Non-Archimedean L-Functions

Associated with Siegel Modular Forms

by

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Introduction

The starting point in the theory of zeta functions is the expansion of the Rieman zeta-function $\zeta(s)$ into the Euler product :

$$\zeta(s) = \prod_p (1-p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s} \quad (\mathrm{Re(s)} > 1)$$

The set of arguments s for which $\zeta(s)$ is defined can be extended to all $s \in \mathbf{C}, s \neq 1$ and considered as the group of continuous quasicharacters

$$\mathbf{C} = \mathrm{Hom}(\mathbf{R}_+^\times, \mathbf{C}^\times), \quad \mathrm{y} \mapsto \mathrm{y}^{\mathrm{s}}.$$

The special values $\zeta(1-k)$ at negative integers are rational numbers:

$$\zeta(1-k) = -\frac{B_k}{k},$$

where B_k - Bernoulli numbers, defined by the formal equality

$$e^{Bt} = \sum_{n=0}^{\infty} \frac{B_n t^n}{n!} = \frac{t\,e^t}{e^t - 1}$$

and we know (by Sylvester-Lipschitz theorem) that

$$c \in \mathbf{Z} \Longrightarrow c^k(c^k - 1)\frac{B_k}{k} \in \mathbf{Z}$$

The theory of non-Archimedean zeta-functions originates in the work of Kubota and Leopoldt [Ku-Le] containing p–adic interpolation of these special values. Their construction turns out to be equivalent to classical Kummer congruences for the Bernoulli numbers, which we recall here in the following form. Let p be a fixed prime number, $c > 1$ an integer prime to p. Put

$$\zeta_{(p)}^{(c)}(-k) = (1-p^k)(1-c^{k+1})\zeta(-k)$$

and let $h(x) = \sum_{i=0}^{n} \alpha_i\, x^i \in \mathbf{Z}_p[x]$ be a polynomial over the ring $\mathbf{Z}_p$ of p-adic integers such that

$$x \in \mathbf{Z}_p \Longrightarrow h(x) \in p^m \mathbf{Z}_p.$$

Then we have that

$$\sum_{i=0}^{n} \alpha_i\, \zeta_{(p)}^{(c)}(-k) \in p^m \mathbf{Z}_p.$$

This property expresses the fact that the numbers $\zeta_{(p)}^{(c)}(-k)$ depend continuously on k in the p-adic sense; it can be deduced from the known formula for the sum of k -th powers:

$$S_k(N) = \sum_{n=1}^{N-1} n^k = \frac{1}{k+1}[B_{k+1}(N) - B_{k+1}]$$

in which $B_k(x) = (x+B)^k = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i}$ denotes the Bernoulli polynomial. Indeed, all summands in $S_k(N)$ depend p-adic analytically on k, if we restrict ourselves to numbers n, prime to p ,so that the desired congruence follows if we express the numbers $\zeta_{(p)}^{(c)}(-k)$ in terms of Bernoulli numbers.

The set, on which p-adic zeta functions are defined, is the p-adic analytic Lie group

$$X_p = \mathrm{Hom}(\mathbf{Z}_p^\times, \mathbf{C}_p^\times)$$

where $\mathbf{C}_p = \widehat{\overline{\mathbf{Q}}}_p$ is Tate field (completion of an algebraic closure of the p-adic field $\mathbf{Q}_p$), so that all integers k can be considered as the characters $x_p^k : y \mapsto y^k$. The construction of Kubota and Leopoldt is equivalent to existence a p–adic analytic function $\zeta_p : X_p \to \mathbf{C}_p$ with a single pole at the point $x = x_p^{-1}$, which becomes a bounded holomorphic function on X_p after multiplication by the elementary factor $(x_p\, x - 1)$ $(x \in X_p)$, and is *uniquely defined* by the condition

$$\zeta_p({x_p}^k) = (1-p^k)\zeta(-k) \qquad (k \ge 1).$$

This result has a very natural interpretation in framework of the theory of non-Archimedean integration (due to Mazur): there exists a p-adic measure $\mu^{(c)}$ on $\mathbf{Z}_p^\times$ with values in $\mathbf{Z}_p$ such that $\int_{\mathbf{Z}_p^\times} x_p^k\, \mu^{(c)} = \zeta_{(p)}^{(c)}(-k)$. Indeed , if we integrate $h(x)$ over $\mathbf{Z}_p^\times$ we exactly get the above congruence. On the other hand, in order to define such a measure $\mu^{(c)}$ it is sufficient for any continuous function $\phi : \mathbf{Z}_p^\times \to \mathbf{Z}_p$ to define its integral $\int_{\mathbf{Z}_p^\times} \phi(x)\, \mu^{(c)}$. For this purpose we approximate $\phi(x)$ by a polynomial (for which the integral is already defined by the above condition), and then pass to the limit.

The important feature of the construction is that it equally works for primitive Dirichlet characters χ modulo a power of p : if we fix an embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ then the character $\chi : (\mathbf{Z}/\mathbf{Z}_{p^N})^\times \to (\overline{\mathbf{Q}})^\times$ becomes an element of the torsion subgroup $X_p^{\mathrm{tors}} \subset X_p$ and the above equality also holds for the special values $L(-k, \chi)$ of the Dirichlet L-series

$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s} = \prod_p (1 - \chi(p)p^{-s})^{-1},$$

so that we have

$$\zeta_p(\chi x_p^k) = i_p[(1-\chi(p)p^{-k})L(-k,\bar{\chi})] \quad (k \ge 1,\ k \in \mathbf{Z},\ \chi \in X_p^{\mathrm{tors}}).$$

The original construction of Kubota and Leopoldt [Ku-Le] was successesfully used by Iwasawa [Iw] for the description of class groups of cyclotomic fields. Since then the class of functions admitting p-adic analogues has gradually extended.

Zeta-functions (of complex variable) can be attached as certain Euler products to various objects such as diophantine equations, representations of Galois groups, modular forms etc., and they play a crucial role in modern number theory. Deep interrelations between these objects discovered in last decades are based on identities for the corresponding zeta functions which presumably all fit into a general concept of Langlands L-functions associated with automorphic representations of a reductive group G over a number field K. From this point of view the study of arithmetic properties of these zeta functions is becoming especially important.

The theory of modular symbols (due to Mazur and Manin, see [Man1]-[Man6], [Maz-SD]) provided a non-Archimedean construction of functions, which correspond to the case of the group $G = \mathrm{GL}_2$ over $K = \mathbf{Q}$. Several authors (including Deligne, Ribet, N.M.Katz, Kurčanov and others, see [De-Ri], [Ka1]-[Ka3], [Kurč1] -[Kurč3], [Sho], [V1], [V2]) investigated this problem for the case $G = \mathrm{GL}_1$ and GL_2 over totally real fields and fields of CM-type (i.e. totally imaginary quadratic extentions of totally real fields). But the case of more general reductive groups remained unclear until the mid-eighties although important complex analytic properties of the Langlands L-functions had been proved. In recent years a general approach to consruction of non-Archimedean L-functions associated with various classes of automorphic forms was developed, in particular, for the case of symplectic groups of even degree over $K = \mathbf{Q}$ and the group $G = \mathrm{GL}_2 \times \mathrm{GL}_2$ over a totally real field K.

The main tool of the appearing theory is the systematic use of the Rankin-Selberg method for obtaining both complex-valued and p-adic distributions as certain integrals involving cusp forms and Eisenstein series. By this method we constructed non-Archimedean analogues of the standard zeta functions attached to Siegel cusp forms of even degree and of sufficiently large weight.

For a Siegel modular form $f(z)$ of degree m and weight k, which is an eigenfunction of the Hecke algebra, and for each prime number p one can define Satake p-parameters of f denoted by $\alpha_i(p)$ with $i = 0, 1, \cdots, m$. Then the standard zeta function of f is the following product

$$\mathcal{D}(s,f,\chi) = $$
$$= \prod_p \left\{ \left(1 - \frac{\chi(p)}{p^s}\right) \prod_{i=1}^{m} \left(1 - \frac{\chi(p)\alpha_i(p)}{p^s}\right) \left(1 - \frac{\chi(p)al_i(p)^{-1}}{p^s}\right) \right\}^{-1},$$

where χ is a Dirichlet character. According to A.N.Andrianov and V.L.Kalinin [An-K], this function can be represented as an integral convolution of f and a theta series with an Eisenstein series as a kernel. The construction of its p-adic analytic continuation is based on explicit formulas for the special values of the standard zeta function and is equivalent to some generalized Kummer congruences for these values. For to give the precise formulation of our results we first introduce the normalized zeta functions

$$\mathcal{D}^-(s,f,\chi) = (2\pi)^{-m(s+k-(m+1)/2)} \prod_{j=1}^{m} \Gamma(s+k-j)\mathcal{D}(s,f,\chi),$$
$$\mathcal{D}^+(s,f,\chi) = \frac{2i^\delta \Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi)^s}\mathcal{D}^-(s,f,\chi),$$
$$\mathcal{D}^*(s,f,\chi) = \pi^{-(s+\delta)/2}\Gamma((s+\delta)/2)\mathcal{D}^-(s,f,\chi),$$

where $\delta = 0$ or 1 according as $\chi(-1) = (-1)^\delta$, and let

$$f(z) = \sum_{\xi > 0} a(\xi) e_m(\xi z) \in \mathcal{S}_k^m$$

be the Fourier expansion of the Siegel cusp form $f(z)$ of weight k, the sum is being taken over all positive definite half integral $m \times m$-matrices , $z \in \mathfrak{H}_m$,

$$\mathfrak{H}_m = \{z \in \mathrm{GL}_m(\mathbf{C}) \,|^t\, z = z, \quad \mathrm{Im}(z) \text{ is positive definite}\}$$

Siegel upper half plane of degree m and $e_m(z) = \exp(\mathrm{tr}(2\pi i z))$. Assume that $k > 2m+2$ and m is even.

Theorem A *(Algebraic properties of the special values of standard zeta functions)*

a) For all integer s with $1 \le s \le k - \delta - m$ an χ^2 non-trivial for $s = 1$ we have that

$$\langle f,\ f\rangle^{-1}\mathcal{D}^+(s, f, \chi) \in K = \mathbf{Q}(f, \Lambda_f, \chi),$$

where $K = \mathbf{Q}(f, \Lambda_f, \chi)$ denote the field generated by Fourier coefficients of f ,by the eigenvalues $\Lambda_f(X)$ of Hecke operators X on f, and by the values of the character χ.

b) For all integers s with $1 - k + \delta + m \le s \le 0$ we have that

$$\langle f,\ f\rangle^{-1}\mathcal{D}^-(s, f, \chi) \in K.$$

Assume also that $a(\xi_0) = 1$ for some $\xi_0 > 0$ with $\det 2\xi_0 = 1$;our essential assumption is that the form f is p-*ordinary* in a sence that $|\ i_p(\alpha_0(p)\ |_p = 1$ for a fixed embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$.

Theorem B *(non-Archimedean interpolation of the standard zeta functions) Under the assumptions as above for each integer $c > 1$ prime to p there exist bounded $\mathbf{C}_p$ - analytic functions*

$$\mathcal{D}^{c+}(x, f), \quad \mathcal{D}^{c-}(x, f) : X_p \to \mathbf{C}_p,$$

which are uniquely defined by the following conditions:

a) for all non-trivial Dirichlet characters $\chi \in X_p^{\mathrm{tors}}$ and for all integers s with $1 \le s \le k - \delta - m$ the following equality holds

$$\mathcal{D}^{c+}(\chi x_p^s, f) = \\ i_p\Big[\frac{G_m(\chi)\, C_\chi^{m(s+k-1-m)}}{\alpha_0(C_\chi)^2}\frac{C_\chi^s}{G(\bar\chi)}(1 - \bar\chi^2(c)c^{-2s})\frac{\mathcal{D}^+(s, f, \bar\chi)}{\langle f,\ f\rangle}\Big],$$

b) for all non-trivial Dirichlet characters $\chi \in X_p^{\mathrm{tors}}$ and for all integers s with $1 - k + \delta + m \le s \le 0$ holds the equality

$$\mathcal{D}^{c-}(\chi x_p^s, f) = \\ i_p\Big[\frac{G_m(\chi)\, C_\chi^{m(s+k-1-m)}}{\alpha_0(C_\chi)^2}(1 - \chi^2(c)c^{2s-2})\frac{\mathcal{D}^-(s, f, \bar\chi)}{\langle f,\ f\rangle}\Big],$$

where

$$G_m(\chi) = \sum_{h \in M_m(\mathbf{Z}) \bmod C_\chi} \chi(\det h) e_m(h/C_\chi)$$

denotes the Gauss sum of degree m of the primitive Dirichlet character $\chi \bmod C_\chi$, $C_\chi = p^{N_\chi}$, $\alpha_O(C_\chi) = \alpha_O(p)^{N_\chi}$, $G(\chi) = G_1(\chi)$.

The standard zeta function $D(s, f, \chi)$ provides an example for the general definition of Langlands L-functions. For a reductive group G over a number field K this definition is based on the notion of the Langlands L–group ${}^LG$;this group is a complex analytic reductive group such that the lattice of characters of its maximal torus and the lattice of its one-parameter subgroups are obtained from the similar objects of the group G by inversion.

The important fact in the representation theory of reductive groups over local fields is that for a place v of K semisimple cojugacy classes h_v of ${}^LG$ correspond to certain infinite dimensional representations π_v of the group $G(K_v)$ over the local field K_v (the completion of K at v). It is known that for groups of the type $\mathbf{A}_n$ and $\mathbf{D}_n$ this construction preserves their types, and interchanges the types $\mathbf{B}_n$ and $\mathbf{C}_n$, so that if $G = \mathrm{GL}_n$ then ${}^LG = \mathrm{GL}_n(\mathbf{C})$, and if $G = \mathrm{GSp}_m$ then ${}^LG = \mathrm{Spin}_{2m+1}(\mathbf{C})$,the universal covering of the orthogonal group $\mathrm{SO}_{2m+1}(\mathbf{C})$. For example, if $G = \mathrm{GL}_2$ and v a non-Archimedean place then ${}^LG = \mathrm{GL}_2(\mathbf{C})$ and for $h_v \ni \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$ the corresponding representation π_v is the representation $\mathrm{Ind}_T^G(\mu_1 \otimes \mu_2)$ induced from the maximal torus $T = \mathrm{GL}_1 \times \mathrm{GL}_1$, $\mu_1, \mu_2 : K_v^\times \to \mathbf{C}^\times$ being unramified quasicharacters of $K_v^\times$ with $\mu_i(\mathfrak{p}_v) = \alpha_i$, $i = 1, 2$.

Let π be an automorphic representation of the group G, which is an irreducible subrepresentation of the smooth regular representation of the adelized group $G(\mathbf{A}_K)$. Then there is the decomposition of π into the infinite tensor product: $\pi = \otimes_v \pi_v$ where π_v is a representation of $G(K_v)$ which correspond to certain classes h_v from ${}^LG$ for almost all v (i.e for $v \notin S$ where S is a finite set of places of K). For a finite dimensional representation $r : {}^L G \to \mathrm{GL}_t(\mathbf{C})$ of the L-group we define automorphic L-functions

$$L(s, \pi, r) = L_S(s, \pi, r) = \prod_{v \notin S} \det(1_t - (\mathbf{N}v)^{-s} r(h_v))^{-1}$$

where $\mathbf{N}v$ is the number of elements of the residue class field of v (which is a power of some prime number),and the product is taken over all non-Archimedean places v, $v \notin S$.

In the Siegel modular case we consider, associated with f, the automorphic representation π_f which is generated by a function on $\mathrm{GSp}_m(\mathbf{A})$ inflated from the cusp form f on $\mathfrak{H}_m$ (as a subrepresentation of the regular representsation of $G(\mathbf{A_Q}) = \mathrm{GSp}_m(\mathbf{A})$). The irreducibility of π_f is equivalent to the fact that f is an eigenfunction of the Hecke algebra $\mathcal{H}^m = \otimes_p \mathcal{H}_p^m$ of the Siegel modular group Γ_m of degree m. In this case the corresponding character of $\mathcal{H}_p$ on f is completely determined by its Satake p-parameters , and for the universal covering $r : \mathrm{Spin}_{2m+1}(\mathbf{C}) \to \mathrm{SO}_{2m+1}(\mathbf{C})$ with $\mathrm{Spin}_{2m+1}(\mathbf{C}) \subset \mathrm{GL}_{2^m}(\mathbf{C})$ we have that the classes h_v and $r(h_v)$ are represented by the matrices

$$\begin{aligned} h_v = Sp(h_v) &\ni \mathrm{diag}\{\alpha_0(p)\alpha_{i_1}(p) \cdots \alpha_{i_r}(p) \mid 0 \le r \le m,\ 1 \le i_1 < \ \cdots\ i_r \le m\} \\ r(h_v) = St(h_v) &\ni \mathrm{diag}\{1, \alpha_1(p), \alpha_2(p), \cdots, \alpha_m(p), \alpha_1(p)^{-1}, \alpha_2(p)^{-1}, \cdots, \alpha_m(p)^{-1}\}, \end{aligned}$$

where Sp and St are called, respectively, *spinor* and *standard* representations of the Langlands group ${}^LG$. Theerefore the standard zeta function $D(s, f, \chi)$ coinside with the L-function $L(s, \pi_f, St)$. The function

$$L(s,\pi_f,Sp)=\prod_p[\prod_{\substack{0\leq r\leq m\\ 1\leq i_1<\cdots i_r\leq m}}(1-\chi(p)\alpha_0(p)\alpha_{i_1}(p)\cdots\alpha_{i_r}(p)p^{-s})]^{-1}$$

is the spinor zeta function of f. Its analytic properties were investigated by A. N. Andrianov in the case $m=2$ but stil nothing is known about the algebraic properties of the function; however, it follows from the general Deligne congecture on critical values of L- functions that the properties analogous to those given in Theorem A could exist only for $s=k-1$.

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Chapter I. Siegel modular forms and the holomorphic projection operator

This chapter contains mainly some preparatory facts which will be used for the construction of non-Archimedean standard zeta-functions in the next chapter. We recall main properties of Siegel modular forms and of the action of the Hecke algebra on them, as well as the definitions of spinor zeta functions and standard zeta functions (§1), see also [An2], [An7]. Then in §2 we give basic facts about theta series with a Dirichlet series [An-M1], [An-M2], [St2] and the definitions of Siegel-Eisenstein series and of Rankin type convolutions of Siegel modular forms and their relation with the standard zeta functions. In §3 we give an exposition of some recent results of Shimura and P.Feit on real analytic Siegel-Eisenstein series and their analytic continuation in terms of confluent hypergeometric functions [Fe], [Shi7], [Shi9]. These results extend previous results of V.L.Kalinin [K] and Langlands [Ll1]. In the final §4 a detailed study of holomorphic projection operator and its basic properties is given. The formula of theorem 4.2 provides an explicit formula for calculating the holomorphic projection onto the space of holomorphic (not necessarily cusp) modular forms of functions belonging to a wide class of (non holomorphic) Siegel modular forms . Ealier the holomorphic projection operator onto the space of cusp form was studied by J.Sturm [St1], [St2], B.Gross and D.Zagier [Gr-Z] under some restrictive assumptions on the growth of modular forms. Theorem 4.6 gives an explicit description of the action of this operator in terms of the (non holomorphic) Fourier expansions. Here we also establish a very explicit formula (3.36) for the special (critical) values of the confluent hypergeometric function.

Notations

Let A be a commutative ring with identity, then $\mathrm{M}_{r,s}(A)$ denote the set of all $r \times s-$matrices with coefficients in A. For $z \in \mathrm{M}_r(\mathbf{C})$ put $e_r(z) = e(\mathrm{tr}(z))$ with $e(u) = \exp(2\pi i u)$ for $u \in \mathbf{C}$. We denote by ${}^tz \in \mathrm{M}_{s,r}$ the matrix, which is transpose to $z \in \mathrm{M}_{r,s}(A)$, and write $\xi[\eta]$ for ${}^t\eta\xi\eta$. For a degenerate square matrix ξ we put $\xi^* = {}^t\xi^{-1}$. If ξ is a hermitian matrix then we write $\xi \geq 0$ or $\xi > 0$ according as ξ is non negative or positive definite.

Let $\mathfrak{H}_m$ denote the Siegel upper half plane on the degree m,

$$\mathfrak{H}_m = \{z \in \mathrm{M}_m(\mathbf{C}) | {}^tz = z = x + iy, \quad y > 0\},$$

so that $\mathfrak{H}_m$ is a complex analytic variety whose demention is denoted by $\langle m \rangle = m(m+1)/2$.

Let the symbol A_m denote the lattice of all half integral symmetric matrices in the vector space $V = \{y \in \mathrm{M}_m(\mathbf{R}) | {}^ty = y\}$, . This lattice is dual to the lattice $L = \mathrm{M}_m(\mathbf{Z}) \cap V$ with respect to the pairing given by $(u, v) \mapsto e_m(uv)$. For a function $f : \mathfrak{H}_m \to \mathbf{C}$ of the form

$$f = \sum_{\xi \in A_m} c(\xi) e_m(\xi z) \quad (z \in \mathfrak{H}_m)$$

and for a positive integer A we use the notations

$$f|V(A)(z) = f(Az) = \sum_{\xi \in A_m} c(\xi) e_m(A\xi z),$$
$$f|U(A)(z) = \sum_{\xi \in A_m} c(A\xi) e_m(\xi z),$$
$$f^{\rho} = \sum_{\xi \in A_m} \overline{c(\xi)} e_m(\xi z)$$

as well as the notations by A.N.Andrianov for the action of the Frobenius elements $\Pi^+(q), \Pi^-(q)$ given in 1.8. Moreover for $A \geq 1$ and an integer k we put

$$f|W(A)(z) = f|_k W(A)(z) = \det(\sqrt{A}z)^k f(-(Az)^{-1}),$$

so that

$$(f|W(A))|W(A) = f\ ,$$

§1. Siegel modular forms and Hecke operators

1.1. Symplectic group and Siegel upper half plane. (see [An2], [An7], [Shi4], [Sie3], [Fr], [Maa]). Let $G = \mathrm{GSp}$ be the algebraic subgroup of GL_{2m} defined by

$$G_A = \left\{\alpha \in \mathrm{GL}_{2m}(A) \mid {}^t\alpha J_m \alpha = \nu(\alpha) J_m, \nu(\alpha) \in A^\times\right\}, \tag{1.1}$$

for any commutative ring A, where

$$J_m = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix}.$$

The elements of G_A are characterized by the conditions

$$b^t a - a^t b = d^t c - c^t d = 0_m, \quad d^t a - c^t b = 1_m, \tag{1.2}$$

and if

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_A \textit{ then } \alpha^{-1} = \nu(\alpha)^{-1} \begin{pmatrix} {}^t d & -{}^t b \\ -{}^t c & {}^t a \end{pmatrix}.$$

The multiplier ν defines a homomorphism $\nu : G_A \to A^\times$ so that $\nu(\alpha)^{2m} = \det(\alpha)^2$ and $\operatorname{Ker}\nu$ is denoted by $\mathrm{Sp}_m(A)$. We also put

$$G_\infty = G_{\mathbf{R}}, G_{\infty+} = \{\alpha \in G_\infty \mid \nu(\alpha) > 0\}, G_{\mathbf{Q}+} = G_{\infty+} \cap G_{\mathbf{Q}}. \tag{1.3}$$

The group G acts transitively on the upper half plane $\mathfrak{H}_m$ by the rule

$$z \to \alpha(z) = (az+b)(cz+d)^{-1} \quad \left(\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\infty+}, z \in \mathfrak{H}_m\right)$$

so that scalar matrices act trivially, and $\mathfrak{H}_m$ can be identified with a homogeneous space of the group $\mathrm{Sp}_m(\mathbf{R})$. Let K_m denote the stabilizer of the point $i1_m \in \mathfrak{H}_m$ in the group $\mathrm{Sp}_m(\mathbf{R})$,

$$K_m = \{\alpha \in \mathrm{Sp}_m(\mathbf{R}) \mid \alpha(i1_m) = i1_m\},$$

then there is a bijection $\mathrm{Sp}_m(\mathbf{R})/K_m \simeq \mathfrak{H}_m$ and $K_m = \mathrm{Sp}_m(\mathbf{R}) \cap \mathrm{SO}_{2m}$. The group G is a maximal compact subgroup of the Lie group $\mathrm{Sp}_m(\mathbf{R})$ which can be identified with the group $\mathrm{U}(m)$ of all unitary $m \times m$-matrices via the map $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + ib$. We adopt also notations

$$\begin{gathered} dx = \prod_{i \le j} dx_{ij}, \quad dy = \prod_{i \le j} dy_{ij}, \quad dz = dx\, dy, \\ d^\times y = \det(y)^{-(m+1)/2} dy, \quad d^\times z = \det(y)^{-(m+1)} dz, \end{gathered} \tag{1.4}$$

where $z = x + iy, \quad x = (x_{ij}) = {}^t x, \quad y = (y_{ij}) = {}^t y > 0$. Then $d^\times z$ is a differential on $\mathfrak{H}_m$ invariant under the action of the group $G_{\infty+}$, and the measure $d^\times y$ is invariant under the action of elements $a \in \mathrm{GL}_m(\mathbf{R})$ on

$$Y = \{y \in \mathrm{M}_m(\mathbf{R}) \mid {}^t y = y > 0\}$$

defined by the rule $y \mapsto {}^t a y a$

1.2. Siegel modular forms. Let us consider Siegel modular group $\Gamma^m = \mathrm{Sp}_m(\mathbf{Z})$ and let $\Gamma \subset G_{\mathbf{Q}+}$ be an arbitrary congruence subgroup. This means that Γ is commeasurable with Γ^m in $G_{\mathbf{Q}+}$ modulo its center (i.e as a group of transformations of $\mathfrak{H}_m$) and $\Gamma \supset \Gamma^m(N)$ for some $N \in \mathbf{N}$,where

$$\Gamma^m(N) = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^m \mid \gamma \equiv 1_{2m}(\mathrm{mod} N)\right\}$$

is the main congruence subgroup of level N in Γ^m. In order to give the general definition of modular forms we consider a rational representation $\rho : \mathrm{GL}_m(\mathbf{Q}) \to \mathrm{GL}_r(\mathbf{Q})$ which will also be denoted by ρ. For $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\infty+}$ and for any complex valued function $f : \mathfrak{H}_m \to \mathbf{C}^r$ we use the notation

$$f \mid_\rho \alpha(z) = \rho(cz+d)^{-1} f(\alpha(z)). \tag{1.5}$$

Definition. A function $f : \mathfrak{H}_m \to \mathbf{C}^r$ is called holomorphic modular form of weight ρ on Γ if the following conditions (1.6)-(1.8) are satisfied:

$$f \mid_\rho = f, \tag{1.6}$$

$$f \text{ is holomorphic on } \mathfrak{H}_m, \tag{1.7}$$

$$\text{if } m = 1, \text{ then } f \text{ is holomorphic at cusps of } \Gamma. \tag{1.8}$$

Let $\mathcal{M}_\rho(\Gamma)$ be the complex vector space of functions satisfying the above conditions. For each $f \in \mathcal{M}_\rho(\Gamma)$ there is the following Fourier expansion

$$f(z) = \sum_\xi c(\xi) e_m(\xi z),$$

where $c(\xi) \in \mathbf{C}^r, \xi$ run over all $\xi = {}^t\xi \in \mathrm{M}_m(\mathbf{Q}), \xi \geq 0$ (for $m > 1$ the last condition automatically follows by the Koecher principle). More precisely, let M be the smallest integer such that

$$\Gamma \supset \left\{ \begin{pmatrix} 1_m & Mu \\ 0 & 1_m \end{pmatrix} \mid u \in \mathrm{M}_m(\mathbf{Z}), \ \ {}^tu = u\right\}$$

and we put

$$A = A_m = \{\xi = (\xi_{ij}) \in \mathrm{M}_m(\mathbf{R}) \mid \xi = {}^t\xi, \ \xi_{ij}, 2\xi_{ii} \in \mathbf{Z},$$
$$B = B_m = \{\xi \in A \mid \xi \geq 0\}, C = C_m = \{\xi \in A \mid \xi > 0\}$$

Then A_m is a lattice in the $\mathbf{R}$-vector space of symmetric matrices $V = \{\in \mathrm{M}_m(\mathbf{R}) \mid {}^tx = x\}$ dual to the lattice $L = \mathrm{M}_m(\mathbf{Z}) \cap V$ with respect to the action $(\xi, x) \mapsto e_m(\xi x)$ and for each $f \in \mathcal{M}_\rho(\Gamma)$ there is the following Fourier expansion

$$f(z) = \sum_{\xi \in M^{-1}B} c(\xi) e_m(\xi z), \tag{1.9}$$

Moreover for each $\sigma \in G_{\mathbf{Q}+}$ we have that $f\,|_\rho\,\sigma \in \mathcal{M}_\rho(\Gamma(\sigma))$,where $\Gamma(\sigma)$ is a congruence subgroup,

$$f(z) = \sum_{\xi \in M_\sigma^{-1}B} c_\sigma(\xi) e_m(\xi z), \tag{1.10}$$

with $c_\sigma(\xi) \in \mathbf{C}^r, M_\sigma \in \mathbf{N}$. A form f is called a cusp form if for all ξ with $\det(\xi) = 0$ in expansion (1.10) one has $c_\sigma(\xi) = 0$ for all $\sigma \in G_{\mathbf{Q}+}$ that is

$$f(z) = \sum_{\xi \in M_\sigma^{-1}C} c_\sigma(\xi) e_m(\xi z),$$

We denote by $\mathcal{S}_\rho(\Gamma) \subset \mathcal{M}_\rho(\Gamma)$ the subspace of cusp forms.

Definition of the vector spaces $\mathcal{M}(N,\psi)$. Let us consider congruence subgroups $\Gamma_1^m(N) \subset \Gamma_0^m(N) \subset \Gamma^m(N) = \mathrm{Sp}_m(\mathbf{Z})$,defined by

$$\Gamma_0^m(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^m(N) \mid c \equiv 0_m (\mathrm{mod} N)\},$$

$$\Gamma_1^m(N) = \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^m(N) \mid c \equiv 0_m (\mathrm{mod} N), \det(a) \equiv 1 (\mathrm{mod} N)\},$$

and let $r = 1, \rho(x) = \rho_k(x) = \det(x)^k (k \in \mathbf{N})$. Then the vector space $\mathcal{M}(\Gamma_1^m(N))$ has already been defined, and we put

$$\mathcal{M}_m^k(N,\psi) = \{f \in \mathcal{M}_\rho(\Gamma_1^m(N)) \mid f\,|_\rho\,\gamma = \psi(\det(a)) f \text{ for all } \gamma = \begin{pmatrix} a\, b \\ c\, d \end{pmatrix} \in \Gamma_0^m(N)\}, \tag{1.11}$$

where ψ is a Dirichlet series modulo N. Elements $f \in \mathcal{M}_m^k(N,\psi)$ admit a Fourier expansion of the form

$$f(z) = \sum_{\xi \in B_m} c(\xi) e_m(\xi z), \tag{1.12}$$

with $z \in \mathfrak{H}_m, c(\xi) \in \mathbf{C}$, and the condition

$$c(u\xi {}^t u) = \psi(\det u) \det u^k c(\xi) \quad (\xi \in B_m, u \in \mathrm{GL}_m(\mathbf{Z})). \tag{1.12a}$$

Put

$$\mathcal{S}_m^k(N,\psi) = \mathcal{M}_m^k(N,\psi) \cap \mathcal{S}_{\rho_k}(\Gamma_1^k(N).$$

The Petersson scalar product. For$f \in \mathcal{S}_m^k(N,\psi)$ and $h \in \mathcal{M}_m^k(N,\psi)$ the Petersson scalar product is defined by

$$\langle f,\, h \rangle_N = \int_{\Phi_0(N)} \overline{f(z)} h(z) \det y^k d^\times z, \tag{1.13}$$

where $\Phi_0(N) = \mathfrak{H}_m / \Gamma_0^m(N)$ is a fundamental domain for the group $\Gamma_0^m(N)$.

The Siegel operator connects the vector spaces $\mathcal{M}_m^k(N,\psi)$ for different values of m. If $f \in c$, $z \in \mathbf{H}_{m-1}$ and $\lambda > 0$ then we have that

$$\begin{pmatrix} z' & 0 \\ 0 & i\lambda \end{pmatrix} \in \mathfrak{H}_m,$$

and it follows from (1.12) that there exists the limit

$$(\Phi f)(z\prime) = \lim_{\lambda\to\infty} f\left(\begin{pmatrix} z' & 0 \\ 0 & i\lambda \end{pmatrix}\right) = \sum_{\xi' \in B_{m-1}} c\left(\begin{pmatrix} \xi' & 0 \\ 0 & 0 \end{pmatrix}\right) e_{m-1}(\xi' x'), \tag{1.14}$$

where $c(\xi)$ are the Fourier coefficients of f. Then

$$\Phi f \in \mathcal{M}_{m-1}^k(N,\psi),$$

(we put $\mathcal{M}_0^k(N,\psi) = 0$). We then have that

$$\mathcal{S}_m^k(N,\psi) \subseteq \mathrm{Ker}\Phi = \{f \in \mathcal{M}_m^k(N,\psi) \mid \Phi f = 0\}$$

(for $N = 1$ both sets coincide , see[Maa]).

Estimates for Fourier coefficients. If $f \in \mathcal{S}_m^k(N,\psi)$ then there is the following upper estimate

$$\mid f(z) \mid = \mathcal{O}((\det y)^{k/2}) \quad (z = x + iy \in \mathfrak{H}_m), \tag{1.15}$$

which provide us also with the estimate

$$\mid c(\xi) \mid = \mathcal{O}(\det(\xi)^{k/2}) \tag{1.15a}$$

For modular (not necessary cusp) forms

$$f(z) = \sum_{\xi\in B_m} c(\xi) e_m(\xi z) \in \mathcal{M}_m^k(N,\psi)$$

there is the upper estimate of their growth:

$$\mid c(\xi) \mid = c_1 \prod_{j=1}^{m} (1 + \lambda_j^k), \tag{1.16}$$

with $\lambda_1, \cdots, \lambda_m$ being eigenvalues of the matrix $y, z = x + iy$ (see [St2],p.335). In this situation one has also the following estimate

$$\mid c(\xi) \mid = c_2(\det(\xi')^{k/2}) \tag{1.16a}$$

in which c_2 is a positive constant depending only on f,$\xi = {}^t u \begin{pmatrix} \xi' & 0 \\ 0 & 0 \end{pmatrix} u$, $u \in \mathrm{SL}_m(\mathbf{Z})$, $\xi' \in B_r, \det \xi' > 0$, $r \le m$.

We refer the reader to [Fo1], [Ki1], [Rag1], [Rag4] for a more detailed discussion of various estimates for Fourier coefficients and for growth of Siegel modular forms and for their applications to quadratic forms .

1.3. The Hecke algebra (see [An7], [Bö], [Sat]). Let q be a prime,$q \not| N$,

$$\Delta = \Delta_q^m =$$
$$\{\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbf{Q}+} \cap \mathrm{GL}_{2m}(\mathbf{Z}[q^{-1}]) | \nu(\alpha)^{\pm} \in \mathbf{Z}[q^{\pm}], c \equiv 0_m (\mathrm{mod} N)\}$$

be a subgroup in $G_{\mathbf{Q}+}$ containing $\Gamma = \Gamma_0^m(\mathrm{N})$. The Hecke algebra

$$\mathcal{L} = \mathcal{L}_q^m(\mathrm{N}) = \mathcal{D}_{\mathbf{Q}}(\Gamma, \Delta)$$

over $\mathbf{Q}$ is then defined as a $\mathbf{Q}$-linear space generated by the double cosets $(g) = (\Gamma g \Gamma)$, $g \in \Delta$ of the group S with respect to the subgroup Γ , for which multiplication is defined by the standard rule (see [An7], [Shi1] and in 1.7 below). We recall the description of the structure of $\mathcal{L} = \mathcal{L}_q^m(\mathrm{N})$, $q \not| N$ which looks as follows: for each $j, 1 \leq j \leq m$ let us denote by w_j an automorphism of the algebra $\mathbf{Q}[x_0^{\pm 1}, x_1^{\pm 1}, \cdots, x_m^{\pm 1}]$ defined on its generators by the rule:

$$x_0 \mapsto x_0 x_j, \ x_j \mapsto x_j^{-1}, x_i \mapsto x_i \quad (1 \leq i \leq m, i \neq j).$$

Then the automorphisms w_j and the permutation group Σ_m of the variables x_i $(1 \leq i \leq m)$ generate together the Weyl group $W = W_m$, and there is the Satake isomorphism:

$$Sat : \mathcal{L} \xrightarrow{\sim} \mathbf{Q}[x_0^{\pm 1}, x_1^{\pm 1}, \cdots, x_m^{\pm 1}]^{W_m} \tag{1.17}$$

where W_m indicates the subalgebra of elements fixed by W_m . For any commutative $\mathbf{Q}$-algebra A the group W_m act on the set $(A^\times)^{m+1}$, therefore any homomorphism of $\mathbf{Q}$-algebras $\lambda : \mathcal{L} \to A$ can be identified with some element

$$(\alpha_0, \alpha_1, \cdots, \alpha_m) \in [(A^\times)^{m+1}]^{W_m}. \tag{1.18}$$

An explicit description of the Satake isomorphism is given below in 1.7.

1.4. Any double coset $(g) = (\Gamma g \Gamma)$ $(g \in \Delta = \Delta_q^m(N))$ can be represented is a disjoint union of the left cosets:

$$(g) = \cup_{i=1}^{t(g)} \Gamma g_i,$$

therefore any element $X \in \mathcal{L}$ of the Hecke algebra $\mathcal{L}$ takes the form of a finite linear combination

$$X = \sum_{i=1}^{t(X)} v_i (\Gamma g_i),$$

with $v_i \in \mathbf{Q}, g_i \in \Delta$. In order to define Hecke operators we put for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta$

$$(f|^{k,\psi} g)(z) = (\det g)^{k-(m+1)/2} \psi(\det a) \det(cz+d)^{-k} f(g(z)) \tag{1.19}$$

(this convenient notation by Petersson and Andrianov is especially useful when dealing with Hecke orerators in their normalized form, compare with (1.5)). Then the automorphy condition can be rewritten as follows

$$(f|_{k,\psi}\gamma)(z) = f \text{ for all } \gamma \in \Gamma = \Gamma_0^m(N). \tag{1.20}$$

In this case for any

$$X = \sum_{i=1}^{t(X)} v_i(\Gamma g_i) \in \mathcal{L}$$

we have that the expression

$$f|X = \sum_{i=1}^{t(X)} v_i f|_{k,\psi}\, g_i, \tag{1.21}$$

is well defined and $f|X \in \mathcal{M}_k^m(N,\psi)$ so that the formula (1.21) gives a representation of the Hecke algebra $\mathcal{L} = \mathcal{L}_q^m(N)$ on the complex vector space $\mathcal{M}_k^m(N,\psi)$ $(q \not| N)$.

1.5. Hecke polynomials. Following A.N.Andrianov[An7], let us consider polynomials

$$\begin{aligned}\tilde{Q} = \tilde{Q}(x_0, x_1, \cdots, x_m; z) = \\ = (1 - x_0 z)\prod_{r=1}^{m} \prod_{1\le i_1<\cdots<i_r\le m} (1 - x_0 x_{i_1}\cdots x_{i_r} z),\end{aligned} \tag{1.22}$$

$$\tilde{R}(z) = \prod_{i=1}^{m}(1 - x_i^{-1}z)(1 - x_i z) \in \mathbf{Q}[x_0^{\pm1}, \cdots, x_m^{\pm1}]. \tag{1.23}$$

It follows from the definition that the coefficients of the powers of the variable z all belong to the subring

$$\mathbf{Q}[x_0^{\pm1}, x_1^{\pm1}, \cdots, x_m^{\pm1}]^{W_m}$$

. Therefore by Satake isomorphism (1.17) we have that there exist uniquely defined polynomials

$$Q(z) = \sum_{i=0}^{2^m}(-1)^i T_i z^i, \quad R(z) = \sum_{i=0}^{2m}(-1)^i R_i z^i \in \mathcal{L}[z] \tag{1.24}$$

over the associative commutative ring $\mathcal{L} = \mathcal{L}_q^m(N)$ such that

$$\tilde{Q}(z) = \sum_{i=0}^{2^m}(-1)^i \tilde{T}_i z^i, \quad \tilde{R}(z) = \sum_{i=0}^{2m}(-1)^i \tilde{R}_i z^i$$

with $\tilde{X} = Sat\, X$, $X \in \mathcal{L}$. As generators of the Hecke algebra one can take the polynomials ${\Delta'_M}^{\pm1}, R_i (1 \le i \le m-1)$ and T_1 for which

$$\tilde{\Delta}'^{\pm1}_M = x_0^2 x_1 \cdots x_m, R_i = S_i(x_1, \cdots x_m; x_1^{-1}, \cdots x_m^{-1}),$$

$$\tilde{T}_1 = x_0 \sum_{i=1}^{m} S_i(x_1, \cdots x_m) = x_0 \prod_{i=1}^{m}(1 + x_i),$$

where S_i denotes the elementary symmetric polynomial of degree i and of the corresponding sets of variables.

1.6. The spinor zeta function and the standard zeta function. Let $f \in \mathcal{M}^k_m(N,\psi)$ be an eigenfunction of all Hecke operators $f \to f|X, X \in \mathcal{L}^m_q(N)$ with q being prime numbers , $q \not| N$, so that $f|X = \lambda_f(X)f$. Then the numbers $\lambda_f(X) \in \mathbf{C}$ define a homomorphism $\lambda_f : \mathcal{L} \to \mathbf{C}$ (see 1.3), which is uniquely determined by a $(m+1)$-tuple of the numbers

$$(\alpha_0, \alpha_1, \cdots \alpha_m) = (\alpha_{0,f}(q), \alpha_{1,f}(q), \cdots \alpha_{m,f}(q)) \in [(\mathbf{C}^\times)^{m+1}]^{W_m} \qquad (1.25)$$

which are called *the Satake q-parameters* of the modular form f.

Now let the variables $x_0, x_1, \cdots, x_m$ in (1.22), (1.23) be equal to the corresponding Satake q-parameters $\alpha_{0,f}(q), \alpha_{1,f}(q), \cdots \alpha_{m,f}(q)$

$$Q_{f,q}(z) = \\ = (1-\alpha_0 z)\prod_{r=1}^{m} \prod_{1\le i_1<\cdots<i_r\le m} (1-\alpha_0\alpha_{i_1}\cdots\alpha_{i_r} z), \qquad (1.26)$$

$$R_{f,q}(z) = \prod_{i=1}^{m}(1-\alpha_i^{-1}z)(1-\alpha_i z) \in \mathbf{Q}[\alpha_0^{\pm1}, \cdots, \alpha_m^{\pm1}]. \qquad (1.26a)$$

It follows then from 1.5 that the coefficients of the polynomials (1.26) can be expressed in terms of the eigenvalues $\lambda_f(X)$ of the Hecke operators $X = T_i, R_i$ from (1.24). Next we put

$$Z^{(q)}(s,f) = Q_{f,q}(q^{-s})^{-1} = \\ = \{(1-\alpha_0(q)q^{-s})\prod_{r=1}^{m} \prod_{1\le i_1<\cdots<i_r\le m} (1-\alpha_0(q)\alpha_{i_1}(q)\cdots\alpha_{i_r}(q)q^{-s})\}^{-1}, \qquad (1.27)$$

and define the spinor zeta function $Z(s,f)$ of the modular form $f \in \mathcal{M}^k_m(N,\psi)$ by

$$Z(s,f) = \prod_{q \not| N} Z^{(q)}(s,f) \qquad (1.27a).$$

Complex analytic properties of zeta functions $Z(s,f)$ were investigated by A. N. Andrianov [An2] in the case $m=2$. For $m=1$ we have that

$$\alpha_0(q) + \alpha_0(q)\alpha_1(q) = a(q), \quad \alpha_0^2(q)\alpha_1(q) = \psi(q)q^{k-1},$$

where $f(z) = \sum_{n=0}^{\infty} a(n)e(nz)$ is the Fourier expansion of the normalized elliptic cusp Hecke eigenform, so that the zeta function

$$Z(s,f) = \prod_{q \not| N} [1-a(q)q^{-s} + \psi(q)q^{k-1-2s}]^{-1} = \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty} a(n)n^{-s}$$

coincide essentially with the Mellin transform

$$L(s,f)=\sum_{n=1}^{\infty}a(n)n^{-s}$$

of f. A remarkable fact is that one can get all the inverse roots of the Hecke polynomial $Q_{f,q}(z)$ by the action of the Weyl group W_m on $\alpha_0(q)$, and

$$\begin{aligned}&Q_{f,q}(z)=\\&1-\lambda_f(q)q^{-s}+\cdots+q^{2^{m-1}(mk-m(m+1)/2)}z^{2^m}=\\&(1-\alpha_0(q)z)\prod_{r=1}^{m}\prod_{1\le i_1<\cdots<i_r\le m}(1-\alpha_0(q)\alpha_{i_1}(q)\cdots\alpha_{i_r}(q)z),\end{aligned}\tag{1.28}$$

with

$$\lambda_f(q)=\lambda_f(T(q)),T((q))=\sum_{\nu(g)=q}(\Gamma g\Gamma)$$

being the Hecke opertor for the group $\Gamma=\Gamma_0^m(N),{}^tgJ_mg=\nu(g)J_m$. According to the generalized Ramanujan-Petersson conjecture for a cusp eigenform $f\in\mathcal{S}_m^k(N,\psi)$ the absolute values of all inverse roots of the polynomial (1.28) should coincide (and therefore be equal to $q^{-(2mk-m(m+1)/4)}$). In the case $m=1$ this conjecture is valid; it was deduced by Deligne [Del] from the Weil conjectures, also proven by him [De2]. However, for larger values of m this conjecture in its original form is not true. Various counterexamples to it are known, see [Kur],[H-PSh1],[H-PSh2] which for the case $m=2$ reflect the fact that some cusp forms of degree 2 (namely those belonging to the Maass subspace) are lifted from elliptic cusp forms (i.e. of degree 1) via the Saito- Kurokawa lifting [An8], [Koj]. According to the modified Ramanujan-Petersson conjecture these properties of the inverse roots and polynomials $Q_{f,q}$ should be valid for the "real" cusp forms, i.e. those which can not be obtained from the forms of smaller degree by a lifting of the type mentioned above.

The standard zeta function $\mathcal{D}(s,f,\chi)$ of $f\in\mathcal{M}_m^k(N,\psi)$ is the product $\mathcal{D}(s,f,\chi)=\prod_{q\nmid N}\mathcal{D}^{(q)}(s,f,\chi)$ with

$$\mathcal{D}^{(q)}(s,f,\chi)=(1-\chi(q)\psi(q)q^{-s})^{-1}R_{f,q}(\chi(q)\psi(q)q^{-s})^{-1},$$

that is

$$\begin{aligned}&\mathcal{D}(s,f,\chi)=\\&=\prod_p\left\{\left(1-\frac{\chi(p)\psi(p)}{p^s}\right)\prod_{i=1}^{m}\left(1-\frac{\chi(p)\psi(p)\alpha_i(p)^{-1}}{p^s}\right)\left(1-\frac{\chi(p)\psi(p)\alpha_i(p)}{p^s}\right)\right\}^{-1},\end{aligned}\tag{1.29}$$

where χ is a Dirichlet character mod M. Analytic properties of the standard zeta functions were investigated by A. N. Andrianov and V. L. Kalinin [An-K] in the case of even degree m, and more recently S. Böcherer [Bö] extended these results to the case of arbitrary degree using a different approach. For $m = 1$ and a normalized cusp eigenform $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in \mathcal{M}_1^k(N, \psi)$ we have that

$$\mathcal{D}(s, f, \chi) = L_{2,f}(s + k - 1, \chi),$$

where

$$L_{2,f}(s, \chi) = L_{NM}(2s - 2k + 2, \chi^2\psi^2) \sum_{n=1}^{\infty} \chi(n)a(n^2)n^{-s} \tag{1.30}$$

is the symmetric square of the modular form f, see [An3], [An-K], [Pa3], [Sch], [Shi2] [Za].

1.7. Non-commutative extensions of Hecke algebra and Satake isomorphism. Now we recall that multiplication in the Hecke algebra $\mathcal{L} = \mathcal{D}_{\mathbf{Q}}(\Gamma, \Delta)$ is defined by use of the larger vector space $\mathcal{V} = \mathcal{V}_{\mathbf{Q}}(\Gamma, \Delta)$ over $\mathbf{Q}$ consisting of all $\mathbf{Q}$-linear combinations of left cosets of the form (Γg), $g \in \Delta = \Delta_q^m(N)$. If $\gamma \in \Gamma$ and

$$X = \sum_{i=1}^{t(X)} v_i(\Gamma g_i) \in \mathcal{V}$$

then the formula

$$X \cdot \gamma = \sum_{i=1}^{t(X)} v_i(\Gamma g_i \gamma) \in \mathcal{V}$$

defines a right action of the group Γ on $\mathcal{V}$ so that the algebra $\mathcal{L} = \mathcal{D}_{\mathbf{Q}}(\Gamma, \Delta)$ coincide s with the subspace of all elements in $\mathcal{V}_{\mathbf{Q}}(\Gamma, \Delta)$ fixed by Γ via the inclusion $\mathcal{L} \to \mathcal{V}$ which sends a double cosets

$$(g) = (\Gamma g \Gamma) \ \ (g \in \Delta = \Delta_q^m(N)) = \cup_{i=1}^{t(g)} \Gamma g_i$$

to the formal sum $\sum_{i=1}^{t(g)} \Gamma g_i$. If

$$X = \sum_{i=1}^{t(X)} a_i(\Gamma g_i), Y = \sum_{j=1}^{t(Y)} b_j(\Gamma h_j) \tag{1.32}$$

are two elements in $\mathcal{D}_{\mathbf{Q}}(\Gamma, \Delta) \in \mathcal{V}$ then the element

$$X \cdot \mathcal{V} = \sum_{i,j=1} a_i b_j (\Gamma g_i h_j) \in \mathcal{V}_{\mathbf{Q}}(\Gamma, \Delta)$$

is well defined and also belongs to $\mathcal{D}_{\mathbf{Q}}(\Gamma, \Delta) \in \mathcal{V}$. This construction can be applied to a large variety of couples (Γ, Δ), for which Γ is a subgroup of a (semi)group Δ such that any double coset $(g) = (\Gamma g \Gamma)$ $(g \in \Delta = \Delta_q^m(N))$ can be represented is a disjoint union of the left cosets:

$$(g) = \cup_{i=1}^{t(g)} \Gamma g_i$$

(the theory of Hecke couples, see [An7],chapter 1). In particular for

$$\Gamma_0 = \Gamma_0^m = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma | \ \ c = 0\},$$
$$\Delta_0 = \Delta_{0,q}^m = \{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_{0,q}^m | \ c = 0\}$$

we thus obtain an associative (but no longer commutative) ring

$$\mathcal{L} = \mathcal{L}_{0,q}^m(N) = \mathcal{D}_{\mathbf{Q}}(\Gamma_0, \Delta_0).$$

It follows from the theory of elementary divisors that $\Gamma\Delta_0 = \Delta$; therefore we get the homomorphism $\varepsilon : \mathcal{L} \to \mathcal{L}_0$ defined as follows: for each

$$X = \sum_{i=1}^{t(X)} v_i(\Gamma g_i) \in \mathcal{L} \subset \mathcal{V}$$

we may assume that

$$g_i = \begin{pmatrix} q^{\nu_i} d_i^* & b_i \\ 0 & d_i \end{pmatrix},$$

where $q^{\nu_i} = \nu(g_i)$, and we put

$$\varepsilon_0(X) = \sum_{i=1}^{t(X)} v_i(\Gamma_0 \begin{pmatrix} q^{\nu_i} d_i^* & b_i \\ 0 & d_i \end{pmatrix}) \tag{1.33}$$

Similar arguments show also that the matrix d_i can be chosen in the form

$$d_i = \begin{pmatrix} q_i^{\delta}(1) & * & \cdots & * \\ 0 & q_i^{\delta}(2) & \cdots & * \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & q_i^{\delta}(m) \end{pmatrix} \ \ \delta_i(m) \le \cdots \le \delta_i(m) \tag{1.34}$$

with the uniquely defined exponents $\delta_i(j)$. Now for

$$X = \sum_{i=1}^{t(X)} v_i(\Gamma g_i) \in \mathcal{L}_0$$

we put by definition

$$\Phi(X) = \sum_{i=1}^{t(X)} v_i x_0^{\nu_i} \prod_{j=1}^{m} (x_j q^{-j}) \delta_i(j). \tag{1.35}$$

We get the desired Satake isomorphism as the composition of homomorphisms (1.33) and (1.35):

$$Sat = \Phi\varepsilon_0 : \mathcal{L} \xrightarrow{\sim} \mathbf{Q}[x_0^{\pm 1}, x_1^{\pm 1}, \cdots, x_m^{\pm 1}]^{W_m} \tag{1.36}$$

As a consequence of the construction we get that for an arbitrary homomorphism of algebras $\lambda : \mathcal{L} \to \mathbf{C}$ $(\lambda = \lambda_q, q \not| N, \mathcal{L} = \mathcal{L}_q^m(N)$,attached to the $(m+1)$-tuple of q-parameters

$$(\alpha_0, \alpha_1, \cdots, \alpha_m) \in [(\mathbf{C}^\times)^{m+1}]^{W_m},$$

and for any element

$$X = \sum_{i=1}^{t(X)} v_i(\Gamma g_i) \in \mathcal{L} \;\; with \; g_i = \begin{pmatrix} q^{\nu_i} d_i^* & b_i \\ 0 & d_i \end{pmatrix}$$

and d_i is chosen in the form (1.34) the following equality holds:

$$\lambda(X) = \sum_{i=1}^{t(X)} v_i \alpha_0^{\nu_i} \prod_{j=1}^{m} (\alpha_j q^{-j}) \delta_i(j). \tag{1.37}$$

If$\lambda = \lambda_{f,q}$ for a modular form $f \in \mathcal{M}_m^k(N, \psi)$ then the numbers $\alpha_0, \alpha_1, \cdots, \alpha_m$ satisfy the following relation

$$\alpha_0^2 \alpha_1 \cdots \alpha_m = \psi(q)^m q^{km - m(m+1)/2}, \tag{1.38}$$

which follows directly from the formula (1.37) and the definition (1.19) applied to the operator $X = (g)$ with $g = q1_{2m}$ (see[An7]).

1.8. Action of Hecke operators on Fourier expansions. Let

$$f(z) = \sum_{\xi \in B_m} c(\xi) e_m(\xi z) \in \mathcal{M}_m^k(N, \psi) \quad (\xi \in B_m)$$

and

$$X = \sum_{i=1}^{t(X)} v_i(\Gamma_0 g_i) \in \mathcal{L}_0 \quad (\Gamma_0 = \Gamma_0^m, \mathcal{L}_0 = \mathcal{L}_{0,q}^m)$$

be an arbitrary element of the extended Hecke algebra $\mathcal{L}_0$ with

$$g_i = \begin{pmatrix} q^{\nu_i} d_i^* & b_i \\ 0 & d_i \end{pmatrix} \in \Delta_{0,q}^m. \tag{1.39}$$

Let coefficients $(c|X)(\xi)$ be defined by the equality

$$(f|X)(z) = \sum_{\xi \in B_m} (c|X)(\xi) e_m(\xi z). \tag{1.40}$$

Then we have that

$$\begin{aligned}(c|X)(\xi) = \sum_{i=1}^{t(X)} v_i q^{\nu_i(mk - \langle m \rangle)} (\det d_i)^{-k} \times \\ \times \psi(q^{m\nu_i} (\det d_i)^{-1}) e_m(q^{-\nu_i}\, {}^t b_i d_i) c(q^{-\nu_i} d_i \xi\, {}^t d_i),\end{aligned} \tag{1.41}$$

(here we assume that $c(\xi) = 0$ for $\xi \notin B_m$ (see [An7], §3.2)). We remark that for $X \in \mathcal{L}_0$ not beloning to the subalgebra $\varepsilon_0(\mathcal{L}) \subset \mathcal{L}_0$ we have , in general, that $f|X \notin \mathcal{M}^k_m(N, \psi)$. However we claim that $f|X \in \mathcal{M}^k_m(\Gamma'_X)$, where

$$\Gamma'_X = \cap_i g_i \Gamma^m_1(N) g_i$$

(see (1.19)).

As an example we consider the Frobenius elements (see [An7], §2.1):

$$\begin{aligned} \Pi_- &= \Pi^m_-(q) = (\Gamma_0 \begin{pmatrix} q1_m & 0 \\ 0 & 1_m \end{pmatrix} \Gamma_0), \\ \Pi_+ &= \Pi^m_+(q) = (\Gamma_0 \begin{pmatrix} 1_m & 0 \\ 0 & q1_m \end{pmatrix} \Gamma_0). \end{aligned} \tag{1.42}$$

Then for $f(z) = \sum_{\xi \in B_m} c(\xi) e_m(\xi z) \in \mathcal{M}^k_m(N, \psi)$ we get from (1.41)

$$(c|\Pi_-)(\xi) = \psi(q)^m q^{mk - \langle m \rangle} c(q^{-1}\xi), \tag{1.43}$$

$$(c|\Pi_+)(\xi) = c(q\xi) \quad (\xi \in B_m \subset A_m) \tag{1.44}$$

Note that the operator $\Pi_+ = \Pi_+(q)$ is defined by the formula (1.44) also for $q|N$, when $\Pi_+(q)$ sends $\mathcal{M}^k_m(N, \psi)$ to itself. However we assume in the next proposition that $q \not| N$.

1.9. Proposition. *If $f \in \mathcal{M}^k_m(N, \psi)$ with q not dividing N then*

$$f|\Pi_-(q), f|\Pi_+(q) \in \mathcal{M}^k_m(qN, \psi). \tag{1.45}$$

Proof. Note that

$$\Pi^m_-(q) = (\Gamma_0 \begin{pmatrix} q1_m & 0 \\ 0 & 1_m \end{pmatrix}),$$

therefore

$$f|\Pi_- = f|_{k,\psi} \begin{pmatrix} q1_m & 0 \\ 0 & 1_m \end{pmatrix},$$

If we now take into account that

$$\Gamma^m_0(qN) \subset \begin{pmatrix} q1_m & 0 \\ 0 & 1_m \end{pmatrix}^{-1} \Gamma^m_0(N) \begin{pmatrix} q1_m & 0 \\ 0 & 1_m \end{pmatrix},$$

then we get that for all $\gamma \in \Gamma^m_0(qN)$ holds the following equality holds

$$f|\Pi_-\gamma = (f|\Pi_-)|_{k,\psi}\gamma = f|\Pi_-.$$

In order to prove the analogous statement about Π_+ we use the antiisomorphism (involition) $X \mapsto X^*$ of the algebra $\mathcal{L}_0$ (see [An7], p.75) with the property $\Pi^*_- = \Pi_+$ and recall that by definition if

$$X = \sum_{i=1}^{t(X)} v_i(\Gamma_0 g_i) \in \mathcal{L}_0 \quad (v_i \in \mathbf{Q}, g_i \in \Delta_0),$$

then

$$X^* = \sum_{i=1}^{t(X)} v_i(\Gamma_0 q^{\nu_i} g_i^{-1}) \quad (q^{\nu_i} = \nu(q).$$

Next, let the equality $X\Gamma' = X$ be valid for a given $X \in \mathcal{L}_0$ where $\Gamma_0 \subset \Gamma'\Gamma$. Put $\Delta' = \Gamma'\Delta_0$, $\Delta_0 \subset \Delta' \subset \Delta$ and consider as in 1.3 the Hecke algebra $\mathcal{L}' = \mathcal{D}_{\mathbf{Q}}(\Gamma', \Delta')$. Then we have again that the homomorphism $\varepsilon' : \mathcal{L}' \to \mathcal{L}_0$, is defined so that its image $\varepsilon_0'(\mathcal{L}')$ coincides with the subalgebra of all elements in $\mathcal{L}_0$ fixed by Γ'. Consequently, for some $Y \in \mathcal{L}'$ we have that $X = \varepsilon_0'(Y)$ and we get $X^* = \varepsilon_0'(Y^*)$ by the definition of the antiautomorphism $*$.In other words, the operator $X^* \in \mathcal{L}'$ turns out to be Γ'-invariant. In particulary we get $\Pi_+\Gamma_0^m(Nq) = \Pi_+$ from the equality $\Pi_-\Gamma_0^m(Nq) = \Pi_-$ which is already proven, and from the fact that $f|\Pi_+ \in \mathcal{M}_m^k(qN, \psi)$.

Now let us consider again Hecke polynomials, defined by (1.22)-(1.24). A. N .Andrianov has dicovered factorization of the polynomials $Q(z), R(z) \in \mathcal{L}[z]$ over the non - commutative algebra $\mathcal{L}_0 = \mathcal{L}_{0,q}^m$ in terms of the Frobenius elements $\Pi_+(q), \Pi_-(q)$ (see 1.8). This factorization is essentially used in our non-Archimedean construction.

1.10. Proposition. *There are the following expansions for the Hecke polynomials introduced above*

$$Q(z) = (\sum_{i=1}^{2^m-1} V_i^+ z^i)(1 - \Pi_+ z) = (1 - \Pi_- z)(\sum_{i=1}^{2^m-1} V_i^- z^i), \tag{1.46}$$

where

$$V_i^+ = \sum_{j=0}^{i} (-1)^j T_j \Pi_+^{i-j},$$
$$V_i^- = \sum_{j=0}^{i} (-1)^j \Pi_-^{i-j} T_j \in \mathcal{L}_0 \tag{1.46a}$$

Proof (see in [An7], §2.2)

Now let

$$f(z) = \sum_{\xi \in B_m} c(\xi) e_m(\xi z) \in \mathcal{M}_m^k(N, \psi)$$

be an eigenfunction of the algebra $\mathcal{L} = \mathcal{L}_p^m(N)$ whose eigenvalue is a homomorphism $\lambda_f : \mathcal{L} \to \mathbf{C}, f|X = \lambda_f(X)f$ for all $X \in \mathcal{L}$ which is given by the $(m+1)$-tuple $\alpha_0 = \alpha_0(p), \alpha_1 = \alpha_1(p), \cdots, \alpha_m = \alpha_m(p)$ of the Satake p-parameters. Put

$$f_0 = \sum_{i=1}^{\tilde m - 1} \alpha_0(p)^{-i} f|V_i^+(p) \quad (\tilde m = 2^m). \tag{1.47}$$

1.11. Proposition. *a). The function f_0 belongs to $\mathcal{M}_m^k(Np^{\tilde m-1}, \psi)$ with* $(\tilde m = 2^m)$.

b) The following equality holds

$$f_0|\Pi_+(p) = \alpha_0(p) f_0.$$

Remark. We know from 1.3 that by the action of the Weyl group the inverse root α_0 can be transformed into each of the 2^m inverse roots

$$\alpha_0\alpha_{i_1}\cdots\alpha_{i_r} \quad (1 \le i_1 < \cdots \le i_r \le m)$$

of the local factor $\mathbf{Z}^{(p)}(s,f)$ of the spinor zeta finction. Therefore the construction in Proposition 1.11 provides also 2^m eigenfunctions $g \in \mathcal{M}^k_m(Np^{\tilde{m}-1},\psi)$ of the algebras $\mathcal{L} = \mathcal{L}^m_q(N)$ for $q \not| pN$ with the same eigenvalues as those of the original modular form $f \in \mathcal{M}^k_m(N,\psi)$ such that for $q = p$ we have that $g|\Pi_+(p)\alpha q$ where α is any of the inverse roots mentioned above.

Proof of the Proposition 1.11. The statement a) follows directly from (1.46a) and from the definition (1.47). For to prove b) we note that there is the equality (with $\tilde{m} = 2^m$)):

$$\sum_{i=0}^{\tilde{m}=2^m} f|T_i(p)\cdot(-z)^i = \tilde{Q}(\alpha_0(p),\alpha_1(p)\cdots,\alpha_m(p);z)f;$$

in addition by definition of the polynomials (1.22) we have the identity:

$$\sum_{i=0}^{\tilde{m}=2^m} (-\alpha_0(p))^{-i} f|T_i(p) = 0. \tag{1.48}$$

We want to prove that $f_0|(\Pi_+(p)-\alpha_0(p)) = 0$. But we have from (1.46),(1.47) that

$$\begin{aligned} f_0|(\Pi_+(p)-\alpha_0(p)) &= -\alpha_0(p)f|(1-\alpha_0^{-1}(p)\Pi_+) = \\ -\alpha_0(p)f|(&\sum_{i=0}^{\tilde{m}=2^m}(-\alpha_0(p))^{-i}f|V_i^+(p))(1-\alpha_0^{-1}(p)\Pi_+(p) = \\ \sum_{i=0}^{\tilde{m}=2^m} f|T_i(p)&(-\alpha_0(p))^{-i} = 0, \end{aligned}$$

and the statement b) follows.

1.12. An analogous statement is valid in the case of a finite set S of prime numbers q coprime with N. For this purpose it is convenient to consider global Hecke algebras

$$\mathcal{L}(N) = \bigotimes_{q \not| N} \mathcal{L}^m_q(N), \quad \mathcal{L}_0(N) = \otimes_{q\not| N}\mathcal{L}^m_{0,q}(N).$$

Then the definition of the operators $\Pi_+(M), \Pi_-(M), V_i^+(q), V_i^-(q) \in \mathcal{L}^m_{0,q}(N)$ and of the numbers $\alpha_0(q)$ $(q \not| N)$ can be extended by multiplicativity to all positive integers M coprime with N; more precisely, operators $V^+(M), V^-(M)$ are defined by the identities

$$\sum_{M|M_0^{\tilde{m}-1}} M_{-s}V^+(M) = \prod_{q\not| N}\Big[\sum_{i=1}^{\tilde{m}-1} q^{-s}V_i^+(q)\Big], \tag{1.49}$$

$$\sum_{M|M_0^{\tilde{m}-1}} M_{-s}V^-(M) = \prod_{q \nmid N} [\sum_{i=1}^{\tilde{m}-1} q^{-s}V_i^-(q)], \tag{1.49a}$$

where the notation

$$V^{\pm}(q^i) = V_i^{\pm}(q), \quad M_0 = \prod_{q\in S} q.$$

is used. We then put

$$f_0 = f_{0,S} = \sum_{M|M_0^{\tilde{m}-1}} \alpha_0(M)^{-1} f|V^+(M).(1.50)$$

1.13. Proposition. *a). Let* $f \in \mathcal{M}_k^m(N,\psi)$, *then*

$$f_0 = f_{0,S} = \in \mathcal{M}_k^m(NM^{\tilde{m}-1},\psi).$$

b). For all positive integers M *with support* $S(M)$ *in the set* S *we have that*

$$f_0|(\Pi_+(M) = \alpha_0(M))f_0. \tag{1.51}$$

c). Let

$$f_{0,S}(z) = \sum_{\xi\in B_m} a_0(\xi)e_m(\xi z) \in \mathcal{M}_m^k(NM^{\tilde{m}-1},\psi) \quad (\xi \in B_m)$$

be Fourier expansion of the function $f_{0,S}(z)$ *then there is the following multiplicativity property of its Fourier coefficients: for all* $M \in \mathbf{N}$ *with* $S(M) \subset S$

$$a_0(M\xi) = \alpha_0(M)a_0(\xi) \quad (\xi \in A_m, \xi \geq 0). \tag{1.52}$$

The proof of the proposition is carried out in a very similar way as that of the previous one if we take into account the formulas (1.44) for the action of the operator $\Pi_+(M)$ on Fourier expansions.

§2. Theta series, Eisenstein series and Rankin zeta function

2.1. Theta series (see [An3],[An-M1],[An-M2],[St2]). Let $F \in 2C_m$ be an even symmetric positive definite matrix, and q_0 its level (i.e. the smallest positive integer such that $q_0F^{-1} \in \mathrm{M}_m(\mathbf{Z})$, and χ a Dirichlet series modulo Q (not necesarily primitive). Put $\nu = 1$ or 0 and define the theta function

$$\theta(\chi) = \theta_F^{(\nu)}(z;\chi) = \sum \xi \in \mathrm{M}_m(\mathbf{Z})\chi(\det\xi)\det\xi^{\nu}e_m(zF[\xi]/2). \tag{2.1}$$

2.2.Proposition. *(a) If*

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^m(q_0Q^2),$$

then the following transformation formula holds

$$\theta_F^{\gamma(\nu)}(z;\chi) = \chi(\det d)\chi_{Q^2F}^{(m)}(\gamma)\det(cz+d)^{(m/2)+\nu}\theta_F^{(\nu)}(z;\chi), \tag{2.2}$$

where $\chi_F^{(m)}(\gamma)$ *is a root of unity of the eighth degree, and if* m *is even, then*

$$\chi_F^{(m)}(\gamma) = \left(\frac{(-1)^{m/2}\det(F)}{\det d}\right). \tag{2.3}$$

(b) Let $J(M)$ *denote for* $M > 0$ *the matrix* $\begin{pmatrix} 0_m & -1_m \\ M1_m & 0_m \end{pmatrix}$. *If* χ *is primitive modulo* Q *then the action of the involution* $J(Q^2q_0)$ *on (2.1) is given by*

$$\begin{aligned}&\theta_F^{(\nu)}(J(Q^2q_0)z;\chi) = \\ &\qquad \chi(-1)^m Q^{m\nu} G_Q(1_m,\chi)\det(F)^{(m/2)+\nu}[\det(-iz)]^{(m/2)+\nu}\theta_{\hat{F}}^{(\nu)}(z;\overline{\chi}),\end{aligned} \tag{2.4}$$

with $\hat{F} = q_0F^{-1}$ and

$$G_Q(\xi,\chi) = \sum_{h\in \mathrm{M}_m(\mathbf{Z})\bmod Q} \chi(\det h)\, e_m({}^t\xi h/Q)$$

being the Gauss sum of degree m *of the character* χ.

2.3. *The proof* of (a) is given in [An-M2] for primitive characters χ, and in [St2] in the general case by use of the generalized theta series

$$\theta_F^{(m)}(z;X,Y) = \sum_{\xi\in \mathrm{M}_m(\mathbf{Z})} e_m((zF[\xi - Y] + 2^{\xi}X - {}^tXY)/2, \tag{2.6}$$

where $z \in \mathfrak{H}_m$, $X, Y \in \mathrm{M}_m(\mathbf{C})$). The series (2.6) satisfy the following properties: for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^m(q_0)$ we have that

$$\begin{aligned}&\theta_F^{(m)}(\gamma(z);X^a + FY^tb, F^{-1}X^tc + Y^td) = \\ &\qquad \chi_F^{(m)}(\gamma)\det(cz+d)^{(m/2)}\theta_F^{(m)}(z;X,Y),\end{aligned} \tag{2.7}$$

$$\theta_F^{(m)}(z;X,Y) = (\det F)^{-(m/2)}[\det(-iz)]^{-(m/2)}\theta_{F^{-1}}^{(m)}(-z^{-1};Y,-X) \tag{2.8}$$

(see [An-M2], theorem 1 and lemma 2). For $\nu = 0$ the statements (a) and (b) immediately follow from (2.7), (2.8). For example , in order to get (b) it suffices to put in (2.8) F equal to Q^2F, $X = 0$, Y equal to $-Q^{-1}Y$ then multiply by $\chi(\det(Y))$ and to carry out summation over $Y \in \mathrm{M}_m(\mathbf{Z})\bmod Q$. From the left hand side of (2.8) we get

$$\begin{aligned}&\sum_{Y\in \mathrm{M}_m(\mathbf{Z})\bmod Q} \chi(\det(Y))\theta_{Q^2F}^{(m)}(z;0,-Q^{-1}Y) = \\ &\qquad \sum_{Y\in \mathrm{M}_m(\mathbf{Z})\bmod Q} \chi(\det(Y)) \sum_{\xi\in \mathrm{M}_m(\mathbf{Z})} e_m(zF[Q\xi + Y]/2),\end{aligned} \tag{2.9}$$

and from the right hand side

$$(\det Q^2F)^{-(m/2)}[\det(-iz)]^{-(m/2)}\theta^{(m)}_{Q^{-2}F^{-1}}(-z^{-1};-Q^{-1}Y,0) =$$
$$(\det Q^2F)^{-(m/2)}[\det(-iz)]^{-(m/2)}\times$$
$$\times \sum_{Y\in \mathrm{M}_m(\mathbf{Z})\mathrm{mod} Q} \chi(\det(Y)) \sum_{\xi\in \mathrm{M}_m(\mathbf{Z})} e_m((-z(Q^2q_0)^{-1}/q_0F^{-1})[\xi] - 2Q^{-1}{}^t\xi Y)/2). \tag{2.10}$$

Now we note that for any Dirichlet character χ the definition (2.1) can be rewritten as follows:

$$\theta(\chi) = \theta^{(\nu)}_F(z;\chi) = \sum_{Y\in \mathrm{M}_m(\mathbf{Z})\mathrm{mod} Q} \chi(\det(Y)) \sum_{\xi\equiv Y \mathrm{mod} Q} (\det\xi)^\nu e_m(zF[\xi]/2), \tag{2.11}$$

and if we take into account the primitivity of χ then we get (see [An5],(5.12))

$$\theta^{(\nu)}_F(z;\chi) = G_Q(1_m,\overline{\chi})^{-1}\times \sum_{Y\in \mathrm{M}_m(\mathbf{Z})\mathrm{mod} Q} \overline{\chi}(\det(Y)) \sum_{\xi\equiv Y \mathrm{mod} Q} (\det\xi)^\nu e_m(zF[\xi] + 2Q^{-1}{}^t\xi Y)/2), \tag{2.12}$$

If we now take into account the standard relation

$$G_Q(1_m,\chi)G_Q(1_m,\overline{\chi}) = \chi(-1)Q^{m^2}. \tag{2.13}$$

for Gauss sums, then (2.10) transforms to

$$\theta^{(0)}_F(z;\chi), \tag{2.14}$$

$$(\det Q^2F)^{-(m/2)}[\det(-iz)]^{-(m/2)}G_Q(1_m,\chi)\theta^{(0)}_{q_0F^{-1}}(J(q_0Q^2)(z);\overline{\chi}) = \chi(-1)(\det F)^{-(m/2)}[\det(-iz)]^{-(m/2)}G_Q(1_m,\overline{\chi})\theta^{(0)}_{q_0F^{-1}}(J(q_0Q^2)(z);\overline{\chi}), \tag{2.15}$$

Put in (2.12) F equal to q_0F^{-1}, z equal to $-(Q^2q_0z)^{-1}$, replace χ by $\overline{\chi}$ and ξ by $-\xi$; comparison of (2.14) and (2.15) provides us with an identity ,which is equivalent to (2.4) with $\nu = 0$.

In order to prove the statements (a) and (b) with $\nu = 1$ we take a matrix $F_1 = {}^tF_1 \in \mathrm{M}_m(\mathbf{R})$ with the condition $f_1 > 0, F_1^2 = F$,and let $\eta = (\eta_{ij}) \in \mathrm{M}_m(\mathbf{C})$ be the matrix variable . If we put in (2.7) $X = 0, Y = F_1^{-1}\eta$ and apply to both parts of the resulting equality the differential operator $L_\eta = \det(\partial/\partial\eta_{ij})$, then after some simplifications we get the equality (2.2) with $\nu = 1$. In this calculation the differential identity

$$L_\eta(e_m((P\,{}^t\eta\eta + 2\,{}^tT\eta + R)/2)) = \det(2\pi i(\eta P + T))e_m((P\,{}^t\eta\eta + 2\,{}^tT\eta + R)/2)), \tag{2.16}$$

is particulary useful , where $P,T,R \in \mathrm{M}_m(\mathbf{C})$, ${}^tP = P$ (see [An7], lemma 5.1).

We give a more detailed proof of (b). Put in (2.8) $X = 0$ and $Y = F_1^{-1}\eta$ and apply the operator L_η. Then the expression in the exponent of the summand corresponding to an integral matrix ξ in the left hand side of (2.8) is equal to

$$\begin{aligned}\pi i \cdot \mathrm{tr}(zF[\xi - F_1^{-1}\eta]) = \\ \pi i \cdot \mathrm{tr}(z\,{}^t\eta\eta - 2\,{}^t(F_1\xi z)\eta + zF[\xi]).\end{aligned}$$

This is easily deduced from the definition (2.6) if we note that

$$\mathrm{tr}(AB) = \mathrm{tr}(BA) = \mathrm{tr}({}^tA\,{}^tB).$$

According to (2.16) after application of L_η this term will be multiplied by

$$\det(2\pi i(\eta z - F_1\xi z)) = (2\pi i)^m \det F^{1/2} \det((F_1^{-1}\eta - \xi)z).$$

Similarly the summand corresponding to a given matrix ξ in

$$(\det F)^{-(m/2)}[\det(-iz)]^{-(m/2)}\theta^{(m)}_{F^{-1}}(-z^{-1}; F_1^{-1}\eta, 0),$$

after application of L_η will get the factor $(\det F)^{-1/2}(2\pi i)^m \det(\xi)$. Next we remember that $Y = F_1^{-1}\eta$, then we see that after application of L_η the expression (2.8) transforms to the following

$$\begin{aligned}\det(z)\sum \xi \in \mathrm{M}_m(\mathbf{Z})\det(Y - \xi)e_m((zF[\xi - Y]/2) = \\ (\det F)^{-(m/2)}[\det(-iz)]^{-(m/2)}\times \\ \times \sum_{\xi\in \mathrm{M}_m(\mathbf{Z})} \det(\xi)e_m((-z^{-1}F^{-1}[\xi] + 2\,{}^t\xi Y)/2.\end{aligned} \tag{2.17}$$

Now again we put in (2.17) F equal to Q^2F, Y equal to $-Q^{-1}Y$, ξ equal to $-\xi$, multiply by $\chi(\det(Y))$ and carry out summation over $Y \bmod Q$ keeping in mind the relation (2.13); as a result we get the equality (2.4) with $\nu = 1$. An explicit calculation of the multiplier $\chi_F^{(m)}(\gamma)$ for even m is given in [An-K].

2.4. Siegel -Eisenstein series. We start with recalling the definition of these series. We call matrices $c, d \in \mathrm{M}(\mathbf{Z})$ coprime if

$$\{G \in \mathrm{M}(\mathbf{Q}) \big| Gc,\ Gd \in \mathrm{M}(\mathbf{Z})\} = \mathrm{M}(\mathbf{Z})$$

A couple (c, d) is called a *symmetric couple* if $c\,{}^td = d^tc$. Two symmetric couples of the coprime matrices are called *equivalent* iff for some unimodular matrix $U \in \mathrm{GL}_m(\mathbf{Z})$ we have $(c_1, d_1) = (Uc_2, Ud_2)$.

Let $\Delta = \Delta_m$ denote the set of equivalence classes of symmetric couples of coprime matrices. Then the set can be identified with the set of right coset classes $\Gamma_0^m\backslash\Gamma^m$ of the group $\Gamma^m = \mathrm{Sp}_m(\mathbf{Z})$ with respect to its parabolic subgroup

$$\Gamma_0^m = \{\gamma = \begin{pmatrix} a & b \\ 0 & d\end{pmatrix} | \gamma \in \Gamma^m\}$$

via the map

$$\Gamma_0^m\backslash\Gamma^m \ni \Gamma_0^m \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \text{class of } (c,d) \in \Delta_m. \tag{2.18}$$

By this map also the set

$$\{(c,d) \in \Delta_m | c \equiv 0(\text{mod}N)\}$$

identifies with the set of cosets $\Gamma_0^m\backslash\Gamma^m(N)$.

Now let k , N be positive integers , s a complex number and χ a Dirichlet character modulo N. For $z \in \mathfrak{H}_m$ define the Siegel-Eisenstein series by

$$E(z,s;k,\chi,N) = E(z,s) = \det(y)^s \sum \chi(\det(d))\det(cz+d)^{-k-|2s|}, \tag{2.19}$$

where the summation is taken over all $(c,d) \in \Delta$ with the condition $c \equiv 0(\text{mod}N)$ and we adopt the convenient notation by Deligne and Ribet [De-R]:

$$z^{-k-|2s|} \overset{\text{def}}{=} z^{-k}|z|^{-2s} \; for \; z \in \mathbf{C}^*$$

The series (2.19) is absolutely convergent for $k + 2\text{Re}(s) > m+1$ and it admits a meromorphic analytic continuation over the whole complex s-plane. Put $j(\alpha, z) = \det(cz+d)$ for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z \in \mathfrak{H}_m$, then it follows from the description of Δ_m given above that

$$E(z,s;k,\chi,N) = \det(y)^s \sum_{\alpha \in P\cap\Gamma\backslash\Gamma} \chi(\det(d_\alpha)j(\alpha,z)^{-k-|2s|}, \tag{2.20}$$

where $\Gamma = \Gamma_0^m(N), \alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix}$, and P denotes the subgroup of $P \subset G_{\infty+}$, consisting of elements α with the condition $c_\alpha = 0$.

2.5. Rankin zeta function. Let f and g be two holomorphic modular forms of weight k and l on the congruence subgroup $\gamma_0^m(N) \subset \Gamma^m$ with Dirichlet characters ψ and ω. More precisely we assume that

$$f(z) = \sum_{\xi \in B_m} a(\xi)e_m(\xi z) \in \mathcal{S}_m^k(N,\psi), \tag{2.21}$$

$$g(z) = \sum_{\xi \in C_m} b(\xi)e_m(\xi z) \in \mathcal{M}_m^l(N,\psi), \tag{2.22}$$

with B_m being the set of half integral non negative matrices of size $m\times m$, and $C_m \subset B_m$ the subset of all positive definite matrices, see §1. Define an equivalence relation on B_m by $\xi_1 \sim \xi_2$ iff $\xi_1 = {}^t u \xi_2 u$ for some matrix $u \in \text{SL}_m(\mathbf{Z})$. Then Rankin zeta function (convolution of Siegel modular forms) is defined as the series [St2] :

$$L(s,f,g) = \sum_{\xi \in \tilde{C}_m} a(\xi)b(\xi)\det(\xi)^{-s} \quad (s \in \mathbf{C}) \tag{2.23}$$

which is well defined due to basic properties of Fourier coefficients of Siegel modular forms; $\tilde{C}_m = C_m \bmod \sim$ denotes the orbit space with respect to $\sim$ and $\mathrm{Re}(s)$ is supposed to be large enough: $\mathrm{Re}(s) \gg 0$. To be more precise, we note that the series

$$\sum_{\xi \in \tilde{C}_m} \det(\xi)^{-s} \tag{2.24}$$

is absolutely convergent for $\mathrm{Re}(s) > m$. It is easily seen from the fact that cardinality of the set

$$\mathrm{SL}_m(\mathbf{Z}) \backslash \{\gamma \in \mathrm{M}_m(\mathbf{Z}) | \det\gamma = a > 0\}$$

is estimated by the finite sum $\sum a_2 a_3^2 \cdots a_m^{m-1}$, hence the Dirichlet series in (2.24) admits the upper bound

$$\prod_{i=0}^{m-1} \zeta(\mathrm{Re}(s) - i),$$

with $\zeta(s)$ being the Riemann zeta function (see [An7], p.133). Keeping in mind estimations (1.15a) and (1.16a) for Fourier coefficients of Siegel modular forms we get

$$a(\xi) = \mathcal{O}(\det(\xi)^{k/2}), \quad b(\xi) = \mathcal{O}(\det(\xi)^l)$$

so that the Dirichlet series (2.23) is absolutely convergent for $\mathrm{Re}(s) > m + k/2 + l + \varepsilon$ ($\varepsilon > 0$).

2.6. Proposition *(integral representation of Rankin zeta functions, see [St2] , proposition 6).*

For s with $\mathrm{Re}(s) > 0$ there is the following integral representation.

$$(4\pi)^{-ms}\Gamma_m(s) L(s, f, g) = \langle f^\rho, g E(z, s - k + (m+1)/2; k - l, \psi\omega, N)\rangle_N \tag{2.25}$$

where the inner product is defined by (1.13),

$$f^\rho(z) = \sum_{\xi \in B_m} \overline{a(\xi)} e_m(\xi z) \in \mathcal{S}_m^k(N, \psi)$$

$\Gamma_m(s)$ denotes the Γ-function of degree m, i.e.

$$\Gamma_m(s) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(s - (j/2))$$

(see also (3.8)).

2.7. The standard zeta function $\mathcal{D}(s, f, \chi)$ as a Rankin convolution. Now let

$$f(z) = \sum_{\xi \in C_m} \overline{a(\xi)} e_m(\xi z) \in \mathcal{S}_m^k(N, \psi)$$

be a cusp form of even degree m which is an eigenfunction of the Hecke algebra $\mathcal{L}_q^m(N)$ for q not dividing N, and

$$(\alpha_0, \alpha_1, \cdots, \alpha_m) = (\alpha_0^{(q)}, \alpha_1^{(q)}, \cdots, \alpha_m^{(q)}) \in [(A^\times)^{m+1}]^{W_m}$$

be the corresponding $(m+1)$-tuple of the Satake q- parameters of f (see (1.18)). Fix a matrix $\xi_0 \in C_m$ such that $a(\xi_0) \neq 0$ and consider the primitive quadratic Dirichlet character χ_{ξ_0} definied for positive integers d with $(d, 2\det(2\xi_0)) = 1$ by the formula

$$\chi_{\xi_0}(d) = \left(\frac{(-1)^{m/2}\det(2\xi_0)}{d}\right),$$

when $\det(2\xi_0)$ is odd , then $\chi_{\xi_0}(2) = 1$ or -1 according to which of the two following quadratic forms

$$x_1x_2 + x_3x_4 + \ldots x_{m-1}x_m$$

or

$$x_1x_2 + \ldots + x_{m-3}x_{m-2} + x_{m-1}^2 + x_{m-1}x_m + x_m^2$$

is equivalent our quadratic form ξ_0 over the field $\mathbf{F}_2$ of two elements. Assume also that $\chi \bmod M$ is chosen so that $(-1)^\nu = \chi(-1)$. The main result of the A.N.Andrianov's work [An6] can be stated as a certain identity expressing the standard zeta function $\mathcal{D}(s, f, \chi)$ as a Rankin zeta function , namely the convolution of the given form f and a theta function with the Dirichlet caracter χ mod M. The precise statement of the result is given in the following proposition.

2.8.Proposition. *Under the notation and assumptions as above for the sufficiently large values of* $\mathrm{Re}(s)$ *the following identity is valid*

$$\begin{aligned} &a(\xi_0)R(s,f,\chi) = \\ &\quad 2^{-1}\det\xi_0^{(s+k-1+\nu)/2}L((s+k-1+\nu)/2, f, \theta^{(\nu)}_{2\xi_0}(z;\chi)), \end{aligned} \tag{2.26}$$

where the function $R(s, f, \chi)$ *is defined by the following equality*

$$\mathcal{D}(s,f,\chi) = L(s+(m/2), \psi\chi_{\xi_0}\chi)\prod_{i=0}^{m-1} L(2s+2i, \psi^2\chi^2)R(s,f,\chi), \tag{2.27}$$

and it is assumed that the modulus M *of the character* χ *is divisible by all prime divisors of the number* $N\det 2\xi_0, \theta^{(\nu)}_{2\xi_0}(z;\chi)$ *being the theta function in* 2.1.

More explicitly , the right hand side of (2.26) can be represented as the series

$$\begin{aligned} &(\det\xi_0)^{(s+k-1+\nu)/2}\sum_{\xi}\chi(\det\xi)\det\xi^\nu a(\xi_0[\xi])(\det\xi_0[\xi])^{-(s+k-1+\nu)/2} = \\ &\sum_{\xi}\chi(\det\xi)a({}^t\xi\xi_0\xi)(\det\xi)^{-(s+k-1+\nu)/2} \end{aligned}, \tag{2.28}$$

where the summation is taken over the set of equivalence classes $\xi \in \mathrm{SL}_m(\mathbf{Z})\backslash \mathrm{M}_m^+(\mathbf{Z})$ of the form $\xi\mathrm{SL}_m(\mathbf{Z})$.

Now we put q_0 to be equal to the level of the quadratic form with the matrix $2\xi_0$ (see 2.1). In order to get the integral representation of the standard zeta function we apply to it the result of 2.6. For this purpose we put in the notation of 2.6 $l = (m/2)+\nu, \omega = \chi\chi_{\xi_0}$ with χ being a Dirichlet character modulo M, and take the number Nq_0M^2 as N. We note also that both parts of (2.27) and (2.26) converge absolutely for $\mathrm{Re}(s) > m$ (see also [An7], p. 133).

2.9. Proposition *(integral representation of the standard zeta function). With the notation and assumption as above for* Re(s) *the following equality holds:*

$$\begin{aligned} &2a(\xi_0)(\det\xi_0)^{-(s+k-1+\nu)/2}(4\pi)^{-ms} \times \Gamma_m((s+k-1+\nu)/2)\, R(s,f,\chi) = \\ &\langle f^\rho, \theta^{(\nu)}_{2\xi_0}(z;\chi)E(z,(s-k+m+\nu)/2)\rangle_{NM^2q_0} \end{aligned}, \qquad (2.29)$$

with the Eisenstein series

$$E(z,s) = E(z,s;k-\nu-(m/2),\chi\chi_{\xi_0}\psi, NM^2q_0).$$

defined by (2.29) in the right hand side.

§3. Formulas for Fourier coefficients of Siegel-Eisenstein series

3.1. Rationality properties of Fourier coefficients. For the full symplectic modular group $\Gamma = \mathrm{Sp}_m(\mathbf{Z})$ Siegel has defined the series [Sie2], [Sie3] (see §2, 2.4)

$$E(z) = E^{(m)}_k(z) = \sum_{\gamma\in P\cap\Gamma^m\backslash\Gamma^m} j(\gamma,z)^{-k}, \qquad (3.1)$$

where $z \in \mathfrak{H}_m$ is the point of the Siegel upper half plane of degree m, $j(\gamma,z) = \det(cz+d)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^m$, P denotes the subgroup of Γ^m consisting of block matrices of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. In the original definition by Siegel the number k is even and $k > m+1$ so that the series (3.1) is absolutely convergent and is referred to as Siegel-Eisenstein series. The rationality property of its Fourier coefficients were established by Siegel himself although it was certainly known earlier in the case $m = 1$:

$$\begin{aligned} &E^{(1)}_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e(nz) = \\ &1 + \sum_{n=1}^{\infty}\left(\frac{2\sigma_{k-1}(n)}{\zeta(1-k)}\right)e(nz), \quad \sigma_{k-1}(n) = \sum_{d|n} d^{k-1}, \end{aligned} \qquad (3.2)$$

where B_k are Bernoulli numbers, $\zeta(s)$ being the Riemann zeta function. After the original Siegel's work his calculation was generalized in various directions : to the case of congruence subgroups of $\Gamma^{(m)}_0(N) \subset \Gamma^m$ [St2] to non-convergent series defined by analytic continuation over an additional parameter (Hecke's method) [He], [Fe],to other classes of algebraic groups and symmetric domains [Ba],[Har2],[Fe],[Shi7],[Shi10].It was discovered that the rationality property remain valid even for more complicated series, which themselves are defined by some inductive process of inducing from other cusp forms of lower degree (Klingen-Eisenstein series).More precisely, let$f \in S^r_k$ be a cusp form of degree r (with respect to the group Γ^r).If $k > m+r+1$ and $m \geq r$ then Klingen-Eisenstein series is defined as the following absolutely convergent series

$$E^{m,r}_k(z,f) = \sum_{\gamma\in\Delta_{m,r}\backslash\Gamma^m} f((\gamma\gamma z)^{(r)})j(\gamma,z)^{-k}, \qquad (3.3)$$

with $z \in \mathfrak{H}_m$, $z^{(r)}$ being the upper left corner of z of size $r \times r$, and $\Delta_{m,r}$ denotes the set of elements in Γ^m having the form $\begin{pmatrix} * & * \\ 0_{m-r,m+r} & * \end{pmatrix}$ [Kl3]. This series turns out to be a modular form of degree m on the group Γ^m . M Harris has proven the validity of Garrett's conjecture : all Fourier coefficients the modular form $E_k^{m,r}(z,f)$ belong to the field $\mathbf{Q}(f)$ generated by Fourier coefficients of f [Har2]. Explicit formulas for Fourier coefficients of the series $E_k^{m,r}(z,f)$ were given by Mizumoto [Kur-Miz],[Miz1],[Miz2]. It turned out that that the most significant term in these formulas involves the special values of the standard zeta function of f twisted with a certain quadratic Dirichlet character attached to the matrix number ξ of a Fourier coefficient; as it was noticed above ,these functions reduce to the (twisted) symmetric squares of the form f if $m = 2$. The formulas of Mizumota can be considered as a vast generalization of the classical formulas (3.2), if we assume that cusp forms of degree 0 to be constants and their zeta functions reduce to the Riemann zeta function and to Dirichlet L-series.

However, in what follows we are interested only in Siegel-Eisenstein series which were defined in §2 by:

$$E(z,s;k,\chi,N) = E(z,s) = \det(y)^s \sum_{\alpha \in P\cap\Gamma\backslash\Gamma} \chi(\det(d_\alpha)j(\alpha,z)^{-k-|2s|}, \tag{3.4}$$

for $k + 1\mathrm{Re}(s) > m+1$, and by analytic continuation on s for the other values of $s \in \mathbf{C}$ (see [K], [Shi10] and 3.3 below). It is assumed in the identity (3.4) that $N > 1, \chi$ is a Dirichlet character mod N (not necessarily primitive, e.g. trivial modulo $N > 1$) , and

$$\alpha = \begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix} \in \Gamma = \Gamma_0^m(N) \subset \Gamma^m.$$

The following investigation of arithmetic properties of Fourier coefficients is based on an explicit calculation of the Fourier expansion of the series

$$E^*(z.s) = E(-z^{-1},s)(\det z)^{-k}, \tag{3.5}$$

obtained from (3.4) by applying the involution

$$J_m = \begin{pmatrix} 0 & -1_m \\ 1_m & 0 \end{pmatrix}.$$

However for $k > m+1$ and $N = 1$ both series coincide and reduce to the series originally studied by Siegel :

$$E(z) = E_k^m(z) = E(z,0) = E^*(z,0).$$

The investigation mentioned above was carried out by Shimura [Shi10] and P.Feit ([Fe], §10) and were given in a more general situation, in particular for the Eisenstein series attached to the group Sp_m over a totally real field. For convenience we reproduce only a specialization of these results to the case of $F = \mathbf{Q}$.

The remarkable summation method by means of the analytic continuation of the function $E(z,f)$ over s to the point $s = 0$ was first discovered by Hecke [He]. Quite recently this method was essentially extended by R.Weissauer [We2] to the case of Klingen-Eisenstein series (3.3) and then used for a construction of Siegel modular forms of relatively small weight ($k = m, m+1$) which play a significant role in studying the geometric invariants of the Siegel modular varieties [We1],[We3].

3.2. Preparation : the confluent hypergeometric function . For a detailed description of the Fourier expansion of the series (3.5) we need some additional notation.Let

$$V = V_m = \{h \in \mathrm{M}_m(\mathbf{R}) | {}^t h = h\} \tag{3.5}$$

be the set of all real symmetric matrices of size $m \times m$, and

$$Y \;=\; V^+\{h \in V | h > 0\} \tag{3.7}$$

the subset of its positive definite elements. For each matrix $T \in \mathrm{M}_m(\mathbf{R})$ let $\delta_+(T)$ denote the product of all positive eigenvalues of T, $\delta_-(T) = \delta_+(-T)$ and $\delta_+(T) = 1$ if T does not have positive eigenvalues.

For $h \in V$ let $p = p(h)$ denote the number of positive eigenvalues of h counted with their multiplicities, and $q = q(h)$ the number of negative eigenvalues. Then $r = r(h) = p + q$ is the rang of h.

Let also

$$\Gamma_m(s) = \pi^{m(m+2)/4} \prod_{j=0}^{m-1} \Gamma(s - (j/2)), \tag{3.8}$$

be the Γ-function of degree m , which generalizes the ordinary Γ-function according to its integral representation

$$\Gamma_m(s) = \int_Y (\det y)^s e^{-\mathrm{tr}(y)} d^\times y, \tag{3.9}$$

which is valid for $s \in \mathbf{C}$ with $\mathrm{Re}(s) > (m-1)/2$, and

$$dy = \prod_{i \le j} dy_{ij}, \quad \det(y)^{-(m+1)/2} dy.$$

Recall that $d^\times y$ is a measure on Y which is invariant with respect to the action of $a \in \mathrm{GL}_m(\mathbf{R})$ given by $d^\times({}^t aya) = d^\times y$. For complex numbers α and β we define the numbers

$$\begin{aligned} \kappa &= (m+1)/2, \\ \tau &= \tau(h, \alpha, \beta) = \\ &= (2p - m)\alpha + (2q - m)\beta + m + (m - r)\kappa + pq/2, \end{aligned} \tag{3.10}$$

$$\begin{aligned} \sigma &= \sigma(h, \alpha, \beta) = \\ p\alpha + q\beta &+ m - r + \{(m - r)(m - r - 1)\}/2. \end{aligned} \tag{3.11}$$

In Shimura's work [Shi8] the function

$$\omega(y, h; \alpha, \beta), \tag{3.12}$$

was constructed which is defined for all $(y,h;\alpha,\beta)\in Y\times V\times \mathbf{C}^2$ and is a holomorphic function $(\alpha,\beta)\in \mathbf{C}^2$. It can be used for writing the Fourier expansion of the series

$$S(z,L;\alpha,\beta)=\sum_{a\in L}\det(z+a)^{-\alpha}\det(\overline{z}+a)^{-\beta}\quad (z\in\mathfrak{H}_m) \tag{3.13}$$

which is obtained by summation over a lattice $L\subset V$ and is absolutely convergent for $\mathrm{Re}(\alpha+\beta)>m$. Let

$$L'=\{h\in V|\mathrm{tr}(hL)\in\mathbf{Z}\}$$

be the lattice dual to L with respect to the pairing given by $(u,v)\mapsto e_m(uv)$. In particular there is the equality

$$\mu(V/L)S(z;\alpha,\beta)=\sum_{h\in L'}\xi(y,h;\alpha,\beta)e_m(hx) \tag{3.14}$$

in which

$$\mu(V/L)=\int_{V/L}dy$$

denotes the volume of the fundamental domain V/L,

$$\begin{aligned}\xi(y,h;\alpha,\beta)=&\\ i^{m\beta-m\alpha}2^r\pi^\sigma\Gamma_{m-r}(\alpha+\beta-\kappa)&\Gamma_{m-q}(\alpha)^{-1}\Gamma_{m-p}(\beta)^{-1}\times\\ \times(\det y)^{\kappa-\alpha-\beta}\delta_+(hy)^{\alpha-\kappa+q/4}&\delta_-(hy)^{\beta-\kappa+q/4}\omega(2\pi y,h;\alpha,\beta),\end{aligned} \tag{3.15}$$

and is additionaly assumed that $\mathrm{Re}(\alpha)>m/2$, $\mathrm{Re}(\beta)>m/2$ (for the regularity of Γ-functions in (3.15) (see [Shi8],(4.34.K)) and we adopt the standard choice of branch for the exponentiation , namely,

$$v^\alpha=e^{\alpha\log(v)},\quad -\pi\le\mathrm{Im}(\log v)<\pi.$$

The function $\xi(y,h;\alpha,\beta)$ admits the following integral representation: for $g\in Y,h\in V,(\alpha,\beta)\in\mathbf{C}^2$

$$\begin{aligned}&\xi(y,h;\alpha,\beta)=\\ &\int_V e_m(-hx)\det(x+ig)^{-\alpha}\det(x-ig)^{-\beta}dx,\end{aligned} \tag{3.16}$$

with the integral being absolutely convergent for $\mathrm{Re}(\alpha+\beta)>2\kappa+1$ (see [Shi8], (1.25)).

Applying the equality (3.14) to the lattice $L=S=V\cap\mathrm{M}_m(\mathbf{Z})$ when $L'=A=A_m$ is the lattice of all symmetric half integral matrices and also to the lattice $L=NS,L'=N{-}1A_m$ with $\alpha=k,\beta=0,k>m$ and $C_m=A_M\cap Y$ we get the classical equality

$$\sum_{a\in S}\det(z+a)^{-k}=(2\pi i)^{mk}\Gamma_m(k)^{-1}\sum_{h\in C_m}(\det h)^{k-\kappa}e_m(hz) \tag{3.17}$$

(see , for example , the book of Maass [Maa]). Indeed, in the equality only the terms with $p=m,q=0$ do not vanish because of the poles of the Γ functions in the denominator of (3.15) , and the function $\omega(2\pi y, h; \alpha, \beta)$ reduces to the exponent $e_m(iyh)$ in view of the formulas:

$$\xi(y, h; \alpha, 0) = \tag{3.18}$$
$$i^{-m\alpha}2^{(1-\kappa)m}(2\pi)^{m\alpha}\Gamma_m(\alpha)^{-1}(\det h)^{\alpha-\kappa}e_m(iyh),$$

$$\xi(y, 0; \alpha, \beta) = \tag{3.19}$$
$$i^{m\beta-m\alpha}2^{m(\kappa+1-\alpha-\beta)}\pi^{m\kappa}\Gamma_m(\alpha+\beta-\kappa)\Gamma_m(\alpha)^{-1}\Gamma_m(\beta)^{-1}$$

$$\lim_{\xi\to 0}\xi(y, h; \kappa+s, s) = i^{-m\kappa}2^{\theta}\pi^{m\kappa}\Gamma_m(\kappa)^{-1}e_m(iyh), \tag{3.20}$$

with $q = 0$ and $\theta = [(m+p)/2]$. The formulas (3.18)-(3.20) are easily deduced from [Shi8], (1.31), (4.35.K); see also [Shi10],(7.11)-(7.14).

The confluent hypergeometric function $\omega(2\pi y, h; \alpha, \beta)$ can be used for an analytic continuation of the Siegel-Eisenstein series [K],[Fe],[Shi10] by means of the term by term analytic continuation of their Fourier coefficients, which can be expressed in terms of the functions (3.12) (see theorem 3.6 below). We list also some other properties of these functions, which are uselul for the analytic continuation (see [Shi8], theorem 4.2):

functional equation

$$\omega(2\pi y, h; \alpha, \beta) = \omega(2\pi y, h; \kappa+(t/2)-\beta, \kappa+(t/2)-\alpha); \tag{3.21}$$

a uniform upper bound on compact subsets

$$|\omega(2\pi y, h; \alpha, \beta)| \le C_1 e^{-\tau(hy)}(1+\mu(hy)^{-C_2}), \tag{3.22}$$

with α, β varying in a fixed compact subset $T \subset \mathbf{C}^2$ and the constants C_1, C_2 depending only on $T, \tau(x)$ being *the sum of eigenvalues* of a matrix x , μ *the minimum of their absolute values.*

3.3. Critical values of the confluent hypergeometric function. Now we give formulas, which express the function $\omega(2\pi y, h; \alpha, \beta)$ in terms of certain polynomials of the entries of the matrix $y = (y_{ij})$ provided $h > 0$ and either $\alpha\kappa \in \mathbf{Z}, \alpha - \kappa \ge 0$ or $\beta \in \mathbf{Z}, \beta \le 0$ $(\kappa = (m+1)/2)$. We call such pairs (α, β) critical: as we will see in the sequel the critical values of s for the standard zeta function correspond to some critical pairs. The following calculation of the special values is based on properties of the function $\zeta(z; \alpha, \beta)$ defined for $z \in \mathfrak{H}'_m = \{z \in \mathrm{M}_m(\mathbf{C}) | iz \in \mathfrak{H}_m\}$ by the integral

$$\zeta(z; \alpha, \beta) = \int_Y e^{-\mathrm{tr}(zx)}\det(x+1_m)^{\alpha-\kappa}\det x^{\beta-\kappa}dx, \tag{3.23}$$

which is absolutely convergent for Re $> \kappa - 1$ and defines a holomorphic function of (z, α, β). Let

$$\omega(z; \alpha, \beta) = \Gamma_m(\beta)^{-1}\det(z)^{\beta}\zeta(z; \alpha, \beta). \tag{3.24}$$

It was established by Shimura ([Shi8],theorem 3.1) that the function (3.24) can be analytically continued to a holomorphic function over $\mathfrak{H}'_m \times \mathbf{C}^2$ satisfying the functional equation

$$\omega(z;\kappa-\beta,\kappa-\alpha)=\omega(z;\alpha,\beta). \tag{3.25}$$

For an arbitrary compact subset $T \subset \mathbf{C}^2$ there exsist positive constants $A, B, > 0$ depending only on T such that

$$|\omega(z;\alpha,\beta)| \le A(1+\mu(y)^{-B}) \; for\; y \in Y \subset \mathfrak{H}'_m, \quad (\alpha,\beta)\subset T. \tag{3.26}$$

It is known also (see [Shi8],, (4.19))that

$$\omega(y,1_m;\alpha,\beta)=2^{m(m+1)/2}e^{-\mathrm{tr}(y)}\omega(2y;\alpha,\beta) \tag{3.27}$$

and that for all $a \in \mathrm{GL}_m(\mathbf{R})$ one has

$$\omega({}^t a^{-1} y a^{-1};\alpha,\beta)=\omega(y;\alpha,\beta), \tag{3.28}$$

$$\omega(y,-h;\alpha,\beta)=\omega(y,h;\beta,\alpha) \tag{3.29}$$

$$\omega(y,h;\alpha,\beta)=1. \tag{3.30}$$

Comparison of (3.27) and (328) shows that for $h > 0$ there is the identity

$$\omega(y,h;\alpha,\beta)=\omega(a(hy)a^{-1},1_m;\alpha,\beta)=2^{m(m+1)/2}e^{-\mathrm{tr}(y)}\omega(2a(hy)a^{-1};\alpha,\beta), \tag{3.31}$$

This is done by taking a matrix $a \in \mathrm{GL}_m(\mathbf{R})$ with the condition $ah^ta = 1_m$.

Now let us consider the differential operator Δ_m on $\Delta \otimes \mathbf{C}$ of degree m defined by the equality:

$$\Delta_m=\det(\partial_{ij}), \quad \partial_{ij}=2^{-1}(1+\delta_{ij})\partial/\partial_{ij}. \tag{3.32}$$

For an integer $n \ge 0$ and a complex number β consider the polynomial

$$R(z;n,\beta)=(-1)^{mn}e^{\mathrm{tr}(z)}\det(z)^{n+\beta}\Delta_m^n[e^{\mathrm{tr}(z)}\det(z)^{-\beta}], \tag{3.33}$$

with $z \in V \otimes \mathbf{C}$ and the exponentiation being well defined by the condition : $\det(y)^\beta = \exp(\log(\det(y)))$fordet $> 0, y \in Y \otimes \mathbf{C}$. According to the definition (3.33) the degree of the polynomial $R(z;n,\beta)$is equal to mn and the term of the highest degree coincides with $\det z^n$. We have also that for $\beta \in \mathbf{Q}$ the polynomial $R(z;n,\beta)$ has rational coefficients.

Proposition. *(See [Shi8], proposition 3.2). For any non negative integer n the functions $\det(z)^n\omega(z;n+\kappa,\beta)$ and $\det(z)^n\omega(z;\alpha,-n)$ are polynomial functions of z. More precisely, we have that*

$$\omega(z;n+\kappa,\beta)=\det{}^{-n}R(z;n,\beta), \tag{3.34}$$

$$\omega(z;\alpha,-n)=\omega(z;n+\kappa,n-\alpha)=\det{}^{-n}R(z;n,\kappa-\alpha). \tag{3.34a}$$

The proof is carried out with help of the following differentiation rule :

$$(-1)^{mn}\Delta_m^n\{e^{-\mathrm{tr}(z)}\det(z)^{-\beta}\omega(z;\alpha,\beta)\}=e^{-\mathrm{tr}(z)}\det(z)^{-\beta}\omega(z;\alpha+n,\beta), \tag{3.35}$$

which follows immediately from the definition (3.23), (3.24) and provides also an analytic continuation of the function $\omega(z;\alpha,\beta)$. The formulas (3.34) and (3.34a) follow then from the identities (3.35) and

$$\omega(z;\kappa,\beta)=\omega(z;\alpha,0)=1,$$

Notice also that for $m=1$ one has

$$R_1(z,n,\beta)=\sum_{k=0}^{n}\binom{n}{k}\beta(\beta+1)\cdots(\beta+k-1)z^{n-k}.$$

In §4, theorem 4.8, we establish the following more explicit expression for the function $R_m(z;n,\beta)$ of arbitrary degree m (see (4.32))

$$R_m(z;n,\beta)=\sum_{r_1,\cdots,r_n=0}^{m} c_{m-r_1}(-\beta)c_{m-r_2}(-\beta-1)\cdots c_{m-r_n}(-\beta-n+1)\lambda_{r_1}(z)\cdots\lambda_{r_n}(z), \tag{3.36}$$

where

$$c_r(z)=\prod_{k=0}^{r-1}(\alpha+(k/2)),$$

and $\lambda_r(z)$ are polynomial functions of entries of the matrix variable $z\in \mathrm{M}_m(\mathbf{C})$ defined by

$$\det(t1_m-X)=\sum_{r=0}^{m}(-1)^r\lambda_r(X)t^{m-r} \tag{3.37}$$

In other words $\lambda_r(z)$ is the sum of all diagonal minors of size $r\times r$ of the matrix z.

If we apply this to functions $\omega(2\pi y,h;\alpha,\beta)$ from 3.2 then we get for $h>0$ the following identity:

$$\begin{aligned}&\omega(2\pi y,h;n+\kappa,\beta)=\omega(2\pi y,h;\kappa-\beta,n)=\\&2^{-m(m+1)/2}e_m(ihy)\omega(4\pi a^{-1}(hy)a\,;n+\kappa,\beta)=\\&2^{-m(m+1)/2}e_m(ihy)\det(4\pi hy)^{-n}R_m(4\pi hy\,;n,\beta).\end{aligned} \tag{3.38}$$

3.4. Proposition. *(Fourier expansion of the Siegel-Eisenstein series) $E^*(z,s)$, see [Fe], §10). For the series defined by (3.5) the following Fourier expansion is valid*

$$E^*(z,s)=\sum_{h\in N^{-1}A_m} b(h,y,s)e_m(hz), \tag{3.39}$$

in which the coefficients have the form of the product

$$b(h,y,s)=N^{-m\kappa}W(y,h,s)\Gamma(h,s)RL^*(h,\chi,k+2s)M(h,\chi,k+2s), \tag{3.40}$$

with the factors described as follows ((a)-(d)):

(a)The confluent hypergeometric function

$$\begin{aligned}W(y,h,s) = i^{-mk}2^{\tau}\pi^{\sigma}\omega(2\pi y, h; k+s, s)\times \\ (\det y)^{\kappa-k-s}\delta_+(hy)^{k+s-\kappa+q/4}\delta_-(hy)^{s-\kappa+p/4},\end{aligned} \tag{3.41}$$

with (comp. (3.15))

$$\begin{aligned}\tau =&(2p-m)(k+s)+(2q-m)s+m+(m-r)+pq/2= \\ &2(r-m)s+(2p-m)k+m+(m-r)(m+1)/2+pq/2, \\ \sigma =&p(k+s)+qs+m-r+\{(m-r)(m-r-1)-pq\}/2= \\ =&rs+pk+\{(m-r)(m-r-1)-pq\}/2\end{aligned};$$

(b) Gamma factor $\Gamma(h,s)$. *Let for the integer* r *the symbol* $\varepsilon(r)$ *denote its parity:* $\varepsilon(r) = 0$, *1 with* $r \equiv \varepsilon(r) \bmod 2$. *Put* $\delta = \varepsilon(k), \mu = \varepsilon((r/2)+q+k)$ *and then define:*
for $\varepsilon(r) = 0$

$$\Gamma(h,s) = \\ = \frac{\Gamma_{m-r}\left(k+2s-\frac{m+1}{2}\right)\Gamma\left(s+\frac{k+\delta}{2}\right)\prod_{i=1}^{[m/2]}\Gamma(k+2s-i)}{\Gamma_{m-q}(k+s)\Gamma_{m-p}(s)\Gamma\left(s+\frac{k-m+r/2+\mu}{2}\right)\prod_{i=1}^{[(m-r)/2]}\Gamma(k+2s-m+i+(r-1)/2)},$$

and for $\varepsilon(r) = 1$

$$\Gamma(h,s) = \\ = \frac{\Gamma_{m-r}\left(k+2s-\frac{m+1}{2}\right)\Gamma\left(s+\frac{k+\delta}{2}\right)\prod_{i=1}^{[m/2]}\Gamma(k+2s-i)}{\Gamma_{m-q}(k+s)\Gamma_{m-p}(s)\prod_{i=1}^{[(m-r-1)/2]}\Gamma(k+2s-m+i+r/2)},$$

(c)The ratio of Dirichlet L -functions RL^* . *Let for a Dirichlet character* χ *modulo* N *of a parity* $\delta = 0$ *or 1*

$$L^*_N(s,\chi) = \Gamma((k+\delta)/2)L_N(s,\chi) = \Gamma((k+\delta)/2)\prod_{q\nmid N}(1-\chi(q)q^{-s})^{-1} \tag{3.42}$$

denote the normalized Dirichlet L-function, which is regular for all $s \in \mathbf{C}, s \neq 1$, *including* $s = 0$ *(due to the condition* $N > 1$*). Next we define an additional quadratic Dirichlet character* χ_h *depending on* $h \in A_m$ *and defined only for even* $r \neq 0$. *Namely, for* $h = 0$ *let* $\chi_h = \chi_0$ *be trivial; for* $h \neq 0$ *we know that for some matrix* $u \in \mathrm{GL}_m(\mathbf{Q})$

$${}^t uhu = \begin{pmatrix} h_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ with } \det h_1 \neq 0, \tag{3.43}$$

then let χ_h *denote the quadratic character attached to the quadratic field* $\mathbf{Q}(\sqrt{\det h_1})/\mathbf{Q}$ *(this definition is independent on the choice of the matrix* u*). Under these notation we put:*

for an even r (i.e. with $\varepsilon(r)=0$)

$$RL^*(h,\chi,k+2s)=$$
$$\frac{L_N^*(k+2s-m+(r/2),\chi\chi_h)\prod_{i=1}^{[(m-r)/2]}L_N^*(2k+4s-2m+r-1+2i,\chi^2)}{L_N^*(k+2s,\chi)\prod_{i=1}^{[m/2]}L_N^*(2k+4s-2i,\chi^2)},$$

and for an odd r (i.e. with $\varepsilon(r)=1$)

$$RL^*(h,\chi,k+2s)=$$
$$\frac{\prod_{i=1}^{[(m-r-1)/2]}L_N^*(2k+4s-2m+r-1+2i,\chi^2)}{L_N^*(k+2s,\chi)\prod_{i=1}^{[m/2]}L_N^*(2k+4s-2i,\chi^2)}.$$

(d) The integral factor

$$M(h,\chi,k+2s)=\prod_{q\in P(h)}M_q(h,\chi(q)q^{-k-2s}) \tag{3.44}$$

is a finite Euler product , extended over prime numbers q from the set $P(h)$ of prime divisors of the number N and of all elementary divisors of the matrix h. The important property of the product is that for each q we have that $M_q(h,t)\in\mathbf{Z}[t]$ is a polynomial with integral coefficients. The explicit form of it is insignificant for our purposes; however , one can find interesting explicit formulas for such polynomials in [Rag3], [Ki2].

3.5. Normalized Siegel-Eisenstein series. We introduce here three types of normalized Siegel-Eisenstein series in order to give a precise statement on their holomorphy properties with respect to the variable s , the properties of positivity of matrices ξ enumerating the summands in their Fourier expansions, and also the algebraicity properties of the Fourier coefficients:

$$G^*(z,s)=G^*(z.s;k,\chi,N)=$$
$$N^*m(k+2s)/2i^{mk}2^{-m(k+1)}\pi^{-m(s+k)}\Gamma(1_m,s)^{-1}\times$$
$$\times L_N^*(k+2s,\chi)\prod_{i=1}^{[m/2]}L_N^*(2k+4s-2i,\chi^2)E(-(Nz)^{-1},s)\det(\sqrt{N}z)^{-k}= \tag{3.45}$$
$$N^{m(k+2s)/2}\tilde{\Gamma}(k,s)L_N(k+2s,\chi)\prod_{i=1}^{[m/2]}L_N(2k+4s-2i,\chi^2)E^*(Nz,s),$$

with

$$E^*(Nz,s) = E(-(Nz)^{-1},s)\det(Nz)^{-k} = N^{-km/2}E|W(N),$$

$$\tilde{\Gamma}(k,s) =$$

$$i^{mk}2^{-m(k+1)}\pi^{-m(s+k)}\Gamma(1_m,s)^{-1}\Gamma((k+2s+\delta)/2)\prod_{j=1}^{[m/2]}\Gamma(k+2s-j) =$$

$$i^{mk}2^{-m(k+1)}\pi^{-m(s+k)} \times \begin{cases} \Gamma_m(k+s)\Gamma(s+(k-(m/2)+\mu)/2), & \text{if } m \text{ is even;} \\ \Gamma_m(k+s), & \text{otherwise.} \end{cases} \tag{3.46}$$

If m is odd then we put $G^+(z,s) = G^-(z,s) = G^*(z,s)$. If m is even then we define (with $\mu = \varepsilon((m/2)+k)$)

$$G^-(z,s) = \Gamma((k+2s-(m/2)+\mu)/2)^{-1}G^*(z,s), \tag{3.47}$$

$$G^+(z,s) = \frac{i^\mu\pi^{(1/2)-k-2s+(m/2)}}{\Gamma((1-k-2s+(m/2)+\mu)/2)}G^*(z,s) =$$

$$\frac{2i^\mu\Gamma(k+2s-(m/2))\cos(\pi(k+2s-(m/2)-\mu)/2)}{(2\pi)^{k+2s-(m/2)}}G^-(z,s). \tag{3.48}$$

We will see in §3 of the next chapter that the normalizing factors in formulas (3.45), (3.47) and (3.48) are closely connected to those of the Dirichlet L-series and the standard zeta functions (for even m).

Now we reformulate proposition 3.4 for the normalized series $G^*(z,s)$

$$G^*(z,s) = \sum_{h\in A_m} b^*(h,y,s)e_m(hz), \tag{3.49}$$

where

$$b^*(h,y,s) = W^*(y,h,s)\Gamma^*(h,s)L_N^*(h,\chi,k+2s)M(h,\chi,k+2s),$$

with

$$\Gamma^*(h,s) = \Gamma^-(1_m,s)^{-1}\Gamma^-(h,s),$$

$$W^*(y,h.s) = N^{m(s+k-\kappa)}i^{mk}2^{-m(k+1)}\pi^{-m(s+k)}e_m(-ihy)W(Ny,N^{-1}h,s) =$$

$$i^{mk}2^{-m(k+1)}\pi^{-m(s+k)}e_m(-ihy)W(y,h,s).$$

The factor $M(h,\chi,k+2s)$ is given by (3.44) ,and for r is even we have that

$$L_N^*(h,\chi,k+2s) = L_N^*(k+2s-m+(r/2),\chi\chi_h)\prod_{i=1}^{[(m-r)/2]} L_N^*(2k+4s-2m+r-1+2i,\chi^2)$$

and for r odd

$$L_N^*(h,\chi,k+2s) = \prod_{i=1}^{[(m-r-1)/2]} L_N^*(2k+4s-2m+r-1+2i,\chi^2)$$

3.6. Theorem *(on holomorphy properties of the Fourier coefficients of the Siegel-Eisenstein series, see [Fe], theorem 9.1) Let $2|N, N>1$. Then*
(a) If $\chi^2 \neq 1$ then the function $G^*(z,s)$ is an entire function of the variable s;
(b) Suppose that χ^2 is trivial, then we have

(b_1) if either $2k \geq m$ and m odd or $2k \geq m$ and m is even , but $(m/2)+k$ is odd (i.e. $\mu = 1$), then the function $G^*(z,s)$ is an entire function of the variable s;

(b_2) if $2k \geq m$ and both numbers m and $(m/2)+k$ are even (i.e. $\mu = \varepsilon(m) = 0$) then the function $G^*(z,s)$ is an entire function of the variable s with the exclusion of a possible simple pole at the point $s = (m+2-2k)/4$;

(b_3) if $m > 2k \geq 0$ then the function $G^*(z,s)$ is an entire function of the variable s with possible exclusion of simple poles at those points s for which $2s$ is an integer and $[(m-2k+3)/2] \leq 2s \leq (m+1-2k)/2$;

(b_4) if $k=0$ then the function $G^*(z,s)$ has a simple pole at the point $s=(m+1)/2$ iff χ is trivial, and in this case we have that the function

$$\mathrm{Res}_{s=(m+1)/2} G^*(z,s;0,1,N)$$

of the variable z is a non-zero constant.

3.7. Theorem. *(On positivity properties of Fourier expansions of the normalized Siegel–Eisenstein series). Assume that $2k > m$ and define the numbers $A(\chi)$, $B(\chi)$, $C(\chi,k)$ as follows:*

(a) If χ^2 is not trivial , m is even and $\mu = \varepsilon(k+(m/2))$ then put

$$A(\chi) = B(\chi) = 1+(m/2), \quad C(\chi,k) = (m-2k+2-2\mu)/4;$$

(b) If χ^2 is trivial , m is even and $\mu = \varepsilon(k+(m/2))$ then put

$$A(\chi) = B(\chi) = (m/2), \quad C(\chi,k) = (m-2k+2-2\mu)/4;$$

(c) If χ^2 is not trivial, m is odd,then put

$$A(\chi) = B(\chi) = (m+3)/2, \quad C(\chi,k) = [(1+m-2k)/4];$$

(d) If χ^2 is trivial, m is odd,then put

$$A(\chi) = (m+5)/2, B(\chi) = (m+1)/2, \quad C(\chi,k) = [(3+m-2k)/4];$$

under these notation and assumptions the following positivity properties of matrices $h \in A_m$ enumerating the Fourier coefficients of the series $G^*(z,s)$ are valid:

1) if $s \leq 0, s \in \mathbf{Z}$ and $k+2s \geq A(\chi)$ then

$$G^*(z,s) = \sum_{A_m \ni h > 0} b^*(h,y,s) e_m(hz), \tag{3.50}$$

2)if $k+s-\kappa \in \mathbf{Z}, k+s-\kappa \geq 0 \quad (\kappa=(m+1)/2), s \leq C(\chi,k)$ *then*

$$G^*(z,s) = \sum_{A_m \ni h \geq 0} b^*(h,y,s) e_m(hz), \tag{3.51}$$

The proof of the theorem 3.7 is contained in the book of P.Feit [Fe], theorems 14.1.A-14.1.F and is based on a detailed investigation of poles and residues of the Γ-factor $\Gamma^*(h,s)$ and of the Dirichlet L-function $L^*(h,\chi,k+2s)$, carried out in [Fe] in terms of the functions $f(n,s) = \Gamma(n+s)/\Gamma(s)$, which for positive integers n turn out to be polynomials with zeros given by $s \in /Z, s \leq 0, s+n > 0$. It was established in [Fe] that the factor $\Gamma^*(g,s) = \Gamma(h,s)/\Gamma(1_m,s)$ is equivalent (up to multiplication by an invertible entire function) to a certain explicitly given polynomial in $\mathbf{C}[s]$ (see [Fe], §11). It follows also from this calculation that

(a) if χ^2 is trivial, m is odd and $s = s_0 = (m+2-2k)/4$ then the function $G^*(z,s)$ has a pole at the point $s = s_0$ such that the residue

$$\mathrm{Res}_{s=s_0} G^*(z,s) \text{ has a non negative Fourier expansion} \tag{3.52}$$

(theorem 14.1.C);

(b) if χ^2 is trivial and m is odd then the function $G^*(z,s)$ is finite at the point $s = s_0$ and has a non negative Fourier expansion.

It is essential for our purposes to reformulate the corresponding statements for the series $G^+(z,s)$ and $G^-(z,s)$ (see(3.47),(3.48)), which are obtained from $G^*(z,s)$ by an additional normalization . The following theorem is an immediate consequence of the theorem 3.6 on holomorphy and the properties (3.50)-(3.52) .

3.8. Theorem *(on Fourier coefficients with positive matrix numbers). Let* m *is even,* $2k > m$*. Then:*

(a) For $2s$ *to be an integer,* $s \leq 0, k+2s \geq 1+(m/2)$ *there is the following Fourier expansion*

$$G^+(z,s) = \sum_{A_m \ni h > 0} b^+(h,y,s) e_m(hz), \tag{3.53}$$

where for $s > (m+2-2k)/4$ *in (3.54) non-zero terms only occur for positive definite* $h > 0$*, and for all* s *from (a) with* $h > 0, h \in A_m$ *the following identity holds*

$$b^+(h,y,s) = W^*(y,h,s) L_N^+(k+2s-(m/2), \chi\chi_h) M(h,\chi,k+2s),$$

with

$$L^+(s,\chi) = \frac{2i^\delta \Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi)^s} L_N(s,\chi)$$

is the normalized Dirichlet L-function , $\delta = 0$ *or* 1 *according to* $\chi(-1) = (-1)^\delta$*, the factor* $M(h,\chi,k+2s)$ *being defined by (3.44),*

$$W^*(y,h,s) = 2^{-m\kappa} \det h_{k+2s-\kappa} \det(4\pi y)^s R(4\pi hy; -s; \kappa-k-s),$$

provided s *is an integer, with* $R(y;n,\beta)$ *defined by (3.33), and* $b^+(h,y,s) = 0$ *if* $s \notin \mathbf{Z}$.

(b)If 2s is an integer with $k+2s \leq m/2, k+s \geq \kappa$ then there is the following Fourier expansion

$$G^-(z,s) = \sum_{A_m \ni h \geq 0} b^-(h,y,s)e_m(hz), \tag{3.53a}$$

and for all s from (b) with $h > 0, h \in A_m$ the following identity holds

$$b^-(h,y,s) = W^*(y,h,s)L_N^-(k+2s-(m/2),\chi\chi_h)M(h,\chi,k+2s),$$

where

$$L^+(s,\chi) = L_N(s,\chi),$$
$$W^*(y,h,s) = 2^{-m\kappa}\det(4\pi y)^{\kappa-k-s}R(4\pi hy;-s;k+s-\kappa,s),$$

provided $s+k-\kappa$ is an integer, and $b^-(h,y,s)=0$ otherwise.

The proof is deduced from the expansions (3.49) if we remember the definition of the normalizing factors and the positivity property from the theorem 3.7. We also note that by (3.41)

$$W^*(y,h,s) = e_m(-ihy)\omega(2\pi y,h;k+s,s)\times$$
$$\times(\det y)^{\kappa-k-s}\delta_+(hy)^{k+s-\kappa+q/4}\delta_-(hy)^{s-\kappa+p/4},$$

and then take into account the formula (3.38) for the critical values of the function ω. In case of the odd parity $2s \in \mathbf{Z}$ we get vanishing of the Fourier coefficients because of the Γ-factors in (3.47), (3.48).

§4. Holomorphic projection operator and the Maass differential operator

4.1.Holomorphic projection operator. We start with describing a vector space on which this operator acts. A function

$$F : \mathfrak{H}_m \to \mathbf{C}, \quad F \in C^\infty(\mathfrak{H}_m)$$

is called a C^∞-*modular form* of weight k on the group $\Gamma_0^m(N)$ with a Dirichlet character $\psi \bmod N$ if the following automorphy condition is satisfied:

$$F((az+b)(cz+d)^{-1}) \;=\; \psi(\det d)\det(cz+d)^k F(z)$$

for all

$$\gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^m(N)$$

(compare with §1, (1.11)). The space of functions F with the above condition is denoted by $\tilde{\mathcal{M}}_m^k(N,\psi)$. For all $F \in \tilde{\mathcal{M}}_m^k(N,\psi)$ there is the following Fourier expansion

$$F(z) = \sum_{h \in A_m} A(y,h)e_m(hx), \tag{4.1}$$

where $A(y,h)$ are some C^∞-functions on Y. The Petersson inner product is defined for an arbitrary holomorphic cusp form $f \in S_m^k(N,\psi)$ and $F \in \tilde{\mathcal{M}}_m^k(N,\psi)$ by

$$\langle f,\, F\rangle_N = \int_{\Phi_0(N)} \overline{f(z)}F(z)\det y^{k-m-1}dx\,dy,$$

where $\Phi_0(N) = \mathfrak{H}_m/\Gamma_0^m(N)$ is a fundamental domain for the group $\Gamma_0^m(N)$.

We call a function $F \in \tilde{\mathcal{M}}_m^k(N,\psi)$ a *function of a bounded growth* if for each $\varepsilon > 0$ the following integral converges:

$$\int_X \int_Y |F(z)| \det y^{k-1-m} e^{-\varepsilon \mathrm{tr}(y)} dy\, dx \ < \ \infty \tag{4.2}$$

where

$$X = \{x \in M_m(\mathbf{R}) | {}^t x = x, |x_{ij}| \le 1/2 \text{ for all } i,j\},$$
$$Y = \{y \in M_m(\mathbf{R}) | {}^t y = y > 0\}.$$

Respectively, we call a function $F \in \tilde{\mathcal{M}}_m^k(N,\psi)$ a *function of a moderate growth* if for all $z \in \mathfrak{H}_m$ and for all sufficiently large values of $\mathrm{Re}(s) \gg 0$ the integral

$$\int_{\mathfrak{H}_m} F(w) \det(\overline{w} - z)^{-k-|2s|} \det \mathrm{Im}(w)^{k+s} d^\times w \tag{4.3}$$

is absolutely convergent and admits an analytic continuation over s to the point $s = 0$. The last definition may differ from a traditional one; its meaning is clarified by the following result (theorem 4.2), which provides a refinement of theorem 1 of the Sturm's paper [St2]. It will follow from the proof that all functions of bownded growth automatically turn out to be of a moderate growth in the sense of definitions (4.2), (4.3) given above.

4.2.Theorem. *Let $F \in \tilde{\mathcal{M}}_m^k(N,\psi)$ and $k > 2m$. Put for $h > 0, h \in A_m$*

$$a(h) = c(k,m)^{-1} \det(4h)^{k-(m+1)/2} \int_Y A(y,h) e_m(ihy) \det y^{k-1-m}\, dy, \tag{4.4}$$

with

$$c(t,m) = \Gamma_m(t-(m+1)/2)\pi^{-m(t-(m+1)/2)},$$

and $A(y,h)$ being coefficients of the expansion (4.1) and suppose that the integral (4.4) is absolutely convergent. Define the function

$$\mathcal{H}ol\, F(z) = \sum_{A_m \ni h > 0} a(h) e_m(hz). \tag{4.5}$$

Then

(a) if the function $F \in \tilde{\mathcal{M}}_m^k(N,\psi)$ is of a bounded growth then that $\mathcal{H}ol\, F(z) \in \mathcal{S}_m^k(N,\psi)$.

(b) If the function $F \in \tilde{\mathcal{M}}_m^k(N,\psi)$ is of a moderate growth and the expansion (4.1) contains only terms with positive definite matrices $h \in A_m$,then we have that $\mathcal{H}ol\, F(z) \in \mathcal{M}_m^k(N,\psi)$.

In both cases the following equality is valid:

$$\langle g, F\rangle_N \ = \ \langle g, \mathcal{H}ol\, F\rangle_N. \tag{4.6}$$

Remark. *The cusp form $\mathcal{H}ol\, F$ is uniquely defined by (4.6) under the assumptions of (a), but in (b) this equality is not sufficient to identify the modular form $\mathcal{H}ol\, F$. For example, (4.6) does not change if we replace this modular form by adding to it an*

Eisenstein series (of Siegel or of Klingen type). the part (a) of the theorem 4.2 was established by Sturm [St2].

4.3. Poincaré series of two variables (of exponential type) of higher level. In order to prove the theorem 4.2 we use this kind of Poincaré series introduced by Klingen [Kl3] and used by Böcherer in [Bö] instead of Poincaré series of one variable from [St2], [Gr-Za]. We consider an element $(\Gamma g \Gamma)$ of the Hecke algebra $\mathcal{L}^m(N)$ with $\Gamma = \Gamma_0^m(N)$,

$$g \in \Delta = \Delta_q^m =$$
$$\{\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbf{Q}+} \cap \mathrm{GL}_2m(\mathbf{Z}[q^{-1}]) | \nu(\alpha)^{\pm} \in \mathbf{Z}[q^{\pm}], c \equiv 0_m (\mathrm{mod} N)\},$$

where ${}^t g J_m g = \nu(g) J_m,\ \nu(g) > 0$.Put

$$P_m^k(z, w, g, s) = \sum_{g \in \Gamma g \Gamma} \psi(\det a) j(\gamma, z)^{-k-|2s|} \det(\gamma(z) + w)^{k-|2s|}, \tag{4.7}$$

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma g \Gamma$ and $u_{k-|2s|} \stackrel{\text{def}}{=} u^{-k}|u|^{-2s}$ for $u \in \mathbf{C}^{\times}, s \in \mathbf{C}$. The series in (4.7) converges absolutely and uniformly on products of the type $V_m(d) \times V_m(d)$ for $k + \mathrm{Re}(2s) > 2m + 1,\ d > 0$,

$$V_m(d) = \left\{ z = x + iy \in \mathfrak{H}_m | y \geq d 1_m, \mathrm{tr}({}^t x\, x) \leq \frac{1}{d} \right\}.$$

We also put

$$\mathcal{P}_m^k(z, w, g, s) = \det \mathrm{Im} z^s \det \mathrm{Im}\, w^s P_m^k(z, w, g, s). \tag{4.8}$$

The following properties of these series were established by Böcherer in [Bö]:

(a) symmetry

$$\mathcal{P}_m^k(z, w, g, s) = \mathcal{P}_m^k(w, z, g, s) \tag{4.9}$$

(b) automorphy with respect to both arguments:

$$\begin{gathered} \mathcal{P}_m^k(\gamma(z), \gamma(w), g, s) = \\ \psi(\det d_1) \psi(\det d_2) j(\gamma_1, z)^k (\gamma_2, w)^k \mathcal{P}_m^k(z, w, g, s), \end{gathered} \tag{4.10}$$

where $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \Gamma,\ i = 1, 2;$

(c) action of Hecke operators:

$$\begin{gathered} \mathcal{P}_m^k(z, 1_m, g, s)|_{k,\psi}(\Gamma g \Gamma)_z = \mathcal{P}_m^k(z, w, g, s) = \\ \mathcal{P}_m^k(z, w, 1_m, s)|_{k,\psi}(\Gamma g \Gamma)_w, \end{gathered} \tag{4.11}$$

where the subscript indicates to which of the variables the Hecke operator is being applied with the action defined by (1.21).

(d) the integral representation: for all $f \in \mathcal{S}_m^k(N, \psi)$ we have that

$$\langle \mathcal{P}_m^k(-\bar{z}, w, g, s), f(w) \rangle_{N,w} = \mu(m, k, s) f|_{k,\psi}(\Gamma g \Gamma)(z), \tag{4.12}$$

with
$$\mu(m,k,s) = 2^{m+\frac{m(m+1)}{2}-2ms+1} i^{-mk} \pi^{\frac{m(m+1)}{2}} \frac{\Gamma_m(k+s-(m+1)/2)}{\Gamma_m(k+s)}.$$

The proof of the properties (4.9), (4.11) is easily deduced from the symmetry relation
$$j(\gamma, z)\det(\gamma(z)+w) = j(\tilde{\gamma}, w)\det(\tilde{\gamma}(w)+z), \tag{4.13}$$
which is valid for all $\gamma \in \mathrm{Sp}_m(\mathbf{R})$ with
$$\tilde{\gamma} = \gamma\left[\begin{pmatrix} 1_m & 0_m \\ 0_m & -1_m \end{pmatrix}\right]$$
and $\Gamma g \Gamma = \Gamma \tilde{g} \Gamma$ for $g \in \Delta$.Then (4.10) is immediately deduced from the definition (4.7). Proof of the integral formula (4.12) is carried out similarly to that in Klingen's article [Kl5]; for this purpose we may admit that $g = 1_m$. The integration in the left hand side of (4.12) is reduced by a standard unfolding procedure to that over the whole Siegel upper half plane
$$\mathfrak{H}_m = \cup_\gamma \gamma(\Phi_0(N)), \quad \gamma \in \Gamma = \Gamma_0^m(N),$$
where $\Phi_0(N) = \mathfrak{H}_m/\Gamma_0^m(N)$ is a fundamental domain for the group $\Gamma_0^m(N)$. The required property follows from the integral representation
$$\begin{aligned}\int_{\mathfrak{H}_m} f(w)\det(\overline{w}-z)^{-k-|2s|}\det\mathrm{Im}(w)^{k+s}d^\times w = \\ i^{mk}2^{m(m+1)-2ms-mk}I_m(k+s-m-1)f(z)\det\mathrm{Im}(z),\end{aligned} \tag{4.14}$$
where
$$\begin{aligned}I_m(s) = \int_{E_m}\det(1_m-\overline{w}w)^s du\,dv = \\ \pi^{\frac{m(m+1)}{2}}2^{-\frac{m(m+1)}{2}}\frac{\Gamma_m(s+1+(m-1)/2)}{\Gamma_m(s+m+1)}\end{aligned} \tag{4.15}$$
denotes the integral investigated by Hua Lo-Ken [HLK], which is absolutely convergent for $\mathrm{Re}(s) > -1$, $f \in \mathcal{S}_m^k(N,\psi)$, the integration in (4.14) being taken over the generalized unit disc
$$E_m = \{w = u+iv \in \mathrm{M}_m(\mathbf{C}) | {}^t w = w, 1_m - \overline{w}w > 0\}$$
of the degree m , the image of $\mathfrak{H}_m$ via the Cayley transform
$$w \;\mapsto\; (w-i1_m)(w+i1_m)^{-1} \quad (w \in \mathfrak{H}_m).$$
In order to prove (4.14) we note that if $f \in \mathcal{S}_m^k(N,\psi)$ then there is the following upper estimate
$$f(z) \le c\det\mathrm{Im}(w)^{-k/2}$$
and the integral in (4.14) is majorated by
$$\int_{\mathfrak{H}_m} |\det(\overline{w}-z)|^{-k-2\mathrm{Re}(s)}\det\mathrm{Im}(w)^{(k/2)+\mathrm{Re}(s)}d^\times w,$$

which provides the absolute convergence of (4.14) in this domain for $(k/2)+\mathrm{Re}(s) > m$. Next we rewrite the integrand in (4.14) in the form

$$g(w)\det(\overline{w}-z)^{-k-s}\det\mathrm{Im}(w)^{(k+s)}$$

with the holomorphic function

$$g(w) = f(w)\det(w-\overline{z})^{-s},$$

which is integrated by use of the Cayley transform , so that (4.14) follows.

4.4. Reduction of the theorem 4.2 to the properties of Poincaré series. We restrict ourselves to the case of functions F satisfying to assumptions of (b). Put in formulas (4.10), (4.12) $g = 1_m$ and define a function of two variables $K^k_m(z,w,s)$ by the equality

$$K^k_m(z,w,s) = \mu(m,k,s)^{-1}\mathcal{P}^k_m(-\overline{z},w,1_2m,s). \tag{4.16}$$

We show that the function

$$\mathcal{H}ol\,F(z) = \langle K^k_m(z,w,s),F(w)\rangle_{N,w}|_{s=0} \tag{4.17}$$

obtained by analytic continuation of the right hand side to the point $s=0$ satisfies to all conditions of the theorem, i.e. it coincides with the function defined by (4.4) ,(4.5); $\mathcal{H}ol\ F \in \mathcal{M}^k_k(N,\psi)$ and the equality

$$\langle g,\,F\rangle_N \;=\; \langle g,\,\mathcal{H}ol\,F\rangle_N.$$

holds for all $g \in \mathcal{S}^m_k(N,\psi)$. For this purpose we note that for sufficiently large value of $\mathrm{Re}(s)$ the right hand side of (4.17) can be rewritten in the form of an integral over the whole Siegel upper half plane $\mathfrak{H}_m$ of degree m:

$$\begin{aligned}&\langle K^k_m(z,w,s),F(w)\rangle_{N,w} =\\ &\mu(m,k,s)^{-1}\det\mathrm{Im}(z)^s\int_{\mathfrak{H}_m} F(w)\det(\overline{w}-z)^{-k-|2s|}\det\mathrm{Im}(w)^{k+s}d^{\times}w\end{aligned} \tag{4.18}$$

(due to the assumption on the growth of F the integral (4.18) is absolutely convergent for all $\mathrm{Re}(s) \gg 0$). Next let us consider the subgroup

$$\Gamma^0_\infty = \left\{\gamma = \pm\begin{pmatrix}1_m & b\\ 0 & 1_m\end{pmatrix}|\gamma\in\Gamma\right\} \subset \Gamma = \Gamma^m_0(N).$$

Then the set

$$\{w = u+iv \in \mathfrak{H}_m | u \in X, v \in Y\}$$

is a fundamental domain for the action of Γ^0_∞ on $\mathfrak{H}_m$, and we see that for $\mathrm{Re}(s) \gg 0$ the right hand side of (4.18) takes the form

$$2\mu(m,k,s)^{-1}\det \mathrm{Im}(z)^s \int_X \int_Y F(w) \sum_{b \in L} \det(\overline{w} - z + b)^{-k-|2s|} \det \mathrm{Im}(w)^{k+s} d^\times w =$$

$$2\mu(m,k,s)^{-1}\det \mathrm{Im}(z)^s \int_X \int_Y F(w) S(\overline{w} - z, L; k+s, s) \det \mathrm{Im}(w)^{k+s} d^\times w, \tag{4.19}$$

where $L = \mathrm{M}_m(\mathbf{Z}) \cap V$ is a lattice in $V = \{x \in \mathrm{M}_m(\mathbf{R}) | {}^t x = x\}$, and the function

$$S(z, L; k+s, s) \;=\; \sum_{b \in L} \det(z+b)^{-k-|2s|}$$

admits an analytic continuation to all $s \in \mathbf{C}$ by means of the fourier expansion of (4.14) and for $k > m$ we have that

$$S(\overline{w} - z, L; k+s, s)|_{s=0} = (-2\pi i)^{mk} \Gamma_m(k)^{-1} \sum_{A_m \ni h > 0} \det h_{k-\kappa} e_m(f(z - \overline{w})) \tag{4.20}$$

Under the growth assumption on F the integral admits an analytic continuation to the point $s = 0$. This analytic continuation can be explicitly given in the form of a Fourier expansion by means of (4.19), (4.20) using the positivity of h . As a result the function $\mathcal{H}ol\, F$ takes the form

$$\mathcal{H}ol\, F(z) = (4\pi)^{m(k-(m+1)/2} \Gamma_m(k - (m+1)/2)^{-1} \times$$
$$\sum_{A_m \ni h > 0} \det h^{k-(m+1)/2} \int_X \int_Y F(z) e_m(h(z - \overline{w})) \det \mathrm{Im} w^k d^\times w.$$

Then the formula (4.4) follows from the obvious equality

$$\int_X \int_Y F(z) e_m(h(z - \overline{w})) \det \mathrm{Im} w^k d^\times w =$$
$$= e_m(hz) \int_Y A(v,h) e_m(ihv) \det v_{k-1-m} dv.$$

In order to prove the remaining statements we note that the function $\mathcal{H}ol\, F(z)$ defined by (4.17) is holomorphic and satisfies the automorphic properties with respect to $\Gamma^m_0(N)$ of weight k with the Dirichlet character $\psi \bmod N$. Indeed , these properties are satisfied by the function $K^k_m(z,w,s)$ and consequently by (4.18) for $\mathrm{Re}(s) \gg 0$. But the identity expressing the automorphy condition (1.11) does not change by analytic continuation , and we get $\mathcal{H}ol\, F \in \mathcal{M}^m_k(N, \psi)$ (for $m > 1$ also the Koecher principle is applicable) the equality (4.6) is then deduced from the definitions (4.16) and (4.17) , and from the automorphy property (4.12) of the Poincaré series. For Re $\gg 0$ we get

$$\langle g, \langle K^k_m(z,w,s), F(w) \rangle_{N,w} \rangle_{N,z} = \langle g,\, F \rangle_N =$$
$$= \langle \overline{K^k_m(z,w,s)}, g(z) \rangle_{N,z}, F(z) \rangle_{N,w}. \tag{4.21}$$

These equalities remain valid by the analytic continuation and we get (4.6). In the equality (4.21) the property

$$\overline{K_m^k(z,w,s)} \quad = \quad K_m^k(z,w,\overline{s}).$$

was taken into account.

4.5. When using the formula (4.4) ,it is convenient to keep in mind the integral representation (3.9) for the Γ-function of degree m ,

$$\Gamma_m(s) = \pi^{m(m+2)/4} \prod_{j=0}^{m-1} \Gamma(s-(j/2)),$$

This integral representation can be rewritten in the equivalent form

$$\begin{aligned}\Gamma_m(\nu-(m+1)/2)\det u^{(m+1)/2-\nu} &= \int_Y (\det y)^{\nu-m-1} e^{-\mathrm{tr}(uy)}\, dy = \\ &= \int_Y (\det y)^{\nu-(m+1)/2} e_m(i(2\pi)^{-1}uy)\, d^\times y.\end{aligned} \tag{4.22}$$

Moreover if $R(y) \in \mathbf{C}[y_{ij}]$ is a polynomial of $y = y_{ij},\ i \le j$ then for all $\nu \in \mathbf{Z}, \nu > m$ we have that

$$\begin{aligned}&\int_Y R(y)(\det y)^{\nu-(m+1)/2} e^{-\mathrm{tr}(uy)}\, d^\times y = \\ &\int_Y [R(\partial/\partial u)e^{-\mathrm{tr}(uy)}](\det y)^{\nu-(m+1)/2}\, d^\times y = \\ &R(\partial/\partial u)[\Gamma_m(\nu-(m+1)/2)\det u^{(m+1)/2-\nu}],\end{aligned} \tag{4.23}$$

where $\partial/\partial u \ = \ \partial_{ij},\ \partial_{ij} = 2^{-1}(1+\delta_{ij}\partial/\partial u_{ij})$. Indeed, it suffices to check the statement (4.23) for monomials of the form

$$R(y) = \prod_{i \le j} y_{ij}^{a(i,j)}, \quad a(i,j) \in \mathbf{Z},\ a(i,j) \ge 0.$$

In this particular case this is done by application of the differential operator

$$\prod_{i \le j} \partial/\partial u_{ij}^{a(i,j)}$$

to both sides of the equality (4.22). We will formulas (4.4) and (4.23) in a special situation described in the theorem below.

4.6.Theorem. *Let C^∞-modular form $F \in \mathcal{M}_k^m(N,\psi)$ be a product of the type $F(z) = g(z)G(z)$, where*

$$g(z) = \sum_{A_m \ni h > 0} B(h) e_m(hz),$$

$$G(z) = \sum_{A_m \ni h \geq 0} C(h) \det(4\pi y)^{-n} R(4\pi hy; n, \beta) e_m(hz),$$

and $F(z)$ satisfy one of the two conditions (a) or (b) of the theorem 4.2 and $R(z; n, \beta)$ denotes the polynomial (3.33) defined for any integer $n \geq 0$, $\beta \in \mathbf{C}$ and $z = {}^t z \in \mathrm{M}_n(\mathbf{C})$ by

$$R(z; n, \beta) = (-1)^{mn} e^{\mathrm{tr}(z)} \det(z)^{n+\beta} \Delta_m^n [e^{\mathrm{tr}(z)} \det(z)^{-\beta}],$$

where

$$\Delta_m = \det(\partial_{ij}), \quad (\partial_{ij} = 2^{-1}(1 + \delta_{ij}) \partial / \partial_{ij} z, \quad i \leq j)$$

being the Maass differential operator. Then the following equality holds

$$\mathcal{H}ol\, F(z) = \sum_{A_m \ni h = h_1 + h_2 > 0} B(h_1) C(h_2) P(h_2, h; n, \beta) e_m(hz), \tag{4.24}$$

where $P(v, u) = P(v, u; n\beta)$ denotes a polynomial of $u = {}^t u = (u_{ij})$ and $v = {}^t v = (v_{ij})$ with the property

$$P(v, u; n, \beta) \equiv \det v^n (\mathrm{mod} \langle u_{ij} \rangle) \tag{4.25}$$

and $P(v, u; n, \beta) \in \mathbf{Q}[u, v]$ for any $\beta \in \mathbf{Q}$.

Proof of the theorem 4.6 is carried out by straightforward application of the integral formula (4.4) for the action of $\mathcal{H}ol$ to each of the Fourier coefficients of the function $F(z)$:

$$A(y, h) = \sum_{A_m \ni h = h_1 + h_2 > 0} B(h_1) C(h_2) \det(4\pi y)^{-n} R(4\pi h_2 y; n, \beta) e_m(ihz),$$

so that we get

$$A(h) = \sum_{A_m \ni h = h_1 + h_2 > 0} B(h_1) C(h_2) P(h_2, h; n, \beta),$$

where

$$P(v,u) = P(v,u;n,\beta) =$$

$$\frac{\det(4\pi u)^{k-(m+1)/2}}{\Gamma_m(k-(m+1)/2)} \int_Y R(4\pi vy;n,\beta)\det(4\pi y)^{-n}\det y^{k-(m+1)/2} e_m(2iuy)d^\times y =$$

$$\frac{\det(4\pi u)^{k-(m+1)/2}}{\Gamma_m(k-(m+1)/2)} \int_Y R(4\pi vy;n,\beta)\det(4\pi y)^{-n+k-(m+1)/2} e_m(2iuy)d^\times y = \tag{4.26}$$

$$\frac{\det(4\pi u)^{k-(m+1)/2}}{\Gamma_m(k-(m+1)/2)} \int_Y R(vy;n,\beta)\det(y)^{-n+k-(m+1)/2} e^{-\mathrm{tr} uy} d^\times y =$$

$$\frac{\Gamma_m(k-n-(m+1)/2)}{\Gamma_m(k-(m+1)/2)} \det u^{k-(m+1)/2} R(v\cdot\partial/\partial u;n,\beta)[\det u^{(m+1)/2-k+n}],$$

with $n \in \mathbf{Z}, n \geq 0, \beta \in \mathbf{C}$. To accomplish the proof it suffices to show that the function $P(v,u) = P(v,u;n,\beta)$ is a polynomial with the desired properties (4.25). This last fact is deduced from the last of the equalities (4.26) and some general properties of the differential operator $\partial/\partial u$ which are given below (see also [Kl5]).

4.7. Let us consider the natural representation

$$\rho_r : \mathrm{GL}_m(\mathbf{C}) \to \mathrm{GL}(\Lambda^r \mathbf{C}^m)$$

of the group $\mathrm{GL}_m(\mathbf{C})$ on the vector space $\Lambda^r\mathbf{C}^m$ and put

$$\rho_r^*(x) = \det(x)\rho_{r-m}{}^t x^{-1} \quad (r = 0,1,\cdots,m)$$

Then the representations ρ and ρ_r^* turn out to be polynomial representations so that for each $x \in \mathrm{M}_m(\mathbf{C})$ the linear operators $\rho_r(x)$, ρ_r^* are well defined. We vconsider the differential operators $\rho_r(\partial/\partial z)$ and $\rho_r^*(\partial/\partial z)$ which associate to each $\mathbf{C}$-valued function on $\mathfrak{H}_m$ a certain $\mathrm{M}_t(\mathbf{C})$ valued function on $\mathfrak{H}_m$ with $t = \binom{m}{r}$. In particular we put

$$\Delta = \rho_m(\partial/\partial z) = \rho_m^*(\partial/\partial z) = \det(2^{-1}(1+\delta_{ij})\partial/\partial_{ij}z).$$

The following differentiation rules are valid (see [Shi8], lemma 9.1):

(a) If f and g are any smooth $\mathbf{C}$-valued functions on $\mathfrak{H}_m$ then

$$\Delta(fg) = \sum_{r=0}^{m} \mathrm{tr}[{}^t\rho_r(\partial/\partial z)f \cdot \rho_{m-r}^*(\partial/\partial z)g] \tag{4.27}$$

(b)

$$\begin{aligned} \rho_r(\partial/\partial z)\det(z)^\alpha &= c_r(\alpha)\det(z)^{\alpha-1}\rho_{m-r}^*(\partial/\partial z), \\ \rho_r^*(\partial/\partial z)\det(z)^\alpha &= c_r(\alpha)\det(z)^{\alpha-1}\rho_{m-r}(\partial/\partial z) \end{aligned} \tag{4.28}$$

for $\alpha \in \mathbf{C}$ with

$$c_r(\alpha) = \prod_{k=0}^{r-1}(\alpha + (k/2)) = \frac{\Gamma_r(\alpha + (r+1)/2)}{\Gamma_r(\alpha + (r-1)/2)} \tag{4.29}$$

Now we define polynomial functions $\lambda_r(X)$ of $X \in \mathrm{M}_m(\mathbf{C})$ by

$$\det(z1_m - X) = \sum_{r=0}^{m}(-1)^r \lambda_r(X) t_{m-r},$$

with a variable t. Then

$$\begin{aligned} &\lambda_r(X) = \mathrm{tr}[\rho_r(X)], \\ &\Delta(e^{\mathrm{tr}(uz)}\det(z)^\alpha) = e^{\mathrm{tr}(uz)}\det(z)^{\alpha-1}\sum_{r=0}^{m} c_{m-r}\lambda_r(uz). \end{aligned} \tag{4.30}$$

To prove (4.30) it is sufficient to use (4.27) with $f = e^{\mathrm{tr}(uz)}, g = \det(z)^\alpha$ and the apply formulas (4.28). Indeed,

$$\begin{aligned} &\Delta(e^{\mathrm{tr}(uz)}\det(z)^\alpha) = \\ &\sum_{r=0}^{m} \mathrm{tr}[{}^t\rho_r(\partial/\partial z)e^{\mathrm{tr}(uz)} \cdot \rho^*_{m-r}(\partial/\partial z)\det(z)^\alpha] = \\ &\sum_{r=0}^{m} c_{m-r}(\alpha)\det(z)^{\alpha-1}\mathrm{tr}[\rho_r(z(\partial/\partial z)e^{\mathrm{tr}(uz)}] = \\ &\sum_{r=0}^{m} c_{m-r}(\alpha)\det(z)^{\alpha-1}\lambda_r(z(\partial/\partial z)e^{\mathrm{tr}(uz)}] = \\ &\sum_{r=0}^{m} c_{m-r}(\alpha)\det(z)^{\alpha-1}\lambda_r(zu) = \\ &\sum_{r=0}^{m} c_{m-r}\lambda_r(uz)(\alpha)\det(z)^{\alpha-1} \end{aligned}$$

(we note that $\mathrm{tr}({}^tab) = \mathrm{tr}({}^tba)$ and that for each $u = {}^tu \in \mathrm{GL}_m(\mathbf{C})$ one has $u = {}^tu \in \mathrm{GL}_m(\mathbf{C})$).

In a very similar way we get the following formula for the action of an iteration of Δ:

$$\begin{aligned} &\Delta^n[e^{\mathrm{tr}(uz)}\det(z)^\alpha] = e^{\mathrm{tr}(uz)}\det(z)^{\alpha-n}\times \\ &\times \sum_{r_1,\cdots,r_n=0}^{m} c_{m-r_1}(\alpha)c_{m-r_2}(\alpha-1)\cdots c_{m-r_n}(\alpha-n+1)\lambda_{r_1}(uz)\cdots\lambda_{r_n}(uz). \end{aligned} \tag{4.31}$$

From this formula we deduce now the following very explicit expression for the polynomial $P(v, u; n, \beta)$, which provides us with all desired informaton about it.

4.8. Theorem. *Under the assumptions and notations as in 4.5 and 4.6 the following equality holds*

$$P(v.u;n,\beta) = \sum_{r_1,\cdots,r_n=0}^{m} C(\beta;r_1,\cdots,r_n)\mathrm{tr}[\rho_{r_1}(v)\rho^*_{m-r_1}(u)]\cdots\mathrm{tr}[\rho_{r_n}(v)\rho^*_{m-r_n}(u)], \tag{4.32}$$

where

$$C(\beta;r_1,\cdots,r_n) = c_{m-r_1}(-\beta)c_{m-r_2}(-\beta-1)\cdots c_{m-r_n}(-\beta-n+1)\times \times(-1)^{mn+r_1+\cdots+r_n}\frac{c_{r_1}(\kappa-k)\cdots c_{r_1}(\kappa-k-n+1)}{c_m(\kappa-k)\cdots c_m(\kappa-k-n+1)},$$

and for $r_1 = r_2 = \ldots = r_n = m$ *we have* $C(\beta;m,\cdots,m) = 1$;

$$\kappa = (m+1)/2,\quad c_r(\alpha) = \prod_{k=0}^{r-1}(\alpha+(k/2)) = \frac{\Gamma_r(\alpha+(r+1)/2)}{\Gamma_r(\alpha+(r-1)/2)} = (-1)^r\frac{\Gamma_r(1-\alpha)}{\Gamma_r(-\alpha)}.$$

It is easily seen from this formula that $P(v,u;n,\beta)\in\mathbf{Q}[u,v]$ *for* $\beta\in\mathbf{Q}$ *and that*

$$P(v,u;n,\beta)\equiv \mathrm{det}v^n(\mathrm{mod}\langle u_{ij}\rangle$$

because of the homogenesity of degree m *of the matrix coefficients of the representations* ρ_r, ρ^*_r.

The proof of the theorem 4.8 is then an immediate consequence of the equalities (4.26) for which in view of (4.31) we have

$$R(z;n,\beta) = \sum_{r_1,\cdots,r_n=0}^{m}(-1)^{r_1+\cdots+r_n}c_{m-r_1}(-\beta)c_{m-r_2}(-\beta-1)\cdots c_{m-r_n}(-\beta-n+1)\times \times\lambda_{r_1}(z)\lambda_{r_2}(z)\cdots\lambda_{r_n}(z),$$

and

$$\frac{\Gamma_m(k-(m+1)/2)}{\Gamma_m(k-n-(m+1)/2)} = (-1)^{mn}c_m(\kappa-k)\cdots c_m(\kappa-k-n+1),$$

and according to formulas (4.28)

$$\lambda_r(v\cdot\partial/\partial u)\mathrm{det}u^\alpha = \mathrm{tr}[\rho_r(v\cdot\partial/\partial u)\mathrm{det}u^\alpha] = c_r(\alpha)\mathrm{tr}[\rho_r(v)\mathrm{det}(u)^{\alpha-1}\rho^*_{m-r}(u)].$$

This completes the proof for both theorem 4.8 and 4.6.

Chapter 2: Non-Archimedean standard zeta functions of Siegel modular forms

In this chapter we give explicit formulas for the special values of the standard zeta function $\mathcal{D}(s,f,\chi)$ of a Siegel cusp form f of even degree m and of weight $k > 2m+2$ and then construct a non-Archimedean interpolation of these special values. Multiplying $\mathcal{D}(s,f,\chi)$ by certain Γ-factors we introduce the normalized standard zeta functions

$$\mathcal{D}^*(s,f,\chi), \quad \mathcal{D}^+(s,f,\chi), \quad \mathcal{D}^-(s,f,\chi),$$

for which we then formulate our results . First we state a theorem on holomorphy properties of the function $\mathcal{D}^*(s,f,\chi)$ (theorem 1.3). This theorem provides a generalization of a result of A.N.Andrianov and V.L.Kalinin [An-K] on analytic properties of the standard zeta function ; it is proved in §3 by use of a detailed study of poles and residues of Eisenstein-Siegel series done by Shimura [Shi10] and P.Feit [Fe]. Then we turn to algebraic properties of the special values

$$\frac{\mathcal{D}^+(s,f,\chi)}{\langle f,f\rangle} \text{ for } s = 1,2,\cdots,k-\nu-m$$

and

$$\frac{\mathcal{D}^-(s,f,\chi)}{\langle f,f\rangle} \; for \; s = 1-k+\nu+m,\cdots,-1,0,$$

where $\langle f,f\rangle$ denotes the Petersson scalar product, $\nu = 0,1$ according as $\chi(-1) = (-1)^\nu$ (Theorem 1.4). The main result of the chapter is contained in Theorem 1.6 establishing the non-Archimedean interpolation of these special values by means of the theory of non-Archimedean integration. We construct the non-Archimedean standard zeta functions $\mathcal{D}^{c+}(x,f)$, $\mathcal{D}^{c-}(x,f)$ as S-adic Mellin transform of certain measures obtained from the special values. In their turn, these measures come from complex valued distributions of §2. After regularization given in §4 these distributions become bounded measures taking (p-adic) algebraic values at certain points and they provide us the non-Archimedean zeta functions of the Theorem 1.6.

§1.Description of the non-Archimedean standard zeta functions

1.1. The set, on which our S-adic zeta functions are defined, is the p-adic analytic Lie group

$$X_S = \mathrm{Hom}_{\mathrm{contin}}(\mathbf{Z}_S^\times, \mathbf{C}_p^\times)$$

where $\mathbf{C}_p = \widehat{\overline{\mathbf{Q}}}_p$ is Tate field (completion of an algebraic closure of the p-adic field $\mathbf{Q}_p$), so that all integers k can be identified with the characters $x_p^k : y \mapsto y^k$. Put

$$U = \left\{x \in \mathbf{Z}_p^\times | x \equiv 1(\mathrm{mod} p^\nu\right\},$$

where $\nu = 1$ or 2 according as $p > 2$ or $p = 2$, then there is the decomposition

$$X_S = X((\mathbf{Z}/p^\nu\mathbf{Z})^\times \times \prod_{q\neq p} \mathbf{Z}_q^\times) \times X(U). \tag{1.1}$$

The analytic structure on X_S is defined by means of the isomorphism

$$\phi : X(U) \xrightarrow{\sim} T = \{t \in \mathbf{C}_p^\times \,|\, |t-1|_p < 1\},$$

with $\phi(x) = x(1+p^\nu), 1+p^\nu$ being topological generator of the multiplicative group U (see §4 of chapter 1). Elements χ of the torsion subgroup $X_S^{\rm tors} \subset X_S$ form a discrete subgroup and can be identified with primitive Dirichlet characters χ with $S(\chi) \subset S$, where $S(\chi)$ is the support $S(C(\chi))$ of the conductor of χ. Recall that every bounded C_p-analytic function F over X_S is uniquely defined by its values $F(\chi_0\chi)$ with χ_0 fixed and χ being taken in X_S with possible exclusion of a finite number of them in each analyticity component of the decomposition (1.1). This condition is satisfied , for example, by the set of characters $\chi \in X_S^{\rm tors}$ with a S-complete conductor (i.e. with $S(\chi) = S$) and even under the additional assumption that χ^2 is non trivial(this remark will be used in the sequel).

Let μ be a bounded $\mathbf{C}_p$-valued measure on $\mathbf{Z}_p^\times$ (see [Man4], [Man6], [V1]), then its non-Archimedean Mellin transform is given by

$$L_\mu(x) = \mu(x) = \int_{\mathbf{Z}_S^\times} x\, d\mu(x) \quad (x \in X_S, \tag{1.2}$$

and defines a bounded $\mathbf{C}_p$-analytic function

$$L_\mu : X_S \to \mathbf{C}_p$$

1.2. Let

$$f = \sum_{\xi > 0} a(\xi) e_m(\xi z)$$

be a Siegel cusp form of the even degree m of weight k on the congruence subgroup

$$\Gamma_0^m(C) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_m(\mathbf{Z}) | c \equiv 0 \bmod C \right\}$$

with a Dirichlet character ψ mod C. Suppose that f is an eigenfunction of the global Hecke algebra

$$\mathcal{L}^m(C) = \otimes_{q \nmid C} \mathcal{L}_q^m(C) \tag{1.3}$$

with the eigenvalue given by a homomorphism $\Lambda : \mathcal{L}^m(C) \to \mathbf{C}$ (i.e. $f|X = \Lambda(X) f$ for all $X \in \mathcal{L}_q^m(C)$). Let

$$\alpha_0(q), \alpha_1(q), \cdots, \alpha_m(q) \tag{1.4}$$

be a $m+1$-tuple of Satake q-parameters, which uniquely determine Λ so that the relation

$$\alpha_0^2(q)\alpha_1(q)\cdots\alpha_m(q) = q^{k-(m(m+1)/2)}\psi(q)^m. \tag{1.5}$$

holds. Recall that the standard zeta function of f with a Dirichlet character χ mod N is defined as the Euler product

$$\mathcal{D}(s, f, \chi) = \prod_{q, q \nmid C} \mathcal{D}^{(q)}(s, f, \chi), \tag{1.6}$$

with

$$D^{(q)}(s,f,\chi)=$$

$$\left\{\left(1-\frac{\chi(q)\psi(q)}{p^s}\right)\prod_{i=1}^{m}\left(1-\frac{\chi(q)\psi(q)\alpha_i(q)^{-1}}{p^s}\right)\left(1-\frac{\chi(q)\psi(q)\alpha_i(q)}{p^s}\right)\right\}^{-1},$$

the product being absolutely convergent for $\mathrm{Re}(s)>1+m$. Together with (1.5) let us consider the following three types of normalized zeta functions

$$\mathcal{D}^*(s,f,\chi)=(2\pi)^{-m(s+k-(m+1)/2)}\Gamma((s+\delta)/2)\prod_{j=1}^{m}\Gamma(s+k-j)\mathcal{D}(s,f,\chi), \qquad (1.7)$$

$$\mathcal{D}^-(s,f,\chi)=(2\pi)^{-m(s+k-(m+1)/2)}\prod_{j=1}^{m}\Gamma(s+k-j)\mathcal{D}(s,f,\chi), \qquad (1.8)$$

$$\mathcal{D}^+(s,f,\chi)=\frac{2\,i^{\delta}\Gamma(s)\cos(\pi(s-\delta)/2)}{2\pi i)^s}\mathcal{D}^-(s,f,\chi) \qquad (1.9)$$

where $\delta=0$ or 1 according as $\psi\chi(-1)=(-1)^{\delta}$.

1.3. Theorem *(on analytic properties of the standard zeta functions). Let χ be a Dirichlet character modulo a positive integer N (not necessarily primitive) and*

$$f=\sum_{A_m\ni\xi>0}a(\xi)e_m(\xi z)\in\mathcal{S}_k^m(C,\psi)$$

be a Siegel cusp form of weight $k\geq m+\nu$ where $\nu=0,1$ with $\chi(-1)=(-1)^{\nu}$ and assume that the condition $C\det 2\xi_0|N$ is satisfied for some matrix ξ_0 such that $a(\xi_0)\neq 0$. Then the function $\mathcal{D}^*(s,f,\chi)$ admits an analytic continuation whioch is holomorphic for all $s\in\mathbf{C}$ with the possible exclusion of a simple pole at the point $s=1$ in case when the character $\chi^2\psi^2$ is trivial.

This theorem is proven in §3 by means of the detailed study of poles and residues of the Siegel- Eisenstein series as functions of the variable s, see 3.7 of chapter 1, and also [Fe], [Shi10].

1.4.Theorem *(Algebraic properties of the special values of standard zeta functions)*

a) For all integers s with $1\leq s\leq k-\delta-m$ and χ^2 non-trivial for $s=1$ we have that

$$\langle f,\ f\rangle^{-1}\,\mathcal{D}^+(s,f,\chi)\in K=\mathbf{Q}(f,\Lambda_f,\psi,\chi),$$

where $K=\mathbf{Q}(f,\Lambda_f,\psi,\chi)$ denote the field generated by Fourier coefficients of f ,by the eigenvalues $\Lambda_f(X)$ of Hecke operators X on f, and by the values of the characters χ and ψ.

b) For all integers s with $1-k+\delta+m\leq s\leq 0$ we have that

$$\langle f,\ f\rangle^{-1}\,\mathcal{D}^-(s,f,\chi)\in K.$$

Remark. *(a) It follows from the definitions (1.7–(1.9) and the theorem 1.3 that under the assumptions of the theorem 1.4 we have that* $\mathcal{D}^+(s,f,\chi)=0$ *for* $s\in\mathbf{N}$, $s\not\equiv\delta(\mathrm{mod}2)$ *and* $\mathcal{D}^-(s,f,\chi)=0$ *for* $s\equiv\delta(\mathrm{mod}2)$, $s\in\mathbf{Z}, s\le 0$.

(b). From the proof of the theorem 1.4 in §3 (see 3.15) one can substract more explicit information about the action of Galois automorphisms $\sigma\in\mathrm{Aut}(\mathbf{C})$ *on the special values (1.10), (1.11): namely, that for some non-zero constant* $\mu(\Lambda,k,\psi)\in\mathbf{C}^\times$ *depending only on* $k>2m+2$, *the character* ψ *and the homomorphism* $\Lambda:\mathcal{L}^m(C)\to\mathbf{C}$ *from the theorem 1.4 we have that*

$$\left[\frac{G(\psi\chi)^{m-1}\mathcal{D}^+(s,f,\chi)}{\mu(\Lambda,k,\psi)}\right]^\sigma=\frac{G(\psi^\sigma\chi^\sigma)^{m-1}\mathcal{D}^+(s,f^\sigma,\chi^\sigma)}{\mu(\Lambda^\sigma,k,\psi^\sigma)} \tag{1.12}$$

$$\left[\frac{G(\psi\chi)^{m-1}\mathcal{D}^-(s,f,\chi)}{\mu(\Lambda,k,\psi)}\right]^\sigma=\frac{G(\psi^\sigma\chi^\sigma)^{m-1}\mathcal{D}^-(s,f^\sigma,\chi^\sigma)}{\mu(\Lambda^\sigma,k,\psi^\sigma)} \tag{1.13}$$

with $G(\psi\chi)$ *being the Gauss sum of the primitive Dirichlet series associated with* $\psi\chi$. *Also, the following equality holds*

$$\mu(\Lambda,k,\psi)^{-1}\langle f,f\rangle_C\ \in\mathbf{Q}(f,\Lambda,\psi),$$

(this means that the Petersson scalar product $\langle f,f\rangle_C$ *differ from the constant* $\mu(\Lambda,k,\psi)$ *only by an algebraic multiple from the field* $K_0=\mathbf{Q}(f,\Lambda,\psi)$ *generated by Fourier coefficients of* f *and the values of* Λ,ψ. *In (1.12), (1.13) we adopted the standard notation*

$$f^\sigma=\sum_{A_m\ni\xi>0}a(\xi)^\sigma e_m(\xi z)\in\mathcal{S}_k^m(C,\psi^\sigma)$$

for the action of $\sigma\in\mathrm{Aut}(\mathbf{C})$.

The theorem 1.4 is proved in §3 (theorem 3.2), and the algebraicity properties , analogous to (1.12), (1.13) are established in 3.15. For some of the special values s in theorem 1.4 these properties where discovered in ealier works of M.Harris [Har1] and J.Sturm [St2].

1.5. Before giving the precise statement of the main result we make some additional assumptions on f. First of all we assume that f is p-ordinary (with respect to a fixed embedding $i_p:\overline{\mathbf{Q}}\hookrightarrow\mathbf{C}_p$). This means that

$$|i_p(\alpha_0(q)|_p=1\ \text{ for }\ q\in S. \tag{1.14}$$

Of course, for $q\ne p$ this condition is automaticaly satisfied because of the relation (1.14), and for $q=p$ we may consider in (1.14) instead of $\alpha_0(q)$ any of the numbers

$$\alpha_0(p)\alpha_{i_1}(p)\cdots\alpha_{i_r}(p)\quad(1\le i_1<\cdots<i_r\le m), \tag{1.15}$$

because these numbers are permutated under the action of Weyl group W_m described in 1.6 of chapter 1. For this purpose it suffices to apply several substitutions of the type

$$\alpha_0(p)\mapsto\alpha_0(p)\alpha_i(p),\ \ \alpha_i(p)\mapsto\alpha_i(p)^{-1},\alpha_j(p)\mapsto\alpha_j(p)\ \ (j\ne i,\ i=1,2,\cdots,m)$$

Recall that these numbers are inverse roots of the characteristic p-polynomial for the spinor zeta function $Z(s,f)$ of the cusp form f (see §1 of chapter 1):

$$Z^{(p)}(s,f)^{-1} = Q_{f,p}(p^{-s}) =$$
$$1 - \lambda_f(p)p^{-s} + \cdots + p^{2^{m-1}(mk-m(m+1)/2)}p^{-2^m s} =$$
$$(1-\alpha_0 p^{-s})\prod_{r=1}^{m}\prod_{1\le i_1<\cdots<i_r\le m}(1-\alpha_0\alpha_{i_1}\cdots\alpha_{i_r}p^{-s}),$$

with

$$\lambda_f(p) = \lambda_f(T(p)),\quad T((p)) = \sum_{\nu(g)=p}(\Gamma g\Gamma)$$

being the Hecke opertor for the group $\Gamma = \Gamma_0^m(C), {}^t g J_m g = \nu(g)J_m$.

For each $q \not| C$ we fix any $(m+1)$-tuple with the condition (1.14) and define by multiplicativity the functions $\alpha_i(n)$ for $n \in \mathbf{N}$, $(n,C)=1$. Moreovere we fix any half integral symmetric matrix ξ_0 such that $a(\xi_0)\neq 0$ and normalize f by the condition $a(\xi_0)=1$. Then $a(\xi_0)\in\overline{\mathbf{Q}}$ for all Fourier coefficients. Suppose also that the fixed set S of prime numbers contains the support (i.e. all prime divisors) of the number $2\det(2\xi_0)$, q_0 being the level of ξ_0 , and our last assumption is that

$$S\cap S(C)=\emptyset,\ \text{i. e.}\ (C,M_0)=1,\ \textit{with}\ M_0=\prod_{q\in S}q. \tag{1.16}$$

Put

$$N_0 = 4q_0M_0^{\tilde m-1}C,\quad \tilde m = 2^m.$$

1.6. Main theorem. *Let*

$$f(z)=\sum_{\xi>0}a(\xi)e_m(\xi z)\in\mathcal{S}_k^m(C,\psi)$$

be a cusp form of weight k with Dirichlet character ψ mod C *which is an eigenfunction of the Hecke algebra $\mathcal{L}^m(C)$ with egenvalues given by a homomorphism $\Lambda:\mathcal{L}^m(C)\to\mathbf{C}$. Suppose that for the cusp form f the conditions (1.14) and (1.16) are satisfied . Then for a positive integer $c>1$ with the condition $(c,N_0)=1$ there exist bounded analytic functions*

$$\mathcal{D}^{c+}(x,f),\mathcal{D}^{c-}(x,f)\ :\ X_S\to\mathbf{C}_p, \tag{1.17}$$

uniquely defined by the following conditions: a) *for all Dirichlet characters $\chi\in X_S^{\rm tors}$ with a S-complete conductor C_χ (i.e. $S(\chi)=S$) and for all integers s with*

$1\le s\le k-\nu-m$ *the following equality holds*

$$\begin{aligned}&\mathcal{D}^{c+}(\chi x_p^s,f)=\\ &i_p\Big[\frac{G_m(1_m,\chi)\,C_\chi^{m(s+k-1-m)}}{\alpha_0(C_\chi)^2}\frac{C(\psi\overline{\chi})^s}{G(\psi\overline{\chi})}(1-(\overline{\chi}\psi)^2(c)c^{-2s})\times\\ &\times\prod_{q|C}\{(1-(\overline{\psi}\chi)_0(q)q^{s-1})/(1-(\overline{\chi}\psi)_0(q)q^{-s})\}\frac{\mathcal{D}^+(s,f,\overline{\chi})}{\langle f,\ f\rangle}\Big],\end{aligned} \tag{1.18}$$

b) for all Dirichlet characters $\chi \in X_S^{\rm tors}$ with $S(\chi) = S$ and for all integers s with $1-k+\nu+m \le s \le 0$ the following equality holds

$$\mathcal{D}^{c-}(\chi x_p^s, f) = \tag{1.19}$$
$$i_p\Big[\frac{G_m(1_m,\chi)\,C_\chi^{m(s+k-1-m)}}{\alpha_0(C_\chi)^2}(1-(\chi\overline{\psi})^2(c)c^{2s-2})\frac{\mathcal{D}^-(s,f,\overline{\chi})}{\langle f,\ f\rangle}\Big],$$

where

$$G_m(\xi,\chi) = \sum_{h\in \mathrm{M}_m(\mathbf{Z}) \bmod C_\chi} \chi(\det h) e_m(\xi h/C_\chi)$$

denotes the Gauss sum of degree m of the primitive Dirichlet character $(\psi\overline{\chi})_0$ mod $C_{\psi\overline{\chi}}$ associated with $\psi\overline{\chi}$ mod $C\,C_\chi$ with the normalized zeta functions $\mathcal{D}^{\pm}(s,f,\chi)$ being defined by (1.8), (1.9).

1.7. *Proof* of the main theorem and of theorems 1.3 and 1.4 is based on the relation of the function $\mathcal{D}(s,f,\chi)$ with a convolution of Rankin type given by

$$2a(\xi_0)\det\xi_0^{-(s+k-1+\nu)/2}\mathcal{D}(s,f,\chi) = L_{C\,C_\chi}(s+\frac{m}{2},\psi\chi_{\xi_0}\chi)\times$$
$$\times\prod_{i=0}^{(m/2)-1} L_{C\,C_\chi}(2s+2i,\psi^2\chi^2))L((s+k-1+\nu)/2,f,\theta_{2\xi_0}^{(\nu)}(\chi), \tag{1.20}$$

where χ is a Dirichlet character, $\nu = 0$ or 1 according as $\chi(-1) = (-1)^\nu$, ξ_0 being a fixed half integral symmetric positive definite matrix. Also the left hand side of (1.20) is an Euler product associated with the homomorphism $\Lambda:\ \mathcal{L}^m(C)\to\mathbf{C}$, and the right hand side of (1.20) is completely determined by the Fourier coefficients of f. The non-Archimedean part of the construction is based on the theory of distributions and S-adic integration.

Using a general criterion of finite additivity we construct complex valued distributions associated with $\mathcal{D}(s,f,\chi)$ by defining their values at Dirichlet characters in §2. We prove in §3 an (Archimedean) integral representation for these values , which enables us to express the distributions in terms of the Fourier coefficients of Siegel-Eisenstein series from §3 of the previous chapter by applying the holomorphy projection operator (see §4 of chapter 1). After a regularization given in §4 these distributions become bounded $\mathbf{C}_p$ -valued measures $\mathcal{D}^{c+}$, $\mathcal{D}^{c-}$ taking algebraic values at compact open subsets of $\mathbf{Z}^S$, and the proof of the main theorem is then completed by application of the non-Archimedean Mellin transform.

§2. Complex valued distributions associated with standard zeta functions of Siegel modular forms

2.1. Let as in 1.2

$$f = \sum_{A_m\ni\xi>0} a(\xi)e_m(\xi z) \in S_k^m(C,\psi) \tag{2.1}$$

be a cusp form of degree m weight k with the Dirichlet character $\chi\ \psi$ modulo C on the congruence subgroup $\Gamma_0^m(C)$, which is an eigenfunction of all local Hecke algebras

$\mathcal{L} = \mathcal{L}_q^m$ with q not dividing the level C of f, so that the Satake q -parameters $\alpha_i = \alpha_i(q)$ $(i = 0, 1, \cdots, m)$ are defined for all $q \not| C$. We extend by multiplicativity definition of the functions $\alpha_i(M)$ to all values of the argument M prime to the level C.

Now for the fixed set of prime numbers $S = \{q\}$ not dividing C we define complex valued distributions on the profinite group

$$G_S = \prod_{q \in S} \mathbf{Z}_q^\times$$

associated with the functions $\mathcal{D}(s, f, \chi)$. The crucial role in the construction is played by the cusp form (1.50) , chapter 1:

$$f_0 = f_{0,S} = \sum_{M | M_0^{\tilde{m}-1}} \alpha_0(M)^{-1} f | V^+(M). \tag{2.3}$$

with $M_0 = \prod_{q \in S} q$, $\tilde{m} = 2^m$. Recall that if

$$f_{0,S}(z) = \sum_{A_m \ni \xi > 0} a_0(\xi) e_m(\xi z) \in \mathcal{M}_m^k(NM^{\tilde{m}-1}, \psi) \tag{2.4}$$

be Fourier expansion of the function $f_{0,S}(z)$ then there is the folowing multiplicativity property of its Fourier coefficients: for all $M \in \mathbf{N}$ with $S(M) \subset S$

$$a_0(M\xi, f_0) = \alpha_0(M) a_0(\xi, f_0) \quad (\xi \in A_m, \xi \geq 0). \tag{2.5}$$

Our construction of the distributions is based on the identity (1.20) expressing $\mathcal{D}(s, f, \chi)$ in terms of the Rankin type convolution:

$$\begin{aligned} 2a(\xi_0)\det\xi_0^{-(s+k-1+\nu)/2}\mathcal{D}(s, f, \chi) &= L_{CM}(s + \frac{m}{2}, \psi\chi_{\xi_0}\chi) \times \\ \times \prod_{i=0}^{(m/2)-1} L_{CM}(2s + 2i, \psi^2\chi^2)) & L((s + k - 1 + \nu)/2, f, \theta_{2\xi_0}^{(\nu)}(\chi)), \end{aligned} \tag{2.6}$$

where χ mod M is a Dirichlet character modulo M $\nu = 0, 1, \chi(-1) = (-1)^\nu, A_m \ni \xi_0$ is some appropriate half integral symmetric positive definite matrix, $s \in \mathbf{C}$ is a complex number with $\mathrm{Re}(s) \gg 0$,q_0 is the level of ξ_0 , $q_0 \in \mathrm{M}_m(\mathbf{Z})$. Put

$$\hat{\xi}_0 = q_0 \xi_o^{-1}, N_0 = 4q_0 C M_0^{\tilde{m}-1}.$$

2.2. Proposition. *Let $s \in \mathbf{C}$, $\mathrm{Re}(s) \gg 0$. Then there exist a complex valued distribution $\mathcal{D}_S$ on G_S which is uniquely determined by its values on Dirichlet characters $\chi \bmod M$ with $S(M) \subset S$ given by*

$$\begin{aligned}
&2a(\xi_0)\det\xi_0^{-(s+k-1+\nu)/2}\mathcal{D}(s,f,\chi) = \\
&\alpha_0(M_0^{\bar{m}-1}M')^{-1}(CM_0^{\bar{m}-1}M')^{m(2s+2k-2-m)/4}C^{m(2\nu+m)/4}\times \\
&L_{N_0}(s+\frac{m}{2},\psi\chi_{\xi_0}\overline{\chi})\prod_{i=0}^{(m/2)-1}L_{N_0}(2s+2i,\psi^2\overline{\chi}^2)\times \\
&\times\det((2q_0)^{-1/2}\xi_0)^{(m/2)+\nu}L((s+k-1+\nu)/2,f_0|V(C),\theta^{(\nu)}_{2\hat{\xi}_0}(\chi_M)|W(N_0M')),
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
&f_0|V(C)(z) = f_0(Cz), \\
&\theta^{(\nu)}_{2\hat{\xi}_0}(\chi_M)|W(N_0M') = \det(\sqrt{N_0M'}z)^{-(m/2)-\nu}\theta^{(\nu)}_{2\hat{\xi}_0}(\chi_M)(-(N_0M')^{-1}),
\end{aligned} \tag{2.8}$$

where M, M' are sufficiently large positive integers with the condition

$$M_0C_\chi|M, \quad MM_0C_\chi^2|M'$$

so that $S(M) = S(M') = S$ with C_χ being the conductor of the character χ and χ_M denoting the Dirichlet character modulo M induced by χ.

Proof. According to a general criterion of finite additivity applied for the family of functions $\mathcal{D}_{s,M} : (\mathbf{Z}/M\mathbf{Z})^\times \to \mathbf{C}$ (see [Ka3], [Maz-SD]) it suffices to check that the right hand side of (2.7) is independent of M and M'. The independence of M obviously follows from the S-completeness of M (i.e. M is divisible by all primes from S.In order to show the independence of M' we put

$$M' = AB,\ A = M_0^2C_\chi^2,\ B = M_1 = M'(M_0C_\chi)^{-2}$$

and use the equality

$$\theta^{(\nu)}_{2\hat{\xi}_0}(\chi_M)|W(N_0M') = M_1^{m(2\nu+m)/4}g|V(M_1), \tag{2.9}$$

in which

$$g(z) = \theta^{(\nu)}_{2\hat{\xi}_0}(\chi_M)|W(N_0M_0^2C_\chi)^{-2}) = \sum_{A_m\ni\xi\geq 0} b(\xi)e_m(\xi z).$$

It follows now from the definition (2.33) of chapter 1 and from (2.9) that

$$\begin{aligned}
&L(s,f_0|V(C),\theta^{(\nu)}_{2\hat{\xi}_0}(\chi_M)|W(N_0M')) = M_1^{m(2\nu+m)/4}L(s,f_0|V(C),g|V(M_1)) = \\
&M_1^{m((2\nu+m)/4-s)}C^{-ms}\sum_{\substack{A_m\ni h>0 \\ \mathrm{mod}\sim}} a(M_1\xi,f_0)b(C\xi)\det\xi^{-s}.
\end{aligned} \tag{2.10}$$

To get the desired independence of M' (or of M_1) we use the multiplicativity property (2.5): $a_0(M\xi,f_0) = \alpha_0(M)a_0(\xi,f_0)$ Then after substititution of (2.10) into (2.7) with s

equal to $(s+k-1+\nu)/2$, $M' = M_0^2 C_\chi^2 M_1$ we easily see that M_1 dissapear , and the proposition follows.

Another way of proving the proposition is to calculate explicitly the integrals $\mathcal{D}_{s,M}(\chi)$ of the Dirichlet characters χ which is done below with some additional technical assumptions.

2.3. Proposition. *Let $\chi \bmod M$ be a primitive Dirichlet character with an S-complete conductor C_χ, $C_\chi|M$ (i.e. $S(\chi) = S(C_\chi) = S$). Assume also that $S(2\det\xi_0) \subset S$. Then there is the following identity*

$$\mathcal{D}_{s,M}(\chi) = \frac{G_{C_\chi}(1_m,\chi)}{C_\chi^{m^2/2}} \cdot \frac{C_\chi^{m(2s+2k-2-m)/4}}{\alpha_0(C_\chi)^2} \mathcal{D}(s,f,\overline{\chi}) \tag{2.11}$$

where

$$G_{C_\chi}(1_m,\chi) = \sum_{h \in \mathrm{M}_m(\mathbf{Z}) \bmod C_\chi} \chi(\det h) e_m({}^t\xi h/C_\chi)$$

is the Gauss sum of degree m of the character χ.

The proof of the proposition 2.3 is based on the transformation formula (2.4) of the previous chapter for the theta function

$$\begin{aligned}&\theta_{\xi_0}^{(\nu)}(\chi)|W(q_0Q^2) = \\ &\chi(-1)^m \frac{G_{C_\chi}(1_m,\chi)}{C_\chi^{m^2}} \det(q_0^{-1/2}\xi_0)^{(m/2)+\nu} \theta_{\hat{\xi}_0}^{(\nu)}(\overline{\chi}),\end{aligned} \tag{2.12}$$

If we now take into account that

$$G_{C_\chi}(1_m,\chi) G_{C_\chi}(1_m,\overline{\chi}) = \chi(-1)^m C_\chi^{m^2}, \quad \theta_{2\xi_0}^{(\nu)}(\chi) = \theta_{\xi_0}^{(\nu)}(\chi)|V(2),$$

then (2.12) transforms to

$$\begin{aligned}&\theta_{2\hat{\xi}_0}^{(\nu)}(\chi)|W(4q_0Q^2) = \\ &\chi(-1)^m \frac{G_{C_\chi}(1_m,\chi)}{C_\chi^{m^2}} \det(q_0^{1/2}\xi_0^{-1})^{(m/2)+\nu} \theta_{2\xi_0}^{(\nu)}(\overline{\chi}),\end{aligned} \tag{2.13}$$

Next we write:

$$N_0 M' = 4q_0 Q^2 N' \ , \ N' = M_0^{\tilde{m}-1} C M' C_\chi^{-2},$$

so that

$$\theta^{(\nu)})|W(N_0M') = (N')^{m(2\nu+m)/4} [\theta^{(\nu)}|W(4q_0Q^2)]|V(N') \tag{2.14}$$

and if we substitute (2.14) into (2.7) we get exactly the desired identity proving the proposition.

Now we obtain integral representations for the values of the distributions $\mathcal{D}_{s,M}$ with $\mathrm{Re}(s) \gg 0$ using the identity (2.29) of the previous chapter from which follows that for all Dirichlet characters χ mod M with the condition $M_0|M$

$$2a(\xi_0)((4\pi)^m \det(\xi_0)^{-(s+k-1+\nu)/2}\Gamma_m((s+k-1+\nu)/2)\,\mathcal{D}(s,f,\chi) =$$

$$L_{N_0}(s+\frac{m}{2},\psi\chi_{\xi_0}\chi)\prod_{i=0}^{(m/2)-1} L_{N_0}(2s+2i,\psi^2\chi^2))\times \tag{2.15}$$

$$\times\,\langle f^\rho(z),\theta^{(\nu)}_{2\xi_0}(z;\chi)E(z,(s-k+m+\nu)/2)\rangle_N,$$

where

$$E(z,s) = E(z,s;k-(m/2(-\nu,\psi\chi_{\xi_0}\chi,N)$$

is the Siegel-Eisenstein series of weight $k-(m/2)-\nu$ of level $N = 4q_0CM^2$ with the Dirichlet character $\psi\chi_{\xi_0}\chi$ introduced in 2.4 of chapter 1.

2.4.Proposition. *For $s \in \mathbf{C}$, $\mathrm{Re} \gg 0$ there is the following integral representation for the distributions of the proposition 2.2:*

$$2a(\xi_0)((4\pi)\det\xi_0)^{-(s+k-1+\nu)/2}\Gamma_m((s+k-1+\nu)/2)\,\mathcal{D}_{s,M}(\chi) =$$

$$C^{m(s+k-1+\nu)/2}\alpha_0(M_0^{\tilde m-1}M')^{-1}(M_0^{\tilde m-1}M')^{m(2s+2k-2-m)/4}\det(q_0^{-1/2}\xi_0)^{(2\nu+m)/2}\times$$

$$\times\, L_{N_0}(s+\frac{m}{2},\psi\chi_{\xi_0}\overline{\chi})\prod_{i=0}^{(m/2)-1} L_{N_0}(2s+2i,\psi^2\overline{\chi}^2))\times$$

$$\times\,\langle f_0^\rho(z)|V(C),\theta^{(\nu)}_{2\hat\xi_0}(\chi_M)|W(N_0M')E(z,(s-k+m+\nu)/2)\rangle_N, \tag{2.16}$$

where M, M' are sufficiently large positive integers with the condition

$$M_0C_\chi|M,\quad MM_0C_\chi^2|M'$$

so that $S(M) = S(M') = S$ with C_χ being the conductor of the character χ ,$C_\chi|M$ and χ_M denote the Dirichlet character modulo M induced by χ.

The proof of the proposition reduces to application of the integral representation (2.15) to the cusp form $f_0|V(C) \in \mathcal{S}_k^m(C^2M_0^{\tilde m-1},\psi)$ of level $C^2M_0^{\tilde m-1}$ defined by (2.7).

The next important ingredient of our construction is an application of the trace operator to modular forms of level $N = CN_0M'$ in the above integral formula, which enables us to reduce all considerations (e.g. the integral formula) to the case of the fixed level CN_0.

2.5. The trace operator. We define the trace operator $\mathrm{Tr}^{CN_0M'}_{CN_0}$ acting on (not necessarily holomorphic) modular forms of the degree m weight k with the Dirichlet character $\overline{\psi}$ on the congruence subgroup $\Gamma_0^m(CN_0M')$ by the following equality

$$F|\mathrm{Tr}^{CN_0M'}_{CN_0} = \sum_{{}^t u=u\in M_m(\mathbf{Z}) \bmod M'} F|\begin{pmatrix}1_m & 0\\ N_0Cu & 1_m\end{pmatrix}. \tag{2.18}$$

Then the scalar product in (2.16) transforms in the obvious way:

$$\langle f_0^\rho, F\rangle_{CN_0M'} = \langle f_0^\rho, F|\mathrm{Tr}^{CN_0M'}_{CN_0}\rangle_{CN_0}. \tag{2.19}$$

Now we give a useful description of the action of the trace operator on Fourier expansions by means of the operator $U(M')$:

$$F|U(M')(z) = (M')^{-m(m+1)/2} \sum_{{}^t u = u \in \mathrm{M}_m(\mathbf{Z}) \bmod M'} F((z+u)/M'). \tag{2.20}$$

(in the notation of the previous chapter this operator coincides with the Frobenius operator $\Pi_+(M')$ extended by multiplicativity to positive integral values of M' prime to the level C: $F|U(M') = F|\Pi_+(M')$). If

$$F(z) = \sum_{\xi \in A_m} a(\xi, y) e_m(\xi z),$$

then

$$F|U(M')(z) = \sum_{\xi \in A_m} a(M'\xi, {M'}^{-1} y) e_m(\xi z). \tag{2.21}$$

The following relation holds

$$F|\mathrm{Tr}_{CN_0}^{CN_0 M'} = (M')^{-m(k-m-1)/2} F|W(CN_0M')U(M')W(CN_0), \tag{2.22}$$

which is immediately implied from the matrix identity

$$\begin{pmatrix} 1_m & 0 \\ CN_0 u & 1_m \end{pmatrix} = (CN_0M')^{-1} \begin{pmatrix} 0 & -1_m \\ CN_0M'1_m & 0 \end{pmatrix} \begin{pmatrix} 1_m & -u \\ 0 & M'1_m \end{pmatrix} \begin{pmatrix} 0 & -1_m \\ CN_0 1_m & 0 \end{pmatrix}.$$

Now let us apply (2.19) and (2.22) to the integral formula (2.16) then we get

$$\begin{aligned} &2a(\xi_0)((4\pi)\det\xi_0)^{-(s+k-1+\nu)/2}\Gamma_m((s+k-1+\nu)/2)\,\mathcal{D}_{s,M}(\chi) = \\ &\alpha_0(M_0^{\tilde{m}-1}M')^{-1}(M_0^{\tilde{m}-1}M')^{m(2s+2k-2-m)/4}\det(q_0^{-1/2}\xi_0)^{(2\nu+m)/2}\times \\ &\times L_{N_0}(s+\frac{m}{2},\psi\chi_{\xi_0}\overline{\chi}) \prod_{i=0}^{(m/2)-1} L_{N_0}(2s+2i,\psi^2\overline{\chi}^2))\times \\ &\times (M')^{m(m+1-k)/2}\langle f_0^\rho(z)|V(C), F(s,\chi)|U(M')W(CN_0)\rangle_{CN_0}, \end{aligned} \tag{2.23}$$

where

$$F(s,\chi) = \theta_{2\hat{\xi}_0}^{(\nu)}(\chi_M)|V(C)\cdot E(z,(s-k+m+\nu)/2)|W(CN_0M'),$$

and we noted that

$$\theta_{2\hat{\xi}_0}^{(\nu)}(\chi_M)|W(N_0M')|W(CN_0M') = C^{-m(2\nu+m)/4}\theta_{2\hat{\xi}_0}^{(\nu)}(\chi_M)|V(C). \tag{2.24}$$

§3. Algebraic properties of special values of normalized distributions

3.1. In this section we consider only the case of even m. In order to give a pricise statements about algebraicity properties of the standard zeta functions and of the corresponding distributions, it is convenient to make some additional normalization of these values because these properties look different for the integral points to the left and to the right of the critical line $\mathrm{Re}(s) = \frac{1}{2}$ (in the same way as for the Riemann zeta function).

Recall that as in §1 we have introduced the following three types of the normalized zeta function: for $\kappa = (m+1)/2$

$$\begin{aligned}&\mathcal{D}^*(s,f,\chi) = \\ &(2\pi)^{-m(s+k-\kappa)}\Gamma((s+\delta)/2)\prod_{j=1}^{m}\Gamma(s+k-j)\mathcal{D}(s,f,\chi),\end{aligned} \tag{3.1}$$

$$\begin{aligned}&\mathcal{D}^-(s,f,\chi) = \Gamma((s+\delta)/2)^{-1}\mathcal{D}^*(s,f,\chi) = \\ &(2\pi)^{-m(s+k-\kappa)}\prod_{j=1}^{m}\Gamma(s+k-j)\mathcal{D}(s,f,\chi),\end{aligned} \tag{3.2}$$

$$\begin{aligned}&\mathcal{D}^+(s,f,\chi) = \frac{2\,i^{\delta}\Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi i)^{s}}\mathcal{D}^-(s,f,\chi) = \\ &\frac{i^{\delta}\pi^{(1-2s)/2}}{\Gamma((1+\delta-s)/2)}\mathcal{D}^*(s,f,\chi),\end{aligned} \tag{3.3}$$

where $\delta = 0$ or 1 according as $\psi\chi(-1) = (-1)^{\delta}$ with $f \in \mathcal{S}_k^m(C,\psi)$ be an eigenfunction of the global Hecke algebra $\mathcal{L}^m(C) = \otimes_{q\not| C}\mathcal{L}_q^m(C)$ with the eigenvalue given as a homomorphism $\Lambda : \Lambda^m(C) \to \mathbf{C}$. The convenience of the $*$-normalization (3.1) is explained by the fact that the standard zeta functions continued holomorphically to the whole complex plane satisfy to a functional equation connecting

$$\mathcal{D}^*(s,f,\chi) \quad \text{and} \quad \pi^{(2s-1)/2}\mathcal{D}^*(s,f^{\rho},\overline{\chi}) \tag{3.4}$$

although the precise form of such an equation is known only in some cases [An-K], [Bö]. The principial difficulty in dealing with the general case is that for $m > 1$ one lacks the correct and reasonable definition of the Euler factors for the standard zeta function at bad primes, i.e. for $q|C$. In the one dimentional case these factors are provided by the Atkin-Lehner theory [At-Le], [At-Li], [La3], [Li1]; however even for $m = 2$ there is no such a theory.

Now we turn to the normalizations (3.3) and (3.3). Their convenience is illustrated by the following result about algebraicity of the special values of (3.2) and (3.3) which is proven in this section together with the corresponding statement about the special values of normalized distributions given below in 3.3.

3.2. Theorem *(Algebraic properties of the special values of standard zeta functions). Assume that the cusp eigenform $f \in \mathcal{S}_k^m(C,\psi)$ is normalized by the condition $a(\xi_0) = 1$ for some $\xi_0 \in A_m, \xi_0 > 0$. Then*

a) For all integers s with $1 < s \le k - \delta - m$ and $s \ne 1$ if the character $\chi^2\psi^2$ is trivial we have that

$$\langle f,\ f\rangle^{-1}\,\mathcal{D}^+(s,f,\chi) \in K = \mathbf{Q}(f,\Lambda_f,\psi,\chi),$$

where $K = \mathbf{Q}(f, \Lambda_f, \psi, \chi)$ *denote the field generated by Fourier coefficients of* f *,by the eigenvalues* $\Lambda_f(X)$ *of Hecke operators* X *on* f*, and by the values of the characters* χ *and* ψ *and* $\mathcal{D}^+(s, f, \chi) = 0$ *for* $s \not\equiv \delta(\bmod 2)$*;*

b) For all integers s *with* $1 - k + \delta + m \le s \le 0$ *we have that*

$$\langle f,\ f\rangle^{-1}\, \mathcal{D}^-(s, f, \chi) \in K$$

and $\mathcal{D}^-(s, f, \chi) = 0$ *for* $s \equiv \delta(\bmod 2)$, $\ s \in \mathbf{Z}, s \le 0$.

The proof of the theorem is completed in 3.15, although we assume there that χ is a Dirichlet character modulo N (not necessarily primitive) such that $2C\det 2\xi_0 | N$ for the fixed $\xi_0 \in A_m, \xi_0 > 0$ with the condition $a(\xi_0) \ne 0$. This restriction is insignificant and is being avoided by multiplying the special values in question by a finite number of Euler factors corresponding to the divisors of $\det 2\xi_0$ which take algebraic values for integer values of s.

3.3. Now we define normalized distributions by the following formula

$$\langle f, f\rangle_C \mathcal{D}^*_{s,M}(\chi) = (2\pi)^{-m(s+k-(m+1)/2)}\Gamma((s+\delta)/2) \prod_{j=1}^{m} \Gamma(s+k-j)\mathcal{D}_{s,M}(\chi), \quad (3.7)$$

$$\mathcal{D}^-_{s,M}(\chi) = \langle f, f\rangle_C^{-1}(2\pi)^{-m(s+k-(m+1)/2)} \prod_{j=1}^{m} \Gamma(s+k-j)\mathcal{D}_{s,M}(\chi), \quad (3.8)$$

$$\mathcal{D}^+_{s,M}(\chi) = \frac{2\, i^\delta \Gamma(s) \cos(\pi(s-\delta)/2)}{2\pi i)^s} \mathcal{D}^-_{s,M}(\chi), \quad (3.9)$$

where $\mathcal{D}_{s,M}(\chi)$ are the values of the distribution $\mathcal{D}_{s,M}$ on the finite group

$$G_S = \mathbf{Z}_S^\times = \prod_{q \in S} \mathbf{Z}_q^\times \xrightarrow{\sim} Gal(\mathbf{Q}(S)/\mathbf{Q})$$

with $Gal(\mathbf{Q}(S)/\mathbf{Q})$ being the Galois group of the maximal abelian extension of $\mathbf{Q}$ unramified outside S and ∞.In the definitions (3.7) - (3.9) is assumed that χ_M is the Dirichlet character modulo M induced by χ and $S(M) \subset S$.

The following proposition is closely connected with theorem 2.2.

3.4. Proposition. *(Algebraicity properties of values of the normalized distributions) Assume that the cusp eigenform* $f \in S_k^m(C, \psi)$ *is normalized by the condition* $a(\xi_0) = 1$ *for some* $\xi_0 \in A_m, \xi_0 > 0$*.Then*

a) For all integers s *with* $1 \le s \le k - \delta - m$ *and* $s \ne 1$ *if the character* $\chi^2\psi^2$ *is trivial we have that*

$$\mathcal{D}^+_{s,M}(\chi) \in K = \mathbf{Q}(f, \Lambda_f, \psi, \chi, \alpha_i(q); i \le m, q \in S),$$

where $K = \mathbf{Q}(f, \Lambda_f, \psi, \chi)$ *denote the field generated by Fourier coefficients of* f *, by the eigenvalues* $\Lambda_f(X)$ *of Hecke operators* X *on* f*, the Satake* q *-parameters and by the values of the characters* χ *and* ψ *.*

b) For all integers s *with* $1 - k + \delta + m \le s \le 0$ *we have that*

$$\mathcal{D}^-_{s,M}(\chi) \in K.$$

In the proof of the properties (a) and (b) we always assume that the set S contains all prime divisors of the number $2\det(2\xi_0)$. According to the explicit formula of proposition 2.3 for Dirichle characters χ with S-complete conductors the values $\mathcal{D}^{\pm}_{s,M}(\chi)$ are expressed in terms of the corresponding special values of the normalized standard zeta functions $\mathcal{D}^{\pm}(s, f^{\rho}, \overline{\chi})$ so that in this case the proposition is an immediate consequence of the theorem 3.2. However for the non-Archimedean construction we have to consider all characters of finite order of G_S because of their unavoidable presence in the generalized Kummer congruences in §4. Very explicit formulas for the corresponding values of the distributions are given in 3.6 and can not be simply reduced to those the theorem 3.2. the proof of the algebraicity properties is completed in 3.15 together with the proof of the theorem 3.2. Application of a fixed embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ to the normalized distributions $\mathcal{D}^{\pm}_{s,M}(\chi)$ provides us with p-adic distributions $i_p[\mathcal{D}^{\pm}_{s,M}(\chi)]$ which becomes bounded p-adic measures after a regularization in §4 . The basic fact used in the proof of proposition 3.4 is an integral representation for $\mathcal{D}^{\pm}_{s,M}(\chi)$ which we state now in a preliminary form in terms of the distribution $\mathcal{D}^{*}_{s,M}(\chi)$.

3.5.Proposition. *Let $f \in S^m_k(C, \psi)$, m be even, χ be a Dirichlet character modulo $M \geq 1$. Then there is the following equality:*

$$\langle f, f\rangle_C \mathcal{D}^{*}_{s,M}(\chi) = \gamma(M')\langle f^{\rho}_0 | V(C), F^{*}_{M'}(s,\chi) | W(CN_0)\rangle_{CN_0}, \qquad (3.10)$$

where

$$\gamma(M') = 2^{m(2k-2-m-\kappa)} i^{-m(k-(m/2)-\nu)} a(\xi_0)^{-1}\alpha_0(M_0)^{\tilde{m}-1} M'^{-1}(CM_0^{\tilde{m}-1})^{(k-1-m)/2},$$

$$F^{*}_{M'}(s,\chi) = (q_0 C)^{-m(2s+m)/4} \det\xi_0^{(s+k-1+\nu)/2} \det(q_0^{-1/2}\xi_0)^{(2\nu+m)/2} \times$$
$$\times [2^{-1}\theta^{(\nu)}_{2\xi_0}(\chi_M) | V(C)\, G^{*}(z, s-k+\nu+m)/2)] | U(M'),$$

where

$$G^{*}(z,s) = G^{*}(z,s; k-(m/2)-\nu, \overline{\chi}\chi_{\xi_0}\psi, N)$$

is the normalised Siegel-Eisenstein series from §3 of the previous chapter with $N = CN_0M' = C^2M_0^{\tilde{m}-1}4q_0M'$.

The right hand side of (3.10) is defined and holomorphic for all $s \in \mathbf{C}$ with possible exclusion of $s = 1$ in case when the character $\overline{\chi}^2\psi^2$ is trivial.

The proof of the proposition follows from the definition of the normalized Eisenstein series $G^{*}(z,s)$ which for k equal to $k-(m/2)-\nu$, s equal to $(s-k+\nu+m)/2$ takes the following form

$$G^*(z,(s-k+m+\nu)/2) =$$

$$(4q_0C^2M')^{m(2s+m)/4}\tilde{\Gamma}(k-(m/2)-\nu,(s-k+\nu+m)/2)\times$$

$$\times L_N(s+(m/2),\overline{\chi}\chi_{\xi_0}\psi)\prod_{j=1}^{m/2} L_N(2s+m-2j,\psi^2\overline{\chi}^2)\times$$

$$\times 2^{m\kappa}E(z,(s-k+\nu+m)/2)|W(N), \tag{3.11}$$

where

$$\tilde{\Gamma}(k-(m/2)-\nu,(s-k+\nu+m)/2) =$$

$$i^{m(k-(m/2)-\nu)}2^{-m(k-(m/2)-\nu+1)}\pi^{-m(s+k-\nu)/2}\Gamma_m((s+k-\nu)/2)\Gamma((s+\delta)/2),$$

and it suffices to take into account the following relation for the gamma factors:

$$(2\pi)^{-m(s+k-\kappa)}\Gamma((s+\delta)/2)\prod_{j=1}^{m}\Gamma(s+k-j) =$$

$$2^{m(2k-2-m-\kappa)}i^{-m(k-(m/2)-\nu)}\tilde{\Gamma}(k-\nu-(m/2),(s-k+\nu+m)/2)\times \tag{3.12}$$

$$\times 2^{m(m+1)/2}2^{m(2s+m)/2}(4\pi)^{-m(s+k-1+\nu)/2}\Gamma_m((s+k-\nu-m)/2),$$

which follows from the duplication formula

$$\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) = 2^{1-s}\sqrt{\pi}\Gamma(s)$$

and the definition

$$\Gamma_m(s) = \pi^{(m-1)m/4}\prod_{j=0}^{m-1}\Gamma(s-(j/2)).$$

The statement about holomorphy follows from the theorem on holomorphy of the Eisenstein series $G^*(z,s)$ (see theorem 3.6 of the previous chapter) and from uniform estimates on $z \in \mathfrak{H}_m$ of the Fourier coeficients which imply that the scalar product in right hand side of (3.10) can be defined for all $s \in \mathbf{C}$ $s \neq 1$,if $\psi^2\overline{\chi}^2$ is trivial.

Now we specialize the integral representation (3.10) to the case of the critical values s (see 3.3 of chapter 1), when the confluent hypergeometric function admit an elementary expression in terms of a certain polynomial. Then we use the holomorphic projection operator $\mathcal{H}ol$ (see §4 of chapter 1 in order to get an integral representation for the distributions $\mathcal{D}^{\pm}_{s,M}(\chi)$ in terms of the holomorphic Siegel modular forms with algebraic (and explicitly given) Fourier coefficients.

3.6. Proposition. *Under the assumptions and notations of Proposition 3.4 the following integral representations are valid* *(a)For all integers s with $1 \leq s \leq k-\delta-m$ and $s=1$ if the character $\chi^2\overline{\psi}^2$ is non-trivial we have that*

$$\langle f,f\rangle_C \mathcal{D}^+_{s,M}(\chi) = \gamma(M')\langle f_0^\rho|V(C), F^+_{M'}(s,\chi)|W(CN_0)\rangle_{CN_0},$$

$$F^+_{M'}(s,\chi) = \sum_{A_m \ni > 0} \sum_{C\hat{\xi}_0[h_1]+h_2=M'h} d^+(s,h_1,h_2)e_m(hz); \tag{3.13}$$

b) For all integers s with $1-k+\delta+m \leq s \leq 0$ we have that

$$\langle f,f\rangle_C \mathcal{D}^-_{s,M}(\chi) = \gamma(M')\langle f_0^\rho|V(C), F^-_{M'}(s,\chi)|W(CN_0)\rangle_{CN_0},$$

$$F^-_{M'}(s,\chi) = \sum_{A_m \ni > 0} \sum_{C\hat{\xi}_0[h_1]+h_2=M'h} d^-(s,h_1,h_2)e_m(hz); \tag{3.14}$$

The functions $F^{\pm}_{M'}(s,\chi) \in \mathcal{M}(CN_0,\psi)$ are holomorphic Siegel modular forms with cyclotomic Fourier coefficients explicitly given by:
if $s \not\equiv \delta(\mathrm{mod}2)$, $1 \leq s \leq k-\nu-m$ then $d^+(s,h_1,h_2)=0$;
if $s \equiv \delta(\mathrm{mod}2)$, $1 \leq s \leq k-\nu-m$ then

$$\begin{aligned} d^+(s,h_1,h_2) = {} & \chi_M(\det h_1)\det h_1^\nu \det h_2^{(2s-1)/2} P(h_2,h,s)\times \\ & \times L^+_N(s,\overline{\chi}\chi_{\xi_0}\psi\xi_{h_2})M(h,\overline{\chi}\chi_{\xi_0}\psi, s+(m/2))\times \\ & \times (q_0C)^{-m(2s+m)/4}\det\xi_0^{(s+k-1+\nu)/2}\det(q_0^{-1/2}\xi_0)^{(2\nu+m)/2}; \end{aligned} \tag{3.15}$$

if $s \equiv \delta(\mathrm{mod}\,2)$, $1-k+\nu+m \leq s \leq 0$ then $d^-(s,h_1,h_2)=0$;
if $s \not\equiv \delta(\mathrm{mod}\,2)$, $1-k+\nu+m \leq s \leq 0$ then

$$\begin{aligned} d^-(s,h_1,h_2) = {} & \chi_M(\det h_1)\det h_1^\nu P(h_2,h,1-s)\times \\ & \times L^-_N(s,\overline{\chi}\chi_{\xi_0}\psi\xi_{h_2})M(h,\overline{\chi}\chi_{\xi_0}\psi, s+(m/2))\times \\ & \times (q_0C)^{-m(2s+m)/4}\det\xi_0^{(s+k-1+\nu)/2}\det(q_0^{-1/2}\xi_0)^{(2\nu+m)/2}; \end{aligned} \tag{3.16}$$

$P(v,u;s) \in \mathbf{Q}[v,u]$ denotes a polynomial of entries of matrix variables $v=(v_{ij}), u=(u_{ij})$ which is defined for $s \equiv \delta(\mathrm{mod}2), 1 \leq s \leq k-m-\nu$ with rational coefficients independent of f, M, χ, M' and satisfies the property

$$P(x,y,s) \equiv \det x^{(k-\nu-m-s)/2} \bmod (\langle y_{ij}\rangle)$$

$M(h,\overline{\chi}\chi_{\xi_0}\psi, s+(m/2)$ denote the integer multiple from the Fourier expansion of the Siegel-Eisenstein series (3.44) of the chapter 1, with

$$\begin{aligned} L_N^+(s,\omega) &= \frac{2i^\delta\Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi)^s}L_N(s,\omega), \\ L_N^-(s,\omega) &= L_N(s,\omega) \quad (\delta=0,1,\ \omega(-1)=(-1)^\delta) \end{aligned}$$

being the normalized Dirichlet L-series with the Euler factors at q, $q|N$ removed from their Euler products. The summation in the inner sums of (3.13) and (3.14) is taken over all pairs (h_1,h_2) of integral matrices with the conditions

$$h_1\in \mathrm{M}_m^+(\mathbf{Z}),\quad h_2>0,\quad h_2\in A_m,\quad C\hat{\xi}_0[h_1]+h_2=M'h$$

(i.e. h_1 is a integral matrix with positive determinant, not necessarily symmetric, $h_2\in A_m$ is a positive definite half integral matrix , and $C\hat{\xi}_0[h_1]$ denotes the matrix given by $C{}^th_1q_0\xi^{-1}h_1 = Cq_0\xi_0^{-1}[h_1]$.)

3.7. The proof of the proposition is carried out in several steps. First we write down a preliminary integral representation using proposition 2.4 and the definitions (3.8), (3.9) of the normalized distributions:

$$\langle f,f\rangle_C\mathcal{D}_{s,M}^{\pm}(\chi)=\gamma(M')\langle f_0^\rho|V(C),\tilde{F}_{M'}^{\pm}(s,\chi)|W(CN_0)\rangle_C N_0, \tag{3.17}$$

where

$$F_{M'}^{\pm}(s,\chi)=(q_0C)^{-m(s+(m/2))}\det\xi_0^{(s+k-1+\nu)/2}\det(q_0^{-1/2}\xi_0)^{(2\nu+m)/2}\times$$

$$\times\,[2^{-1}\theta_{2\hat{\xi}_0}^{(\nu)}(\chi_M)|V(C)\cdot G_{M'}^{\pm}(z,(s-k+m+\nu)/2)]|U(M').$$

We already know the Fourier expansions of the functions in (3.17): by the definition of theta functions from §2 of the previous chapter we have that

$$\theta_{2\hat{\xi}_0}^{(\nu)}(\chi_M)|V(C)=\sum_{h_1\in\mathrm{M}_m(\mathbf{Z})+}\chi_M(\det h_1)\det h_1^\nu e_m(C\hat{\xi}_0[h_1]z). \tag{3.18}$$

The Fourier expansions of the series $G_{M'}^{\pm}(z,s)$ is explicitly written in §3 of chapter 1. If we put in the expansions (3.53), (3.53a) of chapter 1 s equal to $(s-k+\nu+m)/2$, k equal to $k-(m/2)-\nu$ we get the equality:

$$G_{M'}^{\pm}(z,(s-k+m+\nu)/2)=\sum_{h\in A_m}b^{\pm}(h,y,(s-k+\nu+m)/2)e_m(hz), \tag{3.19}$$

in which

$$b^{\pm}(h,y,(s-k+\nu+m)/2)=$$

$$L^{\pm}(s,\overline{\chi}\chi_{\xi_0}\chi_h\psi)W^*(h,y,(s-k+\nu+m)/2)M(h,\overline{\chi}\chi_{\xi_0}\psi,s+(m/2)).$$

For the critical values of s the function $W^*(h,y,(s-k+\nu+m)/2)$ is explicitly given by formulas (3.31), (3.34) of chapter 1 in terms of the polynomial

$$R(z;n,\beta) = (-1)^{mn} e^{\operatorname{tr}(z)} \det(z)^{n+\beta} \Delta_m^n [e^{\operatorname{tr}(z)} \det(z)^{-\beta}], \tag{3.20}$$

where

$$\Delta_m = \det(\partial_{ij}), \quad \partial_{ij} = 2^{-1}(1+\delta_{ij})\partial/\partial_{ij}.$$

with an integer $n \geq 0$ and a complex number β. the degree of $R(z;n,\beta)$ is equal to mn, and the the term of the highest degree coincides with $\det z^n$. If $\beta \in \mathbf{Q}$ then $R(z;n,\beta) \in \mathbf{Q}[z_{ij}]$. Put

$$Q(y,s) = R(y;(k-\nu-m-s)/2,(1-s-k+\nu+m)/2). \tag{3.21}$$

then for $s \in \mathbf{Z}$, $1-k+\nu+m \leq s \leq k-\nu-m$, the coefficients of (3.19) transform to the following:
if $\leq s \leq k-\nu-m$, $s \not\equiv \delta(\bmod 2)$and $\overline{\chi}^2\psi^2$ is non trivial for $s=1$ then $b^+(h,y,(s-k+\nu+m)/2)=0$;
if $\leq s \leq k-\nu-m$, $s \equiv \delta\,(\bmod 2)$and $\overline{\chi}^2\psi^2$ is non trivial for $s=1$ then

$$b^+(h,y,(s-k+\nu+m)/2) =$$

$$(\det h^{s-\frac{1}{2}} \det(4\pi y)^{(s-k+\nu+m)/2} Q(4\pi hy,s) \times \tag{3.22}$$

$$\times L_M^+(s,\overline{\chi}\chi_{\xi_0}\chi_h\psi) M(h,\overline{\chi}\chi_{\xi_0}\psi, s+(m/2));$$

if $s \equiv \delta(\mathrm{mod} 2)$, $1 \leq s \leq k-\nu-m$ and $\psi^2\overline{\chi}^2$ is non trivial for $s=1$ then $b^-(h,y,(s-k+\nu+m)/2)=0$;
if $s \not\equiv \delta(\mathrm{mod} 2)$, $1-k-s+\nu+m \leq s \leq 0$ then

$$b^-(h,y,(s-k+\nu+m)/2) =$$

$$\det(4\pi y)^{(1-s-k+\nu+m)/2} Q(4\pi hy,1-s) \times \tag{3.23}$$

$$\times L_M^-(s,\overline{\chi}\chi_{\xi_0}\chi_h\psi) M(h,\overline{\chi}\chi_{\xi_0}\psi, s+(m/2));$$

It is assumed that in (3.22), (3.23) $h>0$, $h \in A_m$. According to the theorem about positivity for matrix indices in Fourier expansions (theorem 3.7 of chapter 1) for these values of s we have that $b^+(h,y,(s-k+\nu+m)/2)=0$ if $h \in A_m$ is not positive definite. Respectively, $b^-(h,y,(s-k+\nu+m)/2)=0$ for the corresponging values of s if $h \in A_m$ is not non negative (i.e. when it contains negative eigenvalues).However we now will see that the Fourier expansions of the functions $\tilde{F}^{\pm}_{M'}(s,\chi)$ from (3.17) involve only those Fourier coefficients of $G^{\pm}_{M'}(z,(s-k+m+\nu)/2)$ in the expansion (3.19) which correspond to terms with positive definite $h \in A_m$.

3.8. Proposition. *(a) If $s \in \mathbf{Z}$, $1 \le s \le k - \nu - m$ and $\overline{\chi}^2\psi^2$ is non trivial for $s = 1$ then we have in (3.17) that*

$$\tilde{F}^{+}_{M'}(s,\chi) = \sum_{A_m \ni >0} \sum_{C\hat{\xi}_0[h_1]+h_2=M'h} \tilde{d}^{+}(s,h_1,h_2)e_m(hz); \tag{3.24}$$

(b) If $1 - k + \nu + m \le s \le 0$, $s \in \mathbf{Z}$ then

$$\tilde{F}^{-}_{M'}(s,\chi) = \sum_{A_m \ni >0} \sum_{C\hat{\xi}_0[h_1]+h_2=M'h} \tilde{d}^{-}(s,h_1,h_2)e_m(hz); \tag{3.25}$$

the functions $\tilde{F}^{+}_{M'}(s,\chi)$, $\tilde{F}^{-}_{M'}(s,\chi) \in \tilde{\mathcal{M}}^m_k(CN_0,\overline{\psi})$ are non holomorphic Siegel modular forms with the Fourier coefficients given by:

$$\tilde{d}^{+}(y,s,h_1,h_2)e_m(hz) = \chi_M(\det h_1)\det h_1^{\nu}\det h_2^{(2s-1)/2}\det(4\pi y)^{(s-k+\nu+m)/2}\times$$

$$\times\, Q(4\pi hy,s)L^{+}_{N}(s,\overline{\chi}\chi_{\xi_0}\psi\xi_{h_2})M(h,\overline{\chi}\chi_{\xi_0}\psi,s+(m/2))\times \tag{3.26}$$

$$\times\,(q_0C)^{-m(2s+m)/4}\det\xi_0^{(s+k-1+\nu)/2}\det(q_0^{-1/2}\xi_0)^{(2\nu+m)/2};$$

$$\tilde{d}^{-}(y,s,h_1,h_2) = \chi_M(\det h_1)\,\det h_1^{\nu}\det(4\pi y)^{1-s-\nu-m+k)/2}\times$$

$$\times\, Q(4\pi hy,1-s)L^{-}_{N}(s,\overline{\chi}\chi_{\xi_0}\psi\xi_{h_2})M(h,\overline{\chi}\chi_{\xi_0}\psi,s+(m/2))\times \tag{3.27}$$

$$\times\,(q_0C)^{-m(2s+m)/4}\det\xi_0^{(s+k-1+\nu)/2}\det(q_0^{-1/2}\xi_0)^{(2\nu+m)/2}$$

with $Q(y,s)$ being the polynomial (3.21) and the summation being extended to all pairs h_1, h_2 of matrices $h_1 \in \mathrm{M}^{+}_{m}(\mathbf{Z}), A_m \ni h_2 > 0$ with the condition

$$Cq_0{}^th_1\xi_0^{-1}h_1 + h_2 = M'h, \quad A_m \ni h > 0.$$

The proof of the proposition is an immediate consequence of the preliminary integral representation (3.17) and the formulas (3.18), (3.19), (3.22), (3.23) for the Fourier coefficients .

The final step of proving proposition 3.6 is to deduce it from the already proven proposition 3.8 by applying to the non holomorphic modular forms $\tilde{F}^{\pm}_{M'}(s,\chi)$ the holomorphic projection operator $\mathcal{H}ol$ and using the formulas from theorem 4.6 of chapter 1 which describe its action on Fourier expansions . In order to justify the applicability of this result (more precisely, the statement (b)) to our situation to calculate $\mathcal{H}ol(\tilde{F}^{\pm}_{M'}(s,\chi))$ we note that the positivity property of the Fourier expansion for the functions $G^{\pm}_{M'}(z,s)$ in proposition 3.8 imply the corresponding property for the functions $\tilde{F}^{\pm}_{M'}(s,\chi)$. On the other hand, the moderate growth condition follows from growth estimates given in [St2] (see also [Fe], [Shi10]). The essence of these estimates is that for critical values of s (i.e. for which the corresponding Fourier expansions of $G^{\pm}_{M'}(z,s)$

contain only terms with positive definite matrix indices) these (non holomorphic) modular forms satisfy the same growth estimates that those valid for holomorphic modular forms (see (1.15),(1.16) of chapter 1). Hence we obtain the necessary growth estimates also for functions $\tilde{F}^{\pm}_{M'}(s,\chi)$ and the moderate growth condition (4.3) of chapter 1 is then easily checked by applying to it the same upper estimate as in the integral formula (4.14) of chapter 1 finishing the proof of proposition 3.6.

Now we will prove the theorems 1.3 about the analytic properties of the special values of standard zeta functions. We start with an integral representation for the normalized zeta-function $\mathcal{D}^*(s,f,\chi)$ which is analoguous to that for the distributions $\mathcal{D}^*_{s,f}$ (see (3.10) in proposition 3.5).

3.9. Proposition. *Let $f \in \mathcal{S}^m_k(C,\psi)$ be a cusp form of weight $k \geq m+1$ where m is even,χ be a Dirichlet character modulo $M \geq 1$.Put $N = 4q_0M^2C$ where q_0 is the level of a quadratic form with the matrix $2\xi_0$ such that $a(\xi_0) \neq 0$. Then we have that*

$$a(\xi_0)\mathcal{D}^*_N(s,f,\chi) = \langle f^\rho, K^*(z,s;\xi_0,\chi)\rangle_N,$$

where

$$K^*(z,s;\xi_0,\chi) = N^{-m(2s+m)/4}2^{m(2k-m+2-\kappa)}i^{-m(k-(m/2)-\nu)}\times$$
$$\times \det\xi_0^{(s+k-1+\nu)/2}\theta^{(\nu)}_{2\xi_0}(\chi)G^*(z,(s-k+\nu+m)/2)|W(N),$$

where the subscript N in the notation $\mathcal{D}^*_N(s,f,\chi)$ indicates that all Euler factors corresponding to q, $q|N$ are removed from the Euler product, and the series

$$G^*(z,s) = G^*(z,s;k-(m/2)-\nu,\chi\chi_{\xi_0}\psi,N)$$

being defined in 3.5 , (3.45), of chapter 1.

The proof is deduced from the integral representation (2.15) rewritten in the form

$$2a(\xi_0)((4\pi)^m\det\xi_0)^{-(s+k-1+\nu)/2}\Gamma_m((s+k-1+\nu)/2)\,\mathcal{D}(s,f,\chi) =$$
$$\langle f^\rho(z), \tilde{K}(z,s;\xi_0,\chi)\rangle_N,$$

where

$$\tilde{K}(z,s;\xi_0,\chi) =$$
$$L_N(s+\frac{m}{2},\psi\chi_{\xi_0}\chi)\prod_{i=0}^{(m/2)-1}L_N(2s+2i,\psi^2\chi^2))\times$$
$$\times\,\theta^{(\nu)}_{2\xi_0}(z;\chi)[E(z,(s-k+m+\nu)/2)|_{k-(m/2)-\nu}W(N)]|_{k-(m/2(-\nu}W(N).$$

in this equality we used the definition (3.1)0f the normalized zeta functions, the definition of the series $G^*(z,s)$ and the relation (3.12) for the Γ-factors. Now the theorem 1.3 follows from the proposition 3.9 and the theorem 3.6 of chapter 1 in which we take k to be equal to $k-(m/2)-\nu$.

Note that the function $\mathcal{D}^*_N(s,f,\chi)$ is obtained from the function $\mathcal{D}^*(s,f,\chi)$ by multiplying it to an elementary holomorphic multiple; however we do not know how deduce from the theorem holomorphy properties of the function $\mathcal{D}^*(s,f,\chi)$ itself and this interesting question needs a further study. But under the assumptions of theorem 1.3 we have that

$$\mathcal{D}^*(s,f,\chi) = \mathcal{D}^*_N(s,f,\chi)$$

hence the proof is completed.

3.10. In order to prove the theorem 3.2 about algebraicity properties we need an integtral representation for the functions $\mathcal{D}^{\pm}(s,f,\chi)$ analoguous to that of the proposition 3.9.

Let χ be a Dirichlet character modulo M and assume that all conditions of the theorem 3.2 are satisfied .Put $N = 4q_M^2C$, then we have the following integral representation

$$a(\xi_0)\mathcal{D}^{\pm}_N(s,f,\chi) = \langle f^\rho, K^{\pm}(z,s;\xi_0,\chi)\rangle_N, \tag{3.28}$$

with

$$K^{\pm}(z,s;\xi_0,\chi) = N^{-m(2s+m)/4}2^{m(2k-2-m-\kappa)}i^{-m(k-(m/2)-\nu)}\times$$

$$\times \det\xi_0^{(s+k-1+\nu)/2}\mathcal{H}ol\,[\theta^{(\nu)}_{2\xi_0}(\chi)|W(N)G^{\pm}(z,(s-k+\nu+m)/2)]|W(N),$$

in which

$$K^{\pm}(z,s;\xi_0,\chi) \in \mathcal{M}^m_k(N,\overline{\psi}),$$

the series

$$G^{\pm}(z,s) = G^{\pm}(z,s;k-(m/2)-\nu,\chi\chi_{\xi_0}\psi,N)$$

are defined by (3.47), (3.48) of the previous chapter and the symbol $\mathcal{H}ol$ denotes the holomorphic projection operator from the theorem 4.6 of chapter 1. The proof of (3.28) is carried out in exactly the same way as that of the proposition 3.9 if we take into account the definitions (3.2) and (3.3) of the functions $\mathcal{D}^{\pm}(s,f,\chi)$, the definition of the series $G^{\pm}(z,\chi)$ and use the relation (3.12) for the Γ-factors. The possibility of applying $\mathcal{H}ol$ to the function

$$\theta^{(\nu)}_{2\xi_0}(\chi)|W(N)G^{\pm}(z,(s-k+\nu+m)/2) \tag{3.29}$$

by formulas of the theorem 3.7 of chapter 1 is justified as in the end of 3.8 bearing in mind positivity properties of Fourier expansions of the series $G^{\pm}(z,s)$ in the theorem 3.7 of chapter 1 and the growth estimates mentioned above. It follows from these estimates that the function (3.29) satisfy the bounded growth condition , and its Fourier expansion contains only terms with positive definite matrix indices.

Remark. *If $k > 2m+2$ then for $s = k-\nu-m$ the series defining the function*

$$G^{\pm}(z,0) = G^*(z,0;k-(m/2)-\nu,\chi\chi_{\xi_0}\psi,N)$$

is absolutely convergent so that this function is holomorphic , and we can omit the symbol $\mathcal{H}ol$ in the integral representation (3.28):

$$a(\xi_0)\Gamma^+(k-\nu-m)\mathcal{D}_N(k-\nu-m,f,\chi) = \langle f^\rho, K^{\pm}(z,k-\nu-m;\xi_0,\chi)\rangle_N, \tag{3.30}$$

with

$$K^{\pm}(z,k-\nu-m;\xi_0,\chi)=$$
$$N^{-m(2k-2\nu-m)/4}2^{m(2k-2-m-\kappa)+m(2k-2\nu-m)/2}i^{-m(k-(m/2)-\nu)}\times \tag{3.31}$$
$$\times\,\theta^{(\nu)}_{2\xi_0}(\chi)[G^{\pm}(z,0)|W(N)],$$

with

$$\Gamma^+(s)=(2\pi)^{-m(s+k-\kappa)}\frac{2i^{\delta}\Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi)^s}\prod_{j=1}^{m}\Gamma(s+k-j) \tag{3.32}$$

being the gamma-factor. With these values of k and s the identity of A.N.Andrianov (proposition 2.8 of the chapter 1 and the equality (2.28)) takes the form:

$$a(\xi_0)\mathcal{D}^+(s,f,\chi_N)=a(\xi_0)\mathcal{D}^+(s,f,\chi)=$$
$$\Gamma^+(s)L_N(s+(m/2),\chi\chi_{\xi_0}\psi)\prod_{i=0}^{(m/2)-1}L_N(2s+2i,\chi^2\psi^2)\times \tag{3.33}$$
$$\times\sum_{h\in \mathrm{SL}_m(\mathbf{Z})\backslash \mathrm{M}_m^+(\mathbf{Z})}\chi_N(\det f)a(\xi_0[h])\det h^{-(s+k-1)},$$

where χ be a Dirichlet character modulo N defined by $\chi_N(d)=\chi(d)$ for $\det 2\xi_0|N$. The series in (3.33) is absolutely convergent for $\mathrm{Re}(s)>1+m$ due to the estimate

$$|a(h)|=\mathcal{O}(\det h^{(k/2)+\varepsilon})$$

for the Fourier coefficients.

3.11. Action of the group Aut(C) on scalar products of modular forms. Recall that the group Aut(C) acts on modular forms

$$f=\sum_{A_m\ni\geq 0}a(\xi)e_m(\xi z)\ \in\ \mathcal{M}_k^m(N_1,\psi)$$

by the following rule:

$$f^{\sigma}=\sum_{A_m\ni\geq 0}a(\xi)^{\sigma}e_m(\xi z)\ \in\ \mathcal{M}_k^m(N_1,\psi),\quad (\sigma\in\mathrm{Aut}(\mathbf{C}))$$

and this action commutes with the action of the Hecke algebra, see [Shi5], [St2]. Consider the global Hecke algebra

$$\mathcal{L}(N_1)=\otimes_{q\nmid N_1}\mathcal{L}_q^m(N_1)$$

and suppose that $f\in\mathcal{M}(N_1,\psi)$ is an eigenfunction of the Hecke algebra $\mathcal{L}(N_1)$ with the eigenvalue given by a homomorphism $\Lambda:\mathcal{L}(N_1)\to\mathbf{C}$, i.e. $f|X=\Lambda(X)f$ for all $X\in\mathcal{L}_q^m(N_1)$ and all $q\nmid N_1$. Let $N_1|N$. We define a Λ-packet of modular forms as the following subspace of $\mathcal{S}_k^m(N,\psi)$:

$$H_k^m(\Lambda,N,\psi)=\{f\in\mathcal{S}_k^m(N,\psi)|f|X=\Lambda(X)f,\ X\in\mathcal{L}_q^m(N_1),\ q\nmid N\},$$

and let

$$H_k^m(\Lambda,\psi) = \cup_{N\equiv(\mathrm{mod}\,N_1)} H_k^m(\Lambda,N,\psi),$$
$$\mathcal{S}_k^m(\psi) = \cup_{N\equiv(\mathrm{mod}\,N_1)} \mathcal{S}_k^m(N,\psi).$$

From the fact that the action of Aut(**C**) commutes with the action of the Hecke algebra follows that for each $\sigma \in \mathrm{Aut}(\mathbf{C})$

$$f \in H_k^m(\Lambda,\psi) \iff f^\sigma \in H_k^m(\Lambda^\sigma,\psi^\sigma), \tag{3.34}$$

where $\Lambda^\sigma(X) = \Lambda(X)^\sigma$. On the other hand if we use normality of Hecke operator with respect to the Petersson scalar product and commutativity of the Hecke algebra $\Lambda(N_1)$ we see (as in the classical case) that for a certain set of homomorphisms $\Lambda = \Lambda_1,\cdots,\Lambda_t$ there is the decomposition of $\mathcal{S}(N,\psi)$ into the orthogonal direct sum of the corresponding Λ -packets:

$$\mathcal{S}(N,\psi) \;=\; \bigoplus_{i=1}^{t} H_k^m(\Lambda_i,N,\psi). \tag{3.34a}$$

the folloing proposition was established by J.Sturm ([St2], theorem 3). We state here this result in a form more suitable for our applications.

3.12.Proposition. *Let m be even then for any integer k with $k > 2m+2$ a Dirichlet character ψ* mod N *and a homomorphism $\Lambda : \mathcal{L}(N) \to \mathbf{C}$ there exists a non zero constant $\mu(\Lambda,k,\psi) \in \mathbf{C}^\times$ depending only on Λ,k,ψ such that*

$$\left[\frac{\langle f^\rho, g\rangle_N}{\mu(\Lambda,k,\psi)}\right]^\sigma = \frac{\langle f^{\sigma\rho}, g^\sigma\rangle_N}{\mu(\Lambda^\sigma,k,\psi^\sigma)} \tag{3.35}$$

for all $f \in H_k^m(\Lambda,N,\psi), g \in \mathcal{M}_k^m(N,\overline{\psi})$, $\sigma \in \mathrm{Aut}(\mathbf{C})$.

Remark. *If we take in equality (3.35) g equal to f^ρ then proposition 3.12 implies that*

$$\langle f,f\rangle_N c^{-1} \in \mathbf{Q}(\Lambda,f),$$

with $\mathbf{Q}(\Lambda,f)$ being the subfield of **C** *generated by the values of the homomorphism Λ and the Fourier coefficients of f.*

We give here a proof based on the Andrianov's identity (3.33) in which the right hand side has the form:

$$\mathcal{D}^+(s,f,\chi_N) = \Gamma^+(s) = \prod_{q|N} R(\Lambda,q,k)(\chi(q)\psi(q)q^{-s}) \tag{3.36}$$

where

$$R(\Lambda,q,k)(t) \;\in\; \mathbf{Q}[\Lambda(X), X\in\mathcal{L}_q^m(N)]\,[t]$$

are polynomials with the property $R(\Lambda,q,k)(0)=1$ depending only on the Λ-packet of the form f and on the numbers q and k. The product in(3.36) converges absolutely for $\mathrm{Re}(s) > 1+m$. Put $s = k-\nu-m$ where $\nu = 0,1,\ k\equiv\nu \bmod 2$, take as χ_N the trivial character modulo N and define

$$\mu(\Lambda,k,\psi) = G(\psi)^{m-1}\Gamma^+(k-\nu-m)\prod_{q\nmid 2B} R(\Lambda,q,k)(\psi(q)q^{-(k-\nu-m)}), \tag{3.37}$$

with the Γ factor being defined by (3.32),$B = B(\Lambda, k, \psi)$ being positive integer such that the product in (3.37) does not vanish; such number exists due to the absolute convergence of the product (see the remark in 3.10), and we can and will assume that $B(\Lambda, k, \psi)^\sigma = B(\Lambda^\sigma, k, \psi^\sigma)$.

Now , with the number $\mu(\Lambda, k, \psi)$ already been defined , we prove first the proposition 3.12 for the special modular form $g = G(\psi)^{m-1}K^+(\xi_0, \psi)$, in which the notation $K^+(\xi_0, \psi) = K^+(z, k-\nu-m; \xi_0, \chi_N)$ is adopted in order to stress the dependence of the function $K^+(z, k-\nu-m; \xi_0, \chi_N)$ on ξ_0 and χ_N (recall that χ_N is the trivial character modulo N. Then the folloing identity holds

$$[G(\psi)^{m-1}K^+(\xi_0, \psi)]^\sigma = G(\psi^\sigma)^{m-1}K^+(\xi_0^\sigma, \psi^\sigma). \tag{3.38}$$

This important fact expresses in a more precise form the result of proposition 3.8 of chapter 1 about the cyclotomicity of the Fourier coefficients of this Siegel modular form. The identity (3.38) will be proved later in 3.14 , and now we deduce from it proposition 3.12. According to the equalities (3.36) and (3.30) , the following relation is valid:

$$\frac{G(\psi)^{m-1}\langle f^\rho, K^+(\xi_0, \psi)\rangle_N}{\mu(\Lambda, k, \psi)} = a(\xi_0) \prod_{\substack{q|2B \\ q\nmid \det 2\xi_0}} R(\Lambda, q, k)(\psi(q)q^{-(k-\nu-m)}), \tag{3.39}$$

with the finite Euler product in the right hand side on which the automorphisms $\sigma \in \mathrm{Aut}(\mathbf{C})$ act term-by-term. Therefore, it follows from (3.38) and (3.39) that for the functions of the type $g = G(\psi)^{m-1}K^+(\xi_0, \psi)$ the relation (3.35) is valid.

In order to deal with the general case we vary $\xi_0 \in A_m$, ξ_0 so that the number $N = N(\xi_0)$ will now depend on ξ_0 , and consider the trace operator

$$\mathrm{Tr}(N_2, N_1, \psi) = \sum_{i=1}^{d} \psi(a(i))F|_k g(i). \tag{3.40}$$

where elements $g(i) = \begin{pmatrix} a(i) & b(i) \\ c(i) & d(i) \end{pmatrix}$ form a complete system of representatives of the right cosets:

$$\Gamma_0^m(N_1) = \bigcup_{i=1}^{d} \Gamma_0^m(N_2)g(i).$$

The important property of this action is that it commutes with the trace operator. This fact is stated more precisely in the following proposition.

3.13. Proposition. *Let $F \in \mathcal{M}_k^m(N_2, \psi)$ be a Siegel modular form with cyclotomic Fourier coefficients,*

$$F(z) = \sum_{\xi \geq 0} A(\xi)e_m((\xi z), \quad A(\xi) \in \mathbf{Q}^{\mathrm{ab}}.$$

Then for all $\sigma \in \mathrm{Aut}(\mathbf{C})$ the following equality holds

$$[f|\mathrm{Tr}(N_2, N_1, \psi)]^\sigma = [f^\sigma|\mathrm{Tr}(N_2, N_1, \psi^\sigma)].$$

The proof is given in [St2], lemma 11 , and it is based on properties of the action of the restricted adele group $G_{\mathbf{A}+}$ on the gradued ring of automorphic forms studied by Shimura [Shi5] , where $G_{\mathbf{A}+}$ denote the subgroup of the adelization

$$G_{\mathbf{A}} = \{\alpha \in \mathrm{GL}_{2m} | {}^t\alpha J_m \alpha = \nu(\alpha) J_m,\ \nu(\alpha) \in \mathrm{GL}_1(\mathbf{A})\}$$

consisting of all elements $\alpha \in G_{\mathbf{A}}$ for which Archimedean component of the idele $\nu(\alpha)$ is positive. For $x \in G_{\mathbf{A}+}$ and a modular form $F \in \mathcal{M}_k^m = \bigcup_M \mathcal{M}_k^m(\Gamma^m(M))$ the action of x on F is denoted by F^x. If $t \in \prod_q \mathbf{Z}_q^+$ an idele whose action on $\mathbf{Q}^{\mathrm{ab}}$ by the class field theory coincides with that of $\sigma \in \mathrm{Aut}(\mathbf{C})$, then we have that

$$F^\sigma = F^{x(t^{-1})} \text{ with } x(t) = \begin{pmatrix} 1_m & 0_m \\ 0_m & t1_m \end{pmatrix} \in G_{\mathbf{A}+}$$

for the action on modular forms mentioned above. Now the proposition is easily deduced from this equality. In view of the strong approximation theorem for the group $\mathrm{Sp}_m(\mathbf{A})$ we can choose for each representative $g(i)$ in (3.40) elements $u(i) \in \mathrm{Sp}_m(\mathbf{A}),\ h(i) \in \mathrm{Sp}_m(\mathbf{Z})$ such that $u(i)_q \equiv 1_{2m}(\mathrm{mod} N_2)$ for all primes $q,\ q|N_2$ and

$$\begin{pmatrix} 1_m & 0_m \\ 0_m & t^{-1}1_m \end{pmatrix} g(i) \begin{pmatrix} 1_m & 0_m \\ 0_m & t1_m \end{pmatrix} = u(i)\, h(i).$$

Let us take into account $F^\sigma = F^{x(t^{-1})}$ then we get the equality

$$[(F^\sigma)|\mathrm{Tr}(N_2, N_1, \psi^\sigma]^{\sigma^{-1}} = [\sum_i \psi(a(i))^\sigma F^\sigma | g(i)]^{\sigma^{-1}}$$

and it follows from the choice of $h(i)$ that

$$\Gamma_0^m(N_1) = \cup_{i=1}^d \Gamma_0^m(N_2) h(i),$$

with $h(i) = \begin{pmatrix} a'(i) & b'(i) \\ c'(i) & d'(i) \end{pmatrix}$ and $a(i)' \equiv a(i) \bmod N_2$, so that the proposition follows.

Now we are able to finish the proof of the proposition 3.12. We let the element ξ_0 in the equality (3.39) vary, and put $N_2 = B^2 N \det^2 2\xi_0,\ N_1 = N$.then

$$\langle f^\rho, K^+(\xi_0, \psi)\rangle_{N_2} = \langle f^\rho, K^+(\xi_0, \psi)|\mathrm{Tr}(N_2, N_1, \overline{\psi})\rangle_{N_1},$$

and it follows from the proposition 3.13 that the equality (3.35) is now valid for all modular forms g from the set

$$\mathcal{V} = \left\{ G(\psi)^{m-1} K^+(\xi_0, \psi)|\mathrm{Tr}(N_2, N_1, \overline{\psi}) \middle| A_m \ni \xi_0 > 0, N_2 = B^2 \det^2 2\xi_0,\ N_1 = N \right\}.$$

Let

$$\mathcal{V}_1 = \left\{ g_1 \in H_k^m(\Lambda, N, \overline{\psi}) \middle| g - g_1 \textit{ is orthogonal to some } g \in \mathcal{V} \right\}.$$

In other words , the set $\mathcal{V}_1$ consists of those elements in $H_k^m(\Lambda, N, \overline{\psi})$ which are orthogonal projections of the special elements $g \in \mathcal{V}$ considered above. We claim that $\mathcal{V}_1$

generates the whole Λ-packet $H_k^m(\Lambda, N, \overline{\psi})$. Indeed, if

$$f_1 = \sum_{\xi>0} a_1(\xi) e_m(\xi z) \in H_k^m(\Lambda, N, \overline{\psi})$$

is such that $\langle f_1, g_1 \rangle_N = 0$ for all $g_1 \in \mathcal{V}_1$ then (3.39) implies that $a(\xi_0) = 0$ for all $\xi_0 \in A_m, \xi_0 > 0$ hence $f_1 \equiv 0$. It follows also that proposition 3.12 is valid for all $g_1 \in H_k^m(\Lambda, N, \overline{\psi})$. For an arbitrary Siegel modular form $g \in \mathcal{M}_k^m(N, \overline{\psi})$ write $g = g_1 + h$ with $g_1 \in H_k^m(\Lambda, N, \overline{\psi})$ and $\langle g_1, h_1 \rangle_N = 0$. Now we combine (3.34) and (3,34a) and get that

$$g^\sigma = g_1^\sigma + h_1^\sigma, \quad g_1^\sigma \in H_k^m(\Lambda^\sigma, N, \overline{\psi}^\sigma), \quad \langle g_1^\sigma, h_1^\sigma \rangle_N = 0. \tag{3.41}$$

To obtain the above equality we used the important fact about the invariance of the subspace of Eisenstein series in $\mathcal{M}_k^m(N, \psi)$ under the action of $\sigma \in \mathrm{Aut}(\mathbf{C})$. In turn, this property follows from the general decomposition theorem [Kl3], describing the subspace of Eisenstein series (orthogonal complement to the subspace of cusp forms) in terms of the Klingen-Eisenstein series, and from the invariance properties of such series under the action of $\sigma \in \mathrm{Aut}(\mathbf{C})$, established by M.Harris and other authors (see [Har2],[Kur-Miz], [Miz1], [Miz2]). This last fact stated in a more precise form comprise the content of the Garrett's conjecture, proven in [Har2]. However we do not use this fact any more and therefore will not go into detail of this interesting research. Returning to the equality (3.41) we get

$$\begin{aligned}
&[\langle f^\rho, g \rangle_N \mu(\Lambda, k, \psi)^{-1}]^\sigma = [\langle f^\rho, g_1 \rangle_N \mu(\Lambda, k, \psi)^{-1}]^\sigma = \\
&\langle f^{\sigma\rho}, g_1^\sigma \rangle_N \mu(\Lambda^\sigma, k, \psi^\sigma)^{-1} = \langle f^{\sigma\rho}, g^\sigma \rangle_N \mu(\Lambda^\sigma, k, \psi^\sigma)^{-1},
\end{aligned}$$

To accomplish the proof of proposition 3.12 we need only to check the property (3.38) which is will be now stated in a more general form.

3.14. Proposition . *Let χ mod N be an arbitrary Dirichlet character, and*

$$K^+(\xi_0, \psi, \chi) = K^+(z, k - \nu - m; \xi_0, \psi, \chi) \in \mathcal{M}_k^m(N, \overline{\psi})$$

denote a modular form in the integral representation (3.30). Then for all $\sigma \in \mathrm{Aut}(\mathbf{C})$ there is the following relation

$$[G(\psi\chi)^{m-1} K^+(\xi_0, \psi, \chi)]^\sigma = G(\psi^\sigma \chi^\sigma)^{m-1} K^+(\xi_0, \psi^\sigma, \chi^\sigma) \tag{3.42}$$

Proof. If we look at the definition of the theta series we immediately see that

$$\theta_{2\xi_0}^{(\nu)}(z, \chi)^\sigma = \theta_{2\xi_0}^{(\nu)}(z, \chi^\sigma).$$

Therefore it suffices to check the following property:

$$\begin{aligned}
&[G(\psi\chi)^{m-1} \det\xi_0^{1/2} G^+(z, 0; \chi\chi_{\xi_0}\psi, k - (m/2) - \nu, N)|W(N)]^\sigma = \\
&G(\psi^\sigma\chi^\sigma)^{m-1}((\det\xi_0^) 1/2)^\sigma G^+(z, 0; (\chi\chi_{\xi_0}\psi)^\sigma, k - (m/2) - \nu, N)|W(N),
\end{aligned} \tag{3.43}$$

because of the equality

$$\det\xi_0^{k-(m+1)/2} = \det\xi_0^{k-(m/2)-1}, \quad m/2 \in \mathbf{Z}.$$

According to formulas in §3 (3.53) of chapter 1 , the Fourier coefficient of the series $G^+(z,0;\chi\chi_{\xi_0}\psi,k-(m/2)-\nu,N)$ by $e_m(hz)$ with $h\in A_m, h>0$ has the form

$$(\det h)^{k-\nu-m-(1/2)}L^+(k-\nu-m,\chi\chi_h\chi_{\xi_0}\psi)M(h,\chi\chi_{\xi_0}\psi,s), \tag{3.44}$$

where

$$M(h,\chi\chi_{\xi_0}\psi,s)=\prod_{q\in P(h)} M_q(h,\chi\chi_{\xi_0}\psi(q),q^{-s})$$

is the finite Euler product (3.44) of the previous chapter, the product is extended to all prime numbers q in the set $P(h)$ of prime divisors of N and the elementatry divisors of the matrix h , with the property $M_q(h,t)\in \mathbf{Z}[t]$. Therefore

$$M(h,\chi\chi_{\xi_0}\psi,k-\nu-(m/2))^\sigma = M(h,(\chi\psi)^\sigma\chi_{\xi_0},k-\nu-(m/2)).$$

Now let us consider the factor $L^+(k-\nu-m,\chi\chi_h\chi_{\xi_0}\psi)$ and recall an elementary result about the special values of the Dirichlet L-functions [Le1], [Shi3], [Wa], [La3]. Let χ mod N be a Dirichlet character of conductor N_0, and χ_0 mod N_0 be the corresponding primitive character,

$$G(\chi)=G(\chi_0)=\sum_{x=1}^{N_0}\chi_0(x)e(x/N_0)$$

be its Gauss sum. Put for a positive integer r

$$P(r,\chi)=G(\chi)^{-1}(2\pi i)^{-r}L(r,\chi).$$

Then for all $\sigma\in \mathrm{Aut}(\mathbf{C})$ and $\chi(-1)=(-1)^r$ we have that

$$P(r,\chi)^\sigma = P(r,\chi^\sigma)(3.45)$$

If we apply this property to normalized Dirichlet series we see that

$$[G(\chi)^{-1}L^+(r,\chi)]^\sigma = G(\chi^\sigma)^{-1}L^+(r,\chi^\sigma) \quad r\in\mathbf{Z}, r>0 \tag{3.46}$$

$$L^-(r,\chi)^\sigma = L^-(r,\chi^\sigma) \quad r\in\mathbf{Z}, r\le 0 \tag{3.47}$$

so that for the values of the "wrong parity" the corresponding values vanish.It follows from the basic property of Gauss sums that

$$G(\chi)^\sigma = \chi^\sigma(v)^{-1}G(\chi^\sigma) \ for\ v\in(\mathbf{Z}/N\mathbf{Z})^\times,\ such\ that\ e(1/N)^\sigma = e(v/N). \tag{3.47}$$

The last property implies the useful relation:

$$\frac{G(\psi\chi)^\sigma}{G(\psi^\sigma\chi^\sigma)}=\frac{G(\psi)^\sigma G(\chi)^\sigma}{G(\psi^\sigma)G(\chi^\sigma)}. \tag{3.48}$$

Let us now apply the properties (3.46) and (3.48) to the coefficients (3.44), then we get

$$\begin{aligned}&[G(\psi\chi)^{-1}(\det\xi_0 h)^{1/2}L^+(k-\nu-m,\chi\chi_h\chi_{\xi_0}\psi]^\sigma = \\ &G(\psi^\sigma\chi^\sigma)^{-1}((\det\xi_0 h)^{1/2})^\sigma L^+(k-\nu-m,(\chi\psi)^\sigma\chi_{\xi_0}\chi_h).\end{aligned} \tag{3.49}$$

In the equality (3.49) we used the following elementary property of Gauss sums:

$$G(\chi_{\xi_0}\chi_h)\,(\det(xi_0h))^{1/2} \in \mathbf{Q}$$

which is due to the fact that $\chi_{\xi_0}\chi_h)$ is an even quadratic character (see [B-Šaf]). To complete the proof of (3.43) and (3.44) we need the following general compatibility property of the action of Aut(**C**) and of the involution $W(N)$ (see [St2],lemma 5, and [Shi8]): for a modular form $f \in \mathcal{M}_k^m(N,\psi)$ with cyclotomic Fourier coefficients and for all $\sigma \in \mathrm{Aut}(\mathbf{C})$ we have that

$$f^\sigma|W(N) = \psi(v^m)^\sigma(f|W(N))^\sigma, \tag{3.50}$$

where $v \in (\mathbf{Z}/N\mathbf{Z})^\times$ is chosen so that $e(1/N)^\sigma = e(v/N)$. Now the proof of proposition 3.12 is finished.

3.15. Now in order to deduce the theorem 3.2 about algebraicity of the normalized standard zeta function and the algebraicity property of the normalized distributions we use the already proved algebraicity properties of Fourier coefficients of the functions $K^\pm(z,s;\xi_0,\chi)|W(N)$ in the corresponding integral representation (3.28) and of the functions $F^\pm(z,\chi)$ in (3.13) and (3.14). We apply these properties in the form given below: let $k > 2m+2$, $f \in H_k^m(\Lambda,N,\psi) \subset \mathcal{S}_k^m(N,\psi)$ be a cusp form ,an eigenfunction of the Hecke algebra $\mathcal{L}^m(N)$ with an eigenvalue given as the homomorphism $\Lambda : \mathcal{L}^m(N) \to \mathbf{C}$. Let us consider a linear functional

$$\mathcal{L}_f : g \mapsto \frac{\langle f^\rho, g|W(N)\rangle_N}{\langle f, f\rangle_N} \tag{3.51}$$

on the vector space $\mathcal{M}_k^m(N,\psi)$ with

$$g = \sum_{h\geq 0} b(h)e_m(hz) \in \mathcal{M}_k^m(N,\psi)$$

being its arbitrary element. Then there exist positive matrices $h_1, h_2, \cdots, h_t \in A_m$ and algebraic numbers $\alpha_1, \alpha_2, \cdots, \alpha_t \in \mathbf{Q}(f,\Lambda,\psi)$ from the field $\mathbf{Q}(f,\Lambda,\psi)$ generated by the Fopurier coefficients of f and values of the homomorphism Λ and the character ψ such that for all $g \in \mathcal{M}_k^m(N,\psi)$ the linear functional is explicitly given by

$$\mathcal{L}_f(g) = \sum_i \alpha_i b(h_i). \tag{3.52}$$

Indeed, we notice that every Siegel modular form of weight $k > 2m$ is uniquely determined by its Fourier coefficients with positive matrix indices $h \in A_m$. This fact is equivalent to saying that for such a weight k there are no singular modular forms (i.e. having only Fourier coefficients with degenerate $h \in A_m, \det h = 0$), which was established by G.L.Resnikov ([Res], [Rag3]). Then proposition 3.12 implies that the number

$$(\langle f^\rho, g|W(N)\rangle_N)^{-1}\mu(\Lambda,k,\psi)^{-1} \in \mathbf{Q}(f,g,\Lambda,\psi)$$

belongs to the field $\mathbf{Q}(f, g, \Lambda, \psi)$ generated by the Fourier coefficients of the forms f and g , by the values of Λ and ψ. Moreover we have that

$$\langle f, f\rangle_N\, \mu(\Lambda, k, \psi)^{-1} \in \mathbf{Q}(f, \Lambda, \psi)^{\times}$$

(see the remark after proposition 3.12), and (3.52) follows. In order to prove theorem 3.2 we use proposition 3.12 and take in it the modular forms

$$K^{\pm}(z, s; \xi_0, \chi)|W(N) \in \mathcal{M}_k^m(N; \psi),$$

as g, which have cyclotomic Fourier coefficients vanishing for degenerate matrix indices $h \in A_m$ such that the action of $\sigma \in \mathrm{Aut}(\mathbf{C})$ on them is described as in 3.14. Hence we obtain also the following more explicit description of the action of $\sigma \in \mathrm{Aut}(\mathbf{C})$ on the special values in question. Put

$$\tilde{\mathcal{D}}^{+}(s, f, \chi) = G(\psi\chi)^{m-1}\mathcal{D}^{+}(s, f, \chi)\mu(\Lambda, k, \psi)^{-1},$$

$$\tilde{\mathcal{D}}^{-}(s, f, \chi) = G(\psi\chi)^{m}\mathcal{D}^{-}(s, f, \chi)\mu(\Lambda, k, \psi)^{-1}.$$

Then under the assumption of the theorem 3.5 for every $\sigma \in \mathrm{Aut}(\mathbf{C})$ we have that

$$\tilde{\mathcal{D}}^{\pm}(s, f, \chi)^{\sigma} = \tilde{\mathcal{D}}^{\pm}(s, f^{\sigma}, \chi^{\sigma}), \tag{3.53}$$

proving ,in particular, theorem 3.2. In order to deduce algebraicity properties 3.4 for the normalized distributions we take in 3.12 f be equal to $f_0|V(C) \in \mathcal{S}_k^m(N_0C, \psi)$, and g be equal to $F^{\pm}(z, \chi)$ from (3.13), (3.14). In this situation the constant $\mu(\Lambda, k, \psi)$ depending only on the Λ-packet of f is the same as that for the original cusp f considered above.

§4. Integrality properties and congruences for the distributions

4.1. The proof of theorem 1.6 is based on a regularization od the $\mathbf{Q}$-valued distributions

$$\mathcal{D}^{+}_{s,M}(s) \;\; (1 \le s \le k - \nu m) \; \textit{and} \; \mathcal{D}^{-}_{s,M}(s) \;\; 1 - k + \nu + m \le s \le 0)$$

from §3.

Let c be a positive integer with $(c, N_0) = 1, c > 1$ and $N_0 = 4q_0 \mathrm{M}_0^{\tilde{m}-1}C$. Then we see as in §2 that there exist distributions $\mathcal{D}^{c-}_{s}$, $\mathcal{D}^{c+}_{s}$ on $\mathbf{Z}_S^{\times}$ uniquely defined by

$$\mathcal{D}^{c-}_{s,M}(\chi) = (1 - (\chi\overline{\psi})^2(c)c^{2(s-1)})\mathcal{D}^{-}_{s,M}(\chi), \tag{4.1}$$

$$\begin{aligned}\mathcal{D}^{c+}_{s,M}(\chi) =& C^{s}_{\overline{\psi}\chi} G(\overline{\psi}\chi)^{-1}(1 - (\overline{\chi}\psi)^2 c^{-2s})\mathcal{D}^{+}_{s,M}(\chi)\times \\ &\times \prod_{q|N_0} \{(1 - (\overline{\psi}\chi)(q)q^{s-1})(1 - (\overline{\chi}\psi)(q)q^{-s})^{-1}\},\end{aligned} \tag{4.2}$$

where $\mathcal{D}^{+}_{s,M}(\chi)$ and $\mathcal{D}^{-}_{s,M}(\chi)$ are defined by (3.8) and (3.9). under the assumption of §1 we have $(C, C_{\chi}) = 1$ where C_{χ} is the conductor of χ and it follows from basic elementary relations for Gauss sums that

$$\frac{C^s_{\overline{\psi}\chi}}{G(\overline{\psi}\chi)} = \frac{C^s_\psi C^s_\chi}{G(\overline{\chi})\chi^{-1}(C_\chi)G(\psi)\psi(C_\chi)} = \\ C(\psi)^s\chi(C_\psi)\frac{C^s_\chi}{G(\overline{\chi})}[G(\psi)\psi(C_\chi)]^{-1}. \tag{4.3}$$

In order to define S-adic measures in the main theorem we put $s = 0$ and apply the embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$, then we get

$$\mathcal{D}^{c+} = i_p(\mathcal{D}^{c+}_0), \quad \mathcal{D}^{c-} = i_p(\mathcal{D}^{c-}_0). \tag{4.4}$$

4.2. Proposition. *(a) For all integers s with the condition $1 \le s \le k - \nu - m$ we have that the distributions $i_p(\mathcal{D}^{c+}_0)$ are bounded and the following equality holds*

$$\int_{\mathbf{Z}_S^\times} \chi\, x_p^s\, d\mathcal{D}^{c+} = i_p(\mathcal{D}^{c+}_{s,M}(\chi)), \tag{4.5}$$

in which both sides vanish for $s \not\equiv \delta(\mathrm{mod} 2)$.

(b) For all integers s with the condition $1 - k + \nu + m \le s \le 0$ we have that the distributions $i_p(\mathcal{D}^{c-}_0)$ are bounded and the following equality holds

$$\int_{\mathbf{Z}_S^\times} \chi\, x_p^s\, d\mathcal{D}^{c-} = i_p(\mathcal{D}^{c-}_{s,M}(\chi)), \tag{4.6}$$

in which both sides vanish for $s \equiv \delta(\mathrm{mod} 2)$ (Recall that

$$\nu, \delta = 0, 1\ , (-1)^\nu = \chi(-1), (-1)^\delta = \psi\chi(-1)).$$

4.3. *The proof* of the proposition 4.2 is based on integral representations (3.13) and (3.14) . Taking into account the reguarizing factors in (4.1) and (4.2) we deduce that for the corresponding values of $s \in \mathbf{Z}$ given in the proposition 4.2 the following hold

$$\langle f, f\rangle_C \mathcal{D}^{c\pm}_{s,M}(\chi) = \gamma(M')\langle f_0^\rho | V(C), F^{c\pm}_{M'}(s, \chi_M) | W(CN_0)\rangle_{CN_0}, \qquad (4.7);$$

here

$$F^{c\pm}_{M'}(s, \chi_M) = \sum_{A_m \ni > 0}\ \sum_{C\hat{\xi}_0[h_1] + h_2 = M'h} d^{c\pm}(s, h_1, h_2) e_m(hz) \tag{4.8}$$

are modular forms from $\mathcal{M}^m_k(CN_0, \psi)$ with cyclotomic Fourier coefficients given by

$$d^{c+}(s,h_1,h_2) = \chi_M(\det h_1)\det h_1^{\nu}\det h_2^{(2s-1)/2}P(h_2,h,s)M(h,\overline{\chi}\chi_{\xi_0}\psi,s+(m/2))\times$$
$$\times C^{s}_{\overline{\psi}\chi}G(\overline{\psi}\chi)^{-1}(1-(\overline{\chi}\psi)^2c^{-2s})L_N^{+}(s,\omega_{\xi_0,h_2})\times$$
$$\times \prod_{q|N_0}\{(1-(\overline{\psi}\chi)(q)q^{s-1})(1-(\overline{\chi}\psi)(q)q^{-s})^{-1}\}\times$$
$$\times (q_0C)^{-m(2s+m)/4}\det\xi_0^{(s+k-1+\nu)/2}\det(q_0^{-1/2}\xi_0)^{(2\nu+m)/2}; \tag{4.9}$$

with $\omega_{\xi_0,h_2} = \overline{\chi}\chi_{\xi_0}\psi\xi_{h_2}$ be a Dirichlet character modulo N_0;

$$d^{-}(s,h_1,h_2) = \chi_M(\det h_1)\det h_1^{\nu}P(h_2,h,1-s)\times$$
$$\times (1-(\chi\overline{\psi})^2(c)c^{2(s-1)})L_N^{-}(s,\overline{\chi}\chi_{\xi_0}\psi\xi_{h_2})M(h,\overline{\chi}\chi_{\xi_0}\psi,s+(m/2))\times$$
$$\times (q_0C)^{-m(2s+m)/4}\det\xi_0^{(s+k-1+\nu)/2}\det(q_0^{-1/2}\xi_0)^{(2\nu+m)/2}; \tag{4.10}$$

where $P(v,u,s) \in \mathbf{Q}[v_{ij},u_{ij}; i \le j]$ denotes the polynomial explicitly given by the formula (4.32) of chapter 1 which is defined for all $s \in \mathbf{Z}, 1 \le s \le k-m-\nu, s \equiv \delta(\bmod 2)$ with coefficients independent of f, M, χ, M' and with the property

$$P(v,u,s) = \det v^{(k-\nu-m-s)/2}(\mathrm{mod}(u_{ij})), \tag{4.11}$$

and where

$$M(h,\chi\chi_{\xi_0}\psi,s) = \prod_{q\in P(h)} M_q(h,\chi\chi_{\xi_0}\psi(q),q^{-s})$$

is the finite Euler product (3.44) of the previous chapter, with the product being extended to all prime numbers q in the set $P(h)$ of prime divisors of N and the elementatry divisors of the matrix h, such that for all these q $M_q(h,t) \in \mathbf{Z}[t]$. The summation in the inner sum of (4.8) is taken over all pairs (h_1,h_2) of integral matrices with the conditions

$$h_1 \in \mathrm{M}_m^{+}(\mathbf{Z}),\quad h_2 > 0,\quad h_2 \in A_m,\quad C\hat{\xi}_0[h_1] + h_2 = M'h$$

(i.e. h_1 is a integral matrix with positive determinant, not necessarily symmetric, $h_2 \in A_m$ is a positive definite half integral matrix , and $C\hat{\xi}_0[h_1]$ denotes the matrix given by $C^t h_1 q_0 \xi^{-1} h_1 = Cq_0\xi_0^{-1}[h_1]$.

Now we notice that according to (4.9) and (4.10) the coefficients

$$d^{-}(s,h_1,h_2) = d^{-}(\chi_M; s,h_1,h_2)$$

does not depend on modulus M for the character χ_M (that is, they satisfy the compatibility crierium for distributions) and define for fixed h_1, h_2 a distribution on $G_S = \mathbf{Z}_S^{\times}$ with values in $\mathbf{Q}^{\mathrm{ab}}$; these distributions will also be denoted by $d^{-}(s,h_1,h_2)$. As we soon will see these distributions turn out to be bounded measures , and the measures of proposition 4.2 will be expressed in terms of them.

4.4. Let us consider the $\mathbf{C}$ -linear functional

$$\mathcal{L}_f : g \mapsto \frac{\langle f^\rho, g|W(N)\rangle_N}{\langle f, f\rangle_N} \tag{4.14}$$

on the vector space $\mathcal{M}_k^m(N,\psi)$ defined by (3.51) whose explicit description is given in 3.15 with

$$g = \sum_{h\geq 0} b(h)e_m(hz) \in \mathcal{M}_k^m(N,\psi)$$

being arbitrary element of the vector space on which the functional is defined: there exist positive matrices $h_1, h_2, \cdots, h_t \in A_m$ and algebraic numbers $\alpha_1, \alpha_2, \cdots, \alpha_t \in \mathbf{Q}(f,\Lambda,\psi)$ from the field $\mathbf{Q}(f,\Lambda,\psi)$ generated by the Fourier coefficients of f and values of the homomorphism Λ and the character ψ such that for all $g \in \mathcal{M}_k^m(N,\psi)$ the linear functional is explicitly given by

$$\mathcal{L}_f(g) = \sum_i \alpha_i b(h_i). \tag{4.15}$$

According to proposition 4.5 of chapter 1 the values of the distributions $\mathcal{D}_s^{c\pm}$ can be represented in terms of the functional $\mathcal{L}$ as follows

$$\mathcal{D}_s^{c\pm} = \gamma(M')\mathcal{L}(F_{M'}^{c\pm}(s,\chi_M)). \tag{4.16}$$

Combining (4.15) and (4.16) we see that

$$\mathcal{D}_s^{c\pm} = \sum_i \alpha_i\, v^{c\pm}(M'h^{(i)}, s, \chi_M), \tag{4.17}$$

with

$$v^{c\pm}(M'h^{(i)}, s, \chi_M) = \sum_{C\hat{\xi}_0[h_1]+h_2=M'h} d^{c\pm}(s, h_1, h_2) \tag{4.17a}$$

being Fourier coefficients of the functions $F_{M'}^{c\pm}(s,\chi_M)$. Therefore the statements (a) and (b) of proposition 4.2 under the assumptions of the main theorem are equivalent to the corresponding statements about the distributions $d^{c\pm}(s,h_1,h_2)$. Indeed, the value of $|i_p(\gamma(M')|_p$ remain unchanged with the varying M' by the definition of $\gamma(M')$ in (3.10),

$$\gamma(M') = 2^{m(2k-2-m-\kappa)} i^{-m(k-(m/2)-\nu)} a(\xi_0)^{-1}\alpha_0(M_0^{\tilde{m}-1}M'^{-1})(CM_0^{\tilde{m}-1})^{(k-1-m)/2}$$

if we remember the condition $|i_p(M_0)|_p = 1$ of the form f to be p-ordinary which implies that the denominators in the linear combination (4.17) are uniformly bounded with varying χ_M and M'. Below in 4.6 is given a more precise argument based on the generalized Kummer congruences. However we show first that the distributions $d^{c\pm}(s,h_1,h_2)$ essentially reduce to the S-adic Mazur measure for the Kubota-Leopoldt zeta function (see [Ku-Le], [Le2], [Maz-SD], [Man4], [Man6], [Wa]) whose properties are recalled in the following 4.5.

4.5. Let ω mod A be a fixed primitive Dirichlet character such that $(A, M_0) = 1$ with $M_0 = \prod_{q\in S} q$. Put $\overline{S} = S \cup S(A)$, $\overline{M} = \prod_{q\in\overline{S}} q$. Then for any positive integer

c with $(c,\overline{M}) = 1,\ c > 1$ there exist $\mathbf{C}_p$-measures $\mu^+(c,\omega), \mu^-(c,\omega)$ on $\mathbf{Z}_S^\times$ which are uniquely defined by the following conditions:

$$i_p^{-1}[\int_{\mathbf{Z}_S^\times} \chi\, x_p^s\, d\mu^+(c,\omega)] = (1-\overline{\chi}\omega(c)c^{-s})\frac{C_{\omega\overline{\chi}}}{G(\omega\overline{\chi})}\times$$
$$\times \prod_{q\in S\setminus S(\chi)} \{(1-\chi\overline{\omega}(q)q^{s-1})/(1-\overline{\chi}\omega(q)q^{-s})L^+_{M_0}(s\}\overline{\chi}\omega) \tag{4.18}$$

for $s \in \mathbf{Z}, s > 0$, and

$$i_p^{-1}[\int_{\mathbf{Z}_S^\times} \chi\, x_p^s\, d\mu^-(c,\omega)] = (1-\chi\overline{\omega}(c)c^{s-1})L^-_{M_0}(s,\overline{\chi}\omega) \tag{4.19}$$

for $s \in \mathbf{Z}, s \leq 0$ where

$$L^+_{M_0}(s,\overline{\chi}\omega) = L_{\overline{M}}(s,\overline{\chi}\omega)\frac{2i^\delta\Gamma(s)\cos(\pi(s-\delta)/2)}{(2\pi)^s}, \tag{4.20}$$

$$L^+_{M_0}(s,\overline{\chi}\omega) = L_{\overline{M}}(s,\overline{\chi}\omega) \tag{4.21}$$

are the normalized Dirichlet L-functions with $\delta = 0,1,\ (-1)^\delta = \overline{\chi}\omega(-1)$. The functions (4.18) and (4.19) satisfy the following functional equation

$$L_{M_0}(1-s,\chi\overline{\omega}) = \prod_{q\in S\setminus S(\chi)} \{(1-\chi\overline{\omega}(q)q^{s-1})/(1-\overline{\chi}\omega(q)q^{-s})\}L^+_{M_0}(s,\overline{\chi}\omega). \tag{4.22}$$

The properties (4.18) - (4.22) easily follow when we remember the definition of the $\overline{S}$-adic Mazur measure μ^c on $\mathbf{Z}_{\overline{S}}^\times$ for which (4.18) and (4.19) are given by the equalities

$$\int_{\mathbf{Z}_S^\times} x\, d\mu^-(c,\omega) = \int_{\mathbf{Z}_{\overline{S}}^\times} x^{-1}\omega\, d\mu^c,$$

$$\int_{\mathbf{Z}_S^\times} d\mu^+(c,\omega) = \int_{\mathbf{Z}_{\overline{S}}^\times} x\, x_p^{-1}\omega^{-1}\, d\mu^c,$$

where we understand that $x \in X_S \subset X_{\overline{S}}$.

4.6. To accomplish the proof of proposition 4.2 we use *the abstract Kummer congruences* ([Ka3],p.258) which give a criterion of boundness of p-adic valued distributions as follows. Let $\{f_i\}$ be a family of continuous functions $f_i \in \mathcal{C}(Y,\mathcal{O}_p)$ in the ring $\mathcal{C}(Y,\mathcal{O}_p)$ of all continuous functions over the compact totally disconnected group $Y = \mathbf{Z}_S^\times$ with values in the ring of integers $\mathcal{O}_p$ of $\mathbf{C}_p$ such that the $\mathbf{C}_p$-linear span of $\{f_i\}$ is dense in $\mathcal{C}(Y,\mathbf{C}_p)$. Let also $\{a_i\}$ be a family of elements $a_i \in \mathcal{O}_p$. Then existence of a $\mathcal{O}_p$ -valued measure μ on Y with the property

$$\int_Y f_i\, d\mu = a_i$$

is equivalent to the validity of the following congruences: for an arbitrary choice of elements $b_i \in \mathbf{C}_p$ almost all of which vanish

$$\sum_i b_i f_i(y) \in p^n \mathcal{O}_p \text{ for all } y \in Y \textit{ implies } \sum_i b_i a_i \in p^n \mathcal{O}_p.$$

In our situation we take the family of functions of the type χx_p^s with s as in proposition 4.2 with $\chi \in X_S^{\mathrm{tors}}$ denoting Dirichlet characters to be the family $\{f_i\}$ with the dense $\mathbf{C}_p$-linear span. For any finite number of Dirichlet characters $\chi \in X_S^{\mathrm{tors}}$ we choose an integer M and a sufficiently large integer M' such that each of these characters is defined modulo M and the formula of the proposition 3.6 is valid for the values of distributions $\mathcal{D}_{s,M}^{c\pm}$. Then we apply the functional $\mathcal{L}$ and use the equality (4.17). As it was mentioned above , the coefficients of the linear combination in (4.17) are p-adically bounded for varying M' and χ hence the proof of the abstract Kummer congruences for the numbers $\mathcal{D}_{s,M}^{c\pm}$ is reduced to checking the correspondent congruences for the numbers $v^{c\pm}(M'h, s, \chi)$ (see (4.17a)). In turn, in order to do this we fix h_1 and use the formulas (4.9) and (4.10). It is seen from the relation $C\hat{\xi}_0[h_1] + h_2 = M'h$ in (4.8) that the following congruence holds:

$$\det h_2 \equiv (-C)^m \det\hat{\xi}_0 \det h_1^2 \pmod{M'} \tag{4.23}$$

with $\det\hat{\xi}_0 = q_0^m \det\xi_0^{-1}$. If we then use the property (4.11) of the polynomial $P(v, u, s)$ and (4.23) then we get for the factor $P(h_2, h, s)$ in (4.9) and (4.10) the following congruence:

$$P(h_2, h, s) \equiv [(-Cq_0)^m \det\xi_0{}^{-1} \det h_1^2]^{(k-\nu-m-s)/2} \pmod{M'} \tag{4.24}$$

where $1 \le s \le k - \nu - m,\ s \equiv \delta(\mathrm{mod} 2)$.

It can happen that the congruence (4.24) is valid only modulo a slightly smaller positive integer than M' (i.e. obtained from M' by dividing it by a divisor independent on the choice of M'). However, with the growing M' we may ignore this divisor when we multiply the numbers (4.9) and (4.10) by a suitable positive integer independent of M'.

Recall that in formulas (4.9) and (4.10) we used the notation $\omega_{\xi_0,h_0} = \omega$ for the primitive Dirichlet character associated with $\chi_{\xi_0}\chi_{h_2}\psi$,and if $\det h_2 \det(\hat{\xi}_0) = a^2 t$ with a square free integer t then we have that $\omega = \chi_t \psi$ where χ_t is the primitive Dirichlet cxharacter associated with the quadratic field $\mathbf{Q}(\sqrt{t})$. The congruence (4.23) imply in particular that $t \equiv 1(\mathrm{mod} 4)$ hence the conductor of χ_t is equal to t. Indeed if $q|M'$ (e.g. $q = 2$) then according to (4.23) we get

$$\left(\frac{\det h_2 \det\hat{\xi}_0}{q}\right) = \left(\frac{\det h_1^2 \det\hat{\xi}_0^2 (-C)^m}{q}\right),$$

in view of the parity of m and it follows that $\omega(q) = \psi(q)$ for $q|M_0$ $(q \in S)$ and $\omega^2 = \psi^2$, and also that $(t, M_0) = 1$.

Now we compare the formulas (4.9) and (4.10) with the corresponding formulas (4.18) and (4.19) for the Dirichlet L-series, and take into account that

$$\begin{aligned} &C_{\omega\overline{\chi}} = tC_{\psi\overline{\chi}},\ \omega = \chi_t\psi,\ \chi_t(C_{\psi\overline{\chi}}) = 1, \\ &G(\omega\overline{\chi}) = G(\chi_t\psi\overline{\chi}) = \chi_t(C_{\psi\overline{\chi}}\psi\overline{\chi}(t)G(\chi_t)G(\psi\overline{\chi}) = \psi\overline{\chi}(t)\sqrt(t)G(\psi\overline{\chi}), \end{aligned} \tag{4.25}$$

$$(1-(\overline{\chi}\psi)(c)^{-2s}) = (1-(\omega\overline{\chi}c^{-s})(1+(\omega\overline{\chi}c^{-s}) \,. \tag{4.26}$$

Next let us apply the embedding i_p to (4.9) and (4.10) keeping in mind (4.25) and (4.26), then (4.9) and (4.9) take the following form

$$\begin{aligned} &i_p[d^{c+}(s,h_1,h_2)] = \\ &\frac{\psi(t)\sqrt{t}}{\chi(t)t^s}(1+(\omega\overline{\chi}c^{-s})\int_{\mathbf{Z}_S^\times}\chi\, x_p^s\, d\mu^-(c,\omega)\times \end{aligned} \tag{4.27}$$

$$\begin{aligned} &\times\ \chi_M(\det h_1)\det h_1^\nu \det h_2^{(2s-1)/2}P(h_2,h,s)M(h,\overline{\chi}\chi_{\xi_0}\psi,s+(m/2))\times \\ &\times (q_0C)^{-m(2s+m)/4}\det\xi_0^{(s+k-1+\nu)/2}\det(q_0^{-1/2}\xi_0)^{(2\nu+m)/2}; \end{aligned}$$

for $1\le s\le k-\nu-m$,

$$\begin{aligned} &i_p[d^-(s,h_1,h_2)] = \\ &(1+(\omega\overline{\chi}c^{-s})\int_{\mathbf{Z}_S^\times}\chi\, x_p^s\, d\mu^-(c,\omega)\times \\ &\times\ \chi_M(\det h_1)\det h_1^\nu P(h_2,h,1-s)M(h,\overline{\chi}\chi_{\xi_0}\psi,s+(m/2))\times \\ &\times (q_0C)^{-m(2s+m)/4}\det\xi_0^{(s+k-1+\nu)/2}\det(q_0^{-1/2}\xi_0)^{(2\nu+m)/2}; \end{aligned} \tag{4.28}$$

for $1-k+\nu+m\le s\le 0$.

Notice that the finite Euler product $M(h,\overline{\chi}\chi_{\xi_0}\psi,s+(m/2))$ is a finite linear combination of terms of the type $\overline{\chi}(b)b^s$ $((b,M')=1)$ whose coefficients are algebraic integers independent of χ. Bearing in mind also the congruences (4.23) we get from (4.27) and (4.28) that for the corresponding values of s the expression for $v^{c\pm}(M'h,s,\chi) \bmod M'$ takes the form

$$\begin{aligned} &\sum_i A_i\chi(y_i)y_i^s\int_{\mathbf{Z}_S^\times}\chi\, x_p^s\, d\mu^\pm(c,\omega) = \\ &\sum_i A_i\int_{\mathbf{Z}_S^\times}(\chi\, x_p^s)(y_iy)\, d\mu^\pm(c,\omega)(y)\quad (y,y_i\in\mathbf{Z}_S^\times), \end{aligned} \tag{4.29}$$

with uniformly p-adically bounded algebraic coefficients $A_i\in i_p(\mathbf{Q}^{\mathrm{ab}})$. It remains to notice that the abstract Kummer congruences are tautologically valid for the expressions of the type (4.29) which obviously satisfy the identities of the form (4.5) and (4.6). Hence the corresponding statements valid also for the distributions $\mathcal{D}_s^{c\pm}$ proving proposition 4.2.

Finally, we get the proof of the main theorem as a combination of all the results obtained above: proposition 4.2, the relation of the values of the normalized distributions with the values of the normalized standard zeta-functions (proposition 2.3), the definitions of the normalizing factors (3.2), (3.3), (3.8), (3.9) and of the regularizing factors (4.1), (4.2).

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