}) \longrightarrow 0.$$

This induces a sequence

$$\begin{aligned} 0 \quad &\longrightarrow H^0(\mathcal{O}_{\mathcal{T}}((b_1-b_2)\mathcal{F})) &&\longrightarrow H^0(\mathcal{O}_{\mathcal{T}}(2\mathcal{H}-b_2\mathcal{F})) \\ &\longrightarrow H^0(\mathcal{O}_Y(2\mathcal{H}-b_2\mathcal{F})) &&\longrightarrow H^1(\mathcal{O}_{\mathcal{T}}((b_1-b_2)\mathcal{F})) \longrightarrow 0. \end{aligned}$$

Since $h^0(\mathcal{O}_{\mathcal{T}}((b_1-b_2)\mathcal{F})) = b_1-b_2+1$ and $h^1(\mathcal{O}_{\mathcal{T}}((b_1-b_2)\mathcal{F})) = 0$, we obtain

$$\begin{aligned} \dim|\mathcal{O}_Y(2\mathcal{H}-b_2\mathcal{F})| &= \dim|\mathcal{O}_{\mathcal{T}}(2\mathcal{H}-b_2\mathcal{F})| - (b_1-b_2+1) \\ &= 5g-6-10b_2+\delta_4-(b_1-b_2+1) \\ &= 5g-8-10b_2+\delta_4-\delta_2. \end{aligned}$$

Summing up, we obtain

$$\begin{aligned} &\dim\mathcal{O}_{\mathcal{T}}(2\mathcal{H}-b_1\mathcal{F}) + \dim\mathcal{O}_Y(2\mathcal{H}-b_2\mathcal{F}) \\ = \ & 5g-6-10b_1+\delta_3+5g-8-10b_2+\delta_4-\delta_2 \\ = \ & 10g-14-10(b_1+b_2)-\delta_2+\delta_3+\delta_4 \\ = \ & 36-\delta_2+\delta_3+\delta_4. \end{aligned}$$

In particular, if g is even, $b_1-b_2=1$ and $e_4 \geq \frac{b_1-1}{2} = \frac{g-6}{4}$, we get 36. We remark that $\frac{g-6}{4}$ is just $\frac{3}{4}$ less than the average values of the e_i. In general there is thus a $(36-\delta_2+\delta_3+\delta_4)$-dimensional set of complete intersections of type $(2\mathcal{H}-b_1\mathcal{F}, 2\mathcal{H}-b_2\mathcal{F})$, provided the section s is uniquely determined. The latter fact follows, for example by the same kind of argument as in [Sc], section (6.2), where scrolls from tetragonal curves are treated (see also the proof of Proposition 8.12 below).

We now assume $b_1 = b_2 (= b = \frac{g-5}{2})$, which can only occur if g is odd. Then $h^0(2\mathcal{H}-b\mathcal{F}) = h^0(\mathbf{P}^1, \mathrm{Sym}^2(\mathcal{E})\otimes\mathcal{O}_{\mathbf{P}^1}(b))$. This number is of the form $20+\delta_3 = 20+\delta_4$, where δ_3 and δ_4 are defined as in the case above. We see that $\delta_3=\delta_4=0$ if and only if $2e_4-b\geq -1$, or equivalently $e_4 \geq \frac{g-7}{4}$. The average value of the e_i is $\frac{g-3}{4}$, which is just one more. The set of complete intersections corresponds to an open set in the Grassmannian $G(2, h^0(2\mathcal{H}-b\mathcal{F}))$, which has dimension $36+2\delta_3$. Hence we get the expected number if $b_1=b_2$, and the e_i are well balanced. Whether $b_1=b_2$ or not, we have now proved:

Proposition 8.10. *The set of complete intersection schemes of type $(2\mathcal{H}-b_1\mathcal{F}, 2\mathcal{H}-(g-5-b_1)\mathcal{F})$ on a smooth rational normal scroll of dimension 4 in $\mathbf{P}^4$ of a fixed scroll type, is either empty or of dimension $36-\delta_2+\delta_3+\delta_4$.*

Corollary 8.11. *The set of complete intersection schemes of type $(2\mathcal{H}-b_1\mathcal{F}, 2\mathcal{H}-(g-5-b_1)\mathcal{F})$ with no or finitely many singularities on a smooth rational normal scroll of dimension 4 in $\mathbf{P}^4$ of a fixed scroll type, is either empty or of dimension at least 36 if $\delta_1 \geq 1$.*

Proof. Set $s = b_1 - \frac{g-5}{2}$. If $s=0$, then $\delta_2=0$, and there is nothing to prove. If $s\geq\frac{1}{2}$, then $\delta_2 = 2s-1$. We split into 4 subcases: $e_1+e_2+e_3+e_4 = 4e+h$, where $h=0,1,2,3$. If $h=0,1$, we have if $\delta_1\geq 1$: $2e_4 \leq 2(\lfloor\frac{g-3}{4}\rfloor - 1) \leq \frac{g-7}{2}$, and $e_3+e_4 \leq \frac{g-3}{2}-1 = \frac{g-5}{2}$. This gives $b_1-2e_4-1 \geq \frac{g-5}{2}+s-\frac{g-7}{2}-1 = s$, and $b_1-(e_3+e_4)-1 \geq \frac{g-5}{2}+s-\frac{g-1}{2}-1 = s-1$. Hence

$$\delta_3 = h^1(\mathbf{P}^1, \mathrm{Sym}^2(\mathcal{E})\otimes\mathcal{O}_{\mathbf{P}^1}(b_1)) \geq s+(s-1) = 2s-1 = \delta_2,$$

and then the result follows from Proposition 8.10.

If $h = 3$, essentially the same method works (look at the three terms $b_1 - 2e_3 - 1$, $b_1 - e_3 - e_4 - 1$ and $b_1 - 2e_4 - 1$).

If $h = 2$, then essentially the same method works (look at the six terms of the form $b_1 - e_i - e_j - 1$, for $i, j = 2, 3, 4$), except in the case $s = 1$, and $(e_1, e_2, e_3, e_4) = (e+2, e, e, e)$. But in that case $b_1 = \frac{g-1}{2} + 1 = 2e + 1$. Using Lemma 7.26 we see that that Z_1 is a factor in every section of $2\mathcal{H} - b_1\mathcal{F}$, so this case simply does not occur. See also the appendix of [Br]. □

If we add the assumption that there exists at least one smooth model S' of a given intersection type, giving rise to a scroll of a given type, then we can conclude that there is a set of dimension $(36 - \delta_2 + \delta_3 + \delta_4)$ parametrizing smooth projective models S' in a scroll of the given type. We see that if the scroll type or (b_1, b_2)-type is unbalanced, the dimension of the set of complete intersections (smooth or singular) can a priori go up, or it can go down from its "expected" value 36. From the last corollary, however, we see that the dimension goes down only if the scroll type is maximally balanced, and the intersection type is not.

If we just start with an arbitrary scroll type (e_1, e_2, e_3, e_4), with $e_4 \geq 1$, $\sum_{i-1}^{c+2} e_i = g - 3$ and "intersection type" $b_1 \geq b_2 \geq 0$, with $b_1 + b_2 = g - 5$, it is an intricate question to decide whether there are any that are smooth projective models of $K3$ surface, or any that are not singular along a curve. This problem is studied in detail in [Br], and we will study the cases for low g in Subsection 8.4.

The question whether a projective model S' can be included in several scrolls of the same type simultaneously is not as simple to answer as in the case $c = 1$. In the case $c \geq 2$ the scroll $\mathcal{T}$ is no longer the intersection of the hyperquadrics containing S'; in fact S' itself is that intersection [SD]. The question is essentially how many divisor classes of elliptic curves E with $E.L = c + 2$ there are on S.

On the other hand it is clear that a projective model cannot be contained in a positive dimensional set of scrolls of the type in question. There is a discrete family of divisor classes on S, so the dimension of the family of smooth projective models of $K3$ surfaces on scrolls of the type in question is now, as for $c = 1$, equal to the sum of the dimension of the set of scrolls of a given type and the dimension of the set of smooth projective models of that type. This sum is

$$\begin{aligned} &(g+1)^2 - 19 - \delta_1) + (36 - \delta_2 + \delta_3 + \delta_4) \\ &= \dim(\mathrm{Aut}(\mathbf{P}^g)) + 18 - \delta_1 - \delta_2 + \delta_3 + \delta_4. \end{aligned}$$

To obtain the dimension of the set of projective equivalence classes, since only a finite number of automorphisms of $\mathbf{P}^g$ fixes a $K3$ surface, we subtract the number $\dim(\mathrm{Aut}(\mathbf{P}^g))$ and get

$$18 - \delta_1 - \delta_2 + \delta_3 + \delta_4.$$

By Theorem 3.8 this number is equal to 18 for at least one scroll type, where there exists smooth complete intersections of that type, and where the fibers of the complete intersection represent a free Clifford divisor on S with $c = 2$. It is clear that if we choose the most balanced scroll type for a fixed g, then $\delta_1 = 0$. If in addition we choose the most balanced (b_1, b_2)-type, then $\delta_2 = \delta_3 = \delta_4 = 0$. Using Lemma 7.26 it is also easy to prove that for all g the most balanced scroll and intersection type (the unique combination for fixed g with $\delta_1 = \delta_2 = 0$) then a general complete intersection will be a smooth projective model of a $K3$ surface.

8.3. **An interpretation of b_1 and b_2.** We will briefly study the case $c = 2$ in an analogous manner as the case $c = 1$ was studied in Remark 8.9 when we showed that a projective model of a $K3$ surface of Clifford index one with smooth asssociated scroll $\mathcal{T}$ must be of divisor type $3\mathcal{H} + (g-4)\mathcal{F}$ in $\mathcal{T}$ without using the resolution from Proposition 6.2.

Define the vector space $W = H^0(\mathcal{J}_{S'}(2))/H^0(\mathcal{J}_{\mathcal{T}}(2))$. In a natural way W represents the space of quadric functions on $\mathcal{T}$ that vanish on S'.

As in Remark 8.9 one obtains $\dim W = h^0(\mathcal{J}_{S'}(2)) - h^0(\mathcal{J}_{\mathcal{T}}(2)) = g-3$, since $h^0(\mathcal{O}_{\mathcal{T}}(2)) - h^0(\mathcal{O}_{S'}(2)) = g-3$ and $h^1(\mathcal{J}_{S'}(2)) = h^1(\mathcal{J}_{\mathcal{T}}(2)) = 0$ (see [SD, Prop. 1.5(i) and Theorem 6.1(ii)]. Assume g is odd. Take $b = \frac{g-5}{2}$ arbitrary fibers F of the ruling on $\mathcal{T}$, that is three-planes. For each three-plane we have two independent linear conditions on the quadric hypersurfaces to contain it. (First, take one point in the three-plane, not on S'. There is only one quadric surface in the threespace containing this point and the intersection with S'. Then take another point outside this quadric surface. To contain these two points and S' is equivalent to containing the threespace and S'.) So, one naively expects there to be $2b = g-5$ conditions to contain all the b three-planes. Hence there should be a pencil, and only a pencil, of elements of W doing so. Intersecting the elements of the pencil, one would expect to get the projective model of the $K3$ surface. If it really were so simple, however, all intersection types (b_1, b_2) would be completely balanced. This is not always true, and one reason is that two different elements of W may intersect $\mathcal{T}$ in a common threedimensional component dominating $\mathbf{P}^1$ in the fibration on $\mathcal{T}$.

We are therefore not able to imitate the reasoning of Remark 8.9, and thereby establish the fact that the ideal of S' in $\mathcal{T}$ is generated as it is, without using Proposition 6.2. On the other hand we may use the knowledge that we have from Proposition 6.2, that S' is indeed of intersection type $(2\mathcal{H} - b_1\mathcal{F}, 2\mathcal{H} - b_2\mathcal{F})$ in its scroll. Make no assumption on the parity of g.

Proposition 8.12. *The invariant b_1 is equal to the largest number k, such that there exists a non-zero element Q of W (a hyperquadric in $\mathbf{P}^g$ containing S', but not $\mathcal{T}$) containing k three-planes in the pencil.*

Moreover, b_2 is the largest number m, such that there exists a non-zero element of W containing m three-planes in the pencil, and intersecting $\mathcal{T}$ in a different 3-dimensional dominant component than Q does.

Proof. Define two elements in W to be congruent if they have the same dominating three-dimensional component (but possibly differ in which three-planes they contain). Define the index of an element of W as the number of three-planes it contains (if necessary, counted with multiplicity). It is clear that if two elements of W are congruent, then they have the same index (they correspond to a well defined divisor class $2\mathcal{H} - i\mathcal{F}$, where i is the index). It follows from a Bezout argument that if the sum of the indices of two elements is larger than $2b = g-5$, then they must be congruent. This shows both assertions. It also shows that the element in $2\mathcal{H} - b_1\mathcal{F}$ which any element in W with index larger than b gives rise to, is the same. (This element is nothing but the congruence class of the element in W.) $\square$

8.4. **Possible scroll types for $c = 2$.** Almost all the information in this subsection can also be found in [Br] and [St], taken together, but we include it for completeness, and present it in our own way, for the sake of the reader.

Also in the case $c = 2$ it is possible to use Lemma 7.26 to obtain useful conclusions for many concrete scroll types. First we will look at possible smooth scroll types for $g \leq 10$.

For $g = 7$ and $g = 8$, the only possible smooth scroll types are $(1,1,1,1)$ and $(2,1,1,1)$, respectively. For $g = 7$, Lemma 7.26 immediately gives $b_1 \leq 2$. Here $b_1 + b_2 = 2$, so $b_2 \geq 0$.

For $g=8$ we see that Z_1 is a factor in every section of $2\mathcal{H}-b_1\mathcal{F}$, for $b_1 \geq 3$. Since a reducible section which is the sum a section $\mathcal{H}-2\mathcal{F}$ and a section $\mathcal{H}-(b-2)\mathcal{F}$ would intersect another section of type $2\mathcal{H}-b_2\mathcal{F}$ in something reducible, alternatively since S' is non-degenerate, we must have $b_1=2$ and $b_2=1$.

For $g=9$ we have $b_1+b_2=4$ and two smooth scroll types $(2,2,1,1)$ and $(3,1,1,1)$. For the latter type Lemma 7.26 gives $b_1=b_2=2$, since any section of $2\mathcal{H}-b\mathcal{F}$, with $b \geq 3$ must be a sum of a section $\mathcal{H}-3\mathcal{F}$ and $\mathcal{H}$. For the type $(2,2,1,1)$ any section of $2\mathcal{H}-3\mathcal{F}$ has a zero scheme containing the subscroll generated by the two linear directrices (a quadric surface Q). If $b_1=3$, then $b_2=1$, and the section $2\mathcal{H}-b_2\mathcal{F}$ intersects Q as a curve of type $(1,2)$ on Q. This is a rational twisted cubic Γ. This corresponds to the fact that $\delta_1=\delta_3=\delta_4=0$ and $\delta_2=1$, so that the set of complete intersections inside $\mathcal{T}$ has dimension at most $36-\delta_2+\delta_3+\delta_4=35$, and taking the union over all $\mathcal{T}$ of the same type we get dimension at most $\dim(\mathrm{Aut}(\mathbf{P}^g))+17$.

Any section of $2\mathcal{H}-4\mathcal{F}$ has a zero scheme, which restricts to two lines in each fiber of $\mathcal{T}$. This is impossible if this scheme shall contain a (necessarily non-degenerate) model S'. We also have $h^0(2\mathcal{H}-b\mathcal{F})=0$ for $b \geq 5$. Hence b_1 is 2 or 3.

For $g=10$, we have $b_1+b_2=5$ and a priori three possible scroll types $(4,1,1,1)$, $(3,2,1,1)$ and $(2,2,2,1)$. But the first cannot occur, since any section of $2\mathcal{H}-b_1\mathcal{F}$ must have total weight $-b_1 \leq -3$, and then Z_1 must be a factor, using Lemma 7.26. Since S' is non-degenerate, this is impossible. Likewise, if $\mathcal{T}$ has type $(3,2,1,1)$ and $b_1 \geq 5$, we conclude that Z_1 must be a factor, again impossible. The cases $b_1=4$ and $b_1=3$ are however possible.

If $\mathcal{T}$ has type $(2,2,2,1)$, then $b_1 \leq 4$, since $h^0(2\mathcal{H}-b\mathcal{F})=0$ if $b \geq 5$. If $b_1=4$, then the zero scheme of any section of $2\mathcal{H}-b_1\mathcal{F}=2\mathcal{H}-4\mathcal{F}$ contains the linear directrix of $\mathcal{T}$ twice (its equation is a homogeneous quadric in Z_1,Z_2,Z_3 involving neither Z_4,t nor u). If we intersect with a section of $2\mathcal{H}-b_2\mathcal{F}=2\mathcal{H}-\mathcal{F}$, and interpret it as the intersection with a quadric containing a fiber, and throw away the fiber, the residual intersection with $\mathcal{T}$ must contain one point of its linear directrix. This must then be a singular point of S'. Hence only $b_1=3$, $b_2=2$ gives a smooth S' for this scroll type. For these invariants $\delta_i=0$, for $i=1,2,3,4$, so we get a family of total dimension $\dim(\mathrm{Aut}(\mathbf{P}^g))+18$.

So far we have studied smooth scroll types for $7 \leq g \leq 10$. At this point we could either proceed with smooth scroll types for $g \geq 11$, or look at singular scroll types for $g \geq 7$. These topics are closely related. Assume we have a singular scroll $\mathcal{T}(c,D,\{D_\lambda\})$ for $g \geq 7$, $D^2=0$, $c=2$. Then the associated smooth scroll $\mathcal{T}_0$ is contained in $\mathbf{P}^{g+4}$, with a resolution as in Proposition 7.14:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}_0}(-4\mathcal{H}_0+(g-1)\mathcal{F}) \longrightarrow \oplus_{k=1}^{2}\mathcal{O}_{\mathcal{T}_0}(-2\mathcal{H}_0+b_k\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0.$$

So, at this point we can use the method of rolling factors to check what scroll types in $\mathbf{P}^{g+4}$ that may contain a surface like S''. Scroll types $(e_1+1,\dots,e_4+1)$ for $\mathcal{T}_0$ correspond to types $(e_1,\dots,e_4)$ for $\mathcal{T}$.

Let us study complete intersection surfaces in smooth scroll types for $g=11$ with this dual viewpoint. Now $b_1+b_2=6$ and $\deg\mathcal{T}=8$ and there are a priori 5 different possible scroll types:

$$(5,1,1,1),\quad (4,2,1,1),\quad (3,2,2,1),\quad (3,3,1,1)\quad \text{and}\quad (2,2,2,2).$$

The type $(5,1,1,1)$ cannot occur, for the same reason that $(4,1,1,1)$ cannot occur for $g=10$. For $(4,2,1,1)$ and $(3,3,1,1)$ we can conclude that Z_1 or Z_2 is a factor in every term of every section of $2\mathcal{H}-3\mathcal{F}$. This gives that the subscroll formed by the two linear directrices is contained in S' (or S''). This is clearly impossible. Hence $b_1 \geq 4$ for these types.

If $b_1 \geq 5$, then Z_1 is a factor in every section of $2\mathcal{H} - b_1\mathcal{F}$, for each of the types $(4,2,1,1), (3,2,2,1)$ and $(2,2,2,2)$, which gives a contradiction. For the type $(3,3,1,1)$ we argue as follows: If $b_1 \geq 5$, then no term of the form Z_iZ_3 or Z_iZ_4 can occur as factor in a monomial of a section of $2\mathcal{H} - b_1\mathcal{F}$, so for each fixed value of (t,u) we get a quadric in Z_1, Z_2 only. This defines a union of two planes in each $\mathbf{P}^3$ which is a fiber of $\mathcal{T}$ (or $\mathcal{T}_0$). Hence each fiber of S' (or S'') is degenerate, a contradiction. So $b_1 = 4$.

We make the same kind of considerations for all $g \leq 14$. We end up with the following a priori possible combinations of smooth scroll type and b_1, for $7 \leq g \leq 14$ (of course $b_2 = g - 5$). For each scroll type and intersection type (b_1, b_2) we indicate whether the general zero scheme of a complete intersection of type $(2\mathcal{H} - b_1\mathcal{F}, 2\mathcal{H} - b_2\mathcal{F})$ is smooth or singular. See also Remark 8.13. In the column with headline " # mod." we give the value of $18 - \delta_1 - \delta_2 + \delta_3 + \delta_4$. This table contains information that can also be found in [Br] and [St], taken together. In [Br] all possible scroll and intersection types for smooth scrolls for all $g \geq 7$ are listed, and in [St] the information on the moduli of the corresponding families are given. We include the list for $g \leq 14$, since it will be useful in the study of projective models on singular scrolls, and for the lattice-theoretical considerations in Section 10.

g	scroll type	b_1	comp. int.	# mod.	g	scroll type	b_1	comp. int.	# mod.
7	$(1,1,1,1)$	1	Smooth	18	12	$(4,3,1,1)$	5	Smooth	16
7	$(1,1,1,1)$	2	Smooth	17	12	$(4,2,2,1)$	4	Singular	15
8	$(2,1,1,1)$	2	Smooth	18	13	$(3,3,2,2)$	4	Smooth	18
9	$(2,2,1,1)$	2	Smooth	18	13	$(3,3,2,2)$	5	Smooth	17
9	$(2,2,1,1)$	3	Smooth	17	13	$(3,3,3,1)$	4	Smooth	17
9	$(3,1,1,1)$	2	Smooth	15	13	$(3,3,3,1)$	6	Smooth	18
10	$(2,2,2,1)$	3	Smooth	18	13	$(4,2,2,2)$	4	Smooth	15
10	$(2,2,2,1)$	4	Singular	17	13	$(4,3,2,1)$	4	Singular	16
10	$(3,2,1,1)$	3	Smooth	16	13	$(4,3,2,1)$	5	Smooth	16
10	$(3,2,1,1)$	4	Singular	17	13	$(4,3,2,1)$	6	Smooth	18
11	$(2,2,2,2)$	3	Smooth	18	13	$(5,3,1,1)$	6	Smooth	17
11	$(2,2,2,2)$	4	Smooth	17	14	$(3,3,3,2)$	5	Smooth	18
11	$(3,2,2,1)$	3	Smooth	17	14	$(3,3,3,2)$	6	Singular	17
11	$(3,2,2,1)$	4	Smooth	17	14	$(4,3,2,2)$	5	Smooth	16
11	$(4,2,1,1)$	4	Singular	15	14	$(4,3,2,2)$	6	Singular	17
11	$(3,3,1,1)$	4	Smooth	16	14	$(4,3,3,1)$	5	Smooth	17
12	$(3,2,2,2)$	4	Smooth	18	14	$(4,4,2,1)$	5	Singular	16
12	$(3,3,2,1)$	4	Smooth	17	14	$(5,3,2,1)$	5	Singular	15
12	$(3,3,2,1)$	5	Smooth	17	14	$(5,3,2,1)$	6	Singular	16

For a perfect Clifford divisor D we get the following possibilities for the corresponding singular $\mathcal{T}$ in $\mathbf{P}^g$, for $7 \leq g \leq 10$ (subtracting 2 from all values of b_i for $\mathcal{T}_0$, for $i = 1,2$):

g	sing. scroll type	b_1	$(S'')_{virt}$	g	sing. scroll type	b_1	$(S'')_{virt}$
7	$(2,1,1,0)$	$1,2$	Smooth	9	$(4,2,0,0)$	4	Smooth
7	$(2,2,0,0)$	2	Smooth	9	$(3,2,1,0)$	$2,3$	Smooth only for $b_1=3$
7	$(3,1,0,0)$	2	Singular	9	$(2,2,2,0)$	2	Smooth
8	$(3,2,0,0)$	3	Smooth	10	$(4,2,1,0)$	$3,4$	Singular
8	$(3,1,1,0)$	2	Singular	10	$(3,3,1,0)$	3	Singular
8	$(2,2,1,0)$	2	Smooth	10	$(3,2,2,0)$	3	Smooth

Remark 8.13. For each smooth scroll $\mathcal{T}$ and intersection type where the general element S' is smooth (on the upper list of types for $7 \leq g \leq 14$) it is clear that we have a smooth projective model of a $K3$ surface. For the remaining cases (on that list) it is natural to interpret them as projective models S' of $K3$ surfaces by non-ample linear systems. The types on the upper list are the only ones for $g \leq 14$ where a general complete intersection $(2\mathcal{H} - b_1\mathcal{F}, 2\mathcal{H} - b_2\mathcal{F})$, with $b_1 + b_2 = g - 5$, is either smooth, or singular in a finite number of points. In Section 10, moreover, we describe all projective models for low g, including those with $c = 2$. All scroll types listed above (smooth or singular) for $g \leq 10$ reappear in the list in Section 10.

To decide which intersections that are in general smooth, which intersections that are in general singular in finitely many points, and which intersections that are in general singular along a curve (or even reducible) one uses rolling factors (Lemma 7.26) as in [Br] and [St]. In particular we have checked with the Appendix in [Br], which gives a list of smooth complete intersection $K3$ surfaces in 4-dimensional smooth rational normal scrolls and also a list of relevant intersections with only finitely many singularities.

On the lower list, concerning singular scrolls for ≤ 10, we have listed all scroll and intersection types which might a priori appear as “images” of scrolls $\mathcal{T}_0$ and surfaces S'' on the upper list.

In the columns with heading $(S'')_{virt}$ we have indicated whether a general complete intersection of type in question on $\mathcal{T}_0$, which contains the exceptional divisor of the map i from $\mathcal{T}_0$ to $\mathcal{T}$, is smooth or singular. We call such a complete intersection $(S'')_{virt}$, since we do not a priori know that it is an S''. If D is perfect and the scroll $\mathcal{T}$ is singular, each occurring projective model S' on $\mathcal{T}$ is of course also singular, but S'' smooth means that all singularities of S' are due to contractions across the fibers; there are no contractions in the individual fibers.

We will illustrate that the issues whether a smooth scroll $\mathcal{T}(2, D)$ and an associated intersection type for a model S' appears on the upper list, is different from the issue whether the scroll and intersection type appears for a $\mathcal{T}_0$ and a S''. If D is perfect, it is a priori possible that all complete intersections, or a general one, represents a model S', but not an S''. As an example, look at the type $(4, 3, 2, 1)$, with $(b_1, b_2) = (6, 2)$. Then S' consists of the common zeroes of two sections of the form

$$\begin{gathered} P_2(t,u)Z_1^2 + P_1(t,u)Z_1Z_2 + c_1Z_2^2 + c_2Z_1Z_3, \qquad \text{and} \\ f(t,u,Z_1,Z_2,Z_3,Z_4) + c_4Z_4^2, \end{gathered}$$

where $f(t, u, Z_1, Z_2, Z_3, Z_4)$ is contained in the ideal generated by Z_1, Z_2, Z_3. The general such intersection is smooth. But in order to be a surface of the form S'', the intersection must contain the last directrix. This forces c_4 to be zero. In that case the intersection is no longer smooth, in fact it contains the directrix in its singular locus, and hence it cannot be an S''. Hence scroll type $(4, 3, 2, 1)$ with $b_1 = 6$ appears on the upper list, but the corresponding "pushed down" type $(3,2,1,0)$ does not appear in combination with (the revised) $b_1 = 4$.

A similar, but slightly different case, is scroll type $(3,2,2,1)$ and intersection type $(4,2)$. Then S' consists of the common zeroes of two sections of the form:

$$P_{2,1}(t,u)Z_1^2 + P_{1,1}(t,u)Z_1Z_2 + P_{1,2}(t,u)Z_1Z_3 + c_1Z_2^2 + c_2Z_2Z_3 + c_3Z_3^2 + c_4Z_1Z_4$$
$$f(t,u,Z_1,Z_2,Z_3) + P_{2,2}(t,u)Z_1Z_4 + P_{1,3}(t,u)Z_2Z_4 + P_{1,4}(t,u)Z_3Z_4 + c_5Z_4^2.$$

If this is an S'', then it contains the last directrix, which means $c_5 = 0$. Even if $c_5 = 0$, the intersection will in general be smooth if $c_4 \neq 0$, and $P_{1,3}$ and $P_{1,4}$ have no common roots. In this example only a subfamily of positive codimension of the $(\dim \operatorname{Aut}(\mathbf{P}^g) + 18 - \delta_1 - \delta_2 + \delta_3 + \delta_4)$-dimensional family of all complete intersections of that type are of the form S''.

Remark 8.14. If we do not assume that D is perfect, then we are not able to eliminate cases that appear in the upper list, from the lower list.

For instance we would also get the possibility of scroll type $(2,2,2,0)$ with $b_1 = 4$ if $\operatorname{Sing}\mathcal{T} \cap S' = \emptyset$. This concludes the proof of Proposition 7.31.

8.5. **Projective models with $c = 3$.** From Proposition 6.2(a) we get that $\mathcal{O}_{S'}$ has a resolution as a $\mathcal{O}_{\mathcal{T}}$-module of the following form:

$$0 \to \mathcal{O}_{\mathcal{T}}(-5\mathcal{H} + (g-6)\mathcal{F}) \to \oplus_{k=1}^5 \mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + b_k\mathcal{F}) \to$$
$$\oplus_{k=1}^5 \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + a_k\mathcal{F}) \to \mathcal{O}_{\mathcal{T}} \to \mathcal{O}_{S'} \to 0.$$

We see from [Sc] that we are in a situation very similar to that of a pentagonal canonical curve, which is natural, since a general hyperplane section of S' is such a curve. We do not intend to say as much about this situation as about the cases $c = 1$ and 2. Study the skew-symmetrical map Φ in the resolution above, already introduced in Corollary 7.27:

$$\Phi : \oplus_{k=1}^5 \mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + b_k\mathcal{F}) \to \oplus_{k=1}^5 \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + a_k\mathcal{F}).$$

Recall that the Pfaffians of this map generate the ideal of S' in $\mathcal{T}$. Assume only that S' is contained in a rational normal scroll $\mathcal{T}$ of degree $g-4$ in $\mathbf{P}^g$ and of type $(e_1, \dots, e_5)$, where $e_5 \geq 1$.

From Proposition 8.1 the dimension of the set of scrolls of type $\mathbf{e}$ in $\mathbf{P}^g$ is equal to $(g+1)^2 - 28 - \delta_1 = \dim(\operatorname{Aut}(\mathbf{P}^g) - 27 - \delta_1$, where $\delta_1 = \sum_{i,j} \max(0, e_i - e_j - 1)$. To obtain the number $18 + \dim(\operatorname{Aut}(\mathbf{P}^g)$ for the dimension of the set of projective models of $K3$ surfaces in scrolls of some type, one expects a 45-dimensional set of such models in a given scroll, provided the scroll type is reasonably well balanced. We will look into this issue, but we will not give a rigorous proof that we can find such a 45-dimensional set.

A given projective model S' is characterized by the ten above-diagonal entries of a five-by-five matrix description of the map Φ. These entries are sections of:

$$\mathcal{H} - (b_2 - a_1)\mathcal{F}, \mathcal{H} - (b_3 - a_1)\mathcal{F}, \mathcal{H} - (b_4 - a_1)\mathcal{F}, \mathcal{H} - (b_5 - a_1)\mathcal{F}, \mathcal{H} - (b_3 - a_2)\mathcal{F},$$
$$\mathcal{H} - (b_4 - a_2)\mathcal{F}, \mathcal{H} - (b_5 - a_2)\mathcal{F}, \mathcal{H} - (b_4 - a_3)\mathcal{F}, \mathcal{H} - (b_5 - a_3)\mathcal{F}, \mathcal{H} - (b_5 - a_4)\mathcal{F}.$$

We have $h^0(\mathcal{T}, \mathcal{H} - (b_i - a_j)\mathcal{F}) = g + 1 - 5(b_i - a_j) + \delta_{2,i,j}$, where $\delta_{2,i,j} = h^1(\mathbf{P}^1, \mathcal{E} \otimes \mathcal{O}_{\mathbf{P}^1}(a_j - b_i))$ and is zero if and only if $e_5 - (b_i - a_j) \geq -1$. In all, there is a set of choices of the ten linear terms, such that this set has dimension

$$10(g+1) - \sum_{i>j} (5(b_i - a_j) + \delta_{2,i,j}).$$

Moreover, we have

$$\begin{aligned}\sum_{i>j}(b_i - a_j) &= b_2 + 2b_3 + 3b_4 + 4b_5 - 4a_1 - 3a_2 - 2a_3 - a_4 \\ &= b_2 + 2b_3 + 3b_4 + 4b_5 - 4(g-6-b_1) - 3(g-6-b_2) \\ &\quad -2(g-6-b_3) - (g-6-b_4) \\ &= 4(b_1 + \cdots + b_5) - 10(g-6) \\ &= 2g - 12,\end{aligned}$$

where we have used the self-duality of the resolution (Proposition 6.2(b)) which gives $a_i = g - 6 - b_i$, for $i = 1, \ldots, 5$) and Proposition 6.2(d) (which gives $b_1 + \cdots + b_5 = 3g - 18$). Inserting this in the expression above we obtain the number

$$10(g+1) - 5(2g-12) + \delta_2 = 70 + \delta_2,$$

where $\delta_2 = \sum_{i>j} \delta_{2,i,j}$ for the dimension of the set of choices of entries in the matrix determining the map Φ.

We see from $a_i = g - 6 - b_i$, for $i = 1, \ldots, 5$, that $\sum_{i=1}^{5} a_i = 2g - 12$, so the average value of the $b_i - a_j$ is $\frac{g-6}{5}$. The average value of the e_i is $\frac{g-4}{5}$, so if both the e_i and the b_j (and therefore the a_j) are maximally balanced, we will in fact have $e_5 - (b_i - a_j) \geq -1$, so $\delta_{2,i,j} = 0$, for each $i > j$.

To obtain the desired value 45 in the maximally balanced situation, one needs to argue that it is correct to subtract 25, in the sence that there is typically a 25-dimensional family of matrix decriptions giving rise to each projective model of a $K3$ surface as described. We do not know how to do this in a rigorous way, but the problem is related to the one mentioned in [B-E, p. 457], where one treats matrix descriptions of maps between two free modules of rank 5 over a ring (see also [Be]). Translating the discussion in [B-E] into our situation, the issue is: Do two matrices A' and A have the same Pfaffian ideal if and only if there is a matrix B, such that $A' = BAB^t$? In an extremely simple case, take $g = 11$ and $a_i = 2$ for all i (and consequently $b_j = 3$ for all j), so that all entries in the matrix representation A of Φ are sections of the same line bundle on $\mathcal{T}$ (in this case $\mathcal{H} - \mathcal{F}$). One can imagine the set of five-by-five matrices acting on the matrix A representing Φ as $A \to BAB^t$, for all B in $GL(5)$. In a situation where the a_i are less balanced, one can imagine an analogous matrix B with entries in suitably manufactured line bundles, so that the "shape" of A is preserved under a similar action. By this we mean that if entry $A_{i,j}$ of A is a section of a line bundle $L_{i,j}$, then entry $A'_{i,j}$ of BAB^t is also a section of $L_{i,j}$. One must then count the sections in the entries of B, control the stabilizers of the action, and show that all A with the same Pfaffian ideal are in the same orbit by the action.

A natural candidate for such a matrix B is one where the entry B_{ij} is chosen as a general section of $\mathcal{O}_{\mathcal{T}}((a_j - a_i)\mathcal{F}) = \mathcal{O}_{\mathcal{T}}((b_i - b_j)\mathcal{F})$, for all (i,j). Since $h^0(\mathcal{O}_{\mathcal{T}}((a_j - a_i)\mathcal{F})) + \mathcal{O}_{\mathcal{T}}((a_i - a_j)\mathcal{F}) = 2 + \max(0, |a_i - a_j| - 1)$, we see that $\sum_{i,j} h^0(\mathcal{O}_{\mathcal{T}}((a_j - a_i)\mathcal{F}) = 25$ if and only if the a_i are chosen in a maximally balanced way. Set $\delta_3 = \sum_{i>j} \max(0, |a_i - a_j| - 1)$. Then the dimension of the set of choices of matrix B as described is $25 + \delta_3$ (We see that $\det B$ is a constant, and we look at the closed subset of those B with non-zero determinant). One checks that BAB^t is antisymmetric, and has entries that are sections in the same line bundles as the corresponding ones for A. This leads to the following:

Conjecture 8.15. *The dimension of the set M of projective models of $K3$ surfaces of Clifford index 3, contained in a maximally balanced smooth rational normal scroll $\mathcal{T}(3, D)$ of dimension 5 is equal to 45. For an arbitrary scroll type, and given combination $(a_1, \ldots, a_5)$*

the corresponding set M is empty, or it has dimension $45+\delta_2-\delta_3$. *We have* $\delta_2 \geq \delta_3$ *if* $\delta_1 \geq 1$.

Remark 8.16. The last statement of the conjecture does not follow directly from purely numerical considerations. As an example, take the case $g=11$, scroll type $(3,1,1,1,1)$ and $(a_1,\dots,a_5)=(1,2,2,2,3)$, which gives $(b_1,\dots,b_5)=(4,3,3,3,2)$. Here $\delta_1=4$, and $\delta_3=1$. For all (i,j) with $i>j$, we have $e_5-(b_i-a_j)\geq -1$, so $\delta_2=0$.

On the other hand the entries outside the diagonal in the first row of a matrix description of Φ are sections of $\mathcal{H}-(b_2-a_1)\mathcal{F},\mathcal{H}-(b_3-a_1)\mathcal{F},\mathcal{H}-(b_4-a_1)\mathcal{F}$ and $\mathcal{H}-(b_5-a_1)\mathcal{F}$, which here are $\mathcal{H}-2\mathcal{F},\mathcal{H}-2\mathcal{F},\mathcal{H}-2\mathcal{F}$ and $\mathcal{H}-\mathcal{F}$. Taking the submaximal minor where we disregard the term $\mathcal{H}-\mathcal{F}$, we see that Z_1 is a factor, since Z_1 is a factor in every section of $\mathcal{H}-2\mathcal{F}$. This is a contradiction, and hence the case does not occur.

8.6. **Speculations for higher values of c.** From Proposition 8.1 we see that the dimension of the set of scrolls of a given type in $\mathbf{P}^g$ is $(g+1)^2-3-(c+2)^2-\delta_1$, where δ_1 is a non-negative number, which is zero if and only if the scroll type is maximally balanced. Since we know that for all c in the range in question there exists a set of dimension $\dim(\mathrm{Aut}(\mathbf{P}^g))+18=(g+1)^2+17$ parametrizing projective model of $K3$ surfaces in $\mathbf{P}^g$ with Clifford-index c fibered by elliptic curves on a scroll of some type, we know that for this type, the set of projective models of $K3$ surfaces of Clifford index c, with smooth associated scrolls $\mathcal{T}$, has dimension at least

$$(g+1)^2+17-((g+1)^2-3-(c+2)^2)=(c+2)^2+20.$$

A scroll type with $\delta_1=0$ is then a natural candidate. It is therefore a challenge to try to prove the following conjecture:

Conjecture 8.17. *Let* $g\geq 5$ *and* $1\leq c<\lfloor\frac{g-1}{2}\rfloor$. *Then the dimension of the set M of projective models of $K3$ surfaces of Clifford index c, contained in a maximally balanced smooth rational normal scroll* $\mathcal{T}(c,D,\{D_\lambda\})$ *of dimension* $c+2$ *is equal to* $(c+2)^2+20$. *For the scroll types with* $\delta_1>0$, *the set M is empty, or*

$$(c+2)^2+20\leq \dim M\leq (c+2)^2+20+\delta_1.$$

Remark 8.18. The inequality $(c+2)^2+20\leq\dim M$ is inspired from Proposition 8.2, Corollary 8.11 and Remark 8.16. (Many examples above reveal that a strict inequality is impossible here.)

The cases $c=1$, scroll type $(7,3,1)$, and $c=2$, scroll types $(3,3,3,1)$ with $b_1=6$, and $(4,3,2,1)$ with $b_1=6$, reveal that a strict inequality $\dim M<(c+2)^2+20+\delta_1$ is impossible. In Section 10 one sees that these cases indeed occur with the fiber D a perfect Clifford divisor. In these cases both the most balanced scroll/intersection type and the mentioned non-balanced types give families of dimension $\dim(\mathrm{Aut}(\mathbf{P}^g))+18$.

9. BN generality and Clifford generality

It is shown in [Mu1] that a general projective model of a $K3$ surface in $\mathbf{P}^g$, for $g=6$, 7, 8, 9 and 10, is a complete intersection in a homogeneous spaces described below.

We recall the following definition of Mukai:

Definition 9.1. [Mu2] *A polarized $K3$ surface (S,L) of genus g is said to be* Brill-Noether (BN) general *if the inequality* $h^0(M)h^0(N)<h^0(L)=g+1$ *holds for any pair (M,N) of non-trivial line bundles such that* $M\otimes N\simeq L$.

Clearly the polarized $K3$ surfaces which are BN general form a 19-dimensional Zariski open subset in the moduli space of polarized $K3$ surfaces of a fixed genus g.

The following theorem is due to Mukai. We use the following convention: For a vector space V^i of dimension i, we write $G(r, V^i)$ (resp. $G(V^i, r)$) for the Grassmann variety of r-dimensional subspaces (resp. quotient spaces) of V.

The variety $\Sigma_{12}^{10} \subseteq \mathbf{P}^{15}$ is a 10-dimensional spinor variety of degree 12. Let V^{10} be a 10-dimensional vector space with a nondegenerate second symmetric tensor λ. Then Σ_{12}^{10} is one of the two components of the subset of $G(V^{10}, 5)$ consisting of 5-dimensional totally isotropic quotient spaces [1].

The variety $\Sigma_{16}^{6} \subseteq \mathbf{P}^{13}$ is the Grassmann variety of 3-dimensional totally isotropic quotient spaces of a 6-dimensional vector space V^6 with a nondegenerate second skew-symmetric tensor σ. It has dimension 6 and degree 16.

Also, $\Sigma_{18}^{5} = G/P \subseteq \mathbf{P}^{13}$, where G is the automorphism group of the Cayley algebra over $\mathbf{C}$ and P is a maximal parabolic subgroup. The variety has dimension 5 and degree 18.

Finally, in the case $g = 12$, let V^7 be a 7-dimensional vector space and $N \subseteq \wedge^2 V^\vee$ a 3-dimensional vector space of skew-symmetric bilinear forms, with basis $\{m_1, m_2, m_3\}$. We denote by $\mathrm{Grass}(3, V^7, m_i)$ the subset of $\mathrm{Grass}(3, V^7)$ consisting of 3-dimensional subspaces u of V such that the restriction of m_i to $U \times U$ is zero. Then $\Sigma_{12}^{3} = \mathrm{Grass}(3, V^7, N) := \cap\, \mathrm{Grass}(3, V^7, m_i)$. It has dimension 3 and degree 12.

Theorem 9.2. [Mu2] *The projective models of BN general polarized $K3$ surfaces of small genus are as follows:*

genus	*projective model of BN general polarized $K3$ surface*
2	$S_2 \longrightarrow \mathbf{P}^2$ *double covering with branch sextic*
3	$(4) \subseteq \mathbf{P}^3$
4	$(2,3) \subseteq \mathbf{P}^4$
5	$(2,2,2) \subseteq \mathbf{P}^5$
6	$(1,1,1,2) \cap G(2, V^5) \subseteq \mathbf{P}^6$
7	$(1^8) \cap \Sigma_{12}^{10} \subseteq \mathbf{P}^7$
8	$(1^6) \cap G(V^6, 2) \subseteq \mathbf{P}^8$
9	$(1^4) \cap \Sigma_{16}^{6} \subseteq \mathbf{P}^9$
10	$(1^3) \cap \Sigma_{18}^{5} \subseteq \mathbf{P}^{10}$
12	$S_{12} = (1) \subseteq \Sigma_{12}^{3}$

We will in this section compare the notions of BN generality with our notion of Clifford generality. For reasons of space, we will leave out the case $g = 12$.

We have

Proposition 9.3. *Let (S, L) be a polarized $K3$ surface of genus $g = 2, 3 \ldots, 7$ or 9. Then (S, L) is BN general if and only if it is Clifford general.*

If $g = 8$ resp. 10, then (S, L) is Clifford general but not BN general if and only if there is an effective divisor D satisfying $D^2 = 2$ and $D.L = 7$ resp. 8, and there are no divisors satisfying the conditions $(*)$ for $c < 3$ resp. 4.

Proof. We must investigate the condition that there exists an effective decomposition $L \sim D + F$ such that $h^0(F)h^0(D) \geq g + 1$, but $\mathrm{Cliff}\, \mathcal{O}_C(D) \geq \lfloor \frac{g-1}{2} \rfloor$ for any smooth curve $C \in |L|$.

[1] A quotient $f : V \to V'$ is totally isotropic with respect to λ if $(f \otimes f)(\lambda)$ is zero on $V' \otimes V'$.

Since $\operatorname{Cliff} \mathcal{O}_C(D) \leq g+1-h^0(D)-h^0(F)$, we easily see that we must be in one of the two cases above. ☐

Since the divisor D in the proposition satisfies $\operatorname{Cliff} L = \operatorname{Cliff} \mathcal{O}_C(D)$, we have $D \in \mathcal{A}^0(L)$, so we get the following from Propositions 2.4 and 2.5:

Corollary 9.4. *Any divisor D as in Proposition 9.3 must satisfy $h^1(D) = h^1(L-D) = 0$, and among all such divisors we can find one satisfying the conditions (C1)-(C5).*

By arguing as in the proof of Proposition 3.9, with the lattice $\mathbf{Z}L \oplus \mathbf{Z}D$, with

$$\begin{bmatrix} L^2 & L.D \\ D.L & D^2 \end{bmatrix} = \begin{bmatrix} 2(g-1) & c+4 \\ c+4 & 2 \end{bmatrix}$$

for $g = 8$ and 10 and $c = \lfloor \frac{g-1}{2} \rfloor = 3$ and 4 respectively, we see that there exists an 18-dimensional family of isomorphism classes of polarized $K3$ surfaces that are Clifford general but not BN general for both $g = 8$ and $g = 10$.

We will now investigate these two cases. A choice of a subpencil $\{D_\lambda\}$ of $|D|$ gives as before a rational normal scroll $\mathcal{T}$ within which $\phi_L(S) = S'$ is contained. Unfortunately, as we will see, we no longer have such a nice result about $V = \operatorname{Sing} \mathcal{T}$ as Theorem 4.7, since the Clifford index c is now the general one. We will however be able to describe these particular cases in a similar manner, too.

Let us first consider the case $g = 8$, where $c = 3$. We have $D.L = 7$, $h^0(L) = 9$ and $h^0(L-D) = 3$. Since $(L-2D)^2 = -6$ and $(L-2D).L = 0$, we have $h^0(L-2D) = 0$ or 1 and $h^0(L-3D) = 0$.

Recall that the type $(e_1, \dots, e_d)$ of the scroll $\mathcal{T}$, with $d = d_0 = \dim \mathcal{T}$, is given by

$$e_i = \#\{j \mid d_j \geq i\} - 1, \tag{40}$$

where

$$d_i := h^0(L-iD) - h^0(L-(i+1)D).$$

We have $d_{\geq 3} = 0$ and $(d_0, d_1, d_2) = (6, 3-h^0(L-2D), h^0(L-2D))$ and the two possible scroll types

$$(e_1, \dots, e_6) = \begin{cases} (1,1,1,0,0,0) & \text{if } h^0(L-2D) = 0 \\ (2,1,0,0,0,0) & \text{if } h^0(L-2D) = 1 \end{cases}$$

We first study the case $h^0(L-2D) = 0$, that is $h^1(L-2D) = 1$. We have $V = \operatorname{Sing} \mathcal{T} = \mathbf{P}^2$. Here we already see that Theorem 4.7 will not apply, since it is clear by the examples given by the lattice above that there are such cases with no contractions across the fibers. Denote the two base points of the pencil $\{D_\lambda\}$ by p_1 and p_2 and their images under ϕ_L by x_1 and x_2. We have the following result:

Lemma 9.5. *Either*
(i) $\mathcal{R}_{L,D} = \emptyset$, or
(ii) $\mathcal{R}_{L,D} = \{\Gamma\}$ and V intersects S' in x_1, x_2 and $y := \phi_L(\Gamma)$, and $V = < x_1, x_2, y >$.

Proof. We first show that $\mathcal{R}_{L,D}$ is either empty or contains at most one curve.

Choose any smooth $D_0 \in |D|$. Set $F := L-D$ as usual. Since $\deg F_{D_0} = c+2 = 5 = 2p_a(D)+1$, one has that F_{D_0} is very ample, and by arguing as in Lemma 5.1, we get that $D.\Delta = 0$ or 1. This shows the assertion.

By arguing as in the proof of Theorem 4.7, we get that V intersects S' in at most three points (two of which must of course be x_1 and x_2), and that these three points are then independent. ☐

By this lemma, there are only two cases occurring for $h^0(L-2D)=0$, which we denote by (CG1) and (CG2), since they are Clifford general:

(CG1) $\mathcal{R}_{L,D}=\emptyset$,
(CG2) $\mathcal{R}_{L,D}=\{\Gamma\}$.

If $h^0(L-2D)=1$, then since $F^2=2$, we have

$$L \sim 2D+\Delta,$$

where Δ is the base divisor of $|F|$ and satisfies $\Delta^2=-6$, $\Delta.L=0$ and $\Delta.D=3$. By arguing as in the proof of Proposition 3.5, we find that L is as in one of the three following cases (where all the Γ and Γ_i are smooth rational curves):

(CG3) $L \sim 2D+\Gamma_1+\Gamma_2+\Gamma_3$, with the following configuration:

$$\begin{tikzcd}
D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma_1 \\
\Gamma_2 & \Gamma_3
\end{tikzcd}$$

and $\mathcal{R}_{L,D}=\{\Gamma_1,\Gamma_2,\Gamma_3\}$,

(CG4) $L \sim 2D+\Gamma+2\Gamma_0+2\Gamma_1+\cdots 2\Gamma_N+\Gamma_{N+1}+\Gamma_{N+2}$, with the following configuration:

$$\begin{tikzcd}
D \arrow[r, dash] \arrow[d, dash] & \Gamma_0 \arrow[r, dash, dashed] & \Gamma_N \arrow[r, dash] \arrow[d, dash] & \Gamma_{N+2} \\
\Gamma & & \Gamma_{N+1} &
\end{tikzcd}$$

and $\mathcal{R}_{L,D}=\{\Gamma,\Gamma_0\}$,

(CG5) $L \sim 2D+3\Gamma_1+2\Gamma_2+2\Gamma_3+\Gamma_4+\Gamma_5$, with the following configuration:

$$\begin{tikzcd}
D \arrow[r, dash] & \Gamma_1 \arrow[r, dash] \arrow[dr, dash] & \Gamma_2 \arrow[r, dash] & \Gamma_4 \\
& & \Gamma_3 \arrow[r, dash] & \Gamma_5
\end{tikzcd}$$

and $\mathcal{R}_{L,D}=\{\Gamma_1\}$.

Defining $\mathcal{Z}_\lambda$ as in (13)-(16), we have by arguing as in the proof of Theorem 4.7 in these three cases that for any $D \in \mathcal{D}$:

$$V=<\mathcal{Z}_\lambda>=\mathbf{P}^3,$$

any subscheme of length 4 spans a $\mathbf{P}^3$ and $V \cap S'$ has support only on this scheme.

For the cases (CG1)-(CG5) we can now argue as in Section 7. In particular, we get a commutative diagram as on page 36, and Propositions 7.6, 7.10, Remark 7.12, Proposition 7.17, and Corollary 7.19 still apply.

In the cases (CG1) and (CG2) the type of $\mathcal{T}_0$ is $(2,2,2,1,1,1)$. We leave it to the reader to use Lemma 7.26 to show that the only possible combinations of the b_i's are

$$(b_1,\dots,b_8)=(4,3,3,3,2,2,2,1), \quad (4,3,3,2,2,2,2,2) \quad \text{and} \quad (3,3,3,3,2,2,2,2).$$

In the cases (CG3), (CG4) and (CG5) the type of $\mathcal{T}_0$ is $(3,2,1,1,1,1)$. We again leave it to the reader to show that $(b_1,\dots.,b_8)=(4,3,3,2,2,2,2,2)$ is the only possibility.

The case $g=10$ is very similar. We have $c=4$, $D.L=8$, $h^0(L)=11$ and $h^0(L-D)=4$. Since $(L-2D)^2=-6$ and $(L-2D).L=2$, we have $h^0(L-2D)=0$ or 1 and $h^0(L-3D)=0$.

This gives as before $d_{\geq 3} = 0$ and $(d_0, d_1, d_2) = (7, 4 - h^0(L-2D), h^0(L-2D))$ and the two possible scroll types

$$(e_1, \dots, e_6) = \begin{cases} (1,1,1,1,0,0,0) & \text{if } h^0(L-2D) = 0 \\ (2,1,1,0,0,0,0) & \text{if } h^0(L-2D) = 1 \end{cases}$$

We now get exactly analogous cases (CG1)' and (CG2)' as for $g = 8$, corresponding to the scroll type $(1,1,1,1,0,0,0)$. If $h^0(L-2D) = 1$, write as usual $F := L - D$ and denote by Δ the base divisor of $|F|$, so that we have

$$L \sim 2D + A + \Delta.$$

for some $A > 0$ satisfying $A.L = (L-2D).L = 2$ and $A.\Delta = 0$. We can now show that $2 = h^1(R) = \Delta.D$, so that $A^2 = -2$ and $A.D = 2$. By arguing as in the proof of Proposition 3.5 again, we find that L is as in one of the two following cases (where all the Γ_i are smooth rational curves such that $\Gamma_i.A = 0$):

(CG3)' $L \sim 2D + A + \Gamma_1 + \Gamma_2$, with $\Gamma_1.D = \Gamma_2.D = 1$, $\Gamma_1.\Gamma_2. = 0$ and $\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2\}$,

(CG4)' $L \sim 2D + A + 2\Gamma_0 + 2\Gamma_1 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$, with all the Γ_i having a configuration as in (E2), $\Gamma_i.A = 0$, $\mathcal{R}_{L,D} = \{\Gamma_0\}$,

Defining $\mathcal{Z}_\lambda$ as in (13)-(16), we have by arguing as in the proof of Theorem 4.7 in these two cases that for any $D \in \mathcal{D}$:

$$V = < \mathcal{Z}_\lambda > = \mathbf{P}^3$$

and $V \cap S'$ has support only on this scheme.

For the cases (CG1)' and (CG2)' the type of $\mathcal{T}_0$ is $(2,2,2,2,1,1,1)$. Again one can use Lemma 7.26 to show that the only possible combinations of the b_i's are

$$(4,3,3,3,3,3,3,3,2,2,2,2,2) \quad \text{and} \quad (3,3,3,3,3,3,3,3,3,2,2,2,2)$$

for the case (CG1)', and

$$(4,3,3,3,3,3,3,3,2,2,2,2,2), \quad (3,3,3,3,3,3,3,3,3,2,2,2,2) \quad \text{and}$$

$$(4,3,3,3,3,3,3,3,3,2,2,2,1)$$

for the case (CG2)'.

The type of $\mathcal{T}_0$ for the cases (CG3)' and (CG4)' is $(3,2,2,1,1,1,1)$. The only possible cases for the b_i's are found to be

$$(4,4,4,3,3,3,2,2,2,2,2,2,2) \quad \text{and} \quad (4,4,3,3,3,3,3,2,2,2,2,2,2),$$

We conclude this section by showing that all the cases (CG1)-(CG5) and (CG1)'-(CG4)' actually exist, by arguing with the help of Propositions 1.4 and 1.5.

We first consider the case $g = 8$.

The case (CG1) can be realized by the lattice just below Corollary 9.4 and therefore has number of moduli 18.

We now show that the case (CG2) can be realized by the lattice $\mathbf{Z}D \oplus \mathbf{Z}F \oplus \mathbf{Z}\Gamma$, with intersection matrix:

$$\begin{bmatrix} D^2 & D.F & D.\Gamma \\ F.D & F^2 & F.\Gamma \\ \Gamma.D & \Gamma.F & \Gamma^2 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 1 \\ 5 & 2 & -1 \\ 1 & -1 & -2 \end{bmatrix}$$

One easily checks that this matrix has signature $(1,2)$, so by Proposition 1.4 there is an algebraic $K3$ surface with this lattice as its Picard lattice.

Set $L := D + F$. By Proposition 1.5 we can assume that L is nef, whence by Riemann-Roch D, $F > 0$.

We first show that L is base point free and that $\operatorname{Cliff} L = 3$. Since $D.L - D^2 - 2 = 3$, we only need to show that there is no effective divisor B on S satisfying either

$$\begin{aligned} B^2 &= 0, \quad B.L = 1,2,3,4, \quad \text{or} \\ B^2 &= 2, \qquad\quad B.L = 6. \end{aligned}$$

Setting $B \sim xD + yF + z\Gamma$, one finds

$$B.L = 7(x+y),$$

which is not equal to any of the values above. Furthermore, D forces (S, L) to be non-BN general. Since one easily sees that we cannot be in any of the cases (CG1), (CG3)-(CG5), we must be in case (CG2).

We can argue in the same way for the cases (CG3)-(CG5), with the obvious lattices. The number of moduli of these cases are 16, 15 and 14, respectively. We leave these cases to the reader.

In the same way, we can show the existence of each of the types (CG1)'-(CG4)' for $g = 10$ and show that their number of moduli 18, 17, 16 and 15 respectively.

These results will all be summarized in the next section, together will all non-Clifford general projective models for $g \leq 10$.

10. Projective models of $K3$ surfaces of low genus

In this section we will use the results obtained to classify all projective models of $K3$ surfaces of low genus. Together with the description of the general models, we are able to give a complete classification and characterization.

We first need to describe the possible perfect Clifford divisors for $c = 1$, 2 and 3, and also in some more detail the cases where $h^1(L - 2D) > 0$, since this last number determines the singular locus of the scroll $\mathcal{T}$ by equation (10).

The reason why we concentrate on perfect Clifford divisors is purely to make the classification simpler. If we did not restrict to perfect Clifford divisors, we would get more projective models, but the extra projective models would also have been possible to describe with a perfect Clifford divisor, whence they would belong to our list.

In the list below, A and R are defined as in Section 5. Furthermore, we have:

- Γ is a smooth rational curve such that $\Gamma.D = 1$ and $\Gamma.A = 0$.
- Γ_1 and Γ_2 are smooth rational curves such that $\Gamma_1.D = \Gamma_2.D = 1$ and $\Gamma_1.A = \Gamma_2.A = \Gamma_1.\Gamma_2 = 0$.
- $\Delta_0 := 2{\Gamma_0}' + 2{\Gamma_1}' + \cdots + 2{\Gamma_N}' + {\Gamma_{N+1}}' + {\Gamma_{N+2}}'$, for $N \geq 0$, with a configuration with respect to D as in (E2) and such that $A.{\Gamma_i}' = 0$ for $i = 0, \ldots, N+2$.

Also we denote the different cases by $\{c, D^2\}$.

Here is the list of all possible perfect Clifford divisors for $c = 1$, 2 and 3, and the cases where $h^1(R) > 0$:

$$\mathbf{c = 1, L^2 \geq 8}$$

$\{1,0\}$ $D^2 = 0$, $D.L = 3$, $\dim \mathcal{T} = 3$.
$\{1,2\}$ $D^2 = 2$, $L^2 = 10$, $L \sim 2D + \Gamma$ as in (E0), $\dim \mathcal{T} = 4$.

Moreover, $h^1(R) \neq 0$ if and only if L is as in the following case:

$\{1,0\}^a$ $L \sim 2D + A + \Gamma$, $A^2 \geq -2$, $D.A = 2$, $L^2 = A^2 + 10 \leq 18$ with equality if and only if $L \sim 6D + 3\Gamma$, $h^1(R) = 1$, $\mathcal{R}_{L,D} = \{\Gamma\}$.

$\mathbf{c = 2, L^2 \geq 12}$

$\{2,0\}$ $D^2 = 0$, $D.L = 4$, $\dim \mathcal{T} = 4$.
$\{2,2\}$ $D^2 = 2$, $D.L = 6$, $L^2 \leq 18$ with equality if and only if $L \sim 3D$, $\dim \mathcal{T} = 5$.
$\{2,4\}$ $D^2 = 4$, $L^2 = 16$, $L \sim 2D$ as in (Q), $\dim \mathcal{T} = 6$.

Moreover, $h^1(R) \neq 0$ if and only if L is as in one of the following cases:

$\{2,0\}^a$ $L \sim 2D + A + \Gamma$, $A^2 \geq -2$, $D.A = 3$, $L^2 = A^2 + 14 \leq 32$ with equality if and only if $L \sim 8D + 4\Gamma$, $h^1(R) = 1$, $\mathcal{R}_{L,D} = \{\Gamma\}$.
$\{2,0\}^b$ $L \sim 2D + A + \Gamma_1 + \Gamma_2$, $A^2 \geq 0$, $D.A = 2$, $L^2 = A^2 + 12 \leq 16$ with equality if and only if $L \sim 4D + 2\Gamma_1 + 2\Gamma_2$, $h^1(R) = 2$, $\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2\}$ [2].
$\{2,0\}^c$ $L \sim 2D + A + \Delta_0$, $A^2 \geq 0$, $D.A = 2$, $L^2 = A^2 + 12 \leq 16$ with equality if and only if $L \sim 4D + 2\Delta_0$, $h^1(R) = 2$, $\mathcal{R}_{L,D} = \{\Gamma_0\}$ [3].
$\{2,2\}^a$ $L \sim 2D + \Gamma_1 + \Gamma_2$ as in (E1), $L^2 = 12$, $h^1(R) = 1$, $\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2\}$.
$\{2,2\}^b$ $L \sim 2D + \Delta_0$ as in (E2), $L^2 = 12$, $h^1(R) = 1$, $\mathcal{R}_{L,D} = \{\Gamma_0\}$.

$\mathbf{c = 3, L^2 \geq 16}$

$\{3,0\}$ $D^2 = 0$, $D.L = 5$, $\dim \mathcal{T} = 5$.
$\{3,2\}$ $D^2 = 2$, $D.L = 7$, $L^2 \leq 22$, $\dim \mathcal{T} = 6$.
$\{3,4\}$ $D^2 = 4$, $L^2 = 18$, $L \sim 2D + \Gamma$ as in (E0), $\dim \mathcal{T} = 7$.

Moreover, $h^1(R) \neq 0$ if and only if L is as in one of the following cases:

$\{3,0\}^a$ $L \sim 2D + A + \Gamma$, $A^2 \geq -2$, $D.A = 4$, $L^2 = A^2 + 18 \leq 50$ with equality if and only if $L \sim 10D + 5\Gamma$, $h^1(R) = 1$, $\mathcal{R}_{L,D} = \{\Gamma\}$.
$\{3,0\}^b$ $L \sim 2D + A + \Gamma_1 + \Gamma_2$, $A^2 \geq 0$,$D.A = 3$, $L^2 = A^2 + 16 \leq 24$, $h^1(R) = 2$, $\mathcal{R}_{L,D} = \{\Gamma_1, \Gamma_2\}$.
$\{3,0\}^c$ $L \sim 2D + A + \Delta_0$, $A^2 \geq 0$, $D.A = 3$, $L^2 = A^2 + 16 \leq 24$, $h^1(R) = 2$, $\mathcal{R}_{L,D} = \{\Gamma_0\}$.
$\{3,2\}^a$ $L \sim 2D + A + \Gamma$, $A^2 = -2$, $D.A = 2$, $L^2 = 16$, $h^1(R) = 1$, $\mathcal{R}_{L,D} = \{\Gamma\}$.

This list is obtained by using the relations $(*)$ and (3) in Section 3 together with Propositions 4.3, 4.5 and 4.6. We now show how it works for $c = 3$.

The three cases $\{3,0\}$, $\{3,2\}$ and $\{3,4\}$ follow directly from the relations $(*)$. If $D^2 > 0$, then by (3) we must have $L^2 \leq 24$. Assume $L^2 = 24$ and consider the divisor $E := L - 3D$. This satisfies $E^2 = 0$ and $E.L = 3$, thus inducing a Clifford index 1 on L, a contradiction. So $L^2 \leq 22$.

Now assume we are in case $\{3,0\}$ and $h^1(R) > 0$. By Propositions 4.5 and 4.6 we have $1 \leq D.\Delta \leq 2$. Since

$$5 = D.L = A.D + \Delta.D,$$

we have the two possibilities:

(a) $\Delta.D = 1$ and $D.A = 4$,
(b) $\Delta.D = 2$ and $D.A = 3$.

In case (a), there has to exist a smooth rational curve Γ in the support of Δ such that $\Gamma.D = 1$ and $\Gamma.A = 0$ (the last equality follows since $\Gamma.(D + A) \leq 1$ by [SD, (2.7.3)]). Write

$$L \sim 2D + A + \Gamma + \Delta'.$$

[2]If $L^2 = 14$, then the moving part of A is a perfect Clifford divisor of type $\{2,2\}$ containing D, and if $L^2 = 16$, then A is a perfect Clifford divisor of type $\{2,4\}$ containing D.
[3]Same comment as above.

Clearly $D.\Delta' = A.\Delta' = 0$, and by $0 = \Gamma.L = 2 - 2 + \Gamma.\Delta'$, we also get $\Gamma.\Delta' = 0$, whence

$$(2D + A + \Gamma).\Delta' = 0,$$

and we must have $\Delta' = 0$ since L is numerically 2-connected. This establishes case $\{3,0\}^a$. From the Hodge index theorem on L and A it follows that $L^2 \leq 50$ with equality if and only if $4L \sim 5A$.

In case (b), there either exist two (and only two) disjoint smooth rational curves Γ_1 and Γ_2 in the support of Δ such that $\Gamma_1.D = \Gamma_2.D = 1$ and $\Gamma_1.A = \Gamma_2.A = 0$, or there exists one and only one smooth rational curve Γ_0 in the support of Δ (necessarily with multiplicity 2) such that $\Gamma_0.D = 1$ and $\Gamma_0.A = 0$. Arguing as in the proof of Proposition 3.5, these two cases give the cases $\{3,0\}^b$ and $\{3,0\}^c$ respectively. Again it follows from the Hodge index theorem on L and A that $L^2 \leq 24$.

Assume we are in case $\{3,2\}$ and $h^1(R) > 0$. By Propositions 4.5 and Proposition 4.6 we have $L^2 = 16$, $D.\Delta = 1$ and $\Delta^2 = -2$. There has to exist a smooth rational curve Γ in the support of Δ such that $\Gamma.D = 1$ and $\Gamma.A = 0$. Arguing as above, we easily find that $L \sim 2D + A + \Gamma$. Since $7 = D.L = 2D^2 + A.D + \Delta.D$, we have $A.D = 2$, and since $16 = L^2 = 18 + A^2$, we must have $A^2 = -2$. This is case $\{3,2\}^a$.

We leave the easier cases $c = 1$ and 2 to the reader, but make a comment on the cases $\{2,0\}^b$ and $\{2,0\}^c$.

From the Hodge index theorem on L and A we get that $L^2 \leq 16$ with equality if and only if $L \sim 2A$. If $A^2 = 2$ or 4, one calculates

$$A.L - A^2 - 2 = 2,$$

$(A-D)^2 \geq -2$ and $(A-D).D = 2$, whence by Riemann-Roch $A \geq D$, so D does not satisfy the condition (C6). However, since A computes the Clifford index of L, we have $h^1(A) = 0$, whence D is perfect by Lemma 5.10. If $L \sim 2A$, one easily sees that A is base point free, whence perfect.

These cases are particularly interesting, since S' is contained in two scrolls of different types.

Note that for $g \leq 10$ (equivalently $L^2 \leq 18$) a polarized $K3$ surface of non-general Clifford index must have $c \leq 3$, so the above cases are sufficient to consider these surfaces. We know that the general $K3$ surface has general Clifford index. The following proposition considers the dimension of the families in the list above.

Proposition 10.1. *The number of moduli of polarized $K3$ surfaces of genus g, with $5 \leq g \leq 10$, and non-general Clifford index $c > 0$ of each of the types $\{1,0\}$, $\{1,2\}$, $\{2,0\}$, $\{2,2\}$ with $g \leq 9$, $\{3,0\}$, $\{3,2\}$ and $\{3,4\}$ is 18, and of each of the types $\{2,2\}$ with $g = 10$ and $\{2,4\}$ is 19.*

Furthermore the general projective model of each of these types satisfies $h^1(L - 2D) = 0$, and the general projective model of each of these types except for the types $\{1,2\}$ and $\{3,4\}$ is smooth.

The number of moduli of each of these types with $h^1(L - 2D) > 0$ is ≤ 17, except for the type $\{1,0\}^a$ for $g = 10$, whose number is 18.

Proof. In the cases $\{2,2\}$ with $g = 10$ and $\{2,4\}$ we have $L \sim 3D$ and $L \sim 2D$ respectively, so it is clear that those cases can be realized with a Picard group of rank 1 and hence live in 19-dimensional families.

In the other cases, one easily sees that L and D are linearly independent, and we will show that these cases can all be realized with a Picard group of rank 2. Arguing as in the proof of Proposition 3.9 we easily see that there is a $K3$ surface S such that $\operatorname{Pic} S \simeq \mathbf{Z}L \oplus \mathbf{Z}D$

such that L^2, $L.D$ and D^2 have the values corresponding to the different cases in question and such that D is a perfect Clifford divisor for L. This has already been done for $D^2 = 0$ in the proof of Proposition 3.9, and a case by case study establishes the proof in the other cases.

Recall now that $h^1(L-2D) > 0$ if and only if there exists a smooth rational curve Γ such that $\Gamma.L = 0$ and $\Gamma.D = 1$, and (since $c > 0$) $\phi_L(S)$ is singular if and only if there exists a smooth rational curve Γ such that $\Gamma.L = 0$ and $\Gamma.D = 0$ or 1.

Assuming that the rank of the Picard group is two, we can write $\Gamma = aL + bD$, for $a, b \in \mathbf{Q}$. The conditions $\Gamma^2 = -2$, $\Gamma.L = 0$ and $\Gamma.D = 0$ or 1 give the equations:

$$\begin{aligned} a^2(g-1) + ab(c+2+D^2) + \frac{b^2D^2}{2} &= -1, \\ a(2g-2) + b(c+2+D^2) &= 0 \text{ and} \\ a(c+2+D^2) + bD^2 &= 0 \text{ or } 1. \end{aligned}$$

A case by case check reveals that we have a solution only when $\Gamma.D = 1$ and then in the following cases:

(a) $\{1,2\}$, with $\Gamma \sim L - 2D$,
(b) $\{3,4\}$, with $\Gamma \sim L - 2D$,
(c) $\{1,0\}$ for $g = 10$, with $L \sim 6D + 3\Gamma$.

One can easily show that case (c) can be realized with a lattice of the form $\mathbf{Z}D \oplus \mathbf{Z}\Gamma$, with $D^2 = 0$, $D.\Gamma = 1$ and $\Gamma^2 = -2$.

This concludes the proof of the Proposition. □

We now would like to study which scroll types are possible for each value of (g, c, D^2) with $5 \le g \le 10$ and $1 \le c \le 3$. Recall that the type $(e_1, \dots, e_d)$ of the scroll $\mathcal{T}$, with $d = \dim \mathcal{T}$, is given by

$$e_i = \#\{j \mid d_j \ge i\} - 1, \tag{41}$$

where

$$\begin{aligned} d = d_0 &:= h^0(L) - h^0(L-D) = c + 2 + \frac{1}{2}D^2, \\ d_1 &:= h^0(L-D) - h^0(L-2D) = d_0 - r, \\ &\vdots \\ d_i &:= h^0(L-iD) - h^0(L-(i+1)D), \\ &\vdots \end{aligned}$$

with

$$r = \begin{cases} D^2 + h^1(L-2D) & \text{if } L \not\sim 2D \text{ (equiv. } D^2 \ne c+2), \\ D^2 - 1 & \text{if } L \sim 2D \text{ (equiv. } D^2 = c+2) \end{cases} \tag{42}$$

In the cases $\{1,2\}$ and $\{3,4\}$, which are both of type (E0), and the case $\{2,4\}$, which is of type (Q), we have $h^0(L-2D) = 1$ and $h^0(L-iD) = 0$ for all $i \ge 3$, so the scroll types are immediately given.

In the case $\{2,2\}$ with $g = 10$, we have $L \sim 3D$, so $h^0(L-2D) = h^0(D) = 2$, $h^0(L-3D) = 1$ and $h^0(L-iD) = 0$ for all $i \ge 3$.

We will now consider one by one the remaining cases, and gather the result in the tables below.

If $c = 1$ or 2 and $D^2 = 0$ the possible scroll types are given in Section 8. We now briefly review these cases.

Let us first consider the case $c = 1$ and $D^2 = 0$ (case $\{1,0\}$).

For $g = 5$ the two possible scroll types are $(1,1,1)$ and $(2,1,0)$. One easily sees that the first case corresponds to $(d_0, d_1, d_2) = (3,3,0)$, whence $h^0(L-2D) = h^1(L-2D) = 0$ and the second corresponds to $(d_0, d_1, d_2) = (3,2,1)$, whence $h^1(L-2D) = 1$ and we are in case $\{1,0\}^a$.

For $g = 6$ we have three possible scroll types: $(2,1,1)$, $(2,2,0)$ and $(3,1,0)$. Comparing with the possible values of the d_i, one finds that the first case corresponds to $h^0(L-2D) = 1$ (and $h^1(L-2D) = 0$). Moreover, the two last cases corresponds to the case $\{1,0\}^a$ with $A \not> D$ and $A > D$ respectively.

For $g = 7$ there are four possible scroll types: $(2,2,1)$, $(3,1,1)$, $(3,2,0)$ and $(4,1,0)$. We see that the two first cases correspond to $h^1(L-2D) = 0$, with $h^0(L-3D) = 0$ and 1 respectively. The two last cases have $h^1(L-2D) = 1$ and therefore correspond to $\{1,0\}^a$ with $A \not> 2D$ and $A > 2D$ respectively.

We now leave the cases $g = 8$, 9 and 10 to the reader.

If $c = 2$ and $D^2 = 0$ (case $\{2,0\}$), then $12 \leq L^2 \leq 18$.

We leave the easiest case $g = 7$ to the reader.

If $g = 8$, we have seen that the four possible scroll types are $(2,1,1,1)$, $(2,2,1,0)$, $(3,1,1,0)$ and $(3,2,0,0)$. The scroll $(2,1,1,1)$ corresponds to $h^1(R) = 0$, whereas the scrolls $(2,2,1,0)$ and $(3,1,1,0)$ correspond to the case $\{1,0\}^a$ with $A \not> D$ and $A > D$ respectively. The type $(3,2,0,0)$ corresponds to a polarized surface that also has a different perfect Clifford divisor, and is hence contained in another scroll as well, by the footnote on page 71.

If $g = 9$, we have seen that the five possible scroll types are $(2,2,1,1)$, $(3,1,1,1)$, $(2,2,2,0)$, $(3,2,1,0)$ and $(4,2,0,0)$. The types $(2,2,1,1)$ and $(3,1,1,1)$ correspond to $h^1(R) = 0$ with $h^0(L-3D) = 0$ and 1 respectively. (One easily sees that the scroll type $(3,1,1,1)$ can be realized by a $K3$ surface S with Picard group $\operatorname{Pic} S \simeq \mathbf{Z}D \oplus \mathbf{Z}\Gamma$, for a smooth rational curve Γ satisfying $\Gamma.D = 2$, and with $L \sim 3D + \Gamma$. Therefore, it has number of moduli 18). The scroll types $(2,2,2,0)$ and $(3,2,1,0)$ correspond to the case $\{1,0\}^a$ with $A \not> D$ and $A > D$ respectively. The type $(4,2,0,0)$ corresponds to a polarized surface that also has a different perfect Clifford divisor, and is hence contained in another scroll as well, by the footnote on page 71.

If $g = 10$, there are again five possible scroll types: $(2,2,2,1)$, $(3,2,1,1)$, $(3,2,2,0)$, $(3,3,1,0)$ and $(4,2,1,0)$. Again the two first correspond to $h^1(R) = 0$ with $h^0(L-3D) = 0$ and 1 respectively. The three last cases correspond to the case $\{1,0\}^a$ with $h^0(A-D) = 1$ and 2 respectively, but $A \not> 2D$ for the two first cases, and $A > 2D$ for the last case.

If $c = 2$ and $D^2 = 2$ (case $\{2,2\}$), then $12 \leq L^2 \leq 16$ (the case $L^2 = 18$ being already treated). We have

$$(L-3D).L = L^2 - 18 < 0.$$

By the nefness of L we must have $h^0(L-3D) = 0$. Since $(L-2D)^2 = L^2 - 16$, we get by Riemann-Roch $h^0(L-2D) = \frac{1}{2}L^2 - 6 + h^1(R)$. This gives $d_{\geq 3} = 0$ and the two possibilities $(d_0, d_1, d_2) = (5, 3, \frac{1}{2}L^2 - 6)$ or $(5,2,1)$, the latter occurring if and only if $L^2 = 12$ and L is of type (E1) or (E2) (the special cases $\{2,2\}^a$ and $\{2,2\}^b$). The corresponding scroll types in the first situation are then $(1,1,1,0,0)$ for $g = 7$, $(2,1,1,0,0)$ for $g = 8$ and $(2,2,1,0,0)$ for $g = 9$. For $g = 7$ and L of type (E1) or (E2), the scroll type is $(2,1,0,0,0)$.

If $c = 3$ and $D^2 = 0$ (case $\{3,0\}$), then $L^2 = 16$ or 18. We have

$$(L - 3D).L = L^2 - 15 \leq 3$$

and

$$(L - 4D).L = L^2 - 20 < 0.$$

This gives immediately $h^0(L - iD) = 0$ for all $i \geq 3$, whence $d_{\geq 4} = 0$. Also, since $c = 3$, we must have $h^0(L-3D) \leq 1$. We also have by Riemann-Roch $h^0(L-2D) = \frac{1}{2}L^2 - 8 + h^1(R)$.

Let us first consider the case $g = 9$. Then we have $(d_0, d_1, d_2, d_3) = (5, 5-h^1(R), h^1(R) - h^0(L - 3D), h^0(L - 3D))$. If $h^1(R) = 0$, then $h^0(L - 2D) = h^0(L - 3D) = 0$ and $(d_0, d_1, d_2, d_3) = (5, 5, 0, 0)$. The corresponding scroll type is $(1, 1, 1, 1, 1)$. The cases with $h^0(R) = h^1(R) > 0$ are $\{3,0\}^a$, $\{3,0\}^b$ and $\{3,0\}^c$. In the first we have $h^0(R) = h^1(R) = 1$, whence $h^0(L - 3D) = 0$ and $(d_0, d_1, d_2, d_3) = (5, 4, 1, 0)$. The corresponding scroll type is $(2, 1, 1, 1, 0)$. In the cases $\{3,0\}^b$ and $\{3,0\}^c$, we have $h^0(R) = h^1(R) = 2$. If $h^0(L-3D) = 0$ (eqv. $A \not> D$), we get $(d_0, d_1, d_2, d_3) = (5, 3, 2, 0)$ and the scroll type is $(2, 2, 1, 0, 0)$. If $h^0(L - 3D) = 1$ (eqv. $A > D$), we get $(d_0, d_1, d_2, d_3) = (5, 3, 1, 1)$ and the scroll type is $(3, 1, 1, 0, 0)$.

If $g = 10$, we have $(d_0, d_1, d_2, d_3) = (5, 5-h^1(R), 1+h^1(R) - h^0(L-3D), h^0(L-3D))$. If $h^1(R) = 0$, then $h^0(L - 2D) = 1$ and $h^0(L - 3D) = 0$ and $(d_0, d_1, d_2, d_3) = (5, 5, 1, 0)$. The corresponding scroll type is $(2, 1, 1, 1, 1)$. The cases with $h^1(R) > 0$ are $\{3,0\}^a$, $\{3,0\}^b$ and $\{3,0\}^c$ as in the case $g = 9$. Arguing as in that case, we get $(d_0, d_1, d_2, d_3) = (5, 4, 2, 0)$ and scroll type $(2, 2, 1, 1, 0)$ in the case $\{3,0\}^a$ (where $h^1(R) = 1$), and we get $(d_0, d_1, d_2, d_3) = (5, 3, 3, 0)$ and scroll type $(2, 2, 2, 0, 0)$ if $h^0(L-3D) = 0$ (eqv. $A \not> D$), and $(d_0, d_1, d_2, d_3) = (5, 3, 2, 1)$ and scroll type $(3, 2, 1, 0, 0)$ if $h^0(L - 3D) = 1$ (eqv. $A > D$) in the two latter cases (where $h^1(R) = 2$).

If $c = 3$ and $D^2 = 2$ (case $\{3,2\}$), then $L^2 = 16$ or 18. We have

$$(L - 3D).L = L^2 - 21 < 0,$$

whence $h^0(L - iD) = 0$ for all $i \geq 3$, whence $d_{\geq 3} = 0$. By Riemann-Roch, $h^0(L - 2D) = \frac{1}{2}L^2 - 8 + h^1(R)$ and we have $(d_0, d_1, d_2) = (6, 4 - h^1(R), \frac{1}{2}L^2 - 8 + h^1(R), 0)$.

If $g = 9$ and $h^1(R) = 0$, then $(d_0, d_1, d_2) = (6, 4, 0)$ and the corresponding scroll type is $(1, 1, 1, 1, 0, 0)$. The case with $h^1(R) > 0$ is given by $\{3,2\}^a$. In this case we have $(d_0, d_1, d_2) = (6, 3, 1)$ and the corresponding scroll type is $(2, 1, 1, 0, 0, 0)$.

If $g = 10$, then we automatically have $h^1(R) = 0$, whence $(d_0, d_1, d_2) = (6, 4, 1)$ and the corresponding scroll type is $(2, 1, 1, 1, 0, 0)$.

We will summarize these results below.

Furthermore, we can prove, by arguing with lattices that all the cases mentioned above exist, and calculate the number of their moduli. In many cases, we can also explicitly find an expression for L in terms of D and some smooth rational curves on the surface. Also, by studyng the Picard lattices, we can find the curves that are contracted by L, and hence find the mildest singularities of the surfaces in the families.

All these informations are also summarized below.

We will first focus on some concrete examples, to give the reader an idea of the proofs, and leave all the other cases to the reader.

Example 10.2. We start with an easy case: $g = 6$, $c = 1$, $D^2 = 0$ and the scroll type $(3, 1, 0)$. This occurs if L is of type $\{1,0\}^a$ with $A > D$ (and also $\mathcal{R}_{L,D} = \{\Gamma\}$). By arguing as in the proof of Proposition 3.5 we find that $L \sim 3D + 2\Gamma + \Gamma_0 + \Gamma_1$, where Γ, Γ_0 and Γ_1

are smooth rational curves, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma \arrow[r, dash] & \Gamma_1 \\ \Gamma_0. & & \end{tikzcd}$$

By Propositions 1.4 and 1.5 there is an algebraic $K3$ surface S with Picard group $\operatorname{Pic} S = \mathbf{Z}D \oplus \mathbf{Z}\Gamma \oplus \mathbf{Z}\Gamma_0 \oplus \mathbf{Z}\Gamma_1$ and intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma & D.\Gamma_0 & D.\Gamma_1 \\ \Gamma.D & \Gamma^2 & \Gamma.\Gamma_0 & \Gamma.\Gamma_1 \\ \Gamma_0.D & \Gamma_0.\Gamma & \Gamma_0^2 & \Gamma_0.\Gamma_1 \\ \Gamma_1.D & \Gamma_1.\Gamma & \Gamma_1.\Gamma_0 & \Gamma_1^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix},$$

and such that $L := 3D + 2\Gamma + \Gamma_0 + \Gamma_1$ is nef (whence by Riemann-Roch D and $\Gamma_0 > 0$).

We have $D.L - D^2 - 2 = 1$. To show that L is base point free and of Clifford index 1, it suffices to show that there is no effective divisor E such that $E^2 = 0$ and $E.L = 1$ or 2.

Set $E \sim xD + y\Gamma + z\Gamma_0 + w\Gamma_1$. Since we can assume $E \in \mathcal{A}^0(L)$, and E base point free, we easily see that

$$E.\Gamma_0 = x - 2z = 0 \quad \text{or} \quad 1,$$

whence

$$E.L = 3x + z = 7z \quad \text{or} \quad 7z + 3,$$

which can never be equal to 1 or 2.

By Riemann-Roch either $\Gamma > 0$ or $-\Gamma > 0$. If the latter is the case, write $\Gamma = -\gamma$, and we then have $D = D_0 + \gamma$ with $D_0 > 0$, since $D.\gamma = -1$. Therefore, we can write

$$L \sim 3(D_0 + \gamma) - 2\gamma + \Gamma_0 + \Gamma_1 = 3D_0 + 2\gamma + \Gamma_0 + \Gamma_1.$$

We can use the same argument if $-\Gamma_1 > 0$, so possibly after a change of basis, we can assume D, Γ, Γ_0 and $\Gamma_1 > 0$. It is then easy to check that D is nef, whence a perfect Clifford divisor.

Example 10.3. Let us consider the case $g = 9$, $c = 2$, $D^2 = 0$ and the scroll type $(3, 2, 1, 0)$. This occurs if L is of type $\{2, 0\}^a$ with $A > D$ (and also $\mathcal{R}_{L,D} = \{\Gamma\}$). An analysis as in the proof of Proposition 3.5 shows that L is one of the following three types:

(a) $L \sim 3D + 2\Gamma + \Gamma_1 + \Gamma_2$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma \arrow[d, dash] \\ \Gamma_2 & \Gamma_1 \end{tikzcd}$$

(b) $L \sim 3D + 2\Gamma + \Gamma_1 + \Gamma_2$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, equal] & \Gamma \arrow[r, dash] & \Gamma_1 \\ \Gamma_2 & & \end{tikzcd}$$

(c) $L \sim 3D + 2\Gamma + \Gamma_1 + \Gamma_2 + \cdots + \Gamma_{n+3}$, for $n \geq 0$ (in general $n = 0$) with the following configuration:

$$\begin{tikzcd} & D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma_2 \arrow[d, dash, dashed] \\ \Gamma_1 \arrow[r, dash] & \Gamma & \Gamma_{n+3}. \end{tikzcd}$$

(Actually case (b) can be looked at as a special case of case (c), with "$n = -1$".)

One can easily show that both cases (a) and (b) do not occur with a Picard group of rank < 4, and case (c) does not occur with a Picard group of rank < 5. We now show that both case (a) and (b) occur with a Picard group of rank 4.

We first consider case (a).

By Propositions 1.4 and 1.5 there is an algebraic $K3$ surface S with Picard group $\operatorname{Pic} S = \mathbf{Z}D \oplus \mathbf{Z}\Gamma \oplus \mathbf{Z}\Gamma_1 \oplus \mathbf{Z}\Gamma_2$ and intersection matrix corresponding to the configuration above, and such that $L := 3D + 2\Gamma + \Gamma_1 + \Gamma_2$ is nef (whence D, Γ_1, $\Gamma_2 > 0$ by Riemann-Roch).

We calculate $D.L - D^2 - 2 = 2$. To show that L is base point free and that $\operatorname{Cliff} L = 2$ with D as a perfect Clifford divisor, it will suffice to show that there are no divisor B on S satisfying $B^2 = 0$, $B.L = 1, 2, 3$ or $B^2 = 2$, $B.L = 6$, and that D is nef.

Write $B \sim xD + y\Gamma + z\Gamma_1 + w\Gamma_2$. Since we can assume $B \in \mathcal{A}^0(L)$, and B base point free, we easily see that

$$B.\Gamma_2 = x - 2w = 0 \text{ or } 1,$$

and

$$B.\Gamma_1 = x + y - 2z = 0, 1 \text{ or } 2.$$

By the Hodge index theorem one also finds

$$B.D = y + z + w = \begin{cases} 1 & \text{if } B^2 = 0, \\ 2 & \text{if } B^2 = 2 \end{cases}$$

Also, we have

$$B.L = 4x + 3z + w = \begin{cases} 1, 2, 3 & \text{if } B^2 = 0, \\ 6 & \text{if } B^2 = 2 \end{cases}$$

One checks by inspection that these four equations have no integer solutions.

By Riemann-Roch, either $\Gamma > 0$ or $-\Gamma > 0$. As in the previous example, possibly after a change of basis one can assume that $\Gamma > 0$ and that D is nef, whence perfect.

We now consider case (b).

Again by Propositions 1.4 and 1.5 there is an algebraic $K3$ surface S with Picard group $\operatorname{Pic} S = \mathbf{Z}D \oplus \mathbf{Z}\Gamma \oplus \mathbf{Z}\Gamma_1 \oplus \mathbf{Z}\Gamma_2$ and intersection marix corresponding to the configuration for (b) above, and such that $L := 3D + 2\Gamma + \Gamma_1 + \Gamma_2$ is nef (whence D and $\Gamma_2 > 0$ by Riemann-Roch).

We calculate $D.L - D^2 - 2 = 2$. To show that L is base point free and that $\operatorname{Cliff} L = 2$ with D as a perfect Clifford divisor, it will again suffice to show that there are no divisor B on S satisfying $B^2 = 0$, $B.L = 1, 2, 3$ or $B^2 = 2$, $B.L = 6$, and that D is nef.

Write $B \sim xD + y\Gamma + z\Gamma_1 + w\Gamma_2$ as before. Again by the Hodge index theorem and since we can assume $B \in \mathcal{A}^0(L)$, and B base point free, we get

$$B.D = y + 2w = \begin{cases} 1 & \text{if } B^2 = 0, \\ 2 & \text{if } B^2 = 2, \end{cases}$$

$$B.\Gamma_2 = 2(x - w) = 0 \text{ or } 2,$$

and

$$B.\Gamma_1 = y - 2z = -1, 0, \quad \text{or} \quad 1$$

(since we do not know whether it is Γ_1 or $-\Gamma_1$ which is effective). Combining these equations with

$$B.L = 2(2x + y + 2w),$$

we find no integer solutions. Again, possibly after a change of basis, we get that D is perfect and that all D, Γ, Γ_1 and $\Gamma_2 > 0$.

We can also check which curves are contracted by L.

In case (a), the only contracted curve is in general Γ, so all surfaces in that family has an A_1 singularity, and the general surface has only such a singularity. Furthermore S'' is then in general smooth.

In case (b), the only contracted curves are in general Γ and Γ_1 , so all surfaces in that family has an A_2 singularity, and the general surface has only such a singularity. Furthermore S'' is then necessarily singular.

By comparing with the table on page 62, we then find that case (a) has $b_1 = 3$ and case (b) has $b_1 = 2$.

Example 10.4. As an easy example we consider the case $g = 10$, $c = 1$, $D^2 = 0$ and the scroll type $(5, 2, 1)$. This occurs if $\mathcal{R}_{L,D} = \emptyset$ and $h^0(L-5D) = 1$. An analysis as in the proof of Proposition 3.5 shows that $L \sim 5D + 3\Gamma_1 + 2\Gamma_2 + \Gamma_3$, with the following configuration:

$$D \longrightarrow \Gamma_1 \longrightarrow \Gamma_2 \longrightarrow \Gamma_3.$$

One can easily show that this cannot be achieved with a Picard group of rank < 4.

By Propositions 1.4 and 1.5 again there is an algebraic $K3$ surface S with Picard group $\operatorname{Pic} S = \mathbf{Z}D \oplus \mathbf{Z}\Gamma_1 \oplus \mathbf{Z}\Gamma_2 \oplus \mathbf{Z}\Gamma_3$ and intersection marix corresponding to the configuration above, and such that $L := 5D + 3\Gamma_1 + 2\Gamma_2 + \Gamma_3$ is nef (whence D and $\Gamma_1 > 0$ by Riemann-Roch).

We calculate $D.L - D^2 - 2 = 1$. To show that L is base point free and that $\operatorname{Cliff} L = 1$ with D as a perfect Clifford divisor, it will suffice to show that there is no divisor E on S satisfying $E^2 = 0$, $E.L = 1, 2$ and that D is nef.

By the Hodge index theorem $36E.D \leq (E + D)^2 L^2 \leq ((E + D).L)^2 \leq 25$, whence $E.D = 0$. Writing $E \sim xD + y\Gamma_1 + z\Gamma_2 + w\Gamma_3$, we get

$$E.D = y = 0,$$

whence

$$E.L = 3x + y = 3x \neq 1 \quad \text{or} \quad 2.$$

Possibly after a change of basis, we get that D is perfect and that all D, Γ_1, Γ_2 and $\Gamma_3 > 0$.

One finds that the only contracted curves with this Picard group are Γ_2 and Γ_3, so the general surface in this family has an A_2 singularity.

Example 10.5. We give a more involved example: $g = 10$, $c = 2$, $D^2 = 0$ and the scroll type $(3, 2, 1, 1)$. This occurs if $\mathcal{R}_{L,D} = \emptyset$ and $h^0(L - 3D) = 1$. By the table on page 62, we must have $b_1 = 3$ or 4, and we will now show that both these cases exist (with the number of moduli 17 and 16 respectively).

One easily sees that there is no way to achieve this situation with a Picard group of rank < 3. We will now show that it is possible with a Picard group of rank 3.

By Propositions 1.4 and 1.5 there is an algebraic $K3$ surface S with Picard group $\operatorname{Pic} S = \mathbf{Z}D \oplus \mathbf{Z}\Gamma_1 \oplus \mathbf{Z}\Gamma_2$ and intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma_1 & D.\Gamma_2 \\ \Gamma_1.D & \Gamma_1^2 & \Gamma_1.\Gamma_2 \\ \Gamma_2.D & \Gamma_2.\Gamma_1 & \Gamma_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 2 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix},$$

and such that $L := 3D + 2\Gamma_1 + \Gamma_2$ is nef (whence by Riemann-Roch D and $\Gamma_1 > 0$).

We have $D.L - D^2 - 2 = 2$. To show that L is base point free and of Clifford index 2 with D as a perfect Clifford divisor, it suffices to show that there is no effective divisor B such that $B^2 = 0$, $B.L = 1, 2, 3$, or $B^2 = 2$, $B.L = 6$.

By the Hodge index theorem one has

$$18(B^2 + 2B.D) = L^2(B + D)^2 \leq ((B + D).L)^2 = (B.L + 4)^2,$$

which gives $B.D \leq 1$.

Writing $B \sim xD + y\Gamma_1 + z\Gamma_2$, we have $B.D = 2y$, whence $y = 0$.

Since either $\Gamma_2 > 0$ or $-\Gamma_2 > 0$ and we can assume $B \in \mathcal{A}^0(L)$, we must have $B.\Gamma_2 = y - 2z = -2z = -1, 0, 1$. We therefore get $z = 0$.

So B is a multiple of D, a contradiction.

Possibly after a change of basis, we get that D is perfect and that also $\Gamma_2 > 0$.

One finds that the only contracted curve with this Picard group is Γ_2, so that all surfaces in this family have at least an A_1 singularity, and the general such surface has such a singularity. By comparing with the table on page 62, we see that we must have $b_1 = 4$.

But there is also another family of surfaces. Again we find that there is an algebraic $K3$ surface S with Picard group $\operatorname{Pic} S = \mathbf{Z}D \oplus \mathbf{Z}\Gamma_1 \oplus \mathbf{Z}\Gamma_2 \oplus \mathbf{Z}\Gamma_3$ and intersection matrix

$$\begin{bmatrix} D^2 & D.\Gamma_1 & D.\Gamma_2 & D.\Gamma_3 \\ \Gamma_1.D & \Gamma_1^2 & \Gamma_1.\Gamma_2 & \Gamma_1.\Gamma_3 \\ \Gamma_2.D & \Gamma_2.\Gamma_1 & \Gamma_2^2 & \Gamma_2.\Gamma_3 \\ \Gamma_3.D & \Gamma_3.\Gamma_1 & \Gamma_3.\Gamma_2 & \Gamma_3^2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix},$$

and such that $L := 3D + \Gamma_1 + \Gamma_2 + \Gamma_3$ is nef.

One can show that $\operatorname{Cliff} L = 2$ with D as a perfect Clifford divisor (again after possibly changing the basis). Furthermore, one finds that with this lattice, there are no contracted curves, whence S' is smooth. By comparing with the table on page 62, we see that we must have $b_1 = 3$.

We will now summarize all our knowledge of birational projective models S' of $K3$ surfaces of genera $5 \leq g \leq 10$. In some cases we are able to give a resolution of S' in its scroll $\mathcal{T}$. When we are not able to do this, we give the vector bundle a section of which cuts out S'' in $\mathcal{T}_0 \simeq \mathbf{P}(\mathcal{E})$ (which is the dual of the vector bundle F_1 in the resolution

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow \mathcal{O}_{\mathcal{T}_0} \longrightarrow \mathcal{O}_{S''} \longrightarrow 0.)$$

This vector bundle is a direct sum of line bundles, which we write as a linear combination of the line bundles $\mathcal{H}$ and $\mathcal{F}$ on $\mathbf{P}(\mathcal{E})$, where $\mathcal{H} = i^*\mathcal{O}_{\mathbf{P}^g}(1)$ and $\mathcal{F} = \pi^*\mathcal{O}_{\mathbf{P}^1}(1)$, with

$$\begin{array}{ccc} \mathbf{P}(\mathcal{E}) & \xrightarrow{i} & \mathcal{T} \subseteq \mathbf{P}^g \\ \big\downarrow{\scriptstyle \pi} & & \\ \mathbf{P}^1. & & \end{array}$$

Also note that we in all cases have $\mathcal{J}_{S'/\mathcal{T}} = i_* S''/\mathcal{T}_0$ by Proposition 7.15, and that we in most cases, by Remark 7.28, the sections of $F_1^{\vee}$ are constant on the fibers of i, whence they also give "equations" cutting out S' in $\mathcal{T}$ set-theoretically.

$\mathbf{g = 5}$

The general projectice model is a complete intersection of three hyperquadrics. The others are as follows:

c	D^2	scroll type	# mod.	type of L	sing.
1	0	$(1,1,1)$	18	$h^0(L-2D)=0$	sm.
1	0	$(2,1,0)$	17	$\{1,0\}^a$, $A^2=-2$	A_1

In these cases $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}+\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

$\mathbf{g = 6}$

The general projective model is a hyperquadric section of a Fano 3-fold of index 2 and degree 5. The others are as follows:

c	D^2	scroll type	# mod.	type of L	sing.
1	0	$(2,1,1)$	18	$h^0(L-2D)=0$	sm.
1	0	$(2,2,0)$	17	$\{1,0\}^a$, $A^2=0$, $A \not> D$	A_1
1	0	$(3,1,0)$	16	$\{1,0\}^a$, $A^2=0$, $A > D$ [(i)]	A_2
1	2	$(2,1,0,0)$	18	(E0)	A_1

In the three first cases $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}+2\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

In the last case, S' has a resolution:

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H}+2\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-4\mathcal{H}+\mathcal{F}) \\ &\longrightarrow \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}+2\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}+\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}) \\ &\longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0 \end{aligned}$$

Comments on the types of L:

(i) $L \sim 3D + 2\Gamma + \Gamma_0 + \Gamma_1$, with the following configuration:

$$\begin{tikzcd}[arrows=dash] D \arrow[r] \arrow[d] & \Gamma \arrow[r] & \Gamma_1 \\ \Gamma_0 & & \end{tikzcd}$$

$\mathbf{g = 7}$

The general projective model is a complete intersection of 8 hyperplanes in Σ_{12}^{10}, as described in the beginning of Section 9.

The other projective models are as follows:

c	D^2	scroll type	# mod.	type of L	sing.
1	0	$(2,2,1)$	18	$h^0(L-2D)=2,\ h^0(L-3D)=0$	sm.
1	0	$(3,1,1)$	16	$h^0(L-2D)=2,\ h^0(L-3D)=1$	sm.
1	0	$(3,2,0)$	17	$\{1,0\}^a,\ A^2=2,\ A>D,\ A \not> 2D$	A_1
1	0	$(4,1,0)$	16	$\{1,0\}^a,\ A^2=2,\ A>2D$ $^{(i)}$	A_3
2	0	$(1,1,1,1)$	18	$h^0(L-2D)=0$	sm.
2	0	$(2,1,1,0)$	17	$\{2,0\}^a,\ A^2=-2$	A_1
2	0	$(2,2,0,0)$	16	$\{2,0\}^b$ or $\{2,0\}^c,\ A \not> D$ $^{(ii)}$	$2A_1$
2	0	$(3,1,0,0)$	15	$\{2,0\}^b$ or $\{2,0\}^c,\ A>D$ $^{(iii)}$	$2A_2$
2	2	$(1,1,1,0,0)$	18	$h^0(L-2D)=0$	sm.
2	2	$(2,1,0,0,0)$	17	(E1) or (E2) $^{(iv)}$	$2A_1$

In the cases $(c,D^2)=(1,0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}+3\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

.

In the cases $(c,D^2)=(2,0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$\begin{aligned} 0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H}+(g-1)\mathcal{F}) &\longrightarrow \\ \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}+b_1\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}+b_2\mathcal{F}) &\longrightarrow \quad \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0, \end{aligned}$$

with $(b_1,b_2)=(1,1)$ or $(2,0)$ for the scroll types $(1,1,1,1)$ and $(2,1,1,0)$ and $(b_1,b_2)=(2,0)$ for the scroll types $(2,2,0,0)$ and $(3,1,0,0)$.

In the cases $(c,D^2)=(2,2)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

$$\oplus_{i=1}^4 \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-b_i\mathcal{F}),$$

where $(b_1,b_2,b_3,b_4)=(1,1,1,0)$ or $(2,1,0,0)$ for the type $(1,1,1,0,0)$, and $(2,1,0,0)$ for the type $(2,1,0,0,0)$.

Comments on the types of L:

(i) $L \sim 4D+3\Gamma+2\Gamma_1+\Gamma_2$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma \arrow[r, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_2 \end{tikzcd}$$

(ii) The number of moduli of the case $\{2,0\}^c$ is 15, with mildest singularity A_3.

(iii) In the case $\{2,0\}^b$ we have $L \sim 3D+2\Gamma_1+2\Gamma_2+\Gamma_1'+\Gamma_2'$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[dr, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_1' \\ & \Gamma_2 \arrow[r, dash] & \Gamma_2', \end{tikzcd}$$

and in the case $\{2,0\}^c$ we have $L \sim 3D+4\Gamma_0+3\Gamma_1+2\Gamma_2+2\Gamma_3+\Gamma_4$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_3 \arrow[d, dash] \\ & \Gamma_2 & & \Gamma_4. \end{tikzcd}$$

The mildest singularity of this latter case is A_5.

(iv) The number of moduli of the case (E2) is 16, with mildest singularity A_3.

$\mathbf{g = 8}$

The general projective model is a complete intersection of 5 hyperplanes in $\mathrm{Grass}(V^6, 2) \subseteq \mathbf{P}^{14}$.

The others are as follows:

c	D^2	scroll type	# mod.	type of L	sing.
1	0	$(2,2,2)$	18	$h^0(L-2D)=3,\ h^0(L-3D)=0$	sm.
1	0	$(3,2,1)$	17	$h^0(L-2D)=3,\ h^0(L-3D)=1$	sm.
1	0	$(4,2,0)$	17	$\{1,0\}^a,\ A^2=4$	A_1
2	0	$(2,1,1,1)$	18	$h^0(L-2D)=1$	sm.
2	0	$(2,2,1,0)$	17	$\{2,0\}^a,\ A^2=0,\ A \not> D$	A_1
2	0	$(3,1,1,0)$	15	$\{2,0\}^a,\ A^2=0,\ A>D$ $^{(i)}$	A_2
2	0	$(3,1,1,0)$	14	$\{2,0\}^a,\ A^2=0,\ A>D$ $^{(ii)}$	A_2+2A_1
2	0	$(3,2,0,0)$	16	$\{2,0\}^b$ or $\{2,0\}^c$, $A^2=2,\ A>D$ $^{(iii)}$	$2A_1$
2	2	$(2,1,1,0,0)$	18	$h^0(L-2D)=1$	sm.
3	2	$(1,1,1,0,0,0)$	18	(CG1) or (CG2) $^{(iv)}$	sm.
3	2	$(2,1,0,0,0,0)$	16	(CG3),(CG4) or (CG5) $^{(v)}$	$3A_1$

In the cases $(c, D^2) = (1,0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}+4\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

.

In the cases $(c, D^2) = (2,0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$\begin{array}{rcl} 0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H}+(g-1)\mathcal{F}) & \longrightarrow & \\ \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}+b_1\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-2\mathcal{H}+b_2\mathcal{F}) & \longrightarrow & \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0, \end{array}$$

with $(b_1, b_2) = (2,1)$, except for the type $(3,2,0,0)$, where $(b_1,b_2) = (3,0)$. In this latter case, S also contains a different perfect Clifford divisor (by the footnote on page 71), so S' can also be described as for the case $(c, D^2) = (2,2)$.

In the cases $(c, D^2) = (2,2)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

$$\oplus_{i=1}^4 \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - b_i\mathcal{F}),$$

where $(b_1,b_2,b_3,b_4) = (3,2,0,0), (3,1,1,0)$ or $(2,2,1,0)$.

In the cases (CG1) and (CG2) then S'' is cut out in $\mathcal{T}_0$ by a section of

$$\begin{array}{lllllll} \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-2\mathcal{F}) & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-\mathcal{F})^2 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^5 & \text{or} & \\ \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-\mathcal{F})^4 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^4 & & \text{or} & & \\ \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-2\mathcal{F}) & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-\mathcal{F})^3 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^3 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}+\mathcal{F}) \end{array}$$

(which is constant on the fibers of i in the first four cases). In the cases (CG3),(CG4), (CG5) then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-2\mathcal{F}) \ \oplus \ \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}-\mathcal{F})^2 \ \oplus \ \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^5$$

Comments on the types of L:

(i) $L \sim 3D + 2\Gamma + \Gamma' + \Gamma_1 + \Gamma_2$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma \arrow[r, dash] & \Gamma'_1 \\ \Gamma_1 & \Gamma_2 & \end{tikzcd}$$

(ii) $L \sim 3D + 2\Gamma + \Gamma' + 2\Gamma_0 + 2\Gamma_1 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$, for $N \geq 0$ (in general $N = 0$) with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma \arrow[r, dash] & \Gamma' & \\ \Gamma_0 \arrow[r, dash] & \Gamma_1 \arrow[r, dashed, dash] & \Gamma_N \arrow[r, dash] \arrow[d, dash] & \Gamma_{N+2} \\ & & \Gamma_{N+1} & \end{tikzcd}$$

(iii) The number of moduli of the case $\{2,0\}^c$ is 15, with mildest singularity A_3.
(iv) The number of moduli of the case (CG2) is 17, with mildest singularity A_1.
(v) The number of moduli of the cases (CG4) and (CG5) are 15 ands 14 respectively, with mildest singularities $A_1 + A_3$ and A_5 respectively.

$\mathbf{g = 9}$

The general projective model is a complete intersection of 4 hyperplanes in Σ^6_{16}, as described in the beginning of Section 9.

The others are as follows:

c	D^2	scroll type	# mod.	type of L	sing.
1	0	$(3,2,2)$	18	$h^0(L-2D) = 4,\ h^0(L-3D) = 1$	sm.
1	0	$(3,3,1)$	17	$h^0(L-2D) = 4,\ h^0(L-3D) = 2,\ h^0(L-4D) = 0$	sm.
1	0	$(4,2,1)$	16	$h^0(L-2D) = 4,\ h^0(L-3D) = 2,\ h^0(L-4D) = 1$ $^{(i)}$	A_1
1	0	$(5,2,0)$	17	$\{1,0\}^a,\ A^2 = 6$	A_1
2	0	$(2,2,1,1)$	18	$h^0(L-2D) = 2,\ h^0(L-3D) = 0$	sm.
2	0	$(3,1,1,1)$	15	$h^0(L-2D) = 2,\ h^0(L-3D) = 1$	sm.
2	0	$(2,2,2,0)$	17	$\{2,0\}^a,\ A^2 = 2,\ A \not> D$	A_1
2	0	$(3,2,1,0)$	16	$\{2,0\}^a,\ A^2 = 2,\ A > D$ $^{(ii)}$	A_1
2	0	$(3,2,1,0)$	16	$\{2,0\}^a,\ A^2 = 2,\ A > D$ $^{(iii)}$	A_2
2	0	$(4,2,0,0)$	17	$\{2,0\}^b$ $(L \sim 4D + 2\Gamma_1 + 2\Gamma_2)$ or $\{2,0\}^c$ $(L \sim 4D + 2\Delta_0$ $^{(iv)}$	$2A_1$
2	2	$(2,2,1,0,0)$	18	$h^0(L-2D) = 2,\ h^0(L-3D) = 0$	sm.
2	4	$(2,1,1,0,0,0)$	19	$L \sim 2D$	sm.
3	0	$(1,1,1,1,1)$	18	$h^0(L-2D) = 0$	sm.
3	0	$(2,1,1,1,0)$	17	$\{3,0\}^a,\ A^2 = -2$	A_1
3	0	$(2,2,1,0,0)$	16	$\{3,0\}^b$ or $\{3,0\}^c,\ A^2 = 0,\ A \not> D$ $^{(v)}$	$2A_1$
3	0	$(3,1,1,0,0)$	14	$\{3,0\}^b$ or $\{3,0\}^c,\ A^2 = 0,\ A > D$ $^{(iv)\ (vi)}$	$2A_2$
3	2	$(1,1,1,1,0,0)$	18	$h^0(L-2D) = 0$	sm.
3	2	$(2,1,1,0,0,0)$	17	$\{3,2\}^a$	A_1

In the cases $(c, D^2) = (1, 0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + 5\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

.

In the cases $(c, D^2) = (2, 0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$\begin{array}{rcl} 0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H} + (g-1)\mathcal{F}) & \longrightarrow & \\ \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + b_1\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + b_2\mathcal{F}) & \longrightarrow & \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0, \end{array}$$

with $(b_1, b_2) = (2, 2)$ or $(3, 1)$ for the scroll type $(2, 2, 1, 1)$; $(b_1, b_2) = (2, 2)$ for the scroll types $(3, 1, 1, 1)$, $(2, 2, 2, 0)$ and $(3, 2, 1, 0)$; $(b_1, b_2) = (3, 1)$ for the scroll type $(3, 2, 1, 0)$ and $(b_1, b_2) = (4, 0)$ for the scroll type $(4, 2, 0, 0)$. In this latter case S also contains a different perfect Clifford divisor (by the footnote on page 71, so S' can also be described as in the case $(c, D^2) = (2, 2)$ (with perfect Clifford divisor $2D + \Gamma_1 + \Gamma_2$ or $D + \Delta_0$).

In the case $(c, D^2) = (2, 2)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 3\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F})^2 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}).$$

In the case $(c, D^2) = (2, 4)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of:

$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F})^3 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^2 \oplus \mathcal{O}_{\mathcal{T}}(2\mathcal{H})^2.$$

We also have that S' is the 2-uple embedding of the quartic $\phi_D(S)$ if and only if D is not hyperelliptic. If D is hyperelliptic, then S' has a $\mathcal{O}_{\mathcal{T}'}$-resolution as follows:

$$\begin{array}{rcl} 0 \longrightarrow \mathcal{O}_{\mathcal{T}'}(-4\mathcal{H} + (g-1)\mathcal{F}) & \longrightarrow & \\ \mathcal{O}_{\mathcal{T}'}(-2\mathcal{H} + b_1\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}'}(-2\mathcal{H} + b_2\mathcal{F}) & \longrightarrow & \mathcal{O}_{\mathcal{T}'} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0, \end{array}$$

for a suitable scroll $\mathcal{T}'$ in $\mathbf{P}^9$. If D is linearly equivalent to a sum of two smooth elliptic curves intersecting twice, then $(b_1, b_2) = (2, 2)$ or $(4, 0)$, and the type of $\mathcal{T}'$ is $(2, 2, 2, 0)$. Otherwise $(b_1, b_2) = (4, 0)$ and the type of $\mathcal{T}'$ is $(4, 2, 0, 0)$ (see Proposition 7.31).

In the cases $(c, D^2) = (3, 0)$ we have an $\mathcal{O}_{\mathcal{T}}$-resolution of $\mathcal{O}_{S'}$ of the following type:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathcal{T}}(-5\mathcal{H} + 8\mathcal{F}) & \longrightarrow & \oplus_{i=1}^{5}\mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + a_i\mathcal{F}) & & & & \\ & \longrightarrow & \oplus_{i=1}^{5}\mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + b_i\mathcal{F}) & \longrightarrow & \mathcal{O}_{\mathcal{T}} & \longrightarrow & \mathcal{O}_{S'} & \longrightarrow & 0, \end{array}$$

with $a_i = 3 - b_i$, for all i. For the smooth scroll type $(1, 1, 1, 1, 1)$ we have $(b_1, b_2, b_3, b_4) = (2, 1, 1, 1, 1)$ or $(2, 2, 2, 0, 0)$. For the scroll type $(2, 1, 1, 1, 0)$ we have $(b_1, b_2, b_3, b_4) = (2, 1, 1, 1, 1)$, $(2, 2, 1, 1, 0)$ or $(2, 2, 2, 0, 0)$. For the remaining two singular scroll types we have $(b_1, b_2, b_3, b_4) = (2, 2, 1, 1, 0)$ or $(2, 2, 2, 0, 0)$.

In the case $(c, D^2) = (3, 2)$ with scroll type $(1, 1, 1, 1, 0, 0)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of:

$$\begin{array}{lllll} \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F}) & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^5 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^2 \quad \text{or} \\ \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^7 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}) & & \end{array}$$

In the case $(c, D^2) = (3, 2)$ with scroll type $(2, 1, 1, 0, 0, 0)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of:

$$\begin{array}{llllll} \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F})^3 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F}) & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^4 & \text{or} \\ \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F})^2 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^3 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^3. & \end{array}$$

Comments on the types of L:

(i) $L \sim 4D + 2\Gamma_1 + \Gamma_2 + \Gamma_3$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_2 \\ \Gamma_3, & & \end{tikzcd}$$

or $L \sim 4D + 3\Gamma_1 + \cdots + 3\Gamma_N + 2\Gamma_{N+1} + \Gamma_{N+2} + \Gamma_{N+3}$, for $N \geq 1$ (in general $N = 0$) with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] & \Gamma_1 \arrow[r, dash, dashed] & \Gamma_N \arrow[r, dash] \arrow[d, dash] & \Gamma_{N+1} \arrow[r, dash] & \Gamma_{N+2} \\ & & \Gamma_{N+3} & & \end{tikzcd}$$

(ii) $L \sim 3D + 2\Gamma + \Gamma_1 + \Gamma_2$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma \arrow[d, dash] \\ \Gamma_2 & \Gamma_1 \end{tikzcd}$$

(iii) $L \sim 3D + 2\Gamma + \Gamma_1 + \Gamma_2$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, equal] & \Gamma \arrow[r, dash] & \Gamma_1 \\ \Gamma_2 & & \end{tikzcd}$$

or $L \sim 3D + 2\Gamma + \Gamma_1 + \Gamma_2 + \cdots + \Gamma_{N+3}$, for $N \geq 0$ (in general $N = 0$) with the following configuration:

$$\begin{tikzcd} & D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma_2 \arrow[d, dash, dashed] \\ \Gamma_1 \arrow[r, dash] & \Gamma & \Gamma_{N+3} \end{tikzcd}$$

(iv) The number of moduli of the case $\{2,0\}^c$ is 16, with mildest singularity A_3.

(v) The number of moduli of the case $\{3,0\}^c$ is 15, with mildest singularity A_3.

(vi) In the case $\{3,0\}^b$ we have $L \sim 3D + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_3 \\ \Gamma_5 & \Gamma_2 \arrow[r, dash] & \Gamma_4. \end{tikzcd}$$

or $L \sim 3D + 2\Gamma_1 + \Gamma_1' + 3\Gamma_2 + \cdots + 3\Gamma_N + 2\Gamma_{N+1} + \Gamma_{N+2} + \Gamma_{N+3}$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma_2 \arrow[r, dash, dashed] & \Gamma_N \arrow[r, dash] \arrow[d, dash] & \Gamma_{N+1} \arrow[r, dash] & \Gamma_{N+2} \\ \Gamma_1 \arrow[r, dash] & \Gamma_1' & \Gamma_{N+3}. & & \end{tikzcd}$$

In the case $\{3,0\}^c$ we have $L \sim 3D + \Gamma + 4\Gamma_0 + 3\Gamma_1 + 2\Gamma_2 + 2\Gamma_3 + \Gamma_4$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_3 \arrow[d, dash] \\ \Gamma & \Gamma_2 & & \Gamma_4. \end{tikzcd}$$

The mildest singularity of this case is A_5.

$\mathbf{g = 10}$

The general projective model is a complete intersection of 2 hyperplanes in the homogeneous variety Σ^5_{18}, as described in Section 9.

The others are as follows:

c	D^2	scroll type	# mod.	type of L	sing.
1	0	$(3,3,2)$	18	$h^0(L-2D)=5$, $h^0(L-3D)=2$, $h^0(L-4D)=0$	sm.
1	0	$(4,2,2)$	16	$h^0(L-2D)=5$, $h^0(L-3D)=2$, $h^0(L-4D)=1$	sm.
1	0	$(4,3,1)$	17	$h^0(L-2D)=5$, $h^0(L-3D)=3$, $h^0(L-4D)=1$	sm.
1	0	$(5,2,1)$	16	$h^0(L-5D)=1$ $^{(i)}$	A_2
1	0	$(6,2,0)$	18	$\{1,0\}^a$, $L \sim 6D+3\Gamma$	A_1
2	0	$(2,2,2,1)$	18	$h^0(L-2D)=3$, $h^0(L-3D)=0$	sm.
2	0	$(3,2,1,1)$	17	$h^0(L-2D)=3$, $h^0(L-3D)=1$ $^{(ii)}$	A_1
2	0	$(3,2,1,1)$	16	$h^0(L-2D)=3$, $h^0(L-3D)=1$ $^{(iii)}$	sm.
2	0	$(3,2,2,0)$	17	$\{2,0\}^a$, $A^2=4$, $h^0(A-D)=1$, $A \not> 2D$ $^{(iv)}$	A_1
2	0	$(3,3,1,0)$	16	$\{2,0\}^a$, $A^2=4$, $h^0(A-D)=2$, $A \not> 2D$ $^{(v)}$	A_2
2	0	$(4,2,1,0)$	16	$\{2,0\}^a$, $A^2=4$, $A > 2D$ $^{(vi)}$	$2A_1$
2	2	$(3,2,1,0,0)$	19	$L \sim 3D$	sm.
3	0	$(2,1,1,1,1)$	18	$h^0(L-2D)=0$	sm.
3	0	$(2,2,1,1,0)$	17	$\{3,0\}^a$, $A^2=0$	A_1
3	0	$(2,2,2,0,0)$	16	$\{3,0\}^b$, $A^2=2$, $A \not> D$	$2A_1$
3	0	$(2,2,2,0,0)$	15	$\{3,0\}^c$, $A^2=2$, $A \not> D$	A_3
3	0	$(3,2,1,0,0)$	15	$\{3,0\}^b$, $A^2=2$, $A > D$ $^{(vii)}$	$2A_1$
3	0	$(3,2,1,0,0)$	15	$\{3,0\}^b$, $A^2=2$, $A > D$ $^{(viii)}$	A_1+A_2
3	0	$(3,2,1,0,0)$	14	$\{3,0\}^c$, $A^2=2$, $A > D$ $^{(ix)}$	A_3
3	0	$(3,2,1,0,0)$	14	$\{3,0\}^c$, $A^2=2$, $A > D$ $^{(x)}$	A_4
3	2	$(2,1,1,1,0,0)$	18	$h^0(L-2D)=1$	sm.
3	4	$(2,1,1,0,0,0,0)$	18	(E0)	A_1
4	2	$(1,1,1,1,0,0,0)$	18	(CG1)' or (CG2)' $^{(xi)}$	sm.
4	2	$(2,1,1,0,0,0,0)$	16	(CG3)' or (CG4)' $^{(xii)}$	$3A_1$

In the cases $(c, D^2) = (1,0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-3\mathcal{H}+6\mathcal{F}) \longrightarrow \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0.$$

.

In the cases $(c, D^2) = (2, 0)$ $\mathcal{O}_{S'}$ has the following $\mathcal{O}_{\mathcal{T}}$-resolution:

$$\begin{array}{rcl} 0 \longrightarrow \mathcal{O}_{\mathcal{T}}(-4\mathcal{H} + (g-1)\mathcal{F}) & \longrightarrow & \\ \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + b_1\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + b_2\mathcal{F}) & \longrightarrow & \mathcal{O}_{\mathcal{T}} \longrightarrow \mathcal{O}_{S'} \longrightarrow 0, \end{array}$$

with $(b_1, b_2) = (3, 2)$ or $(4, 1)$ for the scroll type $(4, 2, 1, 0)$, $(b_1, b_2) = (3, 2)$ for the scroll types $(2, 2, 2, 1)$, $(3, 2, 1, 1)$ (smooth), $(3, 2, 2, 0)$ and $(3, 3, 1, 0)$, and $(b_1, b_2) = (4, 1)$ for the scroll type $(3, 2, 1, 1)$ (A_1-sing.).

In the case $(c, D^2) = (2, 2)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 4\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 3\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F}) \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}).$$

or of

$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 3\mathcal{F})^3 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H}).$$

In the cases $(c, D^2) = (3, 0)$ we have an $\mathcal{O}_{\mathcal{T}}$-resolution of $\mathcal{O}_{S'}$ of the following type:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathcal{T}}(-5\mathcal{H} + 9\mathcal{F}) & \longrightarrow & \oplus_{i=1}^{5}\mathcal{O}_{\mathcal{T}}(-3\mathcal{H} + a_i\mathcal{F}) & & \\ & \longrightarrow & \oplus_{i=1}^{5}\mathcal{O}_{\mathcal{T}}(-2\mathcal{H} + b_i\mathcal{F}) & \longrightarrow & \mathcal{O}_{\mathcal{T}} \quad \longrightarrow & \mathcal{O}_{S'} & \longrightarrow \quad 0, \end{array}$$

with $a_i = 4 - b_i$ for all i.

For the smooth scroll type $(2, 1, 1, 1, 1)$ we have $(b_1, b_2, b_3, b_4) = (2, 2, 2, 1, 1)$. For the scroll type $(2, 2, 2, 0, 0)$ we have $(b_1, b_2, b_3, b_4) = (2, 2, 2, 1, 1)$ or $(2, 2, 2, 2, 0)$. For the remaining two singular scroll types we have $(b_1, b_2, b_3, b_4) = (b_1, ..., b_4)$ are $(3, 3, 2, 0, 0)$, $(3, 3, 1, 1, 0)$, $(3, 2, 2, 1, 0)$, $(2, 2, 2, 1, 1)$ or $(2, 2, 2, 2, 0)$.

In the case $(c, D^2) = (3, 2)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F})^3 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^4 \oplus 2\mathcal{O}_{\mathcal{T}_0}(\mathcal{H}).$$

In the case $(c, D^2) = (3, 4)$ then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

$$\mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F})^3 \oplus \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^4 \oplus \mathcal{O}_{\mathcal{T}}(2\mathcal{H})^5.$$

In the cases (CG1)' and (CG2)' then S'' is cut out in $\mathcal{T}_0$ by a section of

$$\begin{array}{lllllll} \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F}) & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^7 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^5 & \text{or} & \\ \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^9 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^4 & \text{or} & & & \\ \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F}) & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^8 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^3 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} + \mathcal{F}), \end{array}$$

where the last option occurs only for the case (CG2)' (the section is constant on the fibers of i in the first four cases).

In the cases (CG3)', (CG4)' then S'' is cut out in $\mathcal{T}_0$ by a section (which is constant on the fibers of i) of

$$\begin{array}{llllll} \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F})^2 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^5 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^6 & \text{or} \\ \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - 2\mathcal{F})^3 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H} - \mathcal{F})^3 & \oplus & \mathcal{O}_{\mathcal{T}_0}(2\mathcal{H})^7 & \end{array}$$

Comments on the types of L:

(i) $L \sim 5D + 3\Gamma_1 + 2\Gamma_2 + \Gamma_3$, with the following configuration:

$$D \text{ --- } \Gamma_1 \text{ --- } \Gamma_2 \text{ --- } \Gamma_3$$

(ii) L is in general of the form $L \sim 3D + 2\Gamma_1 + \Gamma_2$, with the following configuration:

$$D = \Gamma_1 \text{ --- } \Gamma_2$$

(iii) L is in general of the form $L \sim 3D + \Gamma_1 + \Gamma_2 + \Gamma_3$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, equal] \arrow[d, dash] \arrow[dr, dash] & \Gamma_1 \\ \Gamma_2 & \Gamma_3 \end{tikzcd}$$

(iv) L is in general of the form $L \sim 3D + 2\Gamma + \Gamma_1$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[dr, equal] & \Gamma \arrow[d, dash] \\ & \Gamma_1 \end{tikzcd}$$

(v) $L \sim 3D + E + 2\Gamma + \Gamma_1$, where E is a smooth elliptic curve, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, equal] & \Gamma \arrow[r, dash] & \Gamma_1 \\ E & & \end{tikzcd}$$

(vi) L is in general of the form $L \sim 4D + 2\Gamma + 2\Gamma_1 + \Gamma_2$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_2 \\ \Gamma & & \end{tikzcd}$$

(vii) L is in general of the form $L \sim 3D+2\Gamma_1+2\Gamma_2+\Gamma_3+\Gamma_4$, with the following configuration:

$$\begin{tikzcd} & D \arrow[r, dash] \arrow[dl, dash] \arrow[d, dash] & \Gamma_1 \arrow[d, dash] \\ \Gamma_3 & \Gamma_2 \arrow[r, dash] & \Gamma_4 \end{tikzcd}$$

(viii) L is in general of the form $L \sim 3D+2\Gamma_1+2\Gamma_2+\Gamma_3+\Gamma_4$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma_1 \arrow[r, dash] & \Gamma_4 \\ \Gamma_3 \arrow[r, dash] & \Gamma_2 & \end{tikzcd}$$

(ix) L is in general of the form $L \sim 3D + 4\Gamma_0 + 2\Gamma_1 + 2\Gamma_2 + \Gamma_3 + \Gamma_4$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma_0 \arrow[r, dash] \arrow[d, dash] \arrow[dr, dash] & \Gamma_3 \\ \Gamma_4 & \Gamma_1 & \Gamma_2 \end{tikzcd}$$

(x) L is in general of the form $L \sim 3D + 4\Gamma_0 + 3\Gamma_1 + 2\Gamma_2 + 2\Gamma_3 + \Gamma_4$, with the following configuration:

$$\begin{tikzcd} D \arrow[r, dash] \arrow[d, dash] & \Gamma_0 \arrow[r, dash] \arrow[dr, dash] & \Gamma_2 \\ \Gamma_4 \arrow[r, dash] & \Gamma_3 \arrow[r, dash] & \Gamma_1 \end{tikzcd}$$

(xi) The number of moduli of the case (CG2)' is 17, with mildest singularity A_1.

(xii) The number of moduli of the case (CG4)' is 15, with mildest singularity $A_1 + A_3$.

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