Nonarchimedean flag domains

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Introduction

We construct flag domains for simply connected groups G defined over a nonarchimedean local field K. The case where G is split over K was treated in [PV] (There the spaces we call flag domains were called symmetric spaces. But flag domains seems to be a better name). Here we concentrate on the absolutely almost simple groups that are not split over K.

Let us first recall the definition of a flag domain. Let X be a projective homogeneous space (not necessarily defined over K). Then $X = G/P$ with $P \subset G \otimes K_s$ a parabolic subgroup and K_s is the separable closure of K. We call an open analytic subspace $Y \subset X$ a *flag domain* if it has the following two properties:

(1) Y is stable under the action of $G(K)$

(2) For every discrete co-compact subgroup $\Gamma \subset G(K)$ the quotient Y/Γ exists and is a proper rigid analytic space defined over K_s.

This definition seems to be the p-adic analog of archimedean flag domains. If G is defined over the field of real numbers $\Re$ then an open $G(\Re)$ - orbit Y in $G(C)/P(C)$ for some parabolic subgroup $P \subset G \otimes C$ is called a flag domain if $Y \cong G(\Re)/H$ for some compact subgroup $H \subset G(\Re)$ (See [GrS] and [WW]).

If $Char(K) > 0$ then discrete co-compact subgroups $\Gamma \subset G(K)$ only exist if G is of inner type A_l (See [Ve] or [M]). So $G = SL_n(D)$ with D a skew field defined over K. To make the notion of a flag domain also meaningful for other groups in positive characteristics, one could replace (2) in the definition above by the following :

(2') There exists a formal scheme $\mathcal{Y}$ on which G(K) acts, with generic fibre Y. The closed fibre of $\mathcal{Y}$ consists of proper components that are in 1–1 correspondance with the vertices of the Bruhat–Tits building of G(K) .

Note that (2') implies (2) in the case of existence of discrete co-compact subgroups of *G(K)*. Our construction of flag domains will be such that they also satisfy (2').

We use the following construction. Let $\mathcal{L}$ be an ample line bundle on X. If the set of stable points coincides with the set of semi–stable points for the action of a maximal K-split torus $S \subset G$ on X with respect to $\mathcal{L}$, then $Y := \bigcap_{g \in G(K)} g \cdot X^s(S, \mathcal{L})$ is a flag domain for *G(K)* . Here $X^s(S, \mathcal{L})$ denotes the set of stable points. This construction is described in detail in [PV].

In section 1 of this article we briefly recall the construction. The sets of stable points on X are studied in section 2. In section 3 the calculations with weights needed in section 2 are performed.

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1 The Construction

1.1

We will briefly recall the construction of flag domains as described in [PV]. We generalize it a little to make it also work for non-split semisimple groups.

1.2

Let K denote a nonarchimedean local field and let K^o denote its ring of integers. Let X be a normal projective (possibly non-connected) variety defined over K^o on which a group defined over K^o acts algebraically. The action being defined over K^o. We assume the group G is connected, semisimple, absolutely almost simple and isotropic. A group G is called *isotropic* if $G(K)$ is non-compact. It is called *almost simple* if $G(K)$ does not contain a proper infinite normal subgroup, and *absolutely almost simple* if this remains true for $G(L)$ with $L \supset K$ any finite extension of fields.

Let $\mathcal{L}$ be an ample line bundle on X. We assume that $\mathcal{L}$ is defined over K^o. We will explain what we mean by this. If $\mathcal{L}$ is ample then for some natural number n the line bundle $\mathcal{L}^{\otimes n}$ is very ample. So $\mathcal{L}^{\otimes n}$ gives an embedding of X into some projective space P^m. The action of G on X induces a G-linearization of $\mathcal{L}^{\otimes n}$. So we can identify P^m with the projectivization $P(V)$ of some G-module V. We call $\mathcal{L}$ *defined over* K^o if there exists a representation $G \longrightarrow GL(V)$ defined over K^o corresponding with the G-linearization of $\mathcal{L}^{\otimes n}$ for some n such that $\mathcal{L}^{\otimes n}$ is very ample.

Let $S \subset T$ be a maximal K^o- rational K^o-split torus.Then $\mathcal{L}$ determines the sets of stable and semistable points for the action of S on X, denoted by $X^s(S,\mathcal{L})$ and $X^{ss}(S,\mathcal{L})$ respectively, both defined over K^o. For a scheme Z defined over K^o we denote by Z_K the same scheme but now defined over K, i.e. $Z_K := Z \times_{spec(K^o)} spec(K)$. Clearly $X^s(S,\mathcal{L})_K$ and $X^{ss}(S,\mathcal{L})_K$ are the sets of (semi-)stable points for the action of S_K on X_K with respect to the ample line bundle $\mathcal{L} \otimes K$.

We define a $G(K)$ -stable analytic subspace $Y \subset X_K$ by:

$$Y := \cap_{g \in G(K)} g \cdot ({X^s}_K(S_K, \mathcal{L} \otimes K)).$$

In [PV] the following theorem is proved:

1.3 Theorem

If $X^s(S,\mathcal{L}) = X^{ss}(S,\mathcal{L})$ then Y/Γ is a proper rigid analytic space for any discrete co-compact subgroup $\Gamma \subset G(K)$.

1.4

In [PV] this theorem is only proved for split groups G acting on a projective homogeneous variety $X = G/P$, where $P \subset G$ is a parabolic subgroup. Inspecting the proof given there, one sees that it only uses general properties of the Bruhat-Tits building (to be found in [BrT] and [T.3]) and properties of sets of (semi-)stable points. Hence the theorem remains valid in this more general set up.

In particular the following remains true:

1.5 Proposition

$X_K - Y$ is the union of a compact set of Zariski closed divisors.

1.6

Suppose $X^s(S,\mathcal{L}) = X^{ss}(S,\mathcal{L})$. Let k be the residue field of K. Let us assume that $X_k := X \times_{spec(K^o)} spec(k)$consists of one single component. Then the pure affinoid covering of Y given in [PV] is such that the reduction of Y consists of components in 1-1 correspondance with the vertices of the Bruhat-Tits building of *G(K)*. This gives rise to a formal scheme $\mathcal{Y}$ associate to Y such that the components of the closed fibre of $\mathcal{Y}$ are in 1-1 correspondance with the vertices of the Bruhat-Tits building.

The analytic space Y/Γ is proper for any discrete co-compact subgroup $\Gamma \subset G(K)$. According to [Lu.2] any formal scheme belonging to a proper rigid analytic space has a closed fibre consisting of proper components.

Therefore we have, assuming X_k consists of one component:

1.7 Theorem

There exists a formal scheme $\mathcal{Y}$ with generic fibre Y such that the closed fibre consists of proper components in 1-1 correspondance with the vertices of the Bruhat-Tits building of G(K) .

1.8

There is also a scheme theoretical way to construct the formal scheme $\mathcal{Y}$ (See [PV] and [Ku]).

The non-algebraicity result proved in [PV] theorem 4.2 remains valid in our more general set up. Indeed the results of Wang [W] and Lütkebohmert [Lu.1] used in the proof are also valid for non split groups.

1.9 Theorem

Let G have only a finite number of orbits on X and let $X^s(S,\mathcal{L}) = X^{ss}(S,\mathcal{L})$. Let $\Gamma \subset G(K)$ be a discrete co-compact subgroup. Then every meromorphic function on the proper rigid analytic space Y/Γ is constant on each connected component of Y/Γ if $codim(X_K - X_K^s(S,\mathcal{L})) \geq 2$.

2 Stable Points

2.1

We assume that G is a simply connected, semisimple, absolutely almost simple group defined over a non-archimedean local field K.

All varieties occuring in this section are defined over a suitable finite separable extension of the field K. So we will not work over K^0 in this section.

2.2

Let $S \subset T \subset G$ be a maximal K-split torus and a maximal torus defined over K, respectively. The torus T and the group G both split over the separable closure K_s of K. Let $\mathcal{X}(T)$ be the character group of T. Let Φ denote the *(absolute) root system* of G. We choose a *simple basis* Δ of Φ.

Let W be the *Weyl group* of Φ. Choosing a W - invariant inner product on $\mathcal{X}(T)\otimes\Re$, the group W is generated by the reflections in the hyperplanes orthogonal to the simple roots $\alpha \in \Delta$. One has $W \cong N_T/Z_T(K_s)$. Here N_T and Z_T are the *normalizer* and *centralizer* of the torus T in G. We denote the simple roots in Δ by α_i, i=1, ... ,ℓ . Here $\ell = dim_{K_s}(T)$ is the *absolute rank* of G. Let $w_i \in W$ be the reflection belonging to α_i.

The simple basis Δ of Φ determines a *Borel subgroup* $B \subset G$. One has $B =< T, U_\alpha | \alpha \in \Phi^+ >$, where Φ^+ is the set of *positive roots* and U_α is the T-stable additive subgroup on which T acts with character α.

For any subset $I \subset \{1,\dots,\ell\}, I \neq \emptyset$ we denote by $W_I \subset W$ the subgroup generated by the reflections $w_i, i \notin I$. Then the *parabolic subgroups* containing B are the groups $P_I := BW_IB$. Any parabolic subgroup of G is conjugated to exactly one of the groups P_I. These parabolic groups P_I are all defined over the splitting field of G.

2.3

Let H be the Galois group $H := Gal(K_s/K)$. We fix an ordering on $\mathcal{X}(S)$. We choose an ordering on $\mathcal{X}(T)$ compatible with the ordering on $\mathcal{X}(S)$. The set of simple roots of G with respect to T vanishing on S is called Δ_0. The *relative root system* of G, i.e. the roots of $\mathcal{X}(S)$, is denoted by Φ_K. The *relative Weyl group* is called W_K. One has $W_K \cong N_S/Z_S(K)$, where N_S and Z_S are the normalizer and the centralizer of S in G. The simple basis of Φ_K is denoted by Δ_K.

The Galois group H acts on $\mathcal{X}(T)$, since T is split over K_s. We will need a twisted action of H on $\mathcal{X}(T)$. For any $h \in H$ the image $h(\Delta)$ of Δ is again a simple basis of Φ. There exists an unique element $w \in W$ such that $wh(\Delta) = \Delta$. We set $h^* = w \circ h$ and call the $*$-action of H on $\mathcal{X}(T)$ the *twisted action* . Let $H^* := \{h^*|h \in H\}$. Then H^* acts on $\mathcal{X}(T)$. Note that H^* is a finite group.

2.4

Let $\mathcal{L}$ be an ample line bundle on $X = G/P$. Then $\mathcal{L}$ is in fact very ample. So $\mathcal{L}$ determines an embedding $X \hookrightarrow P(V)$. Here $P(V)$ is the projectivization of a G -module V. When $Char(K) = 0$ the module V is irreducible. If $Char(K) > 0$ then this might not be the case (See [Ke] and [MR]).

However, the module V is uniquely characterized by its highest weight. We will denote this G-module with highest weight λ by V_λ.

Next we describe the weights λ such that there exists an ample line bundle $\mathcal{L}$ on $X = G/P_I$ corresponding to the G-module V_λ. Let ω_i be the fundamental weight determined by $2(\omega_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$. Here (-,-) is a W-invariant inner product on $\mathcal{X}(T) \otimes \Re$. The ample line bundles $\mathcal{L}$ on $X = G/P_I$ correspond to the modules V_λ with highest weight $\lambda = \sum_{i \in I} n_i \cdot \omega_i$ with $n_i > 0$.

We need to know which representations of G can be defined over K. These can be found in [T.2]. Let Λ_r denote the *root lattice*, i.e. the sublattice of $\mathcal{X}(T)$ generated by the roots $\alpha \in \Phi$. We restate the theorems 3.3 and 7.2 of [T.2] in a form suitable for our purposes:

2.5 Theorem

The representation ρ_λ of G into $GL(\bigoplus_{\sigma \in H^*} V_{\sigma(\lambda)})$ can be defined over K if $\lambda \in \Lambda_r$ (otherwise if $\lambda \notin \Lambda_r$ it can be defined over some skew field D defined over K).

2.6

If $Char(K) = 0$ then this representation is irreducible over K and the theorem is actually proved in [T.2]. Otherwise if $Char(K) > 0$, the representation might not be irreducible, but it follows from the proof given in [T.2] that the theorem above remains true.

The theorem above gives us for each weight $\lambda = \sum_{i \in I} n_i \cdot \lambda_i \in \Lambda_r$ a representation ρ_λ defined over K. Let $v_\lambda \in \bigoplus_{\sigma \in H^*} V_{\sigma(\lambda)}$ be a vector contained in $V_\lambda \oplus < 0 >$, whose component in V_λ is a heighest weight vector. The image $X^\dagger{}_\lambda$ of the orbit $G \cdot v_\lambda$ in $P(\bigoplus_{\sigma \in H^*} V_{\sigma(\lambda)})$ is isomorphic to G/P_I. In $P(\bigoplus_{\sigma \in H^*} V_{\sigma(\lambda)})$ we have a variety $X^\dagger$ defined over K, whose connected components are $X^\dagger{}_{\sigma(\lambda)}$, $\sigma \in H^*$. The connected components are all isomorphic. Moreover the very ample line bundle $\mathcal{L}^\dagger$ associated with this embedding, gives on each connected component $X^\dagger{}_{\sigma(\lambda)}$ the line bundle $\mathcal{L}$ associated with the weight $\sigma(\lambda)$.

So for each $X = G/P_I$ defined over K_s and ample line bundle $\mathcal{L}$ on X corresponding to some weight $\lambda \in \Lambda_r$, we can construct a variety $X^\dagger$ and an ample line bundle $\mathcal{L}^\dagger$,both defined over K , such that one connected component of $X^\dagger$ is isomorphic to X and the restriction of $\mathcal{L}^\dagger$ to this component is $\mathcal{L}$. So we can forget about X and $\mathcal{L}$ being defined over K, since we can always construct suitable $X^\dagger$ and $\mathcal{L}^\dagger$ defined over K, if $\lambda \in \Lambda_r$.

The fact that we have to assume that $\lambda \in \Lambda_r$ is unimportant for us, since for any weight λ we can find an integer $n > 0$, such that $n \cdot \lambda \in \Lambda_r$. If $\mathcal{L}$ is a line bundle on X corresponding to λ, then $\mathcal{L}^{\otimes n}$ corresponds to the weight $n \cdot \lambda$. Hence λ and $n \cdot \lambda$ define the same sets of (semi-)stable points on X.

From now on we will always tacitly assume that $\lambda \in \Lambda_r$.

2.7

The sets of (semi-)stable points of X for the action of T w.r.t. $\mathcal{L}$ can be determined using the criteria given in [MF]. Let $\mathcal{L}$ give an embedding of X into $P(V)$ for some G-module V. Then we have an unique decomposition $V = \bigoplus V_\beta$, $\beta \in \mathcal{X}(T)$ of V into eigenspaces V_β on which T acts with character β. Let π_β be the projection $V \longrightarrow V_\beta$. For $x \in X$ we denote by $\mu(x) \subset \mathcal{X}(T) \otimes \Re$ the polyhedron given by the convex hull of $\{\beta | \pi_\beta(v) \neq 0\}$, where $v \in V$ is some original of $x \in X \subset P(V)$. One has:

$$x \in X^s(T, \mathcal{L}) \Longleftrightarrow 0 \in int\ \mu(x)$$

$$x \in X^{ss}(T,\mathcal{L}) \Longleftrightarrow 0 \in \mu(x)$$

Using [GS] the following is proved in [PV] theorem 1.1:

2.8 proposition

Let $\mathcal{L}$ be the ample line bundle on $X = G/P$, corresponding to the weight λ. Let $T \in G$ be a maximal torus. Then we have:

a) *For any point $x \in X$ the vertices of $\mu(x)$ are contained in the set $\{w(\lambda)|w \in W\}$. The edges of $\mu(x)$ are parallel to the roots $\alpha \in \Phi$.*

b) *$X^s(T,\mathcal{L}) = X^{ss}(T,\mathcal{L})$ if and only if λ is not contained in a hyperplane (through 0) spanned by roots.*

2.9

Using the restriction map r: $\mathcal{X}(T) \longrightarrow \mathcal{X}(S)$ one gets a map $r \circ \mu$ with which one can determine the sets of (semi-)stable point for the action of S on X with respect to the line bundle $\mathcal{L}$. Since all characters of S are stable for the non-twisted action of the Galois group H, we have $r(h(\lambda)) = r(\lambda)$ for any $\lambda \in \mathcal{X}(T)$ and $h \in H$. In [BoT] proposition 6.7 it is proved that $r(h^*(\lambda)) = r(h(\lambda))$, where h^* denotes the twisted action of $h \in H$. This fact will be very useful. We now state some properties of the map $r \circ \mu$.

2.10 proposition

Let $\mathcal{L}$ be an ample line bundle on $X = G/P$, corresponding to the wheight λ and let $S \subset T \subset G$ be as before. Then we have:

a) *For any point $x \in X$ the vertices of $r \circ \mu(x)$ are contained in the set $\{r(w(\lambda))|w \in W\}$. The edges of $r \circ \mu(x)$ are paralell to roots $\alpha \in \Phi_K$.*

b) *$X^s(S,\mathcal{L}) = X^{ss}(S,\mathcal{L})$ if for all $w \in W$ $r(w(\lambda))$ is not contained in a hyperplane (through 0) spanned by roots.*

c) *$X^s(S,\mathcal{L}) \neq X^{ss}(S,\mathcal{L})$ if λ is contained in a hyperplane V spanned by roots $\alpha \in \Phi$ and moreover there exists an element $w \in W$ such that $r(w(V))$ is a hyperplane (spanned by roots $\beta \in \Phi_K$).*

2.11 Proof

Since r maps Φ into $\Phi_K \cup \{0\}$, part (a) of the proposition is clear. Using (a) one concludes that (b) must hold.

Part (c) follows from [PV] 1.4. There one constructs for λ contained in a hyperplane V spanned by roots a point $x \in X$ such that one has $\mu(x) = conv(\{w(\lambda) | w \in W_V\})$. Here W_V is the Weyl group of the root system $\Phi \cap V$. Then $0 \in \mu(x) \subset V$. Taking $w(V)$ such that $r(w(V))$ is a hyperplane in $\mathcal{X}(S) \otimes \Re$, one has $0 \in r(w(\mu(x))) = r \circ \mu(w(x)) \subset r(V)$. This shows that $w(x) \in X^{ss}(S, \mathcal{L}) - X^s(S, \mathcal{L})$. This proves (c).

2.12

The classification of absolutely almost simple groups over a non-archimedean local field K can be found in [Sa], [T.1] and [T.3]. In [Sa] and [T.1] the groups are given by their index. A simply connected semisimple group over a non-archimedean local field is essentially determined by its index.

The *index* of G is the following. One takes the Dynkin diagram $\mathcal{D}$ of the absolute root system Φ. The vertices of $\mathcal{D}$ represent the simple roots $\alpha \in \Delta$. One draws a circle around each vertex that represents a simple root α that does not vanish on S. These circled vertices are called the *distinguished vertices.* One indicates the action of H^* on the simple roots by arrows joining the vertices corresponding to roots that are in the same H^* orbit.

A group G is called an *inner form* if $H^* = \{id.\}$. Otherwise it is called an *outer form* . A group G is called *quasi-split* if no root $\alpha \in \Delta$ vanishes on the K-split torus S.

2.13 Theorem

There exists an ample line bundle $\mathcal{L}$ on $X = G/P$ such that $X^s(S, \mathcal{L}) = X^{ss}(S, \mathcal{L})$ if and only if one of the following holds:

1) *$P = B$ and G is any group*

2) *$P = P_J$, $J = \{1, \ldots, \ell - 1\}$ and G a non split form of C_ℓ. Here α_ℓ is the unique long root in Δ.*

3) *$P = P_I$ and $G = SL_{s+1}(D)$, where D is a skew field defined over K. Here $g.c.d.(i \in I, s+1) = 1$ and $G/P_{\{i\}} = Gr(i, (s+1) \cdot d)$, where d^2 denotes the dimension of D over K.*

2.14

The theorem above is in almost all cases a direct consequence of proposition 2.10 and the calculations done in 3.2 till 3.17. The only exceptions are the outer forms of type A_ℓ with $P_I \neq B$. Then the hyperplane V constructed in 3.11 containing λ_I is not spanned by roots. But, since the polyhedron μ constructed in remark 3.12 in fact is the polyhedron $\mu(x)$ for some $x \in X = G/P_I$, we still have $X^s(S,\mathcal{L}) \neq X^{ss}(S,\mathcal{L})$ in this case.

2.15

Associated to the variety X and the ample line bundle $\mathcal{L}$ both defined over K_s one has a variety $X^\dagger$ and an ample line bundle $\mathcal{L}^\dagger$ both defined over K. By construction one has:

$$X^s(S,\mathcal{L}) = X^{ss}(S,\mathcal{L}) \Longleftrightarrow (X^\dagger)^s(S,\mathcal{L}^\dagger) = (X^\dagger)^{ss}(S,\mathcal{L}^\dagger)$$

Since all components of $X^\dagger \otimes K_s$ are isomorphic to $X \otimes K_s$, one sees that X and $\mathcal{L}$ can be defined over K if and only if $X^\dagger \otimes K_s = X \otimes K_s$. Using proposition 3.18 and theorem 2.5 one easily proves:

2.16 Theorem

There exists an ample line bundle $\mathcal{L}$ defined over K on $X = G/P$ (defined over K) such that $X^s(S,\mathcal{L}) = X^{ss}(S,\mathcal{L})$ if and only if P is as in theorem 2.13 and G is not a quasi-split group with $\sharp H^* = 2$ or an outer form with $\Phi = A_\ell$.

2.17 Proposition

Let G be a non split group and suppose that $X^s(S,\mathcal{L}) = X^{ss}(S,\mathcal{L})$. Then $codim(X - X^s(S,\mathcal{L}) \geq 2$.

2.18 Proof

Since $S \subset T$ one has $X^s(T,\mathcal{L}) \subset X^s(S,\mathcal{L})$ and $X^{ss}(T,\mathcal{L}) \subset X^{ss}(S,\mathcal{L})$. Hence $codim(X - X^s(S,\mathcal{L}) \geq codim(X - X^s(T,\mathcal{L})$.

Let $\Phi \neq A_\ell$, then in [PV] lemma 4.12 it is proved for $X = G/B$ that $codim(X - X^{ss}(T,\mathcal{L})) \geq 2$. If $P \neq B$, then we have a G-equivariant map $\phi : G/B \longrightarrow G/P$ and for any ample line bundle $\mathcal{L}$ on G/P one can find an ample line bundle $\mathcal{L}^b$ on G/B such that $(G/B)^{ss}(T,\mathcal{L}^b) = \phi^{-1}((G/P)^{ss}(T,\mathcal{L}))$. Hence we have always $codim(X - X^{ss}(S,\mathcal{L})) \geq 2$ in this case.

If $\Phi = A_\ell$ and $X = G/P$ with P a maximal parabolic subgroup, then in [PV] lemma 4.5 it is shown that $codim(X - X^s(S,\mathcal{L})) \geq 2$ for $X \neq P^\ell, Gr(2,4)$. Since we assume $X^s(S,\mathcal{L}) = X^{ss}(S,\mathcal{L})$ the case $X = Gr(2,4)$ does not occur. If $X = P^\ell$ then $G = SL_{s+1}(D)$ with $d(s+1) = l+1$. Then one easily that $codim(X - X^s(S,\mathcal{L})) = d > 1$. For general X one has $codim(X - X^s(T,\mathcal{L})) = 1$ if and only if for a map ϕ as above one has $X^s(T,\mathcal{L}) = \phi^{-1}(\text{set of stable points in } P^\ell \text{ or } Gr(2,4))$. Now one easily concludes that $codim(X - X^s(S,\mathcal{L})) > 1$ for all X.

2.19

Assume $X^s(S,\mathcal{L}) = X^{ss}(S,\mathcal{L})$. Since G and $X^\dagger$can in fact be defined over K^0, we are allowed to use the construction of paragraph 1 of this article. Applying theorem 1.9 to $Y^\dagger := \bigcap_{g\in G(K)}(X^\dagger)^s(S,\mathcal{L})$ one sees that every meromorphic function on $Y^\dagger/\Gamma$ is constant on each connected component. Since the connected components of $Y^\dagger/\Gamma$ are all isomorphic to Y/Γ, where $Y := \bigcap_{g\in G(K)} X^s(S,\mathcal{L})$ one has:

2.20 Theorem

Let G be a non split group and let $X^s(S,\mathcal{L}) = X^{ss}(S,\mathcal{L})$. Let $\Gamma \subset G(K)$ be a discrete co-compact subgroup. Then any meromorphic function on Y/Γ is constant.

3 Weights

3.1

In this section we perform the calculations with weights that are needed to prove the last two propositions of the previous paragraph.

As before ℓ will denote the absolute rank of G. The set I will always denote a non-empty subset of the set $\{1,\ldots,\ell\}$. And λ_I will denote a weight of the form $\lambda_I = \sum_{i\in I} n_i \cdot \omega_i$, where the ω_i are the fundamental weights of the root system Φ.

First we will determine for each group G the sets I such that there do or do not exist weights λ_I with $r(w(\lambda_I))$ contained in a hyperplane spanned by roots $\alpha \in \Phi_K$ for some $w \in W$. In proposition 3.18 we will determine the sets I such that for some weight λ_I stable under the action of H^* no $r(w(\lambda_I))$ is contained in a hyperplane as above.

3.2 Proposition

Let $\Phi \neq A_\ell$ and let G be different from a non split form of C_ℓ. Then for every weight λ_I , $I \neq \{1,\ldots,\ell\}$, there exists an element $w \in W$ such that $w(\lambda_I)$ is contained in a hyperplane V spanned by roots $\alpha \in \Phi$ and such that $r(V)$ is a hyperplane (spanned by roots $\alpha \in \Phi_K$).

3.3 Proof

Let β_1 be the highest root of Φ. If Φ contains roots of different length we call the highest short root β_2. Since $\Phi \neq A_\ell$, there exists an unique simple root $\gamma_i \in \Delta$ such that $(\beta_i, \gamma_i) \neq 0$. So the hyperplane ${\beta_i}^\perp \subset \mathcal{X}(T) \otimes \Re$ is spanned by the roots $\alpha \in \Delta$, $\alpha \neq \gamma_i$.

Since the β_i are uniquely determined by the simple basis Δ , they are stable under the twisted action of the Galois group H. The same is true of the simple roots γ_i, since $\Phi \neq A_\ell$.

If $r(\gamma_i) \neq 0$ then $r({\beta_i}^\perp)$ is the hyperplane in $\mathcal{X}(S) \otimes \Re$ spanned by the roots $\alpha \in \Delta_K$, $\alpha \neq r(\gamma_i)$.

We will show that there exists an element $w \in W$ such that $w(\lambda) \in {\beta_1}^\perp$ or $w(\lambda) \in {\beta_2}^\perp$. Since $I \subset \{1,\ldots\ell\}$, $I \neq \{1,\ldots,\ell\}$, There exists a root $\alpha \in \Phi$, such that $(\lambda_I, \alpha) = 0$. Since the Weyl group acts transitively on the long (short) roots in Φ, we can find an element $w \in W$ such that $w(\lambda_I)$ is contained in ${\beta_1}^\perp$ or ${\beta_2}^\perp$.

3.4 Proposition

For every group G there exist weights λ_I, $I = \{1, \ldots, \ell\}$ such that no $r(w(\lambda_I))$, $w \in W$ is contained in a hyperplane spanned by roots $\alpha \in \Phi_K$.

3.5 Proof

Let $\mathcal{A}$ denote the union of hyperplanes in $\mathcal{X}(S) \otimes \Re$ that are spanned by roots. Then $\mathcal{B} := \cup_{w \in W} w(r^{-1}(\mathcal{A}))$ is the union of a finite number of hyperplanes in $\mathcal{X}(T) \otimes \Re$. Since the fundamental weights ω_i span $\mathcal{X}(T) \otimes \Re$, we can find a weight λ_I avoiding all these hyperplanes.

3.6

Next we treat the non split groups with absolute root system Φ of type C_ℓ. They are inner forms of C_ℓ.

We will use the following description of the root system C_ℓ. Let e_i, $i = 1, \ldots, \ell$ be an orthonormal basis of $\Re^\ell$. Then the root system C_ℓ consists of the vectors $\pm e_i \pm e_j$, $i \neq j$ and $\pm 2 \cdot e_i$, $i, j = 1, \ldots, \ell$. As a simple basis we take $\Delta = \{\alpha_i | i = 1, \ldots, \ell\}$, where $\alpha_i = e_i - e_{i+1}$, $i = 1, \ldots, \ell - 1$ and $\alpha_\ell = 2 \cdot e_\ell$.

Let J denote the set $J = \{1, \ldots, \ell - 1\}$.

3.7 Proposition

Let G be a non split group with absolute root system C_ℓ.

a) *For every weight I, $I \not\supseteq J$ there exists an element $w \in W$ such that $w(\lambda_I)$ is contained in a hyperplane V spanned by roots $\alpha \in \Phi$ and such that $r(V)$ is a hyperplane (spanned by roots $\alpha \in \Phi_K$).*

b) *There exist weights λ_I, $I \supseteq J$ such that no $r(w(\lambda_I))$, $w \in W$ is contained in a hyperplane spanned by roots $\alpha \in \Phi_K$.*

3.8 Proof

a) The proof the same as that of proposition 3.2. Now only $r(\beta_2) \neq 0$.

b) Since the fundamental weights ω_i, $i \in J$ span $\alpha_\ell{}^\perp = e_\ell{}^\perp$, it is sufficient to prove that $\alpha_\ell{}^\perp \not\subseteq \mathcal{B}$, where $\mathcal{B}$ is as in 3.5. So we only have to prove that $r(w(e_\ell{}^\perp))$ is not contained in a hyperplane spanned by roots $\alpha \in \Phi_K$.

Now $w(e_\ell{}^\perp) = e_i{}^\perp$ for some $i = 1, \ldots, \ell$. One verifies that we have:

$e_1{}^\perp = < \alpha_2, \alpha_3, \ldots, \alpha_\ell >$

$e_i{}^\perp = < \alpha_1, \ldots, \alpha_{i-2}, \alpha_{i-1} + \alpha_i, \alpha_{i+1}, \ldots, \alpha_\ell >$, $i = 2, \ldots, \ell - 1$.

$e_\ell{}^\perp = < \alpha_1, \ldots, \alpha_{\ell-2}, 2 \cdot \alpha_{\ell-1} + \alpha_\ell >$

Inspecting the indices one finds that $r(e_i{}^\perp)$ spans Φ_K. This proves (b).

3.9

Next we treat the outer forms with absolute root system $\Phi = A_\ell$. Then G is a special unitary group.

3.10 Proposition

Let G be an outer form with absolute root system A_ℓ. Then for every weight λ_I , $I \neq \{1, \ldots, \ell\}$, there exists an element $w \in W$ such that $r(w(\lambda_I))$ is contained in a hyperplane spanned by roots $\alpha \in \Phi_K$.

3.11 Proof

Let β be the highest root of Φ. Then we have :

$\beta^\perp = < \alpha_1 - \alpha_\ell, \alpha_i | i = 1, \ldots, \ell - 1 >$

Since $r(\alpha_1) = r(\alpha_\ell) \neq 0$, we have:

$r(\beta^\perp) = < r(\alpha_i) | i = 1, \ldots, \ell - 1 >$

Therefore there exists an element $w \in W$ such that $r(w(\lambda_I))$ is contained in the hyperplane $r(\beta^\perp)$, which is spanned by roots $\alpha \in \Phi_K$.

3.12 Remark

In the proposition above $w(\lambda_I) \in V = \beta^\perp$, but $\beta^\perp$ is not spanned by roots $\alpha \in \Phi$. Let ϕ^b be $V \cap \Phi$ and let W^bbe the Weyl group of Φ^b. In general 0 is not contained in the polyhedron μ, which is the convex hull of $\{w^b(w(\lambda_I)) | w^b \in$

$W^b\}$. But 0 is contained in $r(\mu)$, since $r(\alpha_1 - \alpha_\ell) = 0$. Furthermore $r(\mu)$ is contained in the hyperplane $r(V)$.

3.13

The only groups left to study are the inner forms of type A_ℓ. They are the groups $A_{\ell,s}$ with $s+1$ dividing $\ell+1$, i.e. the groups $G = SL_{s+1}(D)$ with D a skew field of dimension d^2, $d = \frac{\ell+1}{s+1}$ over K. Since $SL(D)$ is compact, we will assume that $s > 0$. If $d=1$ then $D=K$ and G is split.

We will use the following description of the root system A_ℓ. Let e_i, $i = 1,\dots,\ell+1$ be an orthonormal basis of $\Re^{\ell+1}$. The root system A_ℓ consists of the vectors $\pm(e_i - e_j)$, $i \neq j$. The simple basis Δ of Φ we will use consists of the roots $\alpha_i = e_i - e_{i+1}$, $i = 1,\dots,\ell$. Furthermore $\mathcal{X}(T) \otimes \Re \subset \Re^{\ell+1}$ is given by $\sum_{i=1}^{\ell+1} x_i = 0$.

The index of G has s distinguished vertices. They are the vertices corresponding to the roots $\alpha_{i\cdot d}$, $i = 1,\dots,s$. The relative root system Φ_K of G is of type A_s. We take an orthonormal basis f_i, $i = 1,\dots,s+1$ of $\Re^{s+1}$ and give the root system A_s as above with e_j replaced by f_j. Now $\mathcal{X}(S) \otimes \Re$ is given by $\sum_{j=1}^{s+1} y_j = 0$.

The restriction map r can be given as follows:

$e_i \longrightarrow f_j$, $i = j\cdot(d-1)+1,\dots,j\cdot d$, $j = 1,\dots,s+1$.

The fundamental weights ω_i of Φ with respect to Δ are:

$\omega_i = e_1 + e_2 + \cdots + e_i - \frac{i}{\ell+1}(e_1 + \cdots + e_{\ell+1})$

3.14 Proposition

Let G be an inner form of type A_ℓ. If g. c. d.($i \in I, s+1$)=1 then there exists a weight λ_I such that for no $w \in W$ one has that $r(w(\lambda))$ is contained in a hyperplane in $\mathcal{X}(S) \otimes \Re$ spanned by roots $\alpha \in \Phi_K$.

3.15 Proof

If *g. c. d.*$(i \in I, s+1) = 1$, then we can choose $n_i > 0$ such that *g. c. d.*$(\sum n_i \cdot i, s+1) = 1$. Let us fix such n_i and let $\lambda_I := \sum_{i\in I} n_i \cdot \omega_i$. For any element $w \in W$ we have $w(\lambda_I) = \sum a_j \cdot e_j - \frac{\sum n_i \cdot i}{\ell+1} \cdot (e_1 + \cdots + e_{\ell+1})$, where the a_j are certain integers.

Now $r(w(\lambda_I)) = \sum b_j \cdot f_j - \frac{d \cdot \sum n_i \cdot i}{\ell+1} \cdot (f_1 + \cdots + f_{s+1})$
$= \sum b_j \cdot f_j - \frac{\sum n_i \cdot i}{s+1} \cdot (f_1 + \cdots + f_{s+1})$
$= \frac{1}{s+1} \cdot \sum_{j=1}^{s+1} (b_j \cdot (s+1) - \sum n_i \cdot i) \cdot f_j$

Here the b_j are certain integers. Now $r(w(\lambda_I))$ is contained in a hyperplane in $\mathcal{X}(S) \otimes \Re$ spanned by roots in Φ_K if and only if there exists a subset $J \subset \{1, \ldots, s+1\}$, $J \neq \emptyset$, $J \neq \{1, \ldots, s+1\}$, such that :

$\sum_{j \in J} (b_j \cdot (s+1) - \sum n_i \cdot i) = 0$

$\Leftrightarrow \sum_{j \in J} b_j \cdot (s+1) = (\sharp J) \cdot \sum n_i \cdot i$

Now *g. c. d.* $(\sum n_i \cdot i, s+1) = 1$ implies that $\sharp J = s+1$ or $\sharp J = 0$. This cannot be.

Therefore no $r(w(\lambda_I))$ is contained in a hyperplane spanned by roots.

3.16 Proposition

Let G be an inner form of type A_ℓ. If g.c.d.$(i \in I, s+1) > 1$ then for every weight λ_I there exists an element $w \in W$ such that $w(\lambda_I)$ is contained in a hyperplane $V \subset \mathcal{X}(T) \otimes \Re$ spanned by roots $\alpha \in \Phi$ and $r(V) \subset \mathcal{X}(S) \otimes \Re$ is a hyperplane spanned by roots $\alpha \in \Phi_K$.

3.17 Proof

Let us first look at $\omega_i = e_1 + e_2 + \cdots + e_i - \frac{i}{\ell+1}(e_1 + \cdots + e_{\ell+1})$. Suppose $n > 1$ divides $g.c.d.(i, s+1)$. Then ω_i is contained in the following hyperplanes in $\mathcal{X}(T) \otimes \Re$ spanned by roots $\alpha \in \Phi$:

$\sum_{j \in J} x_j + \sum_{f \in F} x_f = 0$, where $J \subset \{1, \ldots, i\}$, $\sharp J = \frac{i}{n}$ and $F \subset \{i+1, \ldots, \ell+1\}$, $\sharp F = \frac{\ell+1-i}{n}$.

Let $I = \{i_1, \ldots, i_g\}$ with $i_1 < i_2 < \cdots < i_g$. Suppose $g.c.d.(i \in I) = n > 1$. We can construct a hyperplane V^b spanned by roots $\alpha \in \Phi$ containing all the weigts ω_i, $i \in I$ and therefore containing λ_I. The hyperplane V^b is defined by $\sum_{j \in J} x_j = 0$. Here J is given by:

$J = \{1, \ldots, \frac{i_1}{n}\} \cup \bigcup_{j=1}^{g-1} \{i_j + 1, \ldots, i_j + \frac{i_{j+1} - i_j}{n}\} \cup \{i_g + 1, \ldots, i_g + \frac{\ell+1-i_g}{n}\}$

It is clear that d divides $\sharp J = \frac{\ell+1}{n}$. One can find an element $w \in W$ such that $w(\lambda_I)$ is contained in the hyperplane V given by $\sum_{j=1}^{m} x_j = 0$, where $m = \sharp J$. Then $r(V)$ is given by $\sum_{j=1}^{m/d} y_j = 0$, which is spanned by roots $\alpha \in \Phi_K$. This proves the proposition.

3.18 Proposition

Let $\Lambda^{H^*} \subset \mathcal{X}(T) \otimes \Re$ denote the set of weights stable under the action of H^*. Then there exists an element $w \in W$ such that $r(w(\Lambda^{H^*}))$ is contained in a hyperplane spanned by roots $\alpha \in \Phi_K$ if and only if G is an outer form of type A_ℓ or a non split quasi-split form with $\sharp H^* = 2$ (Then $\Phi = A_\ell, D_\ell$ or E_6.).

3.19 Proof

We first remark that $\Lambda^{H^*} \otimes \Re$ is always spanned by roots. Therefore $r(w(\Lambda^{H^*} \otimes \Re))$ is always spanned by roots $\alpha \in \Phi_K$.

If G is an inner form then $\sharp H^* = 1$ and therefore $\Lambda^{H^*} = \Lambda$ and $r(w(\Lambda^{H^*}))$ is never contained in a hyperplane.

Let us now assume that G is quasi-split with $\sharp H^* = 2$ and that $\Phi \neq A_\ell$. Since G is quasi-split we have $rank(\Lambda^{H^*}) = rank(\Phi_K)$. We can find $\alpha, \beta \in \Delta$, $\alpha \neq \beta$, such that $\{\alpha, \beta\}$ is a H^*-orbit and $(\alpha, \beta) = 0$. Let $w = r_\alpha$ be the reflection in the hyperplane orthogonal to α. Then $\alpha + \beta \in \Lambda^{H^*}$ and $w(\alpha + \beta) = -\alpha + \beta$. Now $r(\alpha + \beta) \neq 0$, but $r(-\alpha + \beta) = 0$. One easily concludes that $r(w(\Lambda^{H^*}))$ is contained in a hyperplane spanned by roots $\alpha \in \Phi_K$.

A similar argument as above also works for the non quasi-split outer form with $\Phi = A_\ell$.

Now we treat the non quasi-split outer form with $\Phi = D_\ell$ and with $\sharp H^* = 2$. Let Φ be $\Phi = \{\pm e_i \pm e_j | i, j = i \dots \ell, i \neq j\}$, where the e_i, $i = 1, \dots, \ell$ form an orthonormal basis of $\Re^\ell$. Let us assume that the simple basis Δ consists of the roots $\alpha_i = e_i - e_{i+1}$, $i = 1, \dots, \ell - 1$ and $\alpha_\ell = e_{\ell-1} + e_\ell$. It is easy to see that Λ^{H^*} spans the hyperplane orthogonal to $\alpha_{\ell-1} - \alpha_\ell = -2e_\ell$. Therefore one has for any $w \in W$ that $w(\Lambda^{H^*} \otimes \Re) = e_i^\perp$ for some $i = 1, \dots, \ell$. Inspecting the indices one sees that $r(e_i^\perp)$ is never a hyperplane. So $r(w(\Lambda^{H^*}))$ is never contained in a hyperplane in this case.

We leave the remaining cases up to the reader. They are outer forms with $\Phi = A_2$ and $\Phi = D_4$.

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