CONTINUOUS COHOMOLOGY OF THE GROUP OF VOLUME-PRESERVING AND SYMPLECTIC DIFFEOMORPHISMS, MEASURABLE TRANSFER AND HIGHER ASYMPTOTIC CYCLES

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Topology of a manifold is reflected in its diffeomorphism group. It is challenging therefore to understand the diffeomorphism group $Diff(M)$ both as a topological and discrete group. Twenty years ago, some work had been done, in connection with characteristic classes of foliations, in constructing continuous cohomology classes for $Diff(M)$. For M closed oriented n-dimensional manifold, a class in $H^{n+1}_{cont}(Diff(M),\mathbb{R})$ had been explicitly written down by Bott [Bo] [Br]. This class is defined as follows. The group $Diff(M)$ acts in the multiplicative group $C^\infty_+(M)$ of positive smooth functions, and on its torsor $A_n(M)$ of volume forms. Hence one gets a cocycle in $H^1_{cont}(Diff(M), C^\infty_+(M))$, defined by $\lambda(f) = \frac{f^*(v)}{v} = Jac_v(f)$, where $\nu \in A_n(M)$ and $f \in Diff(M)$. The Bott class is

$$\int_M \log\lambda \cup \underbrace{d\log\lambda \cup \ldots \cup d\,\log\lambda}_{n}$$

The nontriviality of Bott class had been shown for $M = S^1$ [Br], and recently for S^n [BCG], $\mathbb{C}P^n$ [Go] by restricting to finite-dimensional Lie groups in $Diff(M)$. In fact, the restriction of the Bott class on $SO(n,1) \subset Diff(S^n)$ gives the hyperbolic volume class, whereas the restriction on $PSL(n+1,\mathbb{C}) \subset Diff(\mathbb{C}P^n)$ gives the Borel class.

By its construction, the Bott class vanishes on the group $Diff_\nu(M)$ of volume-preserving diffeomorphisms. Moreover, since it is defined by an invariant closed $(n+1)$ -form in the space $A_n(M)$ where $Diff(M)$ acts, and by a theorem of Brooks [Br] there are no more invariant forms there, one gets just one class in dimension $(n+1)$ for a fixed manifold M. This contrasts sharply the usual intuition coming from the study of finite-dimensional semisimple group, where there is a range of continuous cohomology classes.

In this paper we construct, for a closed manifold M^n with a volume form ν, a series of continuous cohomology classes in $H^\kappa_{cont}(Diff_\nu(M),\mathbb{R})$ for all $\kappa = 5, 9, \ldots$. The classes will be shown nontrivial already for a torus T^n. We also will construct, for a symplectic manifold (M,w), a series of classes in $H^{2\kappa}(Sympl(M),\mathbb{R})$

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for $\kappa = 1, 3, \ldots$. Again, these are nontrivial for a torus T^n with standard symplectic structure.

Working harder, we will show that for the smooth moduli space of stable vector bundles over a Riemann surface $\mathcal{M}$ with its Kähler structure, our class in $H^2(Symp(\mathcal{M}_g), \mathbb{R})$ is nontrivial and restricts to a generator of $H^2(Map_g, \mathbb{R})$, where Map_g is the mapping class group:

Theorem (3.6). *$H^2(Sympl(\mathcal{M}_g), \mathbb{R})$ is nontrivial. Moreover, the homomorphism $Map_g \to Sympl(\mathcal{M}_g, \mathbb{R})$ induces a nontrivial map in the second real cohomology.*

In both cases, our classes arise from action on a "principal homogeneous space" X which in the case of $Diff_\nu(M)$ will be the space of Riemannian metrics with volume form ν, and in the case of $Sympl(M)$ will be the twistor variety, introduced in [Re] [Rel]. In that paper we have studied the symplectic reduction of X with respect to the Hamiltonian action of subgroups of $Sympl(M)$ with a primal interest in integrable systems arising on Teichmüller space and universal Jacobian. A lenghty computation from [Re] [Rel] related to the existence of the moment map will be used here to prove a vanishing result in 5.4.

There is quite another way to look at our classes, from the stand point of the transfer map. The subgroup $Diff_\nu^0(M)$ of $Diff_\nu(M)$ which fixes a point $p \in M$, has the tangential representation to $SL_n(\mathbb{R})$ and one can pull the Borel classes back on $Diff_\nu^0(M)$. The transfer map [G] [Gu] will send these classes to $H^*_{cont}(Diff_\nu(M))$. We will not however prove a rigorous comparison theorem relating these two types of construction in the present paper. However we do use the transfer map to define a new source of classes in $H^*(Diff_\nu(M))$ coming from the fundamental group of M. Namely, a map

$$S : H^\kappa(\pi_1(M), \mathbb{R}) \to H^\kappa(Diff_\nu^\sim(M), \mathbb{R})$$

will be constructed where $Diff_\nu^\sim(M)$ is the connected component of $Diff_\nu(M)$. For $\kappa = 1$, the dual of this map, a character

$$S^\vee : Diff_\nu^\sim(M) \to H_1(M, \mathbb{R})$$

has been known for forty years [Sch] and called the asymptotic cycle map. One can view our map S as "higher" asymptotic cycle map.

For M a closed surface with an area form, the groups $Diff_\nu(M)$ and $Sympl(M)$ coincide. The two previously described constructions produce a class in $H^2_{cont}(Diff_\nu(M))$ which we will show to lie in bounded cohomology group $H^2_b(Diff(M), \mathbb{R})$. For $f, g \in Diff_\nu(M)$ we give an explicit formula fora cocycle $\ell(f, g)$ representing this class. For any lamination on M [Th] one can exhibit quite a different formula, using the expression for Euler class from [BG].

The following application of dynamical nature will be proven. Let F_2 be a free group in two generators, and let, for some words h_i, k_i in F_2, a sum $\sum_{i=1}^\infty a_i(h_i, k_i)$, $\Sigma|a_i| < \infty$ be a cycle for ℓ^1 -homology of F_2. This homology has dimension $2^{\aleph_0}$, as shown in [ ]. Let M be a closed surface with an area form ν. Given $f, g \in Diff_\nu(M)$ one has a homomorphism $F_2 \to Diff_\nu(M)$, so the words h_i, k_i may be viewed as diffeomorphisms in $Diff_\nu(M)$.

Theorem (4.2). *Suppose* $\sum_{i=1}^{\infty} a_i\, \ell(h_i, k_i) \neq 0$. *Then the group generated by* f, g *in* $Diff_\nu(M)$ *is not amenable.*

The significance of Theorem stems from the fact that the condition $\Sigma\, a_i \ell(h_i, k_i) \neq 0$ is C^1 -open on f, g. Therefore one gets a domain in $Diff_\nu(M) \times Diff_\nu(M)$, such that any pair (f, g) in it generate a "big" group in $Diff_\nu(M)$. One can see this result as a step towards "Tits alternative" for the infinite-dimensional Lie group $Diff_\nu(M)$.

We will show in the next paper that this theorem holds for M symplectic of higher dimension. For that purpose we ill use Lagrangian measurable foliations and Lyon-Vergne Maslov class to show that our class in $H^2(Sympl(\mathcal{M}_g, \mathbb{R})$ is bounded. See also the end of [BG].

In [Re2] we defined the "symplectic Chern-Simons" classes $K_{2i-1}^{alg}(Sympl(M)) = \pi_{2i-1}((B\,Symp)^\delta(M))^+) \to \mathbb{R}/A$, where A is the group of periods of the Cartan form in $\Omega_{cl}^{2i-1}(\mathrm{Sympl}^{top}(M))$, introduced in [Re2], on the Hurewitz image of $\pi_{2i-1}(\mathrm{Sympl}^{top}(M))$ in $H_{2i-1}(\mathrm{Sympl}^{top}(M), \mathbb{R})$. The real classes introduced in the present paper seem to be in the same relation to the symplectic Chern-Simons classes as Borel classes in $H^*_{cont}(SL_n(K), \mathbb{R})$ are to proper Chern-Simons classes ($K = \mathbb{R}, \mathbb{C}$). The "symplectic Chern-Simons classes" of [Re2] have remarkable rigidity property: for a continuous family of representations of a f.p. group Γ into $\mathrm{Sympl}(M)$, the pull-back of these classes are constant in $H^*(\Gamma)$. This contrasts strikingly the famous non-rigidity of the Bott class, proved by Thurston. In fact, Thurston exhibited a family of homomorphism $\pi_1(S) \to Diff(S^1)$, where S is a closed surface of genus two, with varying Godbillon-Vey class (which coincides with the Bott class for $Diff(S^1)$).

We do not know if the real classes constracted in the present paper in $H^*(Diff_\nu(M))$ and $H^*(\mathrm{Sympl}(M))$ are rigid. However, we introduce a new "Chern-Simons" class in $H^3(Diff_\nu(S^3), \mathbb{R}/\mathbb{Z})$ which is rigid and restricts to usual Chern-Simons class on $H^3(SO(4), \mathbb{R}/\mathbb{Z})$. This uses the invariant scalar product on Lie $(Diff_\nu(S^3))$ in much the same way we used invariant polynomials on Lie $(\mathrm{Sympl}(M))$ in [Re2].

1. Forms on the space of metrics

We work with the manifold M with the fixed volume form ν. Define the space $\mathcal{P}$ as the Frechet manifold of C^∞ -Riemannian metrics on M, whose volume form is ν. Obviously, $Diff_\nu(M)$ acts on $\mathcal{P}$. We can look at $\mathcal{P}$ as a space of sections of a fibration $\mathbb{R} \to M$ with a fiber $SL_N(\mathbb{R})/SO(N)$, where $N = \dim M$. Clearly, $\mathcal{M}$ is contractible. For any $n = 5, 9, \ldots$ fix the Borel form: a $SL_N(\mathbb{R})$ -invariant closed n-form on $SL_N(\mathbb{R})/SO(N)$, normalized as in [Bo]. For a vector space V of dimension N with a volume form ν this gives a canonical choice of a closed form on the space $\mathcal{P}^V$ of Euclidean metrics on V with determinant ν. Call this form ψ_n^V. Now, we define a form on $\mathcal{P}$ by $\psi_n = \int_M \psi_n^{T_x M} d\nu(x)$. That means the following: let $g \in \mathcal{P}$ a Riemannian metric on M. Let $h_1, \ldots, h_n \in T_g\mathcal{P}$ be symmetric bilinear smooth 2-forms. Define $\psi_n(h_1, \ldots, h_n) = \int_M \psi_n^{T_x(M)}(h_1(x), \ldots, h_n(x)) d\nu$.

Lemma (1.1). *The form* $\psi \in \Omega^n(\mathcal{P})$ *is closed and* $Diff_\nu(M)$ *-invariant.*

Proof. The invariance is obvious from definition. To prove the closedness, observe first that a form $\psi_n(x_1, \ldots, x_m)(h_1, \ldots, h_n) = \sum_{j=1}^m \lambda_j \psi_n^{T_{x_j}(M)}(h_1(x_j), \ldots, h_n(x_j))$

is closed as a pull-back of a closed form under the map $\mathcal{P} \mapsto \prod_{j=1}^{m} \mathcal{P}^{T_{x_j}(M)}$. Now one approximates ψ by $\psi_n(x_1, \dots, x_m)$ to show that ψ is closed.

1.2 The definition of the classes. We will now apply a general theory of regulators, as presented in [Re1], section 3. For a Frechet-Lie group $\mathfrak{G}$, acting smoothly on a contractible smooth manifold Y, preserving a closed form ψ_n, this theory prescribes a class in $H^n(\mathfrak{G}^\delta, \mathbb{R})$, called $r(\psi_n)$ in [Re1].

Definition (1.2). Consider the action of $Diff_\nu(M)$ on the contractible manifold $\mathcal{P}$ with the invariant form ψ_n as above. A class $\gamma_n \in H^n(Diff_\nu^\delta(M), \mathbb{R})$ is defined as $r(\psi_n)$.

Theorem (1.3). *The class γ_n lies in the image of the natural map*

$$H^n_{cont}(Diff_\nu(M), \mathbb{R}) \to H^n(Diff_\nu^\delta(M), \mathbb{R}).$$

The proof follows from Proposition 1.3 below.

1.3 Simplices in $\mathcal{P}$ and a Dupont-type construction. Fix two metrics g_1, g_2 in $\mathcal{P}$. We can join them by a segment in two different ways. First, there is a straight line segment $I_{g_1,g_2}(t) : t \mapsto t \cdot g_1 + (1-t)g_2$. Second, there is a geodesic segment $J_{g_1,g_2}(t) : t \mapsto (x \mapsto c(t, g_1(x), g_2(x)))$. Here $t \in [0,1], x \in M, g_1(x), g_2(x) \in \mathcal{P}^{T_x}(M)$ and $c(t, g_1(x), g_2(x))$ is a geodesic segment in the homogeneous metric of symmetric space on $\mathcal{P}^{T_x(M)} \approx SL_N(\mathbb{R})/SO(N)$. Now, having n metrics $g_1, \dots, g_n$ in $\mathcal{P}$ we define two singular simplices $I_{g_1 \dots g_n} : \sigma \to \mathcal{P}$ and $J_{g_1 \dots g_n} : \sigma \to \mathcal{P}$ by induction as a joint of g_1 and $I_{g_2, \dots, g_n}$, (resp. g_1 and $J_{g_2 \dots g_n}$) using straight line segments (resp. geodesic segments, comp [Th2]).

Now fix a reference metric g in $\mathcal{P}$. Define

$$\gamma_n^I(g_1, \dots, g_n) = \int_{I(\qquad)} \psi_n$$

and

$$\gamma_n^J(g_1, \dots, g_n) = \int_{J(\qquad)} \psi_n$$

Proposition (1.3). *Both γ_n^I and γ_n^J are continuous cocycles, representing γ_n.*

Proof. The proof mimics the finite-dimensional case, cf. [Du], and is therefore omitted.

2. Non-triviality

We will prove that the class γ_n in discrete group cohomology, and consequently classes of γ_n^I and γ_n^J in continuous cohomology are non-trivial in general. For that purpose, consider a torus $T^N = \mathbb{R}^N/\mathbb{Z}^N$ with a standard volume form $dx_1 \dots dx_N$. We have an inclusion

$$SL(N, \mathbb{Z}) \hookrightarrow Diff_\nu(T^N)$$

Proposition (2.1). *The class γ_n restricts to the Borel class in $H^n(SL(N,\mathbb{Z}),\mathbb{R})$ and is therefore nontrivial for N big enough.*

Proof. Let $\mathcal{P}_0$ be the space of left-invariant metrics on T^N with the determinant ν; as a manifold, $\mathcal{P}_0 \approx SL_N(\mathbb{R})/SO(N)$. The embedding $\mathcal{P}_0 \hookrightarrow \mathcal{P}$ is $SL_N(\mathbb{Z})$ -invariant, and the pull-back of the form ψ_n on $\mathcal{P}_0$ is the Borel form on $\mathcal{P}_0$. Now by [Re1], section 3, $r(\psi_n)$ coincides with the Borel class.

3. Cohomology of symplectic diffeomorphisms

We will now adapt the theory for the group Sympl(M) of symplectic diffeomorphisms of a compact symplectic manifold M. For this purpose, we will introduce a new (∞ -dimensional) contractible manifold Z, on which $Sympl(M)$ acts, preserving some differential forms of even degree.

3.1 Principal transformation space. Let $\mathfrak{F}$ be the fibration over M^{2n}, whose fiber over $x \in M$ consists of complex structures in T_xM, say J, such that ω_x is J-invariant and the symmetric form $\omega(J\,\cdot,\cdot)$ is positive definite. Alternatively, $\mathfrak{F}$ is a $Sp(2n,\mathbb{R})/U(n)$ fiber bundle over M, associated to the $Sp(2n,\mathbb{R})$ -frame bundle. The principal transformation space Z is defined as a space of C^∞ -sections of $\mathfrak{F}$. So a point in Z is just an almost-complex structure on M, tamed by ω, in the sense of Gromov [Gr]. Since the Siegel upper half-plane $Sp(2n,\mathbb{R})/U(n)$ is contractible, the space Z is contractible, too.

3.2 Forms on Z. Fix an $Sp(2n,\mathbb{R})$ -invariant form on $Sp(2n,\mathbb{R})/U(N)$. This induces a form φ^{T_xM} on $\mathcal{F}_x$ for each $x \in M$ and a form

$$\varphi = \int_M \varphi^{T_xM} \cdot \omega^n$$

as in 1.1. Obviously, this form φ is Sympl(M) -invariant. Recall that the ring of $Sp(2n,\mathbb{R})$ -invariant forms on $Sp(2n,\mathbb{R})/U(n)$ is generated by forms in dimensions $2,6,\dots$ [Bo].

Correspondingly, we have Sympl(M) -invariant closed forms, in same dimensions.

We single out the symplectic (Kähler) form on $Sp(2n,\mathbb{R})/U(n)$, which may be described as follows. For $J \in Sp(2n,\mathbb{R})/U(n)$, the tangent space $T_JSp(2n,\mathbb{R})/U(n)$ consists of operators $A:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ satisfying $AJ=-JA$ and $\langle Ax,y\rangle = \langle Ay,x\rangle$, where $\langle\cdot,\cdot\rangle$ is the symplectic structure. Alternatively, A is self-adjoint in the Euclidean scalar product $\langle J\cdot,\cdot\rangle$ and skew-commutes with J. The Kähler form on $T_J\,Sp(2n,\mathbb{R})/U(n)$ is given by $\langle A,B\rangle = Tr\,JAB$.

3.3 Simplices on Z. For two almost-complex structures J_1, J_2, tamed by ω, we define a segment $\mathcal{J}(t): t \mapsto (c(t,J_1(x),J_2(x))$ where $c(t,J_1(x),J_2(x))$ is the geodesic segment in the Hermitian symmetric space of nonpositive curvature $Sp(2n,\mathbb{R})/U(n)$, joining $J_1(x)$ and $J_2(x)$. For a collection $J_1,\dots,J_n$ define a singular simplex $K(J_1,\dots,J_n)$ as in 1.3.

3.4 Continuous cohomology classes in Sympl(M): *a definition.* For any generator of the ring of $Sp(2n,\mathbb{R})$ -invariant form on $Sp(2n,\mathbb{R})/U(n)$ we define a continuous cohomology class in $H_{cont}(\text{Sympl}(M),\mathbb{R})$ by the explicit formula

$$\delta(h_1,\dots,h_n) = \int_{K(\quad)} \varphi$$

where J_0 is any fixed tamed almost-complex structure, and φ is a form of 3.2.

3.5 Non-trviality. Let M be a flat torus $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with a standard symplectic structure $dx_1 \wedge dx_2 + \dots + dx_{2n-1} \wedge dx_{2n}$. As in 2.1, we have an $Sp(2n,\mathbb{Z})$ -invariant embedding $Sp(2n,\mathbb{R})/U(n) \hookrightarrow X$, and the classes of 3.4 on Sympl(M) restrict to Borel classes on $Sp(2n,\mathbb{Z})$, nontrivial for big n [B].

3.6 Application to moduli spaces. Let S be a closed Riemann surface of genus $g \geq 2$, and let $\mathcal{M}_g$ be a component of the representation variety $Hom(\pi_1(S), SO(3))/SO(3)$ with Stiefel-Whitney class 1. This is known to be a smooth compact simply-connected symplectic manifold [Go2] of dimension $6g-6$. By a famous theorem of [NS], $\mathcal{M}_g$ is identified with the moduli space of stable holomoprhic vector bundles of rank 2 and . The mapping class group Map_g acts symplectically on $\mathcal{M}_g$, so we have an injective homomorphism $Map_g \to Sympl(\mathcal{M}_g)$. Now we claim the following

Theorem (3.6). *$H^2(Sympl(\mathcal{M}_g),\mathbb{R})$ is nontrivial. Morover, the homomorphism $Map_g \to Sympl(\mathcal{M}_g,\mathbb{R})$ induces a nontrivial map in second real cohomology.*

Proof. By the main theorem of [NS] there is a holomorphic embedding of the Teichmüller space T_g to the space of complex structures in $\mathcal{M}_g$, tamed by Goldman's symplectic form. In particular, we have a Map_g-invariant holmorphic embedding $T_g \xrightarrow{\alpha} Z(\mathcal{M}_g)$. Let Ω be the Kähler form of $Z(\mathcal{M}_g)$, then $\alpha^*(\Omega)$ is a Map_g - equivariant Kähler form on T_g. We know there exist holomorphic maps $Y \xrightarrow{\pi} S$, where S is a closed Riemann surface, Y is a compact complex surface and π is a smooth fibration by complex curves of genus g, such that the corresponding holomorphic map $\tilde{S} \to T_g$ is nontrivial. We may form a flat holomorphic fibration $\mathcal{F} \to S$ with T_g as a fiber, associated to the homonmorphism $\pi_1(S) \to Map_g$, coming from π. The Borel regulator of the flat fibration $\mathcal{F} \to S$, corresponding to the form $\alpha^*(\Omega)$ on T_g, will coincide with the pullback of the class in $H^2(Sympl(\mathcal{M}_g),\mathbb{R})$ under the composite map $\pi_1(S) \to Map_g \to Sympl(\mathcal{M}_g)$. The variation of complex structure $Y \xrightarrow{\pi} S$ gives a holomorphic section of $\mathcal{F} \to S$ which is not horizontal. Therfore the pullbak of $\alpha^*(\Omega)$ on S using this section will have positive integral over S. By [Rel], section 3, this precisely means that the class we get in $H^2(S,\mathbb{R})$ is nontrivial. Therefore the map $Map_g \to Sympl(\mathcal{M}_g)$ induces a nontrivial map in H^2. Q.E.D.

4. Bounded cohomology for area-preserving diffeomorphisms

4.1. Let M^2 be a compact oriented surface of any genus and let ν be an area from on M. Then $Diff_\nu M =$ Sympl(M). The construction of 3.4 gives a class in $H^2_{cont}(Diff_\nu M,\mathbb{R})$.

Theorem (4.1). *The cocyle $\delta(h_1,h_2)$ of 3.4 is bounded. The class $[\delta]$ lives therefore in the image of the natural map*

$$H^2_b(Diff_\nu(M),\mathbb{R}) \to H^2(Diff^\delta_\nu(M),\mathbb{R})$$

Proof. Fix a tame almost-complex structure J_0. Then $\delta(h_1, h_2)$ is given by $\int_M$ area$_h($ $\quad)\cdot$ ω, where area$_h(x,y,z)$ is the hyperbolic area in $SL_2(\mathbb{R})/SO(2) \approx \mathcal{H}^2$ of the geodesic triangle, spanned by x, y, z. Therefore $|\delta(h_1, h_2)| \leq \pi \cdot \omega(M)$.

4.2 Non-amenability of two-generated subgroups of $Diff_\nu(M)$. We will apply theorem 4.1 to the following problem: given two area-preserving maps $f, g : M \to M$, when the group $\phi(f,g) \in Diff_\nu(M)$ is "big" (say, free)? When $Diff_\nu(M)$ is replaced by a finite-dimensional Lie group, this problem has been studied extensively, see e.g. [Re4], and references therein. In [Re4] we showed how the value of a (twisted) Euler class forces 2κ elements $f_1, \ldots, f_{2\kappa}$ of $SL_2(\mathbb{R})$ to generate a free group. Here we will give a criterion for $\phi(f,g)$ as above to be non-amenable. For that, denote $F(f,g)$ a free group in two generators f, g. Consider the ℓ^1 -homology Banach space $H_2^{\ell^1}(F,\mathbb{R})$ [ ]. An element of this space has a representive $\sum_{j=1}^{\infty} a_j(h_j, k_j)$ with $h_j, k_j \in F, \Sigma|a_j| < \infty$ and $\sum a_j(h_jk_j - h_j - h_j) = 0$ in $\ell^1(F)$. A bounded cocycle ℓ induces a continuous functional

$$\sum a_j\, \ell(h_i, k_i) : H_2^{\ell_1}(F,\mathbb{R}) \to \mathbb{R}$$

which vanishes if $[\ell] = 0$ in $H_b^2(F,\mathbb{R})$.

Theorem (4.2). *Let $\sum a_j(h_j,k_j)$ be any ℓ^1 -cycle in $H_2^{\ell_1}(F,\mathbb{R})$. If $\sum a_j\,\delta(h_j,k_i) \neq 0$, then the group $\phi(f,g)$ is non-amenable. The set of pairs $(f,g) \in Diff_\nu(M) \times Diff_\nu(M)$ satisfying this inequality, is open in C^1 -topology.*

Proof. Consider the following maps:

$$H_b^2(Diff_\nu(M),\mathbb{R}) \to H_b^2(\phi(f,g),\mathbb{R}) \to H_b^2(F(f,g),\mathbb{R}) \to (H_2^{\ell_1}(F(f,g),\mathbb{R}))^*$$

If $\phi(f,g)$ is amenable, then $H_b^2(\phi(f,g),\mathbb{R}) = 0$ [Gr2], so the image of δ in $(H_2^{\ell_1}(F(f,g),\mathbb{R}))^*$ is zero and $(\delta, \sum a_i(h_j,k_j)) = 0$, a contradiction. The last statement of the theorem is checked directly from the definition of δ.

4.3 Constructing ℓ^1-cycles. The cardinality of $\dim_{\mathbb{R}} H_2^{\ell_1}(F(f,g),\mathbb{R})$ is $2^{\aleph_0}$ by [ ]. To apply the theorem 4.2 it is useful to have explicit formulas for ℓ^1 -cycles. One way is described in [ ].

5. Lie algebra cohomology

We will give the Lie algebraic analogues of the above constructed classes in $Diff_\nu(M)$ and Sympl(M). Observe that some odd-dimensional classes in the Lie algebra of *Sympl*(M) were constructed in [Re2] they induce, in general, nontrivial classes in cohomology of Sympl(M) as a topological space. The even-dimensional classes constructed here always induce trivial classes in $H^*(\text{Sympl}^{top}(M),\mathbb{R})$.

5.1 Formulas for $Diff_\nu(M)$. Let $X_1, \ldots, X_{2\kappa+1} \in$ Lie$(Diff_\nu(M))$. Fix a Riemannian metric g with volume form V. Let

$$\psi(X_1, \ldots, X_{2\kappa+1}) = \int_M Alt\, Tr \prod_{j=1}^{2\kappa+1} (\nabla X_j + (\nabla X_j)^*) \cdot \nu$$

Theorem (5.1). *ψ defines a cocycle for $H^{2\kappa+1}(Lie(Diff_\nu(M))$.*

Proof. Consider a $Diff_\nu(M)$ -equivariant evaluation map $Diff_\nu(M) \to M : f \mapsto (f^*)^{-1}(g)$. Then the $Diff_\nu(M)$ -invariant forms on M, constructed in 1.1 induce left-invariant closed forms on $Diff_\nu(M)$, whose restriction on $T_e\, Diff_\nu(M)$ will be a Lie algebra cocycle. The derivative of the evaluation map Lie($Diff_\nu(M)$) $\to T_g\, M$ is given by $X \mapsto \mathcal{L}_X g = g(\nabla X + (\nabla X)^* \cdot, \cdot)$. Accounting the formula for Borel classes (see e.g. [Re3]), one arrives above-written formula for ψ.

5.2 Formulas for Sympl(M). Let $X_1, \ldots, X_{2\kappa} \in$ Lie(Sympl(M)). Fix a tame almost-complex structure J. Let

$$\varphi_{2\kappa}(X_1, \ldots, X_{2\kappa}) = \int_M Alt\, Tr\, J \cdot \prod_{j=1}^{2\kappa} \mathcal{L}_{X_j} J \cdot \omega^n$$

Theorem (5.2). *φ defines a cocycle for $H^{2\kappa}(Lie(Sympl(M))$.*

Proof. Same as for 5.1.

5.3 Vanishing for φ_2 for flat torus.

Proposition (5.3). Let $M = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ be a torus with standard symplectic structure. Then for any choice of a tame almost-complex structure, the cohomology class of φ_2 in $H^2(\text{Lie(Sympl}(M)), \mathbb{R})$ is zero.

Proof. The cohomology class of φ_2 does not depend on the choice of J, since X is connected. Choose J to be the standard complex structure. We need to work on the formula for φ_2. Let g be a metric, defined by $g(J\cdot, \cdot) = \omega$ (flat in our case). We then have $\mathcal{L}_X J = [\nabla X, J]$ since g is Kähler and $\nabla_X J = 0$. So

$$\varphi_2(X, Y) = \int_M Tr\, J([\nabla X, J][\nabla Y, J] - [\nabla Y, J][\nabla X, J]]) \cdot \omega^n$$

Let X be Hamiltonian, so that $X = J\, grad\, f$. Then $\nabla X = J\, H_f$, where H_f is the Hessian of f. If Y is also Hamiltonian, say $Y = J\, grad\, h$, we have

$$\varphi_2(X, Y) = -\int_M Tr\, J[H_f, J][H_h, J] \cdot \omega^n$$

Direct computation shows that the last expression is zero for flat torus. Now, Lie(Sympl(M)) is a semidirect product of the ideal of Hamiltonian vector fields and an abelian subalgebra of constant vector fields, generated by (multivalued) linear Hamiltonians. Clearly, $\varphi_2(X, Y)$ is zero for all choices for X and Y.

5.4 Vanishing of φ_2 for a symplectic surface.

Proposition (5.4). *Let (M, ω) be a compact surface with a symplectic form. Then for any choice of a tame almost-complex structure, the cohomology class of φ_2 in $H^2(Lie(Ham(M), \mathbb{R}))$ is zero.*

Proof. Let g be as above. Again we have

$$\varphi_2(X,Y) = -\int_M Tr\, J[H_f, J][H_h, J] \cdot \omega$$

The proposition follows now from the following remarkable identity.

Theorem (5.4). *On a compact Riemannian surface (M, g) the following identity holds:*

$$\int_M Tr\, J[H_f, J][H_h, J] \cdot d\,\text{area} = -\int K(g)\{f, h\} \cdot d\,\text{area}, \qquad (*)$$

where $K(g)$ is the curvature of g.

Proof. We were only able to prove this identity by a direct (very) long computation ([Rel]), which we will sketch here. Let $g = e^{A(x,y)}(dx^2 + dy^2)$ in local conformal coordinates. Then $\Gamma^x_{xx} = \frac{1}{2}A_x, \Gamma^y_{yy} = \frac{1}{2}A_y$, $\Gamma^x_{xy} = \frac{1}{2}A_y, \Gamma^y_{xy} = \frac{1}{2}A_x$, $\Gamma^y_{xx} = -\frac{1}{2}A_y$, $\Gamma^x_{yy} = -\frac{1}{2}A_x$. Next, $H_f = \nabla(Grad\, f)$ and to the matrix of H_f is

$$\begin{pmatrix} e^{-a} f_{xx} + \frac{1}{2}e^{-A}(A_y f_y - A_x f_x) & e^{-A} f_{xy} - \frac{1}{2}e^{-A}(A_y f_x + A_x f_y) \\ e^{-A} f_{xy} - \frac{1}{2}e^{-A}(A_y fx + A_x f_y) & e^{-A} f_{yy} + \frac{1}{2}e^{-A}(A_x f_x - A_y f_y) \end{pmatrix}$$

and the same for h. Substituting to the left side of (*) one gets

$$-2\Big[\int (e^{-A} f_{xy} - \frac{1}{2}e^{-A}(A_y f_x + A_x f_y)) \cdot (h_{xx} - h_{yy} + A_y h_y - A_x h_x) -$$

$$-\int (e^{-A} h_{xy} - \frac{1}{2}(A_y h_x + A_x h_y))(f_{xx} - f_{yy} + A_y f_y - A_x f_x)\Big]\, dxdy$$

Twice integrating by parts, one finds this equal to

$$\int e^{-A}[-A_{xxy} f_x + A_y A_{xx} f_x - A_{yyy} f_x +$$
$$+ A_y A_{yy} f_x + A_{yyx} f_y - A_x A_{yy} f_y + A_{xxx} f_y - A_x A_{xx} f_y]dxdy$$

On the other hand, the right hand side is

$$\int_M \{f_x h_y - f_y h_x\} \cdot (A_{xx} + A_{yy}) e^{-A} dxdy.$$

Again integrating by parts, one gets the same expression as above. q.e.d.

6. Chern-Simons-type class in $H^3(Diff_\nu(M^3), \mathbb{R}(\mathbb{Z})$

This section is best read in conjunction with [Re2]. In that paper, we constructed secondary classes in $Hom(\pi_{2i-1}(B\,\mathrm{Sympl}^\delta(M)^+, \mathbb{R}/A)$ where M^{2n} is a compact simply-connected symplectic manifold and A is a group of periods of a biinvariant $(2i-1)$ -form on $\mathrm{Sympl}(M)$, whose restriction on the Lie algebra is $f_1, \dots, f_{2i-1} \to \mathrm{Alt} \int_M \{f_1, f_2\} f_3 \dots f_{2i-1} \cdot \omega^n$. In particular, it implied the following results.

6.1 Theorem ([Re2]) (Chern-Simons class extends to Sympl(S^2)). *There exists a rigid class in $H^3(Sympl(S^2, can), \mathbb{R}/\mathbb{Z})$ whose restriction on $SO(3)$ is the standard Chern-Simons class.*

6.2 Theorem ([Re2]) (Chern-Simons class extends to Sympl($\mathbb{C}P^2$)). *There exists a class in $H^3(Sympl(\mathbb{C}P^2, can), \mathbb{R}/\mathbb{Z})$ whose restriction on $SU(3)$ is the standard Chern-Simons class.*

6.3 Theorem ([Re2]). *There exists a class in $H^3(Sympl((S^2, a_1 \cdot can) \times S^2(a_2 \times can)), \mathbb{R}/\mathbb{Z}), a_1 \neq a_2$, whose restriction on $SO(3) \times SO(3)$ is the sum of standard Chern-Simons classes.*

Let M^3 be a rational homology sphere, say $f \cdot H_1(M, \mathbb{Z}) = 0, f \in \mathbb{Z}$.

6.4 The definition of the ChS class. Fix a point $p \in M$ and consider the evaluation (at p) map

$$Diff_\nu(M) \to M.$$

The pull-back of ν under this map is a closed left-invariant form ν_p on $Diff_\nu(M)$, having integral periods. The general theory of [Re3] and [Re2] produces a regulator

$$\pi_3(B\,Diff^\delta_\nu(M)^+) \to \mathbb{R}/\mathbb{Z} \tag{*}$$

A different choice of a point $p' \in M$ will give another left-invariant form $\nu_{p'}$ such that $\nu_p - \nu_{p'} = d\mu$ for a left-invariant form μ. It follows from [Re3] that the regulator (*) does not depend on p. In fact, one has a biinvariant 3-form ω on $Diff_\nu(M)$, whose values on the Lie algebra are given by $\omega(X, Y, Z) = \int_M \nu(X(p), Y(p), Z(p)) d\nu(p)$. The form ω gives the same regulator as above.

To extend the regulator to $H^3(Diff^\delta_\nu(M), \mathbb{R}/\mathbb{Z})$, we need to alter the scheme of [Re3] as follows. Since $MSO_3(B\,Diff^\delta(M)) \approx H_3(B\,Diff^\delta(M), \mathbb{Z})$ any class in $H_3(B\,Diff^\delta(M), \mathbb{Z})$ is represented by a map $X \xrightarrow{\varphi} B\,Diff(M)$, or equivalently, by a representation $\pi_1(X) \xrightarrow{\rho} Diff_\nu(M)$. Now, for M a flat bundle $M \to \mathcal{E} \to X$, associating to ρ. The form ω extends to the closed form on $\mathcal{E}$ whose periods on fibers are 1. That gives an element λ in $H^3(\mathcal{E}, \mathbb{R}/\mathbb{Z})$. The spectral sequence of $\mathcal{E}$ with $\mathbb{R}/\mathbb{Z}$ -coefficients looks like

$$\begin{array}{cccc}
\mathbb{R}/\mathbb{Z} & H^1(X, \mathbb{R}/\mathbb{Z}) & H^2(X, \mathbb{R}/\mathbb{Z}) & H^3(X, \mathbb{R}/\mathbb{Z})\dots \\
0 & 0 & 0 & 0 \\
H^0(X, \underline{W}) & H^1(X, \underline{W}) & H^2(X, \underline{W}) & H^3(X, \underline{W})\dots \\
\mathbb{R}/\mathbb{Z} & H^1(X, \mathbb{R}/\mathbb{Z}) & H^2(X, \mathbb{R}/\mathbb{Z}) & H^3(X, \mathbb{R}/\mathbb{Z})\dots
\end{array}$$

where W is the local system whose stalk at p is $H^1(M, \mathbb{R}/\mathbb{Z}) \approx \widehat{H_1(M, \mathbb{Z})}$. The element λ lies in the kernel of the wedge map $H^3(\mathcal{E}, \mathbb{R}/\mathbb{Z}) \to H^3(M, \mathbb{R}/\mathbb{Z})$. Now, the group $H^2(X, \underline{W})$ has exponent a divisor of f, and the image of the transgression $d^2 : H^1(X, \underline{W}) \to H^3(X, \mathbb{R}/\mathbb{Z})$ has the same property. Therefore, $f \cdot \lambda$ induces a well-defined class in $H^3(X, \mathbb{R}/\mathbb{Z} \cdot \frac{1}{f})$. If M is a $\mathbb{Z}$-homology sphere, we get a class in $H^3(X, \mathbb{R}/\mathbb{Z})$.

If $Y \to B\,Diff^\delta(M)$ is a map, bordant to φ, then the same argument as in [Re2] proves that the value of the corresponding class in $H^3(Y, \mathbb{R}/\mathbb{Z} \cdot \frac{1}{f})$ on $[Y]$ is the same as for X. So we constructed a well-defined map

$$H_3(Diff^\delta(M), \mathbb{Z}) \to \mathbb{R}/\mathbb{Z} \cdot \frac{1}{f}$$

6.5 Invariant scalar product on $Lie(Diff_\nu(M))$, the Cartan form and rigidity of ChS class. Here we will prove that the ChS class

$$H_3(Diff^\delta_\nu(M), \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$$

of the previous section is rigid for $M \approx S^3$. For that purpose we need to work with principal flat bundles rather then with flat associated bundles. The clue is that the form ω constructed above on $Diff_\nu(M)$ can be viewed as a Cartan form, associated with an invariant scalar product on $\mathrm{Lie}(Diff_\nu(M))$.

We are going to prove similar results for the group $Diff_\nu(M^3)$ of volume-preserving diffeomorphisms of a compact oriented three-manifold. Throughout this section, M is assumed to be a rational homology sphere, that is, $H_1(M, \mathbb{Z})$ is torsion.

Let $X \in \mathrm{Lie}(Diff_\nu(M))$ a vector field with $\operatorname{div} X = 0$. The form $X \rfloor \nu$ is closed, whence exact: $d\mu = X \rfloor \nu$. Put $\langle X, X \rangle = \int_M \mu \cdot (X \rfloor \nu)$. An immediate computation shows that $\langle X, X \rangle$ does not depend on the choice of μ. Moreover $X \mapsto \langle X, X \rangle$ is a quadratic form, invariant under the adjoint action of $Diff_\nu(M)$. By Arnold [A], $\langle X, X \rangle$ is the asymptotic self-linking number of trajetories of X. We need the following elementary lemma (the proof of left to the reader)

Lemma (6.5). *For any $X, Y, Z \in Lie(Diff_\nu(M))$,*

$$\Omega(X, Y, Z) = \omega(X, Y, Z)$$

that is, the forms Ω and w coincide.

Now, as in [Re2] we define a biinvariant form Ω on $Diff_\nu(M)$ by $\Omega(X, Y, Z) = \langle [X, Y], Z \rangle$ on the Lie algebra.

Lemma (6.6). *Let $M = S^3/\Gamma$ where S^3 is considered as a compact Lie group and the finite subgroup Γ acts from the right. Then the pullback of Ω by the natural map $S^3 \to Diff_\nu(M)$ is $\frac{1}{|\Gamma|} \cdot$ (volume form of S^3).*

Proof. It is clearly enough to check this for $\Gamma = \{1\}$. Let $v \in \mathrm{Lie}(S^3)$ and X is the corresponding right-invariant vector field. Let μ be a right-invariant 1-form, defined by $(v, \cdot)$ on Lie S^3). Then $d\mu = X \rfloor \nu$ and $\mu \wedge (X \rfloor \nu) = \nu$. q.e.d.

6.6 Theorem (Chern-Simons class in $Diff_\nu(S^3)$). *There exists a rigid class in $H^3(Diff_\nu(S^3), \mathbb{R}/\mathbb{Z})$ whose restriction on $SO(4) \approx S^3 \times S^3/\mathbb{Z}_2$ coincides with the sum of standard Chern-Simons classes. Moreover, for $M = S^3/\Gamma$ there exists a class in $H^3(Diff_\nu(M), \mathbb{R}/\mathbb{Z})$ whose restriction on S^3 is $|\Gamma|$ times the standard Chern-Simons class.*

Proof. By the general theory of regulators, developed in [Re3], section 3, and [Re2], the invariant form Ω gives rise to a map

$$\pi_3(B\, Diff_\nu^\delta(M)^+) \to \mathbb{R}/A$$

where A is the group of periods of Ω on the Hurewitz image of $\pi_3(Diff_\nu(M))$ in $H_3(Diff_\nu(M), \mathbb{Z})$. Moreover, if $Diff_\nu(M)$ is homotopically equivalent to S^3 or $SO(4)$ this extends to a map

$$H_3(B\, Diff^\delta(M)) \to \mathbb{R}/A$$

By Hatcher [H] and Ivanov [I] this is exactly the case for $M = S^3/\Gamma$. Moreover, periods of Ω are $2\pi^2 \cdot \mathbb{Z}$ and $2\pi^2 \cdot \frac{1}{|\Gamma|}\mathbb{Z}$, respectively. Since Ω is a Cartan form, associated to an invariant polynomial in Lie($Diff_\nu(M)$), it is rigid by Cheeger-Simons [Che-S].

6.6 Case of Seifert manifolds. Let Γ be a uniform lattice in $\widetilde{SL_2}(\mathbb{R})$, then $M = \widetilde{SL_2}(\mathbb{R})/\Gamma$ is a Seifert manifold. There is a cohomology class $\beta \in H^3(\widetilde{SL_2}(\mathbb{R})^\delta, \mathbb{R})$, called the Seifert volume class [BGo], such that for any $\Gamma \subset \widetilde{SL_2}(\mathbb{R})$, the restriction of β on Γ is vol($\widetilde{SL_2}(\mathbb{R})/\Gamma$) times the fundamental class. Then the computation of 6.4 gives the class in $H^3(Diff_\nu(M), \mathbb{R})$, whose restriction on $\widetilde{SL_2}(\mathbb{R})$ is β, subject to the condition that $Diff_\nu(M)$ is contractible. It is not known to the author if this is true for all such M, comp. [FJ].

7. Measurable transfer and higher asymptotic cycles

We will first outline here an alternative approach in defining the classes of 1.2 in $Diff_\nu(M)$. For M a locally symmetric space of nonpositive curvature, this approach also leads to new classes in $H^*_{cont}(Diff_\nu(M), \mathbb{R})$, different from those of 1.2.

Let $\mathfrak{G} = Diff_\nu(M)$ and $\mathfrak{G}_0 \subset \mathfrak{G}$ is a closed group, stabilizing a fixed point $p \in M$. Let $\mathfrak{G}^\sim$ be the connected component of $\mathfrak{G}$ and let $\mathfrak{G}_0^\sim = \mathfrak{G} \cap \mathfrak{G}_0$. Fix a measurable section $s : M \to \mathfrak{G}$ such that $s(q)p = q$. We will always assume that $\overline{s(M)}$ is compact.

7.1 Ergodic cocycle in non-abelian cohomology [Gu]. Define a map $\psi : \mathfrak{G} \times M \to \mathfrak{G}_0$ by $g\, s(q) = s(gq)\psi(g, q)$. We will view it as a map $\mathfrak{G} \xrightarrow{\psi} \mathcal{F}(M, \mathfrak{G}_0)$. Here $\mathcal{F}(M, \mathfrak{G}_0)$ is the group of measurable functions from M to $\mathfrak{G}_0$ with compact closure of the image. $\mathfrak{G}$ acts on $\mathcal{F}(M, \mathfrak{G}_0)$ by the argument change and ψ is a cocycle for the non-abelian cohomology $H^1(\mathfrak{G}, \mathcal{F}(M, \mathfrak{G}_0))$.

7.2 Measurable transfer [Gu]. Now let $f : \mathfrak{G}_0 \times \ldots \mathfrak{G}_0 \to \mathbb{R}$ be a locally bounded (say, continuous) cocycle. Define $F : \mathfrak{G} \times \ldots \times \mathfrak{G} \to \mathbb{R}$ as $F = \int_M f(\psi(g_1, m),$

$\psi(g_2, m) \ldots \psi(g_n, m))d\nu(m)$. This defines a cohomology class in $H^n(\mathfrak{G}, \mathbb{R})$, independent of the choices of s and f [Gu].

Now, we have the tangential representation $\mathfrak{G}_0 \to SL(T_p(M))$. Pulling back the usual Borel classes on $\mathfrak{G}_0$, we construct cohomology classes in $H^i(\mathfrak{G}_0, \mathbb{R})$ for $i = 5, 9, \ldots$. The transfer will map these to classes in $H^i(\mathfrak{G}, \mathbb{R})$, which we have constructed in 1.2. We do not prove the comparison theorem here, however.

7.3 Supertransfer. We will now define a map

$$H^\kappa(\pi_1(M), \mathbb{R}) \xrightarrow{S} H^\kappa(Diff_\nu(M), \mathbb{R})$$

in the following way. We know that $\pi_0(\mathfrak{G}_0^\sim) \approx \pi_1(M)/\pi_1(\mathfrak{G}^\sim)$. This defines a homomorphism $\mathfrak{G}_0^\sim \to \pi_0(\mathfrak{G}_0^\sim) \to \pi_1(M)/\pi_1(\mathfrak{G}^\sim)$, and a map $H^\kappa(\pi_1(M)/\pi_1(\mathfrak{G}^\sim), \mathbb{R}) \to H^\kappa(\mathfrak{G}_0^\sim, \mathbb{R})$.

In many interesting cases one knows that $\pi_1(\mathfrak{G}^\sim) = 1$. If M is a surface of genus $g \geq 2$, a result of Earle and Eells says that $\mathfrak{G}^\sim$ is contractible. For M locally symmetric of rank ≥ 2 [FJ]. For any M such that $\pi_1(\mathfrak{G}^\sim) = 1$, we get $\pi_0(\mathfrak{G}_0) \approx \pi_1(M)$ so that there is a map

$$H^k(\pi_1(M)) \to H^\kappa(\pi_0(\mathfrak{G}_0^\sim)) \to H^\kappa(\mathfrak{G}_0^\sim).$$

Now, composing with the measurable transfer $H^\kappa(\mathfrak{G}_0^\sim) \to H^\kappa(\mathfrak{G}^\sim)$ we arrive to a desired map

$$S : H^\kappa(\pi_1(M), \mathbb{R}) \to H^\kappa(\mathfrak{G}^\sim, \mathbb{R})$$

7.4 Higher asymptotic cycles. The dual to the above-constructed map S is

$$S^\vee : H_\kappa(\mathfrak{G}^\sim, \mathbb{R}) \to H_\kappa(\pi_1(M), \mathbb{R}).$$

As we will see now, this is higher version of the classical asymptotic cycle character

$$\mathfrak{G}^\sim \xrightarrow{\tau} H_1(M, \mathbb{R})$$

[Sch]. Indeed, for $\kappa = 1$ the map $S^\vee$ will act as follows: let $g \in \mathfrak{G}^\sim$ be a volume-preserving map, isotopic to identity. Fix an isotopy $g(t, x)$ such that $g(0, \cdot) = \mathrm{id}$ and $g(1, \cdot) = g$. For $x \in M, g(t, x)$ is a path from x to $g(x)$ and may be considered as a $1-$ current. Now, the integral

$$\int_M [g(t, x)] d\nu(x)$$

is a closed current, defining an element in $H_1(M, \mathbb{R})$. This will be $S^\vee(g)$.

Now, the definition of the asymptotic cycle map [Sch] gives the following recepy: for an element $z \in H^1(M, \mathbb{Z})$ let $f : M \to S^1$ be a representing map. The map $f \circ g - f : M \to S^1$ is zero-homotopic, so it comes from the map $F : M \to \mathbb{R}$. Now, $\int_M F(\mathrm{mod}\, \mathbb{Z})$ is the image of $\tau(f)$ on z. If f is isotopic to identity, $\tau(f)$ lifts to $H_1(M, \mathbb{R})$. It is easy to check that $(df, \int_M [g(t, x)] d\nu) = (\tau(f), z)$, which proves $S^\vee = \tau$ in dimension 1.

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