A NOTE ABOUT BIFURCATION POINTS

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ABSTRACT. Let $X \subset k^n$ be a smooth affine variety of dimension $n-r$ and let $f = (f_1, ..., f_m) : X \to \mathbb{C}^m$ be a polynomial dominant mapping. It is well-known that the mapping f is a locally trivial fibration outside a small closed set $B(f)$. It can be proved (using a general Fibration Theorem of Rabier) that the set $B(f)$ is contained in the set $K(f)$ of generalized critical values of f. In this note we give a direct, short proof of the last statement.

1. **Introduction.**

Let X be a smooth affine variety over $k = \mathbb{R}$ or $k = \mathbb{C}$ of dimension $n-r$ and let $f : X \to k^m$ be a polynomial dominant mapping. It is well-known, that the mapping f is a locally trivial fibration outside a bifurcation set $B(f)$, which has a measure 0.

Let us recall that in general the set $B(f)$ is bigger than $K_0(f)$ - the set of critical values of f. It contains also the set $B_\infty(f)$ of bifurcations points at infinity. Briefly speaking the set $B_\infty(f)$ consists of points at which f is not a locally trivial fibration at infinity (i.e., outside a compact set). To control the set $B_\infty(f)$ one can use the set of *asymptotic critical values at infinity of* f (see [5]):

$$K_\infty(f) = \{y \in k^m : \text{ there is a sequence } x_l \to \infty \text{ s.t. } f(x_l) \to y \text{ and } \|x_l\| \nu(res_{T_{x_l} X} df(x_l)) \to 0\},$$

where we consider on X the induced Euclidean metric and ν is the function defined by Rabier (see Definition 2.1 below). Let $K(f) = K_0(f) \cup K_\infty(f)$ be the set of generalized critical values of f.

In [2] (see also [1] and [3]) we have proved that the set $K(f)$ is a proper algebraic subset of $\mathbb{C}^m$ (or proper semi-algebraic in the real case) which can be computed effectively (we can also estimate the degree of an algebraic hypersurface which contains $K(f)$). Moreover it can be proved (e.g., by a general Fibration Theorem of Rabier [5]), that $B(f) \subset K(f)$. These two facts together allows us to construct effectively an Zariski open dense subset $U \subset k^m$ over which the mapping f is a locally trivial fibration.

In this note I prove directly that $B(f) \subset K(f)$ in the case when $X \subset k^n$ is a smooth submanifold and $f : X \to k^m$ is a smooth mapping. This fact follows from a very general Theorem of Rabier (see [5]), but it is so important (e.g., in the study of polynomial mappings), that (as I believe) it is worth to have a simple direct proof of it in a special case of a submanifolds of an Euclidean space.

2. **About Rabier function** ν

In this section we give several equivalent expressions for ν. A part of these results is also contained (with slightly different proofs) in [2].

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Let $X \cong k^n$, $Y \cong k^m$ be finite dimensional vector spaces (over k). Let us denote by $\mathcal{L}(X,Y)$ the set of linear mappings from X to Y and by $\Sigma(X,Y) \subset \mathcal{L}(X,Y)$ the set of non-surjective mappings. Let us recall the following ([5]):

Definition 2.1. Let $A \in \mathcal{L}(X,Y)$. Set

$$\nu(A) = \inf_{||\phi||=1} ||A^*(\phi)||,$$

where $A^* : \mathcal{L}(Y^*, X^*)$ is adjoint operator and $\phi \in Y^*$.

In [3] the following characterization of ν is given: $\nu(A) = dist(A, \Sigma) = \inf_{B\in\Sigma} ||A - B||$. Moreover, we have following useful characterization ([5] and [3]):

Proposition 2.1. *Let $A \in \mathcal{L}(X,Y)$. Then*

a) $\nu(A) = \sup\{r > 0 : B(0,r) \subset A(B(0,1))$, where $B(0,r) = \{x \in X : ||x|| \le r\}$.

b) if $A \in GL(X,Y)$ then $\nu(A) = ||A^{-1}||^{-1}$.

Proposition 2.2. *Let $A = (A_1, ..., A_m) \in \mathcal{L}(X,Y)$ and let $\overline{A_i} = grad\ A_i$. Let*

$$\kappa(A) = \min_{1\le \mathrm{i} \le \mathrm{m}} dist(\overline{A_i}, < (\overline{A_j})_{j\ne i} >),$$

be the Kuo number of A. Then $\nu(A) \le \kappa(A) \le \sqrt{m}\nu(A)$.

We say that $\nu(A)$ and $\kappa(A)$ are equivalent and write $\nu(A) \sim \kappa(A)$. The sign $X \sim Y$ means that there are positive constants C_1, C_2 such that $C_1 X \le Y \le C_2 X$.

Definition 2.2. Let $A \in \mathcal{L}(X,Y)$ and let $H \subset X$ be a linear subspace. We set

$$\nu(A,H) = \nu(res_H A),\ \kappa(A,H) = \kappa(res_H A),$$

where $res_H A$ denotes the restriction of A to H.

From Proposition 2.2 we get immediately:

Corollary 2.1. *Let $A \in \mathcal{L}(X,Y)$ and let $H \subset X$ be a linear subspace. Then*

$$\nu(A,H) \sim \kappa(A,H).$$

We have the following:

Proposition 2.3. *Let $A = (A_1, ..., A_m) \in \mathcal{L}(X,Y)$ and let $H \subset X$ be a linear subspace. Assume that H is given by a system of linear equations $B_j = 0, j = 1, ..., r$. Then*

$$\kappa(A,H) = \min_{1\le \mathrm{i} \le \mathrm{m}} dist(\overline{A_i}, < (\overline{A_j})_{j\ne i}; (\overline{B_j})_{j=1,...,r} >),$$

where $\overline{A_i} = grad\ A_i$ and $\overline{B_j} = grad\ B_j$.

Proof. Indeed, every vector $\overline{A_i}$ we can write as $a_i + b_i$, where a_i is orthogonal to the subspace $B =< \overline{B_j})_{j=1,...,r} >$ (what means that $a_i \in H$) and $b_i \in B$. Hence

$$dist(\overline{A_i}, < (\overline{A_j})_{j\ne i}; (\overline{B_j})_{j=1,...,r} >) = dist(a_i, < (a_j)_{j\ne i} >)$$

and since $grad(res_H A_i) = a_i$ the proof is finished. □

We need also:

Definition 2.3. Let $A \in \mathcal{L}(X,Y)$ (where $n \ge m + r$) and let $H \subset X$ be a linear subspace given by system of independent linear equation $B_i = \sum b_{ij} x_{ij},\ i = 1, ..., r$. Let $\mathbf{a} = [a_{ij}]$ be the matrix of A. Let $\mathbf{c} = [c_{kl}]$ be a $((m+r) \times n)$ matrix given by rows $A_1, ..., A_m; B_1, ..., B_r$ (we identify $A_i = \sum a_{ij} x_j$ with the vector $(a_{j1}, ..., a_{jn})$, similarly for B_j). Let M_I, where $I = (i_1, ..., i_{m+r})$, denote a $((m + r) \times (m + r))$ minor of $\mathbf{c}$ given by columns indexed by I. Let $M_J(j)$ denote a $((m + r - 1) \times (m + r - 1)$ minor given by columns indexed by J and by deleting the jth row , where $1 \le j \le m$. Then by the generalized Gaffney function of A with respect to a linear subspace H, we mean the number

$$g(A,H) = \frac{(\sum_I |M_I|^2)^{1/2}}{(\sum_{J,1\le j\le m} |M_J(j)|^2)^{1/2}}.$$

(If this number is not defined we put $g(A,H)=0$.)

Remark 2.1. It is easy to see that $g(A,H)$ depends on A and H only.

We have:

Proposition 2.4. *Let $A \in \mathcal{L}(X,Y)$ (where $n \ge m$) and let $H \subset X$ be a linear subspace. Then $g(A,H) \sim \kappa(A,H)$.*

Proof. By basic properties of the Gram determinant (see e.g., [4]) we have that

$$dist(\overline{A_i}, <(\overline{A_j})_{j\ne i}, \overline{B_j})_{j\in 1,...,r\}} >) = \frac{G(\{\overline{A_i}\}_{i\in\{1,...,m\}}, \{\overline{B_j}\})_{j\in\{1,...,r\}})^{1/2}}{G(\{\overline{A_j}\}_{j\ne i}, \{\overline{B_j}\}_{j\in\{1,...,r\}})^{1/2}} = \frac{(\sum_I |M_I|^2)^{1/2}}{(\sum_J |M_J(i)|^2)^{1/2}}.$$

Thus $g(A,H) \le \kappa(A,H)$. On the other hand there is a number i_0, such that the sum $\sum_J |M_J(i_0)|^2)^{1/2}$ is maximal. Since

$$(\sum_{J,j} |M_J(j)|^2)^{1/2} = (\sum_r (\sum_J |M_J(r)|^2))^{1/2} \le \sqrt{m}(\sum_J |M_J(i_0)|^2)^{1/2},$$

we have

$$g(A,H) \ge C\frac{(\sum_I |M_I|^2)^{1/2}}{(\sum_J |M_J(i_0)|^2)^{1/2}} = Cdist(\overline{A}_{i_0}, <(\overline{A}_j)_{j\ne i_0}, \{\overline{B_j}\}_{j\in\{1,...,r\}} >) \ge C\kappa(A,H),$$

where $C = 1/\sqrt{m}$. □

At the end of this section we introduce another important function (see [1]):

Definition 2.4. Let $A \in \mathcal{L}(k^n,k^m)$ (where $n \ge m$) and let $H \subset k^n$ be a linear subspace given by system of independent linear equation $B_i = \sum b_{ij}x_{ij},\ i=1,...,r$. Let $\mathbf{a}=[a_{ij}]$ be the matrix of A. Let $\mathbf{c}=[c_{kl}]$ be a $((m+r)\times n)$ matrix given by rows $A_1,...,A_m; B_1,...,B_r$. Let M_I, where $I=(i_1,...,i_{m+r})$, denote a $((m+r)\times(m+r))$ minor of $\mathbf{c}$ given by columns indexed by I. Let $M_J(j)$ denote a $((m+r-1)\times(m+r-1)$ minor given by columns indexed by J and by deleting the jth row A_j, where $1 \le j \le m$. Then we define the number

$$g'(A,H) = \max_I\{\min_{\{J\subset I,\ 1\le j\le m\}} \frac{|M_I|}{|M_J(j)|}\},$$

(where we consider only numbers with $M_J(j) \ne 0$, if all numbers $M_J(j)$ are zero, we put $g'(A,H)=0$).

We have:

Proposition 2.5. *We have $g'(A,H) \sim g(A,H)$.*

Proof. First we prove that there is a constant $C>0$ such that $g'(A,H) \le Cg(A,H)$. Let us fixed index $I=(i_1,...,i_{m+r})$ such that $|M_I| \ne 0$ and consider numbers $\frac{|M_I|}{|M_J(s)|}$, where $J \subset I$ and $1 \le s \le m$. For simplicity we can assume that $I=(1,...,m+r)$. Let the subspace H be given by system of independent linear equation $B_i = \sum b_{ij}x_{ij},\ i=1,...,r$, and let $\mathbf{a}=[a_{ij}]$ be the matrix of A.

Consider a system of linear equations:

$$\begin{array}{rcl}
\sum_{j=1}^n a_{1j}x_j & = & y_1, \\
\vdots & & \vdots \\
\sum_{j=1}^n a_{mj}x_j & = & y_m, \\
\sum_{j=1}^n b_{j1}x_j & = & 0, \\
\vdots & & \vdots \\
\sum_{j=1}^n b_{rj}x_j & = & 0, \\
x_{m+r+1} & = & 0, \\
\vdots & & \vdots \\
x_n & = & 0.
\end{array}$$

Using the Cramer rules we can solve this system. We have

$$\begin{array}{rcc}
x_1 & = & \sum_{k=1}^m (-1)^{1+k} y_k M_{1k}/M_I, \\
\vdots & & \vdots \\
x_n & = & \sum_{k=1}^m (-1)^{m+k} y_k M_{mk}/M_I, \\
x_{m+r+1} & = & 0, \\
\vdots & & \vdots \\
x_n & = & 0.
\end{array}$$

In particular we have $||x|| \le (\max\frac{|M_J(i)|}{|M_I|})||y||$. Consequently we see that the image of a unit ball in a subspace $H' = \{x \in H : x_{m+r+1} = 0, ..., x_n = 0\}$ by the mapping A contain a ball of radius $\min_{\{J\subset I,\ 1\le j\le m\}} \frac{|M_I|}{|M_J(j)|}$. Now by Proposition 2.1 a), we have that $\min_{\{J\subset I, 1\le j\le m\}} \frac{|M_I|}{|M_J(j)|} \le \nu(A, H') \le \nu(A, H)$. Finally we get

$$\nu(A, H) \ge \max_I\{\min_{\{J\subset I,\ 1\le j\le m\}} \frac{|M_I|}{|M_J(j)|}\} = g'(A, H).$$

In particular there is a constant C such that $Cg(A, H) \ge g'(A, H)$.

On the other hand, there exists I_0, such that the minor M_{I_0} has a maximal norm. Since

$$g(A, H) = \frac{(\sum_I |M_I|^2)^{1/2}}{(\sum_{J,j} |M_J(j)|^2)^{1/2}} \le \left(\begin{array}{c} n \\ m+r \end{array}\right)^{1/2} \frac{|M_{I_0}|}{(\sum_{J,j} |M_J(j)|^2)^{1/2}} \le$$

$$\le \left(\begin{array}{c} n \\ m+r \end{array}\right)^{1/2} \{\min_{\{J\subset I_0, 1\le j\le m\}} \frac{|M_{I_0}|}{|M_J(j)|}\} \le \left(\begin{array}{c} n \\ m+r \end{array}\right)^{1/2} g'(A, H),$$

we have that there is a constant $C' > 0$ such that $g(A, H) \le C'g'(A, H)$. □

Corollary 2.2. *We have $g'(A, H) \sim \nu(A, H)$.*

3. Main Result

In this section we give a short, direct proof of the fact $B(f) \subset K(f)$ for smooth mapping $f : X \to k^m$, where X is a smooth submanifold of k^m and f is a smooth mapping. Our approach is a modification of our proof from [1]. Let us recall the following basic definition:

Definition 3.1. Let $k = \mathbb{C}$ or $k = \mathbb{R}$ and let X be a smooth submanifold of k^n. Let $f : X \to k^m$ be a $k-$smooth mapping. Then we define the set of generalized critical values $K(f) = K_0(f) \cup K_\infty(f)$, where $K_0(f)$ is the set of critical values of f and

$$K_\infty(f) = \{y \in k^m : \textit{there is a sequence } x_l \to \infty \textit{ s.t. } f(x_l) \to y \textit{ and } \|x_l\|\nu(df(x_l), T_{x_l}X) \to 0\}$$

is the set of critical values at infinity

Remark 3.1. Note, that by virtue of results of section 2, in the place of the function ν above we can put also κ, g, or g'.

We have the following simple observation (see [1], [5]):

Proposition 3.1. *Let $k = \mathbb{C}$ or $k = \mathbb{R}$ and let X be a smooth affine variety over k. Let $f : X \to k^m$ be a $k-$smooth mapping. Then the set $K(f) = K_0(f) \cup K_\infty(f)$ is closed.*

We need also the following lemma, which is a slight modification of the lemma from [1]. For the sake of completeness we give a short proof.

Lemma 3.1. *Let $U \subset k^n$ be an open set and $V : U \to k^n$ be a smooth mapping. Let $y \in U$ and let*

$$x'(t) = V(x), \ \text{with } x(0) = y,$$

be a differential equation. Let $x(y,t),\ t \in [0, t_0)$ be a solution of this equation. Assume that for $||x(t)||$ large enough, we have $||V(x(y,t))|| < M||x(y,t)||$. Then this trajectory is bounded. In particular this trajectory either is defined for every $t > 0$ or it intersects the boundary ∂U of U.

Proof. By our assumption we have $||V(x(y,t))|| < M||x(y,t)||$ for sufficiently large $||x(y,t)||$, say for $||x(y,t)|| > R$. Let $r(t) = ||x(y,t)||^2$. We show that the function r is bounded. Assume by contradiction that there is a sequence of points $t_i \to t_0$ such that $r(t_i) \to \infty$ as $t_i \to t_0$.

Let N be any regular value of r with $N > R$. The preimage $r^{-1}(N)$ is at most countable set $W = \{w_1, w_2, ...\}$. For every $w_i \in W$ two cases are possible:

a) $r(t) \le N$ for every $t \in [w_i, w_{i+1})$,

b) $r(t) \ge N$ for every $t \in [w_i, w_{i+1})$,

(if w_{i+1} is not defined then we put $w_{i+1} = t_0$). Of course only the case b) is interesting. Hence assume that b) holds. By the assumption for a number $t \in [w_i, w_{i+1})$ we have $||V(x(t))|| < M||x(t)||$. Consequently for every such t, we have $r'(t) \le 2||x||||V(x)|| < 2M||x||^2 = 2Mr(t)$. It means that $ln(r(t))' \le 2M$ and finally $r(t) \le r(w_i)exp(2M(t - w_i)) \le Nexp(2M(t - w_1))$. This means that the set $\{r(t),\ w_1 \le t < t_0\}$ is bounded, a contradiction. □

Now we give a short, direct proof of the fact that $B(f) \subset K(f)$, which is a particular version of very general result of Rabier [5].

Theorem 3.1. *Let $k = \mathbb{C}$ or $k = \mathbb{R}$ and let $X \subset k^n$ be a smooth submanifold (i.e., X is smooth for $k = \mathbb{R}$ or Stein for $k = \mathbb{C}$). Let $f : X \to k^m$ be a $k-$smooth mapping (i.e., f is smooth for $k = \mathbb{R}$ or holomorphic for $k = \mathbb{C}$). Then*

$$B(f) \subset K(f) = K_0(f) \cup K_\infty(f),$$

i.e., the mapping f is a locally trivial fibration outside the set $K(f)$.

Proof. It is well-known that we can assume that f can be extended to a $k-$smooth mapping $\overline{f}$ on the whole of k^n (in real case it is an easy exercise, in complex it follows from the theory of Stein manifolds).

First assume that X is a global complete intersection, i.e. $X = \{b_1 = 0, ..., b_r = 0\}$ and rank $\{d_x b_1, ..., d_x b_r\} = r$ for every $x \in X$.

Let $a \notin K(f)$. Without loss of generality we can assume that $a = 0$. We have that $a \notin K_0(f)$ and $a \notin K_\infty(f)$. This implies that there are $R > 0, \epsilon > 0, \eta > 0$, such that for every $x \in X$ with $||x|| \geq R$ and $||f(x)|| < \eta$, we have

$$max_I\{min_{\{J\subset I\}}||x||\frac{|M_I|}{|M_J(j)|}\} > \epsilon. \tag{3.1}$$

Moreover, there is $\omega > 0$, such that for every $x \in X$ with $||x|| \leq R$ and $||f(x)|| < \eta$, we have $\max_I|M_I(x)| \geq \omega$.

Let $U = \{y \in k^m : ||y|| < \eta\}$ and let $\Gamma = f^{-1}(0)$. We show that $f^{-1}(U) \cong \Gamma \times U$ and f is a projection $\Gamma \times U \ni (\gamma, u) \to u \in U$. Indeed, let us define a set $U_I = \{x \in \overline{f}^{-1}(U) : if\ ||x|| \geq R\ then\ min_{\{J\subset I\}}||x||\frac{|M_I|}{|M_J(j)|} \geq \epsilon,\ if\ ||x|| \leq R\ then\ |M_I(x)| \geq \omega\}$. Further, let $V_I = \{x \in \overline{f}^{-1}(U) :\ if\ ||x|| \geq R\ then\ min_{\{J\subset I\}}||x||\frac{|M_I|}{|M_J(j)|} \leq \epsilon/2,\ if\ ||x|| \leq R\ then\ |M_I(x)| \leq \omega/2\}$. The sets V_I and U_I are disjoint. Consequently there is a C^∞ function $\delta_I : k^n \to [0,1]$, which is equal to 1 on U_I and to 0 on V_I. It is easy to see that sets $H_I = \{x : \delta_I > 0\}$ cover the set $f^{-1}(U)$. Now take $\delta := \sum_I \delta_I$ and let $\Delta_I = \delta_I/\delta$.

Take $y = (y_1, ..., y_n) \in U$. Take the index $I = (1, ..., m+r)$ and let consider a (formal) system of differential equations:

$$\begin{array}{rcl}
\sum_{j=1}^n \frac{\partial \overline{f}_1}{\partial x_j}(x(t))x_j(t)' & = & y_1, \\
\vdots & & \vdots \\
\sum_{j=1}^n \frac{\partial \overline{f}_m}{\partial x_j}(x(t))x_j(t)' & = & y_m, \\
\sum_{j=1}^n \frac{\partial b_1}{\partial x_j}(x(t))x_j(t)' & = & 0, \\
\vdots & & \vdots \\
\sum_{j=1}^n \frac{\partial b_r}{\partial x_j}(x(t))x_j(t)' & = & 0, \\
x_{m+r+1}(t)' & = & 0, \\
\vdots & & \vdots \\
x_n(t)' & = & 0.
\end{array}$$

Using the Cramer rules we can solve this system (at least in U_I). Let $M_J(i) := M_{ki}$ for $J = I \setminus \{k\}$. We have

$$\begin{array}{rcl}
x_1(t)' & = & \sum_{k=1}^m (-1)^{1+k} y_k M_{1k}/M_I, \\
\vdots & & \vdots \\
x_{m+r}(t)' & = & \sum_{k=1}^m (-1)^{m+k} y_k M_{mk}/M_I, \\
x_{m+r+1}(t)' & = & 0, \\
\vdots & & \vdots \\
x_n(t)' & = & 0.
\end{array}$$

We can write this system shortly as

$$x(t)' = V_I(y, x(t)).$$

By the Cramer rules, we have $df(V_I(y,x)) = y$ and $db(V_I(y,x)) = 0$. In an analogous way we can define V_I for an arbitrary index $I = (i_1, ..., i_m)$.

Now consider a vector field $V(y,x) = \sum_I \Delta_I V_I(y,x)$ in a domain $\overline{f}^{-1}(U)$. By the construction, we have $||V(x)|| \leq 2m\eta/\epsilon||x||$ for $||x|| \geq R$ and $x \in X$. Let us consider the differential equation

$$x(t)' = V(y, x(t)), \quad x(0) = \gamma, \tag{3.2}$$

where $\gamma \in \Gamma$. Let us note that

$$df(V(y,x)) = df(\sum_I \Delta_I V_I(y,x)) = \sum_I df(\Delta_I V_I(y,x)) = \sum_I \Delta_I df(V_I(y,x)) = (\sum_I \Delta_I)y = y.$$

Similarly $db(V(x,y) = 0$. Consequently, if $x(t,y,\gamma)$ is a solution of system 3.2, then the trajectory is contained in X and $yt = \overline{f}(x(t),y,\gamma) = f(x(t),y,\gamma)$. Since $y \in U$, we have that the trajectory $x(t,y,\gamma), t \in [0,t_0)$ does not cross the border $\partial f^{-1}(U)$ for every $0 \le t_0 \le 1+\delta$, for some $\delta > 0$. Consequently by Lemma 3.1 the trajectory $x(t,y,\gamma)$ is defined on the whole of $[0,1]$ and it is contained in X. Since $f(x(t,y,\gamma)) = yt$, we have that the phase flow $x(t,y,\gamma),\ t \in [0,1]$ transforms $f^{-1}(0) = \Gamma$ into $f^{-1}(y)$ (in fact, by the symmetry, it transforms Γ onto $f^{-1}(y)$). Let

$$\Phi : \Gamma \times U \ni (\gamma, y) \to x(1,y,\gamma) \in f^{-1}(U).$$

It is easy to see that Φ is a diffeomorphism. Thus $0 \notin B(f)$.

In the general case we can choose a locally finite cover $\{U_i\}$ of k^n, such that in each U_i the manifold $X \cap X_i$ is a complete intersection. Now we can construct vector fields V_i on U_i (construction is as above) and then glue them to one field V by a partition of unity subordinate to the cover $\{U_i\}$. The rest of the proof is the same as above. □

At the end of this note we give two simple examples.

Example 3.1. Let us consider a Stein curve $\Gamma = \{(x,y) \in \mathbb{C}^2 : \exp(xy) = 2\}$. Let us consider the projection $f : \Gamma \ni (x,y) \to y \in \mathbb{C}$. Using generalized Gaffney function we see that

$$K_0(f) = f(\{(x,y) \in \Gamma : y\exp(xy) = 0\}) = \emptyset$$

and

$$K_\infty(f) = \{\lim f(x_n,y_n) = y_n;\ where\ (|x_n| + |y_n|) \to \infty\ \ and\ \ |y_n| \to 0\} = \{0\}.$$

Hence finally $K(f) = \{0\}$ and indeed we can check directly that in this case $B(f) = K(f) = \{0\}$ (in fact f is a topological covering outside 0). Note that the mapping f has no usual critical values.

Example 3.2. Let us consider a smooth mapping

$$f : \mathbb{C}^3 \ni (x,y,z) \to (x\exp(z), y\exp(z)) \in \mathbb{C}^2.$$

Using the function g' we can easily computed that $K(f) = \{0\}$. But the function f is a global fibration of $\mathbb{C}^3$ - in fact it gives a fibration

$$\mathbb{C}^2 \times \mathbb{C} \ni ((x,y),z) \to ((x\exp(-z), y\exp(-z), z) \in \mathbb{C}^3.$$

Thus in general $B(f) \ne K(f)$.

Remark 3.2. In the case of Stein submanifolds in general it may happen that $K(f) = \mathbb{C}^m$. The latter equality is impossible for algebraic submanifolds and polynomial mappings - see [2], [3].

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