STABILITY OF DEGENERATE ABELIAN VARIETIES

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Dedicated to Professor Kunihiko Kodaira

Abstract. We compactify canonically the moduli scheme of abelian schemes over $\mathbf{Z}[\zeta_N, 1/N]$ by introducing the noncommutative level structures. Any degenerate abelian scheme on the boundary of the compactification is one of our models – projectively stable quasi-abelian schemes. We prove that a degenerate abelian scheme has Kempf-stable Hilbert points if and only if it is a projectively stable quasi-abelian scheme.

0. Introduction.

The purpose of the present article is to compactify canonically the moduli scheme of abelian varieties in a way similar to the Deligne-Mumford compactification by stable curves of the moduli scheme of curves [DM69]. The problem has been considered in the complex analytic case by [Namikawa76], Ueno [unpublished] and [Nakamura75]. Recently modifying these degenerate abelian varieties slightly, Alexeev and the author [AN96] defined (torically) stable quasi-abelian varieties over any field and proved a stable reduction theorem over any complete discrete valuation ring.

In the present article we introduce the notion of polarized projectively stable quasi-abelian schemes by modifying slightly those defined in [AN96] and prove Kempf-stability of their normalized Hilbert points (Theorem 0.2) and that Kempf-stability characterizes these schemes conversely (Theorem 0.3). We also prove the existence of the projective *reduced-fine-moduli scheme* of projectively stable quasi-abelian schemes with certain level structures over $\mathbf{Z}[\zeta_N, 1/N]$ where ζ_N is a primitive N-th root of unity and $N \geq 3$. This is a natural geometric compactification of a moduli scheme of abelian varieties with level structures (Theorem 0.4). See [Alexeev96] for the case of principally polarized (torically) stable quasi-abelian varieties.

If we are given Faltings-Chai's degeneration data [FC90] there are two canonical choices of flat projective degenerating families of abelian varieties $(P, \mathcal{L})$ and $(Q, \mathcal{L})$ where $(Q, \mathcal{L})$ is the most naive choice and $(P, \mathcal{L})$ is the

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normalization of $(Q, \mathcal{L})$ after some base change. The following is the stable reduction theorem of abelian varieties proved in [AN96].

Theorem 0.1. *Let R be a complete discrete valuation ring with the fraction field $k(\eta)$ and $(G_\eta, \mathcal{L}_\eta)$ a polarized abelian variety over $k(\eta)$. Then after a suitable finite ramified cover* $\mathrm{Spec}\, R' \to \mathrm{Spec}\, R$ *it can be completed to a flat projective scheme $(P, \mathcal{L})$ (or $(Q, \mathcal{L})$) over R' with a relatively ample invertible sheaf $\mathcal{L}$ extending $\mathcal{L}_\eta$.*

See Section 10 for the Stable reduction theorem we need in the present article. We call the closed fibre $(P_0, \mathcal{L}_0)$ a torically stable quasi-abelian variety (abbr. SQAV), while we call $(Q_0, \mathcal{L}_0)$ a projectively stable quasi-abelian scheme (abbr. PSQAS). We note $(P_0, \mathcal{L}_0) \simeq (Q_0, \mathcal{L}_0)$ if the dimension is less than five. $(P_0, \mathcal{L}_0)$ is always reduced, while $(Q_0, \mathcal{L}_0)$ can be nonreduced if the dimension is greater than four. $(Q_0, \mathcal{L}_0)$ determines $(P_0, \mathcal{L}_0)$ uniquely but the converse is unknown.

By applying [Kempf78, Corollary 5.1] we will prove

Theorem 0.2. *Let $(Q_0, \mathcal{L}_0)$ be a projectively stable quasi-abelian scheme over the residue field $k := R/I$, I the maximal ideal of R. Suppose that k is algebraically closed field and the characteristic of k and $N := \deg \mathcal{L}_0/(g!)$ are coprime. If $\Gamma(Q, \mathcal{L}) \otimes k$ is very ample on Q_0, then the n-th normalized Hilbert point of $(Q_0, \mathcal{L}_0)$ has a closed* $\mathrm{SL}(N, k)$*-orbit, and it is Mumford-semistable for any large n.*

Theorem 0.3 shows that no similar theorem for $(P_0, \mathcal{L}_0)$ except semistability can be expected when $P_0 \neq Q_0$. See also Remark 8.4. See Section 8 for the definition of normalized Hilbert points.

Let (K, e_K) a finite symplectic abelian group, H a maximally isotropic subgroup of K with respect to e_K and N the order of H. Then $\mathrm{ord}\, K = N^2$. By the uniqueness of H up to isomorphism, we define $e_{\min}(K) = e_1$ and $e_{\max}(K) = e_g$ if $H \simeq \oplus_{i=1}^g (\mathbf{Z}/e_i\mathbf{Z})$, $e_i | e_{i+1}$ for some positive integers e_i.

Theorem 0.3. *Let k be an algebraically closed field, (K, e_K) a finite symplectic abelian group with $e_{\min}(K) \geq 3$ and $N = \sqrt{\mathrm{ord}\, K}$. Assume that the characteristic of k and N are coprime. Suppose that a k-scheme (Z, L) is smoothable into an abelian variety (A, M) with $(\ker \lambda(M), e^M) \simeq (K, e_K)$ where e^M is the Weil pairing of M. Then the following are equivalent;*

1. *(Z, L) is a projectively stable quasi-abelian scheme,*
2. $\mathrm{Aut}(Z, L)$ *contains a subgroup of* $\mathrm{SL}_{\pm}(N, k)$ *(conjugate) isomorphic to $G(K)$,*
3. *the n-th normalized Hilbert point of (Z, L) has a closed* $\mathrm{SL}(N, k)$*-orbit for any large n*

where $\mathrm{SL}_{\pm}(N, k) = \{g \in \mathrm{GL}(N, k); \det g = \pm 1\}$ *and $G(K)$ is the finite Heisenberg group.*

Compare [Gieseker82] and [Mumford77]. We say that the n-th normalized Hilbert point of $(Q_0, \mathcal{L}_0)$ is *Kempf-stable* if it has a closed $\mathrm{SL}(N, k)$-orbit.

Let $M = e_{\max}(K)$. The group $G(K)$ is noncommutative of order MN^2 and it is a central extension of K by μ_M with commutator form e_K where μ_M is the group of all M-th roots of unity. The group algebra $k[H^\vee]$ of the dual group $H^\vee$ over k is an irreducible $G(K)$-module of weight one, called the Schrödinger representation of $G(K)$. The k-module $\Gamma(Q,\mathcal{L})\otimes k$ is isomorphic to $k[H^\vee]$ as a $G(K)$-module. If $e_{\min}(K) \geq 3$, any PSQAS $(Q_0, \mathcal{L}_0)$ over k is embedded into the projective space $\mathbf{P}(k[H^\vee])$ (Theorem 6.3). Hence any PSQAS $(Q_0, \mathcal{L}_0)$ over k has a linearlized action of $G(K)$. Thus we are led to the notion of rigid $G(K)$-structures and level $G(K)$-structures on PSQAS's generalizing the classical level structures on abelian varieties in order to formulate the moduli problem for PSQAS's.

Theorem 0.4. *Let K be a symplectic finite abelian group with $e_{\min}(K) \geq 3$. Let $M = e_{\max}(K)$. The functor of g-dimensional projectively stable quasi-abelian schemes with level $G(K)$-structures is reductively-represented by a projective scheme $SQ_{g,K}$ over $\mathbf{Z}[\zeta_M, 1/M]$.*

We prove Theorem 0.4 with the help of Theorem 0.3 and Schur's lemma for irreducible $G(K)$-modules of weight one.

One dimensional PSQAS's with rigid $G((\mathbf{Z}/3\mathbf{Z})^{\oplus 2})$-structures are Hesse cubic curves coupled with the Schrödinger representation of $G((\mathbf{Z}/3\mathbf{Z})^{\oplus 2})$. The well known family of Hesse cubic curves over $\mathbf{P}^1_{\mathbf{Z}[\zeta_3,1/3]}$

$$\mu_0(x_0^3 + x_1^3 + x_2^3) - \mu_1 x_0 x_1 x_2 = 0$$

is the universaly family of PSQAS's with rigid $G((\mathbf{Z}/3\mathbf{Z})^{\oplus 2})$-structures. In particular, $SQ_{1,(\mathbf{Z}/3\mathbf{Z})^{\oplus 2}} \simeq \mathbf{P}^1_{\mathbf{Z}[\zeta_3,1/3]}$. The Hesse cubic curves exhaust all the possible Kempf-stable cubic curves up to PGL(3)-equivalence. Theorems 0.3 and 0.4 generalize these facts.

In Section 1 we will discuss Delaunay decompositions. In Section 2 we recall the basic facts about a degenerating family of abelian varieties. In Sections 3 and 4 we recall the degeneration data of a degenerating family of abelian varieties following [FC90] and the construction of $(P_0, \mathcal{L}_0)$. In Section 5 we will describe the structures of $(Q_0, \mathcal{L}_0)$. In Section 6 we prove that $\Gamma(Q,\mathcal{L}) \otimes k$ is very-ample if $e_{\min}(K) \geq 3$, which is a natural generalization of Lefschetz's theorem for abelian varieties. In Section 7 we define group schemes $\mathcal{G}(K)$, $G(K)$ and a $G(K)$-module $V(K)$ and discuss central extensions of K by $\mathbf{G}_m$. In Section 8 we prove Theorem 0.2. In Sections 9 we define rigid $G(K)$-structures and level $G(K)$-structures on a PSQAS or their family. In Section 10 we prove Stable reduction theorem for PSQAS's with rigid $G(K)$-structures. In Sections 11 and 12 we prove Theorem 0.3 and Theorem 0.4. We will present two basic examples of the reduced-fine-moduli schemes of one dimensional PSQAS's in Section 13.

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1. Delaunay decompositions

Definition 1.1. Let $\mathbf{Z}_0$ (resp. $\mathbf{Z}_+$, $\mathbf{Q}_0$, $\mathbf{R}_0$) be the set of all nonnegative integers (resp. positive integers, nonnegative rational numbers, nonnegative real numbers). Let X be a lattice of rank g, $X_{\mathbf{R}} = X \otimes \mathbf{R}$, and let $B : X \times X \to \mathbf{Z}$ be a positive definite symmetric integral bilinear form $B : X \times X \to \mathbf{Z}$, which determines a distance $\| \ \|_B$ on the Euclidean space $X_{\mathbf{R}}$ by $\|x\|_B := \sqrt{B(x,x)}$ $(x \in X_{\mathbf{R}})$. For an arbitrary $\alpha \in X_{\mathbf{R}}$ we say that a lattice element $a \in X$ is α-nearest if

$$\|a - \alpha\|_B = \min\{\|b - \alpha\|_B; b \in X\}$$

We define a (closed) *B-Delaunay cell* σ (or simply a Delaunay cell if B is understood) to be the closed convex hull of all lattice elements which are α-nearest for some $\alpha \in X_{\mathbf{R}}$. Note that for a given Delaunay cell σ, α is uniquely defined only if σ has the maximal possible dimension, equal to g. In this case we call α *the centre* of σ. Together all the B-Delaunay cells constitute a locally finite decomposition of $X_{\mathbf{R}}$ into infinitely many bounded convex polyhedra which we call the *B-Delaunay decomposition* Del_B. It is clear from the definition that the Delaunay decomposition is invariant under translation by the lattice X and that the 0-dimensional cells are precisely the elements of X.

Let $\mathrm{Del} := \mathrm{Del}_B$, $\mathrm{Del}(c)$ the set of all the Delaunay cells containing $c \in X$, and $\mathrm{Del}^{(i)}(c)$ the the set of all i-dimensional Delaunay cells in $\mathrm{Del}(c)$. Let $\sigma \in \mathrm{Del}$ be a Delaunay cell, and $n\sigma := \{x \in X_{\mathbf{R}}; x/n \in \sigma\}$. Let $\mathrm{Star}\,(0)$ be the union of all $\sigma \in \mathrm{Del}$ such that $\sigma \in \mathrm{Del}(0)$, and $n\,\mathrm{Star}\,(0) := \{x \in X_{\mathbf{R}}; x/n \in \mathrm{Star}\,(0)\}$. For $\sigma \in \mathrm{Del}(0)$, let $C(0,\sigma)$ be the closed cone over $\mathbf{R}_+$ generated by $\sigma \cap X$.

A g-dimensional cone $\sigma \in \mathrm{Del}$ is called *totally generating* if the semi group $C(0,\sigma) \cap X$ is generated by Delaunay vectors $\sigma \cap X$. If a g-dimensional cone $\sigma \in \mathrm{Del}$ is the union of totally generating g-dimensional Delaunay cones with disjoint interiors, then we call σ *simplicially generating.* See [AN96].

Lemma 1.2. *Let B be a symmetric positive definite integral bilinear form on $X \times X$, Del_B the corresponding Delaunay decomposition of $X_{\mathbf{R}}$ and z $(\neq 0) \in X$.*

1. *Let $\sigma \in \mathrm{Del}_B$ be a Delaunay cell. Then $\sigma \cap (nz + \sigma) = \emptyset$ for $n \geq 2$.*
2. $\mathrm{Star}\,(0) \cap (nz + \mathrm{Star}\,(0)) = \emptyset$ *if $n \geq 3$.*

Proof. Let $\alpha(\sigma)$ be the centre of σ. Suppose that $c \in \sigma \cap X$ and $d = c + nz \in \sigma$ for some nonzero $z \in X$. Since σ is convex, $c + z \in \sigma$. Hence $\|c - \alpha(\sigma)\|_B =$

$\|c+z-\alpha(\sigma)\|_B = \|c+nz-\alpha(\sigma)\|_B$. It follows $(n^2-n)B(z,z)=0$, a contradiction. This proves (1).

Let $z \in X$. We choose all the maximal dimensional Delaunay cells σ_i $(i \in I)$ in Del_B containing 0 such that z belongs to the closed cone $C(0,\sigma_i)$ $(i \in I)$. Let C be the union of $C(0,\sigma_i)$ $(i \in I)$. Then z belongs to the interior of C and $z+C \subset C$ because $z+C(0,\sigma_i) \subset C(0,\sigma_i)$. Hence $nz+C \subset C$ for any $n \geq 1$. Let σ be any maximal dimensional Delaunay cell in Star (0). Then $z \in z+\sigma$, hence the interior of $z+\sigma$ contains an interior point of C. This implies that $z+\sigma \subset C$, since C is the union of (infinitely many) Delaunay cells. It follows that $z+\mathrm{Star}\,(0) \subset C$, hence $nz+\mathrm{Star}\,(0) \subset (n-1)z+C$ for any $n \geq 1$. Meanwhile we have

$$\mathrm{Star}\,(0) \cap C = \cup_{i\in I}\,\mathrm{Star}\,(0) \cap C(0,\sigma_i) = \cup_{i\in I}\sigma_i$$
$$\mathrm{Star}\,(0) \cap (2z+C) = \mathrm{Star}\,(0) \cap C \cap (2z+C) = \cup_{i\in I}\sigma_i \cap (2z+C)$$

$$\begin{aligned}\sigma_i \cap (2z+C) &= \sigma_i \cap C(0,\sigma_i) \cap (2z+C)\\ &= \sigma_i \cap (2z+C(0,\sigma_i)) = \sigma_i \cap (2z+\sigma_i) = \emptyset\end{aligned}$$

by (1). Therefore $\mathrm{Star}\,(0) \cap (3z+\mathrm{Star}\,(0)) \subset \mathrm{Star}\,(0) \cap (2z+C) = \emptyset$. Similarly $\mathrm{Star}\,(0) \cap (nz+\mathrm{Star}\,(0)) \subset \mathrm{Star}\,(0) \cap ((n-1)z+C) = \emptyset$ for $n \geq 3$. $\square$

Lemma 1.3. *Let $\sigma \in \mathrm{Del}_B$ be a maximal Delaunay cell containing the origin 0. Let $\sigma \cap X = \{a_i; 0 \leq i \leq r\}$, $a_0 = 0$ and $y = \sum_{i=1}^r n_i a_i$, $n_i \in \mathbf{Z}_0$. Then*

1. *$B(a_i,a_j) \geq 0$. $B(a_i,a_j) = 0$ if and only if $a_i + a_j \in \sigma \cap X$.*
2. *$B(y,y) \geq \sum_{i=1}^r n_i^2 B(a_i,a_i) \geq \sum_{i=1}^r n_i B(a_i,a_i) = 2B(\alpha(\sigma),y)$.*
3. *$B(y,y) = 2B(\alpha(\sigma),y)$ holds if and only if $y \in \sigma \cap X$.*
4. *$B(y,y) > 2B(\alpha(\sigma),y)$ if some $n_i \geq 2$.*

Proof. Since $B(a_i+a_j,a_i+a_j) \geq 2B(\alpha(\sigma),a_i+a_j) = B(a_i,a_i)+B(a_j,a_j)$, equality holds if and only if $a_i+a_j \in \sigma \cap X$. (2) follows from (1). And (3) and (4) follow from (2). $\square$

Lemma 1.4. *Let $\sigma \in \mathrm{Del}(0)$ and $b \in C(0,\sigma) \cap X$. If $b \notin \sigma \cap X$, then there is $a \in \sigma \cap X$ such that $B(b-a,a) > 0$.*

Proof. Let $b \in C(0,\sigma) \cap X$. We assume $B(b-a,a) \leq 0$ for any $a \in \sigma \cap X$. Then we prove $b \in \sigma \cap X$. For this let $b = \sum_{i=1}^r r_i a_i$ for $a_i \in \sigma \cap X$ and some $r_i \in \mathbf{Q}_0$. We see

$$B(b,b) = \sum_{i=1}^r r_i B(b,a_i) \leq \sum_{i=1}^r r_i B(a_i,a_i) = 2B(\alpha(\sigma),b)$$

whence $B(b,b) = 2B(\alpha(\sigma),b)$. It follows $b \in \sigma \cap X$. $\square$

1.5. **Reduction theory.** Let $\mathbf{R}_+$ be the set of nonnegative real numbers. We denote by $\mathcal{Y}$ (resp. $\mathcal{Y}_{\geq 0}$, $\mathcal{Y}_+$) the set of all symmetric (resp. symmetric positive semi-definite, symmetric positive definite) $g \times g$ matrices. The group

$\mathrm{GL}(g, \mathbf{R})$ acts on $\mathcal{Y}$ and keeping $\mathcal{Y}_{\geq 0}$ and $\mathcal{Y}_+$ stable as $y \mapsto uyu^t$. It is known that the action on $\mathcal{Y}_+$ is transitive.

For $g \leq 4$, [Igusa67, Lemmas 4,5] gives a fundamental domain F of $\mathcal{Y}_+$ with respect to the action of $\mathrm{GL}(g, \mathbf{Z})$ explicitly. If $g = 2, 3$, then let

$$C = C_g := \{y = (y_{ij}) \in \mathcal{Y}_+; \sum_{j=1}^{g} y_{ij} \geq 0, y_{ij} \leq 0 \ (i \neq j)\}$$

Then C is a fundamental domain of $\mathcal{Y}_+$ with respect to $\mathrm{GL}(g, \mathbf{Z})$, and any Delaunay decomposition is subdivided by (a $\mathrm{GL}(g, \mathbf{Z})$-conjugate of) $\mathrm{Del}(C)$. Since any maximal-dimensional cell $\sigma \in \mathrm{Del}(C)$ containing 0 has volume $1/(g+1)!$, it is totally generating. Hence any maximal-dimensional cell of any g $(= 2, 3)$-dimensional Delaunay decomposition Del_B is simplicially generating.

If $g = 4$, then with the notation of [Igusa67, Lemmas 4,5] the fundamental domain F is covered with C, F_{ab}, $F_{ab,cd}$ or their transforms by $\mathrm{GL}(4, \mathbf{Z})$. Let Del be one of the Delaunay decompositions $\mathrm{Del}(C)$, $\mathrm{Del}(F_{ab})$ and $\mathrm{Del}(F_{ab,cd})$. Then by [Voronoi09] any maximal-dimensional Delaunay cell $\sigma \in \mathrm{Del}$ has volume $1/5!$, whence it is totally generating. If B is a symmetric positive definite integral 4×4-matrix, then we may assume B to belong to one of the closed convex cones C, F_{ab} and $F_{ab,cd}$ up to $\mathrm{GL}(4, \mathbf{Z})$-conjugacy. If B is not in the interior of the cones, then one of $\mathrm{Del}(C)$, $\mathrm{Del}(F_{ab})$ and $\mathrm{Del}(F_{ab,cd})$ is a subdivision of Del_B, hence Del_B is simplicially generating.

Theorem 1.6. *If the dimension is less than five, any Delaunay decomposition is simplicially generating, in particular the nilpotency* [AN96, Definition 1.15] *of any Delaunay decomposition is equal to one.*

2. The flat closure of $K(\mathcal{L}_\eta)$

The purpose of this section is to recall the Raynaud sequence and the flat closure of a certain finite commutative group scheme $K(\mathcal{L}_\eta)$ for a degenerating family of abelian varieties. We quote these from Faltings-Chai [FC90, II.4.1,5.1] and Moret-Bailly [MB85, IV-VI]. We follow mainly the notation of [FC90].

2.1. **Grothendieck's stable reduction.** Let R be a complete discrete valuation ring, I the maximal ideal of R and $S = \mathrm{Spec}\ R$. Let η be the generic point of S, $k(\eta)$ the fraction field of R and $k = R/I$ the residue field.

Suppose that we are given an abelian scheme G_η over $k(\eta)$. Then by Grothendieck's stable reduction theorem [SGA7] (See also [SGA7, Expóse I pp. 1-24]) G_η can be extended to a semiabelian S-scheme G as the connected Néron model of G_η by taking a finite extension K' of $k(\eta)$ if necessary, [1] in other words, by taking the integral closure of R in K' and by repeating the same construction. Moreover by taking a finite extension of $k(\eta)$ if

[1]We mean by the connected Néron model the identity connected component of the Néron model (with connected closed fibre).

necessary there exists $H \in \mathrm{Pic}^0(G_\eta)$ such that $\mathcal{L}_\eta \otimes H$ is symmetric, namely $i^*(\mathcal{L}_\eta \otimes H) = \mathcal{L}_\eta \otimes H$ for the involution $i = [-1]_{G_\eta}$ of G_η.

Therefore we may assume from the start that $\mathcal{L}_\eta$ is symmetric, ample and rigidified along the unit section. The invertible sheaf $\mathcal{L}_\eta$ extends to G because it is associated with some Cartier divisor, which always extends to a smooth scheme G. The extension of $\mathcal{L}_\eta$ is unique because G_0 is irreducible. See [MB85, II, 1.1]. On the other hand by [Raynaud70, p.158 XI, 1.2 and p.170 XI 1.13] $\mathcal{L}_\eta^{\otimes n}$ extends to G as an ample invertible sheaf for some $n > 0$ if $\mathcal{L}_\eta$ is symmetric and ample. Since $\mathcal{L}_\eta$ satisfies the condition in this case, the extension $\mathcal{L}$ is symmetric and ample.

Thus for a given an abelian scheme G_η over $k(\eta)$ of relative dimension g we have a semiabelian S-scheme G with generic fibre G_η (with a chosen unit section), $\mathcal{L}$ a rigidified relatively ample invertible sheaf on G. The special fibre G_0 is a semiabelian scheme over k, namely an extension of an abelian scheme A_0 of relative dimension g' by a torus T_0 of relative dimension g'', $g' + g'' = g$. We may assume T_0 to be split by taking a finite base change of S.

2.2. **The Raynaud sequence.** Associated to G and $\mathcal{L}$ are the formal scheme $G_{\mathrm{for}} = \lim G \otimes R/I^n$ and an invertible sheaf $\mathcal{L}_{\mathrm{for}} = \lim \mathcal{L} \otimes R/I^n$. The scheme G_{for} fits into an exact sequence

$$1 \to T_{\mathrm{for}} \to G_{\mathrm{for}} \stackrel{\pi_{\mathrm{for}}}{\to} A_{\mathrm{for}} \to 0 \tag{1}$$

where A_{for} is a formal abelian scheme. By the theory of cubical structures [Breen83] [MB85, II, 1.1 (ii)] there exists a unique cubical structure on $\mathcal{L}$ (viewed as a $\mathbf{G}_m$-torsor), which induces a cubical structure of the sheaf $\mathcal{L}_{\mathrm{for}}$. Then $\mathcal{L}_{\mathrm{for}}$ is descended to a unique cubical ample invertible sheaf $\mathcal{M}_{\mathrm{for}}$ on A_{for} because T_{for} is a split torus, that is, $\mathcal{L}_{\mathrm{for}} = \pi_{\mathrm{for}}^*(\mathcal{M}_{\mathrm{for}})$. Since there exists an ample sheaf on A_{for}, A_{for} is algebraizable. In other words by [EGA, III, 5.4.5] there exists an abelian S-scheme A with an ample invertible sheaf $\mathcal{M}$ such that the formal completion of $(A, \mathcal{M})$ along a closed fibre is $(A_{\mathrm{for}}, \mathcal{M}_{\mathrm{for}})$.

By our assumption that T_0 is a k-split torus, T_{for} is a formal S-split torus by [SGA3, IX, 3.6], [FC90, Chap. I, 2.2]. Let $X := \mathrm{Hom}_{\mathbf{Z}}(T_{\mathrm{for}}, \mathbf{G}_{m,\mathrm{for}})$ be the character group of T_{for} and $T := \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{G}_{m,S})$. Then T algebraizes T_{for}.

The sequence $1 \to T_{\mathrm{for}} \to G_{\mathrm{for}} \to A_{\mathrm{for}} \to 0$ is algebraizable because we have $\mathrm{Ext}(A_{\mathrm{for}}, T_{\mathrm{for}}) \simeq \mathrm{Ext}(A, T)$ [FC90, p.34]. The dual abelian scheme G_η^t is also extended to a semiabelian S-scheme G^t in a similar manner. Moreover there is a similar exact sequence for G_{for}^t; $1 \to T_{\mathrm{for}}^t \to G_{\mathrm{for}}^t \to A_{\mathrm{for}}^t \to 0$. Then G_{for}^t, T_{for}^t and A_{for}^t are algebraizable. Let $Y := \mathrm{Hom}(T^t, \mathbf{G}_{m,S})$. Thus there exist two group schemes $\widetilde{G}$ and $\widetilde{G}^t$ with $(\widetilde{G})_{\mathrm{for}} \simeq G_{\mathrm{for}}$ and $(\widetilde{G}^t)_{\mathrm{for}} \simeq G_{\mathrm{for}}^t$ and the following exact sequences (called Raynaud sequences) which algebraize the above sequences of the formal group schemes

$$1 \to T \to \widetilde{G} \xrightarrow{\pi} A \to 0 \tag{2}$$

$$1 \to T^t \to \widetilde{G}^t \xrightarrow{\pi^t} A^t \to 0 \tag{3}$$

plus the homomorphisms $c : X \to A^t(S)$, $c^t : Y \to A(S)$ decoding them. In other words, $c \in \mathrm{Hom}(X, A^t(S)) \simeq \mathrm{Ext}(A, T)$ and $c^t \in \mathrm{Hom}(Y, A(S)) \simeq \mathrm{Ext}(A^t, T^t)$ describe the extension classes of semiabelian schemes $\widetilde{G}$ and $\widetilde{G}^t$ respectively. Moreover A^t is a dual abelian scheme of A by [FC90, p. 35]. See also [MB81, Appendice, Théoréme].

Let $\lambda(\mathcal{L}_\eta) : G_\eta \to G^t_\eta$ be the polarization morphism. Then by the universal property of the (connected) Néron model G^t of G^t_η we have an extension $\lambda : G \to G^t$ of $\lambda(\mathcal{L}_\eta)$. Since G is semi-stable, λ is surjective, flat and quasi-finite by [MB85, IV, 3.4]. This gives rise to a formal surjective flat morphism $\lambda_{\mathrm{for}} : G_{\mathrm{for}} \to G^t_{\mathrm{for}}$, which induces a formal morphism $\widetilde{\lambda}_{\mathrm{for}} : \widetilde{G}_{\mathrm{for}} \to \widetilde{G}^t_{\mathrm{for}}$ as well as surjective formal morphisms $\lambda_{T_{\mathrm{for}}} : T_{\mathrm{for}} \to T^t_{\mathrm{for}}$ and $\lambda_{A_{\mathrm{for}}} = \lambda(\mathcal{M}_{\mathrm{for}}) : A_{\mathrm{for}} \to A^t_{\mathrm{for}}$. Hence we have an injective homomorphism $\phi : Y \to X$ and we can algebraize $\lambda_{T_{\mathrm{for}}}$ into a morphism $\lambda_T : T \to T^t$. We also have a morphism $\lambda_A := \lambda(\mathcal{M}) : A \to A^t$ which algebraizes $\lambda_{A_{\mathrm{for}}}$. Since $\widetilde{\lambda}_{\mathrm{for}}$ is a formal morphism, it follows that $c_{\mathrm{for}}\phi = \lambda_{A_{\mathrm{for}}} c^t_{\mathrm{for}}$, hence $c\phi = \lambda_A c^t$. Thus $\widetilde{\lambda}_{\mathrm{for}}$ is algebraized into a morphism $\widetilde{\lambda} : \widetilde{G} \to \widetilde{G}^t$ compatible with λ_T and λ_A, that is, $\lambda_T = \widetilde{\lambda}_{|T}$, $\lambda_A\pi = (\pi^t)\widetilde{\lambda}$.

2.3. **Flat closures.** Let ${}_nG := \ker(n \cdot \mathrm{id}_G : G \to G)$, and ${}_nT := \ker(n \cdot \mathrm{id}_T : T \to T)$. Since G is smooth, $n\,\mathrm{id}_G$ is S-flat for any positive integer n, therefore ${}_nG$ is a flat closed subscheme of G. Hence ${}_nG$ is the flat closure of ${}_nG_\eta$ in G ([EGA, IV, 2.8.5]). There exists the maximal finite flat subgroup scheme ${}_nG^f$ of ${}_nG$ by [EGA, II, 6.2.6]. In fact, since ${}_n\widetilde{G}$ is S-finite and ${}_nG_{\mathrm{for}} \simeq {}_n\widetilde{G}_{\mathrm{for}}$, ${}_nG_{\mathrm{for}}$ is a finite formal subgroup scheme ${}_nG_{\mathrm{for}}$ of G_{for}. There exists by [EGA, III, 5.1.8] an S-subscheme ${}_nG^f$ of a quasi-projective S-scheme G such that $({}_nG^f)_{\mathrm{for}} \simeq {}_nG_{\mathrm{for}} \simeq {}_n\widetilde{G}_{\mathrm{for}}$. It is clear that ${}_nG^f$ is maximal.

Similarly we see that there exists a finite flat subgroup scheme ${}_nG^m$ of ${}_nG$ such that ${}_nG^m_{\mathrm{for}} \simeq {}_nT_{\mathrm{for}} \subset {}_n\widetilde{G}_{\mathrm{for}}$ where ${}_nT_0 := T_0 \cap {}_nG$. We see that ${}_nG^m$ is maximal among finite flat subgroup S-schemes H of G with $H_0 \subset {}_nT_0$. In fact, since H_{for} is S-finite (hence S-prjective), the image $\bar{H}_{\mathrm{for}}$ of H_{for} in A_{for} is also an S-finite projective group subscheme of A_{for}. Since $\bar{H}_0$ is a reduced single point of A_0 by $H_0 \subset T_0$, $\Gamma(O_{\bar{H}_{\mathrm{for}}})$ is an R-module of rank at most one, hence $\bar{H}_{\mathrm{for}}$ is trivial. It follows that $H_{\mathrm{for}} \subset T_{\mathrm{for}}$. The Cartier dual of H_{for} is therefore a finite abelian group, a quotient group of $X = \mathrm{Hom}(T_{\mathrm{for}}, \mathbf{G}_m)$ of finite order, where we recall that the Cartier dual of H_{for} is given by $\mathrm{Hom}_R(H_{\mathrm{for}}, \mathbf{G}_{m,\mathrm{for}})$. See [Oort66, Theorem 16.1]. Hence there exists a sublattice Z of X of finite index such that $H_{\mathrm{for}} = \mathrm{Hom}(X/Z, \mathbf{G}_{m,S}) = \mathrm{Spf}\ R[w^x, x \in X]/(w^z - 1, z \in Z)$. Since $nX \subset Z$ by the assumption $H_0 \subset {}_nG_0^m$, we see $H_{\mathrm{for}} \subset {}_nG_{\mathrm{for}}^m$, hence $H \subset {}_nG^m$. This proves that ${}_nG^m$ is maximal. See also [SGA7, IX, 6.1].

Let $K(\mathcal{L}_\eta)$ be the kernel of $\lambda(\mathcal{L}_\eta)$, $K_S(\mathcal{L}_\eta)$ the flat closure of $K(\mathcal{L}_\eta)$ in G, and $K_S(\mathcal{L}_\eta)^f$ the maximal flat finite (hence S-proper) subscheme of $K_S(\mathcal{L}_\eta)$. The subscheme $K_S(\mathcal{L}_\eta)^f$ is flat because $K_S(\mathcal{L}_\eta)^f$ is open and closed in $K_S(\mathcal{L}_\eta)$ by the construction of flat closures in [EGA, IV, 2.8.5]. Hence $K_S(\mathcal{L}_\eta)^f$ is the unique maximal finite flat subgroup S-scheme of $K_S(\mathcal{L}_\eta)$. There also exists a unique finite flat subgroup scheme $K_S(\mathcal{L}_\eta)^m$ of $K_S(\mathcal{L}_\eta)$ such that $(K_S(\mathcal{L}_\eta)^m)_0 = T_0 \cap K_S(\mathcal{L}_\eta)$ [SGA7, IX, 6.1]. See also the argument for maximality of ${}_nG^m$. Clearly $K_S(\mathcal{L}_\eta)^m \subset K_S(\mathcal{L}_\eta)^f$.

Since there exists a positive integer n such that $K_S(\mathcal{L}_\eta) \subset {}_nG$ we see $K_S(\mathcal{L}_\eta)^m = K_S(\mathcal{L}_\eta) \cap ({}_nG)^m$ by maximality of $K_S(\mathcal{L}_\eta)^m$ and $({}_nG)^m$. Hence the sequence

$$1 \to K_S(\mathcal{L}_\eta)^m = K_S(\mathcal{L}_\eta) \cap ({}_nG)^m \to ({}_nG)^m \to G^t \tag{4}$$

is exact, and therefore $1 \to K_S(\mathcal{L}_\eta)^m_{\mathrm{for}} \to ({}_nG)^m_{\mathrm{for}} \to G^t_{\mathrm{for}}$ is exact. Since $({}_nG)^m_{\mathrm{for}} \simeq ({}_nT)_{\mathrm{for}}$, we have by [EGA, III, 5.4.5] a unique finite flat subgroup scheme $K_S(\mathcal{L}_\eta)^\natural$ of T such that $K_S(\mathcal{L}_\eta)^\natural_{\mathrm{for}} \simeq K_S(\mathcal{L}_\eta)^m_{\mathrm{for}}$. We also have an exact sequence

$$1 \to K_S(\mathcal{L}_\eta)^\natural \to T \stackrel{\lambda_T}{\to} T^t \to 1 \tag{5}$$

It follows from [EGA, III, 5.4.1] that

$$X(K_S(\mathcal{L}_\eta)^m) \simeq X((K_S(\mathcal{L}_\eta)^m_{\mathrm{for}}) \simeq X(K_S(\mathcal{L}_\eta)^\natural) \simeq X/\phi(Y) \tag{6}$$

where $X(H) := \mathrm{Hom}_S(H, \mathbf{G}_{m,S})$ is the Cartier dual of H.

Since G_0 is connected, $K_S(\mathcal{L}_\eta)^{fn} := K_S(\mathcal{L}_\eta)^f \cap G = K_S(\mathcal{L}_\eta)^f$ with the notation of [MB85, IV, 2.4]. We define $K_S(\mathcal{L}_\eta)^{ab} := K_S(\mathcal{L}_\eta)^{fn}/K_S(\mathcal{L}_\eta)^m$. Let $e^{\mathcal{L}_\eta} : K(\mathcal{L}_\eta) \times K(\mathcal{L}_\eta) \to \mathbf{G}_{m,\eta}$ be the Weil pairing of $\mathcal{L}_\eta$ and $e_S^{\mathcal{L}_\eta} : K_S(\mathcal{L}_\eta) \times K_S(\mathcal{L}_\eta) \to \mathbf{G}_{m,S}$ the natural extension of $e^{\mathcal{L}_\eta}$. By [MB85, IV, 2.4] $e_S^{\mathcal{L}_\eta}$ is an alternating bilinear form.

By [MB85, IV, 2.4, 4.3] $K_S(\mathcal{L}_\eta)^{fn} = (K_S(\mathcal{L}_\eta)^m)^\perp$, the orthogonal complement of $K_S(\mathcal{L}_\eta)^m$ with respect to $e_S^{\mathcal{L}_\eta}$, while we have $K_S(\mathcal{L}_\eta)^{ab} \simeq K_S(\mathcal{M}) := \ker \lambda_A(\mathcal{M})$, a finite subgroup scheme of $(A, \mathcal{M})$. Thus $e_S^{\mathcal{L}_\eta}$ induces a perfect pairing on $K_S(\mathcal{M})$, which is just the Weil pairing $e^{\mathcal{M}}$ of $(A, \mathcal{M})$ [MB85, IV, 2.4 (v)]. Since $K_S(\mathcal{L}_\eta)^m$ and $K_S(\mathcal{M})$ are finite and flat over S, their structure sheaves are finite R-free modules, whose ranks are defined to be d_t and d_a^2 respectively. Let $d := d_a d_t$.

Since $e^{\mathcal{L}_\eta}$ is a perfect pairing of $K(\mathcal{L}_\eta)$, the quotient $K(\mathcal{L}_\eta)/K_S(\mathcal{L}_\eta)^{fn} \otimes k(\eta)$ is dual to $K_S(\mathcal{L}_\eta)^m \otimes k(\eta)$. Hence we have

$$\mathrm{ord}\, K(\mathcal{L}_\eta) = \mathrm{ord}\, K_S(\mathcal{L}_\eta)^m\, \mathrm{ord}(K_S(\mathcal{L}_\eta)^{fn}). \tag{7}$$

On the other hand, $\mathrm{ord}\, K_S(\mathcal{L}_\eta)^{fn} = \mathrm{ord}\, K_S(\mathcal{M}) \cdot \mathrm{ord}\, K_S(\mathcal{L}_\eta)^m = d_a^2 d_t$. Hence $\mathrm{ord}\, K(\mathcal{L}_\eta) = d^2$. In particular, if $d^2 = \mathrm{ord}\, K(\mathcal{L}_\eta)$ is prime to the characteristic of $k := R/I$, then d_a and d_t are prime to it too.

By [MB85, V, 1.4.2] $\Gamma(G,\mathcal{L})$ is a finite torsion-free R-module, hence R-free therefore we have $\mathrm{rank}_R\,\Gamma(G,\mathcal{L}) = \mathrm{rank}_{k(\eta)}\,\Gamma(G_\eta,\mathcal{L}_\eta) = d$ by [Mumford74, §23, p. 234]. See also [Mumford74, §23, pp.221-235].

2.4. **The Heisenberg group.** Let A be an abelian scheme over S, and L an ample invertible sheaf on A. We define $F(L)$ to be the set of all pairs (x,ϕ) where $x \in A(S)$ and ϕ is an isomorphism $L \to T_x^*(L)$. The group law on $F(L)$ is defined to be $(x,\phi)\cdot(y,\psi) = (x+y, T_y^*(\phi)\circ\psi)$ where $T_y^*(\phi)$ is understood as an isomorphism of $T_y^*(L)$ onto $T_{x+y}^*(L)$. For any S-scheme S' we define also $F(L_{S'})$ similarly. Then we define a functor $\underline{G}(L)$ from the category of noetherian S-schemes to the category of groups by

$$\underline{G}(L)(S') := F(L_{S'}) \tag{8}$$

We recall that for any S-scheme S', $K(L)(S')$ is the set of all $x \in A(S')$ such that $T_x^*(L)\otimes L^{-1} \in \mathrm{Pic}(S')$. Therefore there is a natural homomorphism $\pi : \underline{G}(L)(S') \to K(L)(S')$, whose kernel is easily seen to be $\Gamma(S', O_{S'}^*)$ because any fibre of A over S is proper and integral.

Now we go back to the previous situation where G is a semi-abelian scheme over S with G_η an abelian scheme over $k(\eta)$, and $\mathcal{L}$ is an ample invertible sheaf on G. Let $\mathcal{L}^{\mathrm{tsr}}$ be the $\mathbf{G}_m$-torsor on G associated with the invertible sheaf $\mathcal{L}$. Let $\mathcal{G}(\mathcal{L}_\eta) := (\mathcal{L}_\eta^{\mathrm{tsr}})_{|K(\mathcal{L}_\eta)}$ (which we call the Heisenberg group of $\mathcal{L}_\eta$). Then $\mathcal{G}(\mathcal{L}_\eta)$ is a group $k(\eta)$-scheme representing the functor $\underline{G}(\mathcal{L}_\eta)$ by [MB81, 3.2]. It is by [MB81, 3.2] a central extension of $K(\mathcal{L}_\eta)$ by the $k(\eta)$-split torus $\mathbf{G}_{m,\eta}$ with the commutator form $e^{\mathcal{L}_\eta}$;

$$e^{\mathcal{L}_\eta}(\bar{g},\bar{h}) = [g,h] := ghg^{-1}h^{-1}, \quad \text{for } \forall g,h \in \mathcal{G}(\mathcal{L}_\eta) \tag{9}$$

where $\bar{g}$, $\bar{h}$ are the images of g and h in $K(\mathcal{L}_\eta)$. Hence the sequence

$$1 \to \mathbf{G}_{m,\eta} \to \mathcal{G}(\mathcal{L}_\eta) \to K(\mathcal{L}_\eta) \to 0 \tag{10}$$

is exact. By [Mumford66, p. 295] $\mathcal{G}(\mathcal{L}_\eta)$ acts on $\Gamma(G_\eta,\mathcal{L}_\eta)$ by

$$U_z(s) = T_{-a}^*(\phi(s)) \quad \text{for } s \in \Gamma(\mathcal{L}_\eta) \tag{11}$$

where $z = (a,\phi) \in \mathcal{G}(\mathcal{L}_\eta)$. By [MB85, V, 2.5.5] (See also [Mumford74, §23]), $\Gamma(G_\eta,\mathcal{L}_\eta)$ is an irreducible $\mathcal{G}(\mathcal{L}_\eta)$-module of weight one, unique up to isomorphism by taking a finite extension of $k(\eta)$ if necessary. The $\mathcal{G}(\mathcal{L}_\eta)$-module structure of $\Gamma(G_\eta,\mathcal{L}_\eta)$ is known by [MB85, V, 3.4] and [Mumford74, §23] regardless of the characteristic of $k(\eta)$ and $\mathrm{ord}\,K(\mathcal{L}_\eta)$.

Since $K_S(\mathcal{L}_\eta)^m$ is totally isotropic, there exists an étale finite cover S' of S such that the cubical sheaf $\mathcal{L}_{S'}$ on $G_{S'}$ descends to a cubical sheaf on the quotient scheme $G_{S'}/K_{S'}(\mathcal{L}_\eta)^m$ [MB85, VI, 1.3] where $K_{S'}(\mathcal{L}_\eta)^m := K_S(\mathcal{L}_\eta)^m \times_S S'$. In fact, $\mathcal{L}_{S'}$ with $K_{S'}(\mathcal{L}_\eta)^m$ action is a descent data. This means that the action of $K_{S'}(\mathcal{L}_\eta)^m$ on $G_{S'}$ lifts to the action on $\mathcal{L}_{S'}$ hence that on $\Gamma(G_{S'},\mathcal{L}_{S'})$. We (may) assume from the start that $K_S(\mathcal{L}_\eta)^m$ acts on $\Gamma(G,\mathcal{L})$. By [MB85, V, 3.4] $\Gamma(G,\mathcal{L})$ is the direct sum of $K_S(\mathcal{L}_\eta)^m$-eigenspaces $\Gamma(G,\mathcal{L})^{\bar{\chi}}$ ($\bar{\chi} \in X/\phi(Y) = X(K_S(\mathcal{L}_\eta)^m)$) and $\mathrm{rank}_R\,\Gamma(G,\mathcal{L})^{\bar{\chi}} = d_a$ for any $\bar{\chi}$.

3. $(P, \mathcal{L})$ - THE TOTALLY DEGENERATE CASE

3.1. **Fourier expansions.** First we consider the totally degenerate case, that is the case where A_0 (and hence A) is trivial. Then $G_{\mathrm{for}} = T_{\mathrm{for}}$ and $\widetilde{G} = T$. We choose and fix the coordinate w^x $(x \in X)$ of T satisfying $w^x w^y = w^{x+y}$ $(\forall x, y \in X)$. The invertible sheaf $\mathcal{L}_{\mathrm{for}}$ is trivial on T_{for}, and therefore

$$\Gamma(G_\eta, \mathcal{L}_\eta) = \Gamma(G, \mathcal{L}) \otimes k(\eta) \hookrightarrow \Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) \otimes k(\eta) \hookrightarrow \prod_{x \in X} k(\eta) \cdot w^x$$

Therefore, any element $\theta \in \Gamma(G_\eta, \mathcal{L}_\eta)$ can be written as a formal Fourier series $\theta = \sum_{x \in X} \sigma_x(\theta) w^x$ with $\sigma_x(\theta) \in k(\eta)$, which converges in the I-adic topology.

Theorem 3.2. [Faltings–Chai90] *There exists a function $a : Y \to k(\eta)^*$ and a bilinear function $b : Y \times X \to k(\eta)^*$ with the following properties:*

1. $b(y, x) = a(x + y)a(x)^{-1}a(y)^{-1}$, $a(0) = 1$ $\quad(\forall x \in X, y \in Y)$
2. $b(y, z) = b(z, y) = a(y + z)a(y)^{-1}a(z)^{-1}$ $\quad(\forall y, z \in Y)$
3. $b(y, y) \in I$ $\quad(\forall y \neq 0)$, *and for every $n \geq 0$, $a(y) \in I^n$ for almost all $y \in Y$*
4. $\Gamma(G_\eta, \mathcal{L}_\eta)$ *is identified with the $k(\eta)$ vector subspace of formal Fourier series θ that satisfy $\sigma_{x+\phi(y)}(\theta) = a(y)b(y, x)\sigma_x(\theta)$ and $\sigma_x(\theta) \in k(\eta)$ $(\forall x \in X)$.*

For $\theta \in \Gamma(G_\eta, \mathcal{L}_\eta)$, $\theta \in \Gamma(G, \mathcal{L})$ if and only if $\theta \in \Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$. Hence $\Gamma(G, \mathcal{L})$ is an R-submodule of $\Gamma(G_\eta, \mathcal{L}_\eta)$ consisting of formal Fourier series θ with $\sigma_x(\theta) \in R$ $(\forall x \in X)$. Moreover $\Gamma(G, \mathcal{L})^{\bar{X}}$ is an R-free module of rank one and any element θ of $\Gamma(G, \mathcal{L})^{\bar{X}} \simeq \Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})^{\bar{X}}$ is written as $\theta = \sum_{x \in \bar{X} + \phi(Y)} \sigma_x(\theta) w^x$ for some $\sigma_x(\theta) \in R$.

Definition 3.3. By taking a finite base change of S if necessary the functions b and a can be extended respectively to $X \times X$ and X so that the previous relations between b and a are still true on $X \times X$. Then we can define the integer-valued functions $A : X \to \mathbf{Z}$, $B : X \times X \to \mathbf{Z}$ and $\bar{b}(y, x) \in R^*$, $\bar{a}(y) \in R^*$ by

$$\begin{aligned} B(y, x) &= val_s(b(y, x)), \quad dA(\alpha)(x) = B(\alpha, x) + r(x)/2 \\ A(x) &= val_s(a(x)) = B(x, x)/2 + r(x)/2 \\ b(y, x) &= \bar{b}(y, x)s^{B(y,x)}, \quad a(x) = \bar{a}(x)s^{(B(x,x)+r(x))/2} \end{aligned}$$

for some $r \in \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{Z})$. We set $a_0 = \bar{a} \mod I$ and $b_0 = \bar{b} \mod I$. Therefore $a_0(x), b_0(y, x) \in k^*$ for any $x, y \in X$. B is positive definite by Theorem 3.2 (3).

Definition 3.4. For simplicity we identify $\phi : Y \to X$ with the inclusion $Y \subset X$. We define

$$\begin{aligned} \widetilde{R} &:= R[a(x)w^x\vartheta; x \in X] \simeq R[\xi_x\vartheta; x \in X], \\ \xi_x &:= s^{B(x,x)/2+r(x)/2}w^x, \quad \xi_{x,c} := \xi_{x+c}/\xi_c \\ \zeta_{x,c} &:= s^{B(\alpha(\sigma),x)+r(x)/2}w^x \quad (x+c \in C(c,\sigma)) \end{aligned}$$

where $\widetilde{R}$ is the graded algebra with $\deg(a(x)w^x\vartheta) = 1$ and $\deg a = 0$ for $a \in R$, while $\sigma \in \mathrm{Del}_B(c)$ is a maximal-dimensional Delaunay cell with $x + c \in C(c,\sigma)$.

Let $\widetilde{Q} := \mathrm{Proj}(\widetilde{R})$, and $\widetilde{P}$ the normalization of $\widetilde{Q}$. We define an action S_y on $\widetilde{Q}$ by

$$S_y^*(a(x)w^x\vartheta) = a(x+y)w^{x+y}\vartheta \quad \text{for } y \in Y \tag{12}$$

S_y induces a natural action on $\widetilde{P}$, which we denote by the same S_y. We denote by $\widetilde{\mathcal{L}}$ $O_{\mathrm{Proj}}(1)$ on $\widetilde{Q}$ as well as its pull back to $\widetilde{P}$.

Our construction of $(P,\mathcal{L})$ is the same as in [Mumford72].

Theorem 3.5. *The quotient $(\widetilde{P}_{\mathrm{for}}, \widetilde{\mathcal{L}}_{\mathrm{for}})/Y$ is a flat projective formal S-scheme. There exists a flat projective S-scheme $(P,\mathcal{L})$ such that the formal completion $(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ of it along the closed fibre is isomorphic to $(\widetilde{P}_{\mathrm{for}}, \widetilde{\mathcal{L}}_{\mathrm{for}})/Y$.*

Proof. This follows from [EGA, III, 5.4.5] and [Mumford72]. □

Definition 3.6. Let $\mathrm{Del}(P_0)$ be the Delaunay decomposition corresponding to P_0. By taking a finite base change of S if necessary, we may assume that $dA(\alpha(\sigma)) \in \mathrm{Hom}(X, \mathbf{Z})$ for any Delaunay cell $\sigma \in \mathrm{Del}(P_0)$. By [AN96] this implies that P_0 is reduced. This also follows from Theorem 3.7. Then we call the closed fibre $(P_0, \mathcal{L}_0)$ of the flat projective family $(P,\mathcal{L})$ a polarized stable quasi-abelian variety over $k := R/I$.

Theorems 3.7 and 3.8 describe the structures of the closed fibres $\widetilde{P}_0$ and P_0 in terms of $\mathrm{Del}(P_0)$. If P_0 is reduced, G is realized as an open subscheme of P. See 4.12. In what follows we always assume that $dA(\alpha(\sigma)) \in \mathrm{Hom}(X, \mathbf{Z})$ for any $\sigma \in \mathrm{Del} := \mathrm{Del}(P_0)$. Hence P_0 is reduced.

Theorem 3.7. *Let $(\widetilde{P}_0, \widetilde{\mathcal{L}}_0)$ be the closed fibre of $(\widetilde{P}, \widetilde{\mathcal{L}})$ and $\bar{\zeta}_{x,c} := \zeta_{x,c} \otimes k$ the restriction to P_0. Then*

1. *$\widetilde{P}_0$ is covered with finitely many open affine subschemes of finite type*
$$U_0(c) := \mathrm{Spec}\ k[\bar{\zeta}_{x,c}; x \in X] \quad (c \in X).$$
2. *$R_0(c) := k[\bar{\zeta}_{x,c}; x \in X]$ is a k-algebra of finite type. Let $x_i \in X$. If x_i's are cellmates (resp. otherwise), then*
$$\bar{\zeta}_{x_1,c} \cdots \bar{\zeta}_{x_m,c} = \bar{\zeta}_{x_1+\cdots+x_m,c} \quad (\textit{resp. } 0)$$

Theorem 3.8. *Let $(P_0, \mathcal{L}_0)$ be the closed fibre of $(P,\mathcal{L})$. Let σ and τ be Delaunay cells in $\mathrm{Del}(P_0)$.*

1. *For each* σ *there exists a* T*-invariant subscheme* $O(\sigma)$ *of the closed fibre* P_0 *which is a torus of dimension* $\dim\sigma$ *over* k*,*
2. $\sigma \subset \tau$ *iff* $O(\sigma)$ *is contained in* $O(\tau)$*,*
3. *For each* σ *there exists a unique closed subscheme* $V(\sigma)$ *which is the closure of* $O(\sigma)$ *and has dimension* $\dim\sigma$ *over* k*,*
4. $\sigma \subset \tau$ *iff* $V(\sigma)$ *is contained in* $V(\tau)$*,*
5. $P_0 = \bigcup_{\sigma\in \mathrm{Del}(P) \mod Y} V(\sigma)$

See [Namikawa76], [AN96].

We will describe $\Gamma(P_0, \mathcal{L}_0^n)$ explicitly. Compare [AN96].

Theorem 3.9. *If* $n > 0$*,* $h^0(P_0, \mathcal{L}_0^n) = [X : Y]n^g$*,* $h^i(P_0, \mathcal{L}_0^n) = 0$ $(i > 0)$*, and*

$$\Gamma(P_0, \mathcal{L}_0^n) = \left\{ \sum_{x\in X} c(x)\eta_n(x); \begin{matrix} c(x+ny) = b_0(y,x)a_0(y)^n c(x) \\ c(x) \in k, \forall x \in X, \forall y \in Y \end{matrix} \right\}$$

Proof. Let $\mathrm{Del}(P_0) = \mathrm{Del}_B$ for some integral matrix B. By Theorem 3.7 $\widetilde{P}_0$ is covered with open affine sets $U_0(c)$ $(c \in X)$. For any Delaunay cell $\sigma \in \mathrm{Del}$, let $V(\sigma)$ be the closure of a torus orbit $O(\sigma)$. It is a closed subscheme of $\widetilde{P}_0$ and a normal complete torus embedding by Theorem 3.7 or Theorem 3.8. Let $V(\sigma, c) := V(\sigma) \cap U_0(c)$ for $c \in \sigma$ and $C(c, \sigma)$ the cone spanned by edges of σ at c. In order to avoid complication we write $\zeta_{x,c}$ instead of $\bar{\zeta}_{x,c}$. Then we easily see $V(\sigma, c) = \mathrm{Spec}\ k\,[\zeta_{x,c}\,;\, x + c \in C(c, \sigma) \cap X]$. We define

$$\eta_n(x) := \zeta_{x-nc,c}\xi_c^n = s^{B(\alpha(\sigma),x-nc)+nB(c,c)/2+r(x)/2}w^x$$

for $\frac{x}{n} \in \sigma \cap \frac{X}{n}$. Note that $(x - nc) + c \in C(c, \sigma)$. The definition of $\eta_n(x)$ depends on x and n, but does not depend on the choice of a Delaunay cell σ containing $\frac{x}{n}$ and a vertex $c \in X$ of σ. If we need to specify dependence on n or $\mathcal{L}^n$, we write $\eta_{\mathcal{L}^n}(x)$ or simply $\eta_n(x)$. We note $\eta_1(x) = \eta_{\mathcal{L}}(x) = \xi_x := s^{(B(x,x)+r(x))/2}w^x$ $(\forall x \in \mathrm{Star}\,(0) \cap X)$ and that $\zeta_{x,c} = s^{B(\alpha(\sigma),x)+r(x)/2}w^x$ if $x + c \in C(c, \sigma)$.

Now we fix n and compute $H^i(V(\sigma), \mathcal{L}_0^n)$. Let $i = 0$. The line bundle $\mathcal{L}$ restricted to $V(\sigma)$ is defined by a one-cocycle $\{\xi_d/\xi_c\}_{c,d\in\sigma\cap X}$ with respect to an affine covering $\{V(\sigma) \cap U_0(c); c \in \sigma \cap X\}$. Let $\phi = \{\phi_c\} \in H^0(V(\sigma), \mathcal{L}_0^n)$ with $\phi_c = (\xi_d/\xi_c)^n\phi_d$. Let $\phi = \phi_c\xi_c^n = \phi_d\xi_d^n$. Then we easily see that

$$\phi = \sum_{\frac{x}{n}\in C(c,\sigma)\cap\frac{X}{n}} c(x)\zeta_{x-nc,c}\xi_c^n = \sum_{\frac{x}{n}\in \bigcap_{d\in\sigma\cap X} C(d,\sigma)\cap\frac{X}{n}} c(x)\zeta_{x-nd,d}\xi_d^n$$

for some constants $c(x) \in k$ independent of $d \in \sigma$. It follows that

$$H^0(V(\sigma), \mathcal{L}_0^n) = \left\{ \sum_{\frac{x}{n}\in\sigma\cap\frac{X}{n}} c(x)\eta_n(x); c(x) \in k \right\}$$

because $\sigma = \cap_{d\in\sigma\cap X} C(d, \sigma)$. It follows that $\mathcal{L}_0^n$ $(n \geq 1)$ is generated by global sections on $V(\sigma)$. We see that the restriction $\mathcal{L}_0$ on $V(\sigma)$ is ample, because $\mathcal{L}_0$ on the inverse image of $V(\sigma)$ in $\widetilde{P}_0/Y'$ is very ample by taking a smaller sublattice Y' of Y if necessary. By [Oda88], $H^i(V(\sigma), \mathcal{L}_0^n) = 0$ for $i > 0$.

Let $\mathrm{Del}^{(i)}$ be the set of all i-dimensional cells of Del. We see that there is an exact sequence on $\widetilde{P}_0$

$$0 \to O_{\widetilde{P}_0} \to \oplus O_{V(\sigma_g)} \xrightarrow{\partial_g} \cdots \xrightarrow{\partial_2} \oplus O_{V(\sigma_1)} \xrightarrow{\partial_1} \oplus O_{V(\sigma_0)} \to 0$$

where $\sigma_i \in \mathrm{Del}^{(i)}$. The homomorphism $\partial_i : \oplus O_{V(\sigma_i)} \to \oplus O_{V(\sigma_{i-1})}$ is defined by $\partial_i(\bigoplus_\sigma \phi_\sigma) = \bigoplus_\tau \sum_{\tau \subset \sigma} [\sigma : \tau] \phi_\sigma$ where the summation runs over the set of all σ containing a fixed τ as a face of codimension one, while the direct sum $\bigoplus_\tau$ is over the set $\mathrm{Del}^{(i-1)}$. Any Delaunay cell σ is here assumed to be oriented and $[\sigma : \tau]$ $(= \pm 1)$ is the incidence number of σ relative to its face τ of codimension one.

From now we assume by choosing a sublattice Y' of Y of finite index if necessary that $S_y(V(\sigma)) \cap V(\sigma) = \emptyset$ for any Delaunay cell σ and $y \in Y \setminus \{0\}$. We prove $H^i(P_0, \mathcal{L}_0^n) = 0$ for $i > 0$ under the assumption. We note that the general case follows from it easily.

From the above sequence tensored by $\mathcal{L}_0^n$, we obtain a complex with a Y-action

$$0 \to \oplus H^0(V(\sigma_g), \mathcal{L}_0^n) \xrightarrow{\partial_g} \cdots \xrightarrow{\partial_1} \oplus H^0(V(\sigma_0), \mathcal{L}_0^n) \to 0$$

Let $W_i := \bigoplus_{\sigma \in \mathrm{Del}^{(g-i)}} H^0(V(\sigma), \mathcal{L}_0^n)$. By the assumption on Y, we have

$$H^i(\oplus_{\sigma_j \bmod Y} \mathcal{L}_0^n \otimes O_{V(\sigma_j)}) \simeq \oplus_{\sigma_j \bmod Y} H^i(V(\sigma_j), \mathcal{L}_0^n) = 0 \quad (i > 0, \forall j)$$

It follows

$$H^i(P_0, \mathcal{L}_0^n) \simeq \mathbf{H}^i(\bigoplus W_*^{Y\text{-inv}}) \simeq \mathbf{H}^i(\bigoplus_{\sigma \in \mathrm{Del}^{(g-i)} \bmod Y} H^0(V(\sigma), \mathcal{L}_0^n)).$$

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Let $\mathrm{Star}\,(x)$ be the union of $\sigma \in \mathrm{Del}$ such that $x \in \sigma$ and $\mathrm{Star}\,(x)^0$ the relative interior of $\mathrm{Star}\,(x)$. Then it is easy to see

$$\mathbf{H}^i(\bigoplus W_*^{Y\text{-inv}}) \simeq \bigoplus_{x \in X \bmod nY} H^i(\mathrm{Star}\,(\frac{x}{n})^0, k)$$

We see $H^i(\mathrm{Star}\,(x)^0, k) \simeq \mathbf{H}^i(\bigoplus_{x \in \sigma} H^0(\sigma, k), \partial_{g-*})$. Therefore we have $H^0(\mathrm{Star}\,(x)^0, k) = k$ and $H^i(\mathrm{Star}\,(x)^0, k) = 0$ for $i > 0$. Hence $H^i(P_0, \mathcal{L}_0^n) = 0$ for $i > 0$, while $H^0(P_0, \mathcal{L}_0^n) = \ker(\partial_g)^{Y\text{-inv}}$.

By the above proof, $\theta \in H^0(P_0, \mathcal{L}_0^n)$ is an (apparently infinitely many) linear combination of $\eta_n(x)$. S_y acts on $\widetilde{R}$ by $S_y^*(\bar{a}(x)\xi_x\vartheta) = \bar{a}(x+y)\xi_{x+y}\vartheta$. We note that $\theta \in H^0(P_0, \mathcal{L}_0^n)$ if and only if $S_y^*(\theta\vartheta^n) = \theta\vartheta^n$. Theorem follows from the formula $S_y^*(\eta_n(x)\vartheta^n) = b_0(y,x)a_0(y)^n\eta_n(x+ny)\vartheta^n$. Therefore $h^0(P_0, \mathcal{L}_0^n) = |X/Y|n^g$. □

Remark 3.10. Since G_0 is irreducible, the extension of $\mathcal{L}_\eta$ to G is unique. Therefore $\Gamma(G_\eta, \mathcal{L}_\eta^n) = \Gamma(G, \mathcal{L}^n) \otimes k(\eta)$. Hence $\Gamma(G_\eta, \mathcal{L}_\eta^n)$ is identified with the $k(\eta)$ vector subspace of $\Gamma(G_{\mathrm{for}}, \mathcal{L}^n_{\mathrm{for}}) \otimes k(\eta)$ consisting of Fourier series $s = \sum_{x\in X} \sigma_x(s) w^x$ such that

$$\sigma_{x+ny}(s) = a(y)^n b(y,x) \sigma_x(s), \quad \sigma_x(s) \in k(\eta) \quad (\forall x \in X, \forall y \in Y)$$

By Theorem 3.9 we see that $\Gamma(P, \mathcal{L}^n) \otimes k = \Gamma(P_0, \mathcal{L}^n_0)$. Hence we easily see

$$\Gamma(P, \mathcal{L}^n) = \left\{ \sum_{x\in X} c(x)\eta_n(x); \begin{matrix} c(x+ny) = \bar{b}(y,x)\bar{a}(y)^n c(x) \\ c(x) \in R, \forall x \in X, \forall y \in Y \end{matrix} \right\}$$

Hence we have $\Gamma(P_\eta, \mathcal{L}^n_\eta) = \Gamma(G_\eta, \mathcal{L}^n_\eta)$ and therefore $G_\eta \simeq P_\eta$.

We note that G is the semi-abelian scheme we started from, while P is the projective scheme we constructed with the degeneration data of G.

4. General case

4.1. **Fourier expansions in the general case.** In this section we will discuss the general case where the abelian part A is nontrivial. We will prove the theorems similar to the totally degenerate case.

We use freely the notation in Section 2. We assume that $T_0 := T \otimes k$ is a split k-torus. Then T is a split S-torus by [SGA3, exp. IX §3]. See also [FC90, 2.2]. Let $X := \mathrm{Hom}(T, \mathbf{G}_m)$, $Y = \mathrm{Hom}(T^t, \mathbf{G}_m)$. The ample cubical sheaf $\mathcal{L}$ on a semi-abelian scheme G induces a sheaf $\mathcal{L}_{\mathrm{for}}$ on G_{for}. This is descended to an ample cubical sheaf $\mathcal{M}_{\mathrm{for}}$ on A_{for}, which is algebraized into an ample cubical sheaf $\mathcal{M}$ on the algebraisation A of A_{for}. The sheaf $\pi_{*\,\mathrm{for}}(O_{G_{\mathrm{for}}})$ is the I-adic direct sum of T_{for}-eigenspaces $O^\chi_{G_{\mathrm{for}}}$

$$\pi_{*\,\mathrm{for}}(O_{G_{\mathrm{for}}}) = \oplus_{\chi\in X}\ O^\chi_{G_{\mathrm{for}}}$$

where $\mathcal{O}^\chi_{G_{\mathrm{for}}} = \{f \in O_{G_{\mathrm{for}}}; f(tx) = \chi(t)f(x)\ \forall \text{functorial points } t \in T_{\mathrm{for}}, x \in G_{\mathrm{for}}\}$.

Let $\widetilde{G}$ be the algebraisation of G_{for}, $0 \to T \to \widetilde{G} \xrightarrow{\pi} A \to 0$ the Raynaud sequence and $\widetilde{\mathcal{L}} := \pi^*(\mathcal{M})$. It is clear that $\widetilde{\mathcal{L}}_{\mathrm{for}} = \pi^*_{\mathrm{for}}(\mathcal{M}_{\mathrm{for}}) = \mathcal{L}_{\mathrm{for}}$.

Since T acts on $\widetilde{G}$, we have $O_{\widetilde{G}} = \oplus_{\chi\in X}\ O^\chi$ with O^χ the subsheaf of $O_{\widetilde{G}}$ consisting of χ-eigenfunctions. Then $\pi_*(O_{\widetilde{G}}) = \oplus_{\chi\in X}\ O_\chi$ where O_χ is the subsheaf of $\pi_*(O_{\widetilde{G}})$ such that $O^\chi = \pi^{\cdot} O_\chi$, the sheaf-theoretic pull back of O_χ. Then $O_\chi \in A^t$.

By definition we have $O^\chi O^{\chi'} = O^{\chi+\chi'}$, $O_\chi O_{\chi'} = O_{\chi+\chi'}$. If we choose isomorphisms $O^\chi \simeq O_\chi \simeq O_A$ on an affine open U of A for a basis χ of X, we have the isomorphisms $O^\chi \simeq O_\chi \simeq O_U$ for any $\chi \in X$. Hence there exists an open affine covering $\{U\}$ of A such that $O^\chi \simeq O_\chi \simeq O_U$ for any $\chi \in X$ on each $U \in \{U\}$.

Next we prove that $\widetilde{G}$ is a T-bundle over A in the Zariski topology. Let $U := \mathrm{Spec}\ B \in \{U\}$ be an affine open subset of A. Since π is an affine morphism, $W := \pi^{-1}(U) = \mathrm{Spec}\ C$ is affine. Then $C = \Gamma(W, O_{\widetilde{G}}) = \Gamma(U, \pi_*(O_{\widetilde{G}})) = \oplus_{\chi\in X} \Gamma(U, O_\chi)$. Since $O_\chi \simeq O_U$ on U, $\Gamma(U, O_\chi)$ hence $\Gamma(U, O^\chi)$ is a B-module of rank one. Let w^χ_U be a generator of $\Gamma(U, O^\chi)$ such that $\Gamma(U, O^\chi) = Bw_U^\chi$. By $O^\chi O^\chi = O^{\chi+\chi'}$, we may assume $w_U^\chi w_U^{\chi'} = w_U^{\chi+\chi'}$, $C = B[w_U^\chi; \chi \in X]$. Therefore W is an S-split torus $U \times T$ over U. It follows that $\widetilde{G}$ is a T-bundle over A in the Zariski topology. Therefore there exist an affine covering $\{U\}$ of A and coordinates w_U^χ of T such that

$$O_{\widetilde{G}|\pi^{-1}(U)} = \oplus_{\chi\in X}\ O_U^\chi = \oplus_{\chi\in X}\ O_U w_U^\chi, \quad O_U^\chi = O_U w^\chi \simeq O_{\chi|U} \quad (\chi \in X).$$

For different U and V w_U^χ and w_V^χ are related by $w_U^\chi = g_{UV}^\chi w_V^\chi$ where $g_{UV}^\chi \in O_{U\cap V}^*$ with $g_{UV}^\chi g_{VW}^\chi = g_{UW}^\chi$. The cocycle $\{g_{UV}^\chi\} \in H^1(A, O_A^*)$ represents the sheaf O_χ.

4.2. **Fourier coefficients.** We define a subsheaf of $\mathcal{M}_{\mathrm{for}}^\chi$ of $\mathcal{L}_{\mathrm{for}}$ by

$$\mathcal{M}_{\mathrm{for}}^\chi = \{f \in \mathcal{L}_{\mathrm{for}}; f(tx) = \chi(t)f(x)\ \forall\, \text{functorial points}\, t \in T_{\mathrm{for}}, x \in G_{\mathrm{for}}\}$$

Since $\mathcal{L}_{\mathrm{for}}$ is descended to $\mathcal{M}_{\mathrm{for}}$, we have $\pi_{\mathrm{for}}^{\cdot}\mathcal{M}_{\mathrm{for}} = \mathcal{M}_{\mathrm{for}}^0$, the subspace of $\mathcal{L}_{\mathrm{for}}$ consisting of T-invariant sections. Then we have $\pi_{*\,\mathrm{for}}(\mathcal{L}_{\mathrm{for}}) = \oplus_{\chi\in X}\mathcal{M}_{\mathrm{for}}^\chi$, $\mathcal{M}_{\mathrm{for}}^\chi = \mathcal{M}_{\mathrm{for}}^0 \otimes O_{\mathrm{for}}^\chi$. Let $\mathcal{M}_\chi := \mathcal{M} \otimes O_\chi$. Then $\mathcal{M}_\chi$ is ample on A with $\mathcal{M}_{\mathrm{for}}^\chi = \pi_{\mathrm{for}}^{\cdot}\mathcal{M}_{\chi,\mathrm{for}}$. We also have $\Gamma(G_{\mathrm{for}}, \mathcal{M}_{\mathrm{for}}^\chi) = \Gamma(A_{\mathrm{for}}, \mathcal{M}_{\chi,\mathrm{for}}) = \Gamma(A, \mathcal{M}_\chi)$.

By the sign change of c and c^t following [FC90, p.43] we have $c\phi = \lambda_A c^t$ where ϕ is the injection of Y into X and $\lambda_A = \lambda_A(\mathcal{M})$. This means that $T_{c^t(y)}^*(\mathcal{M}) \otimes \mathcal{M}^{-1} \simeq O_{\phi(y)}$, which we denote simply by O_y. As rigidified sheaves $\mathcal{M}$ and O_χ at $0 \in A$ we have $T_{c^t(y)}^*(\mathcal{M}) = \mathcal{M} \otimes O_y \otimes \mathcal{M}(c^t(y))$ and $T_{c^t(y)}^*(\mathcal{M}_\chi) = \mathcal{M}_{\chi+y} \otimes \mathcal{M}_\chi(c^t(y))$. By the isomorphism $\Gamma(G_{\mathrm{for}}, \mathcal{M}_{\mathrm{for}}^\chi) \simeq \Gamma(A, \mathcal{M}_\chi)$, we have an R-homomorphism

$$\sigma_\chi : \Gamma(G, \mathcal{L}) \subset \Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) \simeq \oplus_{\mu\in X}\Gamma(G_{\mathrm{for}}, \mathcal{M}_{\mathrm{for}}^\mu) \to \Gamma(A, \mathcal{M}_\chi).$$

Two homomorphisms σ_χ and $\sigma_{\chi+y}$ will be related as follows. Since $T_{c^t(y)}^*(\mathcal{M}_\chi) = \mathcal{M}_{\chi+y} \otimes \mathcal{M}_\chi(c^t(y))$, we obtain a composition map

$$T_{c^t(y)}^* \circ \sigma_\chi : \Gamma(G, \mathcal{L}) \to T_{c^t(y)}^*(\Gamma(A, \mathcal{M}_\chi)) = \Gamma(A, \mathcal{M}_{\chi+y}) \otimes \mathcal{M}_\chi(c^t(y)).$$

For instance, for a given element $m \in \mathcal{M}_\chi(c^t(y))^{-1}$, we have a multiplication map

$$m \cdot T_{c^t(y)}^* \circ \sigma_\chi : \Gamma(G, \mathcal{L}) \to \Gamma(A, \mathcal{M}_{\chi+y}).$$

Theorem 4.3 asserts that there exists $m(y, \chi) \in \mathcal{M}_\chi(c^t(y))^{-1} \otimes k(\eta)$ such that $\sigma_{\chi+y} = m(y, \chi)\cdot T_{c^t(y)}^* \circ \sigma_\chi$ $(y \in Y, \chi \in X)$. The constant $m(y, \chi)$ will be written as a product $m(y, \chi) = \tau(y, \chi)\psi(y)$ where $\tau(y, \chi) \in O_\chi(c^t(y))^{-1} \otimes k(\eta) \simeq k(\eta)$ and $\psi(y) \in \mathcal{M}(c^t(y))^{-1} \otimes k(\eta) \simeq k(\eta)$. See [FC90, pp. 43-44].

Theorem 4.3. [FC90] *The collection of R-module homomorphisms $\sigma_x : \Gamma(G, \mathcal{L}) \to \Gamma(A, \mathcal{M}_x)$ have the following properties*

1. $\sigma_x \neq 0 \quad (\forall x \in X)$
2. *There exist a cubic trivialization $\psi : 1_Y \to (c^t)^*\mathcal{M}_\eta^{-1}$ and a trivialization of biextension (Poincaré torsor) $\tau : 1_{Y\times X} \to (c^t \times c)^*\mathcal{P}_\eta^{-1}$ such that*

$\sigma_{x+y}(s) = \psi(y)\tau(y,x)T^*_{c^t(y)}(\sigma_x(s))$ $(\forall s \in \Gamma(G,\mathcal{L}), \forall y \in Y, \forall x \in X)$,

In particular, $\psi(0)=1$, $\tau(0,x)=\tau(y,0)=1$.

3. $\tau(y,x)$ *is bilinear in* y *and* x,
4. $\tau(y,z)=\tau(z,y)=\psi(y+z)\psi(y)^{-1}\psi(z)^{-1}$ $(\forall y,z\in Y)$
5. $\tau(y,y)\in I$ *if* $y\neq 0$,
6. *For every* $n\geq 0$, $\psi(y)\in I^n$ *for all but finitely many* $y\in Y$,
7. $\Gamma(G_\eta,\mathcal{L}_\eta)$ *is identified with the* $k(\eta)$ *vector subspace of Fourier series* θ *satisfying* $\sigma_{x+y}(\theta)=\psi(y)\tau(y,x)T^*_{c^t(y)}(\sigma_x(\theta))$ *with* $\sigma_x(\theta)\in\Gamma(A_\eta,\mathcal{M}_{x,\eta})$.

4.4. **Extension of τ and ψ.** In the previous paragraphs, we have $c: X\to A^t$, $c^t: Y\to A$, and $\phi: Y\to X$ such that $\lambda_A c^t = c\phi$. By the sign change of c and c^t we have $c(\chi)=O_\chi$, $T^*_{c^t(y)}(O_\chi)\simeq O_{\chi+y}$ and $T^*_{c^t(y)}(\mathcal{M})\simeq\mathcal{M}_y$ for any $\chi\in X$ and $y\in Y$.

Now we extend the homomorphism $c^t: Y\to A$ to $c^t_X: X\to A$ so that $\lambda_A c^t_X = c$ on X, which extends the relation $\lambda_A\cdot c^t = c\cdot\phi$ on Y. In fact, this is possible if we take a finite base change of S because A is divisible. It is also easy to extend the functions τ and ψ respectively to $X\times X$ and X by taking a finite base change of S if necessary so that the following are true;

1. τ is symmetric bilinear on $X\times X$,
2. $\psi(x+y)=\psi(x)\psi(y)\tau(x,y)$ $(x,y\in X)$.

This also implies that there are a cubical trivialization $\psi: 1_X\to(c^t_X)^*\mathcal{M}_\eta^{-1}$ and a trivialization of the biextension (an isomorphism of the biextensions) $\tau: 1_{X\times X}\simeq(c^t\times c)^*\mathcal{P}_\eta^{-1}$ on $X\times X$. In what follows we denote c^t_X simply by c^t.

4.5. **A rigidification.** Let $\mathcal{P}$ be the Poincaré sheaf on $A\times A^t$, which is by definition the universal invertible sheaf for the functor Pic^0 on A. Let $m: A\times A\to A$ be the multiplication of A, p_{13} and p_{23} the projections of $A\times A\times A^t$. By the theorem of cube there is an (let us say, canonical) isomorphism $\Phi:(m\times 1)^*\mathcal{P}\simeq p_{13}^*\mathcal{P}\otimes p_{23}^*\mathcal{P}$. By this we have a canonical isomorphism $\Phi_{a,\chi}:=\Phi_{|A\times a\times\chi}: T_a^*(O_\chi)=O_\chi\otimes O_\chi(a)$ for any $\chi\in X$ and for any functorial point a of A. By the theorem of the cube we see $\Phi_{a,\chi}T_a^*(\Phi_{a',\chi})=\Phi_{a+a',\chi}$. Let $\phi_{a,\chi}:=\Phi_{a,\chi}T_a^*$, though it is rather formal. Hence $\phi_{0,\chi}=\phi_{a,0}=1$ and

$$\phi_{a,\chi+\chi'}=\phi_{a,\chi}\otimes\phi_{a,\chi'},\quad \phi_{a+a',\chi}=\phi_{a,\chi}\phi_{a',\chi}.$$

Let $\Lambda(\mathcal{M}):=m^*(\mathcal{M})\otimes p_1^*(\mathcal{M})^{-1}\otimes p_2^*(\mathcal{M})^{-1}$. With the notation in [MB85, p. 14] $\Lambda(\mathcal{M})=\mathcal{D}_2(\mathcal{M})$. By the theorem of square $(1_A\times\lambda_A)^*(\mathcal{P})\simeq\Lambda(\mathcal{M})$ as rigidified sheaves. We identify both sheaves via this (canonical) isomorphism. This induces the canonical isomorphism $\rho_{y,0}: T^*_{c^t(y)}(\mathcal{M})=\mathcal{M}\otimes\mathcal{M}(c^t(y))\otimes O_y$ as well as the canonical isomorphisms $\rho_{y,x}=\rho_{y,0}\otimes\Phi_{c^t(y),x}: T^*_{c^t(y)}(\mathcal{M}_x)=T^*_{c^t(y)}(\mathcal{M})\otimes T^*_{c^t(y)}(O_x)=\mathcal{M}_y\otimes\mathcal{M}(c^t(y))\otimes O_x\otimes O_x(c^t(y))$.

Definition 4.6. Modifying [Chai85, p.102] slightly, we define

$$\widetilde{R} := \text{the } R\text{-algebra generated by } \psi(x)\mathcal{M}_x\vartheta \quad (\forall x \in X)$$

where ϑ is an indeterminate with $\deg(\vartheta) = 1$, while $\deg(R) = \deg(\mathcal{M}_x) = 0$.

Let $\widetilde{Q} := \mathrm{Proj}(\widetilde{R})$ and $\widetilde{\mathcal{L}} := O_{\widetilde{Q}}(1)$. Let $\widetilde{R}^* := \widetilde{R}\otimes O_A$. We note $\mathrm{Proj}(\widetilde{R}) \simeq \mathrm{Proj}(\widetilde{R}^*)$ by [EGA, Chap. II, (2.4.7)]. Let $I(\mathcal{M}_x) := \psi(x)\mathcal{M}_x \subset \mathcal{M}_x\otimes_R k(\eta)$ and $\rho_y := \rho_{y,x}$ on $I(\mathcal{M}_x)$. We define the action S_y^* $(y \in X)$ on $\widetilde{R}$ as follows

$$\begin{aligned} S_y^*(\sigma_x\psi(x)\vartheta) &= \rho_y T_{c^t(y)}^*(\sigma_x)\psi(x)\psi(y)\tau(y,x)\vartheta \\ &= \rho_y T_{c^t(y)}^*(\sigma_x)\psi(x+y)\vartheta \in I(\mathcal{M}_{x+y})\vartheta \end{aligned}$$

where $\sigma_x\psi(x) \in I(\mathcal{M}_x)$. S_y^* is well-defined as an endomorphism of the graded algebra $\widetilde{R}$. Thus we have an endomorphism S_y of $(\widetilde{Q}, \widetilde{\mathcal{L}})$. In what follows since we always consider the canonical isomorphisms, we omit $\Phi_{a,\chi}$ and ρ_y if no confusion is possible.

Remark 4.7. Let $R[\mathcal{M}] := R \oplus \oplus_{k\geq 1} \mathcal{M}^k$ and $S(\mathcal{M}) = R[M] \otimes O_A$. It is well known that $\mathrm{Proj}(R[\mathcal{M}]) \simeq \mathrm{Proj}(S(\mathcal{M}))$ by [EGA, Chap. II, (2.4.7)].

Since $\mathcal{M}$ is an invertible sheaf on A, it is clear that $A \simeq \mathrm{Proj}(S(\mathcal{M}))$. Since $\widetilde{R} \otimes O_A$ is an O_A-algebra, we have a morphism $\pi : \widetilde{Q} \to A$ by the isomorphism $\widetilde{Q} \simeq \mathrm{Proj}(\widetilde{R} \otimes O_A)$. Then S_y induces a translation of A_η by $c^t(y)$ because

$$\begin{aligned} S_y^*(\mathrm{Proj}(R[\mathcal{M}]) \otimes k(\eta) &= \mathrm{Proj}(R[T_{c^t(y)}^*(\mathcal{M})\psi(y)] \otimes k(\eta)) \\ &= \mathrm{Proj}(O_A[\psi(y)\mathcal{M}_y] \otimes k(\eta)) \simeq A_\eta. \end{aligned}$$

Since A is the Néron model of A_η, S_y induces a translation of A. It is clear that $(\widetilde{Q}, \widetilde{\mathcal{L}}) \otimes k(\eta) = \mathrm{Proj}(R[\oplus_{y\in X}\psi(y)\mathcal{M}_y\vartheta] \otimes k(\eta)) \simeq (\widetilde{G}, \widetilde{\mathcal{L}}) \otimes k(\eta)$.

Definition 4.8. By a suitable finite base change of S if necessary we can construct a projective flat scheme $(P, \mathcal{L})$ over S with P_0 reduced. Let I the maximal ideal of R, s a generator of I and val_s the valuation of R with $val_s(s) = 1$. As in 4.4, by a finite base change of S if necessary, we choose and fix the extensions ψ, τ and c^t respectively to X, $X \times X$ and X so that Theorem 4.3 (3) and (4) are true for $\forall y, z \in X$ and $\lambda_A \cdot c^t = c$ on X. For $x, y \in X$ we define

$$\begin{gathered} B(x,y) = val_s(\tau(x,y)), \quad dA(\alpha, x) = B(\alpha, x) + r(x)/2 \\ A(x) = val_s(\psi(x)) = B(x,x)/2 + r(x)/2 \\ \bar{\tau}(x,y) = s^{-B(x,y)}\tau(x,y), \quad \bar{\psi}(x) = s^{-A(x)}\psi(x) \\ \bar{\tau}_0(x,y) = \bar{\tau}(x,y) \otimes k, \quad \bar{\psi}_0(x) = \bar{\psi}(x) \otimes k \end{gathered}$$

where there exists an $r \in \mathrm{Hom}_{\mathbf{Z}}(X, \mathbf{Z})$ as above. Since B is positive definite on $X \times X$ we have a Delaunay decomposition Del_B of $X_{\mathbf{R}}$. We denote a Delaunay cell by σ, τ and so on, and the centre of σ by $\alpha(\sigma)$.

We choose an affine U of A, the generators w_U^χ of O_U^χ and ϑ_U of $\mathcal{M}_U\vartheta$. Then we define $\xi_x = s^{A(x)}w_U^x$, $\xi_{x,c} = \xi_{x+c}/\xi_c$ and $\zeta_{x,c} = s^{B(\alpha(\sigma),x)+r(x)/2}w_U^x$

if $x + c \in C(c, \sigma)$ and $c \in \sigma \cap X$. Let $\widetilde{R}^*_U := \Gamma(O_U)[\psi(x)w^x_U\vartheta_U; x \in X]$. Then $\widetilde{Q}$ is covered with $\operatorname{Proj}\widetilde{R}^*_U$ ($\simeq \widetilde{Q}\times_A U$), U affine open subsets of A. In terms of $\zeta_{x,c}$, we can describe the structure of $(\widetilde{Q}, \widetilde{\mathcal{L}})$ and its normalization $(\widetilde{P}, \widetilde{\mathcal{L}})$ over U as in the totally degenerate case.

Theorem 4.9. *Let $A_0 := A \otimes \operatorname{Spec} k$, $g' = \operatorname{rank} X$, $g'' = \dim A_0$ and $g = g' + g''$. Suppose that $dA(\alpha(\sigma)) \in X^*$ for any g'-dimensional Delaunay cell $\sigma \in \operatorname{Del}_B$. Then A_0 is an abelian variety over k and $\widetilde{P}_0$ is a g-dimensional reduced scheme of locally of finite type over k. $\widetilde{P}_0$ is covered with affine A_0-schemes $U_0(c) = \operatorname{Spec} R_0(c)$ of finite type over A_0 and the following holds.*

1. *$R_0(c)$ is an O_{A_0}-module with the basis $\{\bar{\zeta}_{x,c}; x \in X\}$. If $x_1, \cdots, x_m$ are cellmates (resp. otherwise) the multiplication of them is defined by*
$$\bar{\zeta}_{x_1,c}\cdots\bar{\zeta}_{x_m,c} = \bar{\zeta}_{x_1+\cdots+x_m,c} \quad (\textit{resp. } 0)$$
2. *Let $T_0 := \operatorname{Spec} k[w^x; x \in X]$. Any irreducible component of $\widetilde{P}_0$ contains a T_0-torsor over A_0 whose class is given by $c_0 := c \otimes k \in \operatorname{Hom}(X, A_0^t)$.*
3. *$\widetilde{P}_0$ has a stratification by algebraic torus-torsors $O(\sigma)$, i.e. an extension of A_0 by a torus $T_0(\sigma)$ ($\sigma \in \operatorname{Del}_B$) where*
$$T_0(\sigma) = \operatorname{Spec} k[\bar{\zeta}^{\pm 1}_{x,c}; x \in C(c, \sigma) \cap X, c \in \sigma \cap X].$$
The extension class of $O(\sigma)$ is given by the restriction of c_0 to the $\mathbf{Z}$-submodule of X spanned by all $x \in X$ such that $x + c \in C(c, \sigma)$, $c \in \sigma \cap X$.
4. *S_y ($y \in Y$) transforms $O(\sigma + y)$ into $O(\sigma)$ where S_y induces a translation of A_0 by $c_0^t(y) := c^t(y) \otimes k \in A_0$, while it induces a morphism of fibres $T_0(\sigma + y)$ into $T_0(\sigma)$ as $S_y^*(\bar{\zeta}_{x,c}) = \bar{\tau}_0(x, y)T^*_{c^t(y)}(\bar{\zeta}_{x,c+y})$ ($c \in \sigma$).* [2]

Proof. (1) and (2) are clear. Any irreducible component of $\widetilde{P}_0$ contains (a copy of) $\widetilde{G}_0$, a T_0-torsor over A_0, whose extension class is given by c_0. See 2.2. For a Delaunay cell σ $O(\sigma)$ is also given by the push-out of G_0 to the subtorus $T_0(\sigma)$ of T_0, therefore its extension class is given by the restriction of c_0 to the $\mathbf{Z}$-submodule spanned by $x \in X$ such that $x + c \in C(c, \sigma)$, $c \in \sigma \cap X$. This proves (3).

Choosing w^X_U as in Definition 4.8 we can prove (4) on a dense open subset $U_0 \cap S_y(U_0)$ of A_0. In fact, by a formal computation we have
$$\begin{aligned} S_y^*(\xi_{x,c}) &= \psi(x + c + y)\psi(c)\psi(c + y)^{-1}\psi(x + c)^{-1}T^*_{c^t(y)}(\bar{\xi}_{x,c+y}) \\ &= \bar{\tau}(x, y)T^*_{c^t(y)}(\xi_{x,c+y}) \end{aligned}$$
which proves the formula (4). See also [Chai85, pp.76-77]. □

[2] If w^X_U are defined on an open set U, then (4) is true only over $U_0 \cap S_y(U_0)$ where $U_0 := U \otimes k$.

Theorem 4.10. *Let $(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ be the quotient formal scheme of $(\widetilde{P}_{\mathrm{for}}, \widetilde{\mathcal{L}}_{\mathrm{for}})$ by the periods Y. Then there exists a flat projective scheme $(P, \mathcal{L})$ over* $\operatorname{Spec} R$ *which algebraizes $(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$. Let $(P_0, \mathcal{L}_0)$ be the closed fibre of $(P, \mathcal{L})$. Then*

1. $(P_0, \mathcal{L}_0)$ *has the stratification by (the images of) $O(\sigma)$ ($\sigma \in \mathrm{Del}_B$ mod Y)*
2. $h^0(P_0, \mathcal{L}_0^n) = |X/Y| \deg(\mathcal{M}_0) n^g$, $h^i(P_0, \mathcal{L}_0^n) = 0$ $(i > 0)$
3. *For any affine open subset U of A such that $G \times_A U \simeq U \times T$, we define $w^x := w_U^x$ and $\eta_n(x) := \zeta_{x-nc,c} \xi_c^n$ if $\frac{x}{n} \in \sigma \cap \frac{X}{n}$ for a Delaunay cell σ and $c \in \sigma \cap X$ as in Theorem 3.9. Then we have*

$$\Gamma(P_0, \mathcal{L}_0^N) = \left\{ \sum_{x \in X} \sigma_x \eta_n(x); \begin{array}{c} \sigma_{x+ny} = \bar{\tau}_0(y,x) \bar{\psi}_0(y)^n T_{c^t(ny)}^*(\sigma_x) \\ \sigma_x \in \Gamma(A_0, \mathcal{M}_0^n \otimes O_x), \forall x \in X, \forall y \in Y \end{array} \right\}$$

where the right hand side is understood as those sections written over U_0.

Proof. Let $V(\sigma)$ be the closure of $O(\sigma)$. $V(\sigma)$ is a torus embedding bundle over A_0. In order to compute $H^i(V(\sigma), \mathcal{L}_0^n)$ we use the spectral sequence $E_2^{p,q} = H^p(A_0, R^q\pi(\sigma)_*\mathcal{L}_0^n)$ where $\pi(\sigma)$ is the projection of $V(\sigma)$ onto A_0. By the same reason as in Theorem 3.9 we see $R^q\pi(\sigma)_*(\mathcal{L}_0^n) = 0$ for $q, n > 0$ and $R^0\pi(\sigma)_*(\mathcal{L}_0^n) \simeq \oplus_{\frac{x}{n} \in \sigma \cap \frac{X}{n}} \mathcal{M}_0^n \otimes O_x$. Hence $H^p(V(\sigma), \mathcal{L}_0^n) = H^p(A_0, R^0\pi(\sigma)_*\mathcal{L}_0^n) = 0$ for $p > 0$. Moreover

$$\begin{aligned} H^0(V(\sigma), \mathcal{L}_0^n) &\simeq \oplus_{\frac{x}{n} \in \sigma \cap \frac{X}{n}} H^0(\mathcal{M}_0^n \otimes O_x) \\ &= \left\{ \sum_{\frac{x}{n} \in \sigma \cap \frac{X}{n}} c(x) \eta_n(x); c(x) \in H^0(A_0, \mathcal{M}_0^n \otimes O_x) \right\} \end{aligned}$$

where the right hand side is understood as the expressions over U_0 of those sections. The rest of the proof is the same as in Theorem 3.9. □

Remark 4.11. By choosing a trivialization of $\widetilde{\mathcal{L}}_{T_{\mathrm{for}}}^n$ induced from that of $\widetilde{\mathcal{L}}_{T_{\mathrm{for}}}$ we see that $\Gamma(P_\eta, \mathcal{L}_\eta^n)$ is identified with the $k(\eta)$ vector subspace of $\Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}^n) \otimes k(\eta)$ consisting of Fourier series $s = \sum_{x \in X} \sigma_x(s)$ such that

$$\begin{aligned} &\sigma_{x+ny}(s) = \tau(y,x) \psi(y)^n T_{c^t(ny)}^*(\sigma_x(s)), \\ &\quad \sigma_x(s) \in \Gamma(A, \mathcal{M}^n \otimes O_x) \otimes k(\eta) \quad (\forall x \in X, \forall y \in Y) \end{aligned}$$

Hence we have $\Gamma(P_\eta, \mathcal{L}_\eta^n) = \Gamma(G_\eta, \mathcal{L}_\eta^n)$ for any $n > 0$. It follows that $P_\eta \simeq G_\eta$.

4.12. **The group scheme G.** By choosing a suitable base change of S we assume $dA(\alpha(\sigma)) \in \mathrm{Hom}(X, \mathbf{Z})$ for any $\sigma \in \mathrm{Del}_B$. Then a closed fibre P_0 of P is reduced. Now we prove that G is realized as an open subscheme of P. By the assumption we choose (and fix) a maximal dimensional Delaunay cell $\sigma \in \mathrm{Del}(0)$ and define $w^x := \zeta_{x,0}$ of $O(\sigma)$ $(x \in C(\sigma, 0))$. This is equivalent to assuming the condition $B(\alpha(\sigma), x) + r(x)/2 = 0$ $(\forall x \in X)$ (in fact, we may assume after taking a finite base change of S). By choosing $\zeta_{x,0}$

(instead of w^x) as the coordinates of T in the construction of $(P, \mathcal{L})$, we obtain the same $(P, \mathcal{L})$. Hence we have an open subscheme $\widetilde{G}(\sigma) \subset \widetilde{P}$ such that $\widetilde{G}(\sigma)_0 = O(\sigma)_0$, a torus torsor over A_0. We now imitate the argument of [Mumford72, p. 297-p. 303]. Let $S_n = \mathrm{Spec}\,(R/I^n)$, $\widetilde{G}^\dagger := \cup_{y\in Y} S_y(\widetilde{G}(\sigma)) \subset \widetilde{P}$ and $G(\sigma)_n := (\widetilde{G}^\dagger \times_S S_n)/Y \subset P_n$. Clearly $G(\sigma)_n \simeq \widetilde{G}(\sigma) \times_S S_n$.

Let $\widetilde{B}(\sigma) := \widetilde{P} \setminus \widetilde{G}^\dagger$ and make it into a reduced closed subscheme. Then $B(\sigma)_{\mathrm{for}}$, the formal completion of $\widetilde{B}(\sigma)/Y$, is algebraized into a reduced closed subscheme $B(\sigma)$ of P. We let $G(\sigma) := P \setminus B(\sigma)$. Then by the argument in [Mumford72, 4.1] $G(\sigma)$ is smooth and a group scheme over S with $G(\sigma)_\eta \simeq P_\eta$. It is clear that $G(\sigma)_0$ is connected and semi-abelian. Moreover

$$G(\sigma)_{\mathrm{for}} \simeq \lim_n G(\sigma)_n \simeq \lim_n \widetilde{G}(\sigma)_n \simeq \widetilde{G}(\sigma)_{\mathrm{for}}. \tag{13}$$

In order to construct $(P, \mathcal{L})$ we started from a semi-abelian scheme $(G, \mathcal{L})$, and then constructed $(\widetilde{G}, \widetilde{\mathcal{L}})$, where $G_{\mathrm{for}} \simeq \widetilde{G}_{\mathrm{for}}$ and $\widetilde{G}(\sigma) = \widetilde{G}$ by the definition. It follows that $G(\sigma)_{\mathrm{for}} \simeq G_{\mathrm{for}}$. By the universal property of G there is a group scheme morphism $\rho : G(\sigma) \to G$ extending the isomorphism $G(\sigma)_\eta \simeq P_\eta \simeq G_\eta$, which keeps the formal isomorphism $G(\sigma)_{\mathrm{for}} \simeq G_{\mathrm{for}}$ unchanged. It follows that ρ is an isomorphism. Hence $G(\sigma) \simeq G$. This proves $G \subset P$.

4.13. **The group scheme $G^\sharp$.** Let $\widetilde{G}^\sharp := \cup_{x\in X} S_x(\widetilde{G}(\sigma)) \subset \widetilde{P}$. Let $\widetilde{B}^\sharp := \widetilde{P} \setminus \widetilde{G}^\sharp$ and make it into a reduced closed subscheme. Then $B^\sharp_{\mathrm{for}}$, the formal completion of $\widetilde{B}^\sharp/Y$, is algebraized into a reduced closed subscheme $B^\sharp$ of P. We let $G^\sharp := P \setminus B^\sharp$. In the same manner as before we see $G^\sharp$ is a semi-abelian group scheme with $G^\sharp_\eta \simeq P_\eta$ such that G is an open group subscheme of $G^\sharp$.

Let e be the unit section of G and $e(x) := S_x(e) \in G^\sharp(S)$ $(x \in X)$. We define $\tilde{e}$ and $\tilde{e}(x) \in \widetilde{G}^\sharp(S)$ by

$$\begin{aligned} \tilde{e} :&\zeta_{z,0} = 1,\ (\forall z \in C(0,\sigma)),\ \zeta_{z,0} = 0\ (\forall z \notin C(0,\sigma)) \\ \tilde{e}(x) :&\zeta_{z,-x} = \bar{\tau}(z,x)\ (\forall z \in C(0,\sigma)),\ \zeta_{z,-x} = 0\ (\forall z \notin C(0,\sigma)) \end{aligned}$$

plus the sections $c^t(0)$ (resp. $c^t(x)$) of A. By definition $e_{\mathrm{for}} = \tilde{e}_{\mathrm{for}}$ and $e(x)_{\mathrm{for}} = \tilde{e}(x)_{\mathrm{for}}$ as the formal sections of $P_{\mathrm{for}} = \widetilde{P}_{\mathrm{for}}/Y$. Since $S_x(U_{-x}) = U_0$ and both S_x, $T_{\tilde{e}(x)}$ are translations by $c^t(x)$ in the direction of A, it follows from $S_x^*(\zeta_{z,-x}) = T^*_{c^t(x)}(\zeta_{z,0})\bar{\tau}(z,x)$ that $S_x = T_{\tilde{e}(x)}$ on $\widetilde{G}^\sharp$. Since $S_xS_y = S_yS_x$ $(\forall y \in Y)$, S_x is descended to an action on P, which we again denote by S_x. On the other hand since $e(x) \in G^\sharp(S)$, the translation $T_{e(x)}$ of $G^\sharp$ by $e(x)$ is well-defined. Since $S_{x,\mathrm{for}} = T_{\tilde{e}(x),\mathrm{for}} = T_{e(x),\mathrm{for}}$ on $G^\sharp_{\mathrm{for}}$, we have $S_x = T_{e(x)}$ on $G^\sharp$.

Lemma 4.14. *The flat closure $K^\sharp_S(\mathcal{L}_\eta)$ of $K(\mathcal{L}_\eta)$ in $G^\sharp$ is finite over S.*

Proof. In fact, $K_S^\sharp(\mathcal{L}_\eta)$ is quasi-finite and there is the maximal flat finite subscheme $K_S^\sharp(\mathcal{L}_\eta)^f$ of $K_S^\sharp(\mathcal{L}_\eta)$ and a flat quasi-finite subscheme Z such that $K_S^\sharp(\mathcal{L}_\eta) = K_S^\sharp(\mathcal{L}_\eta)^f \amalg Z$ where $Z \cap G_0^\sharp = \emptyset$. By definition $K_S(\mathcal{L}_\eta) \subset K_S^\sharp(\mathcal{L}_\eta)$, $K_S(\mathcal{L}_\eta)^f \subset K_S^\sharp(\mathcal{L}_\eta)^f$ and $K_S(\mathcal{L}_\eta)^{fn} = K_S^\sharp(\mathcal{L}_\eta)^{fn} := K_S^\sharp(\mathcal{L}_\eta)^f \cap G$. Moreover $K_S(\mathcal{L}_\eta)^m = K_S^\sharp(\mathcal{L}_\eta)^m$, the maximal flat finite subgroup S-scheme of $K_S^\sharp(\mathcal{L}_\eta)^{fn}$ such that its intersection with G_0 is a subgroup scheme of T_0. See Subsection 2.3. Hence we have $\operatorname{ord} K_S^\sharp(\mathcal{L}_\eta)^m = d_t$, $\operatorname{ord} K_S^\sharp(\mathcal{L}_\eta)^{fn} = d_a^2 d_t$, and $\operatorname{ord}_{k(\eta)} K(\mathcal{L}_\eta) = d_a^2 d_t^2$ by Subsection 2.4.

Let H^* be the étale finite flat subgroup scheme of $G^\sharp$ consisting of $e(x)$ $(x \in X/Y)$. Since $T_{e(x)}^*(\mathcal{L}) = S_x^*(\mathcal{L}) \simeq \mathcal{L}$ by Subsection 4.13, H^* is a subgroup scheme of $K_S^\sharp(\mathcal{L}_\eta)$. Since H^* is étale finite, the subscheme

$$H^* \cdot K_S^\sharp(\mathcal{L}_\eta)^{fn} := \cup_{x \in X/Y} S_x \cdot K_S^\sharp(\mathcal{L}_\eta)^{fn} \simeq H^* \oplus K_S^\sharp(\mathcal{L}_\eta)^{fn}$$

is a finite flat subgroup scheme of $K_S^\sharp(\mathcal{L}_\eta)$ of order $d_a^2 d_t^2$. This shows that $Z = \emptyset$, $K_S^\sharp(\mathcal{L}_\eta) = K_S^\sharp(\mathcal{L}_\eta)^f = H^* \cdot K_S^\sharp(\mathcal{L}_\eta)^{fn} = H^* \oplus K_S^\sharp(\mathcal{L}_\eta)^{fn}$. Hence $K_S^\sharp(\mathcal{L}_\eta)$ is finite. □

Definition 4.15. Let $\mathcal{L}^{\mathrm{tsr}}$ be the $\mathbf{G}_m$-torsor on $G^\sharp$ associated with the invertible sheaf $\mathcal{L}_{G^\sharp}$. Let $\mathcal{G}_S^\sharp(\mathcal{L}_\eta) := \mathcal{L}^{\mathrm{tsr}}_{|K_S^\sharp(\mathcal{L}_\eta)}$ and $e_S^\sharp$ an extension of $e^{\mathcal{L}_\eta}$. By [MB85, IV, 7.1 (ii)] $\mathcal{G}_S^\sharp(\mathcal{L}_\eta)$ is a group scheme over S. See [MB85, I, 3.1, 3.2] for the definition of the group scheme structure of $\mathcal{G}_S^\sharp(\mathcal{L}_\eta)$. By the natural projection of $\mathcal{L}^{\mathrm{tsr}}$ onto $\mathcal{G}^\sharp$, it is a central extension of $K_S^\sharp(\mathcal{L}_\eta)$ by $\mathbf{G}_{m,S}$ with $e_S^\sharp$ the commutator form as is given in Subsection 2.4. The bilinear form $e_S^\sharp$ on $K_S^\sharp(\mathcal{L}_\eta)$ is nondegenerate alternating by [MB85, IV, 2.4] and by Lemma 4.14. We note $G^\sharp = A^{[1]}$ with the notation of [MB85, IV, 7.1].

By [MB85, I, 3.1, 3.2] $\mathcal{G}_S^\sharp(\mathcal{L}_\eta)$ acts on $\Gamma(G^\sharp, \mathcal{L})$ in the natural manner. Moreover $\Gamma(G^\sharp, \mathcal{L})$ is an irreducible $\mathcal{G}_S^\sharp(\mathcal{L}_\eta)$-module by [MB85, V, 2.4.2; VI, 1.4.7].

5. $(Q, \mathcal{L})$

5.1. **The totally degenerate case.** We consider the totally degenerate case.

From Section 3 we recall

$$\begin{aligned}
&\widetilde{R} := R[a(x)w^x\vartheta; x \in X] \simeq R[\xi_x\vartheta; x \in X]\\
&\xi_x := s^{B(x,x)/2 + r(x)/2} w^x, \quad \xi_{x,c} := \xi_{x+c}/\xi_c\\
&S_y^*(a(x)w^x\vartheta) = a(x+y)w^{x+y}\vartheta
\end{aligned}$$

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Let $\widetilde{Q} := \operatorname{Proj} \widetilde{R}$ and $\widetilde{\mathcal{L}} := O_{\widetilde{Q}}(1)$.

Lemma 5.2. 1. *If $\xi_{x_1}\cdots\xi_{x_m}s^{\alpha}=1$ for some $x_i\in X$ and $\alpha\geq 0$, then $x_i=\alpha=0$.*
2. *If $B(a,b)>0$, then ξ_{a+b} is divisible by $s\xi_a\xi_b$. If $B(a,b)<0$, then $\xi_a\xi_b$ is divisible by $s\xi_{a+b}$.*
3. *For any $x\in X$ there exist $x_i\in \mathrm{Star}\,(0)\cap X$ with $x=x_1+\cdots+x_m$ such that ξ_x is a product of $\xi_{x_1}\cdots\xi_{x_m}$ and some power of s.*
4. *If $x\notin \mathrm{Star}\,(0)$, then ξ_x is divisible by s.*

Proof. We will prove (1). By assumption $x_1+\cdots+x_m=0$. Hence

$$B(x_1,x_1)+\cdots+B(x_m,x_m)+2\alpha=0$$

from which it follows $x_i=\alpha=0$. This proves (1).

If $B(a,b)>0$, then $B(a+b,a+b)=B(a,a)+B(b,b)+2B(a,b)>B(a,a)+B(b,b)$. Hence ξ_{a+b} is divisible by $s\xi_a\xi_b$. This proves (2).

We prove (3) by the induction on $B(x,x)$. If $x\in C(0,\sigma)\cap X\setminus\sigma$ for some $\sigma\in \mathrm{Del}(0)$, then we can choose $a\in\sigma\cap X$ such that $B(x-a,a)>0$ by Lemma 1.4. Hence ξ_x is divisible by $s\xi_{x-a}\xi_a$ and $B(x,x)>B(x-a,x-a)+B(a,a)>B(x-a,x-a)$. Hence by the induction hypothesis, ξ_{x-a} is a product of s and ξ_{a_i}, $a_i\in\mathrm{Star}\,(0)\cap X$. (4) follows from the proof of (3). □

Theorem 5.3. *Let $(\widetilde{Q},\widetilde{\mathcal{L}})=(\mathrm{Proj}\ \widetilde{R},O_{\mathrm{Proj}\ \widetilde{R}}(1))$. Then*
1. *$\widetilde{Q}$ is covered with open affine subschemes $W(c):=\mathrm{Spec}\ S(c)$ where*
$$S(c):=R[\xi_{x,c};x\in X]\quad (c\in X).$$
2. *The coordinate ring $S(c)$ of $W(c)$ is an R-algebra of finite type generated by $\xi_{x,c}$ ($x\in\mathrm{Star}\,(0)\cap X$). All the ring $S(c)$ are isomorphic to each other.*
3. *Let $x_i,y_j\in X$ ($i\in I,j\in J$). If $p:=\sum_{i\in I}\ A(x_i)-\sum_{j\in J}\ A(y_j)\geq 0$, then*
$$\prod_{i\in I}\xi_{x_i,c}=s^p\prod_{j\in J}\xi_{y_j,c}$$

The assertion (2) follows from Lemma 5.2.

Definition 5.4. We set

$$v(x)=\min\{\mathrm{val}_s(\xi_{x_1}\cdots\xi_{x_m});x=x_1+\cdots+x_m,x_i\in X,m\geq 1\}$$

Then we define $\xi(x)=s^{v(x)}w^x\in\Gamma(W(0),O_{\widetilde{Q}})$. Let $v(x,c)=v(x)+B(x,c)$, $\xi(x,c)=s^{v(x)+B(x,c)}w^x$ and $\bar{\xi}(x,c):=\xi(x,c)\otimes k$. In particular $v(x)=v(x,0)$ and $\xi(x)=\xi(x,0)$. We note that $S_d^*(\xi(x,c))=\bar{\tau}(x,d)\xi(x,c+d)$. It is clear that

$$\Gamma(W(c),O_{\widetilde{Q}})\otimes k=\oplus_{x\in X}\ k\cdot\xi(x,c)$$

Lemma 5.5. *Let $\sigma\in\mathrm{Del}(0)$ and define $\mathrm{Semi}(0,\sigma)$ to be the semigroup generated by $\sigma\cap X$. Let $\bar{\xi}_x:=\xi_x\otimes k\in S(0)\otimes k$ ($x\in X$).*
1. *If $x\notin\mathrm{Star}\,(0)\cap X$, then $\bar{\xi}_x=0$.*

2. *If* $x_1, \cdots, x_m \in \mathrm{Star}\,(0)$ *are not cellmates, then the product* $\bar{\xi}_{x_1} \cdots \bar{\xi}_{x_m}$ *is either zero or nilpotent.*
3. *If* $x_1, \cdots, x_m \in \mathrm{Star}\,(0)$ *are not cellmates and if* $x := x_1 + \cdots + x_m \in \mathrm{Semi}(0, \sigma)$ *for some* $\sigma \in \mathrm{Del}(0)$*, then* $\bar{\xi}_{x_1} \cdots \bar{\xi}_{x_m}$ *is zero.*
4. *If the product* $\bar{\xi}_{x_1} \cdots \bar{\xi}_{x_m}$ *is nonzero, then* $B(x_i, x_j) \geq 0$ *for any* i, j*.*

Proof. (1) follows from Lemma 5.2 (4). Next we prove (2).

Let $x = x_1 + \cdots + x_m$ and $\sigma \in \mathrm{Del}^{\max}(0)$ a maximal-dimensional Delaunay cell containing x, and $\alpha(\sigma)$ the centre of σ. Then there exist some $n \in \mathbf{Z}_+$, $n_i \in \mathbf{Z}_0$ and an $a_i \in \sigma \cap X$ such that $nx = n_1 a_1 + \cdots + n_r a_r$. We also have

$$
\begin{aligned}
n \sum_{i=1}^{m} B(x_i, x_i) &\geq 2nB(\alpha(\sigma), \sum_{i=1}^{m} x_i) = 2B(\alpha(\sigma), nx) \\
&= 2 \sum_{i=1}^{r} n_i B(\alpha(\sigma), a_i) = \sum_{i=1}^{r} n_i B(a_i, a_i)
\end{aligned}
$$

Since x_i are not cellmates, there is at least an x_i such that $x_i \notin \sigma$, hence $B(x_i, x_i) > 2B(\alpha(\sigma), x_i)$. Therefore the above inequality is strict. This implies that $(\bar{\xi}_{x_1} \cdots \bar{\xi}_{x_m})^n$ is zero in O_{Q_0}. This proves (2).

Suppose $x := x_1 + \cdots + x_m \in \mathrm{Semi}(0, \sigma)$. Then we see

$$
2B(\alpha(\sigma), x) = \sum_{i=1}^{m} 2B(\alpha(\sigma), x_i) = \sum_{i=1}^{m} B(x_i, x_i).
$$

Since x_i are not cellmates, there is an x_i such that $2B(\alpha(\sigma), x_i) < B(x_i, x_i)$. Therefore $\xi_{x_1} \cdots \xi_{x_m}$ is divisible by $s \cdot s^{2B(\alpha(\sigma), x)} w^x$. On the other hand, since $x \in \mathrm{Semi}(0, \sigma)$, there are $a_i \in \sigma \cap X$ such that $x = \sum_{j=1}^{r} n_i a_j$. Then $s^{2B(\alpha(\sigma), x)} w^x = \prod \xi_{a_j}^{n_j}$. This proves (3). (4) is clear from Lemma 5.2 (2). □

Corollary 5.6. *Let* σ *be a Delaunay cell with* $0 \in \sigma$*, and* $\mathrm{Semi}(0, \sigma)$ *the semi-group generated by* $\sigma \cap X$*. If* $x \in \mathrm{Semi}(0, \sigma)$*, then* $\bar{\xi}(x, c)$ *is a product of* $\bar{\xi}_{a,c}$ $(a \in \sigma \cap X)$*, which is not nilpotent. If* $x \in C(0, \sigma) \cap X \setminus \mathrm{Semi}(0, \sigma)$*, then* $\bar{\xi}(x, c)$ *is nonzero and nilpotent.*

Theorem 5.7. *Let* $(\widetilde{Q}_0, \widetilde{\mathcal{L}}_0)$ *be the closed fibre of* $(\widetilde{Q}, \widetilde{\mathcal{L}})$*. Then*

1. $\widetilde{Q}_0$ *is covered with finitely many open affine subschemes of finite type*
$$
W_0(c) := W(c) \otimes k = \mathrm{Spec}\; k[\bar{\xi}_{x,c}; x \in X] \quad (c \in X).
$$
2. *The coordinate ring* $S_0(c) := k[\bar{\xi}_{x,c}; x \in X]$ *of* $W_0(c)$ *is a* k*-algebra of finite type possibly with nilpotent elements. Let* $x_i, y_j \in X$ $(i \in I, j \in J)$*. Then*
$$
\prod_{i \in I} \bar{\xi}_{x_i,c} = \begin{cases} \prod_{j \in J} \bar{\xi}_{y_j,c} & \text{if } \sum_{i \in I} A(x_i) = \sum_{j \in J} A(y_j) \\ 0 & \text{if } \sum_{i \in I} A(x_i) > \sum_{j \in J} A(y_j) \end{cases}
$$
3. $S_0(c)$ *is generated by* $\bar{\xi}_{x,c}$ $(x \in \mathrm{Star}\,(0) \cap X)$*.*

4. *Let* $S_0^{\mathrm{red}}(c)$ *be the coordinate ring of* $W_0(c)_{\mathrm{red}}$. *Then* $S_0^{\mathrm{red}}(c)$ *is generated by* $\bar{\xi}_{x,c}$ $(x \in \mathrm{Star}\,(0) \cap X)$ *as a* k*-algebra. It is naturally a* k*-submodule of* $R_0(c)$ *generated by* $\bar{\xi}(x,c)$ *for* $x \in \mathrm{Semi}(0,\sigma), \sigma \in \mathrm{Del}(0)$. *The product of* $\bar{\xi}_{x_i,c}$ $(1 \le i \le m)$ *in* $S_0^{\mathrm{red}}(c)$ *is nonzero if and only if* $x_i \in \mathrm{Star}\,(0) \cap X$ *are cellmates.*

Proof. The assertions (1) and (2) are clear from Theorem 5.3. (3) follows from Lemma 5.2. (4) follows from Corollary 5.6. See also Theorem 3.7. □

The construction of the quotient $(Q,\mathcal{L}) := (\widetilde{Q},\widetilde{\mathcal{L}})/Y$ is quite similar to [Mumford72]. See also [AN96] and Theorem 3.5.

Theorem 5.8. *The quotient* $(\widetilde{Q}_{\mathrm{for}},\widetilde{\mathcal{L}}_{\mathrm{for}})/Y$ *is a flat projective formal* S*-scheme. There exists a flat projective* S*-scheme* $(Q,\mathcal{L})$ *such that* $(Q_{\mathrm{for}},\mathcal{L}_{\mathrm{for}}) \simeq (\widetilde{Q}_{\mathrm{for}},\widetilde{\mathcal{L}}_{\mathrm{for}})/Y$. *Moreover* $(P_\eta,\mathcal{L}_\eta) \simeq (Q_\eta,\mathcal{L}_\eta) \simeq (G_\eta,\mathcal{L}_\eta)$ *and* $(P,\mathcal{L})$ *is the normalization of* $(Q,\mathcal{L})$.

Proof. By the definition of P and Q P_{for} is finite over Q_{for} hence there is a finite surjective morphism $f: P \to Q$. Hence $\Gamma(Q_\eta,\mathcal{L}_\eta^n) \subset \Gamma(P_\eta,\mathcal{L}_\eta^n) = \Gamma(G_\eta,\mathcal{L}_\eta^n)$ by Remark 4.11. It is clear from Remark 4.11 and the definition of $\widetilde{Q}$ that any element of $\Gamma(G_\eta,\mathcal{L}_\eta^n)$ has some R-multiple contained in $\Gamma(\widetilde{Q}_{\mathrm{for}},\widetilde{\mathcal{L}}_{\mathrm{for}}^n)^{Y-\mathrm{inv}}$. Therefore

$$\Gamma(G_\eta,\mathcal{L}_\eta^n) \subset \Gamma(\widetilde{Q}_{\mathrm{for}},\widetilde{\mathcal{L}}_{\mathrm{for}}^n)^{Y-\mathrm{inv}} \otimes k(\eta) = \Gamma(Q,\mathcal{L}^n) \otimes k(\eta) \subset \Gamma(Q_\eta,\mathcal{L}_\eta^n)$$

It follows $\Gamma(P_\eta,\mathcal{L}_\eta^n) = \Gamma(Q_\eta,\mathcal{L}_\eta^n)$ hence $(P_\eta,\mathcal{L}_\eta) \simeq (Q_\eta,\mathcal{L}_\eta) \simeq (G_\eta,\mathcal{L}_\eta)$. Hence P is the normalization of Q. We note that this proves Theorem 0.1. □

Definition 5.9. We call the closed fibre $(Q_0,\mathcal{L}_0)$ of $(Q,\mathcal{L})$ *a projectively stable quasi-abelian scheme* over $k := R/I$.

The Delaunay decomposition describes the closed fibre by the following

Theorem 5.10. *Let* Q_0 *be the closed fibre of the flat family* $(Q,\mathcal{L})$ *and* Del *the Delaunay decomposition corresponding to* $(Q,\mathcal{L})$. *Let* σ,τ *be Delaunay cells in* Del.

1. *For each* σ *there exists a* T*-invariant subscheme* $O_Q(\sigma)$ *of the closed fibre* Q_0 *which is a torus of dimension* $\dim\sigma$ *over* k *such that* Q_0 *is the disjoint union of* $O_Q(\sigma)$
2. $\sigma \subset \tau$ *iff* $O_Q(\sigma)$ *is contained in* $O_Q(\tau)$,
3. *For each* σ *there exists a unique closed reduced subscheme* $Z(\sigma)$ *which is the closure of* $O_Q(\sigma)$ *and has dimension* $\dim\sigma$ *over* k, *where the scheme structure of* $Z(\sigma)$ *is defined by*

$$Z(\sigma) \cap W(c) = \mathrm{Spec}\; k[\bar{\xi}_{x-c,c}; x \in \sigma \cap X] \quad (c \in \sigma \cap X)$$

4. $\sigma \subset \tau$ *iff* $Z(\sigma)$ *is contained in* $Z(\tau)$,
5. $Q_0 = \bigcup_{\sigma \in \mathrm{Del} \mod Y} Z(\sigma)$

The general case where the abelian part A_0 is nontrivial is discussed in a manner similar to Section 4. We omit the details.

Definition 5.11. With the notation in Definition 4.15 we define

$$\begin{aligned} K(Q,\mathcal{L}) &= K(P,\mathcal{L}) = K_S^{\sharp}(\mathcal{L}_\eta)\\ \mathcal{G}(Q,\mathcal{L}) &= \mathcal{G}(P,\mathcal{L}) = \mathcal{G}_S^{\sharp}(\mathcal{L}_\eta)\\ K(G_\eta,\mathcal{L}_\eta) &= K(\mathcal{L}_\eta) = K(Q,\mathcal{L})\otimes k(\eta)\\ \mathcal{G}(G_\eta,\mathcal{L}_\eta) &= \mathcal{G}(\mathcal{L}_\eta) = \mathcal{G}(Q,\mathcal{L})\otimes k(\eta)\\ K(Q_0,\mathcal{L}_0) &= K(P_0,\mathcal{L}_0) := K(Q,\mathcal{L})\otimes k,\\ \mathcal{G}(Q_0,\mathcal{L}_0) &= \mathcal{G}(P_0,\mathcal{L}_0) = \mathcal{G}(Q,\mathcal{L})\otimes k \end{aligned}$$

We call $K(Q,\mathcal{L})$ (resp. $\mathcal{G}(Q,\mathcal{L})$) the abelian Heisenberg group scheme (resp. the Heisenberg group scheme) of $(Q,\mathcal{L})$. We call $K(Q_0,\mathcal{L}_0)$ (resp. $\mathcal{G}(Q_0,\mathcal{L}_0)$) the abelian Heisenberg group scheme (resp. the Heisenberg group scheme) of $(Q_0,\mathcal{L}_0)$. We will see in Lemma 5.14 that $K(Q_0,\mathcal{L}_0)$ is independent of the choice of $(Q,\mathcal{L})$. Therefore by Definition 4.15 and by [MB85, I, 3.1, 3.2] $\mathcal{G}(Q_0,\mathcal{L}_0)$ as well as its group scheme structure is uniquely determined by $(Q_0,\mathcal{L}_0)$.

Lemma 5.12. *Suppose $G(\sigma) = G \subset P$ for some $\sigma \in \mathrm{Del}^{\max}(0)$. Let H^* be the subgroup consisting of all $e(x)$ $(x \in X/Y)$ in Lemma 4.14 and $G^\sharp := H^*\cdot G$. Then*

1. $\Gamma(Q,\mathcal{L}) = \Gamma(P,\mathcal{L}) = \Gamma(G^\sharp,\mathcal{L})$,
2. $\Gamma(Q,\mathcal{L})$ *is an irreducible $\mathcal{G}(Q,\mathcal{L})$-module of weight one, that is by definition, any $\mathcal{G}(Q,\mathcal{L})$-submodule of $\Gamma(Q,\mathcal{L})$ of weight one is of the form $J\Gamma(Q,\mathcal{L})$ for some ideal J of R,*
3. $\Gamma(Q,\mathcal{L})\otimes k$ *is an irreducible $\mathcal{G}(Q_0,\mathcal{L}_0)$-module of weight one.*

Proof. Since $K_S^\sharp(\mathcal{L}_\eta)$ is finite over S by Lemma 4.14, $\Gamma(G^\sharp,\mathcal{L})$ is an irreducible $\mathcal{G}_S^\sharp(\mathcal{L}_\eta)$-module by [MB85, V, 2.4.2; VI, 1.4.7]. To fix an embedding $G = G(\sigma)\subset P$, we choose σ with $B(\alpha(\sigma),x)+r(x)/2 = 0$ for any $x \in X$. See 4.12. Since $\Gamma(P,\mathcal{L})$ is R-free, we have $\Gamma(P,\mathcal{L}) \hookrightarrow \Gamma(P_\eta,\mathcal{L}_\eta) = \Gamma(G_\eta^\sharp,\mathcal{L}_\eta) = \Gamma(G_\eta,\mathcal{L}_\eta)$. Hence $\Gamma(P,\mathcal{L})\subset\Gamma(G^\sharp,\mathcal{L})$. Since both $\Gamma(P,\mathcal{L})$ and $\Gamma(G^\sharp,\mathcal{L})$ are $\mathcal{G}(P,\mathcal{L})$-modules, there is an ideal J of R such that $\Gamma(P,\mathcal{L}) = J\Gamma(G^\sharp,\mathcal{L})$ by [MB85, V, 2.4] because $\Gamma(G^\sharp,\mathcal{L})$ is an irreducible $\mathcal{G}(P,\mathcal{L})$-module. If $J\neq R$, then the image of $\Gamma(P,\mathcal{L})\otimes k$ in $\Gamma(G^\sharp,\mathcal{L})\otimes k$ is zero. However if $x\in \mathrm{Semi}(0,\sigma)\cap X$, then $\eta_1(x) = \xi_x = w^x$ and $\theta\otimes k = \sigma_0 w^x + \cdots$ is nontrivial if we choose a nontrivial $\sigma_0\in\Gamma(A_0,\mathcal{M}_0)$. Hence $J = R$ and $\Gamma(P,\mathcal{L}) = \Gamma(G^\sharp,\mathcal{L})$. Since P is the normalization of Q, we have $\Gamma(Q,\mathcal{L})\subset\Gamma(P,\mathcal{L})$. By the same reason as before there exists an ideal J of R such that $\Gamma(Q,\mathcal{L}) = J\Gamma(P,\mathcal{L})$. By Theorems 3.9 and 4.10 we have a basis of $\Gamma(P,\mathcal{L})$ very explicitly. Any element of $\Gamma(P,\mathcal{L})$ is easily seen to belong to $\Gamma(Q,\mathcal{L})$. Thus as before we see $J = R$, $\Gamma(Q,\mathcal{L}) = \Gamma(P,\mathcal{L})$. (2) follows from (1) and [MB85, V, 2.4.2; VI, 1.4.7]. (3) is a corollary to (2). □

Definition 5.13. Let $k = R/I$, I maximal. Let $(Q_0, \mathcal{L}_0)$ be a projective stable quasi-abelian k-scheme. We call a k-submodule V of $\Gamma(Q_0, \mathcal{L}_0)$ *the theta k-module* of $(Q_0, \mathcal{L}_0)$ if $V = \Gamma(Q, \mathcal{L}) \otimes k$. We note that by Theorem 4.10 $\Gamma(Q, \mathcal{L}) \otimes k$ is the unique submodule of $\Gamma(Q_0, \mathcal{L}_0)$ generated by a sum of monomials ξ_a with $\Gamma(A_0, M_a)$ coefficients $(a \in X)$ invariant under the action of S_y $(y \in Y)$. This is restated as

$$V = \Gamma(Q_0, \mathcal{L}_0) \cap (\oplus_{a \in X} \Gamma(A_0, M_a)\xi_a) \tag{14}$$

Lemma 5.14. *We define a morphism $\lambda(\mathcal{L}_0) : G_0^\sharp \to \mathrm{Pic}^0(Q_0)$ by*

$$\lambda(\mathcal{L}_0)(a) = T_a^*(\mathcal{L}_0) \otimes \mathcal{L}_0^{-1}$$

for any U-valued point a of $G_0^\sharp$, U any k-scheme. Then $K(Q_0, \mathcal{L}_0) = \ker \lambda(\mathcal{L}_0)$.

Proof. First we recall $\mathrm{Pic}^0(Q_0)$ is a k-scheme by [FGA, 232, Corollaire 6.6]. By Definition 5.11 and the proof of Lemma 4.14 $K(Q_0, \mathcal{L}_0) := K_S^\sharp(\mathcal{L}_\eta)_0 = H_0^* \oplus K_S^\sharp(\mathcal{L}_\eta)_0^{fn}$ and moreover the sequence

$$1 \to K_S^\sharp(\mathcal{L}_\eta)_0^m \to K_S^\sharp(\mathcal{L}_\eta)_0^{fn} \to K(A_0, \mathcal{M}_0) \to 0 \tag{15}$$

is exact. We note $G_0^\sharp = \cup_{x \in X/Y} S_x(G_0) \simeq H_0^* \oplus G_0$, and $S_x = T_{e(x)_0}$ on $G_0^\sharp$ by Subsection 4.13.

Let U be any connected k-scheme. Suppose $a \in G_0^\sharp(U)$ such that $T_a^*(\mathcal{L}_0) \simeq \mathcal{L}_0$. Then there exists $x \in (X/Y)(U)$ and $a = b \cdot e(x)_{0,U}$ for some $b \in G_0(U)$. Since $T_{e(x)}^*(\mathcal{L}_{0,U}) = S_x^*(\mathcal{L}_{0,U}) \simeq \mathcal{L}_{0,U}$, we see $T_b^*(\mathcal{L}_{0,U}) \simeq \mathcal{L}_{0,U}$. Since $\mathcal{L}_{0|G_{0,U}} \simeq \pi_{0,U}^*(\mathcal{M}_{0,U})$, we have $T_{\pi_{0,U}(b)}^*(\mathcal{M}_{0,U}) \simeq \mathcal{M}_{0,U}$, hence $\pi_{0,U}(b) \in K(A_0, \mathcal{M}_0)(U)$. Hence there is $c \in K_S^\sharp(\mathcal{L}_\eta)_0^{fn}(U)$ such that $\pi_{0,U}(c) = \pi_{0,U}(b)$ and $T_c^*(\mathcal{L}_{0,U}) \simeq \mathcal{L}_{0,U}$. Thus $d := b \cdot c^{-1} \in T_0(U)$ satisfies $T_d^*(\mathcal{L}_{0,U}) \simeq \mathcal{L}_{0,U}$ where $T_0 = G_0^m \subset G_0$. It follows that there exists $\alpha \in \mathrm{Hom}(X, \mathbf{G}_m)(U)$ such that $T_d^*(w^x) = \alpha(x)w^x$. T_d induces an endomorphism of $\Gamma(Q_{0,U}, \mathcal{L}_{0,U})$, which is spanned by Fourier series $\sum_{y \in Y} \sigma_{x+y}\eta_1(x+y)$ in view of Lemmas 4.10 (3) and 5.12. Hence $T_d^*(\eta_1(x+y))/\eta_1(x+y) = T_d^*(\eta_1(x))/\eta_1(x)$ because $T_d^*(\sigma_x) = \sigma_x$, therefore $\alpha(y) = 1$ $(\forall y \in Y(U))$. It follows that $\alpha \in \mathrm{Hom}(X/Y, \mathbf{G}_m)(U) = (X/Y)^\vee(U)$, that is, $d \in K_S^\sharp(\mathcal{L}_\eta)_0^m(U)$ $(\simeq (X/Y)^\vee(U))$. This proves Lemma. ☐

6. Projective embedding

Definition 6.1. Let k be an algebraically closed field and $(Q_0, \mathcal{L}_0)$ be a PSQAS over k. Then we define

$$e_{\min}(K(Q_0, \mathcal{L}_0)) = \max\{n > 0; \mathrm{ord}\ker(n \cdot \mathrm{id}_{K(Q_0, \mathcal{L}_0)}) \geq n^{2g}\}$$
$$e_{\max}(K(Q_0, \mathcal{L}_0)) = \min\{n > 0; n \cdot \mathrm{id}_{K(Q_0, \mathcal{L}_0)} = 0\}$$

If the order of $K(Q_0, \mathcal{L}_0)$ and the characteristic of k are coprime, then $K(Q_0, \mathcal{L}_0) \simeq \oplus_{i=1}^g (\mathbf{Z}/e_i\mathbf{Z})^{\oplus 2}$ for some positive integers e_i with $e_i | e_{i+1}$. From this it follows that $e_{\min}(K(Q_0,\mathcal{L}_0)) = e_1$ and $e_{\max}(K(Q_0,\mathcal{L}_0)) = e_g$. See Definition 7.5.

Theorem 6.2. *Let A be an abelian variety over an algebraically closed field k, L an ample invertible sheaf on A. If $e_{\min}(K(A,L)) \geq 3$, L is very ample.*

Proof. This is a corollary to [Mumford74, p.231, Theorem 3]. By the assumption $K(A,L) \supset A_n$, all the n-division points of A for some $n \geq 3$. By [Mumford74, p.231, Theorem 3] there exists an invertible sheaf M on A such that $L = M^n$. Since L is ample, M is ample. By Lefschetz's theorem, L is very ample because $n \geq 3$. □

Theorem 6.3. *Let $(Q_0,\mathcal{L}_0)$ be a polarized projectively stable quasi-abelian scheme over an algebraically closed field k, and $K(Q_0,\mathcal{L}_0)$ the abelian Heisenberg group. If $e_{\min}(K(Q_0,\mathcal{L}_0)) \geq 3$, then $\Gamma(Q,\mathcal{L})\otimes k$ is very ample, a fortiori $\mathcal{L}_0$ is very ample.*

Proof. We consider first the totally degenerate case. By Theorem 5.12 $\Gamma(Q,\mathcal{L}) = \Gamma(P,\mathcal{L})$. Let $V := \Gamma(Q,\mathcal{L})$ and $V^0 := \Gamma(Q,\mathcal{L})\otimes k$.

We define

$$\theta(x) := \sum_{y\in Y} a_0(x+y)\xi_{x+y} \quad \text{for } x \in X/Y \tag{16}$$

We note $\theta(x+z)\vartheta = S_z^*(\theta(x)\vartheta)$ $(\forall z \in X)$. Hence it suffices to prove the assertion on $W_0(c) := W(c)\otimes k$ for $c \in \mathrm{Star}\,(0)$ where $\mathrm{Star}\,(0)$ is the union of Delaunay cells containing 0.

It follows from Lemma 5.5 (1) that on $W_0(c)$

$$\theta(x+c)/\xi_c = \sum_{y\in Y} a_0(x+y+c)\xi_{x+y+c}/\xi_c \tag{17}$$

$$= \sum_{y\in(-x+\mathrm{Star}\,(0))\cap Y} a_0(x+y+c)\xi_{x+y,c} \tag{18}$$

By the assumption $Y \subset nX$ for $n \geq 3$. Hence $\mathrm{Star}\,(c) \cap (y + \mathrm{Star}\,(c)) = \emptyset$ for $y \in Y$ by Lemma 1.2. It follows that $(\theta(x)/\xi_c)\otimes k$ consists of at most a single term. Therefore

$$\theta(x+c)/\xi_c = \sum_{y\in(-x+\mathrm{Star}\,(0))\cap Y} a_0(x+y+c)\xi_{x+y,c} \tag{19}$$

$$= \begin{cases} a_0(x+y+c)\xi_{x+y,c} & \text{if } \exists y \in Y\cap(-x+\mathrm{Star}\,(0)) \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

By Theorem 5.10 Q_0 is a disjoint union of the torus orbits $O_Q(\sigma) \simeq \mathrm{Spec}\,k[\xi^{\pm}_{z,c}; z+c \in \sigma\cap X]$ $(\sigma \in \mathrm{Del}(c), c \in X/Y)$. For any point of $O_Q(\sigma)$, $\theta(x+c)/\xi_c = a_0(x+y+c)\xi_{x+y,c}$ is nonvanishing if $\{x+y+c\} = (x+c+Y)\cap(\sigma\cap X)$. Therefore *the linear system V^0* is base point free.

Let ϕ_{V^0} be the morphism defined by V^0. Suppose that we are given two distinct points q and q' of $(Q_0)_{\mathrm{red}}$. Suppose that $q \in O_Q(\sigma)$ and $q' \in O_Q(\sigma')$

and $\sigma \neq \sigma' \mod Y$. Then choose $c \in \sigma \cap X \setminus (\sigma' + Y)$. Then $\theta(c)$ is nonvanishing at q while $\theta(c)$ is zero at q'. Hence $\phi_{V^0}(q) \neq \phi_{V^0}(q')$. If $q, q' \in O_Q(\sigma)$, then choose $z + c, c \in \sigma \cap X$ such that $\xi_{z,c}(q) \neq \xi_{z,c}(q')$. Since $(\theta(c)/\xi_c)_{|W_0(c)} = a_0(c) \in k$ is nonzero, this implies that $(\theta(z+c)/\theta(c))(q) \neq (\theta(z+c)/\theta(c))(q')$. Thus ϕ_{V^0} separates two distinct points of $W_0(c)_{\mathrm{red}}$.

The ring $S_0(c)$ is generated by $\xi_{x,c}$ ($x \in \mathrm{Star}\,(0) \cap X$), while $\xi_{x,c} = 0$ in $S_0(c)$ by Lemma 5.5 if $x \notin \mathrm{Star}\,(0)$. Hence ϕ_{V^0} is an immersion locally on $W_0(c)$ by Theorem 5.7 (3) and Equation (20). This proves Theorem in the totally degenerate case.

The general case easily follows from the above and Theorem 6.2. With the notation in Lemma 4.14 and Definition 5.11 let $K(Q, \mathcal{L}) := K(P, \mathcal{L}) = K^{\sharp}_S(\mathcal{L}_\eta)$ and let H^* be an étale subgroup scheme of $K(Q, \mathcal{L})$ generated by $e(x) := S_x(e)$ ($x \in X/Y$) and $K_{ab} := (H^* \oplus K_S(\mathcal{L}_\eta)^m)^{\perp}$. Let $e^{\sharp}_S : K^{\sharp}_S(\mathcal{L}_\eta) \times K^{\sharp}_S(\mathcal{L}_\eta) \to \mathbf{G}_{m,S}$ be the extension of $e^{\mathcal{L}_\eta}$. Hence $e^{\sharp}_S$ is also an extension of $e^{\mathcal{L}_\eta}_S$. By [MB85, IV, 2.4] and Lemma 4.14 $e^{\sharp}_S$ is a perfect pairing. Since $X(K_S(\mathcal{L}_\eta)^m) \simeq H^*$, we see that $K(Q, \mathcal{L}) \simeq (H^* \oplus K_S(\mathcal{L}_\eta)^m) \oplus K_{ab}$. It is clear that $K_{ab} \simeq K_S(\mathcal{L}_\eta)^{ab} \simeq K(A, \mathcal{M})$, hence $K_{ab} \otimes k \simeq K(A_0, \mathcal{M}_0)$. Hence $e_{\min}(K(A_0, \mathcal{M}_0)) \geq 3$ and $Y \subset nX$ for some $n \geq 3$. Hence we can apply Theorem 6.2 and Theorem in the totally degenerate case.

We need only to consider $W_0(0)$. We take $\theta = \sum_{y \in Y} \sigma_y \xi_y \in \Gamma(Q_0, \mathcal{L}_0)$. Then by the assumption and by the same argument we have $\theta_{|W(0)} = \sigma_0 \in \Gamma(A_0, \mathcal{M}_0)$. By Theorem 6.2 there is $\sigma_0 \in \Gamma(A_0, \mathcal{M}_0)$ nonvanishing at any given point of A_0. This shows that the linear system V^0 is base point free. Suppose that we are given two distinct points q and q' of $W_0(0)$. If moreover their images in A are different, then $|\mathcal{M}_0|_A$ separates these points. Hence V^0 separates q and q'. Suppose their images in A are the same, say a. For any point a of A, we choose an affine open subset U of A and coordinates w^x_U ($x \in X$) of the split U-torus $\widetilde{G}_U := \widetilde{G} \times_A U$.

Then by Theorem 4.10, Lemma 5.12 and by the totally degenerate case a general global section $\theta \in \Gamma(Q_0, \mathcal{L}_0)$ is given on $W_0(0)$ by $\theta_{|W_0(0)} = \sum_{x \in \mathrm{Star}\,(0) \cap X} \sigma_x \xi_x$. We have a basis of $\Gamma(Q_0, \mathcal{L}_0)$ whose restriction to $W_0(0)$ consists of at most a single term as in Equation (20) by Lemma 1.2. Since $\mathcal{M}_x$ is very ample, we can choose a section $\sigma_x \in \Gamma(A, \mathcal{M}_x)$ for any $x \in \mathrm{Star}\,(0) \cap X$ such that σ_x is nonvanishing at a. Therefore V^0 is base point free. Then by the argument as in the totally degenerate case, we can separate the points q and q'. We can prove similarly that ϕ_{V^0} is an immersion. This completes the proof. □

7. Central extension

Definition 7.1. Let R be an integral domain, $k(R)$ its fraction field and $S = \mathrm{Spec}\,R$. Let K be a finite constant reduced group scheme over S. A group S-scheme $\mathcal{G}$ is said to be *a central extension* of K by $\mathbf{G}_{m,S}$ if $\mathbf{G}_{m,S}$ is contained in the center of $\mathcal{G}$ and the quotient of $\mathcal{G}$ by $\mathbf{G}_{m,S}$ is K. Hence we have an exact sequence of group S-schemes

$$1 \to \mathbf{G}_{m,S} \xrightarrow{i} \mathcal{G} \xrightarrow{\tau} K \to 0$$

A central extension $\mathcal{G}$ is said to be *totally split* if $K(R) \simeq K(k(R)) \simeq K(\overline{k(R)})$ and $\tau(R) : \mathcal{G}(R) \to K(R)$ is surjective.

Each section $P_i \in K(R)$ is a subscheme of K isomorphic to S. There is an irreducible component $\mathcal{G}_i$ of $\mathcal{G}$ mapped onto P_i. Although $\mathcal{G}_i$ may have no sections over S, by taking a finite base change if necesary, there is a section Q_i of $\mathcal{G}$ such that $P_i = \tau(Q_i)$ and $\mathcal{G}_i = \mathbf{G}_{m,S} \cdot Q_i \simeq \mathbf{G}_{m,S}$. Thus any central extension $\mathcal{G}$ of K by $\mathbf{G}_{m,S}$ is totally split by a finite base change.

Now we recall a well known lemma.

Lemma 7.2. *The set of equivalence classes of totally split central extensions is the cohomology group $H^2(K(R), \mathbf{G}_{m,S}(R)) = H^2(K(R), R^*)$.*

Proof. Suppose that $\mathcal{G}$ is totally split. Then $\mathcal{G}(R) = \{\alpha{\cdot}Q_i; \alpha \in \mathbf{G}_{m,S}(R), P_i \in K(R)\}$. We denote $\alpha \cdot Q_i$ by (α, k) where $P_i = k \in K(R)$. Then in the same manner as in [Mumford74, p. 221] we can express the twisted group law of $\mathcal{G}$ as a two cocycle $\{f(k,k'); k, k' \in K(R)\}$

$$(\alpha, k) \cdot (\alpha', k') = (\alpha \cdot \alpha' \cdot f(k,k'), k + k') \tag{21}$$

where α, α' are R-valued points of $\mathbf{G}_{m,S}$, k, k' are R-valued points of K and K is written additively. $f(\cdot,\cdot)$ satisfies the cocycle condition

$$f(\lambda + \nu, \mu) f(\lambda, \nu) = f(\lambda, \nu + \mu) f(\nu, \mu) \tag{22}$$

If another cocycle h describes an isomorphic cohomologous to f, that is, h/f is a coboundary in the sense that

$$h(\lambda, \nu) = f(\lambda, \nu) g(\lambda + \nu) g(\lambda)^{-1} g(\nu)^{-1} \tag{23}$$

for some g where $\lambda, \nu \in K(R)$. □

Remark 7.3. The commutator form $e_{\mathcal{G}}(\lambda, \nu)$ for $\mathcal{G}$ is given by the formula [Mumford74, p. 222]

$$e_{\mathcal{G}}(\lambda, \nu) = f(\lambda, \nu)/f(\nu, \lambda) \tag{24}$$

even if $f(0,0) \neq 1$. It is clear $e_{\mathcal{G}}(\lambda, \nu) = h(\lambda, \nu)/h(\nu, \lambda)$ if h is cohomologous to f.

Conversely we prove

Lemma 7.4. *Let $\mathcal{G}$ and $\mathcal{G}'$ be totally split central extensions of K by $\mathbf{G}_{m,S}$. Suppose that their commutator forms are equal. Then $\mathcal{G} \simeq \mathcal{G}'$ after a suitable finite base change.*

Proof. Let f and h be two-cocycles associated with the extensions $\mathcal{G}$ and $\mathcal{G}'$ respectively. Then we prove that after a suitable finite base change there exists g satisfying (23) if $e_{\mathcal{G}} = e_{\mathcal{G}'}$. Since $H^2(K(R), \mathbf{G}_{m,S}(R))$ is a group, it suffices to prove that if a cocycle f determine a central extension $\mathcal{G}$ with its commutator form trivial, then the pull back of $\mathcal{G}$ by a finite base change is trivial, that is, the pull back of f is a coboundary. We prove the assertion by induction on the minimal number of generators of K. Now we write the group law of $\mathbf{G}_{m,S}$ additively. Thus our conditions on f are

$$f(\lambda+\nu,\mu)+f(\lambda,\nu)=f(\lambda,\nu+\mu)+f(\nu,\mu) \tag{25}$$

$$f(\lambda,\nu)=f(\nu,\lambda) \tag{26}$$

where $f(\lambda,\nu)\in\mathbf{G}_m(R)=R^*$, $\lambda,\nu\in K(R)$. First we may assume $f(0,0)=0$. In fact, otherwise, let $g(\lambda)=-f(0,0)$ and $h(\lambda,\nu):=f(\lambda,\nu)-g(\lambda+\nu)+g(\lambda)+g(\nu)$. Then $h(0,0)=0$, $h(\lambda,\nu)=h(\nu,\lambda)$ and h is cohomologous to f. Now we assume $f(0,0)=0$. The condition (25) implies $f(\lambda,0)+f(\lambda,0)=f(\lambda,0)+f(0,0)$ by setting $\nu=0$. Hence $f(\lambda,0)=0$ for any $\lambda\in K(R)$.

Let $N=\operatorname{ord}K:=\operatorname{rank}_R B$. Hence by [OT70] $N\lambda=0$ for any $\lambda\in K(R)$. We choose and fix any λ. We define (formally at this moment)

$$g(\lambda)=-(1/N)\sum_{i=0}^{N-1}f(\lambda,i\lambda) \tag{27}$$

$$g(k\lambda)=kg(\lambda)+\sum_{i=0}^{k-1}f(\lambda,i\lambda) \tag{28}$$

First we note $g((k+N)\lambda)=g(k\lambda)$. We see readily $f(\lambda,k\lambda)=g((k+1)\lambda)-g(k\lambda)-g(\lambda)$ for any k. Next by induction on ℓ we prove

$$f(\ell\lambda,k\lambda)=g((\ell+k)\lambda)-g(\ell\lambda)-g(k\lambda) \tag{29}$$

In fact, if $\ell=1$, clear. Assume (29) for any $k\in\mathbf{Z}$ and $1\le\ell\le n$. Since

$$f((n+1)\lambda,k\lambda)=-f(\lambda,n\lambda)+f(n\lambda,(k+1)\lambda)+f(\lambda,k\lambda)$$

we have $f((n+1)\lambda,k\lambda)=g((n+k+1)\lambda)-g((n+1)\lambda)-g(k\lambda)$. Therefore if $K(R)$ is generated by λ and if R contains $g(\lambda)$ defined by (27), then the pull back of f is cohomologous to zero.

Next we consider the case where $K(R)=H(R)\oplus\mathbf{Z}\lambda$ for some $\lambda\in K(R)$. For $\nu,\mu\in H(R)$ and $k,\ell\in\mathbf{Z}$, we write $\nu+k\lambda\in K(R)$ and $\mu+\ell\lambda\in K(R)$. We define

$$g(\rho)=-(1/N)\sum_{i=0}^{N-1}f(\rho,i\rho) \tag{30}$$

$$g(\nu+k\lambda)=f(\nu,k\lambda)+g(\nu)+g(k\lambda) \tag{31}$$

for $\rho\in H(R)$ or $\rho\in\mathbf{Z}\lambda$, and $\nu\in H(R)$, $k\in\mathbf{Z}$. By induction hypothesis on H we have $f(\mu,\nu)=g(\mu+\nu)-g(\mu)-g(\nu)$ for $\mu,\nu\in H(R)$. It follows from the relation (25), the definition (31) and symmetry of f that

$$\begin{aligned}f(\mu+k\lambda,\nu)&=-f(k\lambda,\mu)+f(k\lambda,\nu+\mu)+f(\mu,\nu)\\&=g(\nu+\mu+k\lambda)-g(\nu)-g(\mu+k\lambda)\end{aligned}$$

Therefore by symmetry of f and the relation (25)

$$f(\mu+k\lambda,\nu+\ell\lambda)=-f(k\lambda,\mu)+f(k\lambda,\ell\lambda+\mu+\nu)+f(\mu,\nu+\ell\lambda)$$

we have $f(\mu + k\lambda, \nu + \ell\lambda) = g(\mu + \nu + (k+\ell)\lambda) - g(\mu + k\lambda) - g(\nu + \ell\lambda)$.

Now we switch into the first multiplicative cocycle $f(\lambda, \nu)$. Let R' be the ring obtained by adding to R N-th roots of $\prod_{i=0}^{N-1} f(\rho, i\rho)$ for all $\rho \in K(R)$. Clearly R' is finite over R. Then by the above argument the pull back to Spec R' of f becomes cohomologous to zero. This completes the proof. □

Definition 7.5. Let N and M be a pair of positive integers such that N is divisible by M, and ζ_M a primitive M-th root of unity and $\mathcal{O}_M := \mathbf{Z}[\zeta_M, 1/M]$. For simplicity we write $\mathcal{O} = \mathcal{O}_M$ if there is no fear of confusion.

Let H be a finite abelian group of order N such that $e_{\max}(H)$, the maximal order of elements in H, is equal to M. Now by abuse of notation we regard H as a constant affine finite abelian group $\mathcal{O}$-scheme of order N. Let $H^\vee := \mathrm{Hom}_{\mathcal{O}}(H, \mathbf{G}_{m,\mathcal{O}})$ be the Cartier dual of H over $\mathcal{O}$. We set $K := K(H) = H \oplus H^\vee$ and define a bimultiplicative (we say simply a *bilinear*) form $e_K : K \times K \to \mathbf{G}_{m,\mathcal{O}}$ by

$$e_K(z \oplus \alpha, w \oplus \beta) = \beta(z)\alpha(w)^{-1} \tag{32}$$

where $z, w \in H$, $\alpha, \beta \in H^\vee$. We denote e_K and K by ℓ_H and $K(H)$ when it is necessary to emphasize dependence on H. However we note that H is a maximally isotropic subgroup of K with respect to e_K, which is uniquely determined by this property up to isomorphism.

Let $\mu_M := \mathrm{Spec}\,\mathcal{O}[x]/(x^M - 1)$ be the group scheme of all M-th roots of unity. We define group $\mathcal{O}$-schemes $\mathcal{G}(K)$ and $G(K)$ by

$$\mathcal{G}(K) := \{(a, z, \alpha); a \in \mathbf{G}_{m,\mathcal{O}}, z \in H, \alpha \in H^\vee\} \tag{33}$$

$$G(K) := \{(a, z, \alpha); a \in \mu_M, z \in H, \alpha \in H^\vee\} \tag{34}$$

endowed with a group scheme structure

$$(a, z, \alpha) \cdot (b, w, \beta) = (ab\beta(z), z + w, \alpha + \beta) \tag{35}$$

where $a, b \in \mathbf{G}_{m,\mathcal{O}}$ or $a, b \in \mu_M$, $z, w \in H$ and $\alpha, \beta \in H^\vee$.

An automorphism ϕ of $\mathcal{G}(K)$ as a group $\mathcal{O}$-scheme is said to be *a central extension automorphism* if

$$\phi(ah) = a\phi(h) \quad \text{for } a \in \mathbf{G}_{m,\mathcal{O}}, h \in \mathcal{G}(K) \tag{36}$$

A level subgroup scheme of $\mathcal{G}(K)$ (resp. $G(K)$) is by definition a subgroup scheme B of $\mathcal{G}(K)$ (resp. $G(K)$) whose natural image in $K(H)$ is a maximal isotropic subgroup scheme with respect to e_K isomorphic to B. Let $\tilde{H} := \{(1, z, 0); z \in H\}$ and $\tilde{H}^\vee := \{(1, 0, \alpha); \alpha \in H^\vee\}$. It is clear that $\tilde{H}$ and $\tilde{H}^\vee$ are level subgroup schemes of $\mathcal{G}(K)$ and $G(K)$.

Let $V(K)$ be the group algebra $\mathcal{O}[H^\vee]$ of $H^\vee$ over $\mathcal{O}$, and an $\mathcal{O}$-basis $v(\chi)$ ($\chi \in H^\vee$) of $V(K)$. The group scheme $G(K)$ as well as $\mathcal{G}(K)$ acts on $V(K)$ by

$$U(K)(a, z, \alpha)(v(\chi)) = a\chi(z)v(\chi + \alpha). \tag{37}$$

where $a \in \mu_M$ or $a \in \mathbf{G}_{m,\mathcal{O}}$, $z \in K$ and $\alpha \in H^\vee$. We define subgroup $\mathcal{O}$-schemes $\bar{\mathcal{G}}(K)$ and $\bar{G}(K)$ of an algebraic group $\mathcal{O}$-scheme $\mathrm{GL}(V(K))$ by

$$\bar{\mathcal{G}}(K) : = \{U(K)(g); g \in \mathcal{G}(K)\} \tag{38}$$

$$\bar{G}(K) : = \{U(K)(g); g \in G(K)\} \tag{39}$$

Lemma 7.6. *Let $\psi : \mathcal{G}(K) \to K$ be the natural surjection. Let S be a subgroup $\mathcal{O}$-scheme of $\mathcal{G}(K)$ containing two level subgroup $\mathcal{O}$-schemes $\tilde{H}_1$ and $\tilde{H}_2$ with $\psi(\tilde{H}_1) \cap \psi(\tilde{H}_2)$ trivial. Then $G(K) \subset S$. In particular, $G(K)$ is the unique minimal subgroup $\mathcal{O}$-scheme among those subgroup $\mathcal{O}$-schemes S with the above property.*

Proof. Let $H_i := \psi(\tilde{H}_i)$. By definition $H_i \simeq \tilde{H}_i$. Let R be any ring over $\mathcal{O}$. It follows that there exists a homomorphism $f_i : H_i \to \mathbf{G}_m$ such that

$$\tilde{H}_i(R) = \{(f_i(z,\alpha), z, \alpha); (z,\alpha) \in H_i(R)\} \tag{40}$$

Since $e_{\max}(K) = M$, we have $f_i(z,\alpha) \in \mu_M(R)$ for $(z,\alpha) \in H_i(R)$. Since $H_1 \cap H_2 = \{1\}$ and H_i are maximally isotropic, we have $K \simeq H_1 \oplus H_2$ and $H_2 \simeq \mathrm{Hom}(H_1, \mathbf{G}_m)$. It follows that $\zeta_M \in e_K(H_1, H_2) = [\tilde{H}_1(R), \tilde{H}_2(R)]$, hence $(f_i(z,\alpha), 0, 0) \in S(R)$. Therefore $G(K)(R) \subset S(R)$. This proves Lemma. □

Lemma 7.7. *Let ϕ be a central extension automorphism of $\mathcal{G}(K)$. Assume that it induces the identity of K. Then $\phi(G(K)) = G(K)$.*

Proof. We see that ϕ is written as

$$\phi : (a,u) \mapsto (\phi(a,u), u) \quad \text{for } a \in \mathbf{G}_{m,\mathcal{O}} \text{ and } u \in K \tag{41}$$

Then ϕ is subject to the relation

$$\phi(ab, u+v) = \phi(a,u)\phi(b,v),\ \phi(a,u) = a\phi(1,u) \tag{42}$$

Hence $\phi(1, ?) \in \mathrm{Hom}_{\mathcal{O}}(K, \mathbf{G}_{m,\mathcal{O}}) = \mathrm{Hom}_{\mathcal{O}}(K, \mu_M)$. It follows $\phi(G(K)) = G(K)$. This Lemma also follows from Lemma 7.6. □

Lemma 7.8. *Let T be an $\mathcal{O}$-scheme, and $\mathcal{G}$ a central extension of K_T by $\mathbf{G}_{m,T}$,*

$$1 \to \mathbf{G}_{m,T} \to \mathcal{G} \to K_T \to 0$$

Suppose that there exists an open affine covering $U_i := \mathrm{Spec}\ A_i$ of T such that $\mathcal{G}_{U_i} \simeq \mathcal{G}(K)_{U_i}$. Then there exists an $\mathcal{O}$-scheme W finite over T such that $\mathcal{G}_W \simeq \mathcal{G}(K)_W$.

Proof. Let $\rho_i : \mathcal{G}_{U_i} \simeq \mathcal{G}(K)_{U_i}$ be the given isomorphisms and $U_{ij} = U_i \cap U_j$. There is a natural isomorphism $g_{ij} : (\mathcal{G}_{U_j})_{U_{ij}} \simeq (\mathcal{G}_{U_i})_{U_{ij}}$. Then $h_{ij} := \rho_i \cdot g_{ij} \cdot (\rho_j)^{-1}$ is an isomorphism of $(\mathcal{G}(K)_{U_j})_{U_{ij}}$ and $(\mathcal{G}(K)_{U_i})_{U_{ij}}$. Therefore $\mathcal{G}$ is obtained by gluing $\mathcal{G}(K)_{U_i}$ by a one-cocycle h_{ij}. By the proof of Lemma 7.7 we see $h_{ij} \in H^1_{\mathrm{Zar}}(T, \mathrm{Hom}(K, \mu_M))$. We prove that h_{ij} is cohomologous zero by a suitable finite base change. Since K is finite abelian, it suffices to prove that any element $\theta \in H^1_{\mathrm{Zar}}(T, \mu_M)$ is cohomologous zero by a suitable finite base change. The cocycle θ determines a $\mathbf{G}_m$-torsor F_θ on T of order at most M. Let $(F_\theta)_{U_i} = \{(\zeta_i, z); \zeta_i \in \mathbf{G}_m, z \in U_i\}$. We may assume that $F_\theta = \cup_i\ (F_\theta)_{U_i}$ is obtained by gluing together by the relations $\zeta_i = \theta_{ij}\zeta_j$. Let $W := \cup\{(\zeta_i, z) \in (F_\theta)_{U_i}; \zeta_i^M = 1\}$. Since $\theta_{ij}^M = 1$, W is a well-defined subscheme of F_θ finite over T. The pull back of θ to W is trivial. This completes the proof. □

Remark 7.9. Lemma 7.2 and Lemma 7.4 are true over any algebraically closed field k by [MB81]. Lemma 7.7 is clearly true over any algebraically closed field k.

Definition 7.10. Let $\operatorname{Spec} k$ be a point of $\operatorname{Spec}\mathcal{O}$. Any $G(K) \otimes k$-module V is of weight one if every $a \in \mathbf{G}_{m,k}$ of $G(K) \otimes k$ acts on V as scalar multiplication $a \cdot \mathrm{id}_V$. In this case we say also that the action of $G(K)$ on V is of weight one.

Lemma 7.11. *Let* $\operatorname{Spec} k$ *be a point of* $\operatorname{Spec}\mathcal{O}$. *Then* $V(K) \otimes k$ *is an irreducible* $G(K) \otimes k$*-module of weight one, unique up to equivalence. Any finite dimensional* $G(K)$*-module of weight one over* k *is a direct sum of copies of* $V(K) \otimes k$.

Proof. We imitate the argument in [Mumford66]. Let V be a nontrivial $G(K)$-k-module of weight one. Let $V(\chi)$ be the maximal k-submodule of V such that H acts on $V(\chi)$ by a character $\chi \in H^\vee$. By definition $V(0) = V^H$, the submodule of H-invariants. Since $(1, 0, -\chi) \cdot V(\chi) = V(0)$, $V(\chi) \neq 0$ if and only if $V(0) \neq 0$. Hence $V(0) \neq 0$. Then we set $v(\chi, w) := (1, 0, \chi) \cdot w \in V(\chi)$ for $w \in V(0) = V^H$. Let V_0 be a k-submodule of V spanned by $v(\chi, w)$ $(\forall \chi \in H, w \in V(0))$. We see

$$\begin{aligned}(a, z, \alpha) \cdot v(\chi, w) &= (a, 0, \alpha) \cdot (1, z, 0) \cdot v(\chi, w)\\ &= a\chi(z)v(\chi + \alpha, w)\end{aligned}$$

This proves that V_0 is a $G(K)$-module. Since V/V_0 is a $G(K)$-module with $(V/V_0)^H = 0$, we have $V = V_0$. Hence $V \simeq V(K) \otimes_k V(0) \simeq (V(K) \otimes k)^{\oplus \dim_k V/\dim_k V_0}$ as $G(K) \otimes k$ modules. □

Definition 7.12. Let R be a complete discrete valuation ring over $\mathcal{O}$ with $k = R/I$ the residue field and $S := \operatorname{Spec} R$. Suppose we are given a PSQAS $(Q, \mathcal{L})$, a SQAV $(P, \mathcal{L})$ and a semi-abelian scheme $(G, \mathcal{L})$ over S such that $(P, \mathcal{L})$ is the normalization of $(Q, \mathcal{L})$ and G is an open subscheme of $(P, \mathcal{L})$. In what follows we need only a pair $(Q, \mathcal{L})$ and $(G, \mathcal{L})$. We define $\mathcal{G}(Q, \mathcal{L})$ as follows. Let $\mathcal{L}^{\mathrm{tsr}}$ be the $\mathbf{G}_m$-torsor on G associated with the invertible sheaf $\mathcal{L}$. Then we define $\mathcal{G}(Q, \mathcal{L})$ to be $(\mathcal{L}^{\mathrm{tsr}})_{|K(Q,\mathcal{L})}$. Since $\mathcal{L}$ is a cubical sheaf on G, $\mathcal{G}(Q, \mathcal{L})$ is shown to be a group scheme [MB81, 3.2]. The natural projection from $\mathcal{L}^{\mathrm{tsr}}$ to G makes $\mathcal{G}(Q, \mathcal{L})$ a central extension of $K(Q, \mathcal{L})$ by $\mathbf{G}_{m,S}$

$$1 \to \mathbf{G}_{m,S} \to \mathcal{G}(Q, \mathcal{L}) \to K(Q, \mathcal{L}) \to 0 \tag{43}$$

with commutator form $e_S^\sharp$.

Remark 7.13. Assume that the order of $K(Q,\mathcal{L})$ and the characteristic of k are coprime. Then $K(Q,\mathcal{L})$ is a reduced flat finite group S-scheme, étale over S. Hence by a finite base change of S we may assume that $K(Q,\mathcal{L})$ is a constant group scheme. The symplectic group scheme $(K(Q,\mathcal{L}),e^{\sharp}_S)$ is then isomorphic to $(K_S,(e_K)_S)=(K(H)_S,\ell_{H,S})$ for some finite group $\mathcal{O}$-scheme $K=K(H)$. By taking a further finite base change if necessary, $\mathcal{G}(Q,\mathcal{L})$ is a totally split central extension of $K(Q,\mathcal{L})$ by $\mathbf{G}_{m,S}$. By Lemma 7.4 $\mathcal{G}(Q,\mathcal{L})\simeq\mathcal{G}(K)_S$ after a finite base change.

Definition 7.14. We choose a central extension isomorphism $\rho:\mathcal{G}(K)_S\simeq\mathcal{G}(Q,\mathcal{L})$ and define $G(Q,\mathcal{L})$ to be the image of $G(K)_S$ by ρ. $G(Q,\mathcal{L})$ is a central extension of $K(Q,\mathcal{L})$ by μ_M. By Lemma 7.7 $G(Q,\mathcal{L})$ is independent of the choice of ρ.

We also define $G(Q_\eta,\mathcal{L}_\eta):=G(Q,\mathcal{L})\otimes k(\eta)$, and $G(Q_0,\mathcal{L}_0):=G(Q,\mathcal{L})\otimes k$. We note that both $\mathcal{G}(Q,\mathcal{L})$ and $G(Q,\mathcal{L})$ are defined only after a base change because $\mathcal{G}(Q,\mathcal{L})$ is defined by choosing an open subgroup scheme $G^{\sharp}$ of P. They act on $\Gamma(Q,\mathcal{L})$ and $\Gamma(Q,\mathcal{L})$ is irreducible by these actions in view of Lemma 7.11.

8. Kempf-Stability

8.1. **Hilbert points.** Let $(Q_0,\mathcal{L}_0)$ be a projectively stable quasi-abelian scheme over k. Suppose that $V^0:=\Gamma(Q,\mathcal{L})\otimes k$ is very ample. Hence $(Q_0,\mathcal{L}_0)$ is a closed subscheme of the projective space $(\mathbf{P},O_{\mathbf{P}}(1))$ where $\mathbf{P}=\mathbf{P}(V^0)$. Let I be the ideal of $O_{\mathbf{P}}$ defining $(Q_0,\mathcal{L}_0)$. Then by Serre vanishing theorem there exists a sufficiently large n_0 such that $H^1(\mathbf{P},I\otimes O_{\mathbf{P}}(n))=0$ and $H^q(Q_0,\mathcal{L}_0^n)=0$ for $n\geq n_0$ and $q\geq 1$. Hence we have an epimorphism

$$\phi_n:S^n\Gamma(Q,\mathcal{L})\otimes k\to\Gamma(Q_0,\mathcal{L}_0^n) \tag{44}$$

The epimorphism ϕ_n determines a point of the Grassman variety. Let $n(g):=n^g\deg(\mathcal{L}_0)$. By taking the Plücker coordinates we obtain a point $\overset{n(g)}{\wedge}\phi_n$ of the projective space $\mathbf{P}(\overset{n(g)}{\wedge}S^n\Gamma(Q,\mathcal{L})\otimes k)$

$$\overset{n(g)}{\wedge}\phi_n:\overset{n(g)}{\wedge}S^n\Gamma(Q,\mathcal{L})\otimes k\to\overset{n(g)}{\wedge}\Gamma(Q_0,\mathcal{L}_0^n)\simeq k. \tag{45}$$

We call $\overset{n(g)}{\wedge}\phi_n$ *the n-th normalized Hilbert point of* $(Q_0,\mathcal{L}_0)$, which we denote by $hilb_n(Q_0,\mathcal{L}_0)$. If $\Gamma(Q_0,\mathcal{L}_0)=\Gamma(Q,\mathcal{L})\otimes k$, for instance if $Q_0=P_0$, then $hilb_n(Q_0,\mathcal{L}_0)$ is just *the n-th Hilbert point of* $(Q_0,\mathcal{L}_0)$ in the usual sense.

We say that $hilb_n(Q_0,\mathcal{L}_0)$ is Kempf-stable if it has a closed $\mathrm{SL}_{\pm}(V^0)$-orbit.

By Lemma 7.11 the following is a corollary to [Kempf78, Corollary 5.1].

Theorem 8.2. *Let $(Q_0,\mathcal{L}_0)$ be a polarized projectively stable quasi-abelian scheme over an algebraically closed field k. Suppose that the characteristic of k and* $\deg\mathcal{L}_0:=(\mathcal{L}_\eta^g)/g!$ *are coprime. If $\Gamma(Q,\mathcal{L})\otimes k$ is very ample, then $hilb_n(Q_0,\mathcal{L}_0)$ is Kempf-stable for all large n. In particular it is Mumford-semistable.*

Proof. Let $V^0 := \Gamma(Q, \mathcal{L}) \otimes k$. Let $\mathrm{SL}_{\pm}(V^0)$ be a subgroup of $\mathrm{GL}(V^0)$ consisting of elements with determinant ± 1. We note that closedness of the orbits for the actions of SL or $\mathrm{SL}_{\pm}$ are equivalent to each other because $[SL_{\pm}(V^0); SL(V^0)]$ is finite. $G(Q_0, \mathcal{L}_0)$ acts on $\Gamma(Q_0, \mathcal{L}_0)$ keeping V^0 stable, hence $hilb_n(Q_0, \mathcal{L}_0)$ is $G(Q_0, \mathcal{L}_0)$-invariant. Since V^0 is an irreducible $G(Q_0, \mathcal{L}_0)$ ($\simeq G(K) \otimes k$)-module by Lemma 7.11, $G(Q_0, \mathcal{L}_0)$ is contained in no parabolic subgroup of $\mathrm{SL}_{\pm}(V^0)$. By applying [Kempf78, Corollary 5.1] to $\mathrm{SL}_{\pm}(V^0)$, we see that $hilb_n(Q_0, \mathcal{L}_0)$ has a closed $\mathrm{SL}_{\pm}(V^0)$ orbit. They are semistable in the sense of Mumford by [Seshadri77, p. 252, Proposition 6 (1)]. $\square$

We remark that Theorem 8.2 is true without the assumption on $\deg \mathcal{L}_0$ by [MB85, V, 2.5.5].

Corollary 8.3. *Let $(P_0, \mathcal{L}_0)$ be a polarized torically stable quasi-abelian variety over an algebraically closed field k. Suppose that the characteristic of k and $\deg \mathcal{L}_0$ are coprime. If $\mathcal{L}_0$ is very ample, then $hilb_n(P_0, \mathcal{L}_0)$ is Kempf-stable for all large n.*

The (normalized) Hilbert points $hilb_n(Q_0, \mathcal{L}_0)$ are not necessarily properly stable, for instance a 3-gon of rational curves. We note $P_0 = Q_0$ by Theorem 1.6 if $g \leq 4$. Corollary 8.3 is an immediate consequence of Theorem 8.2 because $(P_0, \mathcal{L}_0)$ is very ample only if $P_0 = Q_0$.

Remark 8.4. There are various notions of stability, which differ one from another. Since the situation is rather confusing, we explain here the relation between them so far as the present article is concerned ([Kempf78, p. 314], [MFK94, p. 36], [Seshadri77, p. 252]). Semi-stability in all these literatures is equivalent to each other by [Seshadri77, Proposition 6 (1)] as well as Proper-stability. Mumford-stability is strictly stronger than Kempf-stability, which is defined to be closedness of the orbit of the point. Kempf-stability is strictly stronger than semi-stability.

Let $V = \mathbf{Z}x_0 + \mathbf{Z}x_1 + \mathbf{Z}x_2$. Let $\mathbf{P}(S^3(V^{\vee}))$ be the projective space of ternary cubic forms on V. Let U_{node} be an open affine subscheme of $\mathbf{P}(S^3(V^{\vee}))$ consisting of cubic curves with at worst nodal singularities. The action of SL(3) on U_{node} is not closed because cubic curves with exactly one or two nodes are only semi-stable, their orbits are not closed. Hence cubic curves with exactly one or two nodes are not Kempf-stable. However a 3-gon of rational curves C_0 – a union of three lines with nodal singularities only – is Kempf-stable by Theorem 8.2. It is not Mumford-stable but Mumford-semistable. This example shows in particular that Kempf-stability is not an open condition. We also note that by Theorem 11.6 $(C, O_C(n))$ is not Kempf-stable for any $n \geq 2$ if C is a 3-gon.

9. Level $G(K)$-structures

9.1. **The ring $\mathcal{O}_M$.** We keep the same notation in Section 7. Let N and M be a pair of positive integers such that N is divisible by M, and ζ_M a primitive M-th root of unity and $\mathcal{O} = \mathcal{O}_M := \mathbf{Z}[\zeta_M, 1/M]$.

Let H be a constant affine finite abelian group $\mathcal{O}$-scheme of order N with $e_{\max}(H) = M$. Let $(K, e_K) := (H \oplus H^{\vee}, \ell_H)$ be the constant symplectic finite abelian group $\mathcal{O}$-scheme of rank N^2 defined in Section 7.

Definition 9.2. Let $\operatorname{Spec} k$ be a (not necessarily closed) point of $\operatorname{Spec}\mathcal{O}$. A triple $(Q_0, \mathcal{L}_0, V_0)$ is called a g-dimensional K-symplectic projectively stable quasi-abelian scheme over k or a K-symplectic PSQAS over k if

(i) $(Q_0, \mathcal{L}_0)$ is a g-dimensional projectively stable quasi-abelian scheme over k, that is, a closed fibre of some $(Q, \mathcal{L})$ in Theorem 5.8,

(ii) $(K(Q_0, \mathcal{L}_0), e_0^{\sharp}) \otimes \bar{k} \simeq (K, e_K) \otimes \bar{k}$.

(iii) V_0 is a $G(Q_0, \mathcal{L}_0)$-submodule of $H^0(Q_0, \mathcal{L}_0)$ such that $V_0 \otimes \bar{k}$ is the theta $\bar{k}$-module of $(Q_0, \mathcal{L}_0) \otimes \bar{k}$

where $\bar{k}$ is the algebraic closure of k. See Definition 5.13 for theta modules. By definition We note that (ii) implies a central extension isomorphism $G(Q_0, \mathcal{L}_0) \otimes \bar{k} \simeq G(K) \otimes \bar{k}$.

If the following conjecture for $n = 1$ is affirmatively solved, the condition (iii) as well as V_0 in the triple is unnecessary

Conjecture: $H^q(Q_0, \mathcal{L}_0^n) = 0$ for $q, n > 0$.

9.3. **Separable polarizations.** Let $\operatorname{Spec} k$ be a point of $\operatorname{Spec}\mathcal{O}$. Hence the characteristic of $k := R/I$ is prime to N. Therefore the condition (ii) in Definition 9.2 implies that $d := \deg \mathcal{L}_\eta/(g!)$ is prime to the characteristic of $k := R/I$ because $d = \sqrt{\operatorname{ord} K} = N$, hence the polarizations $\mathcal{L}_\eta$ and $\mathcal{L}_0$ are separable. By the discussion in Subsection 2.3 $d_t := \operatorname{ord} K(\mathcal{L}_\eta)^m$ and $d_a = \operatorname{ord} K(\mathcal{M})$ are prime to the characteristic of k. In particular, $\mathcal{M}$ is also a separable polarization of the abelian scheme A, the abelian part of the Raynaud sequence of G.

Suppose $e_{\min}(K) \geq 3$ in what follows.

Definition 9.4. Let Spec k be a point of $\operatorname{Spec}\mathcal{O}$, (Z, L, V) a K-symplectic PSQAS over k. A level $G(K)$-structure (ϕ, ρ) on (Z, L, V) is a pair of a closed k-immersion $\phi : Z \to \mathbf{P}(V(K) \otimes k)$ and a central extension isomorphism $\rho : G(K) \otimes_{\mathcal{O}} k \to G(Z, L)$ which induce ρ-equivariant isomorphisms

(i) $\phi^*(O_{\mathbf{P}(V(K)\otimes k)}(1)) \simeq L$,

(ii) $\phi^* : H^0(O_{\mathbf{P}(V(K))}(1)) \otimes k \simeq V(K) \otimes k \simeq V$

for some action of $G(K)$ on $V(K)$ of weight one (possibly different from $U(K)$).

The level $G(K)$-structure (ϕ, ρ) is called a rigid $G(K)$-structure if in addition to (i) and (ii) the following is true,

(iii) $\rho = G(\phi^*) \cdot (U(K) \otimes_{\mathcal{O}} k)$

where $G(\phi^*)(g) = (\phi^*)g(\phi^*)^{-1}$ for any $g \in \bar{G}(K)$, that is, the action of $G(K)$ on $V(K)$ is just $U(K)$.

Evidently L and V are uniquely determined by ϕ. We denote the sextuplet $(Z, L, G(Z, L), V, \phi, \rho)$ by $(Z, \phi, \rho)_{\mathrm{LEV}}$. If moreover (iii) is true, we denote it by $(Z, \phi, \rho)_{\mathrm{RIG}}$.

Definition 9.5. Let $(Z_i, L_i, G(Z_i, L_i), V_i, \phi_i, \rho_i)$ be k-PSQAS's with level $G(K)$-structures $(i = 1, 2)$. They are defined to be isomorphic as k-PSQAS's with level $G(K)$-structures if there is a k-isomorphism $f : Z_1 \simeq Z_2$ such that

(i) $L_1 \simeq f^*L_2$,
(ii) $V_1 = f^*V_2$ as k-linear subspaces of $H^0(Z_1, L_1)$,
(iii) $\rho_1 = G(f^*) \cdot \rho_2$

where $G(f^*)(g) = f^*g(f^*)^{-1}$ for any $g \in G(Z_2, L_2)$.

In this case we write $(Z_1, \phi_1, \rho_1)_{\mathrm{LEV}} \simeq (Z_2, \phi_2, \rho_2)_{\mathrm{LEV}}$. By (iii) we have $G(Z_1, L_1) = G(f^*)G(Z_2, L_2)$. We define $(Z_i, \phi_i, \rho_i)_{\mathrm{RIG}}$ to be isomorphic if so are $(Z_i, \phi_i, \rho_i)_{\mathrm{LEV}}$. For any PSQAS $(Z, \phi, \rho)_{\mathrm{RIG}}$, we see

$$(Z, \phi, \rho)_{\mathrm{RIG}} \simeq (\phi(Z), i, U(K))_{\mathrm{RIG}} \tag{46}$$

where $i : \phi(Z) \to \mathbf{P}(V(K) \otimes k)$ is the natural inclusion map.

Lemma 9.6. *Let* $\operatorname{Spec} k$ *be a point of* $\operatorname{Spec} \mathcal{O}$. *Let* $(Z_i, \phi_i, \rho_i)_{\mathrm{RIG}}$ *be k-PSQAS's with rigid $G(K)$-structures $(i = 1, 2)$. Then the following are equivalent;*

1. $(Z_1, \phi_1, \rho_1)_{\mathrm{RIG}} \simeq (Z_2, \phi_2, \rho_2)_{\mathrm{RIG}}$
2. *there is a k-isomorphism $f : Z_1 \simeq Z_2$ with $\phi_1 = \phi_2 \cdot f$.*

Proof. First we prove that (2) implies (1). Definition 9.5 (ii) is clear from the uniqueness of theta modules. (i) follows from it. See Definition 5.13 for theta modules. Definition 9.5 (ii) is clear.

Next we prove that (1) implies (2). By Definition 9.5 (i) and by the very-ampleness of V_i there is an $h \in GL(V(K) \otimes k)$ such that $\phi_1^* \cdot h = f^* \cdot \phi_2^* \in \operatorname{Hom}(V(K) \otimes k, V_1)$. Hence $G(\phi_1^*)G(h) = G(f^*)G(\phi_2^*)$. It follows from Definition 9.5 (ii) that $G(h)$ is the identity, *i.e.*, $h \cdot g = g \cdot h$ for any $g \in \bar{G}(K)$. By Schur's lemma, h is a scalar multiple of the identity. Hence $\phi_1 = \phi_2 \cdot f$. □

Corollary 9.7. *If* $e_{\min}(K) \geq 3$, *then* $\operatorname{Aut}((Z, \phi, \rho)_{\mathrm{LEV}})$ *is trivial.*

Lemma 9.8. *Let* $\operatorname{Spec} k$ *be a point of* $\operatorname{Spec} \mathcal{O}$. *Let (Z, L, V) be a K-symplectic PSQAS over k. For any level $G(K)$-structure (ϕ, ρ) on (Z, L, V) there exists a unique rigid $G(K)$-structure $(\phi(\rho), \rho)$ such that*

$$(Z, \phi(\rho), \rho)_{\mathrm{LEV}} \simeq (Z, \phi, \rho)_{\mathrm{LEV}} \tag{47}$$

Proof. We choose and fix a symplectic isomorphism $\sigma : (K, e_K) \otimes k \simeq (K(Q_0, \mathcal{L}_0), e_0^\sharp)$ and a central extension isomorphism $\rho : G(K) \otimes k \simeq G(Q_0, \mathcal{L}_0)$ in a compatible way by Remark 7.9. See Definition 4.15 for $e^\sharp$. By Lemma 7.11 there is a k-isomorphism $\phi(\rho)^* : V(K) \otimes k \to V = \Gamma(Q, \mathcal{L}) \otimes k$ such that

$$U(\rho(g))(\phi(\rho)^*(w)) = \phi(\rho)^* U(K)(g)(w) \quad (\forall g \in G(K), \forall w \in V(K)) \tag{48}$$

where U is the action of $G(Q_0, \mathcal{L}_0)$ on V. Hence $\rho = G(\phi(\rho)^*)U(K)$. By Theorem 6.3 $\phi(\rho)$ is a closed k-immersion $\phi(\rho) : Z \to \mathbf{P}(V(K) \otimes_{\mathcal{O}} k)$.

Uniqueness of ϕ follows from Lemma 9.6. □

Remark 9.9. Let $(Z, \phi, \rho)_{\mathrm{LEV}}$ be a k-PSQAS with a level $G(K)$-structure. The stabilizer group $\mathrm{Stab}(\phi(\rho)(Z))$ in $\mathrm{GL}(V(K) \otimes k)$ contains $\bar{G}(K) \otimes k$. There is a unique $h \in \mathrm{GL}(V(K) \otimes k)$ such that $\phi = h \cdot \phi(\rho)$. Then $\mathrm{Stab}(\phi(Z))$ contains $h(\bar{G}(K) \otimes k)h^{-1}$. Thus we see

$$G(\phi(Z), O_{\mathbf{P}(V(K))}(1)_{\phi(Z)}) = h(\bar{G}(K) \otimes k)h^{-1} \tag{49}$$

Lemma 9.10. *Let T be an irreducible reduced $\mathcal{O}$-scheme, $\pi : (A, L) \to T$ a polarized abelian T-scheme, $A^t := \mathrm{Pic}^0(A/T)$ and $\lambda(L) : A \to A^t$ the polarization morphism. Assume that L_s is a separable polarization for any geometric point $s \in T$ and assume that $(\ker \lambda(L_s), e^{L_s}) \simeq (K, e_K) \otimes k(s)$ for a geometric point s of T. Then there exists a finite étale covering $f : T^* \to T$ and a constant finite reduced symplectic abelian group $\mathcal{O}$-scheme K such that $(\ker \lambda(L_{T^*}), (e^L)_{T^*}) \simeq (K, e_K)_{T^*}$ where e^L is the Weil pairing on $\ker \lambda(L)$.*

Proof. We have an exact sequence of group schemes

$$0 \to \ker \lambda(L) \to A \stackrel{\lambda(L)}{\to} A^t \to 0.$$

By the assumption $\lambda(L)_s = \lambda(L_s)$ is étale for any geometric point s of T. Hence $\lambda(L)$ is étale. Therefore $\ker \lambda(L)$ is étale and finite over T. Let T' be one of the irreducible components of $\ker \lambda(L)$ with $T' \not\simeq T$. Then $\ker \lambda(L)_{T'}$ has a new section over T'. By repeating the same argument we see that there exists an irreducible reduced $\mathcal{O}$-scheme T^* étale and finite over T such that $H := \ker \lambda(L)_{T^*}$ is a disjoint union of copies of T^*. The multiplication morphism of H as a group scheme as well as its inversion maps each connected component of $H \times H$ into a connected component of H. Since T^* is reduced, it follows that H is a trivial group T^*-scheme, say $H = K_{T^*}$ for some finite abelian group K (regarded as a reduced group $\mathcal{O}$-scheme). The Weil pairing $(e^L)_{T^*}$ is a symplectic bilinear form on $H \times H$ with values in $(\mu_M)_{T^*}$, which is constant on each component of $H \times H$ because T^* is reduced. Hence there exists a symplectic form e_K on K such that $(H, e^L_{T^*}) \simeq (K, e_K)_{T^*}$. This proves Lemma. $\square$

Lemma 9.11. *Let T be an irreducible reduced $\mathcal{O}$-scheme, $\pi : (A, L) \to T$ a polarized abelian T-scheme. Suppose that there exists a constant finite reduced symplectic abelian group $\mathcal{O}$-scheme K such that $(\ker \lambda(L), e^L) \simeq (K, e_K)_T$. Then after a suitable finite base change of T if necessary $\mathcal{G}(A, L) \simeq \mathcal{G}(K)_T$.*

Proof. Let $\mathcal{G}(A, L) = L^{\mathrm{tsr}}_{|K(A,L)}$ be the Heisenberg group of (A, L), a central extension of $K(A, L) := \ker \lambda(L)$ by $\mathbf{G}_{m,T}$ with commutator form e^L. By the assumption there exists a constant subgroup scheme H of K such that $(K, e_K) \simeq (K(H), \ell_H)$.

We prove that after a finite base change of T there are an open affine covering U_i of T and isomorphisms $\rho_i : \mathcal{G}(K)_{U_i} \simeq \mathcal{G}(A, L)_{U_i}$ as group U_i-schemes.

In fact, let $U_i := \mathrm{Spec}\ A_i$ be an affine covering of T and $k(\eta)$ the field of a generic point of T. We note $k(\eta) = k(A_i)$, the filed of fractions of A_i. By choosing a finite extension $k(\eta')$ of $k(\eta)$ for $k(\eta)$ and integral closures of A_i in $k(\eta')$ for A_i, we may assume from the start that $\mathcal{G}(A,L)_{U_i}$ is a totally split central extension and described by two cocycles $f_i(\lambda,\nu)$ $(\lambda,\nu\in K(A_i))$.

Hence by the argument in the proof of Lemma 7.4, we can construct a field $k(\eta')$ by adding to $k(\eta)$ N-th roots of some elements of $k(\eta)$ as in Equation (30). Then by taking integral closures A'_i of A_i in $k(\eta')$ we may assume that $\mathcal{G}(K)_{U'_i}\simeq\mathcal{G}(A,L)_{U'_i}$ where $U'_i:=\operatorname{Spec}A'_i$.

The affine open sets U'_i form an open covering of the normalization T' of T in $k(\eta')$. Therefore by Lemma 7.8 by taking a finite base change W of T' we infer that $\mathcal{G}(K)_W\simeq\mathcal{G}(A,L)_W=\mathcal{G}(A_W,L_W)$. □

Remark 9.12. By Lemma 7.7 and Lemma 9.11, after a finite base change we have a subgroup W-scheme $G(A_W,L_W)$ of $\mathcal{G}(A_W,L_W)$, a central extension of K_W by $(\mu_M)_W$.

Lemma 9.13. *Let T be an irreducible reduced $\mathcal{O}$-scheme, $\pi:(A,L)\to T$ a polarized abelian T-scheme. Suppose that there exists a constant reduced symplectic finite abelian group $\mathcal{O}$-scheme K such that $(\ker\lambda(L),e^L)\simeq(K,e_K)_T$. Then after a suitable finite base change of T if necessary there exist an invertible O_T-module M with trivial $G(K)_T$-action and an isomorphism $\rho^{-1}:G(A,L)\simeq G(K)_T$ such that $\pi_*(L)$ is ρ-equivariant isomorphic to $M\otimes_{\mathcal{O}}V(K)$.*

Proof. We may assume $(K,e_K)=(K(H),\ell_H)$ for some contant reduced finite group $\mathcal{O}$-scheme H. By Lemma 9.11 we may assume that $G(A,L)\simeq G(K)_T$. Let s be a closed point of T, U an affine open subset of T with $s\in U$. It follows that through ρ $\pi_*(L)$ is a $G(K)_T$-module of weight one, that is, the centre of $G(K)_T$ acting on $\pi_*(L)$ by scalar multiplication. Since $\operatorname{rank}\pi_*(L)=\operatorname{ord}H$, we have by Lemma 7.11 $\pi_*(L)\otimes k(s)\simeq V(K)\otimes_{\mathcal{O}}k(s)$ as $G(K)_T\otimes k(s)$-modules for any closed point $s\in T$. Since any eigenvalue of the action of $H_T(T)$ on $\pi_*(L)\otimes k(s)$ belongs to $\Gamma(T,O_T)$, the action of the subgroup scheme H_T on $\pi_*(L)$ is diagonalized with respect to some $\mathcal{O}_T$-module basis of $\pi_*(L)$ locally at s. By the proof of Lemma 7.11 this implies that there exists an open affine covering $\{U_i\}$ of T such that $\pi_*(L)\otimes_{O_T}O_{U_i}\simeq O_{U_i}\otimes_{\mathcal{O}}V(K)$ as $G(K)_{U_i}$-O_{U_i}-modules. Let $U(K)_i=1_{O_{U_i}}\otimes U(K)$ be the action of $G(K)_{U_i}$ on $O_{U_i}\otimes_{\mathcal{O}}V(K)$. It follows that there is a one-cocycle $\ell_{jk}\in H^1(\{U_{jk}\},GL(O_T\otimes_{\mathcal{O}}V(K)))$ such that

$$U(K)_j(g)\ell_{jk}=\ell_{jk}U(K)_k(g)\quad(\forall g\in G(K)_T(T))$$

on $U_{jk}:=U_j\cap U_k$. Hence $U(K)(g)\ell_{jk}=\ell_{jk}U(K)(g)$. Therefore ℓ_{jk} is a scalar matrix by Schur's lemma. Hence $\ell_{jk}\in H^1(\{U_{jk}\},O_T^*)$, which defines an invertible O_T-module M. This proves Lemma. □

Corollary 9.14. *Let anything be as in Lemma 9.13. Suppose $e_{\min}(K)\geq 3$. Then by taking a suitable finite base change of T if necessary there exist a closed T-immersion $\phi:A\to\mathbf{P}(V(K)\otimes_{\mathcal{O}}O_T)$, a central extension isomorphism $\rho:G(K)_T\to G(A,L)$ and an invertible O_T-module M with trivial $G(K)_T$-action such that*

1. $L \simeq \phi^*(M \otimes_{\mathcal{O}} O_{\mathbf{P}(V(K))}(1))$,
2. $\pi_*(L) \simeq M \otimes_{\mathcal{O}} V(K)$,

where the isomorphisms (1) and (2) are ρ-equivariant and $\Gamma(O_{\mathbf{P}(V(K))}(1))$ is identified with $V(K)$.

Proof. Since $e_{\min}(K) \geq 3$, L is relatively very ample by Theorem 6.3. Hence any choice of O_T-module basis of $\pi_*(L)$ defines a projective embedding of A over T. Corollary follows from Lemma 9.13. □

Corollary 9.14 motivates the following definition.

Definition 9.15. Let T be a noetherian $\mathcal{O}_M$-scheme. Then a sextuplet $(Q, \mathcal{L}, G(Q, \mathcal{L}), \mathcal{V}, \phi, \rho)$ is called a projectively stable quasi-abelian T-scheme of relative dimension g with a level $G(K)$-structure if

(i) Q is a proper flat T-scheme with a relatively ample invertible sheaf $\mathcal{L}$,
(ii) ϕ is a closed T-immersion of Q into $\mathbf{P}(V(K) \otimes_{\mathcal{O}} O_T)$,
(iii) $G(Q, \mathcal{L})$ is a finite flat reduced group T-scheme acting on $(Q, \mathcal{L})$
(iv) $\rho : G(K)_T \to G(Q, \mathcal{L})$ is an isomorphism of group T-schemes
(v) $\mathcal{V}$ is a locally O_T-free subsheaf of $\pi_*(\mathcal{L})$
(vi) $\mathcal{L} \simeq \phi^*(O_{\mathbf{P}(V(K))}(1) \otimes_{\mathcal{O}} M)$ and $\mathcal{V} \simeq \pi_*(\phi^* H^0(O_{\mathbf{P}(V(K))}(1))) \otimes_{\mathcal{O}} M$ are ρ-equivariant isomorphisms for some action of $G(K)_T$ on $V(K)_T$
(vii) for any geometric point s of T, (Q_s, ϕ_s, ρ_s) is a PSQAS of dimension g over $k(s)$ with a level $G(K)$-structure,

where M is some invertible O_T-module with trivial $G(K)_T$-action and $\pi : Q \to T$ is the structure morphism.

We denote $(Q, \mathcal{L}, G(Q, \mathcal{L}), \mathcal{V}, \phi, \rho)$ by $(Q, \phi, \rho)_{\mathrm{LEV}}$ for brevity. If any (Q_s, ϕ_s, ρ_s) in (vii) is a PSQAS over $k(s)$ with a rigid $G(K)$-structure, then we call (Q, ϕ, ρ) a T-PSQAS with a rigid $G(K)$-structure and denote it by $(Q, \phi, \rho)_{\mathrm{RIG}}$.

We note $\mathcal{V} \simeq V(K) \otimes_{\mathcal{O}} M$ by (vi). (vii) implies $G(Q_s, \mathcal{L}_s) = G(Q, \mathcal{L}) \otimes k(s)$ for any point $s \in T$.

Definition 9.16. Let $(Q_i, \phi_i, \rho_i)_{\mathrm{LEV}} := (Q_i, \mathcal{L}_i, G(Q_i, \mathcal{L}_i), \mathcal{V}_i, \phi_i, \rho_i)$ $(i = 1, 2)$ be T-PSQAS's with level $G(K)$-structures. We define them to be isomorphic if there exist a T-isomorphism $f : Q_1 \to Q_2$ and an invertible O_T-module M with trivial $G(K)$-action such that

(i) $\mathcal{L}_1 \simeq f^* \mathcal{L}_2 \otimes_{\mathcal{O}} M$ is a $G(K)$-equivariant isomorphism,
(ii) $\mathcal{V}_1 = f^*(\mathcal{V}_2 \otimes_{\mathcal{O}} M)$ as $G(K)$-O_T-submodules of $\pi_*(\mathcal{L}_1)$,
(iii) $\rho_1 = G(f^*) \cdot \rho_2$.

If (Q_i, ϕ_i, ρ_i) are rigid $G(K)$-structures, then (iii) is equivalent to the following

(iv) $\phi_1 = \phi_2 \cdot f$

which implies (i) and (ii) and that M in (i) and (ii) is necessarily trivial.

The above equivalence of (iii) and (iv) is proved in the same manner as in Lemma 9.6.

Definition 9.17. We define the functor $\mathcal{SQ}_{g,K}$ as follows. For any noetherian $\mathcal{O}_{N,M}$-scheme T, we set

$$\begin{aligned}\mathcal{SQ}_{g,K}(T) = &\text{ the set of projectively stable quasi-abelian}\\ &\ T\text{-schemes } (Q,\phi,\rho)_{\mathrm{LEV}} \text{ of relative dimension } g\\ &\ \text{with level } G(K)\text{-structures modulo } T\text{-isom}\end{aligned}$$

$$\begin{aligned}\mathcal{SQ}^{\mathrm{RIG}}_{g,K}(T) = &\text{ the set of projectively stable quasi-abelian}\\ &\ T\text{-schemes } (Q,\phi,\rho)_{\mathrm{RIG}} \text{ of relative dimension } g\\ &\ \text{with rigid } G(K)\text{-structures modulo } T\text{-isom}\,.\end{aligned}$$

10. Stable reduction theorem – summary

10.1. **The rigid $G(K)$-structure we start from.** Let R be a complete discrete valuation ring with the fraction field $k(\eta)$ and k the residue field. Let $(G_\eta,\mathcal{L}_\eta)$ be a polarized abelian variety over $k(\eta)$ with separable polarization $\mathcal{L}_\eta$ and $K(\mathcal{L}_\eta) := \ker\lambda(\mathcal{L}_\eta)$. Let $e^{\mathcal{L}_\eta}$ be the Weil pairing of $K(\mathcal{L}_\eta)$. Then after a suitable finite ramified cover of Spec R we may assume that $K(\mathcal{L}_\eta)(k(\eta)) = K(\mathcal{L}_\eta)(\overline{k(\eta)})$. Hence there exists a finite symplectic abelian group (K,e_K) (regarded as a group $\mathbf{Z}$-scheme) such that $(K(\mathcal{L}_\eta),e^{\mathcal{L}_\eta})\simeq(K,e_K)\otimes k(\eta)$.

By Lemma 7.4 or Remark 7.9, after a suitable finite ramified cover of Spec R we may assume that $\mathcal{G}(K)\otimes k(\eta)\simeq\mathcal{G}(G_\eta,\mathcal{L}_\eta)$ and $\rho_\eta : G(K)\otimes k(\eta)\simeq G(G_\eta,\mathcal{L}_\eta)$.

Then $\Gamma(G_\eta,\mathcal{L}_\eta)$ is a unique irreducible $G(G_\eta,\mathcal{L}_\eta)$-module of weight one by Lemma 7.11, hence so as a $\mathcal{G}(G_\eta,\mathcal{L}_\eta)$-module because we may of course assume $\zeta_M\in R$. By Lemma 9.6 $(G_\eta,\mathcal{L}_\eta,\Gamma(G_\eta,\mathcal{L}_\eta))$ has a unique rigid $G(K)$-structure for the above ρ_η.

We summarize the previous results in the separably polarized case as follows.

Theorem 10.2. *Let R be a complete discrete valuation ring with the fraction field $k(\eta)$ and k the residue field. Let $(G_\eta,\mathcal{L}_\eta)$ be a polarized abelian variety over $k(\eta)$ and $K(\mathcal{L}_\eta) := \ker\lambda(\mathcal{L}_\eta)$. Assume that the charateristic of k and the order of $K(\mathcal{L}_\eta)$ are coprime. Then after a suitable finite base change if necessary, there exist flat projective schemes $(P,\mathcal{L})$ and $(Q,\mathcal{L})$, semiabelian group schemes G and $G^\sharp$, the flat closure $K(Q,\mathcal{L})$ of $K(\mathcal{L}_\eta)$ in $G^\sharp$, and the finite and infinite Heisenberg group schemes $G(Q,\mathcal{L})$ and $\mathcal{G}(Q,\mathcal{L})$ of $(Q,\mathcal{L})$, all of these being defined over S, such that*

1. *$\mathcal{L}$ is relatively ample on P and Q,*
2. *P is reduced over S*
3. *$(G,\mathcal{L})$ and $(G^\sharp,\mathcal{L})$ are open subschemes of $(P,\mathcal{L})$,*
4. *$G^\sharp = K(Q,\mathcal{L})\cdot G$,*
5. *$(G_\eta,\mathcal{L}_\eta)\simeq(G^\sharp_\eta,\mathcal{L}_\eta)\simeq(P_\eta,\mathcal{L}_\eta)\simeq(Q_\eta,\mathcal{L}_\eta)$,*

6. $(K(Q,\mathcal{L}),e^{\sharp})$ *is a constant finite flat reduced subgroup scheme of* $G^{\sharp}$ *with a constant symplectic form* $e^{\sharp}$ *extending* $(K(\mathcal{L}_\eta),e^{\mathcal{L}_\eta})$
7. $G(Q,\mathcal{L})$ *and* $\mathcal{G}(Q,\mathcal{L})$ *are respectively totally split central extensions of* $K(Q,\mathcal{L})$ *by* $\mu_{M,S}$ *and* $\mathbf{G}_{m,S}$ *with commutator form* $e^{\sharp}$
8. $\Gamma(G^{\sharp},\mathcal{L})\simeq\Gamma(P,\mathcal{L})\simeq\Gamma(Q,\mathcal{L})$ *and they are irreducible* $\mathcal{G}(Q,\mathcal{L})$*-module of weight one, unique up to equivalence*

where $M=e_{\max}(K(Q_\eta,\mathcal{L}_\eta))$.

Proof. By the assumption $\mathcal{L}_\eta$ is a separable polarization, hence we can start from Subsection 10.1. (1) and (5) follow from Theorems 3.5, 4.10, 5.8, Remark 3.10 and Subsection 4.13. (2) follows from [AN96] and Definition 3.6. (3) and (4) follow from Subsections 4.12, and 4.13. (6) follows from Lemma 4.14 and Definition 5.11. (7) follows from Subsection 2.4, Definitions 5.11, 7.12, 7.14 and Lemma 7.4. (8) follows from Lemma 5.12. □

As a corollary to the proof of Theorem 10.2 we obtain,

Corollary 10.3. *Under the same assumptions and notation as in Theorem 10.2 there exists a finite symplectic abelian group* $\mathbf{Z}$*-scheme* (K,e_K) *such that*

1. $(K(Q,\mathcal{L}),e^{\sharp})\simeq(K,e_K)_S$,
2. $G(Q,\mathcal{L})\simeq G(K)_S$ *and* $\mathcal{G}(Q,\mathcal{L})\simeq\mathcal{G}(K)_S$
3. $\Gamma(Q,\mathcal{L})\simeq V(K)\otimes_{\mathcal{O}_M}R$

where $M=e_{\max}(K)$. *All the isomorphisms in (1) and (2) are compatible and coincide with those given in Subsection 10.1 over* $k(\eta)$. *The isomorphism in (3) is (2)-equivariant with respect to some action of* $\mathcal{G}(K)_S$ *on* $V(K)\otimes_{\mathcal{O}_M}R$.

The following is the stable reduction theorem we need in Sections 11 and 12.

Theorem 10.4. *Let* (K,e_K) *be a finite reduced symplectic abelian group* $\mathbf{Z}$*-scheme with* $M=e_{\max}(K)$ *and* $e_{\min}(K)\geq 3$. *Let* R *be a complete discrete valuation ring over* $\mathcal{O}_M$ *with the fraction field* $k(\eta)$ *and* k *the residue field. Assume that* M *and the charateristic of* k *are coprime. For a given abelian variety* $(G_\eta,\phi_\eta,\rho_\eta)_{\mathrm{RIG}}$ *over* $k(\eta)$ *with a rigid* $G(K)$*-structure, by a finite base change of* Spec R *if neccesary, there exists a PSQAS* $(Q,\phi,\rho)_{\mathrm{RIG}}$ *over* Spec R *with a rigid* $G(K)$*-structure extending* $(G_\eta,\phi_\eta,\rho_\eta)_{\mathrm{RIG}}$.

Proof. Since $e_{\min}(K)\geq 3$, $\mathcal{L}_\eta$ is very ample by Theorem 6.2. By assumption we have a rigid $G(K)$-structure (ϕ_η,ρ_η) on $(G_\eta,\mathcal{L}_\eta,\Gamma(G_\eta,\mathcal{L}_\eta))$. By Theorem 10.2 and Corollary 10.3 we have an S-PSQAS $(Q,\mathcal{L},\Gamma(Q,\mathcal{L}))$, an isomorphism $\rho:G(K)_S\simeq G(Q,\mathcal{L})$ extending ρ_η and a ρ-equivariant isomorphism $\psi^*:V(K)\otimes_{\mathcal{O}}R\simeq\Gamma(Q,\mathcal{L})$ with respect to some action of $G(K)_S$ on $V(K)\otimes_{\mathcal{O}}R$. It follows from Lemma 9.8 that there exists a rigid $G(K)$-structure (ϕ,ρ), uniquely determined for a given ρ, on an S-PSQAS $(Q,\mathcal{L},\Gamma(Q,\mathcal{L}))$ such that $(Q,\phi,\rho)_{\mathrm{LEV}}\simeq(Q,\psi,\rho)_{\mathrm{LEV}}$. By the uniqueness of a rigid $G(K)$-structure for ρ_η, we have $\phi\otimes k(\eta)=\phi_\eta$. □

11. The scheme $SQ_{g,K}$

11.1. $Z_{g,K}$ **and** $H_{g,K}$**.** Let $\mathcal{O} = \mathcal{O}_M$ and let (K, e_K) be a constant reduced finite symplectic abelian group $\mathcal{O}$-scheme with $e_{\min}(K) \geq 3$, $M = e_{\max}(K)$ and $N := \sqrt{\operatorname{ord}_{\mathcal{O}} K}$. Let $P(n) := n^g N$, $S_{\mathcal{O}} := \operatorname{Spec}\mathcal{O}$ and let $H_{g,K} := \operatorname{Hilb}_{\mathbf{P}}^{P(n)}$ be the Hilbert scheme parametrizing all projective subschemes of $\mathbf{P} := \mathbf{P}(V(K))$ with their Hilbert polynomial $P(n)$, $(Z_{g,K}, L_{g,K})$ the universal subscheme over $H_{g,K}$ and $\pi : Z_{g,K} \to H_{g,K}$ the natural morphism. Let $i : (Z_{g,K}, L_{g,K}) \to \mathbf{P}(V(K)) \times_{S_{\mathcal{O}}} H_{g,K}$ be the natural (given) closed immersion of the universal subscheme $Z_{g,K}$ over $H_{g,K}$. We remark that $H_{g,K}$ and $Z_{g,K}$ are $\mathcal{O}$-schemes by [FGA, 221, Théorème 3.1]. In this section we will define an $\mathcal{O}$-subscheme $SQ_{g,K}$ of $H_{g,K}$ which ought to be the moduli scheme. See [MFK94, Proposition 7.3, pp.132-134].

Since $e_{\min}(K) \geq 3$, any K-symplectic PSQAS $(Q_0, \mathcal{L}_0, V_0)$ over k is a closed point of $H_{g,K}$ by choosing any isomorphism $V_0 \simeq V(K) \otimes_{\mathcal{O}} k$ in view of Theorem 6.3.

Let U be the maximal subscheme of $H_{g,K}$ such that π is smooth, which is an open $\mathcal{O}$-subscheme of $H_{g,K}$ by [EGA, IV, Corollaire 6.8.7]. Let U_1 be the union of all connected components (as $\mathcal{O}$-subschemes) of U containing at least a geometric point s of U, over which a fibre of π is an abelian variety with $\ker\lambda(L_{g,K,s}) \simeq K \otimes_{\mathcal{O}} k(s)$. Let $Z_1 := Z_{g,K} \times_{H_{g,K}} U_1$. By the base change U_2 of U_1 we may assume $Z_2 := Z_1 \times_{U_1} U_2$ has a section e over U_2. For instance choose $U_2 = Z_1$. Then by [MFK94, Theorem 6.14] Z_2 is an abelian scheme over U_2 with e unit section. It follows from Lemma 9.10 that any geometric fibre of Z_2, a fortiori, any geometric fibre of Z_1 is an abelian variety with $\ker\lambda(L_{g,K,s}) \simeq K \otimes_{\mathcal{O}} k(s)$.

Let

$$U_3 := U_2^{\bar{\mathcal{G}}(K)} = U_2^{\bar{G}(K)} = \{s \in U_2; g(s) = s \quad \text{for } \forall g \in \bar{G}(K)\} \tag{50}$$

It is clear that U_3 is an $\mathcal{O}$-scheme. Let $(Z_3, L_3) := (Z, L) \times_{H_{g,K}} U_3$. The relative polarized-Aut-scheme $\operatorname{Aut}_{U_3}(Z_3, L_3)$ is a closed subgroup $\mathcal{O}$-scheme of $\operatorname{PGL}(V(K))_{U_3}$ finite over U_3 because L_3 is very ample. Meanwhile the relative Aut-scheme $\operatorname{Aut}^0_{U_3}(Z_3)$ is quasi-projective. Let

$$\operatorname{Aut}^0_{U_3}(Z_3, L_3) := \operatorname{Aut}_{U_3}(Z_3, L_3) \cap \operatorname{Aut}^0_{U_3}(Z_3) \tag{51}$$

It is an open and closed normal subgroup $\mathcal{O}$-scheme of $\operatorname{Aut}_{U_3}(Z_3, L_3)$. By the definition of U_3, $\bar{G}(K)_{U_3}$ is a subgroup scheme of $\operatorname{Aut}_{U_3}(Z_3, L_3)$. Let

$$U_4 := \{s \in U_3; \bar{G}(K) \otimes k(s) \subset \operatorname{Aut}^0_{U_3}(Z_3, L_3) \otimes k(s)\} \tag{52}$$

Definition 11.2. Let $U_{g,K} = U_1$, and let $W_{g,K}$ be the closure of $(U_{g,K})_{\mathrm{red}}$ in $H_{g,K}$ with reduced structure. Let $A_{g,K} := U_4$. By (52) $A_{g,K}$ is an $\mathcal{O}$-subscheme of $H_{g,K}$.

We define $SQ_{g,K}$ to be the closure of $A_{g,K}$ in $H_{g,K}$, *i.e.*, the minimal (reduced) closed $\mathcal{O}$-subscheme of $H_{g,K}$ containing $(A_{g,K})_{\mathrm{red}}$. Let $Z^{SQ}_{g,K} := Z_{g,K} \times_{H_{g,K}} SQ_{g,K}$, and let $\pi_{g,K} : Z^{SQ}_{g,K} \to SQ_{g,K}$ be the natural projection.

We note that for any abelian scheme $(A, L) \in U_{g,K}(k)$ over a closed field k, there is by Lemma 9.8 a closed immersion $\phi : A \to \mathbf{P}(V(K) \otimes k)$ such that $(\phi(A), O_{\mathbf{P}}(1)_{|\phi(A)}) \in A_{g,K}(k)$.

11.3. **The level structure over $A_{g,K}$.** There is a natural isomorphism

$$\rho_{g,K} : G(K)_{A_{g,K}} \simeq G((Z_{g,K}, L_{g,K}) \times_{H_{g,K}} A_{g,K}) \tag{53}$$

In fact, any element h of $\bar{G}(K) \otimes k(s)$ is a translation $T_{a(s)}$ by some point $a(s)$ of an abelian variety Z_s for any $s \in U_4$ because $h \in \mathrm{Aut}^0_{U_3}(Z_3)$. Since h keeps L_s invariant, $a(s)$ belongs to $K(Z_s, L_s) = \ker \lambda(L_{g,K,s})$. Therefore

$$G((Z_{g,K}, L_{g,K}) \times_{H_{g,K}} A_{g,K}) \otimes k(s) = \bar{G}(K) \otimes_{\mathcal{O}} k(s) \tag{54}$$

for any geometric point s of $A_{g,K}$ because both group $k(s)$-schemes have order N^2M. The isomorphism $U(K) : G(K) \to \bar{G}(K)$ induces a natural morphism $\rho_{g,K}$ of group $\mathcal{O}$-schemes in (53), which is an isomorphism by the fibrewise isomorphism (54).

Theorem 11.4. *Assume $e_{\min}(K) \geq 3$. Then $SQ_{g,K}$ is a projective $\mathcal{O}$-subscheme of $W_{g,K}$ pointwise fixed by $\bar{G}(K)$ such that for any geometric point* Spec k *of* Spec $\mathcal{O}$,

1. $A_{g,K}(k) = \left\{ (Z, L) \in W_{g,K}(k);\ \begin{array}{l} (Z, i, U(K)) \textit{ is an abelian variety} \\ \textit{over } k \textit{ with a rigid } G(K)\textit{-structure} \end{array} \right\}$
2. $SQ_{g,K}(k) = \left\{ (Z, L) \in W_{g,K}(k);\ \begin{array}{l} (Z, i, U(K)) \textit{ is a PSQAS over } k \\ \textit{with a rigid } G(K)\textit{-structure} \end{array} \right\}$

where i is the natural inclusion of Z into $\mathbf{P}(V(K) \otimes k)$.

Proof. By Equality (54) $A_{g,K}$ is an $\mathcal{O}$-subscheme of $W_{g,K}$ such that any $(Z, L) \in A_{g,K}(k)$ is an abelian variety with $G(Z, L) = i^*\bar{G}(K)$, hence $(Z, i, U(K))$ is an abelian variety with a rigid $G(K)$-structure. Conversely if $(Z, i, U(K))$ is an abelian variety with a rigid $G(K)$-structure and $(Z, L) \in H_{g,K}$, then (Z, L) is fixed by the action of $\bar{G}(K)$. Hence $(Z, L) \in U_3$. Any automorphism of Z induced from $\bar{G}(K)$ is a translation by the definition of rigid $G(K)$-structures, hence $(Z, L) \in U_4 = A_{g,K}$. This proves (1).

$A_{g,K}(k)$ is pointwise fixed by the action of $\bar{G}(K)$ on $H_{g,K}$ because $i^*\bar{G}(K) = G(Z, L) \subset \mathrm{Aut}(Z, L)$. It follows from it that $SQ_{g,K}$ is an $\mathcal{O}$-subscheme of $W_{g,K}$ pointwise fixed by $\bar{G}(K)$. It remains to prove (2).

Let R be a complete discrete valuation ring with fraction field $k(\eta)$, η the generic point of $S := \mathrm{Spec}\, R$. Suppose that we are given a flat S-subscheme (Z, L) of $(Z_{g,K} \times_{H_{g,K}} SQ_{g,K})_S$ such that (Z_η, L_η) is a polarized abelian scheme. Let i be the inclusion immersion of (Z, L) into $\mathbf{P}(V(K))_S$, $G(Z, L) := i^*(\bar{G}(K)_S)$ and $\chi := G(i^*) \cdot (U(K)_S)$. Then (i, χ) is a rigid $G(K)$-structure on $(Z, L, i^*V(K)_S)$ over S.

In view of Lemma 7.4, Lemma 9.8 and Theorem 10.4, by choosing a suitable finite ramified cover of S if necessary we may assume that we have a projective flat family (W, M) with a rigid $G(K)$-structure (ϕ, ρ) such that $(W_\eta, M_\eta) \simeq (Z_\eta, L_\eta)$ and (W_0, M_0) is a PSQAS. To be more precise,

(i) $\phi : (W, M) \to \mathbf{P}(V(K))_S$ is a closed S-immersion
(ii) $\rho : G(K)_S \simeq G(W, M)$ is an isomorphism with $\rho = G(\phi^*)(U(K)_S)$
(iii) $(W_\eta, \phi_\eta, \rho_\eta)_{\mathrm{LEV}} \simeq (Z_\eta, i_\eta, U(K)_\eta)_{\mathrm{LEV}}$,
(iv) (W_0, ϕ_0, ρ_0) is a PSQAS with a rigid $G(K)$-structure

Hence there is by Lemma 9.6 a $k(\eta)$-isomorphism $f : Z_\eta \to W_\eta$ such that $i_\eta = \phi_\eta \cdot f$, that is, $i_\eta(Z_\eta) = \phi_\eta(W_\eta)$. Since i and ϕ are closed S-immersions respectively, they induce natural morphisms $\mathrm{Hilb}(i)$ and $\mathrm{Hilb}(\phi)$ from $\mathrm{Spec}\, R$ into $SQ_{g,K}$ by the universal property of $H_{g,K}$. It follows from $i_\eta = \phi_\eta \cdot f$ that we have $\mathrm{Hilb}(i_\eta) = \mathrm{Hilb}(\phi_\eta) : \mathrm{Spec}\, k(\eta) \to SQ_{g,K}$. Since $SQ_{g,K}$ is projective (separated), we have $\mathrm{Hilb}(i) = \mathrm{Hilb}(\phi)$ [EGA, II, 7.2.3]. This implies that $i(Z) = \phi(W)$, *a fortiori*, $i_0(Z_0) = \phi_0(W_0)$. Hence $(Z_0, L_0) \simeq (W_0, M_0)$, and $(Z_0, i_0, U(K))$ is a k-PSQAS with a rigid $G(K)$-structure. In fact, $G(Z_0, L_0) = \bar{G}(K)$. This proves (2). ☐

Definition 11.5. Now we define the relative n-th Hilbert point $hilb_n$. Let $\pi : Z_{g,K} \to H_{g,K}$ be the natural morphism, and let

$$\mathcal{V} := O_{H_{g,K}} \otimes_{\mathcal{O}} V(K), \quad \mathcal{V}_n := \pi_*(O_{Z_{g,K}}(n)).$$

We note that $\mathcal{V}_n$ is locally free of rank $n(g) := n^g N$ for sufficiently large $n \geq n_0$. Let $\phi_n : S^n\mathcal{V} \to \mathcal{V}_n$ be the natural epimorphism for $n \geq n_0$. Thus we have a morphism $hilb_n := \overset{n(g)}{\wedge} \phi_n$ of $H_{g,K}$ into the projective space $\mathbf{P}_{\mathrm{large}} := \mathbf{P}(\overset{n(g)}{\wedge} S^n V(K))$. For any large $n \geq n_0$ $hilb_n$ is a closed $\mathcal{O}$-immersion of $H_{g,K}$ into $\mathbf{P}_{\mathrm{large}}$. For a geometric fibre (Z, L) of π we call $hilb_n(Z, L)$ *the n-th Hilbert point of* (Z, L).

Theorem 11.6. *Let* $\mathrm{Spec}\, k$ *be a geometric point of* $\mathrm{Spec}\, \mathcal{O}$ *and* $(Z, L) \in H_{g,K}(k)$. *Suppose* $e_{\min}(K) \geq 3$ *and that* (Z, L) *is smoothable into an abelian variety* (A, M) *with* $\ker \lambda(M) \simeq K \otimes k$. *Then the following are equivalent.*

1. (Z, L) *is a K-symplectic PSQAS over k,*
2. $\mathrm{Aut}(Z, L)$ *contains a subgroup of* $\mathrm{SL}_{\pm}(V(K) \otimes k)$ *conjugate to* $\bar{G}(K) \otimes k$,
3. *the normalized Hilbert points* $hilb_n(Z, L)$ *are Kempf-stable for any large* $n \geq n_0$.

Proof. (2) follows from (1) by Lemma 9.8. (3) follows from (2) by [Kempf78, Corollary 5.1]. It remains to prove that (3) implies (1). We choose and fix a large n so that $hilb_n$ may be a closed immersion of $W_{g,K}$. Suppose (3) and that (Z, L) is smoothable into an abelian variety with $\ker \lambda \simeq K \oplus K^\vee$. It follows $(Z, L) \in W_{g,K}(k)$. By (3) $hilb_n(Z, L)$ is Mumford-semistable. By [Seshadri77, p. 269, Remark 8] there is a categorical quotient

$$\bar{W}_{g,K} := W_{g,K} \cap hilb_n^{-1}(\mathbf{P}_{\mathrm{large}}(\mathrm{SemiStable}))/\,\mathrm{SL}(V(K))$$

which is a projective $\mathcal{O}$-scheme. By Lemma 9.8 and Theorem 10.4 $\bar{W}_{g,K}$ is set-theoretically $\mathrm{SL}(V(K) \otimes k)$-orbits of PSQAS's modulo some relation between orbit closures. By [Seshadri77, p. 269, Theorem 4] the closure of the $\mathrm{SL}(V(K) \otimes k)$-orbit of $hilb_n(Z, L)$ intersects the $\mathrm{SL}(V(K) \otimes k)$-orbit of a PSQAS $hilb_n(W, M)$. By the assumption (3) the orbit of $hilb_n(Z, L)$

is closed. Therefore the orbit of $hilb_n(Z,L)$ is that of $hilb_n(W,M)$, hence $(Z,L)\simeq(W,M)$ is a PSQAS. $\square$

12. Reduced-fine-moduli schemes

Definition 12.1. Let $\mathcal{O}_M := \mathbf{Z}[\zeta_M, 1/M]$. When no confusion is possible, we write $\mathcal{O}=\mathcal{O}_M$. For a contravariant functor F over $\mathcal{O}$ a reduced $\mathcal{O}$-scheme W is said to be a reduced-fine-moduli scheme over $\mathcal{O}$ of F or we say that F is reductively-represented over $\mathcal{O}$ by W if the following conditions are satisfied;

(a) $f_W(T): F(T)\to \mathrm{Hom}_S(T,W)$ is a bijection for a reduced $\mathcal{O}$-scheme T
(b) $f_W(T)\cdot F(f)=\mathrm{Hom}(f,W)\cdot f_W(U)$ for an $\mathcal{O}$-morphism $f:T\to U$ of reduced $\mathcal{O}$-schemes T,U

It is easy to see that a reduced-fine-moduli scheme is unique if there exists.

Theorem 12.2. *If $e_{\min}(K)\geq 3$, then $\mathcal{SQ}_{g,K}$ is reductively-represented by a projective scheme $SQ_{g,K}$ over $\mathbf{Z}[\zeta_M,1/M]$ where $M=e_{\max}(K)$.*

Proof. First we prove $\mathcal{SQ}_{g,K}\simeq\mathcal{SQ}^{\mathrm{RIG}}_{g,K}$. Suppose $(Q,\phi,\rho)_{\mathrm{LEV}}$ is a T-PSQAS with a level $G(K)$-structure. By Definition 9.15 (iv), we have an isomorphism $\rho: G(K)_T\to G(Q,L)$. Let $\{U_i=\mathrm{Spec}\ R_i\}$ be an open affine covering, $\mathcal{V}_i:=\mathcal{V}_{U_i}\simeq V_i\otimes_{\mathcal{O}}O_{U_i}$ for some free R_i-module V_i, $\phi_i:=\phi_{U_i}$ and $\rho_i:=\rho_{U_i}$. By Lemma 7.11, by choosing a finer open covering of $\{U_i\}$ if necessary, there is a collection of $A_i\in\mathrm{GL}(\mathcal{V}_i)$ such that $\rho_i(g)=A_iG(\phi_i^*)U(K)(g)A_i^{-1}$ $(\forall g\in G(K)_{U_i}(U_i))$. Let $\phi(\rho)_i^*:=A_i\phi_i^*$. Then $\phi(\rho)_i^*$ induces a closed U_i-immersion of Q_{U_i} into $\mathbf{P}(V(K)\otimes_{\mathcal{O}}O_{U_i})$ such that $\rho_i(g)=G(\phi(\rho)_i^*)U(K)(g)$. On $U_i\cap U_j$, we have $\rho_i=\rho_{U_i\cap U_j}=\rho_j$, hence $G(\phi(\rho)_i^*)(g)=G(\phi(\rho)_j^*)(g)$ for any $g\in\bar{G}(K)$. Hence by Schur's lemma there is a scalar $\ell_{ij}\in\mathrm{GL}(V(K)\otimes_S O_{U_{ij}})$ such that $\phi(\rho)_i^*=\ell_{ij}\phi(\rho)_j^*$.

It follows that there are a closed T-immersion $\phi(\rho)$ of Q into $\mathbf{P}(V(K))_T$ and $M:=\{\ell_{ij}\}\in H^1(O_T^*)$ such that $\mathcal{V}\simeq\phi(\rho)^*(V(K)\otimes_{\mathcal{O}}M)$ and $\rho=G(\phi(\rho)^*)U(K)$. By Lemma 9.8 $(Q,\phi(\rho),\rho)_{\mathrm{RIG}}$ is a unique T-PSQAS with a rigid $G(K)$-structure such that $(Q,\phi(\rho),\rho)_{\mathrm{LEV}}\simeq(Q,\phi,\rho)_{\mathrm{LEV}}$ where the isomorphism is induced from the identity of Q. It follows $\mathcal{SQ}_{g,K}\simeq\mathcal{SQ}^{\mathrm{RIG}}_{g,K}$.

It remains to prove that $\mathcal{SQ}^{\mathrm{RIG}}_{g,K}$ is *reductively*-represented by $SQ_{g,K}$. Let T be a reduced $\mathcal{O}_M$-scheme. Suppose we are given a projectively stable quasi-abelian T-scheme $(Z,\phi,\rho)_{\mathrm{RIG}}$ with a rigid $G(K)$-structure. Then $\phi: Z\to\mathbf{P}(V(K)\otimes_{\mathcal{O}}O_T)$ is a closed immersion of Z. Hence we have a natural T-morphism $\mathrm{Hilb}(\phi): T\to H_{g,K}$ which factors through $SQ_{g,K}$ by Theorem 11.4 (2). Moreover $Z=\mathrm{Hilb}(\phi)^*(Z^{SQ}_{g,K})=Z^{SQ}_{g,K}\times_{SQ_{g,K}}T$ by the universal property of $Z_{g,K}$. Hence ϕ is clearly the pull back of the universal inclusion map of $Z^{SQ}_{g,K}$ into $\mathbf{P}(V(K))_{SQ_{g,K}}$ by $\mathrm{Hilb}(\phi)$. Since $\rho=G(\phi^*)U(K)$ by definition, it follows that the map $(Z,\phi,\rho)_{\mathrm{RIG}}\mapsto\mathrm{Hilb}(\phi)$ is a bijection between $\mathcal{SQ}_{g,K}(T)$ and $\mathrm{Hom}_S(T,SQ_{g,K})$. This proves Definition 12.1 (a).

The second condition (b) in Definition 12.1 is clear. This proves that $\mathcal{SQ}_{g,K}$ is reductively-represented by $SQ_{g,K}$. $\square$

Remark 12.3. Modifying [FC90, Chapter IV, Section 6] slightly we can construct a family of PSQAS's over the Voronoi compactification $A_{g,K}^{\mathrm{Vor}}$ of $A_{g,K}$ if $e_{\min}(K) \geq 3$. Hence we have a natural morphism of $A_{g,K}^{\mathrm{Vor}}$ onto $SQ_{g,K}$, which is an isomorphism over $A_{g,K}$, the moduli scheme of nonsingular abelian varieties with level $G(K)$-structures. We conjecture that $A_{g,K}^{\mathrm{Vor}} \simeq SQ_{g,K}$. The reduced-coarse-moduli scheme is defined in a manner similar to Definition 12.1. For smaller K the functor $\mathcal{SQ}_{g,K}$ is defined as in Definition 9.15 except one point that ϕ is no longer a closed immersion but ϕ^* is a ρ-equivariant isomorphism of k-vector spaces.

The reduced-coarse-moduli scheme for $\mathcal{SQ}_{g,K}$ will be obtained by taking a quotient $SQ_{g,K_n}/G(B_n)$ for $n \geq 3$ where B_n and K_n are symplectic finite abelian groups fitting into the following exact sequece

$$0 \to B_n \ (\simeq (\mathbf{Z}/n\mathbf{Z})^{2g}) \to K_n \to K \to 0 \tag{55}$$

One of the difficulities we encounter with in constructing the reduced-coarse-moduli scheme for $\mathcal{SQ}_{g,K}$ is that if SQ_{g,K_n} is nonnormal, then the quotient $SQ_{g,K_n}/G(B_n)$ may depend on n, though all of these $SQ_{g,K_n}/G(B_n)$ are naturally bijective and there are morphisms from larger n to smaller n. We would like to discuss these topics somewhere else.

13. Examples in dimension one

13.1. **Cubic curves.** Let $V = \mathbf{Z}x_0+\mathbf{Z}x_1+\mathbf{Z}x_2$. Let $\mathbf{P} := \mathbf{P}(S^3(V^\vee))$ be the projective space of ternary cubic forms on V. Let U be an open subscheme of $\mathbf{P}$ consisting of curves with at worst a unique nodal singularity. The categorical quotient of U by SL(3) is $\mathbf{P}^1_{\mathbf{Z}}$, which is the coarse moduli scheme of the functor of smooth elliptic curves and a rational curve with a node. This is the moduli scheme corresponding to the j-invariant of ellitic curves. This is also regarded as the coarse moduli scheme of one-pointed stable curves of genus one. However as we saw in Section 12 and as we also see in Subsection 13.2, we have a different kind of compactification $SQ_{1,(\mathbf{Z}/3\mathbf{Z})^{\oplus 2}}$.

13.2. **The case of $K = (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$.** Let $\mathcal{O} := \mathcal{O}_3 = \mathbf{Z}[\zeta_3, 1/3]$. Let $H = \mathbf{Z}/3\mathbf{Z}$ and $K = H \oplus H^\vee = (\mathbf{Z}/3\mathbf{Z})^{\oplus 2}$. The order of $G(K)$ is $3^3 = 27$. We define $\chi_i \in H^\vee$ by $\chi_i(1) = \zeta_3^i$. Let

$$V := \mathcal{O}[H^\vee] = \mathcal{O}[(\mathbf{Z}/3\mathbf{Z})^\vee] = \mathcal{O}x_0 + \mathcal{O}x_1 + \mathcal{O}x_2 \tag{56}$$

where $x_i := v(\chi_i)$. The group $\bar{G}(K)$ $(\simeq G(K))$ is generated by σ and τ

$$\sigma : (x_0, x_1, x_2) \mapsto (x_0, \zeta_3 x_1, \zeta_3^2 x_2) \tag{57}$$

$$\tau : (x_0, x_1, x_2) \mapsto (x_1, x_2, x_0) \tag{58}$$

The group $G(K)$ has 9 one-dimensional representations $\rho_{i,j}$ $(i,j \in (\mathbf{Z}/3\mathbf{Z}))$ and two three-dimensional representations ρ_3 and ρ'_3 where

$$\rho_{i,j}(a \oplus b) = \zeta_3^{ia+jb}, \quad (a,b) \in H \oplus H^\vee \tag{59}$$

$$\rho_3 \simeq V, \quad \rho'_3 \simeq \mathrm{Hom}_{\mathcal{O}}(V, \mathcal{O}) \tag{60}$$

which checks that $|G(K)| = 27 = 9 \cdot 1^2 + 3^2 + 3^2$.

Let $S_n = S^n V$. We note $V = S_1 \simeq \rho_3$. Then we see

$$S_2 = 2\rho'_3, \quad S_3 = 2\rho_{0,0} \oplus \oplus_{(i,j)\neq(0,0)} \rho_{i,j} \tag{61}$$

Let $V' = \{x_0^2, x_1^2, x_2^2\}$ and $V'' = \{x_1x_2, x_2x_0, x_0x_1\}$ be the subspaces of S_2 spanned by those polynomials. We see $V' \simeq V'' \simeq \rho'_3$. The only possible nontrivial subschemes of $\mathbf{P}^2$ with nontrivial moduli defined by quadric polynomials are therefore

$$t_0 x_i^2 - t_1 x_{i+1} x_{i+2} = 0 \quad (i \in \mathbf{Z}/3\mathbf{Z}) \tag{62}$$

which are clearly zero-dimensional. These subschemes are Kempf-stable by [Kempf78, Corollary 5.1].

Our case of cubic curves arises from $2\rho_{0,0}$ in S_3. Let $\pi : (E, L) \to T$ be any non-isotrivial cubic curve (or a one-dimensional PSQAS) with a rigid $G(K)$-structure over some connected $\mathcal{O}$-scheme T. By the fibrewise Riemann-Roch theorem, the O_T-module $\pi_*(nL)$ is locally free of rank $3n$ and the natural homomorphisms $\pi_n : S^n(\pi_* L) \to \pi_*(nL)$ are surjective for any $n \geq 1$. It follows that $\ker \pi_3$ is locally free of rank one. Since $\ker \pi_3$ has a $G(K)$-action, it is generated by a $\bar{G}(K)$-(semi)invariant cubic polynomial with nontrivial parameters. Since $\rho_{i,j}$ $(i,j) \neq (0,0)$ has multiplicity one in S_3, the equation f_E defining E on $\mathbf{P}(V)_T$ lies in $2\rho_{0,0}$. It follows that

$$f_E = A_0(x_0^3 + x_1^3 + x_2^3) - A_1 x_0 x_1 x_2 \tag{63}$$

for some $(A_0, A_1) \in \mathbf{P}^1_T$. Combining this with the subsequent observation on the curves defined by cubic polynomials with character $\rho_{i,j}$ for $(i,j) \neq (0,0)$, the above argument reproves that the universal subscheme $Z := Z^{\mathrm{SQ}}_{1,K}$ is given by the Hesse cubic

$$\mu_0(x_0^3 + x_1^3 + x_2^3) - \mu_1 x_0 x_1 x_2 = 0 \tag{64}$$

and that $SQ_{1,K} \simeq \mathbf{P}(\mathcal{O}[\mu_0, \mu_1]) \simeq \mathbf{P}^1_{\mathcal{O}}$.

Let k be any closed field of char $k \neq 3$. Let $f := x_0^3 + \zeta_3 x_1^3 + \zeta_3^2 x_2^3$ and let C be a curve defined by $f = 0$. By the natural isomorphism $G(K)/\mu_3 \simeq H \oplus H^\vee$, we see that $kf \simeq \rho_{0,1}$. Let $y_i = \zeta_9 x_i$. Then C is isomorphic to the Fermat cubic $y_0^3 + y_1^3 + y_2^3 = 0$. Via the isomorphism of C with the Fermat cubic, we see that $G(C, O_C(1))$ is generated by two elements σ and τ_C where

$$\tau_C : (x_0, x_1, x_2) \mapsto (\zeta_9 x_1, \zeta_9 x_2, \zeta_9^{-2} x_0) \tag{65}$$

Let $\rho : (x_0, x_1, x_2) \mapsto (\zeta_9^{-2} x_0, \zeta_9 x_1, \zeta_9 x_2)$. Then $\rho = \tau^{-1}\tau_C$. Since $\rho \in \mathrm{Aut}(C, O_C(1))$ is of order 3 and has 3 fixed points on C, it is neither a translation of C nor an endomorphism of C. This shows that $\rho \notin G(C, O_C(1))$ and

$\tau \in \bar{G}(K) \setminus G(C, O_C(1))$. It follows that $(C, O_C(1)) \in H^{\bar{G}(K)}_{1,K}(k) \setminus S_{1,K}(k)$ with the notation in Subsection 11.1. The same is true for the other $\rho_{i,j}$ $(i,j) \neq (0,0)$. This proves that any isotrivial cubic curve E with a rigid $G(K)$-structure is also defined by a cubic polynomial (63). This completes the proof of the universality of $Z^{\mathrm{SQ}}_{1,K}$ and $SQ_{1,K} \simeq \mathbf{P}(\mathcal{O}[\mu_0, \mu_1])$.

We see that the fibre Z_μ of Z is singular only for $(\mu_0, \mu_1) = (0,1)$ or $(1, -3\zeta_3^i)$ $(i = 0,1,2)$. Let $G^\sharp$ be the Néron model of Z. Then $G^\sharp$ is a semi-abelian scheme with fibres elliptic curves or extensions of a torus $\mathbf{G}_m$ by H. Let $Z_3 := \ker \lambda(O_Z(1))$ be the set of all flexes of the Hesse cubic Z,

$$Z_3 = \{(x_0, x_1, x_2) \in Z; x_0x_1x_2 = 0\} \tag{66}$$

Then Z_3 is regarded as the set of all 3-torsion points of $G^\sharp$. Let $e_0 = (1, -1, 0)$, $e_1 = (1, -\zeta_3, 0)$ and $e_2 = (-1, 0, 1)$. If we choose $e_0 \in Z_3$ as the identity of $G^\sharp$, then e_1 and e_2 generate Z_3 $(\simeq K)$ as an abelian group. Moreover σ and τ are translations of $G^\sharp$ by e_1 and e_2 respectively because $e_1 = \sigma(e_0)$ and $e_2 = \tau(e_0)$.

13.3. **The case of $K = (\mathbf{Z}/4\mathbf{Z})^{\oplus 2}$.** Let $\mathcal{O} := \mathbf{Z}[\zeta_4, 1/2]$. Let $H = \mathbf{Z}/4\mathbf{Z}$ and $K = H \oplus H^\vee$. We define $\chi_i \in H^\vee$ by $\chi_i(1) = \zeta_4^i$. Let

$$V := \mathcal{O}[H^\vee] = \mathcal{O}x_0 + \mathcal{O}x_1 + \mathcal{O}x_2 + \mathcal{O}x_3 \tag{67}$$

where $x_i := v(\chi_i)$. $G(K)$ has 16 one-dimensional representations, 4 two-dimensional representations V^i $(i = 0,1,2,3)$ and 2 four-dimensional representations V and its dual where we check $|G| = 64 = 16 + 4 \cdot 2^2 + 2 \cdot 4^2$.

Let V' (rresp. V'') be the subspace of S^2V spanned by the pair $x_0^2 + x_2^2, x_1^2 + x_3^2$ (resp. x_1x_3, x_0x_2). Then $V' \simeq V'' \simeq V^0$ and $S^2V = 2V^0 + V^1 + V^2 + V^3$. Let Z be the curve in $\mathbf{P}^1_{\mathcal{O}} \times \mathbf{P}(V)$

$$\mu_0(x_0^2 + x_2^2) - \mu_1 x_1 x_3 = 0 \tag{68}$$

$$\mu_0(x_1^2 + x_3^2) - \mu_1 x_0 x_2 = 0 \tag{69}$$

over $\mathbf{P}^1_{\mathcal{O}}$. We see as before that any non-isotrivial elliptic curve (or a non-isotrivial one-dimensional PSQAS) E with a rigid $G(K)$-structure over some connected $\mathcal{O}$-scheme T is defined by the equations on $\mathbf{P}^1_T \times \mathbf{P}(V)_T$

$$A_0(x_0^2 + x_2^2) - A_1 x_1 x_3 = 0 \tag{70}$$

$$A_0(x_1^2 + x_3^2) - A_1 x_0 x_2 = 0 \tag{71}$$

for some $(A_0, A_1) \in \mathbf{P}^1_T$. It follows that $(Z, i, U(K))_{\mathrm{RIG}}$ is the universal PSQAS with a rigid $G(K)$-structure where $i : Z \to \mathbf{P}^1_{\mathcal{O}} \times \mathbf{P}(V)$ is the natural inclusion. In particular, $SQ_{1,K} \simeq \mathbf{P}(\mathcal{O}[\mu_0, \mu_1]) \simeq \mathbf{P}^1_{\mathcal{O}}$. The curves defined by V^i $(i = 1,2,3)$ belong to $H^{\bar{G}(K)}_{1,K} \setminus S_{1,K}$ as in Subsection 13.2.

We define a morphism of $\mathbf{P}^1_{\mathcal{O}}$ to $\mathbf{P}^1_{\mathcal{O}}$ by $h^*\mu_0 = \nu_0^2$ and $h^*\mu_1 = \zeta_4\nu_1^2$. The pull back h^*Z of Z has exactly 16 4-torsion points $x_0x_1x_2x_3 = 0$ rational over $\mathbf{P}(\mathcal{O}[\nu_0, \nu_1])$, while it has no rational points over $\mathbf{P}(\mathcal{O}[\mu_0, \mu_1])$. It follows that Z is not isomorphic to $S(4)$, the universal generalized elliptic curve with a classical level 4-structure. This will be discussed in detail in [NT98].

Notation/Terminology

$a(x), b(x,y)$	degeneration data, Theorem 4.3
$A_{g,K}$	Definition 11.2
$\mathrm{Del}, \mathrm{Del}(P_0), \mathrm{Del}(Q_0)$	Definitions 1.1, 3.6,
e_K	Definition 7.5
$e_S^{\mathcal{L}_\eta}, e^{\mathcal{L}_\eta}$	Eq.(9)
G	semi abelian scheme, Subsections 2.1, 4.12
$G^\sharp$	Subsection4.13
$G(Q,\mathcal{L})$	Definition 7.14
$\mathcal{G}(Q,\mathcal{L})$	Definitions 5.11, 7.12
$G(K), \bar{G}(K)$	Heisenberg group, Definition 7.5
$\mathcal{G}(K), \bar{\mathcal{G}}(K)$	Definition 7.5
Hilbert point	Introduction, Section 8
$H, K, K(H)$	Definition 7.5
Kempf-stable	Introduction, Section 8
(K, e_K)	Definition 7.5
$K_S(\mathcal{L}_\eta), K_S^\sharp(\mathcal{L}_\eta)$	Subsection 2.3, Lemma 4.14
$K_S(\mathcal{M})$	Subsection 2.3
K-symplectic PSQAS	Definitions 9.2
$K(Q,\mathcal{L})$	Definitions 5.11
$\lambda, \lambda(\mathcal{L}_0)$	Subsection 2.2
level $G(K)$-structure	Definitions 9.4, 9.15, Lemma 5.14
μ_M	Definition 7.5
$\mathcal{O}, \mathcal{O}_M$	Definition 7.5
$(P,\mathcal{L}), (P_0,\mathcal{L}_0)$	SQAV, Theorem 3.5
$(Q,\mathcal{L}), (Q_0,\mathcal{L}_0)$	PSQAS, Theorem 5.8, Definition 5.9
$(Q,\phi,\rho)_{\mathrm{LEV}}, (Z,\phi,\rho)_{\mathrm{LEV}}$	Definitions 9.4, 9.15
$(Q,\phi,\rho)_{\mathrm{RIG}}, (Z,\phi,\rho)_{\mathrm{RIG}}$	Definitions 9.2, 9.4, 9.15
Raynaud sequence	Subsection 2.2
rigid $G(K)$-structure	Definitions 9.2, 9.4, 9.15
$S_{g,K}$	Definition 11.2
S_y	Eq.(12)
theta submodule	Definition 5.13, Eq (14)
$V(K)$	Definition 7.5
$W_{g,K}$	Subsection 11.2
ξ_x	Definition 3.4
$\zeta_{x,c}$	Definition 3.4
(Z,L,V)	Definition 9.2

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