Boundaries of singularity sets, removable singularities, and CR-invariant subsets of CR-manifolds

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BOUNDARIES OF SINGULARITY SETS, REMOVABLE SINGULARITIES, AND CR-INVARIANT SUBSETS OF CR-MANIFOLDS

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ABSTRACT. Let H be a smooth hypersurface in $\mathbb{C}^n$, $n \geq 3$, and let M be a generic submanifold of H of real codimension one. We describe classes of compact removable singularities K for L^p-solutions of the tangential Cauchy-Riemann equations on H under the conditions $K \subset M$, $1 \leq p \leq \infty$. The classical theory gives results only in the case $p > 1$. But even for $p > 1$ removable singularities for L^p-solutions of the tangential Cauchy-Riemann operators may be metrically much more massive than the classical theory predicts.

There is a relation of this problem with the problem of describing envelopes of holomorphy of suitable neighbourhoods of $H \setminus K$. The most complete results for the last problem are obtained in the case when H is the boundary $\partial\Omega$ of a strictly pseudoconvex domain Ω in $\mathbb{C}^n$, $n \geq 3$. As before K is assumed to be contained in a generic submanifold M of $\partial\Omega$. The results use the decomposition of the manifold M into CR-orbits. One section is devoted to the geometry of the decomposition of a CR-manifold into CR-orbits. The minimal obstructions for the germ of envelopes of holomorphy of suitable neighbourhoods of $\partial\Omega \setminus K$ to be equal to $\overline{\Omega} \setminus K$ are of two kinds. Either K is the boundary of a complex variety of codimension one in Ω or it is an exceptional minimal CR-invariant subset of M. The latter case may occur as is shown by examples. Further examples show that the mentioned germ of envelopes of holomorphy may be multisheeted. A couple of open problems is discussed.

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0. Introduction and background

During the last few years the problem of describing removable singularities of analytic functions of several variables or of their boundary values became very popular. We mention here especially the paper [St] which gives a survey of most of the relevant articles up to 1988 and contains a large number of related references. References of more recent papers are, for example, [An–Ci], [Či–St], [Duv], [Fo–St], [Jö2], [Jö3], [Law], [Lu]. The popularity of this subject, maybe, is based on the fact that there are various connections of this subject with other ones and a number of "variations on the theme " itself.

We will not give here a survey of known results but we will recall some aspects of the theory of removable singularities and related subjects. One problem is the following: *Let $\Omega \subset \mathbb{C}^n, n \geq 2$, be a bounded domain of holomorphy. Characterize singularities of analytic functions, i.e. those closed subsets A of the closure $\overline{\Omega}$ of Ω, for which the set $\Omega \backslash A$ is a domain of holomorphy or the union of domains of holomorphy.*

On the opposite side, we ask *which closed sets $A \subset \overline{\Omega}$ are removable.* This means, *each function which is analytic in $\Omega \backslash A$ is the restriction to $\Omega \backslash A$ of an analytic function in Ω.* In particular, a removable set does not contain any singularity set.

The most general problem, which includes both problems above, is to *describe the envelope of holomorphy of the set $\Omega \backslash A$ for domains of holomorphy Ω and closed subsets A of the closure $\overline{\Omega}$.* (In case $\Omega \backslash A$ is not connected the disjoint union of the envelopes of holomorphy of the connected components of $\Omega \backslash A$ is meant.) It turns out that although Ω is a domain of holomorphy the envelope of holomorphy of $\Omega \backslash A$ may not be a subset of $\mathbb{C}^n$, i.e. it may be multisheeted.

A central question is to describe *the possible intersection $K = \partial\Omega \cap A$ of singularity sets or of removable sets A with the boundary $\partial\Omega$.* In other words, we are essentially interested in envelopes of holomorphy of certain one-sided neighbourhoods of $\partial\Omega \backslash K$. We take here the following definition of one-sided neighbourhoods. *Let H be a hypersurface of class C^1 in* $\mathbb{C}^n$ (i.e. in a neighbourhood of each of its points H is the level set of a real function of class C^1 with non-vanishing gradient, or equivalently, (see below) H is a proper sub-manifold of $\mathbb{C}^n$ of class C^1 and of real codimension one.) *Let p be a point of H. A one-sided neighbourhood of p (with respect to H) is one of the connected components of $U \backslash H$ for any neighbourhood U of p in $\mathbb{C}^n$ which is divided by H into two connected components. A one-sided neighbourhood of H is a set which contains a one-sided neighbourhood of each point of H (with respect to H).*

The envelope of holomorphy depends in general on the choice of the one-sided neighbourhood of $\partial\Omega \backslash K$. At least for bounded strictly pseudoconvex domains $\Omega \subset \mathbb{C}^n$ and one-sided neighbourhoods of $\partial\Omega \backslash K$ $(K \neq \partial\Omega)$ which are contained in Ω the corresponding envelopes of holomorphy stabilize (i.e. they will not decrease for one-sided neighbourhoods contained in a *suitable* fixed one-sided neighbourhood of $\partial\Omega\backslash K$.) We will speak on the germ of envelopes of holomorphy of one-sided neighbourhoods of $\partial\Omega\backslash K$.

For avoiding the inconvenience of the described kind the problem is sometimes slightly modified: For example, one considers the question of analytic extendability of analytic functions on $\partial\Omega\backslash K$ (i.e. of functions which are analytic in certain neighbourhood of $\partial\Omega\backslash K$ depending on the function) or the question of analytic extendability of continuous CR-functions on $\partial\Omega\backslash K$. (As usual, a CR-function (CR-distribution, respectively) is the weak solution of the tangential Cauchy-Riemann equations.) All these questions are closely related one to the other and sometimes one of them is reduced to another one by varying the domain (letting fixed the boundary part K). The last setting of the problem, namely, the question of analytic extendability of continuous CR-functions on $\partial\Omega\backslash K$ seems to be nice in each sense, nevertheless it has also some inconveniences: it consists of two problems, the problem of analytic extension of continuous CR-functions to a one-sided neighbourhood of $\partial\Omega\backslash K$ and the problem of describing the envelope of holomorphy for a special class of analytic functions (those with continuous (but not necessarily bounded) boundary values on $\partial\Omega\backslash K$) see [Jö3] for the discussion of the problem for domains in $\mathbb{C}^2$.

We prefer to take here the setting of envelopes of holomorphy of one-sided neighbourhoods of $\partial\Omega\backslash K$. In particular, *for a bounded domain of holomorphy $\Omega \subset \mathbb{C}^n$ a compact subset K of $\partial\Omega$ is called removable if $\partial\Omega\backslash K$ is connected and the envelope of holomorphy of any domain $\Omega' = \Omega\backslash A$ with $A = \overline{A} \subset \overline{\Omega}$ and $A \cap \partial\Omega = K$ is equal to Ω.*

Thinking on these problems as on a generalization of Hartogs' classical theorem a more precise term could be "Hartogs negleagible" or "boundary trace of a removable set", but the shorter term "removable" is established now.

If one considers instead of all analytic functions those with boundary values on $\partial\Omega\backslash K$ being distributions of finite order or being bounded functions or L^p-functions we come after the generalization to not necessarily pseudoconvex domains and an equivalent reformulation to the following settings.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with boundary of class C^2 and let $1 \leq p \leq \infty$. A compact subset K of $\partial\Omega$ is called $(L^p, \overline{\partial}_b)$-removable if each function $f \in L^p(\partial\Omega)$, which satisfies the tangential Cauchy–Riemann equations $\overline{\partial}_b f = 0$ (in the distributional sense) on $\partial\Omega\backslash K$ satisfies the equation $\overline{\partial}_b f = 0$ on the whole boundary $\partial\Omega$. If $\partial\Omega$ is of class C^∞ and if for any distribution f on $\partial\Omega$ for which $\overline{\partial}_b f = 0$ on $\partial\Omega\backslash K$ there exists a distribution g

on $\partial\Omega$ with support on K such that $\overline{\partial}_b(f-g)=0$ on the whole boundary $\partial\Omega$ then K is called $(\mathcal{E}',\overline{\partial}_b)$-removable.

In this reformulation we are concerned with removable singularities of (bounded or distributional) solutions of partial differential equations in the classical sense (see, for example, [Ha–Po].)

It is reasonable to speak on removable singularities in case of not necessarily closed hypersurfaces in $\mathbb{C}^n$. *Let H be an orientable connected hypersurface in $\mathbb{C}^n$, H not necessarily closed.* Fix a side of H. (Call it the positive side of H.) If a function f is analytic in a one-sided neighbourhood of H which is situated on the positive side of H we will say that f is analytic on the positive side of H.

Let now K be a relatively closed subset of H. K is called removable (with respect to H and to the positive side of H) if the following holds. Let f be an arbitrary function which is analytic on the positive side of $H\backslash K$. Then there exists a connected one-sided neighbourhood $\mathcal{O}_f$ of H ($\mathcal{O}_f$ must not necessarily be situated on the positive side of H) which contains the germ of one-sided neighbourhoods of $H\backslash K$ situated on the positive side of $H\backslash K$ and an analytic function on $\mathcal{O}_f$ which coincides with f on the mentioned germ of one-sided neighbourhoods of $H\backslash K$. For short, *K is removable, if each function which is analytic on the positive side of $H\backslash K$ has analytic extension to a one-sided neighbourhood of H.*

$(L^p,\overline{\partial}_b)$-removability and $(\mathcal{E}',\overline{\partial}_b)$-removability with respect to H can be defined in an obvious way.

The most complete information on the problem we have for the smallest possible dimension $n=2$. We have a good understanding of removable sets and of the connection of removability with other problems in case of domains of holomorphy Ω contained in $\mathbb{C}^2$. (See, among others, [St], [Jö3].) For example, for a compact set K in the boundary $\partial\mathbb{B}^2$ of the unit ball $\mathbb{B}^2$ in $\mathbb{C}^2$ the germ of the envelopes of holomorphy of one-sided neighbourhoods of $\partial\mathbb{B}^2\backslash K$ which are contained in $\mathbb{B}^2$ is always contained in $\mathbb{C}^2$ and is equal to $\mathbb{B}^2\backslash\hat{K}$. $\hat{K}$ denotes the polynomially convex hull of K. So, a compact set $K\subset\partial\mathbb{B}^2$ is removable iff it is polynomially convex. The point of view of describing removable sets gave new progress in getting sufficient geometric conditions for polynomial convexity ([Duv], [Fo–St],[Jö1]). Moreover, the detailed study of singularity sets for special domains in $\mathbb{C}^2$ led to a new proof of the well-known Corona theorem for the unit disc in the plane ([Be–Ra], [Sło]).

In this paper we will consider the question for domains in $\mathbb{C}^n$, $n\geq 3$. There exists a complete characterization of removable sets in boundaries of strictly pseudoconvex domains [Lu]. It is based on generalizing the following observation made for the dimension $n=2$. The original motivation for considering polynomially convex sets comes from approximation theory. Under some condition on a compact set $K\subset\mathbb{C}^2$, excluding for example closed hypersurfaces in $\mathbb{C}^2$, polynomial convexity is equivalent to the possibility of approximating analytic functions near K by polynomials. More exactly *K is polynomially convex iff* *the* $\overline{\partial}$*-cohomology group* $H^{0,1}_{\overline{\partial}}(K)$ *is trivial and holomorphic functions near* K *can be approximated by polynomials uniformly on* K. The condition $H^{0,1}_{\overline{\partial}}(K) = 0$ guarantees that the inclusion map from K into the spectrum $sp(\mathcal{O}(K))$ of the algebra $\mathcal{O}(K)$ is bijective. $\mathcal{O}(K)$ is the space of functions holomorphic near K.

The theorem for $n > 2$ is a natural generalization of this statement: *the condition* $H^{0,1}_{\overline{\partial}}(K) = 0$ *(or equivalently* $H^{2,1}_{\overline{\partial}}(K) = 0$*) has to be replaced by the condition* $H^{n,n-1}_{\overline{\partial}}(K) = 0$ *and instead of approximating holomorphic functions by polynomials it is required that smooth* $\overline{\partial}$*-closed* $(n, n-2)$*-forms near* K *can be approximated uniformly together with all derivatives by smooth* $\overline{\partial}$*-closed* $(n, n-2)$*-forms defined on the whole* $\mathbb{C}^n$. *Another equivalent condition for removability is* $H^{0,1}_{\overline{\partial}}(\mathbb{C}^n \setminus K) = 0$*; this condition being in terms of the complement of* K *rather than in terms of* K *itself.*

In the same way as it is difficult to give geometric conditions for polynomial convexity it seems to be difficult to understand the geometric meaning of this condition. The more general problem of the description of envelopes of holomorphy of one-sided neighbourhoods of $\partial\Omega \setminus K$ (Ω a domain in $\mathbb{C}^n$, $n \geq 3$) is even more difficult. For example, let $\mathbb{B}^3$ be the unit ball in $\mathbb{C}^3$ and let K be a compact subset of the boundary. The germ of the envelopes of holomorphy of one-sided neighbourhoods of $\partial\mathbb{B}^3 \setminus K$ which are contained in $\mathbb{B}^3$ is not necessarily a subset of $\mathbb{C}^n$. It may be multisheeted. The reason for this difference is the fact that the problem is really a problem concerning the operator $\overline{\partial}_b$ rather than the operator $\overline{\partial}$. In $\mathbb{C}^2$ we have to deal with a single operator while in $\mathbb{C}^n$, $n > 2$, we have an overdetermined system of differential operators.

We wish to mention here a simple heuristic principle which gives some understanding of the problem of the removal of compact sets in the boundary of strictly pseudoconvex domains, namely, the analogy between Oka's characterization principle for polynomially convex hulls in $\mathbb{C}^2$ on the one hand and analytic condition along one-parameter families of analytic varieties via the Kontinuitätssatz on the other hand. Recall that by Oka's characterization principle [Sto1] for polynomially convex hulls a point $z \in \mathbb{C}^n \setminus K$ is not in the hull $\hat{K}$ iff there exists a continuous one-parameter family of algebraic hypersurfaces $\mathcal{H}_t$ in $\mathbb{C}^n$ not intersecting K with $z \in \mathcal{H}_1$ and $dist\,(0, \mathcal{H}_t) \to \infty$ for $t \to \infty$. For $n = 2$ we get a family of one-dimensional analytic varieties and this is what is needed for analytic continuation via the Kontinuitätssatz. The only additional thing consits in monodromy considerations to prove that the envelope of holomorphy is contained in $\mathbb{C}^2$. See [Jö3] where the heuristic principle is performed into a rigorous proof. Monodromy considerations are based on the fact that we have to deal with analytic sets of complex codimension one. This heuristic principle also suggests results for $n > 2$. For example, the germ of envelopes of holomorphy of one-sided neighbourhoods of $\partial\mathbb{B}^n \setminus K$ $(n > 2)$ contains $\mathbb{B}^n \setminus \hat{K}$. Moreover, for a suitable small one-sided neighbourhood $\mathcal{O} \subset \mathbb{B}^n$ of $\partial\mathbb{B}^n \setminus K$ each analytic function in $\mathcal{O}$ is the restriction to $\mathcal{O}$ of a well defined analytic function in $\mathbb{B}^n \setminus \hat{K}$. The one-one correspondence between connected components of $\partial\mathbb{B}^n \setminus K$ and connected components of $\mathbb{B}^n \setminus \hat{K}$ is the same as for

$n = 2$. But we have no chance to obtain the whole germ of envelopes of holomorphy by analytic continuation along analytic varieties of complex dimension $n - 1$. It is enough to use varieties of dimension one. So, in general, the envelope germ is expected to be much larger than $\mathbb{B}^n \setminus \hat{K}$. In fact, in general $\mathbb{B}^n \setminus \hat{K}$ is not pseudoconvex, if $n > 2$. Moreover, the envelope germ may be multisheeted. We will explain here an example with K situated on a generic manifold of codimension one in $\partial\mathbb{B}^3$.

Note that the connection between *hulls* characterized by a generalization of *Oka's principle* using *analytic varieties of higher complex codimension* on the one hand and between the *approximation property* like that in Lupacciolu's theorem for *forms of suitable degree* on the other hand is *not clear* at the moment.

The aim of this paper is to *characterize large classes of removable sets in geometric terms.* We will give here detailed proofs of the results announced in [Jö2] and present further results. We will neither try to give a maximum of results nor we will try to give the most general formulation which can be obtained. We choose the statements by trying to keep some balance in geometric clearness and generality and develope some general methods for proving removablility results. Note, that we do not know how to derive the results of the present paper from Lupacciolu's results.

The main results in this paper concern the following situation: *Ω is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$, $n > 3$, with boundary $\partial\Omega$ of class C^2, or H is a hypersurface of class C^2 in $\mathbb{C}^n$, $n > 3$, which is strictly pseudoconvex from one side.* As in [Jö1] (where the case $n = 2$ is considered) the basic assumption is that *K is contained in a manifold M of class C^2 contained in $\partial\Omega$ (or in H, respectively).*
Recall that the most complete results on removable sets which are known up to now, concern the case of sets K contained in manifolds (sometimes in manifolds of real codimension one). This concerns the case of removable singularities in strictly pseudoconvex boundaries in $\mathbb{C}^2$ ([Jö1],[Fo–St],[Duv]) as well as the well-known classical Painlevé problem on removable singularities of bounded analytic functions [Pai] of one complex variable: A compact set contained in a rectifiable curve is removable iff it has zero one-dimensional Hausdorff measure (see, for example, [Ma]). For general sets in the plane a complete geometric characterization is not known.
By results in [Lu–St] the smallest dimension of a C^2 manifold $M \subset \partial\Omega$ which contains non-removable sets is $2n - 3$. So we have to consider only manifolds of real dimension $2n - 3$ and $2n - 2$.

For manifolds of real dimension $2n - 3$ we give a complete characterization of removable compact sets. This result was independently found by E. Čirka [Či–St]. (In [Či–St] the smoothness conditions for M are weakened.)

For manifolds M of real dimension $2n - 2$ we give a sufficient geometric criterion for a compact set $K \subset M$ to be removable. Probably it is also necessary but we are not able to prove this at the moment.

Note, that the assumption of strict pseudoconvexity of $\partial\Omega$ (or of H) can be considerably weakened and still the mentioned conditions are sufficient for removability. But if Ω is far from being strictly pseudoconvex these conditions may be far from being necessary. So, in case Ω is not pseudoconvex or $\partial\Omega$ contains enough analytic manifolds there may occur new reasons for a set to be removable (i.e. such ones which are not covered by the previous formulations).

We will state *examples* of results without the assumption of strict pseudoconvexity. It is left to the reader to add further statements or to generalize the present results. It is also not hard to see that the methods can be applied to more general situations than hypersurfaces in $\mathbb{C}^n$.

1. Terminology and statement of the results

For the formulation of our main results we need several times the notion of manifolds and submanifolds. Since the meaning of these terms in different papers is not always identical we give here the definition we will work with throughout the paper.

A *manifold* of real dimension r is a Hausdorff space each point of which has a neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^r$. A Hausdorff space each point of which has a neighbourhood which is homeomorphic to either an open subset of $\mathbb{R}^r$ or to a set of the form $\{x = (x_1, \dots, x_m) \in U, x_1 \geq 0\}$ where U is an open subset of $\mathbb{R}^r$ is called a *manifold with boundary*. A *compact manifold* (or, equivalently, a *closed manifold*) is a *manifold* which is a compact topological space. A *compact manifold with boundary* is a *manifold with boundary* which is a compact topological space. A manifold (or a manifold with boundary) is said to be of class C^k, $k = 1, \dots, \infty$ (or, real analytic, respectively) if an atlas of class C^k (a real analytic atlas, resp.) is given, i.e. there is a covering of the manifold with relatively open sets U_α, homeomorphisms φ_α of U_α onto open subsets $\tilde{U}_\alpha$ of $\mathbb{R}^r$ (or onto sets of the form $\tilde{U}_\alpha \cap \{x_1 \geq 0\}$ with $\tilde{U}_\alpha$ open in $\mathbb{R}^r$) such that for any pair, say U_1 and U_2, the mapping $\varphi_2\varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is of class C^k (real analytic, respectively). On manifolds of even real dimension (not on manifolds with boundary) one may introduce a complex analytic structure in an analoguous way and call them complex analytic manifolds. All manifolds we will consider here are assumed to be paracompact.

Let M be a manifold of class C^k. A subset $\Gamma \subset M$ equipped with the structure of a manifold of class C^k is called a submanifold of M of class C^k $(k \geq 1)$ if the inclusion map from Γ into M is a C^k map whose differential is everywhere injective. In particular, the inclusion map is continuous. But in this terminology a submanifold of M is not required to be a topological subspace of M, in other words, the manifold topology on a C^k submanifold Γ does not necessarily coincide with the topology on Γ induced from M (it may be more fine). For example, a submanifold Γ of M is allowed to be dense in M. If the manifold topology and the induced topology coincide on a C^k submanifold Γ of M we will call Γ

a proper submanifold of M. Note that in [Go–Gui] submanifolds in our sense are called immersed submanifolds and proper submanifolds in our sense are called there submanifolds. Our terminology is close to that in [Su].

Let Γ be a C^k submanifold of M. Note, that for each point $p \in \Gamma$ there exists a neighbourhood U_p of p in the manifold topology of Γ such that on U_p the two topologies coincide. Therefore the topologies coincide on relatively compact subsets of Γ (in the manifold topology).

For other classes than C^k submanifolds are understood in an analoguous way. We will not explane this here explicitely.

Let now Γ be a connected submanifold of $\mathbb{R}^m$ of class C^1. Then Γ carries a natural metric d_Γ. It is defined in the following way. For two points z_1 and z_2 in Γ the distance $d_\Gamma(z_1, z_2)$ is equal to the infimum of the Euclidean length of all C^1 curves γ which are contained in Γ and join the points z_1 and z_2. (The Euclidean length of the curve γ is the length of γ as a curve in $\mathbb{R}^m$ with the usual Euclidean metric.) It is clear that convergence in the metric d_Γ implies convergence in $\mathbb{R}^m$ but the converse is not true: the Euclidean distance between two points in Γ may be small while the distance in the metric d_Γ is large.

As usual we will say that Γ is complete in the metric d_Γ if every Cauchy sequence in (Γ, d_Γ) has a limit in Γ in the topology defined by d_Γ. Sometimes for a submanifold Γ of $\mathbb{R}^m$ we will speak on metrical completeness (omitting the phrase "in the metric d_Γ") having always in mind completeness in this sense. For example, the infinite spiral

$$\Gamma = \{re^{2\pi i\varphi(r)} : 1 < r < 2\} \subset \mathbb{C}^1,$$

φ being a strictly increasing C^∞ function defined on (1,2) with $\varphi(r) \to -\infty$ for $r \to 1+$ and $\varphi(r) \to \infty$ for $r \to 2-$, is complete in the metric d_Γ (but, of course, not in $\mathbb{C}^1$). the "half-infinite" spiral

$$\Gamma_+ = \{re^{2\pi i\varphi(r)} : \frac{3}{2} < r < 2\} \qquad (\varphi \text{ as above})$$

is not complete in d_{Γ_+}.

In this paper we need also the notion of CR-manifolds. We will consider here only CR-manifolds immersed into $\mathbb{C}^n$. So we have always in mind the following definition.

Let M be a submanifold of $\mathbb{C}^n$ of class C^k. For $p \in M$ the tangent space T_pM can be identified with a real linear subspace of $T_p\mathbb{C}^n$. Usually we identify $T_p\mathbb{C}^n$ with $\mathbb{C}^n$. Denote by J the operator of multiplication with the imaginary unit in $\mathbb{C}^n$ (considered as a real linear space). *If the dimension of the real linear subspace $T_p^JM = T_pM \cap JT_pM$ does not* *depend on* $p \in M$ *and is positive, then* M *is called a CR-manifold of class* C^k *immersed into* $\mathbb{C}^n$.

The space $T_p^J M$ can be considered as a complex linear subspace of $\mathbb{C}^n$. To avoid confusion we will write $T_p^c M$ for $T_p^J M$ considered as a complex linear subspace of $\mathbb{C}^n$. The complex dimension of $T_p^c M$ is called the CR-dimension of M and is denoted by $\dim_{CR} M$.

Let M be a CR-manifold. A submanifold Γ of M of class C^k is called a CR-submanifold of M (of class C^k) if for each point $p \in \Gamma$ the inclusion

$$T_p\Gamma \supset T_p^J M \tag{1.1}$$

holds. (We usually identify the tangent space $T_p\Gamma$ with a subspace of T_pM .) The inclusion (1.1) implies that Γ is a CR-manifold (immersed into $\mathbb{C}^n$) of the same CR-dimension as M .

A CR-manifold M immersed into $\mathbb{C}^n$ will be called generic, if the CR-dimension is the minimal possible one, or in other words, if

$$T_pM + JT_pM = T_p\mathbb{C}^n \quad \text{(in the sense of real linear spaces)} \tag{1.2}$$

for all $p \in M$. A CR-manifold M immersed into $\mathbb{C}^n$ of odd dimension $2l+1$ will be called maximally complex if, conversely, the CR-dimension of M is the maximal possible one, namely l .

In the paper as usual, we will consider a CR-manifold M which is *imbedded* into $\mathbb{C}^n$. This means, that M is a proper submanifold of $\mathbb{C}^n$ with constant dimension of the complex tangent space. Its real dimension will be denoted by $\dim_r M$ and the number $\dim_r M - 2\dim_{CR} M$ will be denoted by $e(M)$.

Now we can formulate our main results. Let Ω be a strongly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 3$. Note first, that the most elementary singularity set which comes into mind is the zero set of an analytic function in Ω which is continuous up to the boundary. In well-behaved cases the intersection of such a set with the boundary $\partial\Omega$ is a connected closed maximally complex CR-manifold of real dimension $2n-3$ contained in $\partial\Omega$. This is, roughly speaking, the only non-removable compact set contained in a connected proper submanifold of $\partial\Omega$ of real dimension $2n-3$.

Theorem 1. *Let* $\Omega \subset \mathbb{C}^n$, $n \geq 3$ *, be a bounded strongly pseudoconvex domain with boundary* $\partial\Omega$ *of class* C^2 *, and suppose* M *is a connected proper submanifold of* $\partial\Omega$ *of class* C^2 *and of real dimension* $2n-3$ *. A compact subset* K *of* M *is not removable iff* $K = M$ *(hence* M *is closed) and* M *is a maximally complex CR-manifold.*

So, if either $M \backslash K$ is not empty or M is not maximally complex then K is removable.

Remark 1. By the theorem of Harvey and Lawson [Ha–La] a connected closed maximally complex CR-manifold of class C^2 and dimension $2n-3$ contained in the boundary $\partial\Omega$

of a strongly pseudoconvex domain $\Omega \subset \mathbb{C}^n$, $n \geq 3$, bounds a complex variety V in Ω of complex dimension $n-1$. (More detailed see section 4.) This fact shows that theorem 1 is the complete analogue of the results for $\Omega \subset \mathbb{C}^2$. Indeed, a compact set K contained in a connected proper submanifold M of $\partial\mathbb{B}^2$ of real dimension one is not removable iff K is not polynomially convex and by a theorem of Stolzenberg [Sto2] this happens iff K bounds a complex variety $V \subset \mathbb{B}^2$ of complex dimension one (hence $K = M$ and M is closed). In dimension $n > 2$ it is even easier to check if a manifold of real dimension $2n-3$ bounds an analytic variety of complex dimension $n-1$ or not. In dimension $n > 2$ this depends only on geometric properties (i.e. on the topology and the CR-geometry) of the manifold.

Now we will formulate the results for the case of manifolds $M \subset \partial\Omega$ of dimension $2n-2$. By [Jöl] compact subsets of sufficiently small diameter of *generic CR-manifolds* M contained in $\partial\Omega$, M of class C^2 and of real dimension $2n-2$, are removable. If a submanifold of $\partial\Omega$ of the same dimension is not generic the conclusion may not be true. So, let $M \subset \partial\Omega$ be a generic CR-manifold of dimension $2n-2$ imbedded into $\mathbb{C}^n$, and let K be a compact subset of M. Theorem 1 describes an obstruction for K to be removable: if K contains a closed maximally complex CR-manifold of dimension $2n-3$ then K is not removable. The most important property of a closed maximally complex CR-submanifold of M of dimension $2n-3$ in this respect is: it is a closed CR-invariant subset of M. *Here a subset S of M is called CR-invariant, if for each $p \in S$ and each C^2 curve $\gamma : [0,1] \to M$ with $\gamma(0) = p$, $\gamma'(t) \neq 0$ and $\gamma'(t) \in T^J_{\gamma(t)}M \setminus \{0\}$, the point $\gamma(1) = q$ belongs to S.* The smallest CR-invariant subset of M containing a given point $p \in M$ is called the CR-orbit through p. By [Su] the CR-orbits are (possibly non proper) CR-submanifolds of M of class C^1. CR-invariant subsets are interesting in connection with the propagation of properties of CR-functions and CR-distributions.

Theorem 2. *Let Ω be as in theorem 1 and let M be a proper submanifold of $\partial\Omega$. Suppose M is a generic CR-manifold of class C^2 and of real dimension $2n-2$. If a compact set $K \subset M$ does not contain a non-empty compact CR-invariant subset of M (and therefore $K \neq M$) then K is removable.*

A compact CR-invariant subset S of M is called minimal, if there is no non-empty compact CR-invariant set S_1 with $S_1 \subset S$, $S_1 \neq S$. Clearly, the intersection of compact CR-invariant subsets is again a compact CR-invariant subset of M, so each compact CR-invariant subset of M contains a minimal compact CR-invariant subset of M. We are interested in a more detailed description of (non-empty) *minimal* compact CR-invariant subsets of manifolds M described in theorem 2. A CR-invariant subset of M is the union of CR-submanifolds of M. Those may be either of real dimension $2n-2$ or $2n-3$ (in view of the strong pseudoconvexity of $\partial\Omega$ no complex manifold is contained in M, so the dimension $2n-4$ is impossible). In case M is foliated into CR-submanifolds of real dimension $2n-3$, the description of minimal compact CR-invariant subsets of M can be given by methods of foliation theory of codimension one (see, for example, [He–Hi]). This are the minimal compact saturated sets. In the general case the approach of Sussmann [Su] combined with methods of foliation theory gives the following theorem. We may even remove the condition that M is contained in a strictly pseudoconvex boundary.

Theorem 3. *Suppose M is a non-compact connected generic CR-manifold of class C^2 and of real dimension $2n-2$ imbedded into $\mathbb{C}^n$. A minimal* **compact** *CR-invariant subset S of M has exactly one of the following two forms:*

1. *S is a proper compact maximally complex CR-submanifold of M of real dimension $2n-3$;*
2. *S is the union of metrically complete maximally complex exceptional CR-submanifolds S_α of M. Each S_α is dense in S.*

Here the metrical completeness of S_α is understood with respect to the metric d_{S_α}. *A metrically complete CR-submanifold of M of real codimension one is called exceptional if it is neither proper nor locally dense in M. A set S of the second kind is called an exceptional minimal compact CR-invariant subset of M.* These objects are the analogues of the exceptional leaves, or, the exceptional minimal sets, respectively, in foliations of codimension one. Note that an exceptional minimal compact CR-invariant set may occur in generic CR-manifolds of dimension $2n-2$ contained in strictly pseudoconvex boundaries and such a set may be the boundary of a minimal singularity set. Indeed, in section 5 we will prove the following

Theorem 4. *There exists a bounded strictly pseudoconvex domain Ω in $\mathbb{C}^3$ with boundary $\partial\Omega$ of class C^∞ and a proper connected compact submanifold M of $\partial\Omega$ of class C^∞ which is a generic CR-manifold of real dimension 4 with the following properties:*

1. *M admits a smooth foliation of codimension one with leaves being CR-submanifolds of M.*
2. *There is an exceptional minimal set S of the foliation but no compact leaf.*
3. *The exceptional minimal set is the boundary of a minimal singularity set in Ω, i.e. there exists a closed subset $A_S = \overline{A}_S$ of $\overline{\Omega}$ with boundary $A_S \cap \partial\Omega$ equal to S such that $\Omega \setminus A_S$ is a pseudoconvex domain. There is no non-empty closed subset $A = \overline{A}$ of A_S such that $\Omega \setminus A$ is pseudoconvex.*

Removing from M a compact set which does not intersect S we get an example of a non-compact generic $(2n-2)$-dimensional CR-manifold with the same properties as the manifold in theorem 4.

We do not know if an exceptional minimal compact CR-invariant subset of a generic CR-manifold M of dimension $2n-2$ contained in a strictly pseudoconvex boundary is always the boundary of a singularity set. This would be a generalization of the theorem of Harvey and Lawson [Ha–La]. This generalization would imply that the condition in theorem 2 is also necessary. See also section 6 for the discussion of further open problems.

We will now formulate variants of the theorems if the assumption of strict pseudoconvexity of Ω is removed.

Theorem 1′. *Let $\Omega \subset \mathbb{C}^n$, $n \geq 3$, be a bounded domain with connected boundary of class C^2. Let M be a connected proper submanifold of $\partial\Omega$ of class C^2 and of real dimension $2n-3$. If a compact subset K of M is not removable (with respect to $\partial\Omega$) then $K = M$ (hence M is closed) and M is a maximally complex CR-manifold.*

Theorem 2′. *Let Ω be as in theorem 1′ and let $M \subset \partial\Omega$ be a proper submanifold which is a generic CR-manifold of class C^2 and of real dimension $2n-2$. If a compact set $K \subset M$ does not contain a non-empty CR-invariant subset of M then K is removable.*

In the following theorems $\partial\Omega$ is replaced by a (not necessarily closed) hypersurface H. We will not give the most general statements.

Theorem 1a. *Let H be an orientable hypersurface of class C^2 in $\mathbb{C}^n$, $n \geq 3$, which is strictly pseudoconvex from one side. Suppose M is a connected proper submanifold of H of dimension $2n-3$ and of class C^2. Let K be a relatively closed subset of H (not necessarily a compact set). If K is contained in M and K is not removable (with respect to H) then $K = M$ (hence M is relatively closed in H) and M is a maximally complex CR-manifold.*

Theorem 2a. *Let H be as in theorem* 1a *and let M be a proper submanifold of H which is a generic CR-manifold of class C^2 and of dimension $2n-2$. Let K be a relatively closed subset of H, $K \subset M$. If K does not contain a non-empty CR-invariant subset of M (and, therefore, $K \neq M$) then K is removable (with respect to H) and, therefore, $H \backslash K$ is connected.*

The minimal relatively closed CR-invariant subsets of M are described by proposition 2.2 below.

The next theorems concern $(L^p, \overline{\partial}_b)$-removablility. No condition (rather than the natural smoothness condition) will be made for the hypersurface H.

Theorem 1 has a reasonable analogue only if $p < 2$. (By classical results, see for example, [Ha–Po], each set in $\mathbb{R}^n$ of finite $(n-2)$-dimensional Hausdorff measure is removable for L^p solutions of first order differential operators if $p \geq 2$.)

Theorem 1b. *Let H be a hypersurface in $\mathbb{C}^n$ of class C^2 and let M be a connected proper submanifold of H of class C^2 and of real dimension $2n-3$. Suppose $p < 2$. If $K \subset M$ is relatively closed in H and K is not $(L^p, \overline{\partial}_b)$-removable then $K = M$ and M is relatively closed in H and maximally complex.*

Theorem 2b. *Let H be a hypersurface in $\mathbb{C}^n$ of class C^2. Suppose M is a proper submanifold of H of real dimension $2n-2$ and of class C^2 which is a generic CR-manifold.*

If the compact subset K of M does not contain $G_J(M)$-invariant sets then K is $(L^p, \overline{\partial}_b)$-removable for each $p, 1 \leq p \leq \infty$.
For $p \geq 2$ a weaker condition is sufficient: If $2 \leq p < \infty$ and the compact subset K of M does not contain compact CR-invariant subsets of M of infinite $(2n-1-p')$ - dimensional Hausdorff measure then K is $(L^p, \overline{\partial}_b)$ -removable. If $K \subset M$ does not contain $G_J(M)$-invariant subsets of positive $(2n-2)$-dimensional measure then K is $(L^\infty, \overline{\partial}_b)$-removable.

Here p' is conjugate to $p : \frac{1}{p} + \frac{1}{p'} = 1$. The minimal compact CR-invariant sets are described by theorem 3.

Note that the property of a compact subset K of a generic CR-manifold M of class C^2, imbedded into $\mathbb{C}^n$ which plays the main role in the preceeding theorems, namely, to be without CR-invariant subsets is stable under small C^2 perturbations of M and K (see proposition 2.4).

The proof of the theorems consists of two parts. The first part is the study of the geometry of the decomposition of a CR-manifold into CR-orbits. This is done in section 2 and may be of interest for itself. The second part is the propagation of wedge-extendability along orbits. This is in principle well-known, but seems to be new in the smoothness class C^2 for generic CR-manifolds of real codimension two. A proof is given in section 3. In section 5 we give a proof of theorem 4. Further examples and open problems are discussed in section 6.

We conclude this section with a simple example which illustrates the theorems 2 and 2b.

Example 1.1. *Let $H = \partial\mathbb{B}^3$ be the unit sphere in $\mathbb{C}^3$, the boundary of the unit ball $\mathbb{B}^3$ in $\mathbb{C}^3$. Let $I = (0, \frac{1}{2})$ be an interval contained in the real axis $\mathbb{R}$. The manifold $M = \{z = (z_1, z_2, z_3) \in \partial\mathbb{B}^3 : z_1 \in I\}$ is a generic CR-manifold of dimension 4 and CR-dimension 1. Moreover, M is foliated, the leaves are the spheres $S_{z_1} = \{z_1\} \times \{(z_2, z_3) : |z_2|^2 + |z_3|^2 = 1 - |z_1|^2\}, z_1 \in I$, and each sphere is a compact CR-manifold of CR-dimension 1 and dimension 3. By theorems 1 and 2 a compact subset K of M is removable iff K does not contain a whole sphere S_{z_1} for some $z_1 \in I$.*
Moreover, K is $(L^\infty, \overline{\partial}_b)$-removable, if the set

$$\{z_1 \in I : S_{z_1} \subset K\}$$

has zero linear measure. This condition is also **necessary** *by Denjoy's theorem (*[Ahl–Beu] , [Den], [Ma]*). Indeed, it is enough to consider functions depending only on the first variable z_1.*

For $p < 2$ K is $(L^p, \overline{\partial}_b)$-removable iff it does not contain any sphere S_{z_1} with $z_1 \in I$. The "only if" part follows easily from the fact, that the function $z \longrightarrow \frac{1}{z}$ is in L^p with respect to planar measure if $1 \leq p < 2$.

Let $p = 2$. K is $(L^2, \overline{\partial}_b)$-removable if K does not contain an infinite number of spheres S_{z_1}, $z_1 \in I$.

For $p > 2$ K is $(L^p, \overline{\partial}_b)$-removable if

$$\Lambda_{2-p'}(\{z_1 \in I : S_{z_1} \subset K\}) < \infty.$$

(Here Λ_α is the Hausdorff measure of dimension α).

Sufficient conditions for a cocmpact set $\bigcup_{z_1 \in E} S_{z_1}$ (E is a compact subset of I) to be a non-removable set can be given in terms of certain capacities (see [Ca]*). In terms of Hausdorff measures one can prove, for example, the following: If $p > 2$ and*

$$\Lambda_\alpha(\{z_1 \in I : S_{z_1} \subset K\}) > 0$$

for some $\alpha > 2 - p'$, then K is not $(L^p, \overline{\partial}_b)$-removable. For this and for more information see [Ca], [Ma–Ha].

Denote for any $K \subset M$ by $H(\partial\mathbb{B}^3 \setminus K)$ the germ of envelopes of holomorphy of one-sided neighbourhoods (contained in $\mathbb{B}^3$) of $\partial\mathbb{B}^3 \setminus K$, and denote by K_1 the largest $G_J(M)$-invariant subset of K. It is quite easy to see that for any $K \subset M$ the set $H(\partial\mathbb{B}^3 \setminus K)$ is one-sheeted, moreover,

$$H(\partial\mathbb{B}^3 \setminus K) = H(\partial\mathbb{B}^3 \setminus K_1) = \mathbb{B}^3 \setminus (\bigcup_{z_1 : S_{z_1} \subset K_1} B_{z_1}),$$

where

$$B_{z_1} = \{z_1\} \times \{(z_2, z_3) : |z_2|^2 + |z_3|^2 < 1 - |z_1|^2\} = \mathbb{B}^3 \cap (\{z_1\} \times \mathbb{C}^2).$$

For deeper examples and open problems the interested reader is referred to section 6 before reading sections 3 and 4.

2. CR-ORBITS IN CR-MANIFOLDS

We will study now the geometry of the decomposition of CR-manifolds into CR-orbits. CR-orbits are propagators of properties of CR-functions (like vanishing in a neighbourhood of points or wedge extendability [Trv], [Trp]). This gives a motivation for this section.

The notion of orbits of families of vector fields goes back to a well written paper of Sussmann [Su]. We will not repeat here the background and motivation for introducing orbits in the general situation, but we have to go into the detail of the construction in the case of CR-manifolds.

So, let M be a CR-manifold of class C^k $(k \geq 2)$ imbedded into $\mathbb{C}^n$. (Sussmann considered only the C^∞ case. His construction works as well in the C^2 case.) Let m be the CR-dimension of M. The CR-structure on M defines in a natural way a family of vector fields: Cover M with relatively open subsets $\{U_j\}_{j=1}^{\infty}$ of sufficiently small diameter and choose for each U_j $2m$ real non-singular vector fields $X_1^{(j)}, \dots, X_{2m}^{(j)}$ of class C^{k-1} such that in U_j their real linear hull coincides with $T^J M$:

$$\operatorname{span}_{\mathbf{R}}(X_1^{(j)}(p), \dots, X_{2m}^{(j)}(p)) = T_p^J M \text{ for each } p \in U_j.$$

The $\{X_1^{(j)}, \dots, X_{2m}^{(j)}\}_{j=1}^{\infty}$ give an everywhere defined set of vector fields on M in the sense of Sussmann. One can make Sussmann's construction with this set of vector fields, but it is more convenient to work from the beginning with their linear span. A vector field X on U_j which is of the form $X = \sum_{l=1}^{2m} a_l X_l^{(j)}$ for real C^{k-1} functions a_l on U_j we will call a CR-vector field on M.

Let X be a CR-vector field of class $C^{k-1}\,(k \geq 2)$ defined in an open subset U of M. For a point $p \in U$ denote by $t \to \gamma_X(p,t)$ the integral curve of X with starting point p,

$$\gamma_X(p,0) = p, \quad \frac{\partial}{\partial t}\gamma_X(p,t) = X(\gamma_X(p,t)),$$

t belongs to the maximal interval of definition I_p. It is well known ([Hart], V.4.1 and V.3.1) that the mapping $(p,t) \to \gamma_X(p,t)$ is of class C^{k-1} and for fixed t the mapping $p \to \gamma_X(p,t)$ is a diffeomorphism of some neighbourhood of p on M onto some neighbourhood of $\gamma_X(p,t)$.

Denote for fixed p the mapping $t \to \gamma_X(p,t)$ by $\rho_{X,p}$ and for fixed t denote the mapping $p \to \gamma_X(p,t)$ by $g_{X,t}$ (both mappings are defined on the natural domain of existence). The local diffeomorphisms $g_{X,t}$ (X is a CR-vector field on some of the U_j, t is a real parameter, $g_{X,t}$ is defined on its domain) generate a pseudogroup of local diffeomorphisms: Composites $g_{X_q,t_q} \circ \dots \circ g_{X_q,t_1}$ and inverses $(g_{X,t})^{-1} = g_{X,(-t)}$ are defined in a natural way on their domain of existence. (The identity corresponds to the time-parameter $t = 0$.) Denote this pseudogroup which is associated to the CR-structure of M by $G_J(M)$.

Two elements p_1 and p_2 of M are called $G_J(M)$-equivalent if there exists an element $g \in G_J(M)$ such that $g(p_1) = p_2$. Equivalence classes for this relation are *called CR-orbits.* In other words, let $p_0 \in M$. The CR-orbit of M through p_0 which will be denoted by $\mathcal{O}(M,p_0)$ consists of all points p of the form $p = g(p_0)$ for some element $g = g_{X_q,t_q} \circ \dots \circ g_{X_1,t_1} \in G_J(M)$. (Here q is a natural number, $X_1, \dots, X_q$ are CR-vector fields of class C^{k-1}, each defined on some U_j. $t_1, \dots, t_q$ are real parameters and the composite g is defined on the natural domain of existence.) The equality $p = g(p_0)$ means the following:

p can be joined with p_0 by a piecewise CR-curve $\gamma = \gamma_q$. This means, there is a continuous mapping $\gamma : [a,b] \to M$ with $\gamma(a) = p_0$, $\gamma(b) = p$ such that the interval $[a,b]$ can be devided into q intervals $[t_l, t_{l+1}]$ $(l = 1, \dots, q)$ with $a = t_1 < \dots < t_{q+1} = b$ and for each l the part $\gamma_l = \gamma|[t_l, t_{l+1}]$ of γ is an integral curve of the vector field X_l (and, therefore, it is of class C^k). In particular each CR-orbit is connected.

Sometimes it is more convenient for us to work with the following definition of CR-curves which is (for curves of small length) equivalent to the preceeding one:
A CR-curve $\gamma : [a,b] \to M$ *is a curve of class* C^k*, of sufficiently small length which satisfies the condition* $\gamma'(t) \in T^J_{\gamma(t)}M\backslash\{0\}$ *for each* $t \in [a,b]$ (at the endpoints a, b derivatives are taken from the right or from the left, respectively). (See also [Jö4].)

Fix now the natural number q, the q-tuple X of CR-vector fields, $X = (X_1, \ldots, X_q)$ and the point $p \in M$. We consider the mapping

$$t = (t_1, \ldots, t_q) \to (g_{X_q,t_q} \circ \cdots \circ g_{X_1,t_1})(p) = g_{X,t}(p) \tag{2.1}$$

on its natural domain of existence $\Omega_{X,p} \subset \mathbb{R}^q$. Denote this mapping by $\rho_{X,p}$. $\rho_{X,p}$ is a mapping of class C^{k-1} from the open subset $\Omega_{X,p}$ of $\mathbb{R}^q$ into M.

Following Sussmann we introduce now on the orbits a natural topology: the strongest one which makes all mappings of the form $\rho_{X,p}$ (for arbitrary $p \in M$, arbitrary natural numbers q and arbitrary q-tuples of CR-vector fields X) continuous. Sussmann's proof gives that with this topology each orbit admits a unique differentiable structure (of class C^{k-1}) such that it is a submanifold of M in the sense described in the introduction (but not necessarily a proper submanifold of M).

For further use we recall two lemmas of Sussmann (lemma 5.1 and lemma 5.2 of [Su]). We formulate the lemmas for C^2 manifolds with our special choice of the system of vector fields.

Lemma A. *Suppose M is a CR-manifold of class C^k, $k \geq 2$, imbedded into $\mathbb{C}^n$ and N is a CR-orbit of M. Let $p_0 \in N$ be an arbitrary point. Suppose q is a natural number, $X = (X_1, \ldots, X_q)$ is a q-tuple of CR-vector fields on M and $t = (t_1, \ldots, t_q)$ is contained in the domain Ω_{X,p_0} of the mapping ρ_{X,p_0}. Denote by $T_t\Omega_{X,p_0}$ the tangent space of Ω_{X,p_0} at t (which we will often identify with $\mathbb{R}^q$). Then*

$$d\rho_{X,p_0}(T_t\Omega_{X,p_0}) \subset T_pN. \tag{2.2}$$

Lemma B. *Let M and N be as in lemma A and let $p \in N$ be an arbitrary point. There exists a point $p_0 \in N$, a natural number q, a q-tuple of CR-vector fields $X = (X_1, \ldots, X_q)$ and a point $t \in \Omega_{X,p_0}$ such that*

$$\rho_{X,p_0}(t)(= g_{X,t}(p_0)) = p \text{ and } (d\rho_{X,p_0})(T_t\Omega_{X,p_0}) = T_pN. \tag{2.3}$$

Note that the lemmas in [Su] are stated with $T_{p_0}N$ replaced by $P(p_0)$. Here P is a distribution on M, i.e. a mapping which assigns to every $p \in M$ a linear subspace $P(p)$

of the tangent space T_pM. Moreover, P is the smallest distribution with the following properties:

- $P(p)$ contains T_p^JM;
- P is $G_J(M)$-invariant, i.e.for each $p \in M$ and each $g \in G_J(M)$ with p in its domain the differential dg maps $P(p)$ into $P(g(p))$.

From the lemmas stated with $P(p_0)$ instead of $T_{p_0}N$ and the rank theorem it follows that for the mapping ρ_{X,p_0} of lemma B the images $\rho_{X,p_0}(\omega)$ of small open sets $\omega \subset \Omega_{X,p_0}$ are integral submanifolds of P, i.e. the $\rho_{X,p_0}(\omega)$ are manifolds such that $T_p\rho_{X,p_0}(\omega) = P(p)$ for all $p \in \rho_{X,p_0}(\omega)$. Moreover, it turns out that sets of the form $\rho_{X,p_0}(\omega)$ for suitable X, p_0 and ω cover the orbit N, they constitute a basis of the topology of the orbit N and define a differentiable structure of class C^{k-1} on N. From this the lemmas A and B follow in the form stated above. We will use the lemmas in this form.

It is clear now from (2.2) that each CR-orbit N of the CR-manifold M is a CR-submanifold of M, i.e. $T_pN \supset T_p^JM$ for each $p \in N$. Moreover, different CR-orbits are disjoint and the manifold M is the union of its CR-orbits. (The decomposition of M into CR- orbits is uniquely determined.)

Together with the CR-orbits of M we will also consider the CR-orbits of open subsets of M. Let $p \in M$ and consider for a neighbourhood U of p on M the U-orbit $\mathcal{O}(U,p)$. It is clear that $\mathcal{O}(U,p)$ is contained in $\mathcal{O}(M,p)$. For a decreasing sequence of neighbourhoods of p the dimensions of the corresponding orbits decrease and stabilize. The CR-manifold germ obtained by considering the U-orbits $\mathcal{O}(U,p)$ for small neighbourhoods U of p on M is called the CR-orbit germ at p and is denoted by $\mathcal{O}^{\text{loc}}(M,p)$.

In the following we need some information about the structure of sets which are invariant under the action of the pseudogroup $G_J(M)$. In other words, we consider subsets S of M with the following invariance property:

$p \in S$ implies $g(p) \in S$ for each $g \in G_J(M)$ with p in the domain of g.

Equivalently, S *is* $G_J(M)$*-invariant iff*

together with a point p the set S contains all points of M which can be joined with p by a (piecewise) CR-curve.

We will call such sets CR-invariant or, more precisely, $G_J(M)$-invariant. If no confusion will arise we will say invariant instead of $G_J(M)$-invariant. It is clear that each invariant subset of M consists of the union of CR-orbits of M.

Now we will formulate our results concerning the geometry of the decomposition of a CR-manifold into CR-orbits. We will see that this decomposition has some properties in common with foliation theory. This concerns especially some results on minimal closed invariant sets. But there are also big differences. For example, different orbits may have different dimensions, moreover, the function $p \to \dim_r \mathcal{O}^{\mathrm{loc}}(M,p)$ is not necessarily constant on a fixed orbit.

We begin with the following simple

Lemma 2.1. *Let M be a CR-manifold of class C^2 imbedded (or immersed) into $\mathbb{C}^n$ and let S be a CR-invariant subset of M. Then the interior $intS$, the closure $\overline{S}$, the boundary $\overline{S}\backslash intS$ and the complement $M\backslash S$ are CR-invariant. (The closure and the interior are taken in M with the manifold topology). If S_1 is another CR-invariant subset of M then the sets $S_1\backslash S$, $S \cup S_1$ and $S \cap S_1$ are also CR-invariant.*

Proof. Let $p \in \mathrm{int}S$ and suppose $g \in G_J(M)$ is a local CR-diffeomorphism which contains p in its domain. There exists a neighbourhood U of p on M which is contained in S and in the domain of g as well. It is clear that $g(U)$ is open in M, it is contained in S by the invariance of S and contains $g(p)$. That means $g(p) \in \mathrm{Int}S$. We proved the invariance of $\mathrm{int}S$.

Let $p \in \overline{S}$ and let p_k, $k = 1, \ldots,$ be points in S which converge to p (in the manifold topology of M). Suppose $g \in G_J(M)$ contains p in its domain. Then for $k \geq k_0$ the point p_k is in the domain of g. Moreover, $g(p_k)$, $(k \geq k_0)$ is contained in S by the invariance of S and $g(p_k)$ converge to $g(p)$. That means, $g(p) \in \overline{S}$. We proved the invariance of $\overline{S}$.

The remaining assertions are obvious. □

For a subset E of M denote by $I(E)$ the smallest invariant subset of M, containing E. In other words, $I(E)$ is the union of all orbits $\mathcal{O}(M,p)$ for $p \in E$. We call $I(E)$ the CR-invariant hull (or invariant hull) of E. In most situations we are interested in invariant hulls of open subsets of M.

Lemma 2.2. *Let M be as in lemma 2.1. If U is open in M then $I(U)$ is open.*

We omit the proof.

The following corollaries of lemmas 2.1 and 2.2 will be usefull.

Corollary 2.1. *Let M be as in lemma 2.1. Suppose K is a relatively closed subset of M, $K \neq M$. If K does not contain a non-empty relatively closed invariant subset then $I(M\backslash K) = M$.*

Proof. $I(M\backslash K)$ is open. If $I(M\backslash K) \neq M$ then the invariant set $M\backslash I(M\backslash K)$ is not empty and relatively closed. Clearly it is contained in K. The contradiction proves the corollary. □

Corollary 2.2. *Let M be as in lemma 2.1. Suppose, moreover, that M does not contain any non-empty relatively closed invariant subset different from M. Then for each open set U in M we have $I(U) = M$.*

Proof. Apply corollary 2.1 to the set $K = M\backslash U$. □

A relatively closed invariant subset S of a CR-manifold M as above is called minimal if there is no non-empty relatively closed invariant subset S_1 of M which is contained in S and does not coincide with S.

Corollary 2.3. *Let M be as in lemma 2.1. Suppose, moreover, that M is connected. If a minimal relatively closed invariant subset S of M is not the whole of M then S has no interior points with respect to M. All orbits, contained in S have dimension strictly smaller than* $\dim_r M$.

Proof. Suppose $S \neq M$ and Int $S \neq \emptyset$. Then $S\backslash \text{Int} S$ is closed in M, non-empty (since $\overline{S} = \text{Int } S \neq \emptyset$ would imply $S = M$ by the connectedness of M) and invariant. This contradicts the minimality of S. The assertion concerning the dimension of the orbits in S is now obvious. □

Lemma 2.3. *Let M be as in lemma 2.1. Suppose S is the union of CR-orbits (for example, S consists of one single CR-orbit of M). Consider the CR-invariant set $\overline{S}\backslash S$ (the closure is taken in M in the manifold topology).* (The invariance of $\overline{S}\backslash S$ follows from lemma 2.1). *Let N be an orbit in $\overline{S}\backslash S$. Then the real dimension* $\dim_r N$ *of N does not exceed the maximum of the real dimensions of the orbits contained in S.*

Proof. Take an arbitrary point $p \in N$ and apply lemma B. Let p_0 and $g_{X,t}$ be the objects which existence is stated in lemma B. Recall that $p_0 = (g_{X,t})^{-1}(p)$. Let $z \in S$ be sufficiently close to p (in the topology of M). Then z is in the domain of $(g_{X,t})^{-1}$. Denote $z_0 = (g_{X,t})^{-1}(z)$. The point z_0 is close to p_0. Since the differential $d\rho_{X,z_0}$ depends continuously on z_0 it follows that for z_0 sufficiently close to p_0 the rank of $d\rho_{X,z_0}$ is not smaller than the rank of $d\rho_{X,p_0}$. But by lemma A the rank of $d\rho_{X,z_0}$ is not greater than the real dimension of the CR-orbit $\mathcal{O}(M, z_0) \subset S$ through z_0 and by lemma B the rank of $d\rho_{X,p_0}$ is equal to the real dimension of N. Lemma 2.3 is proved. □

Let now M be a CR-manifold of class C^2 *imbedded* into $\mathbb{C}^n$. Suppose Γ is a submanifold of M which is not metrically complete in its natural metric d_Γ. So Cauchy sequences in (Γ, d_Γ) may not have a limit in Γ in the topology defined by d_Γ. Nevertheless since for two points p_1 and p_2 in Γ the Euclidean distance between p_1 and p_2 in $\mathbb{C}^n$ does not exceed the distance $d_\Gamma(p_1, p_2)$, each Cauchy sequence in (Γ, d_Γ) has a limit in the Euclidean metric in $\mathbb{C}^n$. Denote by $b\Gamma$ the following set:

$b\Gamma$ = {Euclidean limits of all Cauchy sequences in (Γ, d_Γ) which have no limit in (Γ, d_Γ) }.

Note that in this definition we did not exclude that to non-equivalent Cauchy sequences $\{p_j\}$ and $\{z_j\}$ in (Γ, d_Γ) (i.e. those with $d_\Gamma(p_j, z_j)$ not tending to zero) corresponds one Euclidean limit in $b\Gamma$. Note that if on a submanifold Γ of M the distance d_M is equivalent to d_Γ then Γ is relatively closed in M iff $M \cap b\Gamma = \emptyset$. So the condition $M \cap b\Gamma = \emptyset$

generalizes the notion of relative closedness in M.

Let now N be a CR-orbit in M. The following proposition shows, in particular, that no point of N can be contained in the set bN.

Proposition 2.1. *Let M be a CR-manifold of class C^2 imbedded into $\mathbb{C}^n$. Suppose N is a CR-orbit of M such that $M \cap bN$ is not empty. Then $M \cap bN$ is CR-invariant and for each CR-orbit Γ of M contained in $M \cap bN$ the real dimension $\dim_r \Gamma$ is strictly smaller than that of N.*

Proof. To prove the invariance of $M \cap bN$ let $p \in M \cap bN$ and let $g \in G_J(M)$ be a local CR-diffeomorphism of M with p in its domain. By the definition of $M \cap bN$ $p = \lim_{j\to\infty} p_j$ (the limit is taken in the Euclidean metric) for a Cauchy sequence of points $\{p_j\}$ in (N, d_N) which does not have a limit in (N, d_N). Choose for each j and l C^1 curves $\gamma_{j,l}$ on N which join the points p_j and p_l and which length do not exceed $2d_N(p_j, p_l)$. For $j, l \geq j_0$ the points p_j and p_l together with the curve $\gamma_{j,l}$ are contained in the domain of g. It is now clear, that $g(p_j)$ $(j \geq j_0)$ are contained in N and converge in the Euclidean metric to $g(p)$. Moreover, for $j, l \geq j_0$ the C^1 curve $g \circ \gamma_{j,l}$ is contained in N and the Euclidean length of $g \circ \gamma_{j,l}$ does not exceed $d_N(p_j, p_l)$ multiplied by a constant depending on g. So $\{g(p_j)\}_{j \geq j_0}$ is a Cauchy sequence in (N, d_N). It is clear that this sequence does not have a limit in (N, d_N) : otherwise the inverse local diffeomorphism g^{-1} would map this limit to a limit in (N, d_N) of the sequence $\{p_j\}$. To see this one has to repeat the arguments above for the inverse local diffeomorphism g^{-1} instead of g. The invariance of $M \cap bN$ is proved.

Let now Γ be a CR-orbit of M contained in $M \cap bN$. Since $M \cap bN$ is contained in $\overline{N}$ (closure in M in the manifold topology) it follows from lemma 2.3 that $\dim_r \Gamma \leq \dim_r N$. Suppose, in contrast to the assumption, that the dimensions are equal. For an arbitrary point $p \in \Gamma$ we apply lemma B. Let p_0 and $g_{X,t}$ be as in lemma B, $g_{X,t}(p_0) = p$ and $(d\rho_{X,p_0})(T_t\Omega_{X,p_0}) = T_p\Gamma$. Let $\{z_j\}_{j=1}^{\infty}$ be a Cauchy sequence in (N, d_N) with Euclidean limit p which does not have a limit in (N, d_N). For $j \geq j_0$ z_j is in the domain of $(g_{X,t})^{-1}$. Denote for $j \geq j_0$ the point $(g_{X,t})^{-1}(z_j)$ by ζ_j. Choose by the rank theorem a small C^1 manifold Q in $\Omega_{X,p_0} \subset \mathbb{R}^q$ of dimension equal to $\dim_r \Gamma$ which is relatively closed in some open subset of $\mathbb{R}^q$, contains t and is such that ρ_{X,p_0} is a diffeomorphism of Q onto a neighbourhood of p on Γ (in the manifold topology of Γ). Shrinking Q if necessary we may assume that Q is diffeomorphic to an open ball in $\mathbb{R}^l$ $(l = \dim_r N)$ and is contained in Ω_{X,ζ_j} for large j. Denote by $\tilde{\rho}_{X,p_0}$ ($\tilde{\rho}_{X,\zeta_j}$, respectively) the restriction of ρ_{X,p_0} (ρ_{X,ζ_j}, respectively) to Q. Then $\tilde{\rho}_{X,\zeta_j}$ is close to $\tilde{\rho}_{X,p_0}$ in the C^1 topology if j is large.

Now we use the assumption $\dim_r N = \dim_r \Gamma = \dim_r Q$. It follows (shrinking Q again, if necessary) that for all sufficiently large j $\tilde{\rho}_{X,j}$ is a diffeomorphism of Q onto a neighbourhood of z_j on N. Let Q_1 be an open connected subset of Q containing t with compact closure in Q and with boundary ∂Q_1 of class C^1.

Claim 2.1. *There exists a positive constant a not depending on j such that for each sufficiently large j and for arbitrary points ζ in $N\backslash\tilde{\rho}_{X,\zeta_j}(Q_1)$ the estimate $d_N(\zeta,\zeta_j)\geq a$ holds.*

Proof. Let $\zeta\in N\backslash\tilde{\rho}_{X,\zeta_j}(Q_1)$ and let γ be a C^1 curve in N which joins ζ and ζ_j. Then the Euclidean length of γ is not smaller than the Euclidean length of the connected part $\tilde{\gamma}$ of γ which joins ζ_j with the first point $\tilde{\zeta}$ of intersection of γ with $N\backslash\tilde{\rho}_{X,\zeta_j}(Q_1)$. It is clear that $\tilde{\zeta}$ is contained in $\tilde{\rho}_{X,\zeta_j}(\partial Q_1)$. If j is large enough then the Euclidean length of $\tilde{\gamma}$ is not smaller than the distance in Q of t from the boundary ∂Q_1 of Q_1 multiplied by a constant not depending on j (the constant depends on the C^1 norm of the inverse of the diffeomorphism $\tilde{\rho}_{X,\zeta_j}$ with $\tilde{\rho}_{X,\zeta_j}$ close to $\tilde{\rho}_{X,p_0}$ in C^1). The claim is proved. □

Now we will finish the proof of proposition 2.1. $\{z_j\}_{j\geq 1}$ is a Cauchy sequence in (N,d_N), so for some j_1 and for all $l\geq 0$

$$d_N(z_{j_1},z_{j_1+l})<\frac{a}{2}.$$

Therefore, by the claim 2.1

$$z_{j_1+l}\in\tilde{\rho}_{X,\zeta_{j_1}}(Q_1).$$

But $\tilde{\rho}_{X,\zeta_{j_1}}(\overline{Q}_1)$ is contained in $\tilde{\rho}_{X,\zeta_{j_1}}(Q)\subset N$ and it is compact in the manifold topology on N and therefore also in the Euclidean topology in $\mathbb{C}^n$, since it is the continuous image of a compact set. It is also clear that on $\tilde{\rho}_{X,\zeta_{j_1}}(\overline{Q}_1)$ the metric d_N is equivalent to the induced Euclidean metric:

$$\text{for } p_1,p_2\in\tilde{\rho}_{X,\zeta_{j_1}}(\overline{Q}_1)\quad |p_1-p_2|\leq d_N(p_1,p_2)\leq c_1|p_1-p_2|.$$

($|z|$ denotes the Euclidean norm of a point $z\in\mathbb{C}^n$). So the Euclidean limit p of $\{z_j\}$ is contained in $\tilde{\rho}_{X,\zeta_{j_1}}(\overline{Q}_1)\subset N$ and p is a limit of $\{z_j\}$ in the metric d_N. The contradiction proves that the equality $\dim_r N=\dim_r\Gamma$ is impossible. □

Not every CR-submanifold of a CR-manifold is CR-invariant (i.e. is the union of CR-orbits). The following lemma gives a sufficient condition for a CR-submanifold to be CR-invariant.

Lemma 2.4. *Let M be a CR-manifold of class C^2 imbedded (or immersed) into $\mathbb{C}^n$. Let Γ be a CR-submanifold of M (of class C^1). If $M\cap b\Gamma=\emptyset$ (in particular, if Γ is metrically complete in its natural metric d_Γ) then Γ is CR-invariant.*

Proof. Suppose Γ is not CR-invariant. Let γ be a CR-curve in M with starting point $p\in\Gamma$ and endpoint not in Γ. Consider the maximal connected part of γ which contains p and is contained in Γ. This part is open in γ (since it is an integral curve of a C^1 vector field on an open subset of M which is tangent to Γ at points of Γ). The endpoint of this part is contained in $M\cap b\Gamma$ in contrast to the assumption. Lemma 2.4 is proved. □

The converse is not always true, i.e. not for each CR-orbit N of M the condition $M \cap bN = \emptyset$ is satisfied. The following lemma gives sufficient conditions for a CR-manifold M to have only CR-orbits with this property.

Lemma 2.5. *Let M be a CR-manifold of class C^2 imbedded into $\mathbb{C}^n$. Let $d \geq 0$ be an integer. Consider the following two conditions:*

(a) *Any CR-submanifold of M of real codimension greater than d is contained in a compact part of M.*

(b) *M does not contain a compact subset which consists of the union of CR-submanifolds of M of real codimension greater than d.*

If condition (b) is satisfied, then there are no CR-orbits of real codimension greater than d which are contained in a compact part of M. Moreover, if a CR-orbit N of real codimension d is contained in a compact part of M then N is metrically complete (in the metric d_N).

If both conditions are satisfied, then M does not contain any CR-orbit of real codimension greater than d. Moreover, for each CR-orbit N of M of real codimension d the condition $M \cap bN = \emptyset$ holds.

Lemma 2.6. 1.) *Condition (b) is always satisfied for a CR-manifold M of class C^2 imbedded (or immersed) into $\mathbb{C}^n$ for which* $e(M) = \dim_r M - 2\dim_{CR} M = d + 1$.
2.) *Let M be as in* 1.) . *If the Leviflat part of M (i.e. that part of M where all eigenvalues of the Leviform are zero) is compact then condition (a) is satisfied. This fact takes place if, for example, M is compact or M is contained in a hypersurface which is strictly pseudoconvex from one side.*

Proof of lemma 2.6. The proof is based on the following observation. Suppose, Γ is a manifold of class C^2 imbedded (or immersed) into $\mathbb{C}^n$ such that for each point $p \in \Gamma$ the tangent space $T_p\Gamma$ (considered as a real linear subspace of $T_p\mathbb{C}^n \simeq \mathbb{C}^n$) is invariant under the operator J of multiplication with the imaginary unit: $JT_p\Gamma = T_p\Gamma$. In other words $T_p\Gamma$ can be considered as a complex linear subspace of $\mathbb{C}^n$. Then Γ is an analytic manifold imbedded (or immersed) into $\mathbb{C}^n$. This fact is elementar. Suppose $p_0 \in \Gamma$. Represent Γ locally near p as the graph of a vector valued function over a part of the tangent space $T_{p_0}\Gamma$ ($T_{p_0}\Gamma$ considered as a real linear subspace of $T_p\mathbb{C}^n \simeq \mathbb{C}^n$). The condition $JT_p\Gamma = T_p\Gamma$ for all $p \in \Gamma$ close to p is equivalent to the Cauchy-Riemann equations for the vector valued function.

Moreover, there is no compact subset of $\mathbb{C}^n$ which consists of the union of analytic manifolds (see [Gr]). This proves part 1.). Part 2.) is clear. □

Proof of lemma 2.5. Suppose (b) is satisfied and in contrast to the assertion there exists a CR-orbit Γ of real codimension greater than d which is contained in a compact part K of M. Then $\overline{\Gamma}$ (closure in M) is also contained in K and therefore $\overline{\Gamma}$ is compact. Moreover, by lemma 2.3 $\overline{\Gamma}$ consists of CR-orbits of real codimension greater than d. This contradicts (b). So there is no such CR-orbit.

Let now N be a CR-orbit of real codimension d in M which is contained in a compact part K of M. Then bN is contained in K and by proposition 2.1 bN consists of CR-orbits of real codimension greater than d. By the preceeding arguments $bN = \emptyset$, i.e. N is metrically complete.

Suppose both conditions (a) and (b) are satisfied. If, in contrast to the assertion there exists a CR-orbit Γ of real codimension greater than d then by (a) it must be contained in a compact subset of M. This is impossible as is shown above.

Let N be a CR-orbit of real codimension d in M. Then by proposition 2.1 $M \cap bN$ consists of CR-orbits of M of real codimension greater than d. So $M \cap bN = \emptyset$ as is shown above. □

Lemma 2.7. *Suppose M is a CR-manifold of class C^2 imbedded into $\mathbb{C}^n$. Let S be a minimal relatively closed (respectively, compact) CR-invariant subset of M. Then all CR-orbits N_α contained in S have equal dimension and have the property $M \cap bN_\alpha = \emptyset$ (are metrically complete, respectively). Moreover, $\overline{N} = S$ (closure in M) for each CR-orbit N of M contained in S.*

Proof. Suppose N is a CR-orbit contained in S with $M \cap bN \neq \emptyset$. Then $\overline{bN}$ (closure in M) is CR-invariant, is contained in S and by proposition 2.1 it consists of CR-orbits of dimension strictly smaller than the dimension of N. So N is not contained in $\overline{bN}$. Since this contradicts the minimality of S, for each orbit N in S the set $M \cap bN$ is empty. Moreover, if S is compact, then the orbits contained in S are metrically complete.

Let N be a CR-orbit of M contained in S with

$$\dim_r N \leq \dim_r \Gamma \tag{2.4}$$

for all CR-orbits Γ contained in S. Then $\overline{N}$ (closure in M) is relatively closed (compact, respectively) and invariant and $\overline{N}$ is contained in S. So by minimality $\overline{N} = S$. The lemma follows from (2.4) and lemma 2.3. □

We will now describe minimal closed CR-invariant subsets of CR-manifolds M under the condition that the real codimension of the CR-orbits of M is either zero or one. (See lemma 2.5 and 2.6 for conditions which imply this fact).

The description of minimal closed CR-invariant subsets in this case follows along the same lines as the description of minimal sets in foliation theory of codimension one ([He–Hi] Part A, p. 45–46, Part B p. 17–19).

The following proposition holds.

Proposition 2.2. *Let M be a connected CR-manifold of class C^k, $k \geq 2$, imbedded into $\mathbb{C}^n$. Suppose all CR-orbits in M have either real codimension one or real codimension zero. Then each minimal closed $G_J(M)$-invariant subset S of M has one (and only one) of the following types.*

1. *$S = M$ and either*
   a) *M is itself a CR-orbit or*

   b) *M is the union of CR-orbits N_α of codimension one, each of them being dense in M and with the property $M \cap bN_\alpha = \emptyset$.*
2. *S consists of one orbit of codimension one which is proper and relatively closed in M.*
3. *S is the union of CR-orbits N_α of codimension one in M such that $M \cap bN_\alpha = \emptyset$ each of them with the following two properties:*
   a) *N_α is not proper.*
   b) *N_α is not locally dense in M (i.e. the closure $\overline{N}_\alpha$ in M has no inner point in M).*

   A CR-orbit N_α of codimension one which satisfies a), b) and the condition $M \cap bN_\alpha = \emptyset$ is called exceptional and a minimal closed invariant subset S of M of the third kind is called exceptional.

Proof. By lemma 2.7 each CR-orbit N of M contained in S satisfies the relation $M \cap bN = \emptyset$.

Suppose S has interior points. Then $S = M$ by corollary 2.3, so S is of the first kind. By lemma 2.7

$$\overline{N} = S = M \quad \text{(closure in } M\text{)} \quad \text{for each CR-orbit} \quad N \text{ of } M. \tag{2.5}$$

If a CR-orbit N of M has itself interior points (with respect to M) then by lemma 2.7 $N = M$. Indeed, M is connected and for an orbit N which is an open subset of M the set $M \cap bN$ is the boundary of N in M. But $M \cap bN = \emptyset$ for such N by lemma 2.7.

If no orbit in $S = M$ has interior points, then all of them have real codimension one and in view of (2.5) 1b) is realized.

Suppose S has no interior point. In this case each orbit in S has codimension one and has no interior point in its closure. If no orbit in S is a topological subspace of M (i.e. no orbit in S is proper) then S is of the third kind.

Suppose on some orbit N in S the topology induced from M coincides with the orbit topology, i.e. N is proper. Prove that $\overline{N} = N$ (closure in M). If not there exists an orbit $N_1 \subset \overline{N}$ with $N_1 \cap N = \emptyset$. Since by minimality $\overline{N}_1 = \overline{N}$ for each $p \in N$ there is a sequence $p_j \in N_1$ with $p_j \to p$ (convergence in M) for $j \to \infty$. By the same reason for each j there exists a sequence $p_{jk} \in N$ with $p_{jk} \to p_j$ (convergence in M) for $k \to \infty$. Choose a small neighbourhood U_p of p on N such that $\overline{U}_p$ (closure in M) is contained in N. For each j and $k \geq k_0(j)$ large enough the points p_{jk} are not in U_p. By a diagonal process we get a sequence $p_{jk(j)}$ of points of $N \backslash U_p$ which converges to p in the Euclidean metric (but not in the manifold topology of N). This contradicts the fact that N is proper. The contradiction shows that N is relatively closed in M. By minimality $S = N$ and the second case is realized. The proposition is proved. □

Proof of theorem 3. Since M is not compact, a minimal compact CR-invariant subset of S can not be equal to M. Moreover, a generic CR-manifold of class C^2 and of real dimension $2n-2$ imbedded into $\mathbb{C}^2$ satisfies property (b) of lemma 2.5 with $d = 1$. So, all CR-orbits of M contained in S have real codimension one. By lemma 2.7 all orbits contained in S are metrically complete. The rest is as in the proof of proposition 2.2: If no orbit in S is proper, then S is exceptional. Otherwise, if an orbit N contained in S is proper then it is compact and by minimality $S = N$. □

To give more inside in the geometric picture we give some reformulation of conditions 1, 2 and 3 of proposition 2.2. This reformulation is in analogy to foliation theory ([He–Hi]).

Proposition 2.3. *Let M be a CR-manifold of class C^2 imbedded into $\mathbb{C}^n$. Let S be a minimal relatively closed CR-invariant subset of M. Suppose all CR-orbits of M contained in S have real codimension one. Let $N \subset S$ be a CR-orbit of M, let $p \in N$ and let l_p be a C^1 curve in M which contains p in its interior and which is transverse to T_pN (T_pN considered as a real linear subspace of T_pM). The following conditions are pairwise equivalent.*

I.) *N is locally dense (i.e. $\overline{N}$ (closure in M) has interior points with respect to M).*
I'.) *$\overline{N \cap l_p}$ contains a neighbourhood of p on l_p.*

II.) *N is proper (and relatively closed in M).*
II'.) *p is an isolated point of $N \cap l_p$.*

III.) *N is exceptional (i.e. N is not proper and not locally dense).*
III'.) *$\overline{N \cap l_p}$ is near p a one-dimensional Cantor set (i.e. for a suitable open part l'_p of l_p containing p the set $\overline{N \cap l'_p}$ is a Cantor set.)*

A Cantor set is as usual a closed set without isolated points and without inner points. The conditions I'), II') and III') *do not depend on the the choice of the point $p \in N$ and the choice of the transverse curve l_p.*

As the proof of proposition 2.1 the **proof** of proposition 2.3 is based on lemma B: For some natural number q there exists a q-tuple X of CR-vector fields, a C^1 manifold Q of dimension equal to $\dim_r N$ contained in $\mathbb{R}^q$ (see the proof of proposition 2.1) and a point p_0 with the following properties. For some $t \in \mathbb{R}^q$ $g_{X,t}(p_0) = p$. Denote for $z \in l_p$ close to p the point $(g_{X,t})^{-1}(z)$ by z_0. There exist a C^1 mapping $\tilde{p}_{X,z_0}(t)$, $(z,t) \in l_p \times Q$, which is a diffeomorphism onto a neighbourhood of p on M and such that if z belongs to an orbit of codimension one then $\tilde{p}_{X,z_0}$ maps Q diffeomorphically onto a neighbourhood z on this orbit. We omit further details. □

Proposition 2.4. *Let K be a compact subset of a generic CR-manifold M of class C^2 imbedded into $\mathbb{C}^n$ such that K does not contain non-empty $G_J(M)$-invariant subsets. Then each sufficiently small C^2 perturbation of K is of the same kind.*

More precisely, let Φ be a C^2 diffeomorphism of M onto a manifold M_1 imbedded into $\mathbb{C}^n$. Let M' be an open part of M with $K \subset M'$ and with compact closure $\overline{M'}$ in M. If the C^2 norm of $\Phi - Id$ (Id is the identity on M) does not exceed a positive constant ε then $M'_1 = \Phi(M')$ is a generic CR-manifold in $\mathbb{C}^n$ and $\Phi(K)$ does not contain non-empty $G_J(M'_1)$-invariant subsets.

Proof. It is clear that $\Phi(M')$ is generic if ε is small. The condition on K means that for each $p \in K$ the CR-orbit $\mathcal{O}(M,p)$ of M through p is not contained in K. Equivalently, for each $p \in K$ there exists a local diffeomorphism $g_p \in G_J(M)$ of a neighbourhood U_p of p onto an open set $g_p(U_p)$, such that $g_p(p)$ is not contained in K. We may choose U_p small enough so that $\overline{g_p(U_p)}$ does not intersect K. Cover K with a finite number of sets U_{p_j} as above, $j = 1, \ldots, l$. We have to prove that for each local diffeomorphism $g \in G_J(M)$ and each connected open set $U \subset M$ with $\overline{U}$ in the domain of g there exists a constant $\varepsilon(g)$ such that if the C^2 norm of Φ-Id over M' does not exceed $\varepsilon(g)$ then there exists $g_1 \in G_J(M_1')$ with $\Phi(\overline{U})$ in the domain of g_1 and with $g_1(\Phi(\overline{U}))$ uniformly close to $g(\overline{U})$. This follows from the fact that $T^J_{\Phi(p)}\Phi(M')$ is close to $T^J_p M'$ for $p \in M'$ if Φ is close to the identity in C^2. Indeed, we may suppose that U is small enough such that in a neighbourhood of $\overline{U}$ on M a family of vector fields $X_1, \ldots, X_{2m}$ is defined with the real linear span of $X_l(p)\,(l = 1, \ldots, 2m;\ p \in \overline{U})$ equal to $T^J_p M'$. There are corresponding vector fields $\tilde{X}_l$ in a neighbourhood of $\Phi(\overline{U})$, $l = 1, \ldots, 2m$, with the real linear span of the $\tilde{X}(\Phi(p))$ equal to $T^J_{\Phi(p)}\Phi(M')$, and, moreover, with $\tilde{X}_l \circ \Phi - X_l$ uniformly small near $\overline{U}$. So, to each CR-vector field X in a neighbourhood of $\overline{U}$ on M corresponds a CR-vector field $\tilde{X}$ in a neighbourhood of $\Phi(\tilde{U})$ with $\tilde{X} \circ \Phi - X$ uniformly small near $\overline{U}$. The existence of g_1 follows now from the fact that by Gronwall's lemma the endpoints of two integral curves $\gamma(t) = \gamma_Y(p,t)$ and $\gamma_1(t) = \gamma_{Y_1}(p,t)$, $t \in [0,T]$, are close to each other if the vector fields Y and Y_1 are close to each other. The proposition is proved. □

3. A propagation result for manifolds of class C^2

Let M be a generic CR-manifold imbedded into $\mathbb{C}^n$, let $p \in M$ and let U be a neighbourhood of p on M. Suppose K is a convex open truncated cone in $T_p\mathbb{C}^n \simeq \mathbb{C}^n$,

$$K = \{\zeta \in \mathbb{C}^n : |\zeta| < a \cdot Re\langle\zeta, \Theta\rangle, \quad |\zeta| < h\}$$

for positive constants a and h and a vector Θ (the "symmetry axis" of K) of unit length not contained in T_pM. ($\langle\zeta,\Theta\rangle = \sum_{k=1}^n \zeta_k\overline{\Theta}_k$ is the usual hermitian scalar product in $\mathbb{C}^n$.) A set $W = W(U,K)$ of the form

$$W(U,K) = \{z + \zeta \in \mathbb{C}^n : z \in U, \zeta \in K\}$$

is called a wedge with edge U.

A continuous function u on M is called wedge-extendable at $p \in M$ if there exists a wedge $W(U,K)$ which edge U is a neighbourhood of p on M and a continuous function on $M \cup W(U,K)$ which coincides with u on M and is analytic in $W(U,K)$.

The following proposition is well-known for C^2 replaced by $C^{2,\alpha}, \alpha > 0$, i.e. in case second order derivatives of the defining functions are Hoelder-continuous of order α.

Proposition 3.1. *Let M be a generic CR-manifold of class C^2 and of dimension $2n-2$ imbedded into $\mathbb{C}^n$. (So $e(M) = 2$.) Then the wedge extendability propagates along the CR-orbits of M. In other words, let u be a continuous CR-function on M and let $\gamma : [0,1] \to M$ be a CR-curve (of class C^2). Suppose u is wedge extendable at all points $\gamma(t)$, $t \in [0,1)$, (to wedges $W_{\gamma(t)}$). Then u is wedge extendable at $\gamma(1)$ to a wedge $W_{\gamma(1)}$ which depends only on M and on the wedges $W_{\gamma(t)}, t \in [0,1)$.*

Suppose M is as in Proposition 3.1. Let $p \in M$ and suppose the local orbit germ $\mathcal{O}^{loc}(M,p)$ at p has dimension $\dim_r M - 1 (= 2\dim_{CR} M + 1)$. Let N_p be a small representative of $\mathcal{O}^{loc}(M,p)$ and put $\mathbf{L}_p = T_pN_p + JT_pN_p$. The real line $l_p = JT_pN_p \ominus T_p^J N_p$ (the orthogonal complement of the complex tangent space $T_p^J N_p$ in the real linear subspace JT_pN_p of $T_p\mathbb{C}^n$ with the induced Euclidean structure) is not contained in T_pM. Indeed, $\mathbf{L}_p$ is invariant under the multiplication with the imaginary unit and has a dimension strictly greater than $T_p^J M$. For the proof of proposition 3.1 we will need the following lemma which will be proved below.

Lemma 3.1. *Suppose M is as in proposition 3.1, $p \in M$ and let as above N_p be a small representative of the local orbit germ $\mathcal{O}^{loc}(M,p)$. Suppose $\dim_r N_p = \dim_r M - 1$. Let H be a small hypersurface of class C^2 in $\mathbb{C}^n$ which contains a neighbourhood of p on M and is transverse to l_p. H divides suitable small neighbourhoods ω of p in $\mathbb{C}^n$ into two connected parts ω_1 and ω_2. For sufficiently small such neighbourhoods ω the following holds: At least for one of the ω_j there exists a (uniquely determined) connected analytic hypersurface X_j in $\mathbb{C}^n$ which is relatively closed in ω_j. Moreover, $\overline{X_j} \cap \omega$ is a C^1 manifold with boundary $N_p \cap \omega$ in ω. If M is contained in the boundary $\partial\Omega$ of a strictly pseudoconvex domain Ω then the analytic hypersurface is contained in Ω and relatively closed in $\Omega \cap \omega$.*

Proposition 3.1 may be proved by using the scheme of [Jö4] but in the C^2 case there arise some difficulties. So we will give the

Sketch of the proof. It consists of two steps. The first step is to deform the manifold by moving points of the edge into the wedge with the aim to produce minimal points. The second step consists in the application of Tumanov's theorem [Tu1] to manifolds which are close in the C^2 topology to a given one. For the proof of the results on removable singularities (see section 4) we need the result of each step separately.

We begin with the deformation of the generic manifold M of class C^2. Let p be a point of M and let $\theta \in T_p^J M$. We will use the notation of a CR-cone $C(p,\theta)$ at (p,θ) from [Jö4], i.e. $C(p,\theta)$ is a subset of M which is in suitable Euclidean coordinates φ on M an open truncated cone with vertex p and "symmetry axis" θ:

$$C(p,\theta) = \varphi(\mathcal{C}) \text{ with } \varphi(0) = p, \quad (d_0\varphi)\mathbf{e}_1 = \theta, \tag{3.1}$$

and

$$\mathcal{C} = \left\{ v \in \mathbb{R}^{d(M)} : |v| < a\ (v, \mathbf{e}_1),\ |v| < h \right\} \tag{3.2}$$

for positive constants a and h. Here $\mathsf{d}(M) = \dim_{\mathrm{r}} M$, $\mathbf{e}_1$ is the first coordinate vector in $\mathbb{R}^{\mathsf{d}(M)}$ and $(v, \mathbf{e}_1)$ denotes the scalar product of the two vectors v and $\mathbf{e}_1$ in $\mathbb{R}^{\mathsf{d}(M)}$. Euclidean coordinates φ in a neighbourhood of $\overline{C}$ on M for which (3.1) holds we will call adapted to $C(p,\theta)$. Consider first the case when M is contained in the boundary of a strictly pseudoconvex domain.

Lemma 3.2. *Let M be as in proposition 3.1. Suppose M is contained in the boundary $\partial\Omega$ of a bounded strictly pseudoconvex domain Ω with boundary of class C^2. Let p be a point of M which is not minimal. Denote by N_p a small representative of the local orbit germ $\mathcal{O}^{\mathrm{loc}}(M,p)$ (which is of real codimension one in M). Let C be a CR-cone at p and let φ be Euclidean coordinates on M adapted to C:*

$$\varphi(C) = \mathcal{C}$$

for a convex truncated cone $\mathcal{C}$ in $\mathbb{R}^{\mathsf{d}(M)}$ with vertex zero. Consider a non-negative function μ of class C^2 in $\mathbb{R}^{\mathsf{d}(M)}$ with support in $\overline{\mathcal{C}}$ and with sufficiently small C^2 norm which is strictly positive on $\mathcal{C}$ (compare with lemma 1 in [Jö4]*). Let Θ be a vector in $T_p\mathbb{C}^n$ ($T_p\mathbb{C}^n$ identified with $\mathbb{C}^n$) which is transverse to the real linear span of T_pM and l_p and directed into the inside of Ω.*

Define the function d on M,

$$(3.3)\qquad \begin{aligned} d(z) &= \mu\circ\varphi^{-1}(z)\cdot\Theta && \textit{for } z \textit{ close to } p,\\ d(z) &= 0 && \textit{away from } p. \end{aligned}$$

The deformed manifold M_d,

$$M_d = \{z + d(z) : z \in M\}$$

is of class C^2, is contained in $M \cup \Omega$ and p is a minimal point of M_d.

Proof of Lemma 3.2. It is clear that M_d is contained in $M \cup \Omega$. Suppose p is not minimal for M_d. Then the local orbit germ $\mathcal{O}^{\mathrm{loc}}(M_d,p)$ has real codimension one in M_d. Moreover, a small representative of it, $\mathcal{N}_p$, contains the intersection of $N_p\backslash\overline{C}$ with a small neighbourhood of p in $\mathbb{C}^n$. Indeed, all points of $N_p\backslash\overline{C}$ are minimal for N_p. It follows that p can be joined with any point q in $N_p\backslash\overline{C}$ by a piecewise CR-curve γ_q in M, γ_q contained in $(N_p\backslash\overline{C}) \cup \{0\}$. Since $(N_p\backslash\overline{C}) \cup \{0\}$ is contained in M_d, the curve γ_q is a piecewise CR-curve for M_d, too, and so any point q of $N_p\backslash\overline{C}$ is contained in $\mathcal{N}_p$.

Apply lemma 3.1 to M_d, $\mathcal{N}_p$ and the hypersurface $\partial\Omega_d \supset M_d$ obtained from $\partial\Omega$ by a suitable deformation, $\partial\Omega_d \subset \Omega \cup \partial\Omega$. Denote by Ω_d the domain bounded by $\partial\Omega_d$. If d is small enough, Ω_d is strictly pseudoconvex. From lemma 3.1 we get a connected relatively closed analytic hypersurface $\mathcal{X}_p$ in $\Omega_d \cap \omega$ with boundary $\mathcal{N}_p$ on $\partial\Omega_d$. (ω is a suitable neighbourhood of p in $\mathbb{C}^n$.) In the same way we get a connected relatively closed analytic hypersurface X_p in $\Omega\cap\omega$ (shrinking ω, if necessary) with boundary N_p on $\partial\Omega$. Note that near $N_p\backslash\overline{C}$ the two manifolds X_p and $\mathcal{X}_p$ coincide. To see this, apply the lemma 3.1 to a neighbourhood of each point q on N_p and on M. Since X_p is closed in $\Omega \cap \omega$ and $\mathcal{X}_p$ is connected and contained in $\Omega_d \cap \omega \subset \Omega \cap \omega$, we get

$$\mathcal{X}_p \subset X_p. \tag{3.4}$$

Indeed, otherwise there would exist a curve $\gamma : [0,1] \to \mathcal{X}_p$ with the following property: for each $t \in [0,1)$ a neighbourhood of $\gamma(t)$ on $\mathcal{X}_p$ is contained in X_p, but this property does not hold for $t = 1$. $\gamma(1)$ is contained in $\Omega_d \cap \omega \subset \Omega \cap \omega$, so in a small neighbourhood of $\gamma(1)$ the analytic manifold X_p is the zero set of an analytic function f which is defined in a neighbourhood V of $\gamma(1)$ in $\mathbb{C}^n$. Consider the analytic function $f|(\mathcal{X}_p \cap V)$. It vanishes in a neighbourhood of $\gamma([0,1)) \cap V$ on $\mathcal{X}_p$, so it vanishes identically. Hence a neighbourhood of $\gamma(1)$ on $\mathcal{X}_p$ is contained in X_p. The contradiction proves the inclusion (3.4).

The inclusion (3.4) implies now the following:

$$\mathcal{N}_p \subset (X_p \cup N_p) \cap \partial\Omega_d. \tag{3.5}$$

Now $\mathcal{N}_p$ is a CR-submanifold of M_d and the deformed cone $C_d = \{z + d(z) : z \in C\}$ is a CR-cone on M_d. So certain CR-curves on $\mathcal{N}_p$ with starting point p are contained in $\overline{C}_d$. Lemma 3.2 follows now immediately from the following claim.

Let $\tilde{X}_p$ be an arbitrary C^1 manifold which contains $X_p \cup N_p$. (Recall that $X_p \cup N_p$ is a manifold with boundary of class C^1 in $\overline{\Omega} \cap \omega$, thus a manifold $\tilde{X}_p$ with the described property always exists.)

Claim 3.1. *C_d does not intersect $\tilde{X}_p$.*

Proof of claim 3.1. Let $\mathcal{S}$ be a real hypersurface in $\mathbb{C}^n$ (i.e. a proper submanifold of $\mathbb{C}^n$ of real codimension one) of class C^1 which contains a neighbourhood of p on $\tilde{X}_p \cup M$. The existence of $\mathcal{S}$ follows easily from the fact, that $T_p\tilde{X}_p$ is spanned by $l_p = JT_pN_p \ominus T_p^J N_p$ and T_pN_p and the real line l_p is not contained in T_pM.

By the choice of Θ the vector Θ is transverse to $\mathcal{S}$. Since $C \subset M \subset \mathcal{S}$ and $C_d = \{z + \mu \circ \varphi^{-1}(z) \cdot \Theta : z \in C\}$ with $\mu \circ \varphi^{-1}$ positive on C, the cone C_d does not intersect $\mathcal{S}$ if μ is small. But $\mathcal{S}$ contains a neighbourhood of p on $\tilde{X}_p$ and the claim is proved. □

For arbitrary manifolds M which satisfy the conditions of proposition 3.1 we use the following two lemmas instead of lemma 3.2.

Lemma 3.3. *Suppose M is as in proposition 3.1 and let $p \in M$. Suppose through p passes a CR-submanifold Γ of M of real codimension 2 in M, i.e. an analytic manifold of complex dimension $\dim_{CR} M$ contained in M. Let C be a small CR-cone on M with vertex p, let φ be Euclidean coordinates adapted to C and let μ be a function as in lemma 3.2 for $C = \varphi^{-1}(C)$. Moreover, let Θ be a vector not contained in T_pM.*

Then the local orbit germ $\mathcal{O}^{loc}(M_d, p)$ *of the deformed manifold*

$$M_d = \{z + d(z) : z \in M\},$$

$$d(z) = \mu \circ \varphi^{-1}(z) \cdot \Theta \text{ for } z \text{ close to } p \text{ and } d(z) = 0 \text{ away from } p,$$

has real dimension at least $2\dim_{CR} M + 1$.

We recall a proof (see also [Jö4]).

Proof. If there would exist a CR-submanifold Γ_d of M which is an analytic manifold and contains p then, by the definition of CR-orbits, Γ_d would contain a large part of Γ near p. So by the uniqueness theorems for analytic manifolds it must contain a neighbourhood of p on Γ which is impossible since the deformation moves the points of Γ, which are contained in the CR-cone C, out off Γ. □

Lemma 3.4. *Let M be a generic CR-manifold of class C^2 of real codimension 2 imbedded into $\mathbb{C}^n$, and let p be a point of M. Let $\gamma : [0,1] \to M$ be a small CR-curve on M with $\gamma(1) = p$. Suppose there is an increasing sequence of numbers $t_k \in (0,1)$ such that $t_k \to 1$ and the local orbit germs $\mathcal{O}^{loc}(M, \gamma(t_k))$ have dimension $2\dim_{CR} M + 1 (= \dim_r M - 1)$. Suppose*

$$\dim_r \mathcal{O}^{loc}(M, p) = 2\dim_{CR} M + 1.$$

(It is clear that $\dim_r \mathcal{O}^{loc}(M,p) \geq 2\dim_{CR} M + 1$. The condition excludes that p is a minimal point.)

Let N_p be a small representative of $\mathcal{O}^{loc}(M,p)$ and let $l_p = JT_pN_p \ominus T_p^J N_p$. Suppose C is a small CR-cone on $M \cap \omega$ at p which contains $\gamma([\tau, 1))$ for some $\tau \in (0,1)$. Let Θ be a real vector in $T_p\mathbb{C}^n \simeq \mathbb{C}^n$ which is transverse to the real linear span of T_pM and l_p. Let φ be Euclidean coordinates on M near p adapted to C. There exists a non-negative function μ of class C^2 in $\mathbb{R}^{d(M)}$ with sufficiently small C^2 norm with support in $\mathcal{C} = \varphi^{-1}(C)$ such that p is a minimal point of the deformed CR-manifold

$$M_d = \{z + d(z) : z \in M\}, \tag{3.6}$$

where

$$d(z) = \mu \circ \varphi^{-1}(z) \cdot \Theta \tag{3.7}$$

for z close to p and $d(z) = 0$ away from p.

Proof. Let C_1 be a CR-cone on M at p which is contained in C and does not intersect $\gamma([\tau, 1))$, say $C_1 = \varphi(\mathcal{C}_1)$ for an open truncated cone in $\mathbb{R}^{d(M)}$ with vertex zero, $\mathcal{C}_1$ contained in $\mathcal{C} = \varphi^{-1}(C)$ and $\mathcal{C}_1$ does not intersect $\varphi^{-1}(\gamma[\tau, 1))$. Since $(\varphi^{-1} \circ \gamma)'(1-)$ is a vector contained in the cone $\mathcal{C}$ such a cone $\mathcal{C}_1$ always exists. Let μ be a function as in lemma 3.2 associated to the cone $\mathcal{C}_1$, supp $\mu = \overline{\mathcal{C}}_1$, $\mu > 0$ on $\mathcal{C}_1$.

For the just defined function μ let d be defined by (3.7) and let M_d be defined by (3.6). *Suppose p is not a minimal point of the obtained manifold M_d.* Denote by $\mathcal{N}_p$ a small representative of the local orbit germ $\mathcal{O}^{\mathrm{loc}}(M_d, p)$. Since $\tau([0,1]) \subset M_d$ the manifold $\mathcal{N}_p$ has dimension $\dim_r M - 1$. The key of the proof consists of applying several times the lemma 3.1. Apply it first to N_p and a small hypersurface $\mathcal{H}$ which is transverse to l_p and contains M. Let ω be a small neighbourhood of p in $\mathbb{C}^n$ with $\omega \backslash \mathcal{H}$ consisting of two connected components ω_1 and ω_2. We get a connected relatively closed analytic manifold X_p in one of the components, say in ω_1, such that N_p is the boundary part of X_p which is contained in H. Set $q_k = \gamma(t_k)$ and fix small representatives N_{q_k} of $\mathcal{O}^{\mathrm{loc}}(M, q_k)$ which are contained in $C \backslash C_1$. Apply for all sufficiently large k the lemma 3.1 to N_{q_k} and the hypersurface H. We get analytic manifolds X_{q_k} contained either in ω_1 or in ω_2 such that the part of the boundary of X_{q_k} which is contained in H is equal to N_{q_k}.

Consider now a hypersurface H_d associated to d, which contains the deformed manifold M_d and is obtained from H by moving a part of H into ω_1 if Θ is directed into ω_1, and into ω_2 if Θ is directed into ω_2. Let ω_1^d and ω_2^d be the corresponding connected components of $\omega \backslash H_d$. Apply the lemma 3.1 once more, now to $\mathcal{N}_p$ and H_d, we get a connected analytic manifold $\mathcal{X}_p$ which is a relatively closed subset of one of the ω_j^d and has boundary $\mathcal{N}_p$ on H_d. *First we will deal with the case when Θ is directed into ω_1, i.e. $\omega_1^d \subset \omega_1$.*

Claim 3.2. *If $\omega_1^d \subset \omega_1$ and p is not minimal for M_d then either for a sequence $k_j \to \infty$ the manifolds $X_{q_{kj}}$ are contained in ω_2^d or $\mathcal{X}_p$ is contained in ω_2^d. Moreover, there exist analytic hypersurfaces $X_{q_{k_j}}^{\circ}$ in $\mathbb{C}^n$, which contain the representatives $N_{q_{k_j}}$ of $\mathcal{O}^{loc}(M_d, q_k) = \mathcal{O}^{loc}(M, q_k)$.*

Proof. Suppose for all sufficiently large k the manifolds X_{q_k} are contained in ω_1 and $\mathcal{X}_p$ is contained in ω_1^d. Then (possibly, after shrinking the X_{q_k} slightly) we have

$$X_p \cap \mathcal{X}_p \supset X_{q_k} \tag{3.8}$$

for all sufficiently large k (since M_d coincides with M near the CR-curve $\gamma([0,1))$, for large k the q_k are in N_p and in $\mathcal{N}_p$ as well, and N_p coincides with $\mathcal{N}_p$ near the q_k). Since, moreover, $\omega_1^d \subset \omega_1$

$$\mathcal{X}_p \subset X_p. \tag{3.9}$$

But this is impossible (compare with the proof of lemma 3.2 and claim 3.1). The contradiction proves the first part of the claim.

Suppose now that either $\mathcal{X}_p$ is contained in ω_2^d or for a sequence $k_j \to \infty$ the manifolds $X_{q_{k_j}}$ are contained in ω_2. In both cases we get for a sequence $k_j \to \infty$ two analytic hypersurfaces $X_{q_{k_j}}$ and $X'_{q_{k_j}}$, $X'_{q_{k_j}} \subset X_p \subset \omega_1$ and $X_{q_{k_j}} \subset \omega_2$, such that both analytic hypersurfaces have the same part of the boundary, $N_{q_{k_j}}$, which is contained in H. Since $X_{q_{k_j}} \cup N_{q_{k_j}}$ and $X'_{q_{k_j}} \cup N_{q_{k_j}}$ are of class C^1 this means that $N_{q_{k_j}}$ is contained in an analytic hypersurface $X^\circ_{q_{k_j}} = X_{q_{k_j}} \cup X'_{q_{k_j}} \cup N_{q_{k_j}}$. We may suppose that $X^\circ_{q_{k_j}}$ is relatively closed in a neighbourhood U_j of q_{k_j} in $\mathbb{C}^n$. The claim 3.2 is proved completely. □

We have to consider the remaining case, *when* Θ *is directed into* ω_2, i.e. $\omega_1 \subset \omega_1^d$. In this case we start with the deformed manifold M_d instead of M. Fix a small number $\alpha \in (0,1)$. The vector $(-\alpha\Theta)$ is directed into ω_1^d. Consider a deformation of M_d defined by d',

$$d'(z + d(z)) = \{z + d(z)\} - \alpha \cdot d(z) \cdot \Theta, \quad z + d(z) \in M_d.$$

In other words

$$(M_d)_{d'} = M_{(1-\alpha)d}.$$

Apply the claim 3.2 to $(M_d)_{d'}$ and $(\omega_j^d)^{d'}, j = 1,2$, instead of M_d and ω_j^d. As in the previous case we get that if p is not minimal for $(M_d)_{d'} = M_{(1-\alpha)d}$ then there exists a sequence of analytic hypersurfaces $X^\circ_{q_{k_j}}$ containing the $N_{q_{k_j}}$.

For producing a minimal point of the deformed manifold we have to consider only the case that p is not minimal for M_d and for $M_{(1-\alpha)d}$, too. We will use now the existence of the analytic manifolds $X^\circ_{q_{k_j}}$ and make another deformation such that for the manifold $M_{\tilde{d}}$ obtained in this way p is a minimal point.

Let $\nu_{k_j} : [0,1] \to N_{q_{k_j}}$ be small CR-curves with starting point $\nu_{k_j}(0) = q_{k_j}$ and $\nu'_{k_j}(0) = i\gamma'(t_k)$. Note that $\nu'_{k_j}(\frac{1}{2}) \in T^J_{\nu_{k_j}(\frac{1}{2})}(M)$ and let C_{k_j} be small disjoint CR-cones at $(\nu_{k_j}(\frac{1}{2}), \nu'_{k_j}(\frac{1}{2}))$, which are contained in $U_j \cap C$ and do not intersect γ (see Fig. 1). We may suppose that $C_{k_j} = \varphi(\mathcal{C}_{k_j})$ for open convex truncated cones $\mathcal{C}_{k_j}$ in $\mathbb{R}^{d(M)}$ with vertex $\varphi^{-1}(\nu_{k_j}(\frac{1}{2}))$. Let μ_j be non-negative C^2 functions in $\mathbb{R}^{d(M)}$ with support in $\overline{\mathcal{C}}_{k_j}$, which are positive in $\mathcal{C}_{k_j}$ and such that the sequence $\{||\mu_j||_{C^2}\}$ of the C^2 norms is summable and its sum is small enough. Put

$$\tilde{\mu} = \sum_{j=1}^{\infty} \mu_j, \tag{3.10}$$

and prove that with this choice of $\tilde{\mu}$ the point p is a minimal point of the deformed manifold $M_{\tilde{d}}$ defined by (3.6) and (3.7) with d and μ replaced by $\tilde{d}$ and $\tilde{\mu}$.

Indeed, suppose not. Denote by $\widetilde{N}_p$ a small representative of the local orbit germ $\mathcal{O}^{loc}(M_{\tilde{d}}, p)$. Let $\widetilde{H}_d$ be the hypersurface obtained by a small deformation of H, the deformation associated to $\tilde{d}$ with the choice of $\tilde{\mu}$ described by (3.10). Apply lemma 3.1 to $\widetilde{N}_p$

and $\widetilde{H}_d$. We get a connected analytic hypersurface $\tilde{\mathcal{X}}_p$ contained and relatively closed in one of the connected components $\widetilde{\omega}_j^d$ of $\omega_p \backslash \widetilde{H}_d$, such that the boundary part of $\widetilde{\mathcal{X}}_p$ which is contained in $\widetilde{H}_d$ is equal to $\widetilde{\mathcal{N}}_p$. For sufficiently large j $\widetilde{\mathcal{N}}_p$ contains a neighbourhood of q_{k_j} on $N_{q_{k_j}}$ (since $\widetilde{\mathcal{N}}_p$ contains q_{k_j} for large j, q_{k_j} is minimal for N_p and $M = M_{\tilde{d}}$ near q_{k_j}). Therefore small parts of $\tilde{\mathcal{X}}_p$ contained in small neighbourhoods of the q_{k_j}, are contained in $X^{\circ}_{q_{k_j}}$. It follows that the whole connected part of $\tilde{\mathcal{X}}_p \cap U_j$ which contains q_{k_j} in its boundary is contained in $X^{\circ}_{q_{k_j}}$. But this is not possible by the choice of the vector Θ. (The $(C_d)_{k_j}$ contain certain CR-curves in $\widetilde{\mathcal{N}}_p$, but they do not intersect $X^{\circ}_{q_{k_j}}$.) The argument is as in claim 3.1. Lemma 3.4 is proved completely. □

We will come now to the second step of the proof of proposition 3.1.

In the following lemma we will state Tumanov's result [Tu1] in a form which is convenient for us. We will not give a proof here, although, may be, it would be desirable to give slightly more details then in [Tu1]. For working out more details one may consult [Tu2] or [Jö4].

Lemma 3.5. *Let M be a generic CR-manifold of class C^2 imbedded into $\mathbb{C}^n$ with $p \in M$ a minimal point. There exist two neighbourhoods U and U' of p on M, $\overline{U} \subset U'$, U' with compact closure in M, and an open truncated cone K in $\mathbb{C}^n$ with vertex zero such that the wedge $W(U,K)$ is contained in the polynomially convex hull $\widehat{\overline{U'}}$*

$$\widehat{\overline{U'}} = \left\{ z \in \mathbb{C}^n : |\mathcal{P}(z)| \le \max_{\overline{U'}} |\mathcal{P}| \text{ for all polynomials } \mathcal{P} \right\}$$

of $\overline{U'}$.

Moreover, let M_1 be close enough to M in the C^2 topology and U_1, U_1' and $\overline{U}_1'$ be close enough to U, U' and $\overline{U'}$. (Say, there exists a C^2 diffeomorphism Φ of M onto M_1 such that Φ-id has small C^2 norm on a large compact set containing $\overline{U'}$, and $\Phi(U) = U_1$, $\Phi(U') = U_1'$. Here id is the identity mapping on M.) Then the wedge $W(U_1, K)$ is contained in the polynomially convex hull $\widehat{\overline{U}_1'}$ of $\overline{U}_1'$.

We need also the approximation theorem of Baouendi and Treves which we state in the following form.

Lemma 3.6. *a) Let M be a generic CR-manifold of class C^2 imbedded into $\mathbb{C}^n$, $p \in M$. There exist two neighbourhoods of p on M, $'U$ and $''U$, such that each continuous CR-function on $''U$ can be uniformly approximated by polynomials on $'U$. Moreover, if M_1, $'U_1$ and $''U_1$ are close enough in the C^2 topology to M, $'U$ and $''U$ (see lemma 3.5 for the exact meaning of this) then each continuous CR-function on $''U_1$ can be uniformly approximated on $'U_1$ by polynomials.*

b) Moreover, if M is only of class C^1 (instead of class C^2), $p \in M$, then there are two neighbourhoods $'U$ and $''U$ of p on M, such that each CR-function **of class $\mathbf{C}^1$** *on $''U$ can be uniformly approximated on $'U$ by polynomials.*

c) Let M, $'U$ and $''U$ be as in a) and let p be a real number, $1 \leq \mathsf{p} < \infty$. Then each function in $L^{\mathsf{p}}(''U)$ which satisfies the tangential Cauchy-Riemann equations (in the weak sense) can be approximated by polynomials in $L^{\mathsf{p}}('U)$. As in a) the size of the two neighbourhoods does not change essentially for manifolds M_1 which are C^2 close to M.

The lemmas imply the proposition 3.1. Indeed, it follows from the lemmas 3.2, 3.3 and 3.4 that if M is as in proposition 3.1, $p \in M$, W_p is a wedge with edge a neighbourhood of p on M, and C is a small CR-cone on M at p then one can obtain a small C^2 deformation M_d of M, $M_d \subset M \cup W_p$ which lets fixed $M \backslash C$ such that p is a minimal point of M_d. Let now $\{t_k\}_{k=1}^{\infty}$ be an arbitrary increasing sequence of points from $(0,1)$ which tend to 1. Choose small disjoint CR-cones C_k at $(\gamma(t_k), i\gamma'(t_k))$ and small C^2 functions d_k on M with support in $\overline{C_k}$ such that $M_{d_k} \subset M \cup W_{\gamma(t_k)}$ and $\gamma(t_k)$ is a minimal point of M_{d_k}. Moreover, the d_k may be choosen so small that the sum of the C^2 norms of the d_k is small. Set $d = \sum_{k=1}^{\infty} d_k$. The manifold $M_d = \{z + d(z) : z \in M\}$ is of class C^2 and $\gamma(1)$ is a minimal point of M_d (since for each small neighbourhood $\mathcal{U}$ of p on M_d the point $\gamma(t_k)$ is in the orbit $\mathcal{O}(\mathcal{U}, p)$ if k is large enough and $\gamma(t_k)$ is minimal). If k_0 is large enough the manifolds M_d and $M_{d'}$, $d' \overset{\text{def}}{=} \sum_{k=1}^{k_0} d_k$ are close in the C^2 topology, so by the lemmas 3.5 and 3.6 there exists a neighbourhood U of $\gamma(1)$ on M and a cone K in $\mathbb{C}^n$ such that each continuous CR-function on $M_{d'}$ extends to an analytic function in the wedge $W(U_{d'}, K)$. For each continuous CR-function on M, which is wedge-extendable to the $W_{\gamma(t)}$, $t \in [0,1)$, there exists a uniquely determined continuous CR-function on $M_{d'}$ which coincides on $M_{d'} \cap M$ with the previous one. Moreover, $U_{d'}$ contains a neighbourhood U' of $\gamma(1)$ on M, so each continuous CR-function on M, which is wedge-extendable to the $W_{\gamma(t)}$, $t \in [0,1)$, is wedge-extendable to the wedge $W_{\gamma(1)} \overset{\text{def}}{=} W(U', K)$. Proposition 3.1 is proved. $\square$

It remains to prove the lemmas 3.1 and 3.6. Start with lemma 3.6. Part a) is known (for some details see also [Jö4]). Part b) will be used in the proof of lemma 3.1. We will sketch the proof.

Sketch of the proof of lemma 3.6.b). We may assume (possibly after a complex linear change of coordinates) that $p = 0$ and in a neighbourhood of zero M is given by the equations

$$t_s = h_s(x_1 + it_1, \ldots, x_m + it_m, x_{m+1}, \ldots, x_n) (= h_s(t, x)), \; s = m+1, \ldots, n. \tag{3.11}$$

Here m is the CR-dimension of M, $z_s = x_s + it_s, s = 1, \dots, n$, are the coordinate functions in $\mathbb{C}^n$ and h_s are C^1 functions such that $h_s(0) = 0, (\nabla h_s)(0) = 0$. ($\nabla h_s$ denotes the gradient of the function h_s.)

We will use the scheme of the proof of [Ba–Tr]. The only thing we have to do, is to check that the smoothness assumptions are sufficient. For this aim we approximate the functions h_s in the C^1-topology by C^∞ functions $(h_s)_l$, $l = 1, 2, \dots$, and consider the CR-manifolds M_l defined by the $(h_s)_l$ instead of h_s. Let

$$(\bar{\partial}_j)_\tau = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=m+1}^{n} \alpha_j^k \cdot \frac{\partial}{\partial \bar{z}_k}, \qquad j = 1, \dots, m,$$

be a complete system of tangential Cauchy-Riemann operators for M (i.e. $(\bar{\partial}_j)_\tau(t_s - h_s(t, x)) = 0,\ s = m+1, \dots, n$) and let

$$L_j = \frac{\partial}{\partial t_j} + \sum_{k=1}^{n} \lambda_j^k(t, x) \frac{\partial}{\partial x_k}, \qquad j = 1, \dots, m, \tag{3.12}$$

be a corresponding system in coordinates (t, x) ($x = (x_1, \dots, x_n)$, $t = (t_1, \dots, t_m)$) on M (i.e. $L_j f_1 = 0, j = 1, \dots, m$, for each C^1 function f_1 in a neighbourhood of M which is the continuation not depending on $t_{m+1}, \dots, t_n$ of a CR-function f of class C^1 on M).Let $(L_j)_l$ be the corresponding operators for M_l. It is clear that the coefficients $(\lambda_j^k)_l$ are of class C^∞ and converge uniformly to the continuous functions λ_j^k.

Consider now as in [Ba–Tr] the determinants

$$\begin{aligned} \Delta_l(t, x) &= \det\left(\frac{\partial z^{(l)}}{\partial x}(t, x)\right), \\ \Delta(t, x) &= \det\left(\frac{\partial z}{\partial x}(t, x)\right), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} z^{(l)} &= \left(x_1 + it_1, \dots, x_m + it_m, x_{m+1} + ih_{m+1}^{(l)}(t, x), \dots, x_n + ih_n^{(l)}(t, x)\right) \\ z &= (x_1 + it_1, \dots, x_m + it_m, x_{m+1} + ih_{m+1}(t, x), \dots, x_n + ih_n(t, x)). \end{aligned} \tag{3.14}$$

The Δ_l are of class C^∞ and converge uniformly to Δ. By lemma 2.1 of [Ba–Tr]

$$(L_j)_l^t \Delta_l = 0, \qquad j = 1, \dots, m;\ l = 1, 2, \dots. \tag{3.15}$$

Now we can follow the scheme of proof in [Ba–Tr]. For each C^1 function Ψ_l and the same j and l as above

$$(3.16)\qquad (L_j)_l^t(\Psi_l\cdot\Delta_l)=\Psi_l\cdot(L_j)_l^t\Delta_l+\Delta_l\cdot(L_j)_l\Psi_l=\Delta_l\cdot(L_j)_l\Psi_l.$$

We do not require that $(L_j)_l\Psi_l=0$. Write equations similar to (2.8) and (2.9) in [Ba–Tr] with h replaced by Ψ_l, Δ replaced by Δ_l and the coefficients λ_k^j replaced by $(\lambda_k^j)_l$. The equations in our case differ from those in [Ba–Tr] by terms which come from integrating terms of the form $(g\cdot(L_j)_l\Psi_l\cdot\Delta_l)$ (g is as in [Ba–Tr] but supposed to be only of class C^1.) Take, similarly as in [Ba–Tr]

$$(3.17)\qquad \Psi_l(s,y)=(E_\nu)_l(t,x;s,y)u(s,y)$$

where u is a CR-function on M in coordinates (s,y), and

$$(3.18)\qquad (E_\nu)_l(t,x;s,y)=\left(\frac{\nu}{\sqrt{\pi}}\right)^n\exp\left\{-\nu^2\left(\sum_{k=1}^n(z_k^{(l)}(t,x)-z_k^{(l)}(s,y))^2\right)\right\}$$

(see (3.14). We get a formula which is similar to (2.12) in [Ba–Tr] (with the replacements as above), but on the right we have to add the term

$$(3.19)\qquad -\int_{\gamma(t)}\left\{\sum_{j=1}^m\int_{\mathbf{R}^n}g(y)\Delta_l(s,y)(E_\nu)_l(t,x;s,y)\left((L_j)_lu\right)(s,y)dy\right\}ds_j.$$

In the formula obtained in this way only first order differentiation of g and u appear, both functions being of class C^1. So in this formula we let l tend to infinity and use that $L_ju=0$ ($j=1,\dots,m$). We get exactly formula (2.12) of [Ba–Tr] for the manifold M. The remaining part of the proof of the approximation theorem in [Ba–Tr] uses only the fact that M is of class C^1 and goes through without changes. Lemma 3.6 b) is proved. □

Sketch of the proof of Lemma 3.6.c. Let M be a generic CR-manifold imbedded into $\mathbb{C}^n$ of real dimension $m+n$ and let L_j, $j=1,\dots,m$, be a complete system of tangential Cauchy Riemann operators which are given by (3.12) in certain coordinates on M. Let z and $\Delta(t,x)$ be defined by (3.13) and (3.14). Let $u=u(t,x)$ be a function in L^p in a neighbourhood of zero in $\mathbb{R}^n\times\mathbb{R}^m$ for which

$$L_ju=0,\qquad j=1,\dots,m$$

in the weak sense. Put as in [Ba–Tr]

$$(3.20)\qquad E_\nu(t,x;s,y)=\left(\frac{\nu}{\sqrt{\pi}}\right)^n\exp\left\{-\nu^2\left(\sum_{k=1}^n(z_k(t,x)-z_k(s,y))^2\right)\right\},$$

and

$$\Psi_\nu(s,y)\,(=\Psi_\nu(t,x;s,y)) = E_\nu(t,x;s,y)\cdot u(s,y). \tag{3.21}$$

The function

$$\mathcal{G}(s)\,(=\mathcal{G}_\nu(t,x;s)) = \int_{\mathbf{R}^n} g(y)\cdot\Psi_\nu(s,y)\cdot\Delta(s,y)dy, \tag{3.22}$$

is in $L^p(Q_t)$ for a cubic neighbourhood Q_t of zero in $\mathbb{R}^m$, $Q_t = (-\sigma,\sigma)^m$ for some small $\sigma > 0$. Here g is a C^2 function with compact support in a neighbourhood $Q_x = (-\delta,\delta)^n$ of zero in $\mathbb{R}^n$ (δ is some small positive number), $g \equiv 1$ in a smaller neighbourhood , say $\frac{1}{2}Q_x \overset{\text{def}}{=} (-\frac{\delta}{2}, -\frac{\delta}{2})^n$, of the origin. The functions

$$\mathcal{F}_j(s)\,(=(\mathcal{F}_j)_\nu(t,x;s)) = \int_{\mathbf{R}^n}\Psi_\nu(s,y)\cdot\Delta(s,y)\cdot\left[\sum_{k=1}^{n}\lambda_j^k(s,y)\frac{\partial}{\partial y^k}g(y)\right]dy, \tag{3.23}$$

$j = 1,\ldots,m$, are also in $L^p(Q_t)$. As in [Ba–Tr] we have

$$\mathcal{F}_j(s) = \frac{\partial}{\partial s_j}\mathcal{G}(s), \qquad j = 1,\ldots,m, \tag{3.24}$$

in the weak sense on Q_t (say, as functionals on compactly supported C^2 functions in Q_t). Take convolutions $\mathcal{G}^\varepsilon = \mathcal{G} * \chi_\varepsilon$, $\mathcal{F}_j^\varepsilon = \mathcal{F}_j * \chi_\varepsilon$ for a compactly supported C^2 function χ in Q_t with $\int_{Q_t}\chi = 1$ and $\chi_\varepsilon(s) = \frac{1}{\varepsilon^m}\chi(\frac{s}{\varepsilon})$. Then

$$\begin{aligned}&\mathcal{F}_j^\varepsilon \to \mathcal{F}_j \text{ in } L^p \text{ for } \varepsilon \to 0, \quad j = 1,\ldots,m\\ &\mathcal{G}^\varepsilon \to \mathcal{G} \text{ in } L^p \text{ for } \varepsilon \to 0,\end{aligned} \tag{3.25}$$

and the formula (3.24) holds pointwise for $\mathcal{F}_j$ replaced by $\mathcal{F}_j^\varepsilon$ and $\mathcal{G}$ replaced by $\mathcal{G}^\varepsilon$ for some $\varepsilon > 0$:

$$\mathcal{F}_j^\varepsilon(s) = \frac{\partial}{\partial s_j}\mathcal{G}^\varepsilon(s). \tag{3.26}$$

Write for the point $t \in \mathbb{R}^m$ which we took in (3.20)

$$t = (t_1,\ldots,t_m),$$

and let

$$T = (T_1,\ldots,T_m) \in \mathbb{R}^m$$

be close to zero.

From (3.26) we get for t and T in Q_t

$$\mathcal{G}^\varepsilon(t_1,\ldots,t_m) - \mathcal{G}^\varepsilon(T_1,\ldots,T_m) = \sum_{j=1}^{m} \int\limits_{I_j} \mathcal{F}_j^\varepsilon(s)ds_j, \tag{3.27}$$

where

$$I_j = \{T_1\} \times \cdots \times \{T_{j-1}\} \times (T_j, t_j) \times \{t_{j+1}\} \times \cdots \times \{t_m\}$$

if $T_j < t_j$ and

$$I_j = \{T_1\} \times \cdots \times \{T_{j-1}\} \times (t_j, T_j) \times \{t_{j+1}\} \times \cdots \times \{t_m\}.$$

if $T_j > t_j$. The term is supposed to be zero if $T_j = t_j$.

Integrate (3.27) with respect to T in the cube $(-\tau,\tau)^m$ in $\mathbb{R}^m$ for some small positive τ $(\tau < \sigma)$:

$$(2\tau)^m \mathcal{G}^\varepsilon(t_1,\ldots,t_m) - \int\limits_{(-\tau,\tau)^m} \mathcal{G}^\varepsilon = \sum_{j=1}^{m} (2\tau)^{m-j} \int\limits_{J_j} \rho_j(s_j) \cdot \mathcal{F}_j^\varepsilon(s) ds_1 \ldots ds_j, \tag{3.28}$$

where

$$J_j = (-\tau,\tau)^{j-1} \times (a_j, b_j) \times \{t_{j+1}\} \times \cdots \times \{t_m\}, \quad a_j = \min(-\tau, t_j), \quad b_j = \max(\tau, t_j),$$

and ρ_j is a non-negative bounded function on the interval (a_j, b_j).

Recall that $\mathcal{G}^\varepsilon(t) = \mathcal{G}^\varepsilon(t,x;t)$, $\mathcal{F}_j^\varepsilon(t) = \mathcal{F}_j^\varepsilon(t,x;t)$. It is now easy to see that one can go to the limit in (3.28) for $\varepsilon \to 0$ in the space $L^p(Q_t \times \frac{1}{2}Q_x)$. Hence,

$$(2\tau)^m \mathcal{G}(t_1,\ldots,t_m) - \int\limits_{(-\tau,\tau)^m} \mathcal{G} = \sum_{j=1}^{m} (2\tau)^{m-j} \int\limits_{J_j} \rho_j(s_j) \cdot \mathcal{F}_j(s) ds_1, \ldots, ds_j. \tag{3.29}$$

Prove now that for $\nu \to \infty$ each term on the right of (3.29) tends to zero in $L^p(Q_t \times \frac{1}{4}Q_x)$, provided, after fixing the small positive number δ and the C^2 function g the positive number σ is choosen small enough. The j-th term on the right in (3.29) is equal to

(3.30)

$$(2\tau)^{m-j} \int\limits_{(-\tau,\tau)^{j-1}\times(a_j,b_j)} \rho_j(s_j) ds_1, \ldots, ds_j \Big(\int\limits_{\mathbb{R}^n} dy E_\nu(t,x; s_{t,j}, y) \cdot u(s_{t,j}, y) \cdot \Delta(s_{t,j}, y) \times \\ \times \Big[\sum_{k=1}^{n} \lambda_j^k(s_{t,j}, y) \frac{\partial}{\partial y_k} g(y) \Big] \Big).$$

Here $s_{t,j} = (s_1, \dots, s_j, t_{j+1}, \dots, t_m)$. The integration with respect to y is over the set $Q_x \backslash \frac{1}{2}Q_x$. For $x \in \frac{1}{4}Q_x$ the function $E_\nu(t,x;s_{t,j},y)$ is estimated as in [Ba–Tr], using (3.20) and (3.14):

$$(3.31) \quad \sum_{k=1}^{n} (z_k(t,x) - z_k(s_{t,j},y))^2 =$$

$$\sum_{k=1}^{j} ((x_k - y_k) + i(t_k - s_k))^2 + \sum_{k=j+1}^{m} (x_k - y_k)^2 + \sum_{k=m+1}^{n} ((x_k - y_k) + i(h_k(t,x) - h_k(s_{t,j},y))^2 .$$

Since $(\nabla h_k)(0) = 0$ and the Euclidean distance from $\frac{1}{4}\overline{Q}_x$ to $\overline{Q}_x \backslash \frac{1}{2}Q_x$ is equal to $\frac{1}{4}\delta$, for small $\delta > 0$ and small enough $\sigma > 0$ (σ depending on δ)

$$(3.32) \qquad |E_\nu(t,x;s_{t,j},y)| \le \left(\frac{\nu}{\sqrt{\pi}}\right)^n \exp\left(-\nu^2 \frac{\delta^2}{32}\right).$$

It follows that the right hand side of (3.29) for $\nu \to \infty$ tends to zero in $L^p(Q_t \times \frac{1}{4}Q_x)$.

Note that for each fixed ν the term

$$(3.33) \qquad \int_{(-\tau,\tau)^m} \mathcal{G}_\nu(t,x;s)ds$$

is an entire function in $z = z(t,x)$ (see (3.20), (3.21) and (3.22)). So, it remains to prove that $\mathcal{G}_\nu(t,x;t)$ tends to $u(t,x)$ in $L^p(Q_t \times \frac{1}{8}Q_x)$.

Write as in [Ba–Tr] the formula (3.14) in the form

$$z = x + i\phi(t,x),$$

and make the change of variables $y \to x - \frac{y}{\nu}$, we get

$$(3.34) \quad \mathcal{G}_\nu(t,x;t) = \pi^{-\frac{n}{2}} \int_{\mathbf{R}^n} \exp\left\{ - \sum_{k=1}^{n} (y_k + i\nu(\phi_k(t,x) - \phi_k(t,x - \tfrac{y}{\nu})) \right\}^2 \cdot \mathcal{K}(t, x - \tfrac{y}{\nu})dy,$$

where

$$(3.35) \qquad \mathcal{K}(t,y) = g(y) \cdot u(t,y) \cdot \Delta(t,y).$$

For $\nu \to \infty$ the integral

(3.36)

$$(\pi)^{-\frac{n}{2}} \int\limits_{\mathbf{R}^n} \exp\left\{ -\sum_{k=1}^{n} \left(y_k + i\nu(\phi_k(t,x) - \phi_k(t, x - \tfrac{y}{\nu}))\right)^2 \right\} \cdot \Delta(t, x - \tfrac{y}{\nu}) \cdot \chi_{\text{supp}g}(x - \frac{y}{\nu}) dy$$

tends to 1 uniformly for $(t,x) \in Q_t \times \frac{1}{4}Q_x$ (see [Ba–Tr]). Here $\chi_{\text{supp}g}$ is the function which equals one on supp$g$ and zero elsewhere. Denote the function under the integral sign in (3.36) by $\mathcal{L}_\nu(x,t;y)$. It is now enough to show that with $v = g \cdot u$

$$\int\limits_{\mathbf{R}^n} \mathcal{L}_\nu(x,t;y) \cdot \left\{ v(t, x - \tfrac{y}{\nu}) - v(t,x) \right\} dy = \int\limits_{|y|<C} + \int\limits_{|y|>C} = \mathcal{J}_1^{(\nu)} + \mathcal{J}_2^{(\nu)}, \tag{3.37}$$

considered as a function of (t,x), tends to zero in $L^{\mathrm{p}}(Q_t \times \frac{1}{4}Q_x)$. Estimate the L^{p} norm of $\mathcal{J}_1^{(\nu)}$, using Hoelder's inequality (p' is conjugate to p).

$$\begin{aligned}
&\int\limits_{Q_t \times \frac{1}{4}Q_x} dtdx \left| \int\limits_{|y|<C} dy \mathcal{L}_\nu(x,t;y) \cdot \left\{ v(t, x - \tfrac{y}{\nu}) - v(t,x) \right\} \right|^{\mathrm{p}} \le \\
&\int\limits_{Q_t \times \frac{1}{4}Q_x} dtdx \left(\int\limits_{|y|<C} |\mathcal{L}_\nu(x,t;y)|^{\mathrm{p}'} dy \right)^{\frac{\mathrm{p}}{\mathrm{p}'}} \int\limits_{|y|<C} \left| v(t, x - \tfrac{y}{\nu}) - v(t,x) \right|^{\mathrm{p}} dy \le \\
&\text{const} \int\limits_{|y|<C} dy \int\limits_{Q_t \times \frac{1}{4}Q_x} dtdx \left| v(t, x - \tfrac{y}{\nu}) - v(t,x) \right|^{\mathrm{p}}.
\end{aligned} \tag{3.38}$$

This implies that $\mathcal{J}_1^{(\nu)}$ tends to zero in $L^{\mathrm{p}}(Q_t \times \frac{1}{4}Q_x)$ for each fixed C. The L^{p} norm of $\mathcal{J}_2^{(\nu)}$ is estimated similarly. Use that if (t,x) is contained in $Q_t \times \frac{1}{4}Q_x$ and if $g(x - \frac{y}{\nu}) \neq 0$ (hence $x - \frac{y}{\nu} \in Q_x$) then

$$\nu \left| \phi(t,x) - \phi(t, x - \tfrac{y}{\nu}) \right| < \tfrac{1}{2}|y| \tag{3.39}$$

(see (2.13) of [Ba–Tr]) if δ and σ are small enough, and, therefore

$$|\mathcal{L}_\nu(x,t;y)| \le \text{const } \exp\left\{ -\tfrac{3}{4}|y|^2 \right\}. \tag{3.40}$$

Hence

(3.41)

$$\int\limits_{Q_t\times\frac{1}{4}Q_x} dtdx|\mathcal{J}_2^{(\nu)}|^p \le \text{const} \int\limits_{Q_t\times\frac{1}{4}Q_x} dtdx \left| \int\limits_{|y|>C} \exp\left\{-\frac{3}{4}|y|^2\right\}\cdot\left(|v(t,x-\frac{y}{\nu})|+|v(t,x)|\right)dy\right|^p \le$$

$$\le \text{const} \int\limits_{Q_t\times\frac{1}{4}Q_x} dtdx \int\limits_{|y|>C} \exp\{-\frac{3}{4}|y|^2\}dy\cdot\left(|v(t,x-\frac{y}{\nu})|^p+|v(t,x)|^p\right) \le$$

$$\le \text{const} \int\limits_{Q_t\times Q_x} |u(t,x)|^p dtdx \cdot \int\limits_{|y|>C} \exp\{-\frac{3}{4}|y|^2\}dy.$$

Thus, $\mathcal{J}_2^{(\nu)}$ is small in $L^p(Q_t\times\frac{1}{4}Q_x)$ uniformly in ν if C is large enough. Lemma 3.6 is proved completely. □

Remark 3.1. If in certain local coordinates (x,t) on M the L_j are given by (3.12), then in the lemma 3.6 one can take $'U=\widetilde{Q}_t\times Q_x$ and $''U=\widetilde{Q}_t\times\frac{1}{4}Q_x$, where $Q_x=(-\delta,\delta)^n\subset\mathbb{R}^n$, $\frac{1}{4}Q_x=(-\frac{\delta}{4},+\frac{\delta}{4})^n\subset\mathbb{R}^n$ as in the proof of part c) and $\widetilde{Q}_t=(\sigma_1',\sigma_1'')\times(\sigma_2',\sigma_2'')\times\ldots(\sigma_m',\sigma_m'')\subset\mathbb{R}^m$ whenever $\delta>0$ is small enough and $\sigma_j'<\sigma_j''$, $\max\{|\sigma_j'|,|\sigma_j''|\}\le\sigma$ for $j=1,\ldots,m$ and for some $\sigma>0$ which is sufficiently small (in dependence of δ). This can be seen by a slight modification of the proof.

Proof of Lemma 3.1. We may assume that $p=0$ and in a neighbourhood of zero M is given by the equations

$$\mathsf{x}=h(\mathsf{w},\mathsf{y}) \tag{3.42}$$

where $\mathsf{z}=\mathsf{x}+i\mathsf{y}\in\mathbb{C}^2$, $\mathsf{w}\in\mathbb{C}^{n-2}$, h is C^2 function (with values in $\mathbb{R}^2$) defined in a neighbourhood of zero in $\mathbb{C}^{n-2}\times\mathbb{R}^2$, $h(0)=0,(\nabla h)=0$. Denote, as usual, by C^β the space of functions which are Hölder continuous of order β ($\beta\in(0,1)$) and by $C^{1,\beta}$ the space of functions with first order derivatives in C^β. Let $A_0^{1,\beta}$ be the space of functions which are analytic in $\mathbb{D}$, are of class $C^{1,\beta}$ in $\overline{\mathbb{D}}$ and vanish at 1.

Since the dimension of the local orbit germ $\mathcal{O}^{\text{loc}}(M,p)$ is equal to $\dim_r M-1>\dim_{CR}M$, according to [Tul] there exist functions w_0 and w_1 in $A_0^{1,\beta}$ (β is some number between $\frac{1}{2}$ and 1) such that the differential of the mapping

$$w\to\hat{h}(w)\overset{\text{def}}{=}\int\limits_0^{2\pi}\frac{h(w(\zeta),y(\zeta))}{|1-\zeta|^2}|d\zeta|,\quad w\in A_0^{1,\beta}, \tag{3.43}$$

at w_0 in the direction w_1 does not vanish. Here y is the solution of Bishop's equation

$$y = T_1 h(w, y), \tag{3.44}$$

$T_1\mathcal{G}$ is the harmonic conjugate, which vanishes at 1, of a function $\mathcal{G}$.

Consider now for q in a neighbourhood of $p = 0$ on the CR-submanifold N_0 of M the family $\{f_q\}$ of analytic discs of class $C^{1,\beta}$, which satisfy the following conditions: The boundaries $f_q(\partial\mathbb{D})$ are contained in M, $f_q(1) = q$ and the orthogonal projection π onto $T_0^J M$ of f_q is equal to $\pi q + w_0 + \alpha w_1$ for a small real number α. By [Tu2] the f_q are uniquely determined (the (y_{n-1}, y_n)-component of such a disc is the solution of Bishop's equation, see (3.44)) and the mapping

$$(q, \zeta) \to f_q(\zeta), \tag{3.45}$$

q in a neighbourhood of zero on N_0, $\zeta \in \overline{\mathbb{D}}$, is of class C^{1,β_1} $(\frac{1}{2} < \beta_1 < \beta)$.

We will prove now that the mapping (3.46) with $\zeta = r \in (1 - \delta, 1]$ for some positive δ defines a diffeomorphism onto an analytic manifold $\widetilde{X}_j$, such that $(\widetilde{X}_j \cup N_0) \cap \omega$ is equal to the required C^1 manifold with boundary $\overline{X}_j \cap \omega$. Indeed, write the $C^{1,\beta}$ manifold N_0 as the graph of a CR-function g (of class $C^{1,\beta}$) over a hypersurface $\mathcal{H}_0$ (of the same class) in $\mathbf{L}_0 = T_0 N_0 + J T_0 N_0$:

$$N_0 = \{z + g(z) : z \in \mathcal{H}_0\}, \tag{3.46}$$

g is a CR-function with values in the orthogonal complement $\mathbf{L}_0^{\perp}$, $g(0) = 0, (\nabla g(0)) = 0$. Consider analytic discs $f_q^{\Pi} : \overline{\mathbb{D}} \to \mathbf{L}_0$ with boundary $f_q^{\Pi}(\partial\mathbb{D})$ contained in $\mathcal{H}_0$, of class C^{1,β_1}, such that $f_q^{\Pi}(1)$ is equal to the orthogonal projection Πq of q onto $\mathbf{L}_0$ and the w-component of f_q^{Π}, (i.e. the projection πf_q^{Π} of f_q^{Π} onto $T^{J_0} M \subset \mathbf{L}_0$) is equal to $\pi q + w_0 + \alpha w_1$. The discs f_q^{Π} are uniquely determined by Bishop's equation (see [Tu2] theorem 1.6). Moreover, by lemma 3.6 the mappings

$$F_q(\zeta) = f_q^{\Pi}(\zeta) + g \circ f_q^{\Pi}(\zeta) \qquad (\zeta \in \partial\mathbb{D}) \tag{3.47}$$

extend to mappings of class C^{1,β_1} on the closed disc $\overline{\mathbb{D}}$ which are analytic in $\mathbb{D}$, and moreover, $F_q(\partial\mathbb{D})$ is contained in $N_0 \subset M$. So by the uniqueness of analytic discs with given parameters and boundaries in M we must have

$$F_q = f_q | \partial\mathbb{D}. \tag{3.48}$$

Hence $f_q(\partial\mathbb{D})$ is contained in N_0. The $(\mathsf{x}_{n-1}, \mathsf{x}_n)$-component of $f_0(r)$ is equal to

$$J(r) = \frac{1-r^2}{2\pi} \int_0^{2\pi} \frac{h(w_0(\zeta) + \alpha w_1(\zeta), y_{w_0+\alpha w_1}(\zeta))}{|r-\zeta|^2} |d\zeta| \tag{3.49}$$

with $y_{w_0+\alpha w_1} = T_1 h(w_0 + \alpha w_1, y_{w_0+\alpha w_1})$. Since w_0, w_1 and $y_{w_0+\alpha w_1}$ are in C^{1,β_1} and vanish at 1, it is not hard to see that

$$J(r) = \frac{1-r^2}{2\pi} \left\{ \hat{h}(w_0 + \alpha w_1) + o(1) \right\}, \tag{3.50}$$

this equality holds uniformly for small real α and $r \to 1$. By the choice of w_0 and w_1 $(\frac{\partial}{\partial r} J)(1)$ is not zero for some suitable small real α. This means (see (3.49)) that with this choice of w_0, w_1 and α the vector

$$\left(\frac{\partial}{\partial r} f_0 \right) (\zeta) \big|_{\zeta=1}$$

is not zero and is not contained in $T_0 M$. Moreover, since by the Cauchy-Riemann equations, applied to the analytic function f_0, this vector is contained in $J T_0 N_0 \subset \mathbf{L}_0$, the mapping

$$(q,r) \to f_q^{\Pi}(r), \quad q \text{ in a neighbourhood of zero on } N_0,\ r \in (1-\delta, 1] \tag{3.51}$$

is a diffeomorphism onto a neighbourhood of zero on the union of $\mathcal{H}_0$ with a one-sided neighbourhood of $\mathcal{H}_0$ in $\mathbf{L}_0$. Using again lemma 3.6 we get that g extends to an analytic function $\widetilde{g}$ on the one-sided neighbourhood of $\mathcal{H}_0$ in $\mathbf{L}_0$ and $\widetilde{X}_j$ is the graph of $\widetilde{g}$ over the mentioned one-sided neighbourhood. Moreover, $\widetilde{g}(f_q^{\Pi}(r))$ is equal to the Poisson integral at the point r of the C^{1,β_1} function $g \circ f_q^{\Pi}$, which is defined on the unit circle $\mathbb{T}$. It follows easily that $\overline{X}_j \cap \omega$ is of class C^1. It is clear from the considerations made above that the analytic hypersurface X_j is contained in the envelope of holomorphy of an arbitrary neighbourhood of N_0 in $\mathbb{C}^n$. So, if M is contained in the boundary of a strictly pseudoconvex domain Ω then the analytic hypersurface is contained in Ω and relatively closed in $\Omega \cap \omega$. Lemma 3.1 is proved. □

4. Removable sets

In this section we will prove theorems 1 and 2, related results for more general hypersurfaces, and results on $(L^p, \overline{\partial}_b)$-removability.

Let H be an orientable hypersurface in $\mathbb{C}^n$. Fix the positive side of H. Let K be a relatively closed subset of H. *A point $p \in H$ will be called $(H \backslash K)$-regular if each function u which is analytic on the positive side of $H \backslash K$ has analytic extension to a one-sided neighbourhood of p.* More precisely, this means the following: *There exists a one-sided neighbourhood $\mathcal{O}_p$ of p and an analytic function u_p in $\mathcal{O}_p$. If $\mathcal{O}_p$ is contained on the* *positive side of H then we require that u_p coincides with u on the intersection of $\mathcal{O}_p$ with a germ of one-sided neighbourhoods of $H\backslash K$. If $\mathcal{O}_p$ is contained on the negative side of H then we require that u_p is analytic in a neighbourhood $\mathcal{O}_p^*$ of $\overline{\mathcal{O}_p}\backslash K$ (in particular u_p is analytic near each point of $H\backslash K$ close to p) and coincides with u on the intersection of $\mathcal{O}_p^*$ with a germ of one-sided neighbourhoods of $H\backslash K$.*

Proof of theorems 2 and 2a. We will prove theorem 2a. Theorem 2 follows easily from theorem 2a. Let H, M and K be as in theorem 2a. The set of $(H\backslash K)$-regular points is open and contains $H\backslash K$. Let M_{reg} be an arbitrary open subset of M consisting of $(H\backslash K)$-regular points. It is not hard to see that each function which is analytic on the pseudoconvex side of $H\backslash K$ has analytic extension to a one-sided neighbourhood of $(H\backslash K)\cup M_{reg}$ (which is necessarily contained on the pseudoconvex side of H). We have to show that all points of K are regular.

The following proposition shows that all points in the $G_J(M)$-invariant hull $I(M\backslash K)$ of the set $M\backslash K$ are regular. By corollary 2.1 this implies the theorem.

Proposition 4.1. *Let H, M and K be as in theorem 2a. Suppose $\gamma : [0,1] \to M$ is a CR-curve. If for $t \in [0,1)$ the points $\gamma(t)$ are $(H\backslash K)$-regular then $\gamma(1)$ is $(H\backslash K)$-regular.*

Proof. Let H_0 be a neighbourhood of $\gamma(1)$ on H and let $U_\gamma \subset H$ be a neighbourhood of $\gamma([0,1))$ consisting of $(H\backslash K)$-regular points. Denote $M\backslash U_\gamma$ by M_γ. Let u be analytic on the pseudoconvex side of $H_0\backslash M_\gamma$, say, u is analytic in the one-sided neighbourhood $\mathcal{O}_\gamma$ of $H_0\backslash M_\gamma$. Take a sequence $t_k \in (0,1)$, $t_k \uparrow 1$ and let $C_k = C_k(\gamma(t_k), i\gamma'(t_k)) \subset M \cap U_\gamma$ be small disjoint CR-cones on M. Apply lemma 3.2 to each C_k. For each k we get a C^2 function d_k on M with small C^2 norm such that

$$\begin{aligned} &\text{supp } d_k \subset C_k; \\ (4.1) \qquad &\text{for } z \in C_k \text{ the point } z + d_k(z) \text{ is contained in } M_\gamma \cup \mathcal{O}_\gamma; \\ &\gamma(t_k) \text{ is a minimal point of } M^{d_k} = \{z + d_k(z) : z \in M\}. \end{aligned}$$

Suppose the C^2 norms of the d_k tend to zero sufficiently fast. Then $d = \sum_{k=1}^{\infty} d_k$ is of class C^2 and $\gamma(t_k)$ are minimal points of the deformed manifold $M^d = \{z + d(z) : z \in M\}$.

Let M_0 be the part of M contained in a small neighbourhood of $\gamma(1)$ on H. Sweep out a small neighbourhood U of $\gamma(1)$ on H by disjoint generic CR-manifolds M_t $(t \in (-\delta, \delta))$ which are close (in C^2) to M_0. (For example, take the M_t to be parallel shifts of M_0 in certain Euclidean coordinates in a neighbourhood of $\gamma(1)$ on H.) Let k_0 be large enough and denote by d' the function $d' = \sum_{k=1}^{k_0} d_k$. Extend the d_k, $k = 1, \dots, k_0$, to C^2 functions on U, denoted also by d_k, such that the extended functions have disjoint support not containing $\gamma(1)$ and the set $\{z + d_k(z) : z \in U\}$ is contained in $U \cup \mathcal{O}_\gamma$ for all k. Denote by the same letter d' the sum of the extended functions for $k = 1, \dots, k_0$. Then for small $\delta > 0$ for $t \in (-\delta, \delta)$ the deformed manifolds $M_t^{d'} = \{z + d'(z) : z \in M_t\}$ are contained in

$M_t \cup \mathcal{O}_\gamma$ and are close (in C^2) to $M_0^{d'}$ and, thus, to M_0^d (see Fig. 2). Let ν be the unit normal to H at $\gamma(1)$ which is directed to the pseudoconvex side of H. For each $t \in (-\delta, \delta)$ we make a small parallel shift of $M_0^{d'}$ to the pseudoconvex side of H: Consider

$$\tilde{M}_t^{d'} \overset{\text{def}}{=} M_t^{d'} + s(t) \cdot \nu \tag{4.2}$$

for a small positive constant $s(t)$ choosen in such a way that $\tilde{M}_t^{d'}$ is contained in $\mathcal{O}_\gamma$.

Decreasing δ if necessary we may apply the lemmas 3.5 and 3.6 to the manifolds $\tilde{M}_t^{d'}, t \in (-\delta, \delta)\backslash\{0\}$, and to the restriction of the function u to these manifolds. We get open sets V_t on M_t, $t \in (-\delta, \delta)$, all V_t close in C^2 to V_0, and open sets

$$\tilde{V}_t^{d'} \overset{\text{def}}{=} \{z + d'(z) + s(t) \cdot \nu : z \in V_t\} \tag{4.3}$$

on $\tilde{M}_t^{d'}$ which are close to $\tilde{V}_0^{d'}$ in the C^2 topology. Moreover, V_0 is a neighbourhood of $\gamma(1)$ on M.

We get also an open convex cone K in $\mathbb{C}^n$ ($\mathbb{C}^n \simeq T_p\mathbb{C}^n$ for $p \in \mathbb{C}^n$) such that for $t \in (-\delta, \delta)\backslash\{0\}$ the function u has an analytic continuation from a neighbourhood of $\tilde{V}_t^{d'}$ to a neighbourhood of the closure of the wedge $W(\tilde{V}_t^{d'}, K)$ (see Fig. 2). Recall, that d' vanishes near $\gamma(1)$ on H. Hence d' vanishes on a neighbourhood $'V_0$ of $\gamma(1)$ on V_0 and for t close to zero, say $|t| < \delta'$, the function d' vanishes on open subsets $'V_t$ of V_t which are close in C^2 to $'V_0$. Hence $'V_t^{d'} = {'V_t}$ and $'\tilde{V}_t^{d'} \overset{\text{def}}{=} {'V_t^{d'}} + s(t) \cdot \nu = {'V_t} + s(t) \cdot \nu$ for $|t| < \delta'$. Since the V_t and the number δ will be needed no more, for notational convenience we will write δ instead of $'\delta$ and V_t instead of $'V_t$. Thus, with the new meaning of V_t and δ we have

$$\tilde{V}_t^{d'} = \tilde{V}_t \overset{\text{def}}{=} V_t + s(t) \cdot \nu, \ (t \in (-\delta, \delta)).$$

The previous arguments hold for all sufficiently small numbers $s(t) > 0$, so for each $t \in (-\delta, \delta)\backslash\{0\}$ there exists an analytic function in $W(V_t, K)$ which coincides with u near V_t. By the pseudoconvexity assumption for H it is clear that the $W(V_t, K)$ are contained on the pseudoconvex side of H.

Denote $U_+ = \bigcup_{t\in(0,\delta)} V_t$. It is now easy to see that we get an analytic function u_+ in the connected set $\mathcal{O}_+ \overset{\text{def}}{=} \bigcup_{t\in(0,\delta)} W(V_t, K)$ which coincides with f near U_+. In an analoguous way we define $\mathcal{O}_-$ and U_- and get an analytic function u_- on $\mathcal{O}_-$ which coincides with u near U_-.

Now $\mathcal{O}_+ \cup \mathcal{O}_-$ cover a one-sided neighbourhood of $\gamma(1)$. Moreover, u_+ coincides with u_- on $W(V_0, K) = \mathcal{O}_+ \cap \mathcal{O}_-$. Indeed, V_0 contains a regular point $\gamma(t)$ for some $t < 1$ close to 1. So there exists a one-sided neighbourhood $\mathcal{O}_{\gamma(t)}$ of $\gamma(t)$ and an analytic function

$u_{\gamma(t)}$ in $\mathcal{O}_{\gamma(t)}$ which coincides with u near $H \cap \overline{\mathcal{O}}_{\gamma(t)}$. We may suppose that $\overline{\mathcal{O}}_{\gamma(t)} \cap U_+$ and $\overline{\mathcal{O}}_{\gamma(t)} \cap U_-$ are connected. Therefore, $u_{\gamma(t)}$ coincides with u_+ at the connected component of $\mathcal{O}_{\gamma(t)} \cap \mathcal{O}_+$ which contains $\overline{\mathcal{O}}_{\gamma(t)} \cap U_+$ in its boundary. Similarly, $u_{\gamma(t)}$ coincides with u_- at points in the connected component of $\mathcal{O}_{\gamma(t)} \cap \mathcal{O}_-$ which contains $\overline{\mathcal{O}}_{\gamma(t)} \cap U_-$ in its boundary. Thus $u_+ = u_-$ on an open subset of $\mathcal{O}_+ \cap \mathcal{O}_-$, hence, since $\mathcal{O}_+ \cap \mathcal{O}_- = W(V_0, K)$ is connected, $u_+ = u_-$ on $\mathcal{O}_+ \cap \mathcal{O}_-$. We got a well defined analytic function in $\mathcal{O}_+ \cup \mathcal{O}_-$ which coincides with u near $U_+ \cup U_- = U \backslash M$. Thus it coincides with u near $U \backslash K$. We proved that $\gamma(t)$ is regular. □

Proof of the theorem 2′. Let Ω, M and K be as in theorem 2′. Put $H = \partial\Omega$. We start with a function u which is analytic on the "inner side" of $H \backslash K$. Let M_{reg} be as above an open subset of M consisting of $(H\backslash K)$-regular points of M. K does not divide the connected hypersurface $\partial\Omega$, otherwise it must contain a compact generic CR-manifold of dimension $2n-2$ which would be a CR-invariant subset of M. It is now easy to see that each function which is analytic on the inner side of $H \backslash K$ has analytic extension to a connected open set which contains a one-sided neighbourhood of each point of $(H\backslash K) \cup M_{reg}$. This time the one-sided neighbourhood is not necessarily contained on the inner side of H, but for each point of $H \backslash M$ we may assume that it is. By the Hartogs-Bochner theorem it is enough to show that each point of M is regular.

The regularity of all points of M is proved using corollary 2.1 and a variant of proposition 4.1 adapted to this case. This variant is proved in the same way as the proposition 4.1 itself: H_0, U_γ and M_γ denote the same as in the proof of proposition 4.1, but the connected set $\mathcal{O}_\gamma$ which contains a one-sided neighbourhood of each point of $H_0 \backslash M_\gamma$ is not necessarily contained on the inner side of H. Choose CR-cones C_k as in the proof of proposition 4.1 and construct a deformation $d = \sum_{k=1}^{\infty} d_k$ with the d_k satisfying conditions (4.1). For this we use the lemmas 3.3 and 3.4 instead of lemma 3.2. The manifolds M_t and the function d' are choosen as in the proof of proposition 4.1 and ν denotes the inner normal to H at $\gamma(1)$. The manifolds $M_t^{d'}$ and the manifolds $\tilde{M}_t^{d'} = M_t^{d'} + s(t) \cdot \nu$ are defined as in (4.2); for $t \in (-\delta, \delta) \setminus \{0\}$ and small $s(t) > 0$ the manifold $\tilde{M}_t^{d'}$ is easily seen to be contained in $\mathcal{O}_\gamma$. The rest of the proof is identical to that of proposition 4.1. Theorem 2′ is proved. □

Proof of theorems 1, 1a and 1′. We will prove theorem 1. The proof of the theorem 1a and 1′ is similar to the proof of the "if"-part of theorem 1. The "if"-part of theorem 1 we will derive from theorem 2. With the same definition of $(\partial\Omega\backslash K)$-regular points as in the preceeding proof the following lemma holds.

Lemma 4.1. *Suppose for the point* $p \in K (\subset M)$

$$\dim_r T_p^J M < 2n - 4 \tag{4.4}$$

(in other words, M is not maximally complex at p and, therefore, $\dim_r T_p^J M = 2n-6$). Then p is $(\partial\Omega\backslash K)$-regular.

Proof. Let U be a small neighbourhood of p on $\partial\Omega$, such that $M_0 = M \cap U$ is connected and relatively closed in U, and, moreover, $\dim_r T_q^J M_0 = 2n-6$ for each $q \in M_0$. Thus, M_0 is a generic manifold of real dimension $2n-3$. Let $M_1 \subset U$ be a generic CR-manifold of real dimension $2n-2$ which is a proper submanifold of $U \subset \partial\Omega$ and contains M_0. Such a manifold M_1 can be obtained, for example, in the following way. Consider Euclidean coordinates φ on U, $\varphi : U \to \mathbb{R}^{2n-1}$, take a vector ν in $\mathbb{R}^{2n-1}$ which is not contained in $T_{\varphi(p)}\varphi(M_0)$, and put

$$M_1 = \left\{ z \in U : z = \varphi^{-1}(\varphi(\zeta) + s\nu),\ \zeta \in M_0,\ s \in (-\delta, \delta) \right\} \tag{4.5}$$

for a small positive number δ. Shrinking U if necessary we get the desired CR-manifold M_1.

Apply now theorem 2a to the hypersurface U, the generic CR-manifold M_1 of dimension $2n-2$ and the relatively closed subset M_0 of U which is contained in M_1. Since M_0 is itself a manifold and for $q \in M_0$ $T_q^J M_0$ does not contain $T_q^J M_1$ (the first space has dimension $2n-6$, the second one has dimension $2n-4$) M_0 does not contain non-empty CR-invariant subsets of M_1. So by theorem 2a each point of M_0 is $(U\backslash M_0)$-regular and therefore also $(\partial\Omega\backslash K)$-regular. □

Continue now the proof of theorem 1. Suppose either K does not coincide with M or M is not a maximally complex CR-manifold. Consider the set

$$K_1 = K \backslash \left\{ p \in K : \dim_r T_p^J M = 2n-6 \right\} = \left\{ p \in K : \dim_r T_p^J M = 2n-4 \right\}. \tag{4.6}$$

$K\backslash K_1$ consists of the points of K for which M is not maximally complex (and therefore generic), so K_1 is closed. Moreover, $K\backslash K_1$ consists of $(\partial\Omega\backslash K)$-regular points by lemma 4.1. Let $\tilde{K}$ be the set of non-regular points of K_1 and suppose $\tilde{K} \neq \emptyset$. Since $\tilde{K}$ is compact and is contained in the set of maximally complex points of M, $\tilde{K}$ does not coincide with M. Let p be a "boundary point" of $\tilde{K} : p \in \tilde{K} \cap \overline{M\backslash \tilde{K}}$. Let again U be a small neighbourhood of p on $\partial\Omega$ such that $M_0 = M \cap U$ is connected and relatively closed in U. Construct as in the proof of lemma 4.1 a generic CR-manifold M_1 of dimension $2n-2$ which is a proper submanifold of $U \subset \partial\Omega$ and contains M_0. The manifold M_1 can be obtained as in the proof of lemma 4.1. The only point is that the vector ν must be choosen accurately to avoid complex tangencies of M_1. If U is sufficiently small this can always be done. Apply theorem 2a to the hypersurface U, to the generic CR-manifold M_1 and the relatively closed subset $\tilde{K} \cap U$ of U, $\tilde{K} \cap U \subset M_0$.

Claim 4.1. *$\tilde{K} \cap U$ does not contain $G_J(M_1)$-invariant subsets of M_1.*

Proof. A non-empty relatively closed $G_J(M_1)$-invariant subset K' of $\tilde{K} \cap U$ is the union of CR-submanifolds of M_1 of dimension $2n-3$, since K' is itself contained in the manifold M_0 of dimension $2n-3$. Being the union of manifolds of that dimension the set K' is relatively open in M_0. But it is also relatively closed in U, so by the connectedness of M_0 we must have $K' = M_0$. But this contradicts the fact that $p \in \tilde{K} \cap \overline{M \setminus \tilde{K}}$ (i.e. M_0 contains points close to p which are not in $\tilde{K}$ and therefore not in K'). □

The "if"-part of theorem 1 follows now from theorem 2a.

The following lemma is needed for the proof of the "only if" part of theorem 1.

Lemma 4.2. *Let Ω be a bounded strictly pseudoconvex domain in $\mathbb{C}^n, n \geq 3$, with boundary $\partial\Omega$ of class C^2. Suppose M is a maximally complex CR-manifold of class C^2 contained in $\partial\Omega$. Then M is orientable.*

Proof. By [Fo], lemma 2.1, for all p in M we have

$$T_pM \not\subset T_p^J\partial\Omega. \tag{4.7}$$

Give $\partial\Omega$ the canonical orientation induced from $\mathbb{C}^n$. Since M is maximally complex, the space T_p^JM has real codimension one in T_pM. For getting an orientation of M it is sufficient to choose a continuous nowhere vanishing vector field on M with values in the orthogonal complement $T_pM \ominus T_p^JM$ of T_p^JM in T_pM. But by (4.7) for each $p \in M$ the real line $T_pM \ominus T_p^JM$ has non-trivial projection onto $T_p\partial\Omega \ominus T_p^J\partial\Omega$. Thus, the orientation of $\partial\Omega$ induces an orientation of M. □

Now we come to the proof of the "only if" part of theorem 1. Suppose $K = M$ is a connected maximally complex CR-manifold of dimension $\dim_r M = 2n-3$ of class C^2 imbedded into $\partial\Omega$. By lemma 4.2 we may suppose that M is oriented. By the theorem 1 of [Ha–La] applied to an arbitrary Stein neighbourhood Ω_1 of $\overline{\Omega}$ there exists a complex analytic variety V of complex dimension $\dim_{\mathbb{C}} V = n-1$ contained in $\Omega_1 \setminus M$ and relatively compact in Ω_1 such that

$$d[V] = \pm[M] \tag{4.8}$$

in the sense of currents. Take instead of Ω a slightly smaller strictly pseudoconvex domain Ω' with $M \subset \partial\Omega'$ and $\overline{\Omega'} \subset M \cup \Omega$. Apply the same arguments to arbitrary Stein neighbourhoods of $\overline{\Omega'}$ we get that

$$\overline{V} \subset \Omega \cup M. \tag{4.9}$$

We have to show that $\Omega \setminus V$ is pseudoconvex. Take a strongly pseudoconvex defining function ρ in a neighbourhood of $\overline{\Omega}$, $\Omega = \{\rho < 0\}$, ρ of class C^2 and $d\rho \neq 0$ on $\partial\Omega$. Consider the relatively compact subset $\{\rho < -\varepsilon\}$ of Ω with $\varepsilon > 0$ a small number. V is locally the zero set of an analytic function (see, for example, [Na] or [Či]). Thus the set $\{\rho < -\varepsilon\}\backslash V$ is pseudoconvex for sufficiently small $\varepsilon > 0$. By the Behnke–Stein theorem (see, for example, [Vla] III 16.10) the domain $\Omega\backslash V = \bigcup_{n\in N}(\{\rho < -\frac{1}{n}\}\backslash V)$ is pseudoconvex. □

Now we will come to the proof of theorem 2b. We need analogs of proposition 3.1 and related results for CR-functions of class L^{p} on hypersurfaces. For simplicity we will formulate these results only for hypersurfaces, not for manifolds of higher codimension. (The statement and the proof which we choosed here, have a natural analogue for manifolds of higher codimension.)

Definition 4.1. *Suppose H is a hypersurface of class C^2 in $\mathbb{C}^n$. Let $p \in H$ and let $\mathcal{O}$ be a one-sided neighbourhood of p (with respect to H) with C^2 boundary $\partial\mathcal{O}$. Suppose $\partial\mathcal{O}$ contains a connected neighbourhood U of p on H. Let u be a function on H which is locally of class $L^{\mathrm{p}}, 1 \leq \mathrm{p} \leq \infty$. We will say, that u has an H^{p}-extension to $\mathcal{O}$ if there exists an analytic function $u_{\mathcal{O}}$ of class H^{p} in $\mathcal{O}$ such that the boundary values of $u_{\mathcal{O}}$ on U coincide with u.*

Here $H^{\mathrm{p}}(\mathcal{O})$ denotes the usual Hardy space in $\mathcal{O}$. For $0 < \mathrm{p} < \infty$ it consists of all analytic functions $\mathcal{F}$ in $\mathcal{O}$ for which

$$||\mathcal{F}||_{H^{\mathrm{p}}(\mathcal{O})} \stackrel{\mathrm{def}}{=} \sup_{\varepsilon>0} \int_{\partial\mathcal{O}_\varepsilon} |\mathcal{F}|^{\mathrm{p}} dm_{2n-1} < \infty$$

for any fixed family of approximating domains $\mathcal{O}_\varepsilon$, $\mathcal{O}_\varepsilon = \{\lambda < -\varepsilon\}$ for a C^2 function λ in a neighbourhood of $\overline{\mathcal{O}}$ with $\mathcal{O} = \{\lambda < 0\}$ and $d\lambda \neq 0$ on $\partial\mathcal{O}$. m_{2n-1} is the $(2n-1)$-dimensional surface measure on $\partial\mathcal{O}_\varepsilon$. $H^\infty(\mathcal{O})$ is the space of all bounded analytic functions in $\mathcal{O}$. See also [Stein1] for more details.

We will say that a function $u \in L^{\mathrm{p}}_{\mathrm{loc}}(H)$ has local H^{p}-extension at p, if u has H^{p}-extension to certain one-sided neighbourhood $\mathcal{O}$ of the kind described in the definition.

We need the following lemma only for $p \in [1, \infty)$, but for completeness reasons we state it also for $p = \infty$.

Lemma 4.3. *Suppose H is a hypersurface of class C^2 in $\mathbb{C}^n$ and p is a minimal point of H. Then each CR-function of class $L^{\mathrm{p}}_{loc}(H), 1 \leq \mathrm{p} \leq \infty$, has local H^{p}-extension at p. Moreover, there exists a fixed one-sided neighbourhood $\mathcal{O}$ of p of the kind described in definition 4.1 such that each CR-function of class $L^{\mathrm{p}}_{loc}(H)$ has H^{p}-extension to $\mathcal{O}$. $\mathcal{O}$ depends only on H and p, not on p.*

Proof of lemma 4.3. By Tumanov's theorem [Tu1], [Tu2] there exists $w \in A_0^{1,\alpha}$, $(\alpha \in (\frac{1}{2}, 2))$, with the following properties:
For q in H, q close to p, denote by f_q the uniquely determined analytic disc

$$f_q(= f_{q,w}) : \overline{\mathbb{D}} \to \mathbb{C}^n, \quad f_q(\mathbb{T}) \subset \mathbb{M},$$

for which $f_q(1) = q$ and the orthogonal projection π of f_q onto the complex linear subspace $T_p^c H$ of $\mathbb{C}^n$ (of complex codimension one) is equal to $\pi q + w$. For certain $\lambda_0 \in (0,1)$ and a neighbourhood U_0 of p on H the mapping

$$(\lambda, q) \to f_q(\lambda), \qquad \lambda \in (\lambda_0, 1],\ q \in U_0,$$

is (for some $\beta \in (\frac{1}{2}, \alpha)$) a $C^{1,\beta}$ diffeomorphism onto $U_0 \cup \mathcal{O}_0$ for some one-sided neighbourhood $\mathcal{O}_0$ of p with $(\partial\mathcal{O}_0) \cap H = U_0$. In particular, by the Cauchy-Riemann equations the derivative $\frac{\partial}{\partial\theta} f_p(e^{i\theta})|_{\theta=0}$ is not zero.

Choose Euclidean coordinates

$$v \to q(v),\ v \in V \subset \mathbb{R}^{2n-1},$$

in a neighbourhood of p on H with the following properties: $V = V_1 \times V' \subset \mathbb{R}^{2n-1}$, $V_1 \subset \mathbb{R}$, $V' \subset \mathbb{R}^{2n-2}, V$ is a neighbourhood of zero in $\mathbb{R}^{2n-1}$ and for each fixed $v_1 \in V_1$ the mapping

$$(\theta, v') \to f_q(e^{i\theta}) = f_{q(v_1,v')}(e^{i\theta}),\ |\theta| < \delta,\ v' \in V'. \tag{4.10}$$

is a diffeomorphism onto a neighbourhood of p on H. We may suppose that V_1 is small enough so that for each $v_1 \in V_1$ the image of the mapping (4.10) contains a fixed connected neighbourhood U_1 of p on H. Moreover, we may suppose that for $v_1 \in V_1$ the image of the diffeomorphism

$$(\theta, v', \lambda) \to f_{q(v_1,v')}(\lambda e^{i\theta}), \quad |\theta| < \delta, \quad v' \in V',\ \lambda \in (\lambda_0, 1] \tag{4.11}$$

contains $U_1 \cup \mathcal{O}_1$ for a fixed one-sided neighbourhood $\mathcal{O}_1$ of p, and the norms of the diffeomorphisms (4.11) and the norms of the inverses are uniformly bounded for $v_1 \in V_1$. We need the following

Lemma 4.4. *With the previous notations let $H_d = \{z + d(z) : z \in H\}$ be a hypersurface of class C^1, which is close to H in C^1 (say, d has small norm in C^1). Suppose $U_2 \subset U_1$ is a neighbourhood of p on H such that $(U_2)_d = \{z + d(z) : z \in U_2\}$ is contained in $U_1 \cup \mathcal{O}_1$. Then for each polynomial $\mathcal{P}$ the following estimate holds*

$$\int_{(U_2)_d} |\mathcal{P}|^p dm_{2n-1} \le C \int_{U_3} |\mathcal{P}|^p dm_{2n-1} \tag{4.12}$$

for a suitable neighbourhood U_3 of p on H. U_3 contains U_2. U_3 is small if U_2 is small enough.

m_{2n-1} denotes, as usual, the $(2n-1)$-dimensional surface measure. p is a real number, $1 \le \mathsf{p} < \infty$, and the constant C depends only on H, U_2, d, and on the exponent p.

Proof of lemma 4.4. Fix $v_1 \in V_1$. There is a function $\Lambda = \Lambda_{v_1}(\theta, v')$ of class C^1 on $(-\delta,\delta) \times V'$ with values in $(0,1]$ such that the C^1 norm of $1-\Lambda$ is small uniformly for $v_1 \in V_1$, and, moreover,

$$\{f_{q(v_1,v')}(\Lambda(\theta,v')e^{i\theta}), \quad |\theta| < \delta, \ v' \in V'\} \tag{4.13}$$

is an open subset of H_d which contains $(U_2)_d$. By the classical H^{p} theory for the unit disc

$$\int_{\gamma_q} |\mathcal{P} \circ f_q|^{\mathsf{p}} dm_1 \le C_1 \int_{\mathbf{T}} |\mathcal{P} \circ f_q|^{\mathsf{p}} dm_1. \tag{4.14}$$

Here $q = q(v_1, v')$ is fixed, γ_q is the curve

$$\gamma_q = \{\Lambda_{v_1}(\theta, v')e^{i\theta} : \ |\theta| < \delta\}$$

contained in $\overline{\mathbb{D}}$, dm_1 is the one-dimensional Hausdorff-measure on a curve, the real number p is contained in $[1,\infty)$. The inequality (4.14) is Carleson's imbedding theorem. Indeed, for each q the one-dimensional Hausdorff-measure on $\gamma_q \cap \mathbb{D}$ is a Carleson measure:

$$m_1(\gamma_q \cap \mathbb{D} \cap \mathbb{B}_\rho(\zeta)) \le A \cdot \rho \tag{4.15}$$

for every ball $B_\rho(\zeta)$ with center $\zeta \in \mathbb{T} = \partial\mathbb{D}$ and radius $\rho > 0$. The constant A depends on the C^1 norm of the function $\theta \to (1 - \Lambda_{v_1}(\theta, v'))$ and thus can be choosen to be independent of $(v_1, v') \in V_1 \times V'$. The inequality (4.14) holds now with

$$C_1 = a \cdot A^{\mathsf{p}} + 1$$

for an absolute constant a. (See, for example [Vi–Ha] or [Koo] for more detailed information on Carleson's imbedding theorem and maximal functions.)

Integrate now (4.14) for fixed v_1 with respect to $v' \in V'$. We get (since the set (4.13) contains $(U_2)_d$)

$$\int_{(U_2)_d} |\mathcal{P}|^{\mathsf{p}} dm_{2n-1} \le C_2 \int_{V'} dv' \int_{\mathbf{T}} |d\zeta| \ |\mathcal{P} \circ f_{q(v_1,v')}(\zeta)|^{\mathsf{p}}. \tag{4.16}$$

This holds for each $v_1 \in V_1$. Integrate over $v_1 \in V_1$ we get

$$(4.17)\qquad \int\limits_{(U_2)_d} |\mathcal{P}|^{\mathrm{p}} dm_{2n-1} \le \frac{C_2}{m_1(V_1)} \int\limits_{V_1\times V'} dv_1 dv' \int\limits_{\mathbb{T}} |d\zeta|\, |\mathcal{P}\circ f_{q(v_1,v')}(\zeta)|^{\mathrm{p}}.$$

For estimating the right hand side of (4.17) we use that for each fixed $\zeta \in \mathbb{T}$ the mapping

$$(4.18)\qquad v \to f_{q(v)}(\zeta),\ \ v\in V,$$

is a diffeomorphism onto its image, provided V is small. The norms of the diffeomorphisms and the inverses are uniformly bounded for $\zeta \in \mathbb{T}$. Indeed, we may consider the mapping

$$(4.19)\qquad q \to f_q(\zeta), \qquad q \text{ in a neighbourhood of } p \text{ on } H$$

instead of (4.18). Consider the projection π of (4.19) onto $T_p^J H$,

$$(4.20)\qquad \pi q \to \pi f_q(\zeta) = \pi q + w(\zeta).$$

Its differential is an isomorphism on $T_p^J H$ for each $\zeta \in \mathbb{T}$. It remains to differentiate Bishop's equation (see [Tu2] §2) in the direction $T_p H \ominus T_p^J H$.

It follows that for each $\zeta \in \mathbb{T}$ and each function $\mathcal{F} \in L^{\mathrm{p}}_{\mathrm{loc}}(H)$

$$(4.21)\qquad \int\limits_{V_1\times V'} dv_1 dv' |\mathcal{F}\circ f_{q(v_1,v')}(\zeta)|^{\mathrm{p}} \le C_3 \int\limits_{U_3} |\mathcal{F}|^{\mathrm{p}} dm_{2n-1}$$

holds for a neighbourhood U_3 of p on H, which contains $f_{q(v_1,v')}(\zeta)$ for all $(v_1, v') \in V_1 \times V'$, $\zeta \in \mathbb{T}$. It is clear that U_3 can be taken small if U_2 is small. Integration with respect to ζ gives the desired result. Lemma 4.4 is proved. □

Continue now the proof of lemma 4.3. Let first $\mathrm{p} < \infty$. Consider a neighbourhood U_2 of p on H, $\overline{U}_2 \subset U_1$. Let s_0 be a small positive number. For $s \in (0, s_0)$ denote by d_s a C^1 function on H which is equal to $s\cdot\nu$ on U_2. Here ν is the unit normal of H at p which is directed into $\mathcal{O}_1$. If s_0 is small enough, we may apply lemma 4.4 to U_2 and $d = d_s$ for each $s \in (0, s_0)$. Integrating inequality (4.12) (written for d_s instead of d) with respect to the parameter s we get

$$(4.22)\qquad \int\limits_{\mathcal{O}_2} |\mathcal{P}|^{\mathrm{p}} dm_{2n} \le C' \int\limits_{U_3} |\mathcal{P}|^{\mathrm{p}} dm_{2n-1}$$

for certain one-sided neighbourhood $\mathcal{O}_2$ of U_2 and all polynomials $\mathcal{P}$. This implies in a standart way that for each compact subset K of $\mathcal{O}_2$

(4.23) $$\max_K |\mathcal{P}| \le C''(K) \int_{U_3} |\mathcal{P}|^{\mathsf{p}} dm_{2n-1}$$

for each polynomial $\mathcal{P}$.

Let now $\mathcal{S}$ be a non-negative C^2 function on U_2 which is zero on a connected neighbourhood U_2' of p on H, $\overline{U_2'} \subset U_2$, and strictly positive on $U_2 \backslash \overline{U_2'}$. Put $d(q) = \mathcal{S}(q) \cdot \nu$ with ν as above.

Let $\mathcal{O}$ be a one-sided neighbourhood of p, $\mathcal{O} \subset \mathcal{O}_2$, with C^2 boundary $\partial\mathcal{O}$, such that a part of $\partial\mathcal{O}$ is contained in $(U_2)_d$ and the other part of $\partial\mathcal{O}$ is a compact subset of $\mathcal{O}_2$. By lemma 4.4 and the preceeding remark

(4.24) $$||\mathcal{P}||_{H^{\mathsf{p}}(\mathcal{O})} \le C(\mathcal{O}) \int_{U_3} |\mathcal{P}|^{\mathsf{p}} dm_{2n-1}$$

for each polynomial $\mathcal{P}$.

Let now u be a CR-function on H of class $L^{\mathsf{p}}_{\text{loc}}$. We may suppose that the set U_3 in (4.24) is small enough to apply the approximation lemma 3.6.c. with $'U = U_3$. Let the polynomials $\{\mathcal{P}_k\}_{k=1}^{\infty}$ approximate u in $L^{\mathsf{p}}(U_3, dm_{2n-1})$. Then for $\{\mathcal{P}_k\}_{k\ge1}$ as well as for $\{\mathcal{P}_k - \mathcal{P}_l\}_{k,l\ge1}$ the estimate (4.24) holds. Hence, $\mathcal{P}_k$ converge in $H^{\mathsf{p}}(\mathcal{O})$ to a function $u_{\mathcal{O}}$ and the boundary values of $u_{\mathcal{O}}$ on U_2' are equal to u. Lemma 4.3 is proved for $\mathsf{p} < \infty$.

For $\mathsf{p} = \infty$ we add the following simple arguments. Note first that $L^\infty \subset L^2_{\text{loc}}$, so each bounded CR-function u on H has local H^2-extension $\tilde{u}$ at p. Moreover, using (4.21) integrated with respect to ζ and the approximation lemma 3.6.c we may choose a sequence of polynomials $\{\mathcal{P}_n\}_{n=1}^{\infty}$ in $\mathbb{C}^n$ such that

$$\sum_{n=1}^{\infty} \int_V dv \int_{\mathbb{T}} |u \circ f_{q(v)}(\zeta) - \mathcal{P}_n \circ f_{q(v)}(\zeta)|^2 |d\zeta| < \infty.$$

It follows that for almost all $q \in H$ close to p the function $u \circ f_q(\zeta)$, $\zeta \in \mathbb{T}$, extends to an H^2-function u_q on the unit disc $\mathbb{D}$. Since u is bounded the extension of $u \circ f_q$ is bounded for almost all q by the essential supremum of $|u|$. Using the existence of non-tangential boundary values almost everywhere for the local H^2-extension $\tilde{u}$ (boundary values on a part of H) and for the functions u_q (boundary values on $\mathbb{T}$) we see by Privalov's uniqueness theorem that for almost all $q \in H$ close to p and for $\zeta \in \mathbb{D}$ close to one $u_q(\zeta)$ coincides with $\tilde{u} \circ f_q(\zeta)$. Thus $|\tilde{u}|$ is bounded almost everywhere by ess sup $|u|$, so by continuity $|\tilde{u}|$ is bounded by ess sup $|u|$ at all points. Lemma 4.3 is proved completely. □

Remark 4.1. Lemma 4.3 is stable under small C^2-deformations: If H_1 is close to H in C^2 the lemma still holds with some $\mathcal{O}_1$ and U_1 close to $\mathcal{O}$ and U in the natural sense.

This follows by slightly varying the family of discs f_q. The L^p-analog of proposition 3.1 for hypersurfaces follows:

Suppose H is a hypersurface of class C^2 in $\mathbb{C}^n$ and N is a CR-orbit of H. Let u be a CR-function of class L^p_{loc} on H, $1 \le p \le \infty$. If u has local H^p-extension at some point $p \in N$, then u has local H^p-extension at all points of N.

The assertion follows from the deformation argument used in the proof of proposition 3.1 (see also [Jö4]) and from lemma 4.3.

Let now N be a CR-orbit of H which is an open subset of H. Then N contains a minimal point, hence by lemma 4.3 and the preceeding remark each CR-function u of class $L^p_{loc}(H)$ has local H^p-extension at each point of H. Consider now the union of suitable one-sided neighbourhoods $\mathcal{O}_p$ of all points of N such that the CR-function u has H^p-extension to each $\mathcal{O}_p$. The set obtained in this way may be not locally connected if for some point p of N it contains one-sided neighbourhoods, say, $\mathcal{O}_p^+$ and $\mathcal{O}_p^-$, on both sides of N.

Claim 4.2. *If u has H^p-extension to $\mathcal{O}_p^+$ and to $\mathcal{O}_p^-$ then (after a correction on a set of measure zero on H) u extends to an analytic function in a neighbourhood of the point p.*

Proof. Suppose the z_1-axis is transverse to H at p. Let $D = D_1 \times D'$ ($D_1 = \{|z_1 - z_1^0| < R\} \subset \mathbb{C}$, $D' \subset \mathbb{C}^{n-1}$) be a small open polydisc centered at some point z^0 of $\mathcal{O}_p^+$, which contains p and which closure is contained in $\mathcal{O}_p^- \cup H \cup \mathcal{O}_p^+$. We define a function $\tilde{u}$ on $\mathcal{O}_p^- \cup H \cup \mathcal{O}_p^+$ by the equalities

$$\tilde{u} = u_{\mathcal{O}_p^+} \text{ on } \mathcal{O}_p^+, \quad \tilde{u} = u_{\mathcal{O}_p^-} \text{ on } \mathcal{O}_p^-, \quad \tilde{u} = u \text{ on } H.$$

Varying, if necessary, $\mathcal{O}_p^+$ and $\mathcal{O}_p^-$ slightly outside a neighbourhood of H, we may assume that $\mathbb{C} \times \{z'\}$ intersects $\mathcal{O}_p^+$ and $\mathcal{O}_p^-$ transversely for $z' \in D'$. Denote for $z' \in D'$ the planar domains $(\mathbb{C} \times \{z'\}) \cap \mathcal{O}_p^+$ and $(\mathbb{C} \times \{z'\}) \cap \mathcal{O}_p^-$ by $\mathcal{O}_p^+(z')$ and $\mathcal{O}_p^-(z')$, respectively.

Both domains may be assumed to be simply connected and have C^2 boundary. Moreover, using a maximal function inequality for H^p-functions (see [Stein1], II.9 corollary of theorem 10) it is easy to see that for almost all z' the function $u_{\mathcal{O}_p^+}|\mathcal{O}_p^+(z')$ belongs to $H^p(\mathcal{O}_p^+(z'))$ and the same for $u_{\mathcal{O}_p^-}$. Moreover, for almost all $r > 0$ such that $D_1^r \times D' \overset{\text{def}}{=} \{z_1 : |z_1 - z_1^0| < r\} \times D'$ is contained in $\mathcal{O}_p^- \cup H \cup \mathcal{O}_p^+$ and contains p, the integral

$$\int_{(\partial D_1^r \times D') \cap \mathcal{O}_p^+} |\tilde{u}|^p dm_{2n-1}$$

is finite. This follows from the maximal function inequality and (4.23). The same is true for $\mathcal{O}_p^-$ instead of $\mathcal{O}_p^+$. Choose the number R above such that with $D_1 = D_1^R$ the integral

$$\int_{\partial D_1 \times D'} |\tilde{u}|^p dm_{2n-1} \tag{4.25}$$

is finite. Let $\triangle(z')$ denote the disc $D_1 \times \{z'\}$ for $z' \in D'$. Using the preceeding remarks together with the definition of the H^p-extension, we get that for almost all z' the function $\tilde{u}|\triangle(z') \cap \mathcal{O}_p^+$ can be represented as the Cauchy integral of $\tilde{u}$ over the boundary

$$\partial(\triangle(z') \cap \mathcal{O}_p^+) = (\mathcal{O}_p^+ \cap (\partial D_1 \times \{z'\})) \cup (H \cap (D_1 \times \{z'\}))$$

of $\triangle(z') \cap \mathcal{O}_p^+$.

Moreover, the Cauchy integral of $\tilde{u}|\triangle(z') \cap \mathcal{O}_p^-$ with pole outside $\mathcal{O}_p^-(z')$ taken over the boundary of $\triangle(z') \cap \mathcal{O}_p^-$, is zero for almost all $z' \in D'$. Thus, the Cauchy integral of $\tilde{u}$ over $\partial\triangle(z') = \partial D_1 \times \{z'\}$ coincides on $\triangle(z') \cap \mathcal{O}_p^+$ with $\tilde{u}|\triangle(z') \cap \mathcal{O}_p^+$. By the same reasons for almost all $z' \in D'$ the Cauchy integral of $\tilde{u}$ over $\partial\triangle(z')$ coincides on $\triangle(z') \cap \mathcal{O}_p^-$ with $\tilde{u}|\triangle(z') \cap \mathcal{O}_p^-$. Hence, $\tilde{u}|\triangle(z')$ coincides outside a set of zero linear measure on $\triangle(z') \cap H$ with an analytic function on $\triangle(z')$.

It follows now in a standard way that $u_{\mathcal{O}_p^+}|D \cap \mathcal{O}_{p^+}$ extends to an analytic function in D and thus $\tilde{u}|D$ coincides with this analytic function after a correction on a set of zero measure on H. Indeed, consider the Taylor series of $u_{\mathcal{O}_p^+}$ with respect to z_1 near z^0:

$$u_{\mathcal{O}_p^+}(z) = \tilde{u}(z) = \sum_{k=0}^{\infty} a_k(z')(z_1 - z_1^0)^k. \tag{4.26}$$

Here $z = (z_1, z')$, $z' \in D'$, $|z_1 - z_1^0| < r$ for some small positive r such that $\{|z_1 - z_1^0| \leq r\} \times \overline{D'}$ is contained in $\mathcal{O}_p^+$. (If necessary we will shrink D'.) The coefficients a_k,

$$a_k(z') = \frac{1}{2\pi i} \int_{|z_1 - z_1^0| = r} \tilde{u}(z_1, z')(z_1 - z_1^0)^{-k} dz_1 \tag{4.27}$$

are analytic functions of z' in a neighbourhood of $\overline{D'}$. For almost all z' the function $z_1 \to u_D(z_1, z')$ is analytic on $\overline{D}_1$, thus for almost all z' formula (4.26) holds for $R > r$ instead of r. Hölders inequality and integration over z' give

$$\int_{D'} |a_k(z')|^p dm_{2n-2}(z') \leq \frac{R^{p-1}}{(2\pi)} \int_{(\partial D_1) \times D'} |\tilde{u}|^p |z_1 - z_1^0|^{-kp} dm_{2n-1}. \tag{4.28}$$

Thus for each k

$$R^{k\mathsf{p}} \int_{D'} |a_k(z')|^{\mathsf{p}} dm_{2n-2}(z') \leq C \tag{4.29}$$

for a constant C not depending on k and R. Applying in a suitable way the mean value theorem to the analytic function a_k, we get that $R^k|a_k|$ is bounded on each relatively compact subset $\tilde{D}'$ of D' by a constant not depending on R and k. Hence, $u_{\mathcal{O}_p}$ extends analytically into the set $D_1 \times \tilde{D}'$ which contains p. □

Since the orbit N is connected it is now easy to see that there exists a connected open set $\mathcal{O}$ containing a one-sided neighbourhood $\mathcal{O}_p$ of each point $p \in N$ (the $\mathcal{O}_p$ as in the definition 4.1) such that for each p, $1 \leq \mathsf{p} \leq \infty$, each CR-function u on N of class $L^{\mathsf{p}}_{\text{loc}}(N)$ has H^{p}-extension to $\mathcal{O}_p$ for each p. (This follows from the claim 4.2 and the fact that the $\mathcal{O}_p$ may be choosen not depending on the function u and the exponent p.)

We are now ready to give the

Proof of theorem 2b. We will call a point $p \in M$ $L^{\mathsf{p}}(H\backslash K)$-regular if each function $u \in L^{\mathsf{p}}_{\text{loc}}(H)$, which satisfies the tangential Cauchy-Riemann equations (in the weak sense) on $H\backslash K$, satisfies the tangential Cauchy-Riemann equations also in a neighbourhood of the point p on H.

We have to prove the following

Lemma 4.5. *Let $\gamma : [0,1] \to M$ be a CR-curve such that $\gamma(t)$ is $L^{\mathsf{p}}(H\backslash K)$-regular for $t \in [0,1), \mathsf{p} \in [1,\infty)$. Then $\gamma(1)$ is $L^{\mathsf{p}}(H\backslash K)$-regular.*

Lemma 4.5 implies theorem 2b. Indeed, let $M\backslash K'$ be the $G_J(M)$-invariant hull of $M\backslash K$. Lemma 4.5 implies that each function $u \in L^1_{\text{loc}}(H)$ which is a CR-function on $H\backslash K$ is a CR-function on $H\backslash K'$. K' is a compact CR-invariant subset of M. Since $L^{\mathsf{p}}_{\text{loc}}(H) \subset L^1_{\text{loc}}(H)$ for each $\mathsf{p} \geq 1$, the $L^{\mathsf{p}}(H\backslash K)$-removability is guaranteed, if K' is empty. Hence, theorem 2b is proved for $1 \leq \mathsf{p} < 2$. If $2 \leq \mathsf{p} < \infty$ and the $(2n-1-\mathsf{p}')$-dimensional Hausdorff measure of K' is finite, or if $\mathsf{p} = \infty$ and the $(2n-2)$-dimensional Hausdorf measure is zero, then by general results on removable singularities for solutions of first order differential equations (see [Ha–Po], theorem 4.1) K' is removable for the corresponding space $L^{\mathsf{p}}_{\text{loc}}$. Theorem 2b is proved. □

Proof of lemma 4.5. Let the CR-curve γ on M be the integral curve of a CR-vector field $\tilde{X}_{2n-2}$ (more precisely, of a $G_J(M)$-vector field) of class C^1 defined in a neighbourhood of $\gamma([0,1])$ on M. Since for $p \in M$ the complex tangent space $T^J_p M$ is contained in $T^J_p H$ we may continue $\tilde{X}_{2n-2}$ to a CR-vector field X_{2n-2} of class C^1 on H (more precisely, to a $G_J(H)$-vector field). Let $X_1, \dots, X_{2n-3}$ be CR-vector fields ($G_J(H)$-vector fields) of class C^1 in a neighbourhood of $\gamma([0,1])$ on H such that the $2n-2$ real vectors $X_1(p), \dots, X_{2n-2}(p)$ span $T^J_p H$ for each point p of the mentioned neighbourhood. (We may always consider instead of γ a small part of γ, $\gamma|[\tau,1]$, τ close to 1, and make a reparametrization. So we work in a sufficiently small neighbourhood of the point $\gamma(1)$.)

M is a generic CR-manifold, hence the subspaces $T_{\gamma(1)}M$ and $T^J_{\gamma(1)}H$ of $T_{\gamma(1)}H$ intersect transversally. Take $t_0 \in (0,1)$ close enough to 1 and consider a small C^1 curve $l : (-\delta, \delta) \to M$ in M through $l(0) = \gamma(t_0)$, which is transverse to $(T^J_{\gamma(t_0)}H) \cap T_{\gamma(t_0)}M$. Now we introduce Euclidean coordinates in a neighbourhood of $\gamma(t_0)$ on H using the mapping $g_{X,T}$ of section 2. (Here $X = (X_1, \dots, X_{2n-2})$ with the CR-vector fields X_j defined above.)

Let $\Omega = \Omega_1 \times \cdots \times \Omega_{2n-2} \subset \mathbb{R}^{2n-2}$ be an open set containing $\{0\} \times \cdots \times \{0\} \times [0,1]$. The mapping

$$(4.30)\qquad (T,s) \to G(T,s) \stackrel{\text{def}}{=} g_{X,T}(l(s)) = g_{X_{2n-2},T_{2n-2}} \circ \cdots \circ g_{X_1,T_1}(l(s)),\ \ T \in \Omega,\ \ |s| < \delta,$$

defines a diffeomorphism of $\Omega \times (-\delta, \delta)$ onto a neighbourhood of $\gamma([0,1])$ on H. Denote the set $G(\Omega \times (-\delta,\delta))$ by H_0. The tangent space at $T = 0$ of the $(2n-2)$-dimensional C^1 manifold

$$(4.31)\qquad Q_0 \stackrel{\text{def}}{=} \{G(T,0),\ T \in \Omega\}$$

is equal to $T^J_{\gamma(t_0)}H$. We work in a small neighbourhood of $\gamma(1)$, in particular, $\gamma(t_0)$ is close to $\gamma(1)$, so for each fixed $s \in (-\delta, \delta)$ the $(2n-2)$-dimensional manifold

$$(4.32)\qquad Q_s \stackrel{\text{def}}{=} \{G(T,s) : T \in \Omega\}$$

is a C^1-manifold which is the graph of a C^1 function over an open subset of $T^J_{\gamma(1)}H$. In particular, M intersects each Q_s transversally in H. Thus, $M \cap Q_s$ is a proper submanifold of Q_s of class C^1 and of dimension $2n-3$. It is clear that the closure of $Q_s \setminus M$ in $M \cap H_0$ is equal to Q_s.

Fix now a small neighbourhood U_γ of $\gamma([0,1))$ on H which consists of $L^p(H\setminus K)$-regular points. Since $l(0) = \gamma(t_0) \in U_\gamma$ we may suppose (shrinking $\Omega_1, \dots, \Omega_{2n-3}$ and $(-\delta,\delta)$ if necessary) that for an open interval $\tilde{\Omega}_{2n-2}$ containing t_0 the set

$$(4.33)\qquad \tilde{H}_0 \stackrel{\text{def}}{=} G(\tilde{\Omega} \times (-\delta,\delta)),\quad \tilde{\Omega} = \Omega_1 \times \cdots \times \Omega_{2n-3} \times \tilde{\Omega}_{2n-2},$$

is contained in U_γ. Set $M_\gamma = M \setminus U_\gamma$.

It is clear that $H_0 \setminus M_\gamma$ is connected. Moreover, each point of H_0 can be joined with a point of $\tilde{H}_0$ by an integral curve of X_{2n-2}, in other words, for a suitable interval I we have the relation

$$(4.34)\qquad \bigcup_{t \in I} g_{X_{2n-2},t}(\tilde{H}_0) = H_0.$$

For each $s, |s| < \delta$, denote by $\mathcal{N}(s)$ the $(H_0 \setminus M_\gamma)$-orbit through $l(s)$. For each orbit $\mathcal{N}(s)$ denote by $m(s)$ the set of all limit points of $\mathcal{N}(s)$ on $M \cap H_0$:

$$(4.35)\qquad m(s) = (\overline{\mathcal{N}(s)} \cap M) \cap H_0.$$

We need the following

Claim 4.3. *For each* $s, |s| < \delta$, *the set* $N(s) \overset{def}{=} \mathcal{N}(s) \cup m(s)$ *is the* H_0*-orbit through* $l(s)$ *and*

$$\dim_r \mathcal{N}(s) = \dim_r N(s).$$

Proof. If an $G_J(H_0\backslash M_\gamma)$-orbit $\mathcal{N}(s)$ contains for some $s' \in (-\delta, \delta)$ some point of $\tilde{Q}_{s'} \overset{\text{def}}{=} G(\tilde{\Omega} \times \{s'\})$ then, obviously, it contains $\tilde{Q}_{s'}$. Since X_{2n-2} is tangent to M at points of M we have for $t \in I$

$$(4.36) \qquad \begin{aligned} &g_{X_{2n-2},t}(M \cap \tilde{H}_0) \subset M, \\ &g_{X_{2n-2},t}(\tilde{H}_0\backslash M) \subset \tilde{H}_0\backslash M. \end{aligned}$$

Thus, all points of $G(\Omega \times \{s'\})\backslash M = Q_{s'}\backslash M$ can be joined with a point in $G(\tilde{\Omega} \times \{s'\})\backslash M = \tilde{Q}_{s'}\backslash M$ by an integral curve of X_{2n-2} which is contained in $Q_{s'}\backslash M$. *Thus, if* $\mathcal{N}(s)$ *contains a point of* $Q_{s'}\backslash M$, *it contains also a point in* $\tilde{Q}_{s'}$ *and thus it contains* $\tilde{Q}_{s'}$, *and, therefore, it contains also* $Q_{s'}\backslash M$. Since each $Q_{s'}$, is the graph of a C^1-function over a subset of $T^J_{\gamma(1)}M$,

$$(4.37) \qquad (\overline{Q_{s'}\backslash M}) \cap M \cap H_0 = Q_{s'} \cap M.$$

Consider first an $(H_0\backslash M_\gamma)$-orbit $\mathcal{N}(s)$ of real codimension one in $H_0\backslash M_\gamma$. In this case $\mathcal{N}(s)$ is an analytic hypersurface which contains the subset $\tilde{Q}_s \cup (Q_s\backslash M)$ of Q_s. At each point p of this subset the tangent space T_pQ_s is equal to T^J_pH. Since Q_s is the graph of a C^1-function over a subset of $T^J_{\gamma(1)}H$ and $Q_s \cap M$ is a proper submanifold of Q_s of dimension less than Q_s by continuity for $p \in Q_s \cap M$ the same equality

$$(4.38) \qquad T_pQ_s = T^J_pH$$

holds. Thus (4.38) holds on Q_s and, therefore, Q_s is an analytic manifold of complex dimension $n-1$, which is a proper, relatively closed submanifold of H_0. Thus Q_s is an H_0-orbit, and the connected $G_J(H_0 \setminus M_\gamma)$-invariant set $Q_s \setminus M_\gamma$, which contains $l(s)$, is a $(H_0 \setminus M_\gamma)$-orbit and is therefore equal to $\mathcal{N}(s)$. In the case $\dim_r \mathcal{N}(s) = 2n-2$ the claim is proved.

Consider now $(H_0 \setminus M_\gamma)$-orbits $\mathcal{N}(s)$ of dimension $2n-1$. Recall that $\mathcal{N}(s)$ contains all sets $\tilde{Q}_{s'} \cup (Q'_s \setminus M)$ which it intersects. Since M is generic and, hence, T_pM does not contain the complex tangent space T^J_pH for $p \in M$ the set $m(s) \subset M$ is contained in the $G_J(H_0)$-invariant hull of $\mathcal{N}(s)$. Moreover, $N(s) = \mathcal{N}(s) \cup m(s)$ is a connected open subset of H_0. Since $\bigcup_{\dim \mathcal{N}(s)=2n-2} N(s)$ is $G_J(H_0)$-invariant, the same is true for the complement. Since each $N(s)$ of dimension $(2n-1)$ is a connected component of the complement $\bigcup_{\dim \mathcal{N}(s)=2n-1} N(s)$ we are done. □

Continue now the proof of lemma 4.5. We need the following claim which concerns $(H_0 \setminus M_\gamma)$-orbits of maximal dimension and is in this case formally slightly sharper than lemma 4.5.

Claim 4.4. *Suppose the $(H_0 \setminus M_\gamma)$-orbit $\mathcal{N}(s)$ is open in H_0. Then each CR-function u of class $L^p(\mathcal{N}(s))$, $1 \le p \le \infty$, defines a CR-function on $N(s)$ (of class L^p).*

Each point of $M \cap H_0$ can be joined with a point in $M \cap \tilde{H}_0$ (i.e. with a regular point) by an integral curve of X_{2n-2} (which is contained in $M \cap H_0$). Hence claim 4.4 is exactly lemma 4.5 (with γ and $\gamma(1)$ replaced by an integral curve of X_{2n-2} and its endpoint) in the particular case, when the $(H_0 \setminus M_\gamma)$-orbit $\mathcal{N}(s)$ has maximal dimension. For notational convenience denote the mentioned integral curve of the vector field X_{2n-2}, as before by γ and its endpoint by $\gamma(1)$. Moreover, denote as before a neighbourhood of $\gamma([0,1))$ on H, consisting of regular points, by U_γ and let $M_\gamma = M \setminus U_\gamma$. We have to prove that u is a CR-function in a neighbourhood of $\gamma(1)$ on H_0. Now, if $\mathcal{N}(s)$ is open in H_0, $\mathcal{N}(s)$ contains a minimal point of H_0. Thus, by lemma 4.3 and claim 4.2 there is a connected open set $\mathcal{O}(s)$ which contains a one-sided neighbourhood of each point of $\mathcal{N}(s)$ and an analytic function $u_{\mathcal{O}(s)}$ in $\mathcal{O}(s)$ which is locally the H^p-extension of u.

We consider now a small C^2-deformation of H_0 which has fixed all points of M and of $H_0 \setminus N(s)$ and moves all points of $N(s) \setminus M$ into $\mathcal{O}(s)$. Denote the obtained manifold by H_0'.

Apply the scheme of the proof of proposition 4.1. Recall that the main point is to obtain a small C^2-deformation of M which fixes M_γ and moves small disjoint CR-cones at some points $\gamma(t_k)$, $t_k \in (0,1)$, $t_k \to 1$, into $\mathcal{O}(s)$ in such a way that $\gamma(1)$ becomes a minimal point of the deformed manifold $M^d = \{z + d(z) : z \in M\}$ (see the lemmas 3.3 and 3.4). Let as in the proof of proposition 4.1 the function d' be close to d and vanish in a neighbourhood of $\gamma(1)$ on M. Say d' moves a finite number of CR-cones on M at $\gamma(t_k)$, $k = 1, \dots, k_0$, into $\mathcal{O}(s)$. Denote by the same letter d' a suitable extension of that function to H_0' such that the deformed manifold $(H_0')^{d'}$ is contained in $H_0' \cup \mathcal{O}(s)$ and thus in $M \cup \mathcal{O}(s) \cup (H_0 \setminus N(s))$. As in the proof of proposition 4.1 we sweep out a small neighbourhood of $\gamma(1)$ on H_0' by disjoint manifolds M_t, $t \in (-\delta, \delta)$, which are close to $M_0 \overset{\text{def}}{=} M$ in the C^2 topology. The deformed manifolds

$$M_t^{d'} = \{z + d'(z) : z \in M_t\}$$

are close to M_0^d in C^2. Hence, by the lemmas 3.5 and 3.6, each continuous CR-function on $M_t^{d'}$, $t \in (-\delta, \delta)$, has analytic extension to a wedge W_t' the edge of which is an open set on $M_t^{d'}$. Moreover, since d' vanishes near $\gamma(1)$ the W_t' contain wedges W_t which edges are open subsets of M_t, and the W_t are close to each other. Recall now that the CR-function u of class $L^p_{\text{loc}}(H_0)$ has an analytic extension $u_{\mathcal{O}(s)}$ to the set $\mathcal{O}(s)$. $\mathcal{O}(s)$ is a neighbourhood of each point in $H_0' \setminus M$ which is close to $\gamma(1)$. Hence, $\mathcal{O}(s)$ contains each M_t, $t \in (-\delta, \delta) \setminus \{0\}$, and so $u_{\mathcal{O}(s)}$ has analytic extension to each W_t, $t \in (-\delta, \delta) \setminus \{0\}$. As in the proof of proposition 4.1 we get analytic extensions u_+ and u_- of $u_{\mathcal{O}(s)}$ to $\mathcal{O}_+ = \bigcup_{t \in (0,\delta)} W_t$ and to $\mathcal{O}_- = \bigcup_{t \in (-\delta,0)} W_t$.

In the same way as in that proof, we see that $u_+ = u_-$ in $\mathcal{O}_+ \cap \mathcal{O}_-$. We obtained an analytic extension of $u_{\mathcal{O}(s)}$ to a one-sided neighbourhood of $\gamma(1)$ with respect to H_0' and therefore, we obtained an analytic extension of u to a one-sided neighbourhood $\mathcal{O}_1$ of $\gamma(1)$ with respect to H_0.

It remains to show that the extended function is of class H^p in a suitably choosen one-sided neighbourhood of $\gamma(1)$ (with respect to H_0). To see this use the deformed manifold M^d obtained above. By Tumanovs theorem ([Tu2]) for $\alpha' \in (\frac{1}{2}, 1)$ there exists an analytic function $w \in A_0^{1,\alpha'}$ with values in $T^c_{\gamma(1)} M^d$, such that

$$\frac{\partial}{\partial \lambda} f_{\gamma(1),w}(\lambda)|_{\lambda=1} \tag{4.39}$$

is transverse to H and directed into $\mathcal{O}(s)$. Here by $f_{q,w}$, $q \in M$ close to $\gamma(1)$, we denote the family of analytic discs of class $C^{1,\alpha}$, $\alpha \in (\frac{1}{2}, \alpha')$, with boundary $f_{q,w}(\mathbb{T}) \subset M^d$ for which $f_{q,w}(1) = q$ and the orthogonal projection π onto $T^J_{\gamma(1)} M^d$ is equal to $\pi f_{q,w} = \pi q + w$. The existence of the w for which (4.39) has the desired properties follows from the construction of M^d: The vectors (4.39) sweep out a cone in the transverse directions to M^d and some of these vectors are directed into $\mathcal{O}_1$ by the construction of the set $\mathcal{O}_1$.

For all manifolds which are C^2 close to M^d there exist similar families of analytic discs. More precisely, we need the following: Let the function d' on M be as above and sufficiently close to d in C^2. Extend this time the function d' to H_0 (not to H_0'). Denote the deformed manifold $\{z + d'(z) : z \in H_0\}$ by $H_0^{d'}$. We may suppose that d' moves small neighbourhoods of $\gamma(t_k)$ on H_0, $k = 1, \dots, k_0$, into $\mathcal{O}(s)$ and vanishes outside this neighbourhoods. Hence each CR-function u on $H_0 \setminus M_\gamma$ of class L^p $(1 \le p < \infty)$ extends to a CR-function $u_{d'}$ of the same class on $(H_0)^{d'} \setminus M_\gamma$. (This follows from lemma 3.6.c, from the fact that u has local H^p-extension at $\gamma(t_k)$ to $\mathcal{O}(s)$ and from Carleson's embedding theorem (see also the proof of lemma 4.4).) Sweep out a neighbourhood of $\gamma(1)$ on $H_0^{d'}$ by disjoint C^2 manifolds $M^{d'}(t)$, $|t| < \sigma$, such that $M^{d'}(0) = M^{d'}$ and each $M^{d'}(t)$, $|t| < \sigma$, is sufficiently close to $M^{d'}$ (and thus to M^d) in the C^2 topology. We may suppose that $(v, t) \in V \times \{|t| < \sigma\}$ define Euclidean parameters on $H_0^{d'}$ for a neighbourhood V of zero in $\mathbb{R}^{2n-2}$. More precisely, there exists a C^2 diffeomorphism φ from $V \times \{|t| < \sigma\}$ onto a neighbourhood of $\gamma(1)$ on $H_0^{d'}$, such that for each t the mapping $v \to \varphi(v,t)$ is a diffeomorphism onto an open subset of $M^{d'}(t)$. For each t we get analytic discs $F_{v,t}$ with boundary $F_{v,t}(\mathbb{T})$ contained in $M^{d'}(t)$ such that $F_{v,t}(1) = \varphi(v,t) \in M^{d'}(t)$ and $\pi F_{v,t} = \pi\varphi(v,t) + w$. By Tumanovs theorem [Tu 2] the mapping

$$(v, t, \lambda) \to F_{v,t}(\lambda), \quad (v, t, \lambda) \in V \times \{|t| < \sigma\} \times (\lambda_0, 1], \tag{4.40}$$

is a diffeomorphism of class $C^{1,\alpha}$ onto $(H_0^{d'} \cap \omega_1) \cup \tilde{\mathcal{O}}$ for a small neighbourhood ω_1 of $\gamma(1)$ and a one-sided neighbourhood $\tilde{\mathcal{O}}$ of $\gamma(1)$ (with respect to $H_0^{d'}$). We may suppose that $\tilde{\mathcal{O}} \subset \mathcal{O}_1$, moreover, $V = V_1 \times V'$ and (after shrinking V_1) for each fixed $v_1 \in V_1$ the mapping

$$(4.41) \qquad (\theta, v', t, \lambda) \to F_{(v_1,v'),t}(\lambda e^{i\theta}), \;\; (\theta, v', t, \lambda) \in \{|\theta| < \delta\} \times V' \times \{|t| < \sigma\} \times (\lambda_0, 1]$$

is also a diffeomorphism onto $(H_0^{d'} \cap \omega_2) \cup \tilde{\mathcal{O}}'$ for another neighbourhood ω_2 of $\gamma(1)$ and another one-sided neighbourhood $\tilde{\mathcal{O}}' \subset \mathcal{O}_1$ of $\gamma(1)$.

Recall now, that to each CR-function u of class $L^{\mathrm{p}}_{\mathrm{loc}}$ $(1 \leq \mathrm{p} < \infty)$ on $H_0 \setminus M_\gamma$ corresponds a CR-function $u_{d'}$ of the same class on $(H_0)^{d'} \setminus M_\gamma$. Since M is a *generic* CR-manifold it is easy to see that for a suitable neighbourhood ω_3 of $\gamma(1)$ in $\mathbb{C}^n$ the set $H_0^{d'} \cap \omega_3 \setminus M^{d'}$ consists of two connected components. After a change of coordinates on H (we have to change only t-variables and do not change the form of the corresponding system (3.12)) and shrinking ω_3, if necessary, we may suppose that both connected components of $(H_0^{d'} \cap \omega_3) \setminus M^{d'}$ are of the form described in remark 3.1. Hence, $u_{d'}$ can be approximated by polynomials in L^{p} on each of this sets. Take now any sequence of hypersurfaces $(H^{d'})_\varepsilon \subset \mathcal{O}'$ which approximates $H^{d'} \cap \omega_1$. There is a function Λ_ε of class C^1 with values in $(0,1]$ and small C^1 norm of $1 - \Lambda_\varepsilon$, such that for each fixed $v_1 \in V_1$

$$(4.42) \qquad (H^{d'})_\varepsilon \subset \{F_{(v_1,v'),t}(\Lambda_\varepsilon((v_1, v'), t, \theta)e^{i\theta}) : (\theta, v', t) \in \{|\theta| < \delta\} \times V' \times \{|t| < \sigma\}\}$$

Use now an inequality like (4.14) for each polynomial $\mathcal{P}$, integrate over $\tilde{V}' \times \{t \in (0, \tilde{\sigma})\}$ for suitable small $\tilde{V}' \subset V'$ and $\tilde{\sigma} < \sigma$ and use the approximation theorem. Do the same for $\{t \in (0, \tilde{\sigma}\}$ replaced by $\{t \in (-\tilde{\sigma}, 0\}$ and use that the image of $\{t = 0\}$ under the mapping

$$(4.43) \qquad (\theta, v', t) \to F_{(v_1,v'),t}(\Lambda_\varepsilon((v_1, v'), t, \theta)e^{i\theta})$$

has $(2n-1)$-dimensional measure zero. Integrate over v_1. We get an estimate of the L^{p}-norm of $u_{\mathcal{O}(s)}|(H^{d'})_\varepsilon \cap \omega_4$ by the L^{p}-norm of u over a suitable subset of H. (ω_4 is a suitable neighbourhood of $\gamma(1)$.) It follows (see also the proof of lemma 4.3) that the analytic extension is of class H^{p} in a suitable one-sided neighbourhood of $\gamma(1)$ with respect to $H^{d'}$. Since $H^{d'}$ contains a neighbourhood of $\gamma(1)$ on H claim 4.4 is proved. □

Remark 4.2. The arguments which we used to prove that the extension is of class L^{p} are sufficient to prove the claim 4.4. We have to apply it twice with two different functions $\omega_1, \omega_2 \in A_0^{1,\alpha_1}$ such that the vectors $\frac{\partial}{\partial \lambda} f_{\gamma(1),\omega_1}(\lambda)|_{\lambda=1}$ and $\frac{\partial}{\partial \lambda} f_{\gamma(1),\omega_2}(\lambda)|_{\lambda=1}$ are linearly independent. Doing so we do not need claim 4.2. Claim 4.2 seems to us interesting for itself, so we included it.

If the $(H_0 \setminus M_\gamma)$-orbit $\mathcal{N}(0)$ through $l(0)$ is an open subset of H then we are done by claim 4.4. Suppose it is not. Denote by $\mathcal{A}_0$ the union of all H_0-orbits of real codimension one, i.e. the union of all $N(s) = \mathcal{N}(s) \cup m(s)$ which are analytic manifolds. Let $\mathcal{A}_1$ be the union of H_0-orbits of real codimension zero.

Claim 4.5. *Let u be a CR-function of class L^{p} on $H_0 \setminus M_\gamma$, $1 \leq \mathsf{p} < \infty$. After changing u on a set of $(2n-1)$-dimensional measure zero, u becomes an analytic function on each analytic manifold $\mathcal{N}(s)$ for which $N(s)$ is contained in $\mathcal{A}_0$.*

Proof. The claim follows from the approximation lemma 3.6.c. Let U be an open subset of $H_0 \setminus M_\gamma$ of sufficiently small diameter and let $\mathcal{P}_n$ be polynomials which approximate u in $L^{\mathsf{p}}(U)$. We may suppose (after a complex linear change of coordinates) that U contains zero and near zero H has the form

$$\mathsf{x} = h(\mathsf{w}, \mathsf{y}) \tag{4.44}$$

for a (real) C^2 function h with $h(0) = 0$, $(\nabla h)(0) = 0$. $(\mathsf{w}, \mathsf{x} + i\mathsf{y}) = (\mathsf{w}_1, \dots, \mathsf{w}_{n-1}, \mathsf{x} + i\mathsf{y})$ are the complex coordinates of $\mathbb{C}^n$. Let $X_1, \dots, X_{2n-2}$ be the CR-vector fields on H near zero for which the orthogonal projection onto $T_0^J H$ is equal to Re W_1, Im $W_1, \dots$,Re W_{n-1}, Im W_{n-1}, respectively. ($W_1, \dots, W_{n-1}$ are the standard complex vector fields in $T_0^J H$ which we identify with $\mathbb{C}^{n-1}$ with complex coordinates $\mathsf{w}_1, \dots, \mathsf{w}_{n-1}$.) Denote by $p(y)$ the point $p(y) = (h(y, 0) + iy, 0)$ on H and define C^1 coordinates on H using the mapping $g_{X,T}$ with the just defined CR-vector fields:

$$(T, y) \to g_{X,T}(p(y)) \stackrel{\text{def}}{=} G(T, y). \tag{4.45}$$

We may suppose that

$$U = \{G(T, y) : \ T \in \mathbb{R}^{2n-2}, \ |T| < \delta, \ y \in \mathbb{R}, \ |y| < \delta\}. \tag{4.46}$$

It is now clear that $\mathcal{A}_0 \cap U$ has the form

$$\mathcal{A}_0 \cap U = \{G(T, y) : |T| < \delta, \ y \in E\} \tag{4.47}$$

for a closed subset E of $(-\delta, \delta)$. Moreover, write $w_j = T_{2j-1} + iT_{2j}$. Then the orthogonal projection of $G(T, y)$ onto $T_0^J H$ is equal to $(w_1, \dots, w_{n-1})$, and for fixed y the set $\{G(T, y) : |T| < \delta\}$ has the form

$$\{(\mathfrak{g}(y, w), w) : |w| < \delta\} \tag{4.48}$$

for a C^1-function $\mathfrak{g}$ on $\{|w| < \delta\} \times \{|y| < \delta\}$, which is for fixed $y \in E$ an analytic function of w.

Now, for each polynomial $\mathcal{P}_n$, each test function $\psi \in L^{\mathsf{p}'}(E)$ (p' is the exponent conjugate to p) and each smooth test function χ with compact support in $\{|w| < \delta\}$ the following equalities hold

$$(4.49)\qquad \int_E dy \int_{|w|<\delta} dm_{2n-2}(w)\ \overline{\psi}(y)\cdot\overline{\chi}(w)\cdot\frac{\partial}{\partial\overline{w}_j}\mathcal{P}_n(\mathfrak{g}(y,w),w)=0,\ \ j=1,\dots,n-1.$$

Integrate by parts with respect to w and take the limit for $n\to\infty$, we get for each $\psi\in L^{p'}(E)$ and $\chi\in C_0^\infty(|w|<\delta)$

$$(4.50)\qquad \int_E \overline{\psi(y)}dy \int_{|w|<\delta} (\overline{\frac{\partial}{\partial w_j}\chi})(w)\cdot u(\mathfrak{g}(y,w),w)dm_{2n-2}(w)=0,\ \ j=1,\dots,n-1.$$

Thus, for almost all $y\in E$ the function $w\to u(\mathfrak{g}(y,w),w)$ satisfies the Cauchy-Riemann equations in the weak sense and, therefore, it coincides almost everywhere on $\{|w|<\delta\}$ with an analytic function of w. Claim 4.5 is proved. □

Assume now that u is analytic on each $\mathcal{N}(s)$ with $N(s)\subset\mathcal{A}_0$. The following two lemmas will imply lemma 4.5, and hence, theorem 2b.

Lemma 4.6. *If $N(s)$ is contained in $\mathcal{A}_0$ then $u|\mathcal{N}(s)$ extends to an analytic function on $N(s)$.*

Lemma 4.7. *Suppose $u\in L^p(H_0)$, $1\le p\le\infty$. If $u|\mathcal{A}_1$ is a CR-function and for each $N(s)\subset\mathcal{A}_0$ the function $u|N(s)$ is analytic, then u is a CR-function on H_0.*

Proof of Lemma 4.6. Fix the analytic manifold $N(s)\subset\mathcal{A}_0$ and consider the hypersurface $\mathcal{M}_0\stackrel{\text{def}}{=}M\cap N(s)$ in $N(s)$. Recall that each point in $M\cap N(s)$ can be joined with a point in $M\cap N(s)\cap U_\gamma$ by an integral curve of the $G_J(M)$-vector field $\tilde{X}_{2n-2}$ (the integral curve contained in $M\cap N(s)$). Note that $u|N(s)$ is analytic on $N(s)\setminus M$ and on $N(s)\cap U_\gamma$. Introduce Euclidean coordinates on $N(s)$ to identify $N(s)$ with an open subset of $\mathbb{C}^{n-1}$ and consider in these coordinates small parallel shifts $\mathcal{M}_t$ of $\mathcal{M}_0\cap N(s)$ to both sides of $\mathcal{M}_0$ (in $N(s)$). Apply to the manifolds $\mathcal{M}_t$ the fact that the analytic extendability of CR-functions on hypersurfaces to one-sided neighbourhoods propagates along CR-orbits. We use that for each $t\neq 0$ $u|\mathcal{M}_t$ is a CR-function which has analytic extension to $U_\gamma\cap N(s)$. By the choice of U_γ and by the fact that the $\mathcal{M}_t$ are parallel shifts of $\mathcal{M}_0$, the set $U_\gamma\cap N(s)$ contains for each t close to zero

1. open subsets Q_t of $\mathcal{M}_t$ with the property that the $G_J(\mathcal{M}_t)$-invariant hull of Q_t is equal to $\mathcal{M}_t$ and the Q_t are parallel shifts of Q_0,
2. open subsets $\mathfrak{D}_t$ of $N(s)$ which are parallel shifts of $\mathfrak{D}_0$, and contain a one-sided neighbourhood in $N(s)$ of each point of Q_t.

Thus, $u|\mathcal{N}(s)$ has analytic extension to a one-sided neighbourhood $\mathcal{U}_t$ of $\mathcal{M}_t$ for each $t\neq 0$ which is small enough, $\mathcal{U}_t$ being a small parallel shift of a one-sided neighbourhood $\mathcal{U}_0$ of $\mathcal{M}_0$. It follows that $u|\mathcal{N}(s)$ has analytic extension to each point of $M\cap N(s)$. □

Proof of Lemma 4.7. A proof for continuous functions instead of L^p-functions was given for example in [Di–Pi] for the case of C^2-hypersurfaces and in [Jö4] for the case of CR-manifolds of higher codimension of class C^3 (in the last case the coordinates of the form (4.45) are of class C^2). We will give a proof for C^2 hypersurfaces in the L^p-case. There are some difficulties in the case of C^2 hypersurfaces (the coordinates (4.45) are ensured only to be C^1), so our proof will be close to that in [Di–Pi]. Let U be a small open subset of H_0, $\overline{U} \subset H_0$. Suppose near $\overline{U}$ the hypersurface H can be described by (4.44). Write the tangential Cauchy-Riemann operators L_j in coordinates (y,w), $\varphi(\mathsf{y},\mathsf{w}) = (h(\mathsf{y},\mathsf{w}) + i\mathsf{y}, \mathsf{w})$. We have to prove the tangential Cauchy-Riemann equations in the weak sense:

$$(4.51) \qquad \int\limits_{|\mathsf{w}|<\delta, |\mathsf{y}|<\delta} L_j^t(\chi \circ \varphi) \cdot u \circ \varphi \; dm_{2n-1}(\mathsf{w},\mathsf{y}) = 0$$

for each smooth function χ with compact support in $U \subset \varphi(\{|\mathsf{w}| < \delta\} \times \{|\mathsf{y}| < \delta\})$. L_j^t is the transpose of L_j in the considered coordinates. Suppose the diameter of U is sufficiently small with respect to δ. To prove (4.51) we will divide the domain of integration. One part will be contained in $\mathcal{A}_1$ and the integral over this part will be zero (see claim 4.7, below). The other part will be a small neighbourhood of $\mathcal{A}_0$. The estimate of the integral over this part is based on the fact that u is analytic on each leaf in $\mathcal{A}_0$ and, therefore, u is close to an analytic function on the mentioned neighbourhood of $\mathcal{A}_0$ in an appropriate sense. We need the following preparation.

Consider the set $E = \{y \in (-\delta,\delta) : G(T,y) \subset \mathcal{A}_0 \text{ for } |T| < \delta\}$ (with G as in (4.45)). Let χ_E be the characteristic function of E, $\chi_E = 1$ on E and $\chi_E = 0$ on $(-\delta,\delta) \setminus E$. We get further a function f on $(-\delta,\delta)$ from the following considerations. The function G of (4.45) defines C^1 coordinates on U:

$$(\mathsf{w}, h(\mathsf{w},\mathsf{y}) + i\mathsf{y}) = G(T,y).$$

The connection between coordinates (w,y) and (T,y) is the following:

$$\mathsf{w}_j = G_j(T,y) = T_{2j-1} + iT_{2j}, j = 1,\dots,n-1; \;\; \mathsf{y} = \operatorname{Im} G_n(T,y),$$

or, equivalently,

$$(4.52) \qquad (\mathsf{w},\mathsf{y}) = (\varphi^{-1} \circ G)(T,y) \text{ with } \varphi^{-1} \circ G \text{ of class } C^1.$$

Denote the continuous function $L_j^t(\chi \circ \varphi)$ which appears in (4.51) by $\Gamma \circ \varphi$ and fix it. (4.51) becomes

$$(4.53) \qquad \int\limits_{|T|<\delta} \int\limits_{|y|<\delta} \Gamma \circ G \cdot u \circ G \cdot |\det d(\varphi^{-1} \circ G)| dm_{2n-1}(T,y) = 0.$$

For $y \in (-\delta, \delta)$ we put now

$$f(y) = \int\limits_{|T|<\delta} (\Gamma \circ G)(y,t) \cdot (u \circ G)(y,T) \cdot |\det d(\varphi^{-1} \circ G)(T,y)|\, dm_{2n-2}(T). \tag{4.54}$$

Denote by E' the set of all points of E which are Lebesgue points for the function f and for the function χ_E as well. It is well-known (see, for example, [Stein2]), that $E \setminus E'$ has linear measure zero. Let $y \in E'$. Since y is a Lebesgue point for f the integral

$$\frac{1}{|I(y)|} \int\limits_{I(y)} dy'(f(y) - f(y')) \tag{4.55}$$

is arbitrarily small, if the length $|I(y)|$ of the interval $I(y) \subset (-\delta, \delta)$, which contains y, is small enough. Since y is a Lebesgue point for χ_E, there are intervals $I(y)$ of arbitrarily small length containing y, such that their end points are contained in E. Call intervals with end points in E admissible.

Recall that for each $y \in E$ the set $Q(y) = G(\{|T| < \delta\} \times \{y\})$ is an analytic manifold (of complex codimension one in $\mathbb{C}^n$), and the function $u|Q(y)$ is analytic. To each $y \in E'$ we associate an analytic function u_y^* in a neighbourhood of $Q(y)$ (in $\mathbb{C}^n$) which coincides with $u|Q(y)$ on $Q(y)$. For each interval $I \subset (-\delta, \delta)$ denote by $\mathcal{B}_I$ the "box"

$$\mathcal{B}_I = \{G(T, y') : |T| < \delta,\ y' \in I\}. \tag{4.56}$$

If $I = I(y) = (y_1, y_2)$ with $y_1 < y < y_2$ and $y_1, y_2 \in E$ we will call the box admissible and denote it by $\mathcal{B}(y) \overset{\text{def}}{=} \mathcal{B}_{I(y)}$. The boundary pieces $\{G(T, y_k) : |T| < \delta\}$, $k = 1, 2$, of an admissible box $\mathcal{B}(y)$ are analytic manifolds and on the rest of the boundary of $\mathcal{B}(y)$ the function Γ vanishes.

The lemma 4.7 follows now from the two claims below, and the covering theorem of Vitali (for example [Stein 2], I.5.5.4, or [Saks]).

Claim 4.6. *Fix a small positive number ε. For each $y \in E'$ there exist arbitrarily "thin" admissible boxes $\mathcal{B}(y) = \{G(T, y') : |T| < \delta,\ y' \in I(y)\}$, $|I(y)|$ arbitrarily small, such that*

$$\Big| \int\limits_{\varphi^{-1}(\mathcal{B}(y))} \Gamma \circ \varphi \cdot u \circ \varphi \ dm_{2n-1}(\mathrm{w}, \mathrm{y}) \Big| \le \varepsilon \cdot C \cdot m_{2n-1}(\varphi^{-1}(\mathcal{B}(y))). \tag{4.57}$$

Claim 4.7. *Let U' be a connected open subset contained in $U \setminus \mathcal{A}_0$ with piecewise smooth boundary $\partial U'$. Suppose $\partial U'$ is the union of two disjoint sets ∂_1 and ∂_2, where ∂_1 is a connected analytic manifold contained in $\mathcal{A}_0$ and χ vanishes near ∂_2. Then*

$$(4.58) \qquad \int\limits_{\varphi^{-1}(U')} L_j^t(\chi \circ \varphi) \cdot u \circ \varphi \; dm_{2n-1}(\mathrm{w}, \mathrm{y}) = 0.$$

First we finish the proof of the lemma 4.7. Fix a sufficiently small $\varepsilon > 0$. By the covering theorem of Vitali (applied to the set E' and small admissible covering intervals $I(y)$ for $y \in E'$) there exists a sequence of disjoint admissible intervals $I(y_k)$, $k = 1, 2, \dots$ which satisfy the conclusion of claim 4.6 such that

$$(4.59) \qquad m_1(E' \setminus \bigcup_k I(y_k)) = 0.$$

Hence, for those $I(y_k)$,

$$(4.60) \qquad | \int\limits_{\varphi^{-1}(\bigcup_k \mathcal{B}(y_k))} L_j^t(\chi \circ \varphi) \cdot u \circ \varphi \; dm_{2n-1}(\mathrm{w}, \mathrm{y})| \le \varepsilon \cdot C \cdot (m_{2n-1}(\varphi^{-1}(\bigcup \mathcal{B}(y_k))).$$

Show, that the integral over the complement is zero. Note first that

$$(4.61) \qquad F \stackrel{\text{def}}{=} \overline{\bigcup_k I(y_k)} \setminus \bigcup_k I(y_k) \subset E.$$

Indeed, let y^* be a limit point of the set $\bigcup_k I(y_k)$ which is not contained in this set itself. Then each interval $I(y_k)$ is either on the right or on the left of y^* and therefore y^* is also a limit point for the endpoints of the $I(y_k)$. The endpoints of the $I(y_k)$ are contained in E, E is closed, and, thus, $y^* \in E$. By (4.59) the set F has measure zero. Hence, the set $(-\delta, \delta) \setminus (\overline{\bigcup_k I(y_k)} \cup E)$ differs from $(-\delta, \delta) \setminus (\bigcup_k I(y_k))$ by a set of zero linear measure, moreover it is the union of intervals with endpoints being either $\{-\delta\}$ or $\{\delta\}$ or contained in E. Thus, the complement $U \setminus \{(\overline{\bigcup \mathcal{B}(y_k)}) \cup \mathcal{A}_0\}$ is the union of boxes of the form

$$\mathcal{B}_I = \{G(\{|T| < \delta\} \times I\}$$

for intervals I with endpoints in E or $\{-\delta\}$ or $\{\delta\}$, $\mathcal{B}_I$ contained in $U \setminus \mathcal{A}_0$. Take in each $\mathcal{B}_I$ a suitable partition of the unit with a finite number of elements and apply claim 4.7. We see that the integral of $L_j^t(\chi \circ \varphi) \cdot u \circ \varphi$ over the complement $\varphi^{-1}(U \setminus (\bigcup \mathcal{B}(y_k)))$ is equal to zero. Since $\varepsilon > 0$ in (4.60) can be taken arbitrarily small, lemma 4.7 is proved. □

Proof of Claim 4.6. Consider admissible boxes $\mathcal{B}(y) = \mathcal{B}_{I(y)}$ with $|I(y)|$ small and (4.55) small enough. Recall that the support of the function Γ is contained in U with $\overline{U} \subset H_0$. Thus for small $I(y)$ the box

$$\tilde{\mathcal{B}}(y) = \{G(T, y') : |T| < \tilde{\delta},\ y' \in I(y)\}$$

($\tilde{\delta}$ is some positive number smaller than δ) which is slightly smaller than $\mathcal{B}(y) = \mathcal{B}_{I(y)}$ in the T-directions, contains $\overline{U} \cap \mathcal{B}(y)$. We may assume that $I(y)$ is so small that u_y^* is defined in a neighbourhood of $\overline{\tilde{\mathcal{B}}(y)}$. The integral on the left hand side of (4.57) is equal to

$$\mathcal{I}_1 = \int_{\{|T|<\tilde{\delta}\}\times I(y)} \Gamma \circ G \cdot u \circ G \cdot |\det d(\varphi^{-1} \circ G)|\ dm_{2n-1}(T, y'). \tag{4.62}$$

We will show that this integral differs from

$$\mathcal{I}_2 = \int_{\{|T|<\tilde{\delta}\}\times I(y)} \Gamma \circ G \cdot u_y^* \circ G \cdot |\det d(\varphi^{-1} \circ G)|\ dm_{2n-1}(T, y') \tag{4.63}$$

by a constant not exceeding the right hand side of (4.57) in modulus, and that the analyticity of u_y^* in a neighbourhood of $\overline{\tilde{\mathcal{B}}(y)}$ implies that

$$\mathcal{I}_2 = 0. \tag{4.64}$$

Write

(4.65)
$$\mathcal{I}_2 = \int_{\varphi^{-1}(\tilde{\mathcal{B}}(y))} \Gamma \circ \varphi \cdot u_y^* \circ \varphi\ dm_{2n-1}(\mathsf{w}, \mathsf{y}) = \int_{\varphi^{-1}(\tilde{\mathcal{B}}(y))} L_j^t(\chi \circ \varphi) \cdot u_y^* \circ \varphi\ dm_{2n-1}(\mathsf{w}, \mathsf{y}).$$

Since u_y^* is analytic, the equality $L_j(u_y^* \circ \varphi) = 0$ holds, thus

$$\mathcal{I}_2 = \int_{\varphi^{-1}(\tilde{\mathcal{B}}(y)} L_j^t(\chi \circ \varphi \cdot u_y^* \circ \varphi)\ dm_{2n-1}(\mathsf{w}, \mathsf{y}). \tag{4.66}$$

$\varphi^{-1}(\tilde{\mathcal{B}}(y))$ has piecewise smooth boundary and on the smooth pieces either $\chi \circ \varphi = 0$ or L_j is tangent to the smooth piece. Thus, by Stoke's formula $\mathcal{I}_2 = 0$.
Estimate now $\mathcal{I}_1 - \mathcal{I}_2$:

$$\mathcal{I}_1 - \mathcal{I}_2 = \int_{\{|T|<\tilde{\delta}\}\times I(y)} \Gamma \circ G \cdot |\det d(\varphi^{-1} \circ G)| \cdot \{u \circ G - u_y^* \circ G\}\ dm_{2n-1}(T, y')$$

$$\stackrel{\text{def}}{=} \int_{\{|T|<\tilde{\delta}\}\times I(y)} \Lambda(T, y')\{\mathrm{u}(T, y') - \mathrm{u}_y^*(T, y')\}\ dm_{2n-1}(T, y'). \tag{4.67}$$

Now

$$\begin{aligned} \Lambda(T, y')\{\mathrm{u}(t, y') &- \mathrm{u}_y^*(T, y')\} = \\ \Lambda(T, y')\mathrm{u}(T, y') &- \Lambda(T, y)\mathrm{u}(T, y) + \\ \Lambda(T, y)\mathrm{u}_y^*(T, y) &- \Lambda(T, y')\mathrm{u}_y^*(T, y') = \mathcal{J}_1 + \mathcal{J}_2. \end{aligned} \tag{4.68}$$

We used here that $\mathrm{u}_y^*(T, y) = \mathrm{u}(T, y)$. Note that

$$\begin{aligned} \Big|\int_{\{|T|<\tilde{\delta}\}\times I(y)} \mathcal{J}_1\Big| &= \Big|\int_{I(y)} (f(y') - f(y))\, dy'\Big| \\ &\leq \varepsilon|I(y)| \leq C \cdot \varepsilon \cdot m_{2n-1}(\varphi^{-1}(\mathcal{B}(y))), \end{aligned} \tag{4.69}$$

if $|I(y)|$ is small enough. (C depends on the small, but fixed constant δ, not on $I(y)$.) Since $\Lambda \cdot \mathrm{u}_y^*$ is a continuous function, the estimate

$$|\mathcal{J}_2| < \varepsilon \tag{4.70}$$

holds for each $(T, y') \in \{|T| < \tilde{\delta}\} \times I(y)$ if $|I(y)|$ is less than some constant depending on y. Thus, the integral of $\mathcal{J}_2$ is estimated in the same way. Claim 4.6 is proved. □

Proof of Claim 4.7. We may suppose that zero is contained in ∂_1 and, moreover, that there is a neighbourhood U'' of zero on H and C^2 coordinates φ_1 on U'', $\varphi_1(v) = \varphi_1(v_1, \ldots, v_{2n-1}) \in U''$, such that

$$\partial_1 = \{\varphi_1(v) \in U'' : v_1 = 0\} \tag{4.71}$$

and

$$U' = \{\varphi_1(v) \in U'' : v_1 > 0\}. \tag{4.72}$$

Let $\tilde{L}_j$ be the tangential Cauchy-Riemann operators written in these coordinates and take the transposes $\tilde{L}_j^t$ in these coordinates. We have to prove

$$\int\limits_{\{v_1>0\}} \tilde{L}_j^t\tilde{\chi}\cdot\tilde{u}\ dm_{2n-1}=0. \tag{4.73}$$

Here $\tilde{\chi}$ and $\tilde{u}$ are the functions χ and u written in these coordinates. For each $\varepsilon>0$ let $\tilde{\chi}_\varepsilon$ be a smooth function defined in $\mathbb{R}^{2n-1}$, which depends only on the first coordinates v_1, $\tilde{\chi}_\varepsilon=1$ on $\{v_1\geq\varepsilon\}$, $\tilde{\chi}_\varepsilon=0$ on $\{v_1\leq\frac{\varepsilon}{2}\}$ and

$$0\leq\frac{\partial}{\partial v_1}\chi_\varepsilon\leq C\varepsilon^{-1} \tag{4.74}$$

for some constant C not depending on ε. $\tilde{\chi}\tilde{\chi}_\varepsilon$ has compact support in U' and u is a CR-function on U', thus

$$\int\limits_{\{v_1>0\}} \tilde{L}_j^t(\tilde{\chi}\tilde{\chi}_\varepsilon)\cdot\tilde{u}\ dm_{2n-1}=0. \tag{4.75}$$

But

$$\begin{aligned}\int\limits_{\{v_1>0\}} \tilde{L}_j^t(\tilde{\chi}(1-\tilde{\chi}_\varepsilon))\cdot\tilde{u}\ dm_{2n-1} &= \int\limits_{\{v_1>0\}}(1-\tilde{\chi}_\varepsilon)(\tilde{L}_j^t\tilde{\chi})\cdot\tilde{u}\ dm_{2n-1}\\ &+\int\limits_{\{v_1>0\}}\tilde{\chi}\tilde{L}_j(1-\tilde{\chi}_\varepsilon)\cdot\tilde{u}\ dm_{2n-1}=I+II.\end{aligned} \tag{4.76}$$

$I\to 0$ for $\varepsilon\to 0$ since $(1-\tilde{\chi}_\varepsilon)\to 0$ pointwise. $\tilde{L}_j\tilde{\chi}_\varepsilon$ vanishes outside the strip $S_\varepsilon=\{\frac{\varepsilon}{2}<v_1<\varepsilon\}$. Write $\tilde{L}_j=\sum\tilde{a}_{k,j}(v)\frac{\partial}{\partial v_k}$. $\tilde{L}_j$ is tangent to $v_1=0$, thus for $v_1=0$ the coefficient $\tilde{a}_{1,j}(v)$ vanishes. On the strip S_ε the estimate

$$|\tilde{a}_{1,j}(v)|\leq \text{Const}\ \varepsilon \tag{4.77}$$

holds, since the coefficients of $\tilde{L}_j$ are of class C^1. Since $\tilde{\chi}_\varepsilon$ depends only on v_1, from (4.74) follows now that $II\to 0$ for $\varepsilon\to 0$. Claim 4.7 is proved. □

5. An example of an exceptional minimal CR-invariant set

In this section we will **prove theorem 4**. We start with a real analytic compact manifold of dimension three which carries a real analytic foliation of codimension one with an exceptional minimal set. Such examples can be obtained by the classical suspension construction (see, for example, [He–Hi], part A, pp, 124/125; part B, pp. 33-35). The manifold we will consider here is diffeomorphic to $B\times\mathbb{T}$, where $\mathbb{T}$ is the unit circle in the plane and B is the oriented surface of genus two. For convenience of the reader we recall briefly the construction of the foliation (for more details, see [He–Hi]). Consider the fundamental group $\pi_1(B,b_0)$

of B with fixed point b_0 and define a representation H of $\pi_1(B, b_0)$ into the group of orientation preserving real analytic diffeomorphisms of the circle $\mathbb{T}$ onto itself in the following way. Choose standard generators $\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{a}_2, \mathfrak{b}_2$ of $\pi_1(B, b_0)$. The loops $\tilde{\mathfrak{a}}_1, \tilde{\mathfrak{b}}_1, \tilde{\mathfrak{a}}_2, \tilde{\mathfrak{b}}_2$ in Figure 3 represent $\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{a}_2, \mathfrak{b}_2$ in $\pi_1(B)$.

Put

$$\begin{aligned} H(\mathfrak{b}_1) &= H(\mathfrak{b}_2) = id, \\ H(\mathfrak{a}_1) &= h_1, \ H(\mathfrak{a}_2) = h_2, \end{aligned} \tag{5.1}$$

where id denotes the identity mapping of $\mathbb{T}$ and h_1 and h_2 are the restrictions to $\mathbb{T}$ of hyperbolic transformations of the Riemann sphere which let invariant the unit disc. (In fact, we have a representation of the free group of rank two, $\mathbb{F}_2$, the fundamental group of the "handle body", bounded by the surface B.)

For $j = 1, 2$ the mapping h_j has two fixed points on the unit circle, one attractive and one repulsive. Suppose all fixed points are different. Then h_1 and h_2 generate a discontinuous group G of transformations of the Riemann sphere with the property that the limit set

$$L_G = \{z : \lim_{n\to\infty} g_n(z') \to z \text{ for some } z' \in \overline{\mathbb{D}} \text{ and distinct elements } g_n \in G\}$$

is a Cantor set. See [Le] (e.g. p. 100-105) for more detailed information. Thus, the group G of diffeomorphisms of $\mathbb{T}$ has an invariant Cantor set, and no finite subset of $\mathbb{T}$ is invariant for G.

The foliation is now constructed in the following way: Let $\tilde{B}$ be the universal covering of B. Identify $\pi_1(B, b_0)$ with the group of covering translations. It acts on $\tilde{B} \times \mathbb{T}$ in the following way:

$$\begin{aligned} A : \pi_1(B, b_0) \times (\tilde{B} \times \mathbb{T}) &\to (\tilde{B} \times \mathbb{T}) \\ (\gamma, \quad (\tilde{b}, \zeta)) &\to (\gamma\tilde{B}, H(\gamma)\zeta). \end{aligned} \tag{5.2}$$

Denote the quotient $\tilde{B} \times \mathbb{T}/_A$ by $\mathfrak{M}$ and the projection from $\tilde{B} \times \mathbb{T}$ onto $\mathfrak{M}$ by P. $\mathfrak{M}$ is a compact real analytic manifold. On the manifold $\tilde{B} \times \mathbb{T}$ we have a canonical real analytic foliation of codimension one, namely, that with leaves $\tilde{B} \times \{\zeta_0\}$, $\zeta_0 \in \mathbb{T}$ fixed. The mapping A maps leaves onto leaves. So we get a real analytic foliation on the quotient $\mathfrak{M} = \tilde{B} \times \mathbb{T}/_A$. We have to prove two assertions:

1. $\mathfrak{M}$ is diffeomorphic to $B \times \mathbb{T}$.
2. We get a foliation on $\mathfrak{M}$ with an exceptional minimal set, but without closed leaves.

To see the second fact we mention that for each fixed $\tilde{b} \in \tilde{B}$ the closed curve $\{\tilde{b}\} \times \mathbb{T}$ is transverse to the leaves of the canonical foliation of $\tilde{B} \times \mathbb{T}$. Thus, the image under P of this curve is transverse to the leaves in $\mathfrak{M}$.

It is easy to see that for an arbitrary element γ of $\pi_1(B, b_0)$ the points $P((\tilde{b}, \zeta))$ and $P((\gamma^{-1}\tilde{b}, \zeta)) = P((\tilde{b}, H(\gamma)\zeta))$ belong to the same leaf in $\mathfrak{M} = P(\tilde{B} \times \mathbb{T})$. For all γ the points $P((\tilde{b}, H(\gamma)\zeta))$ are the intersections of the leaf through $P((\tilde{b},\zeta))$ with the transversal $P(\{\tilde{b}\} \times \mathbb{T})$ to the foliation of $\mathfrak{M}$. Recall now that the group G of diffeomorphisms of $\mathbb{T}$, generated by $h_1 = H(\mathfrak{a}_1)$ and $h_2 = H(\mathfrak{a}_2)$, has an invariant Cantor set but no finite invariant set. The exsistence of an exceptional minimal set and the absence of a closed leaf in $\mathfrak{M}$ are now clear from the characterization of exceptional leaves (closed leaves, respectively) by their intersections with transversals (see [He–Hi] or section 2 of the present paper).

To see that the first assertion is true it is enough to have a diffeomorphism of $\tilde{B} \times \mathbb{T}$ onto itself which transforms the following mapping A_0 into the mapping A:

$$\begin{aligned} A_0 : \pi_1(B, b_0) \times (\tilde{B} \times \mathbb{T}) &\to (\tilde{B} \times \mathbb{T}) \\ (\gamma, \quad (\tilde{b}, \zeta)) &\to (\gamma\tilde{b}, \zeta). \end{aligned} \tag{5.3}$$

The quotient space $\tilde{B} \times \mathbb{T}/_{A_0}$ is obviously equal to $B \times \mathbb{T}$. Thus, we have to construct a smooth mapping

$$u : \tilde{B} \times \mathbb{T} \to \mathbb{T} \tag{5.4}$$

which is for fixed $\tilde{b} \in \tilde{B}$ a diffeomorphism of $\mathbb{T}$ onto itself, such that

$$u(\gamma\tilde{b}, \zeta) = H(\gamma)u(\tilde{b}, \zeta) \tag{5.5}$$

for $(\tilde{b}, \zeta) \in \tilde{B} \times \mathbb{T}$ and $\gamma \in \pi_1(B, b_0)$. This can be done in the following way. Consider the normal polygon $\mathfrak{P}$ of the surface B with the symbol $\mathfrak{a}_1\mathfrak{b}_1\mathfrak{a}_1^{-1}\mathfrak{b}_1^{-1}\mathfrak{a}_2\mathfrak{b}_2\mathfrak{a}_2^{-1}\mathfrak{b}_2^{-1}$ (see Figure 4, for more detailed information we refer to the book of Springer [Sp]). Construct first a suitable mapping $\tilde{u}$ which satisfies (5.5) for $\zeta \in \mathbb{T}$ and for $\tilde{b}$ belonging to the boundary $\partial\mathfrak{P}$ of the normal polygon. Extend the mapping $\tilde{u}$ to the inside of the normal polygon and then, using (5.5), to the whole universal covering. It is not difficult to see that this can be done explicitly (for more details see also [He–Hi]).

Now, we will use the manifold $\mathfrak{M}$ with the real analytic foliation to construct the example required in theorem 4. First we will obtain a *real analytic imbedding* φ *of the compact manifold* $\mathfrak{M}$ *as a totally real submanifold* $\mathcal{M} \stackrel{\text{def}}{=} \varphi(\mathfrak{M})$ *of* $\mathbb{C}^3$. Start with a *smooth* totally real imbedding of $B \times \mathbb{T}$ into $\mathbb{C}^3$. For example, suppose B is already realized as a smooth proper submanifold of $\mathbb{R}^3$. For each point $b \in B$ let $\nu(b)$ be the unit normal which corresponds to a fixed orientation of B. Consider $\mathbb{R}^3$ as the real subspace of $\mathbb{C}^3$. The following mapping from $B \times \mathbb{T}$ into $\mathbb{C}^3$,

$$(b, \zeta) \to b + \nu(b) \cdot \varepsilon\zeta, \quad b \in B, \quad \zeta \in \mathbb{T}, \tag{5.6}$$

is a smooth totally real imbedding of $B \times \mathbb{T}$ into $\mathbb{C}^3$ (ε is a small positive number). Recall that $B \times \mathbb{T}$ is diffeomorphic to $\mathfrak{M}$.

By results of Bruhat-Whitney and Grauert (see e.g. [Grau]) the smooth totally real imbedding of the compact real analytic manifold $\mathfrak{M}$ may be approximated in C^1 by a real analytic imbedding φ (which is totally real as before). Thus, we get a compact totally real, real analytic manifold $\mathcal{M} \stackrel{\text{def}}{=} \varphi(\mathfrak{M})$ contained in $\mathbb{C}^3$. On $\mathcal{M}$ a real analytic codimension one foliation with a minimal exceptional set is defined, namely the foliation which is conjugate to the foliation on $\mathfrak{M}$ (under the real analytic mapping φ). So we have a real analytic atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$, where $\{U_i\}$ is a covering of $\mathcal{M}$ with open sets, $\varphi_i : U_i \to V_i \subset \mathbb{R}^3$ are real analytic homeomorphisms such that the coordinate transformations

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} \tag{5.7}$$

are real analytic mappings of the form

$$\varphi_{ij}(x_1, x_2, x_3) = (\varphi_{ij}^{(1)}(x_1, x_2, x_3),\ \varphi_{ij}^{(2)}(x_1, x_2, x_3),\ \varphi_{ij}^{(3)}(x_3)). \tag{5.8}$$

The connected components of the sets $\{p \in U_i :\ \varphi_i^{(3)}(p) = \text{const}\}$ are the connected components of the intersection of the leaves with U_i.

The strictly pseudoconvex domain Ω in the assertion of theorem 4 will now be a small tubular neighbourhood of $\mathcal{M}$

$$\Omega = \Omega_\delta \stackrel{\text{def}}{=} \{z \in \mathbb{C}^3 :\ \text{dist}(z, \mathcal{M}) < \delta\} \tag{5.9}$$

for some small positive δ. The closed CR-manifold M in the assertion of theorem 4 will be the intersection of the boundary $\partial\Omega$ with a smooth hypersurface $\mathcal{N}$ in $\mathbb{C}^n$ defined in the following way.

Let $\{\tilde{U}_i\}$ be a covering of $\mathcal{M}$ with open sets, the $\tilde{U}_i$ being relatively compact subsets of the U_i. Consider the real analytic mappings

$$\tilde{\varphi}_i = \varphi_i | \tilde{U}_i \to \tilde{V}_i \subset \mathbb{R}^3. \tag{5.10}$$

The inverses

$$\tilde{\varphi}_i^{-1} : \tilde{V}_i \to \tilde{U}_i,$$

which map the open subset $\tilde{V}_i$ of $\mathbb{R}^3$ onto the open subset $\tilde{U}_i$ of the totally real manifold $\mathcal{M}$ contained in $\mathbb{C}^3$, can be extended to a complex analytic homeomorphism $({}^c\tilde{\varphi}_i)^{-1}$ of a complex neighbourhood ${}^c\tilde{V}_i$ of $\tilde{V}_i$ in $\mathbb{C}^3$ onto a complex neighbourhood ${}^c\tilde{U}_i$ of $\tilde{U}_i$ in $\mathbb{C}^3$. If the ${}^c\tilde{V}_i$ are close enough to the $\tilde{V}_i$, then ${}^c\tilde{U}_i \cap {}^c\tilde{U}_j$ is close to a relatively compact subset of $U_i \cap U_j$ for each pair i and j. Hence, the real analytic mappings φ_{ij} (see (5.7)) extend to complex analytic mappings

$$^c\tilde{\varphi}_{ij} = {}^c\tilde{\varphi}_i \cdot ({}^c\tilde{\varphi}_j)^{-1} : {}^c\tilde{\varphi}_j({}^cU_i \cap {}^cU_j) \to \mathbb{C}^3 \tag{5.11}$$

of the form

$$^c\tilde{\varphi}_{ij}(z_1, z_2, z_3) = ({}^c\tilde{\varphi}_{ij}^{(1)}(z_1, z_2, z_3),\ {}^c\tilde{\varphi}_{ij}^{(2)}(z_1, z_2, z_3),\ {}^c\tilde{\varphi}_{ij}^{(3)}(z_3)). \tag{5.12}$$

Note now, that for $z \in {}^c\tilde{V}_i$ with real third component $z_3 = x_3$ the third component ${}^c\tilde{\varphi}_{ij}^{(3)}$ of the mapping ${}^c\tilde{\varphi}_{ij}$ is real (see (5.8)). Denote by ${}^r\tilde{V}_i$ the set

$$^r\tilde{V}_i = \{z \in \tilde{V}_i : z_3 = x_3 \in \mathbb{R}\}. \tag{5.13}$$

If the ${}^c\tilde{V}_i$ are close enough to $\tilde{V}_i$ and δ is small enough, then the sets

$$ {}^r\tilde{U}_i = {}^c\tilde{\varphi}_i^{-1}({}^r\tilde{V}_i) \cap \Omega_{2\delta} \tag{5.14}$$

cover a real analytic five-dimensional manifold $\mathcal{N}$ which is relatively closed in $\Omega_{2\delta}$. Moreover, the mappings ${}^c\tilde{\varphi}_{ij}$ (see (5.11)) restricted to ${}^c\tilde{\varphi}_j({}^rU_i \cap {}^rU_j)$ define in view of (5.12) a real analytic codimension one foliation on $\mathcal{N}$. The connected components of the intersections of the leaves in $\mathcal{N}$ with the sets ${}^r\tilde{U}_j$ are the connected components of the complex analytic manifolds

$$ {}^c\tilde{\varphi}_i^{-1}(\{z \in {}^c\tilde{V}_i : z_3 = x_3^\circ\}) \cap \Omega_{2\delta}, \quad x_3^\circ \text{ a fixed real number.} \tag{5.15}$$

The complex dimension of these complex analytic manifolds is two.

The codimension one foliation of $\mathcal{N}$ has an exceptional minimal set $S_{\mathcal{N}}$, but no relatively closed leaf. This can be easily seen by looking on the intersection of leaves with transversals. Since each point of $\mathcal{N}$ can be connected with a point in $\mathcal{M}$ by a curve contained in one single leaf, it is enough to consider arbitrary points p of $\mathcal{M}$ and look on the intersection of the leaf through p with a transversal contained in $\mathcal{M}$. Now the assertion is clear from the property of the foliation of $\mathcal{M}$.

Note now that the set $A_S \stackrel{\text{def}}{=} S_{\mathcal{N}} \cap \Omega_\delta$ is relatively closed in Ω_δ and it is the union of analytic manifolds of complex codimension one in Ω_δ. Moreover, $A_S = S_{\mathcal{N}} \cap \Omega_\delta$ is contained in the real hypersurface $\mathcal{N} \cap \Omega_\delta$, but its does not coincide with this hypersurface. It follows that $\Omega_\delta \setminus S_{\mathcal{N}} = \Omega_\delta \setminus A_S$ is connected and pseudoconvex.

Consider now the set $S \stackrel{\text{def}}{=} S_{\mathcal{N}} \cap \partial\Omega_\delta$. Note that the tangent space $T_p\mathcal{N}$ of $\mathcal{N}$ at a point p of the subset $\mathcal{M}$ of $\mathcal{N}$ is spanned by $T_p\mathcal{M}$ and the tangent space $T_p\mathcal{L}_p^c$ of the (complex) leaf $\mathcal{L}_p^c$ through p in the foliated manifold $\mathcal{N}$ (see (5.13), (5.14) and (5.15)). The space $T_p\mathcal{L}_p^c$ is the (real) linear hull of $T_p\mathcal{L}_p^r$ and $JT_p\mathcal{L}_p^r$. Here J is the multiplication with the imaginary unit and $\mathcal{L}_p^r$ is the (real) leaf through p in the foliated (totally real) manifold $\mathcal{M}$. Hence, $T_p\mathcal{N}$ is spanned by $T_p\mathcal{M}$ and two linearly independent real vectors from $JT_p\mathcal{L}_p^r$ which are transerse to $\mathcal{M}$. From these argument it is clear that for sufficiently small positive δ the boundary $\partial\Omega_\delta$ intersects $\mathcal{N}$ transversally at each point. Moreover, $\partial\Omega_\delta$ intersects each complex leaf contained in $\mathcal{N}$ transversally. Thus $M = \mathcal{N} \cap \partial\Omega_\delta$ is a compact generic CR-manifold with a codimension one foliation, the leaves being maximally complex CR-manifolds of real dimension three. Looking on small transversals to the leaves in M (which are transversals also for the leaves of $\mathcal{N}$) we see, that M has an exceptional minimal set but no closed leaves. Theorem 4 is proved. □

6. Examples and open problems

Here we will collect a few open problems and examples.

The first problem is the question whether a generalization of the theorem of Harvey and Lawson holds. We consider instead of compact maximally complex CR-manifolds contained in strictly pseudoconvex boundaries exceptional minimal compact CR-invariant subsets of CR-manifolds contained in strictly pseudoconvex boundaries.

6.1. Problem. *Suppose Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 3$, with boundary of class C^∞. Let M be a proper submanifold of $\partial\Omega$ of class C^∞ which is a generic CR-manifold of real codimension two in $\mathbb{C}^n$. Suppose M contains an exceptional minimal compact CR-invariant set S (precisely $G_J(M)$-invariant set S). Is S the boundary of a singularity set in Ω? More precisely, does there exist a closed subset $A = \overline{A}$ of $\overline{\Omega}$ such that $A \cap \partial\Omega = S$ and $\Omega \setminus A$ is pseudoconvex? If such a singularity set A exists, it is clearly minimal, i.e. there is no non-empty closed subset $A_1 = \overline{A}_1$ of A not coinciding with A, such that $\Omega \setminus A_1$ is pseudoconvex.*

The problem may be reformulated. Since each smooth maximally complex CR-manifold contained in a strictly pseudoconvex boundary bounds locally an analytic manifold we get a relatively closed subset X_S of a "ring domain" $\Omega \setminus \overline{\Omega}_1$ (Ω_1 a relatively compact open subset of Ω) with $\overline{X}_S \cap \partial\Omega = S$, such that X_S is the union of (non-proper) submanifolds of $\Omega \setminus \overline{\Omega}_1$ which are analytic manifolds of complex dimension $n-1$. Thus, each boundary point of $\Omega \setminus (\overline{\Omega}_1 \cup X_S)$ which is not contained in $\partial\Omega_1$ is a pseudoconvex boundary point. So, the problem is equivalent to the question whether a certain plurisubharmonic function of a special kind defined in a "ring domain" $\Omega \setminus \overline{\Omega}_1$, has plurisubharmonic extension to the whole domain Ω. This is not true for general plurisubharmonic functions ([Fo-Si]).

6.2. Problem. *Let Ω and M be as in problem 1. Do there exist simple topological conditions on M which exclude the existence of*

1. *compact maximally complex CR-manifolds of dimension $2n-3$ contained in M,*
2. *exceptional minimal compact CR-invariant subsets of M?*

(Compare with the problem posed in [Či-St]). The problem is motivated by results concerning the corresponding problem in $\mathbb{C}^2$: totally real discs in strictly pseudoconvex boundaries in $\mathbb{C}^2$ are removable ([Jöl],[Fo-St], [Duv]).

We give the following discussion. To study the removability (or, equivalently, the convexity with respect to suitable function spaces) of totally real manifolds M in strictly pseudoconvex boundaries $\partial\Omega$ in $\mathbb{C}^2$ it is useful to consider the characteristic foliation. For each $p \in M$ the intersection $T_pM \cap T_p^J\partial\Omega$ is a real line. Let M be diffeomorphic to an open planar disc. We get a non-singular vector field on M. The associated foliation is the characteristic foliation. Let K be a compact subset of M.

By Oka's characterization principle for hulls the leaves of the characteristic foliation are transverse to the trace of the essential hull $\hat{K}_{tr} \stackrel{\text{def}}{=} \overline{(\hat{K} \setminus K)} \cap K$ of K on M. The theory of Poincaré and Bendixson implies now that each compact subset of M is convex with respect to a suitable space of analytic functions and hence it is removable ([Jöl],[Duv]). Let M now be a sufficiently smooth generic orientable CR-manifold of codimension 2 in $\mathbb{C}^2$ contained in a strictly pseudoconvex boundary $\partial\Omega$ in $\mathbb{C}^3$, $n \geq 3$. We may also consider a characteristic flow on M. Indeed, the linear space $T_pM \cap T_p^J\partial\Omega$ has codimension 2 in $T_p\partial\Omega$ for each $p \in M$.

Moreover, for each $p \in M$ it contains the linear space $T_p^J M$ which has codimension 3 in $T_p\partial\Omega$. Consider the orthogonal complement $(T_pM \cap T_p^J\partial\Omega) \ominus T_p^J M$ of $T_p^J M$ in $T_pM \cap T_p^J\Omega$. This is a real line for each $p \in M$, and thus we get a non-singular vector field on M. By a theorem of Forstnerič [Fo] the vector field is transverse to each CR-submanifold of M. (Indeed, the tangent space of a CR-submanifold of M contains $T_p^J M$ but it is not contained in $T_p^J\partial\Omega$.) So, in principle, CR-invariant subsets could be studied by looking at the characteristic foliation (i.e. by looking at a foliation which is transverse to the set we are interested in). But an analogue of the Poincaré-Bendixson theory fails for flows on manifolds of dimension greater than two. Even if the manifold is diffeomorphic to an Euclidean ball, in general there may be, for example, cycles or limit cycles.

On the other hand we get some more information than in case of dimension $n = 2$. We have a *geometric* understanding of the obstructions for removability (see theorems 2 and 3).

6.2.a. Question. *Do the theorems 2 and 3 give any suggestions which are helpful for a geometric understanding of corresponding problems in $\mathbb{C}^2$? In particular, are there any suggestions related to the well-known open question which totally real discs in $\mathbb{C}^2$ are polynomially convex? (See for example [Jö1],[Duv-Si].*

We wish to add two remarks concerning the discussion of problem 6.2.

First remark: For finding topological conditions of M which exclude the existence of compact maximally complex CR-submanifolds of M with *finite fundamental group* (in particular, *simply connected* maximally complex CR-submanifolds) stability theorems like Reeb's stability theorem are helpful ([He–Hi, B, p. 97]). But even if Ω is the unit ball $\mathbb{B}^3$ in $\mathbb{C}^3$ there are compact maximally complex CR-manifolds of dimension 3 contained in the boundary $\partial\mathbb{B}^3$ with infinite fundamental group. We ask the following concrete

6.2.b. Question. *Is there a smooth (C^∞) generic CR-manifold M which is diffeomorphic to the real $(2n-2)$-ball b^{2n-2} and properly imbedded into the boundary $\partial\mathbb{B}^n$ of the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, which contains a compact maximally complex CR-manifold of dimension $2n-3$?*

We conclude the first remark with the following example of a compact $(2n-3)$-dimensional maximally complex CR-manifold contained in the boundary of the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ with *infinite* fundamental group. The example was told to the author by Alex Dimca ([Di]).

6.2.c. Example. *If $\varepsilon > 0$ is small and the natural number n exceeds two, then the set*

$$N = \{(z_1, z_2, \dots, z_n) \in \partial\mathbb{B}^n : z_1 \cdot z_2 \cdot \dots \cdot z_n = \varepsilon\} \tag{6.1}$$

is a (connected) smooth compact maximally conplex CR-manifold of dimension $2n-3$, which is diffeomorphic to the product $S^{n-2} \times (S^1)^{n-1}$. ($S^1$ is the unit circle, S^{n-2} is the $(n-2)$-dimensional unit sphere in $\mathbb{R}^{n-1}$). Thus N has infinite fundamental group.

Indeed, for $j = 1, \dots, n-1$ write $z_j = r_j\zeta_j$ with $r_j \in (0, \infty)$ and $\zeta_j \in S^1$ and put

$$z_n = \frac{\varepsilon}{r_1 \cdot \dots \cdot r_{n-1} \cdot \zeta_1 \cdot \dots \cdot \zeta_{n-1}} \tag{6.2}$$

We see that N is diffeomorphic to the direct product of the set

$$N' = \left\{ (r_1, \dots, r_{n-1}) \in \{(0, +\infty)\}^{n-1} : r_1^2 + \dots + r_{n-1}^2 + \frac{\varepsilon^2}{r_1^2 \cdot \dots \cdot r_{n-1}^2} = 1 \right\} \tag{6.3}$$

and the set

$$\left\{ (\zeta_1, \dots, \zeta_{n-1}, (\zeta_1 \cdot \dots \cdot \zeta_{n-1})^{-1}) : \zeta_j \in S^1 \text{ for } j = 1, \dots, n-1 \right\}. \tag{6.4}$$

The last set is diffeomorphic to $(S^1)^{n-1}$. The set N' is a level set of the smooth function g,

$$g(r_1, \dots, r_{n-1}) = r_1^2 + \dots + r_{n-1}^2 + \frac{\varepsilon^2}{r_1^2 \cdot \dots \cdot r_{n-1}^2}. \tag{6.5}$$

A simple calculation shows that g has exactly one critical point, namely the point $(\varepsilon^{\frac{1}{n}}, \dots, \varepsilon^{\frac{1}{n}})$. Using the relation between the arithmetic and the geometric mean, it is easy to see that on the set $\{r_1^2 + \dots + r_{n-1}^2 = s\}$ the function g takes the smallest value if all r_j are equal. This implies easily that g has a minimum g_{min} at $(\varepsilon^{\frac{1}{n}}, \dots, \varepsilon^{\frac{1}{n}})$. Thus, for each number $t > g_{min}$ the set $\{g = t\}$ is diffeomorphic to S^{n-2}. We got that for small $\varepsilon > 0$ the boundary $\partial \mathbb{B}^n$ intersects the analytic manifold $\{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n \cdot z_1 z_2 \cdot \dots \cdot z_n = \varepsilon\}$ transversally, hence N is a compact maximally complex CR-manifold. Moreover, N is diffeomorphic to $S^{n-2} \times (S^1)^{n-1}$.

The **second remark** is the following. For excluding the existence of compact CR-invariant subsets of a CR-manifold M (M as in problem 6.2) it is *not enough to exclude the existence of compact CR-submanifolds of all CR-manifolds M_1 which are sufficiently close to M in some C^k, $k \geq 2$.* One has to deal with exceptional minimal compact CR-invariant subsets separately. Indeed, with a suitable choice of the hyperbolic transformations h_1 and h_2 the example of section 5 is stable under small perturbations, i.e. any foliation which is close enough to that of section 5 is topologically conjugate to it (personal communication by E. Ghys [Gh]).

6.3.a. Problem. *Prove the analogue of theorem 2b in the following situation:*

1. *replace hypersurfaces by sufficiently smooth generic CR-manifolds $\mathfrak{M}$ imbedded into $\mathbb{C}^n$ of arbitrary codimension not exceeding $(n-2)$;*
2. *replace M by a generic (proper) submanifold of $\mathfrak{M}$ of real codimension one in $\mathfrak{M}$.*

(The condition on the codimension of $\mathfrak{M}$ is equivalent to the fact that the CR-dimension of $\mathfrak{M}$ is positive, i.e. M is a CR-manifold.)

6.3.b. *Suppose the hypersurface H and the CR-manifold M in theorem 2b are of class C^∞. Is the analogue of theorem 2b for distributions true? In other words, is each compact subset K of M removable for CR-distributions (see the definition in section 0), if it does not contain CR-invariant subsets of M?*

Note, that this is true if H is strictly pseudoconvex from one side. In this case CR-distributions on open parts of H have analytic extension (in the distribution sense) to the pseudoconvex side. The scheme of the proof of theorem 2 shows that CR-distributions on $H \setminus K$ have analytic extension to a one-sided neighbourhood of H. It remains to give growth estimates of the analytic extension near the hypersurface H.

6.3.c. *Let $\mathfrak{M}$ be a generic CR-manifold embedded into $\mathbb{C}^n$ of dimension $(n+1)$, and let M be a generic submanifold of $\mathfrak{M}$ of codimension one in $\mathfrak{M}$. In other words, CR-$\dim \mathfrak{M} = 1$ and M is totally real. If $n = 2$ $\mathfrak{M}$ is a hypersurface.*

If an orientable (not necessarily closed) hypersurface in $\mathbb{C}^2$ is strictly pseudoconvex from one side then closed totally real discs in $\mathfrak{M}$ are $(\mathcal{E}', \overline{\partial}_b)$-removable and $(L^\infty, \overline{\partial}_b)$-removable ([Jö1], the analogs of corollaries 2 and 3 of theorem 2′). See also the L^1-result in [An-Ci]. *What is the right analogue of these results for $n > 2$?*

The case of CR-$\dim \mathfrak{M} = 1$ seems to be especially difficult to handle. On the totally real manifold M there is no obvious structure associated to the obstructions for removability (as, for example, in theorem 2b). On the other hand, it is difficult to get some information using one-dimensional foliations (flows) which are transverse to these obstructions, since the theory of Poincaré and Bendixson fails for $\dim M > 2$ (see the discussion of problem 6.2).

Before stating the **fourth problem** we give two examples:

6.4.a. Example. *Let M be a connected compact maximally complex CR-manifold of class C^2 and of dimension $2n-3$ contained in $\partial\mathbb{B}^n$, $n \geq 3$. Then the germ of envelopes of holomorphy of one-sided neighbourhoods (contained in $\mathbb{B}^n$) of $\partial\mathbb{B}^n \setminus M$ (denote it by $H(\partial\mathbb{B}^n \setminus M)$ for short) is one-sheeted over $\mathbb{C}^n$ and is equal to $\mathbb{B}^n \setminus V$, where V is the analytic variety in $\mathbb{B}^n$ with "boundary" M which exists by the theorem of Harvey and Lawson (compare with the proof of theorem 1).*

Indeed, V is locally the zero set of an analytic function: There is a covering $\{U_i\}$ of $\mathbb{B}^n$ with open sets and analytic functions f_i in U_i such that $V \cap U_i = \{z \in U_i : f_i(z) = 0\}$. Moreover, we may assume that in $U_i \cap U_j$ the equality

$$f_i = h_{ij} f_j \tag{6.6}$$

holds for an analytic function h_{ij} without zeros in $U_i \cap U_j$ (for more detailed information on analytic varieties see also [Či]). Since the cohomology $H^2(\mathbb{B}^n, \mathbb{Z})$ vanishes, there are analytic functions h_i in U_i without zeros, such that

$$h_{ij} = h_i h_j^{-1} \tag{6.7}$$

(see for example [Hö]). Hence, the function f, which is equal to $f_i h_i^{-1}$ on U_i, is a correctly defined analytic function in $\mathbb{B}^n$ with the property that $V = \{z \in \mathbb{B}^n : f(z) = 0\}$. It follows now by the same methods as for sets in $\mathbb{C}^2$ (see for example [Jö3]) that $H(\partial\mathbb{B}^n \setminus M)$ is equal to $\mathbb{B}^n \setminus V$.

Now, we give an example of a compact subset K_1 of a CR-manifold M_1 contained in $\partial\mathbb{B}^3$ for which the germ of the envelopes of holomorphy of one-sided neighbourhoods of $\partial\mathbb{B}^3 \setminus K_1$, is not a domain in $\mathbb{C}^3$, i.e. $H(\partial\mathbb{B}^3 \setminus K_1)$ is multisheeted. This is a perturbation of the example 1.1.

6.4.b. Example. *Let $I = (0, \frac{1}{2})$ and let M denote the CR-manifold of example 1.1, $M = \{z \in \partial\mathbb{B}^3 : z_1 \in I\}$. Let $K \subset M$ be the compact set*

$$K = \{z \in \partial\mathbb{B}^3 : z_1 \in \overline{I}_1\}, \tag{6.8}$$

where $I_1 = (\frac{1}{8}, \frac{3}{8})$ is an interval which closure is contained in $(0, \frac{1}{2})$. K is the union of spheres $S_{z_1} = \{z_1\} \times \{(z_2, z_3) : |z_2|^2 + |z_3|^2 = 1 - |z_1|^2\}$, $z_1 \in \overline{I}_1$. Each S_{z_1} is a CR-orbit.

Let U be a small neighbourhood on M of the curve $\{(z_1, \sqrt{1-|z_1|^2}, 0) : z_1 \in I_1\}$, which does not intersect the spheres S_{z_1} for $z_1 \in I \setminus I_1$, and suppose for $z_1 \in I_1$ the set $U \cap S_{z_1}$ is connected and its closure does not coincide with the whole sphere S_{z_1}. Suppose $M_1 \subset \partial\mathbb{B}^3$ is a sufficiently small perturbation of M, say of class C^∞, which fixes all points of $M \setminus U$ and moves all points of U into some set U_1 which is contained in $\partial\mathbb{B}^3 \setminus M$ (the manifold M has codimension one in $\partial\mathbb{B}^3$, so this can always be done). Put

$$K_1 = \{(\mathit{Int}\ K \setminus U) \cup U_1\} \cup S_{\frac{1}{8}} \cup S_{\frac{3}{8}} = \mathit{Int}\ K_1 \cup S_{\frac{1}{8}} \cup S_{\frac{3}{8}}. \tag{6.9}$$

Then $H(\partial\mathbb{B}^3 \setminus K_1)$ is at least twosheeted.

The assertion in the example is not quite obvious. To prove it we start with the following

6.4.c. Claim. *If M_1 is close enough to M then the CR-invariant subset Int K_1 of M_1 consists of one single CR-orbit of M_1, or, equivalently, Int K_1 does not contain a CR-orbit of codimension one.*

Proof of the claim. Since Int K_1 is a connected component of $M_1 \setminus (S_{\frac{1}{8}} \cup S_{\frac{3}{8}})$ and $S_{\frac{1}{8}} \cup S_{\frac{3}{8}}$ is CR-invariant, the CR-invariance of Int K_1 is clear (lemma 2.1). The equivalence of the two assertions of the claim follows from the connectedness of Int K_1.

Let now p be in Int $K_1 \setminus \overline{U}_1$, i.e. p is in the unperturbed part of K. Suppose $p \in S_{z_1}$, $z_1 \in I_1$. The $G_J(M_1)$-orbit through p contains a large open part of S_{z_1}. S_{z_1} lies on the analytic manifold $\{z_1\} \times \mathbb{C}^2$ in $\mathbb{C}^3$. If the $G_J(M_1)$-orbit through z_1 would have codimension one it would locally bound an analytic manifold (immersed in $\mathbb{B}^3$). By uniqueness this manifold must be contained in $\{z_1\} \times \mathbb{C}^2$. Hence, the $G_J(M_1)$-orbit through p must be contained in S_{z_1}. Since K_1 is a compact CR-invariant subset of M_1 the $G_J(M_1)$-orbit through p must be metrically complete (lemmas 2.5 and 2.6). Hence, it must coincide with S_{z_1}. But M_1 does not contain S_{z_1} (recall that for each $\zeta_1 \in I_1$ certain points of S_{ζ_1} are moved out off M). This contradiction proves that the $G_J(M_1)$-orbit through each point $p \in$ Int $K_1 \setminus \overline{U}_1$ has codimension zero.

It remains to see, that the orbits through points in Int $K_1 \cap \overline{U}_1$ have codimension zero. This is implied by the following arguments. Let U' be a small open neighbourhood of $\overline{U}$ on M. Each point $q \in \overline{U}$ can be joined with a point in $K \setminus U'$ by a piecewise $G_J(M)$-curve γ_q. If M_1 is close to M (in C^2), then to each pair of points $q \in M$ and $q_1 \in M_1$ with small Euclidean distance $|q - q_1|$ in $\mathbb{C}^n$ and to each piecewise $G_J(M)$-curve γ_q with starting point q corresponds a piecewise $G_J(M_1)$-curve $\gamma_{q_1}^{(1)}$ with starting point q_1, such that $\gamma_{q_1}^{(1)}$ is close to γ_q (see section 2, the proof of proposition 2.4). Since $\overline{U}$ and $K \setminus U'$ are compact the arguments apply uniformly for suitable pairs (q, q_1) with q_1 running over the whole set $\overline{U}_1$. Hence each point $q_1 \in \overline{U}_1$ can be joined with a point in $M_1 \setminus \overline{U}_1 = M \setminus \overline{U}$ by a piecewise $G_J(M_1)$-curve. By the CR-invariance of Int K_1 each point $q_1 \in \text{Int}K_1 \cap \overline{U}_1$ can be joined with a point in Int $K_1 \setminus \overline{U}$, thus the $G_J(M_1)$-orbit through each point of Int K_1 has codimension zero. The claim is proved. □

Now, we prove that $H(\partial\mathbb{B}^3 \setminus K_1)$ **is at least twosheeted.**

Since Int K_1 consists of one single CR-orbit, each continuous CR-function on Int K_1 is wedge-extendable at each point of Int K_1. (It is enough to use here the propagation of wedge-extendability along orbits and the fact, that Int K_1 contains a minimal point, since it is contained in $\partial\mathbb{B}^3$ and hence does not contain analytic manifolds.) Consider CR-manifolds M_1^+ and M_1^- which are contained in $\partial\mathbb{B}^3 \setminus M_1$, are close enough in some C^k, $k \geq 2$, to M_1 and are locally situated on different sides of M_1 in $\partial\mathbb{B}^3$. Fix a suitable small one-sided neighbourhood $\mathcal{O}$ of $\partial\mathbb{B}^3 \setminus K$, $\mathcal{O} \subset \mathbb{B}^3$. Consider $rM_1^{\overset{+}{(-)}} = \{rz : z \in M_1^{\overset{+}{(-)}}\}$ for some $r < 1$ close enough to 1, such that rM_1^+ and rM_1^- are contained in $\mathcal{O}$.

By proposition 2.4 there are relatively open subsets N_1^+ and N_1^- of rM_1^+ and rM_1^-, respectively, which are close (in C^k) to Int K_1 and do not contain compact $G_J(rM_1^+)$-invariant subsets ($G_J(rM_1^-)$-invariant subsets, respectively). Hence, there is a $G_J(rM_1^+)$-orbit which contains N_1^+ and a $G_J(rM_1^-)$-orbit which contains N_1^-. Therefore, each continuous CR-function on rM_1^+ (rM_1^-, respectively), in particular the restriction to this set of each analytic function in $\mathcal{O}$, is wedge-extendable at each point of rM_1^+ and rM_1^-. If r is close enough to 1 and M_1^+ and M_1^- are close enough to M_1, by continuity (see for example lemma 3.5) the corresponding wedges W_1^+ and W_1^- with edges in rM_1^+ and rM_1^-, respectively, overlap.

It remains to see that for some analytic function in $\mathcal{O}$ the analytic continuation from rM_1^+ into W_1^+ does not coincide with the analytic continuation from rM_1^- into W_1^- on the overlapping of this wedge. Consider a branch of the function $\{(\frac{3}{8} - z)(z - \frac{1}{8})^{-1}\}^{\frac{1}{2}}$ on the set $\hat{\mathbb{C}} \setminus \overline{I}_1$. ($\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere). The Riemann surface $\mathcal{R}$ of this function is a twosheeted covering of the set $\hat{\mathbb{C}} \setminus (\{\frac{1}{8}\} \cup \{\frac{3}{8}\})$. Denote the corresponding analytic function on $\mathcal{R}$ by f. Since

$$(6.10) \qquad K_1 \supset \partial\mathbb{B}^3 \cap \left[\left(\left\{\frac{1}{8}\right\} \cup \left\{\frac{3}{8}\right\}\right) \times \mathbb{C}^2\right]$$

it is not hard to see that there is an embedding ι of $\partial\mathbb{B}^3 \setminus K_1$ into $\mathcal{R} \times \mathbb{C}^2$ such that for the canonic projection π of $\mathcal{R} \times \mathbb{C}^2$ onto $(\hat{\mathbb{C}} \setminus (\{\frac{1}{8}\} \cup \{\frac{3}{8}\})) \times \mathbb{C}^2$ the superposition $\pi \circ \iota$ is the identity map on $\partial\mathbb{B}^3 \setminus K_1$. Denote by the same letter ι a continuation of the considered imbedding to a suitable one-sided neighbourhood $\mathcal{O}' \subset \mathbb{B}^3$ of $\partial\mathbb{B}^3 \setminus K_1$, such that as before $\pi \circ \iota$ is the identity map (on $\mathcal{O}'$). Consider on $\mathcal{R} \times \mathbb{C}^2$ the function $\mathcal{F}$ which depends only on the first variable $z \in \mathcal{R}$ and is equal to $f(z)$ for fixed values of the second and third variables. It is now clear that the function $\mathcal{F} \circ \iota$ defines an analytic function on $\mathcal{O}'$ with different values of the analytic continuation to the intersection of the wedges W_1^+ and W_1^- mentioned above. We proved that $H(\partial\mathbb{B}^3 \setminus K_1)$ is at least twosheeted. □

Now, we may formulate the fourth problem.

6.4.d. Problem. *Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$, $n \geq 3$, and let K be a compact subset of a $(2n-2)$-dimensional generic submanifold M of $\partial\Omega$. Denote the germ of envelopes of holomorphy of one-sided neighbourhoods of $\partial\Omega \setminus K$, contained in Ω, by $H(\partial\Omega \setminus K)$. Continue the study of the question whether $H(\partial\Omega \setminus K)$ is multisheeted (see the examples 6.4.a and 6.4.b and the discussion in section 0). What is the number of sheets?*

We add the following more concrete questions.

6.4.e. Question. *Let Ω be the domain of theorem 4 and let S be the exceptional minimal set of that theorem. Is $H(\partial\Omega \setminus S)$ onesheeted and equal to the pseudoconvex domain $\Omega \setminus A_S$ (which appeared in the proof of theorem 4)?*

6.4.f. Question. *Let Ω be an arbitrary strictly pseudoconvex domain in $\mathbb{C}^n$ and let K be a compact CR-invariant subset of a $(2n-2)$-dimensional generic submanifold M of $\partial\Omega$. Is $H(\partial\Omega \setminus K)$ onesheeted, if*

i) *K is the union of (proper) compact maximally complex CR-manifolds of real dimension $2n-3$,*
or more generally,
ii) *K is the union of minimal compact CR-invariant subsets of M (i.e. K is the union of a set as in i) and certain exceptional minimal sets),*
or yet more generally,
iii) *K is the union of CR-orbits of codimension one in M (i.e. K is the union of a set like in ii) and certain locally dense orbits)?*

6.4.g. Question. *Let Ω and M be as in question 6.4.f and let K be a compact CR-invariant subset of M.*
Is $H(\partial\Omega \setminus K)$ always multisheeted if K contains CR-orbits of full dimension?
What is the image of $H(\partial\Omega \setminus K)$ under the canonical projection π?

The last problem concerns generalizations of the present theorems to abstract CR-manifolds or even to operators different from Cauchy-Riemann operators.

6.5.a. Problem. Certain statement of results and problems on removable singularities (e.g. theorems 2b and 3, problems 6.3.a, 6.3.b and 6.3.c and the $(\mathcal{E}', \overline{\partial}_b)$-analog as well as the $(L^\infty, \overline{\partial}_b)$-analog of theorem 2′ in [Jöl]) *do not use* the fact that we have to do with certain hypersurfaces (or CR-manifolds of higher codimension) *imbedded into* $\mathbb{C}^n$. *Prove theorems on removable singularities for general CR-manifolds (which are not necessarily even locally imbedded into some $\mathbb{C}^n$).*

6.5.b. Look on the statements mentioned in problem 6.5.a in the following way. Suppose we have a CR-manifold on which we may introduce global Euclidean coordinates. We get an open subset of some $\mathbb{R}^N$ and a first order differential operator or a system of such operators defined in this set. This operator or system of operators has large removable sets for some spaces of functions or distributions: The removable singularities may be metrically much more massive than the general theory [Ha-Po] predicts. *Understand this phenomenon in* *operator theoretic terms, say in terms of the symbols of the operators. Is there some notion of convexity corresponding to individual operators which helps to understand the phenomenon? What can be done for more general operators?*

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After this paper was finished the author knew about the preprint of S. Berhanu and G.A. Mendoza "Orbits and global unique continuation for systems of vector fields", where things related to our section 2 are treated. There is some intersection of that work with our section 2.

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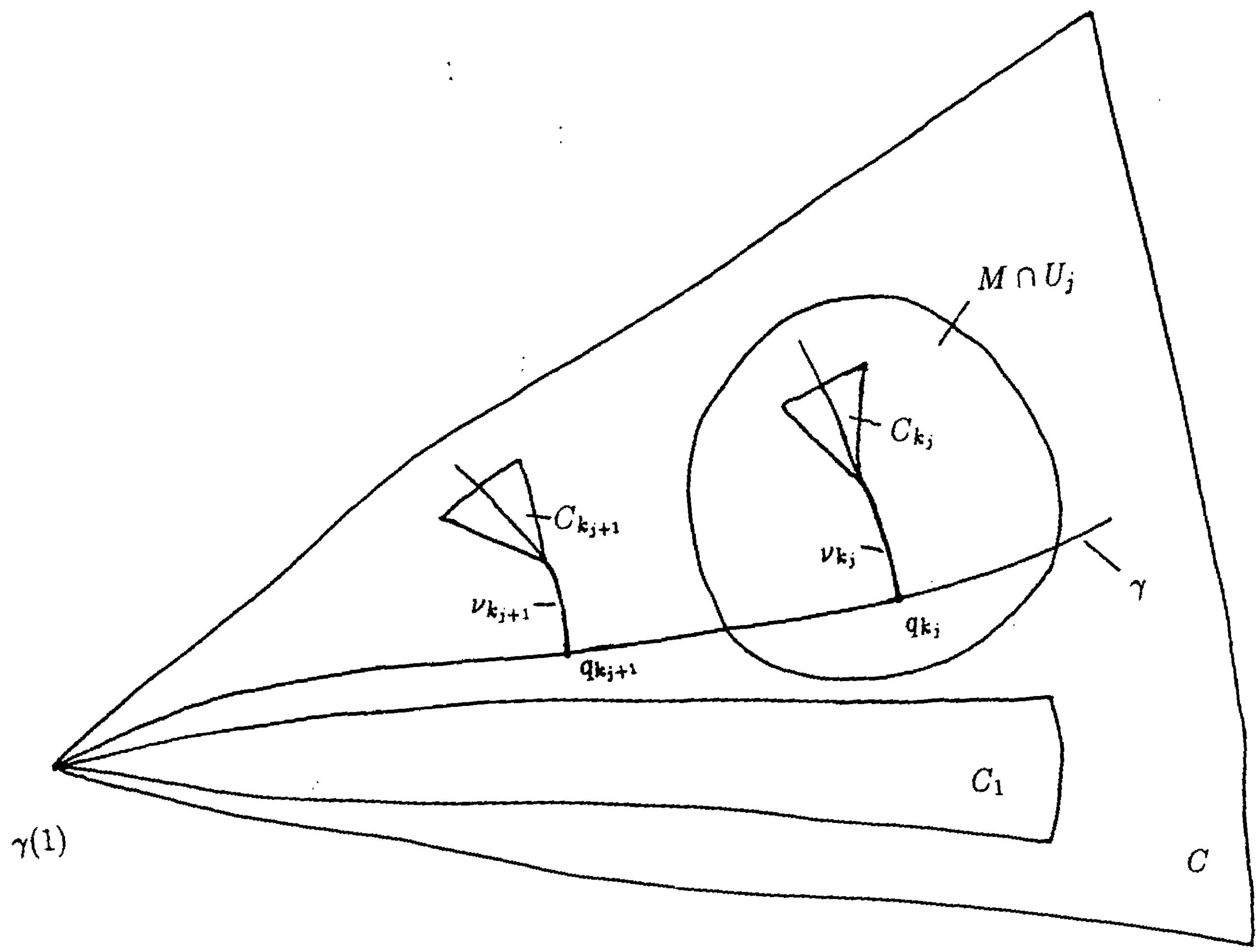

Figure 1

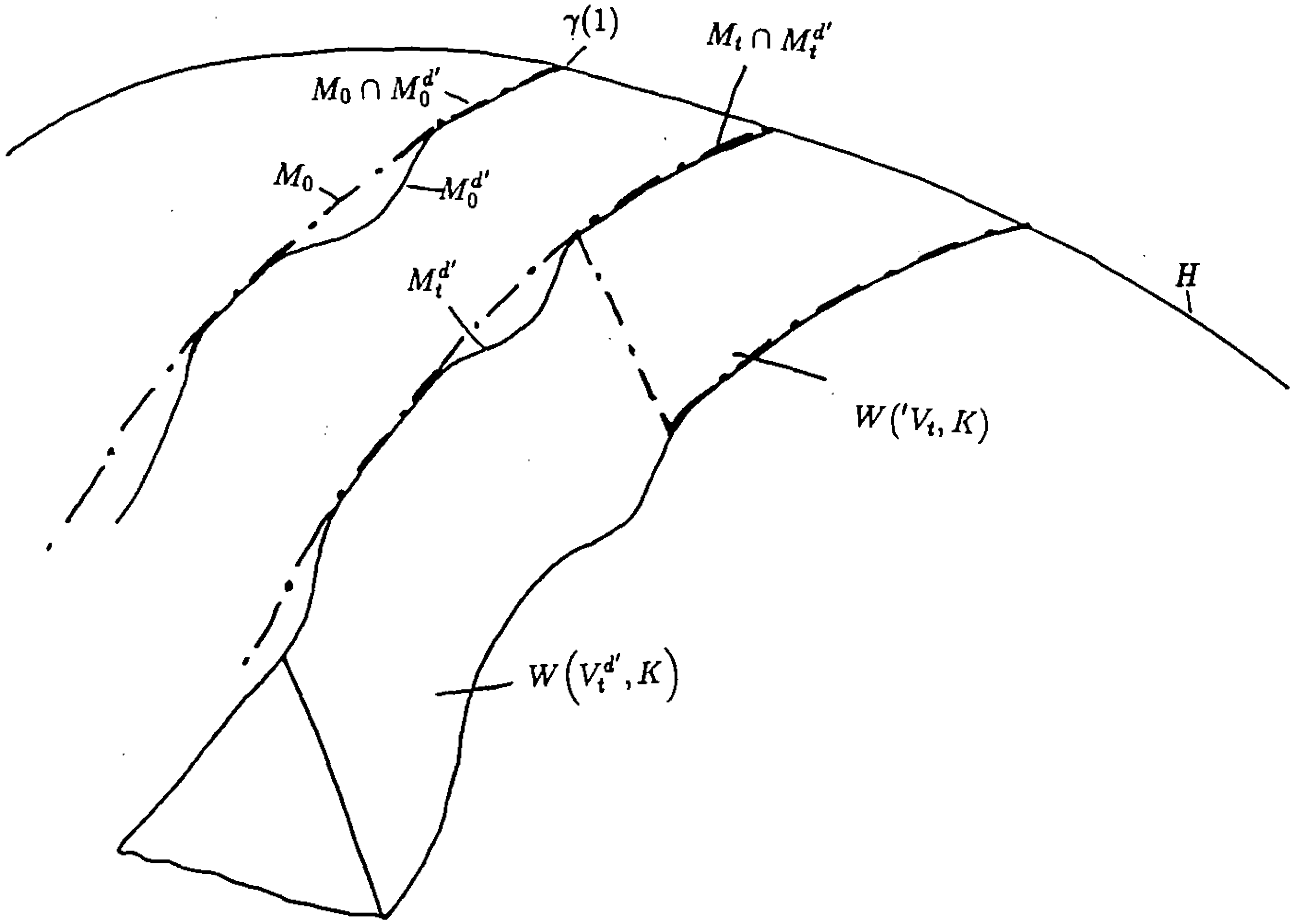

Figure 2

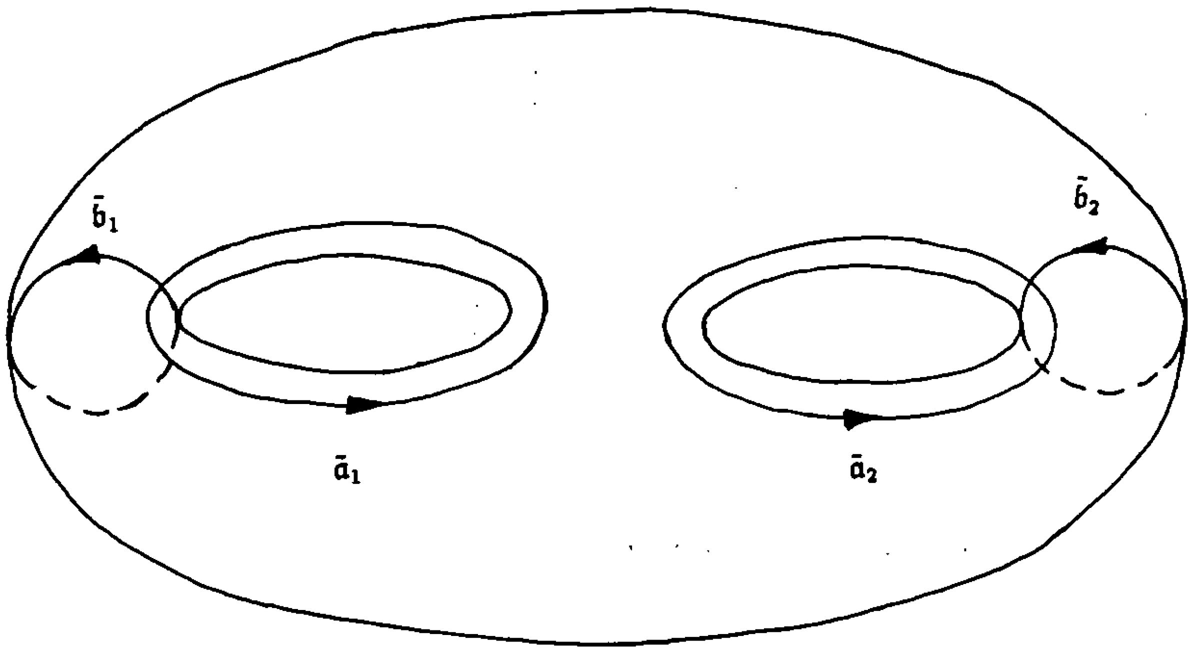

Figure 3

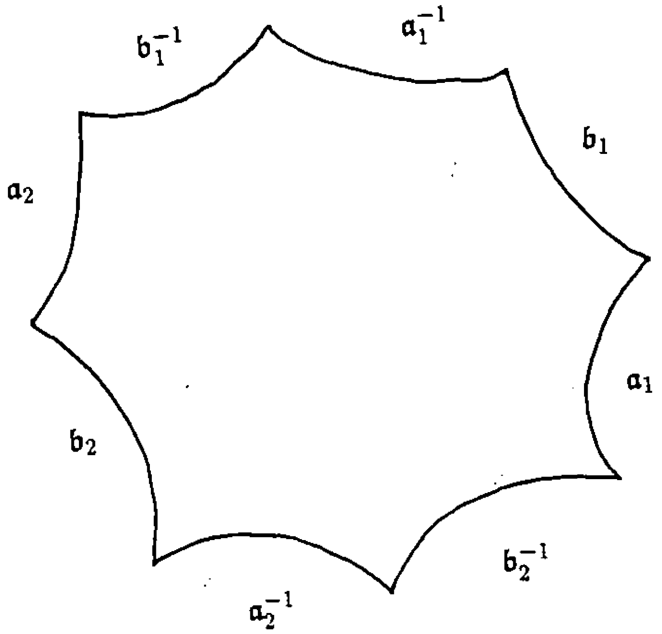

Figure 4