Induced resolutions and Grothendieck groups of polycyclic-by-finite groups

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Induced resolutions and Grothendieck groups of polycyclic-by-finite groups

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§1. Introduction.

Let Γ be a group, let R be a commutative Noetherian ring, and let $G_0(R\Gamma)$ denote the Grothendieck group of finitely generated $R\Gamma$-modules. Let $\mathbf{X}$ be a class of groups, and let $G_0(R\Gamma, \mathbf{X})$ denote the subgroup of $G_0(R\Gamma)$ generated by the classes of modules of the form $M \otimes_{RH} R\Gamma$, where H is a $\mathbf{X}$-subgroup of Γ and M is a finitely generated RH-module. Let $\mathbf{F}$ be the class of finite groups.

Suppose Γ is torsion-free polycyclic-by-finite. Then $\langle 1 \rangle$ is the only $\mathbf{F}$-subgroup of Γ, so $G_0(R\Gamma, \mathbf{F})$ is the image of the induction map $G_0(R) \to G_0(R\Gamma)$. When $R = \mathbf{Z}$, the Cartan homomorphisms $K_0(\mathbf{Z}) \to G_0(\mathbf{Z})$, $K_0(\mathbf{Z}\Gamma) \to G_0(\mathbf{Z}\Gamma)$ are isomorphisms, since $\mathbf{Z}\Gamma$ has finite global dimension. A result of Farrell and Hsiang [5] asserts that $K_0(\mathbf{Z}) \to K_0(\mathbf{Z}\Gamma)$ is also an isomorphism. Hence $G_0(\mathbf{Z}\Gamma) = G_0(\mathbf{Z}\Gamma, \mathbf{F})$ in this case.

The situation when Γ has torsion is somewhat more complicated. However, we shall prove the following result. (A commutative Noetherian ring is *regular* if all its finitely generated modules have projective resolutions of finite length, and is *Hilbert* if each of its prime ideals is an intersection of maximal ideals.)

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THEOREM A *Let* Γ *be a finitely generated group with an abelian normal subgroup of finite index* a. *Let* h *be the Hirsch number of* Γ. *Let* R *be a commutative Noetherian regular Hilbert ring of finite Krull dimension* d. *Then* $G_o(R\Gamma)/G_o(R\Gamma, F)$ *is periodic, of exponent dividing* a^{h+d}.

We know of no example where $G_o(R\Gamma, F) \subsetneq G_o(R\Gamma)$. The restriction to abelian-by-finite groups is essential for our proof of Theorem A, but most of our preliminary results hold for a polycyclic-by-finite group Γ. We have stated these results in their most general form.

When Γ is abelian-by-finite, some insight into the structure of $G_o(\mathbb{Z}\Gamma)$ may be obtained from the action of crystallographic groups on Euclidean space. In fact, something along the same lines is true for polycyclic-by-finite groups in general.

THEOREM B *Let* Γ *be a polycyclic-by-finite group with Hirsch number* h. *Then* Γ *acts smoothly and simplicially on some smooth triangulation of Euclidean space* $\mathbb{R}^h$, *with compact quotient and finite isotropy groups.*

There is nothing essentially new in Theorem B. The ingredients are readily available in the literature. Indeed, something akin to Theorem B seems to be implicit in [16]. Nevertheless, it seems to be worthwhile to include a proof here, in view of the following interesting algebraic consequence, which, when Γ is torsion-free, is just the well-known fact that $\mathbb{Z}$ has a finite free $\mathbb{Z}\Gamma$-resolution of length h.

COROLLARY C *Let* Γ *be as in Theorem B. Then there exists an exact sequence*

$$(*) \qquad 0 \to Q_h \to \dots \to Q_1 \to Q_o \to \mathbb{Z} \to 0$$

of right $\mathbb{Z}\Gamma$*-modules, where each* Q_i *is a finite direct sum of modules of the form*

$$\mathbb{Z} \otimes_{\mathbb{Z}H} \mathbb{Z}\Gamma$$

for various finite subgroups H *of* Γ .

The paper is organised as follows. In §2 we prove Theorem B and Corollary C, and in §3 a stronger form of the latter is deduced (Theorem 3.2). This result provides the starting point in §4 for an inductive proof of Theorem A. A result on the uniform dimension of prime factor rings, which is needed in the proof of Theorem A and which may have some independent interest, is proved in §5.

As applications of Theorem A, we offer

COROLLARY D *Let* R *and* Γ *be as in Theorem A. Suppose* R *is a Dedekind ring for which the Jordan-Zassenhaus Theorem holds.*

(i) $G_0(R\Gamma) = T \times F$, *where* T *is the torsion subgroup, and* F *is free abelian of finite rank,* t *say. Further,* T *contains a finite subgroup* T_0 *such that* T/T_0 *has exponent dividing* a^{h+d} . *If* R *is a field, then*

$$t \leq \sum_{H \in \mathbf{H}} |\mathrm{irr}(RH)| \quad ,$$

where $\mathbf{H}$ *is a set of representatives of the conjugacy classes of maximal finite subgroups of* Γ , *and* $|\mathrm{irr}(RH)|$ *is the number of isomorphism classes of irreducible* RH*-modules.*

(ii) *Suppose* R *is a field of characteristic* p . *The cokernel of the Cartan homomorphism of* $K_0(R\Gamma)$ *into* $G_0(R\Gamma)$ *is torsion of exponent dividing* $p^r a^h$, *where* p^r *is the maximal order of a* p*-subgroup of* Γ .

These follow immediately from Theorem A and [19, proof of Theorem 3.8] and [4, Theorem 21.22] respectively.

§2. Topology.

Throughout this section Γ is an arbitrary polycyclic-by-finite group of Hirsch number h. The proof of Theorem B follows that of Theorem 1 of [1] in constructing a smooth action of Γ on $\mathbb{R}^h$ with finite isotropy groups and compact quotient. (The only difference being that, in [1], Γ is assumed to be torsion-free, so that the quotient $\mathbb{R}^h/\Gamma$ is a $K(\Gamma,1)$-space).

Explicitly, there exists a commutative diagram

$$\begin{array}{ccccccccc} 1 & \to & \Delta & \to & \Gamma & \to & G & \to & 1 \\ & & \Big| & & \Big| & & \Big\| & & \\ 1 & \to & D(\Delta) & \to & \Gamma D(\Delta) & \to & G & \to & 1 \end{array}$$

with exact rows and vertical monomorphisms, where Δ is a torsion-free subgroup of finite index in Γ, and $D(\Delta)$ is a soluble Lie group containing Δ as a discrete co-compact subgroup. As in [1], let K be a maximal compact subgroup of $\Gamma D(\Delta)$. Then $K \backslash \Gamma D(\Delta)$ is diffeomorphic to $\mathbb{R}^h$, and Δ acts freely and smoothly on the right, so that the quotient space M is a smooth manifold. Moreover, the finite group $G = \Gamma/\Delta$ acts smoothly on M, so by [9] there exists a smooth G-equivariant triangulation T_0 of M. This lifts to a smooth Γ-equivariant triangulation T of $\mathbb{R}^h$, and the proof of Theorem B is complete.

For the proof of the Corollary, take (*) to be the simplicial chain complex of T. Thus each Q_i is a free abelian group, with basis the i-simplices of T. As a $\mathbb{Z}\Gamma$-module, Q_i is a permutation module (since Γ

permutes simplices), with finite stabilizers (since Γ has finite isotropy groups), and finitely generated (since T/Γ is a finite complex). But such a module has precisely the form stated in the Corollary.

§3. Modules

We shall need the following version of Frobenius reciprocity [19, Theorem 2.2].

LEMMA 3.1 *Let R be a commutative ring and let H be a subgroup of a group G. Let W be a finitely generated RH-module and let X be a finitely generated RG-module. Then, as RG-modules,*

$$(W \otimes_{RH} RG) \otimes_R X \cong (W \otimes_R X|_H) \otimes_{RH} RG,$$

where each tensor product over R is equipped with the diagonal group action.

Proof It is routine to check that the map $(w \otimes g) \otimes x \longmapsto (w \otimes xg^{-1}) \otimes g$ is well-defined and gives an isomorphism of RG-modules.

THEOREM 3.2 *Let R be a commutative Noetherian ring. Let Γ be a polycyclic-by-finite group, and let V be an $R\Gamma$-module which is finitely generated as an R-module. Then $[V] \in G_0(R\Gamma, F)$.*

Proof By corollary C there is an exact sequence

$$0 \to Q_n \to \ldots \to Q_1 \to Q_0 \to \mathbb{Z} \to 0 \qquad (1)$$

of $\mathbb{Z}\Gamma$-modules, with each Q_i a finite direct sum of modules

$$\mathbb{Z} \otimes_{\mathbb{Z}H} \mathbb{Z}\Gamma$$

for various finite subgroups H of Γ.

Apply the functor $(-) \otimes_{\mathbb{Z}} R$ to (1) to get a sequence

$$0 \to P_n \to \ldots \to P_1 \to P_0 \to R \to 0 \qquad (2)$$

of $R\Gamma$-modules. Then (2) is exact, since (1) consists of free $\mathbb{Z}$-modules. Furthermore each P_i is a finite direct sum of modules

$$(\mathbb{Z} \otimes_{\mathbb{Z}H} \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}} R \quad \cong \quad R \otimes_{RH} R\Gamma$$

for various finite subgroups H of Γ.

Finally, apply the functor $(-) \otimes_R V$ to (2) (where each term is given the $R\Gamma$-module structure with diagonal Γ-action). The resulting sequence is exact, since the modules in (2) are free R-modules. By Lemma 3.1, its terms are finite direct sums of modules

$$(R \otimes_{RH} R\Gamma) \otimes_R V \;\cong\; V|_H \otimes_{RH} R\Gamma \;.$$

The sequence obtained is a sequence of $R\Gamma$-modules, since if $f_i : P_i \to P_{i-1}$, then $(f_i \otimes 1) : P_i \otimes V \to P_{i-1} \otimes V$ takes $(\pi \otimes v)g = \pi g \otimes vg$ to

$$\begin{aligned} f_i(\pi g) \otimes vg &= f_i(\pi)g \otimes vg \\ &= (f_i(\pi) \otimes v)g \\ &= [(f_i \otimes 1)(\pi \otimes v)]g \end{aligned}$$

The result now follows from the fact that V is finitely generated as an R-module.

§4. Abelian-by-finite groups

We begin this section by recalling some well-known facts and definitions concerning a Noetherian ring S. Let M be an S-module. We say that M is *uniform* if it is non-zero and, if X and Y are any two non-zero submodules of M, then $X \cap Y \neq 0$. The *uniform dimension* of M, u-dim(M) is 0 if $M = 0$, t if M contains an essential direct sum of t uniform submodules, and ∞ if no such finite direct sum exists; if M is finitely generated, then u-dim$(M) < \infty$. See [14, Ch. 10, §4] for details. An element m of M is *torsion* if $mc = 0$ for some regular element c of S; and M is *torsion free* if it contains no non-zero torsion elements.

Let P be a prime ideal of S. By Goldie's theorem [14, Theorem 10.4.10], S/P has a simple Artinian quotient ring Q; Q is a ring of $t \times t$ matrices over a division ring, where t is the uniform dimension of S/P (as right or left module). Let U and V be uniform right ideals of S/P. Thus $U \otimes_{S/P} Q$ and $V \otimes_{S/P} Q$ are both irreducible right Q-modules, and so isomorphic. It follows easily that U and V are subisomorphic as S-modules: in other words each is isomorphic to a submodule of the other. More generally, if X and Y are finitely generated torsion free S/P-modules of the same uniform dimension, then X embeds in Y and the cokernel is torsion as an S/P-module. These facts are clear when S/P is a finite module over its centre (the only case we require here). For the general case, one may consult [10, Lemma 2.2.13], for example.

Let M be a finitely generated S-module. Choose a uniform submodule U_1 of M whose annihilator P_1 is maximal among annihilators of non-zero submodules of M. It is easy to see that P_1 is prime. Repeat this process for M/U_1, and so on: we get a chain (finite, since M is Noetherian) $0 \subset U_1 \subset U_2 \subset \ldots \subset U_n = M$ of submodules whose factors U_i/U_{i-1} are uniform, every non-zero submodule of U_i/U_{i-1} having annihilator P_i. Suppose now that S is finitely generated as a module over its centre. Let U be a finitely generated uniform S-module all of whose non-zero submodules have prime annihilator P. We can form the quotient ring Q of S/P by inverting the non-zero elements of the centre of S/P (since the resulting partial quotient ring is a finite dimensional algebra over a field and hence Artinian). Thus, if $c + P$ is a regular element of S/P, $(cS + P)/P$ must have non-zero intersection with the centre of S/P. It follows that U is a torsion free S/P-module, and so, by the previous paragraph, U is (isomorphic to) a uniform right ideal of S/P. To sum up:

PROPOSITION 4.1 *Let S be a Noetherian ring which is a finitely generated module over its centre.*

(i) *Let M be a finitely generated S-module. Then M has a finite series of submodules with successive factors isomorphic to uniform right ideals of prime factor rings of S.*

(ii) *Let P be a prime ideal of S, with u-dim$(S/P) = t$. Let U be a uniform right ideal of S/P. Each of $U^{(t)}$ and S/P embeds in the other, the cokernel having annihilator strictly containing P.*

We shall use induction arguments involving the Krull dimension, k-dim(M), of an S-module M. Details may be found in [6]; but it is almost enough to know that, for S and M as in 4.1,

($K1$) k-dim(S) is the supremum of the lengths of descending chains of prime ideals of S;

($K2$) If $0 \to A \to B \to C \to 0$ is an exact sequence of S-modules, k-dim(B) = max {k-dim(A), k-dim(C)};

($K3$) [6, Theorem 9.2 and 18, Lemma 8] If R is a commutative Noetherian ring of finite Krull dimension d and Γ is a finitely generated abelian-by-finite group of Hirsch number h, then k-dim$(R\Gamma) = h+d$.

The next result contains the crux of the inductive step in the proof of Theorem A. Let R be a commutative Noetherian ring and let G be a polycyclic-by-finite group. For a non-negative integer n, define subgroups of $G_0(RG)$ as follows:

$$G_0(RG)_n = \langle [V] : \text{k-dim}(V) \leq n \rangle,$$

$$G_0(RG)_{n-} = \langle [V] : \text{k-dim}(V) < n \rangle.$$

PROPOSITION 4.2 *With the above notation, let P be a prime ideal of RG*

and set $n = k\text{-}\dim(RG/P)$. *Then, in* $G_0(RG)$,

$$[RG/P] \in G_0(RG,F) + G_0(RG)_{n-}.$$

Proof Let H and N be subgroups of G, with N normal. If M is a finitely generated RH-module with $[M] \in G_0(RH,F)$, then $[M \otimes_{RH} RG] \in G_0(RG,F)$. Hence, in view of Theorem 3.2, inflation from $R(G/N)$- to RG- modules maps $G_0(R(G/N),F)$ to $G_0(RG,F)$. Moreover, under this map $G_0(R(G/N))_{n-}$ is sent to $G_0(RG)_{n-}$. Thus, in proving the proposition we may assume that $\{g \in G : (g-1) \in P\} = 1$.

In the notation of [17], set $H = nio(G)$, an orbitally sound normal subgroup of finite index in G. We argue by induction on $|G : H|$. Suppose first that $G = H$, or more generally that $P \cap RH = Q$ is a prime ideal of RH. Then $\{g \in G : (g-1) \in Q\} = 1$, so $Q = (Q \cap R\Delta)RH$, by [17, Theorem C1], where Δ is the FC-subgroup (see [14, §4.1] for definition) of H and of G. By [12, Lemma 2.1], $QRG = (Q \cap R\Delta)RG$ is a prime ideal of RG which is contained in P by construction. Since $QRG \cap RH = P \cap RH = Q$, we conclude that $QRG = P$ by incomparability [13, Theorem 1.2]. Now, if Δ_0 denotes the (finite) torsion subgroup of Δ, then Δ/Δ_0 is free abelian of finite rank and so $R\Delta$ is an iterated skew Laurent extension of $R\Delta_0$. The so-called twisted Grothendieck Theorem [15, Exercise following Theorem 8, §6], or [20, Proposition 4.1 (2)] therefore implies that $G_0(R\Delta) = G_0(R\Delta,\Delta_0) = G_0(R\Delta, F)$. Consequently, $[RG/P] = [(R\Delta/Q \cap R\Delta) \otimes_{R\Delta} RG] \in G_0(RG,F)$.

We may therefore assume that $P \cap RH = \bigcap_{x \in G} Q^x$ for some prime ideal Q of RH with $B = \{g \in G : Q^g = Q\}$ a proper subgroup of G. Let $J = \{x_1, \ldots, x_t\}$ be a right transversal to B in G. By [12, Theorem 1.7], there is a unique prime ideal L of RB with $L \cap RH = Q$ and $\bigcap_{i=1}^{t} L_i RG = P$, where L_i denotes L^{x_i} for $i = 1, \ldots, t$. Let θ denote the natural embedding of RG/P in $\sum_i^{\oplus}(RG/L_iRG)$ as right RG-modules, with cokernel Y. We claim that

$$k\text{-}dim(Y) < n. \qquad (1)$$

Let us complete the proof, assuming that (1) is true. By definition, $[Y] \in G_0(RG)_{n-}$, and so, setting $B_i = B^{x_i}$,

$$[RG/P] = \sum_i [(RB_i/L_i) \otimes_{RB} RG] - [Y].$$

By [2, Lemma 4.2], $k\text{-dim}(RB_i/L_i) = n$ for all i. By induction on $|G : H|$, $[RB_i/L_i] \in G_0(RB_i.F) + G_0(RB_i)_{n-}$ for all i. By [18, Lemma 8] and the first paragraph of the proof,

$$[(RB_i/L_i) \otimes RG] \in G_0(RG.F) + G_0(RG)_{n-}$$

for all i. Thus (1) shows that $[RG/P]$ is in the required subgroup of $G_0(RG)$. It remains therefore to prove (1).

For $j = 1,\ldots,t$, let π_j: $\Sigma^{\oplus}RG/L_iRG \longrightarrow RG/L_jRG$ be the projection map. Since $Q_i := Q^{x_i} = L_i \cap RH$,

$$0 \neq \pi_j \circ \Theta((\bigcap_{i \neq j} L_iRG \cap RH)RB_j + P/P) := X \subseteq RB_j + L_jRG/L_jRG \cong RB_j/L_j.$$

Further, $\bigcap_{i \neq j} (L_iRG \cap RH)$ is the annihilator in RH of $Q_j/(P \cap RH)$, and as such is invariant under conjugation by B_j. Hence, X is a non-zero two-sided ideal of RB_j/L_j. Since L_j is a prime ideal, $k\text{-dim}((RB_j/L_j)/X) < k\text{-dim}(RB_j/L_j)$, by [6], (or by (K1) if we assume G is abelian-by-finite). Therefore, identifying RB_j/L_j with $RB_j + L_jRG/L_jRG$,

$$k\text{-dim}_{RB_j}((RB_j/L_j)/\text{im}(\pi_j \circ \Theta) \cap (RB_j/L_j)) < k\text{-dim}_{RB_j}(RB_j/L_j) = n,$$

and so, by [18, Lemma 8],

$$k\text{-dim}_{RB_j}((RG/L_jRG)/\text{im}(\pi_j \circ \Theta)) < k\text{-dim}_{RB_j}(RG/L_jRG) = n .$$

By [18, Lemma 8] once more,

$$k\text{-dim}_{RG}((RG/L_jRG)/\text{im}(\pi_j \circ \Theta)) < k\text{-dim}_{RG}(RG/L_jRG) = n . \qquad (2)$$

Since (2) holds for all $j = 1, \ldots, t$, (1) is proved.

Let p be a prime. A finite group is *p-hyperelementary* if it has the form $\langle x \rangle \rtimes P$, with x an element of order prime to p, and P a p-group. Let G be a finite group. We denote by H the class of finite groups are p-hyperelementary for some prime p. We need a version of the Brauer-Berman-Witt induction theorem:-

THEOREM 4.3 *Let R be a commutative Noetherian ring. Let* G *be a finite group. Then* $G_0(RG, H) = G_0(RG)$.

Proof The theorem is true when R is a field by [4, Theorems 21.6 and 21.15] and [19, Corollary 2.9], using the fact [3, Proposition 5, p.23 and Proposition 1c, p. 17] that every field of characteristic p arises as the residue field of some discrete valuation ring of characteristic 0. The proof proceeds by induction on k-dim(R). Since a finitely generated RG-module M has a finite series of submodules $0 = M_0 \subset \ldots \subset M_i \subset \ldots \subset M_n = M$ with each factor M_i/M_{i+1} having prime annihilator in R, we may assume that R is prime. Let K be the quotient field of R. There is a commutative diagram

$$\begin{array}{ccccccc}
\sum\limits_{0 \neq p \in R}^{\oplus} G_0((R/pR)G) & \longrightarrow & G_0(RG) & \longrightarrow & G_0(KG) & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
\sum\limits_{0 \neq p \in R}^{\oplus} G_0((R/pR)G, \mathbb{H}) & \longrightarrow & G_0(RG, \mathbb{H}) & \longrightarrow & G_0(KG, \mathbb{H}) & \longrightarrow & 0,
\end{array}$$

in which the vertical arrows are inclusions, and by [19, Theorem 1.6] the rows are exact. By the induction hypothesis and the result for fields the outer maps are surjections. Hence, so is the middle map.

A commutative Noetherian ring R is *regular* if every finitely generated R-module has a finite resolution by projective R-modules. Let G be a finite group. The abelian group with generators $[M]$, where M is a finitely generated RG-module which is R-projective, and relations given by short exact sequences, is denoted by $G_0^R(RG)$. If R is regular, $G_0^R(RG) = G_0(RG)$ [19, Theorem 1.2]. The group $G_0^R(RG)$ can be given a ring structure by setting $[M][N] = [M \otimes_R N]$, [19, Theorem 1.5]. Note that $[R]$ is the identity element of this ring.

THEOREM A *Let R be a commutative Noetherian regular Hilbert ring of finite Krull dimension d, and let Γ be a finitely generated abelian-by-finite group with Hirsch number h. Let A be a maximal Abelian normal subgroup of Γ, and set $G = \Gamma/A$; put $|G| = a$. Then $G_0(R\Gamma)/G_0(R\Gamma, F)$ is periodic, with exponent dividing a^{h+d}.*

Proof The ring $R\Gamma$ is Noetherian and is a finite module over its centre, [14, Corollary 10.2.8 and proof of Lemma 4.1.10], so we can make use of the facts and concepts given at the start of §4.

Step 1. Reduction to the case where G is hyper-elementary.

Let $\hat{H}(\Gamma)$ be the set of inverse images in Γ of the H-groups in G. By Theorem 4.3,

$$G_0(RG,\mathbf{H}) = G_0(RG). \qquad (3)$$

As pointed out above, the hypotheses on R ensure that $G_0(RG)$ is a ring with identity element. Viewing $G_0(R\Gamma)$ as a $G_0(RG)$-module via inflation and $-\otimes_R-$,

$$G_0(R\Gamma).G_0(RG,\mathbf{H}) \subseteq G_0(R\Gamma,\hat{\mathbf{H}}(\Gamma)), \qquad (4)$$

by Lemma 3.1. By (3) and (4),

$$G_0(R\Gamma) = G_0(R\Gamma, \hat{\mathfrak{H}}(\Gamma)). \qquad (5)$$

It follows from (5) that there is a surjection induced by induction.

$$\sum_{X \in \hat{\mathfrak{H}}(\Gamma)}^{\oplus} G_0(RX)/G_0(RX,\mathbf{F}) \longrightarrow G_0(R\Gamma)/G_0(R\Gamma,\mathbf{F}). \qquad (6)$$

Since $|X/A|^{h+d}$ divides a^{h+d} for all $X \in \hat{\mathfrak{H}}(\Gamma)$, (6) shows that we may replace Γ by one of the groups X in $\hat{H}(\Gamma)$ in proving the theorem.

Step 2. The induction set-up.

We shall deduce the theorem from the following more precise set of statements:

Let P be a prime ideal of $R\Gamma$, with

k-dim $(R\Gamma/P) = m$. Let $\omega = \max\{1, a^{m-1}\}$. (7;m)

Then $\omega.[R\Gamma/P] \in G_0(R\Gamma,\mathbf{F})$.

Since R is Hilbert, every finitely generated Artinian $R\Gamma$-module is finitely generated as an R-module [11, Theorem 31]. Thus Theorem 3.2 shows that (7;0) is true.

We claim that

if $m \geq 0$ and if $(7;i)$ is true for all $i \leq m$, (8;m)

then $a^m.(G_0(R\Gamma)_m) \subseteq G_0(R\Gamma,\mathbf{F})$.

Again, Theorem 3.2 allows us to assume that $m > 0$ and that $(8;\ell)$ is true for all $\ell < m$. By Proposition 4.1 and (K2), $G_0(R\Gamma)_m$ is generated by $[M]$, where M is a uniform right ideal of $R\Gamma/P$ and P ranges over the set of all prime ideals for which k-dim$(R\Gamma/P) \leq m$. Let M be one such right ideal, of $R\Gamma/P$, say. Let t = u-dim$(R\Gamma/P)$. By Proposition 4.1 (ii) there is an exact sequence

$$0 \longrightarrow R\Gamma/P \longrightarrow M^{(t)} \longrightarrow X \longrightarrow 0. \qquad (9)$$

By Proposition 4.1 (ii) and (K1), (K2), k-dim$(X) = \ell < m$. Hence, by $(8;\ell)$, $a^{m-1}.[X] \in G_0(R\Gamma,F)$. By $(7;m)$ and (9), $ta^{m-1}.[M] \in G_0(R\Gamma,F)$. Thus $(8;m)$ follows from this, Proposition 5.2 (ii), and Step 1.

Proposition 4.2 shows that the statements (8;i), for $i = 0,\ldots,m-1$, together imply (7;m). Thus the proof is complete.

§5 Uniform dimension of prime factors

Our aim here is to prove Propostion 5.2, part of which was used in the proof of Theorem A.

Let H be a subgroup of a group G and let R be a commutative ring. Let Q be an ideal of RH. Then Q^G denotes the biggest ideal of RG inside QRG, so $Q^G = \bigcap_{g\in G} (QRG)^g$; see [12].

LEMMA 5.1 *Let R be a commutative ring, and let G be a polycyclic-by-finite group containing a subgroup H of finite index.*

(i) Let P be a prime ideal of RG and let $Q_1,\ldots,Q_r$ be the prime ideals of RH minimal over $P \cap RH$. Then there exist positive integers $z_1,\ldots,z_r$ such that $\text{u-dim}(RG/P) = \Sigma_i z_i.\text{u-dim}(RH/Q_i)$.

(ii) Let Q be a prime ideal of RH and let $P_1,\ldots,P_s$ be the primes of RG minimal over Q^G. Then there exist positive integers $w_1,\ldots,w_s$ such that $\Sigma_j w_j.\text{u-dim}(RG/P_j) = |G:H|\cdot\text{u-dim}(RH/Q)$.

Proof By factoring by $P \cap R$ in (i) and by $Q \cap R$ in (ii), and then inverting the non-zero elements of R, we reduce to the case where R is a field. Thus all rings involved here are Noetherian. In particular there are indeed finite sets of primes lying over P and Q in (i) and (ii) respectively.

(i) This simply expresses the fact that the inclusion of rings $RH \subset RG$ satisfies the additivity principle [21, Corollary 2 and preceding remarks].

(ii) Put $I = Q^G$ and $V = RG/QRG$. Thus V is an $(RH\text{-}RG)$-bimodule with right annihilator I, and

$$RH|V \cong \Sigma^{\oplus}(RH/Q)\bar{x} \cong (RH/Q)^{(t)} ,$$

where $T = \{x_1, \ldots, x_t\}$ is a right transversal for H in G, and $\bar{x}$ denotes the image of x in V. Thus

$$A := RG/I \subseteq B := \mathrm{End}_{RH}(V) \cong M_t(RH/Q),$$

and *(II)* will follow if we can show that the inclusion $A \subseteq B$ satisfies the additivity principle. By [21, Corollary 2], it suffices to show that B is finitely generated as a right and as a left A-module.

For this, fix a normal subgroup N of G with $N \subseteq H$ and $|G : N| < \infty$, and set $A_o = RN/I \cap RN \subseteq A$. Under the embedding $A \subseteq M_t(RH/Q)$, A_o corresponds to the subring

$$D = \left\{ \begin{bmatrix} \overline{r^{x_1}} & & 0 \\ & \ddots & \\ 0 & & \overline{r^{x_t}} \end{bmatrix} : r \in A_o \right\}$$

of $M_t(\bar{A}_o)$, where $\bar{\ }$ denotes images in RH/Q. Clearly the elementary matrices $\{E_{ij} : 1 \leq i,j \leq t\}$ generate $M_t(\bar{A}_o)$ as a left and as a right D-module: and $M_t(RH/Q)$ is finitely generated as a left and as a right module over $M_t(\bar{A}_o)$, since this holds for RH/Q over $\bar{A}_o$. Therefore B is finitely generated on both sides over A_o, and hence over A, as required.

For any ring S, set

$$u(S) := \sup\{\, u\text{-dim}\,(S/P) \mid P \text{ a prime ideal of } S \,\},$$

a positive integer or ∞. Then, in the situation of Lemma 5.1, we have

$$u(RH) \leq u(RG) \leq [G:H].u(RH) .$$

To derive this from Lemma 5.1 one uses the fact that each prime ideal of RH is minimal over $P \cap RH$ for a suitable prime ideal P of RG, and each prime ideal of RG is minimal over Q^G for some prime Q of RH. The details are fairly routine and are left to the reader.

We can now state and prove the main result of this section.

PROPOSITION 5.2 *Let N be a normal subgroup of finite index a in a polycyclic-by-finite group Γ. Let R be a commutative ring, and let P be a prime ideal of $R\Gamma$. Let p be the characteristic of $R/P \cap R$. Let Q be a prime ideal of RN minimal over $P \cap RN$, with $P \cap RN = \bigcap_{\gamma\in\Gamma} Q^{\gamma}$. Then*

(i) $\text{u-dim}(RN/Q) \mid \text{u-dim}(R\Gamma/P)$.

(ii) Let $\hat{C}$ denote the algebraic closure of the centre C of the simple Artinian ring of quotients F of RN/Q. Let $\hat{C}N$ denote the subring of $\hat{C} \otimes_C F$ generated by $\hat{C}$ and RN/Q. If either $p \mid a$ or Γ/N is p-soluble, then

$$\text{u-dim}(R\Gamma/P) \mid a \,.\, \text{u-dim}(\hat{C}N/\hat{Q}) \quad ,$$

where $\hat{Q}$ is a suitable prime ideal of $\hat{C}N$ lying over the zero ideal of RN/Q.

In particular if N is Abelian, and either $p \mid a$ or Γ/N is p-soluble, then

$$\text{u-dim}(R\Gamma/P) \mid a \quad .$$

Proof As in Lemma 5.1 we reduce at once to the case where R is a field. By [17, Lemma 5] we have $P \cap RN = \bigcap_{\gamma\in\Gamma} Q^{\gamma}$. Since $\text{u-dim}(RN/Q) = \text{u-dim}(RN/Q^{\gamma})$ for all $\gamma \in \Gamma$, (i) is a special case of Lemma 5.1(i).

(ii) We argue by induction on a. For $a = 1$, the claim is that $\text{u-dim}(R\Gamma/P) \mid \text{u-dim}(\hat{C}\Gamma/\hat{P})$ for some prime ideal $\hat{P}$ of $\hat{C}\Gamma$ with $\hat{P} \cap R\Gamma = P$. But, for *any* such $\hat{P}$, $R\Gamma/P \subseteq \hat{C}\Gamma/\hat{P}$ is a centralizing extension of prime Noetherian rings, so the assertion follows from [21, Theorem 3], for example. Thus we may assume that (ii) is true for all proper subgroups H of Γ with $N \subseteq H$.

Let M be a proper normal subgroup of Γ with $N \subseteq M$ and such that $P \cap RM$ is not prime. Then $P = P_1{}^{\Gamma}$ for some prime ideal P_1 of RM_1, where $M \subseteq M_1 \subset \Gamma$, by [12, Theorem 1.7]. By Lemma 5.1(ii),

$$\text{u-dim}(R\Gamma/P) \mid \text{u-dim}(RM_1/P_1).|G:M_1|,$$

so the inductive hypothesis applied to M_1 yields the result. (Note that the primes of RN minimal over $P_1 \cap RN$ are minimal over $P \cap RN$, and hence Γ-conjugate to Q.) Thus we may assume that, for all normal subgroups M of Γ with $N \subseteq M$, $P \cap RM$ is prime.

In particular, $Q = P \cap RN$ is prime. Standard arguments along the lines of [12, Lemma 1.5] show that the set C of regular elements of RN/Q forms an Ore set of regular elements in $R\Gamma/QR\Gamma$ and in $R\Gamma/P$. By localising at C we obtain the classical rings of quotients of the rings under consideration:

$$A := (RN/Q)C^{-1} = Q(RN/Q) \subseteq B := (R\Gamma/P)C^{-1} = Q(R\Gamma/P)$$

$$\downarrow \qquad\qquad \nearrow$$

$$(R\Gamma/QR\Gamma)C^{-1} = Q(R\Gamma/QR\Gamma)$$

(Here we are abusing notation by writing C for its image in $R\Gamma/QR\Gamma$ and $R\Gamma/P$.) Note that B has the structure of a crossed product over A, $B \cong A*G$ with $G = \Gamma/N$. Let Γ_{inn} be the normal subgroup of Γ consisting of those elements acting by inner automorphisms on the simple Artinian ring A, and set $G_{inn} = \Gamma_{inn}/N \subseteq G$. By our assumption, $P \cap R\Gamma_{inn}$ is prime. Let $\bar{\ }$ denote images modulo $QR\Gamma$. Thus $T := (\overline{P \cap R\Gamma_{inn}})C^{-1}$ is a prime ideal of $A*G_{inn} \subseteq B$, and, by [13, Theorem 2.5(i)], $P' := \bar{P}C^{-1} = T \cdot B$. Lifted back to $R\Gamma$, this yields $P = (P \cap R\Gamma_{inn})R\Gamma$, and so if $\Gamma_{inn} \neq \Gamma$ the induction hypothesis and Lemma 5.1(ii) again give the result. We may therefore assume that $\Gamma = \Gamma_{inn}$.

Let E denote the centraliser of A in B, and let C be the centre of

A. Then $E \cong C^tG$ is a twisted group algebra of G over the field C, with $B \cong A \otimes_C E$, and moreover, $P' = (P' \cap E)B$: see [13, §2]. Let $\hat{C}$ denote the algebraic closure of C and choose a prime ideal P^* of $\hat{E} := \hat{C} \otimes_C E \cong \hat{C}^tG$ with $P^* \cap E = P' \cap E$. (Simply take P^* to be maximal among ideals I of $\hat{E}$ with $I \cap E = P' \cap E$.) Then

$$\hat{E}/P^* \cong M_v(\hat{C}) \qquad (1)$$

where

$$v|a, \qquad (2)$$

by Lemma 5.3 below.

$$S := B/P' \cong A \otimes_C (E/P' \cap E) \subseteq A \otimes_C (\hat{E}/P^*) \cong M_v(A \otimes_C \hat{C}).$$

It follows from the additivity principle [21, Lemma 1] that the composition length of S divides $\text{u-dim}(M_v(A \otimes_C \hat{C}))$; that is, that

$$\text{u-dim}(R\Gamma/P) \mid v.\text{u-dim}(A \otimes_C \hat{C}). \qquad (3)$$

Now $A \otimes_C \hat{C}$ is a simple ring, and the map from RN to $A \otimes_C \hat{C}$ yields a map from $\hat{C}N$ to $A \otimes_C \hat{C}$ whose kernel $\hat{Q}$ is a prime ideal of $\hat{C}N$ with $\hat{Q} \cap RN = Q$ and $\text{u-dim}(\hat{C}N/\hat{Q}) = \text{u-dim}(A \otimes_C \hat{C})$. With (2) and (3), this completes the proof of the proposition, except that we still have to establish

LEMMA 5.3 . *Let G be a finite group of order a, let K be an algebraically closed field of characteristic p, and let K^tG be a twisted group algebra of G over K. Assume that either*

(i) $p \nmid a$

or (ii) G is p-soluble.

Then for any simple K^tG-module V, $\dim_K(V) \mid a$.

Proof Case (i) is essentially covered by [4, Proposition 11.44], where we can replace the hypothesis that char K = 0 by condition (i), by using the generalized form of Ito's Theorem [7, Satz V.12.11] at the appropriate point in the proof.

For (ii), note that there is a finite central extension H of G such that V is a simple KH-module [4, Theorem 11.40(i)]. Since H is also p-soluble, the Fong-Swan-Rukolaine theorem [4, Theorem 22.1] ensures that V can be "lifted to characteristic zero". Hence, by Ito's theorem [4, Theorem 11.33], $\dim_K(V)$ divides $|H/Z(H)|$, (where $Z(H)$ denotes the centre of H). Thus, $\dim_K(V)$ divides $|G|$, as required.

Lemma 5.3, (and so also Proposition 5.2 (ii)) is false without the hypothesis (i) or (ii), even for ordinary group algebras. For example, if K is algebraically closed with $\operatorname{char} K = 7$, then $G = SL(2,7)$ has a 5-dimensional simple module over K, and $5 \nmid 336 = |G|$ (cf. [8, p. 41]). Also $\hat{C}N/\hat{Q}$ cannot in general be replaced by RN/Q in the situation of Proposition 5.2 (ii). An explicit counterexample is as follows. Take $G = Q_8 \times C_3$, the direct product of the quaternion group of order 8 and the cyclic group of order 3, and let $R = \mathbb{R}$ be the field of real numbers. Viewing Q_8 and C_3 as multiplicative subgroups of the quaternions $\underline{H}$ and the complex numbers $\mathbb{C}$ respectively, we obtain surjections $\phi : \mathbb{R}[Q_8] \to \underline{H}$ and $\psi : \mathbb{R}[G] \to \underline{H} \otimes_{\mathbb{R}} \mathbb{C} \cong M_2(\mathbb{C})$. Thus, with $Q = \operatorname{Ker} \phi$, $P = \operatorname{Ker} \psi$, and $N = Q_8$, the hypotheses of Proposition 5.2 (ii) are satisfied, yet $\text{u-dim}(\mathbb{R}[G]/P) = 2$ does not divide $[G:N].\text{u-dim}(\mathbb{R}[N]/Q) = 3$.

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