Asymptotically quasiconformal four manifolds

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Introduction

For topological manifolds, there are several kinds of additional structures. Of particular interest is the inclusion top $\subset$ quasiconformal $\subset$ smooth, since the situation drastically changes according to the dimension. One of the most important work was done by Donaldson and Sullivan ([DS]) where an example of topological four manifold was given which cannot admit any quasi conformal structure. On the other hand the equality top = quasi conformal holds for dimension larger than 5 ([S]). Donaldson's polynomial $Q(M)$ is defined over symmetric power of $H_2(M : \mathbf{Z})$ which is smooth invariant. In [DS], Q is constructed over quasiconformal four manifolds. In particular it turns out that it is invariant under quasiconformal homeomorphisms. Notice any smooth manifold admits a unique quasiconformal structure compatible with the smooth one.

For a four manifold M, let us take an open subset $R \subset M$ and a smooth disk D, where closure $\bar{R} = \mathrm{cl}R \in M$ is homeomorphic to D. In order to compare smooth structures on $\check{M} = M \backslash \bar{R}$ and $\hat{M} = M \backslash D$, between smooth open four manifolds, we will introduce a notion, asymptotically quasiconformal homeomorphism between these. It is consisted by a family of quasiconformal mappings. In this paper we will use gauge theory as an analytic tool for the family of maps. Like homology theory, we will reconstruct ASD moduli spaces by using underlying Banach spaces which are much larger than W^2_k Hilbert spaces. Two basic constructions are as follows; we construct ASD moduli spaces over cylindrical manifolds where we use uniformly bounded $L^{2+\epsilon}$ or $L^{4+2\epsilon}$ function spaces as the underlying Banach spaces. Then we will formulate an asymptotic ASD connection, which is consisted by a family of approximately ASD connections. It turns out that the moduli space of asymptotic ASD connections contains the ordinary ASD moduli space. If it could bubble at infinity, then this new moduli space also contains such element. This extended moduli spaces will be used to compare two cylindrical manifolds which are asymptotically quasiconformally homeomorphic.

Let $D, D' \subset \mathbf{R}^4$ be a compact domain with Riemannian metrics g and g'

respectively. Then we say that a homeomorphism $f : (D, g) \cong (D', g')$ is quasiconformal and has constant bounded by K if the following numbers are pointwisely bounded:

$$H(\varphi)(x) = \lim \text{Sup}_{r \to 0} \frac{\max\{|\varphi(y) - \varphi(x)| : |x - y| = r\}}{\min\{|\varphi(y) - \varphi(x)| : |x - y| = r\}} < K, \quad \varphi = f, f^{-1}.$$

The equality holds $H(f)(x) = exp(d([g], [f^*(g')]))$ where [ ] denotes the conformal class $(4.A)$. Thus $H(f)(x)$ measures distance at x , between one metric and its pull back by the quasiconformal homeomorphism. Usually one uses the cylindrical metric g on $\hat{M}$, namely $g|\text{end}$ is isometric to $g|S^3 + dt^2$, where $g|S^3$ is a fixed smooth metric on S^3. In relation with the construction of gauge theory passing through quasiconformal mappings, we will benefit from the Gehring's theorem ([G]) which states; if $I : (K, g) \cong (K', g')$ is a quasiconformal homeomorphism, then $\nabla I \in L^{4+\delta}$ and $\delta > 0$ is essentially determined by ess sup$H(I)$. Notice that the pull backed Riemannian metric $I^*(g')$ can be approximated by a smooth family g_i in L^2 (lemma 4.2). If we deonte $I_i =$ id: $(D, g_i) \cong (D, I^*(g'))$, then ess sup$H(I_i)$ are uniformly bounded. In particular $H(I_i)^2 - 1 \to 0$ in all L^n. ¿From these properties, we will introduce the following notion, asymptotically quasiconformal structure. Let us equip a complete Riemannain metric g of bounded geometry on $\hat{M}$.

An *asymptotically quasiconformal homeomorphism* (ACH) between $\check{M}$ and $\hat{M}$ consists of the following data:

(1) exaustion by compact subsets $K_0 \subset\subset K_1 \subset\subset \cdots \subset \check{M}$,

(2) a family of Riemannian metrics (K_l, g_l) of uniformly bounded geometry, $(|g_l - g_{l'}|K_l|C^\infty$ are uniform),

(3) quasi conformal embeddings $I_l : (K_l, g_l) \hookrightarrow (\hat{M}, g)$

so that these data satisfies the following:

(4) for any $x \in \hat{M}$, there exists $l(x) \geq 0$ such that image of I_l contains x for all $l \geq l(x)$, and $I_l(m) \to \infty$ as $m \to \infty$,

(5) for all $m \in \check{M}$, $d(I_l(m), I_{l'}(m)) \to 0$ as $l, l' \to \infty$,

(6) $|H(I_l^{\pm 1})^2 - 1|L^N_{\text{loc}} \to 0$ for a sufficiently large $N >> 0$,

(7) $|\nabla(I_{l'}^{\pm 1} - I_l^{\pm 1})|L^{4+\delta}_{\text{loc}}(K_l) \to 0$ for $l' \geq l$ and some $\delta > 0$.

It is known that there exists an open subset $R' \subset \mathbf{R}^4$, homeomorphic to $\mathbf{R}^4$ such that the induced smooth structure on R' is different from $\mathbf{R}^4$ ([DF]). Then one notices the following; there exists R', an exotic $\mathbf{R}^4$ which admits an exaustion $K_0 \subset K_1 \subset \cdots \subset R'$ and a family of C^∞ embeddings $I_l : K_l \hookrightarrow \mathbf{R}^4$ satisfying (4) and (5) above.

In the following theorem, one may take $M = \sharp_l X$, $X = CP^2$ or $S^2 \times S^2$ for some $l \geq 1$.

Theorem 0.1 *There exists a pair (M, R) as above, such that $\check{M} \equiv M \backslash \bar{R}$ cannot be asymptotically quasiconformally homeomorphic to $\hat{M} = (M \backslash D, g)$.*

In particular the pair (M, R) has the property that any homeomorphism between $(\check{M}, h)$ and $(\hat{M}, g)$ cannot be locally boundedly conformal with respect to their smooth Riemannian structures.

Using the result in [DS] (following the argument in [FU] $p20$), one may have an example of $R \subset M = 3(S^3 \times S^3)$ satisfying the conclusion above. This method essentially depends on the non-existence result of quasi conformal structure on manifolds with non standard definite intersection form, like $|-E_8|\sharp|-E_8|$. Here one gets the result by constructing directly Yang Mills moduli spaces over $\hat{M}$.

There may be intrinsic difference between smooth structures of a closed manifold and the complement of one point, $\hat{M} = M \backslash$ pt. Recall that by [Q], any topological four manifold admits a smooth structure on any complement of one point. Suppose there are two smooth M and M', each are homeomorphic. In order to compare smooth structures of the ends between $\hat{M}$ and $\hat{M}'$, we will introduce morphisms between $\hat{M}$ and $\hat{M}'$ as follows.

Let $K_0(') \subset K_1(') \subset \cdots \subset \hat{M}(\hat{M}')$ be two exaustions as before. A (locally bounded) *asymptotic morphism* from $(\hat{M}, g)$ to $(\hat{M}', g')$ consists of a family of quasiconformal homeomorphisms $I_l : (K_l, g) \cong (K_l', g')$ and $\delta > 0$ satisfying:

(1) $|H(I_l)^2 - 1| L^N_{\text{loc}} \to 0$ for a sufficiently large $N >> 0$,

(2) $d(I_{l'}(x), I_l(x)) \to 0$ as $l, l' \to \infty$ and

(3) $|\nabla(I_{l'} - I_l)| L^{4+\delta}_{\text{loc}}(K_l) \to 0$,

(4) for any $x \in \hat{M}'$, there exists $l(x) \geq 0$ such that image of I_l contains x for all $l \geq l(x)$, and $I_l(m) \to \infty$ as $m \to \infty$.

A *uniformly bounded* asymptotic morphism consists of an asymptotic morphism with the addtional conditions:

(5) $d(x, I_l(x))$ are uniformly bounded, where we identify both end$\hat{M}$ and end$\hat{M}'$ as $S^3 \times [0, \infty)$,

If the family $\{I_l^{-1}\}_l$ consists of another asymptotic morphism, then we say that $\hat{M}$ and $\hat{M}'$ are asymptotically q.c. equivalent.

In this paper we shall introduce asymptotic Banach spaces. Using them as the underlying spaces, we construct asymptotic ASD moduli spaces $As\mathfrak{M}(\hat{M},g)$. It contains the ordinary ASD moduli space $\mathfrak{M}(\hat{M})$. If these donot coincide, then by the construction it implies the existence of some type of bubblings on the end of $\mathfrak{M}(\hat{M})$. The type is determined by the underlying Banach spaces. $As\mathfrak{M}(\hat{M},g) = As\mathfrak{M}(\hat{M},g;\mathbf{a},\mathbf{T})$ is determined by choosing some families of indices $\mathbf{a} = \{\mathbf{a}(\mathbf{l},\mathbf{j})\}$ and $\mathbf{T} = \{\mathbf{T}(\mathbf{l},\mathbf{j})\}$. Then by gathering, one gets the parametrized asymptotic ASD moduli spaces:

$$PAs\mathfrak{M}(\hat{M},g) = \cup_{\mathbf{a},\mathbf{T}} As\mathfrak{M}(\hat{M},g;\mathbf{a},\mathbf{T}).$$

Theorem 0.2 *(1) Suppose there exists a locally bounded equivalence as above, such that for any* $[A_l] \in As\mathfrak{M}(\hat{M}('),g(');\mathbf{a},\mathbf{T})$,

$$lim_{r\to 0} lim\ sup_{x\to\infty}\ sup_{k,l}|I_l(^{-1})^*(F_{A_k})|L^2(B_r(x)) = 0. \quad *$$

Then it induces a homeomorphism:

$$\{I_l^*\}_l : PAs\mathfrak{M}(\hat{M},g) \cong PAs\mathfrak{M}(\hat{M}',g')$$

where g and g' are sufficiently near the cylindrical ones.

(2) There is a natural embedding:

$$\mathfrak{M}(\hat{M},g) \hookrightarrow As\mathfrak{M}(\hat{M},g;\mathbf{a},\mathbf{T}).$$

Thus the connected components of these spaces may land on different kinds of another by $\{I_l^*\}$. Notice that we donot assume genericity of Riemannian metrics. This is verified in section 7. As we will see in $2.A$, the underlying Banach spaces are different from the ordinary ones. It would be very interesting to use non uniformly bounded Banach spaces (ϵ collapsed one, $3.C$). Fredholm property works for the spaces, however gauge group action seems to be problematic.

The author expects that one can construct larger asymptotic ASD moduli spaces $\bar{A}s\mathfrak{M}$ so that it contains the above asymptotic ASD moduli space $As\mathfrak{M}$, and it could contain the information on the local bubbling phenomena by mapping by I_l. In particular if one uses $\bar{A}s\mathfrak{M}$ in theorem 0.2, one would not need to exclude local bubbling situation ($*$ above).

When we consider bundles with small $p_1(E)$, then the condition $*$ is automatically satisfied. Let $\{I_l\}_l$ be a locally bounded asymptotic equivalence between $(\hat{M},g)$ and $(\hat{M}',g')$. Let us take isomorphic $SO(3)$ bundles E and E' over $\hat{M}$ and $\hat{M}'$ with $I = -2p_1(E) - 3(1+b_2^+(M)) > 0$.

Theorem 0.3 *Suppose* $1 \leq I \leq 12$. *Then the induced morphism:*

$$\{I_l^*\}_l : PAs\mathfrak{M}(E,\hat{M},g) \cong PAs\mathfrak{M}(E',\hat{M}',g')$$

gives a homeomorphism.

Moreover for $-p_1(E) \leq 3$, $As\mathfrak{M}(E, \hat{M}, g)$ is homeomorphic to the compact ASD moduli space $\mathfrak{M}(E, \hat{M}, g)$.

Corollary 0.1 *If the Donaldson's invariants of M and M' are different from each other, then they cannot be asymptotically q.c. equivalent.*

These notions all depend on the Riemannian metric g on $\hat{M}$. However these properties hold under C^0 perturbations of g in the following sense. Let $\varphi : \hat{M} \mapsto Gl(T\hat{M})$ be any bounded continuous section. Then for any Riemannian metrics g and g', one has the equality $d([g],[g']) = d([\varphi^*(g)],[\varphi^*(g')])$. In particular if $\varphi = d\Psi$ for some diffeomorphism Ψ near id, and if there is an ACH from $\check{M}$ to $(\hat{M}, g)$, then one gets a perturbed ACH from $\check{M}$ to $(\hat{M}, \varphi^*(g))$.

Here we point out that asymptoticity of $*$ homomorphisms are elegantly introduced in the C^* algebras world in [CH].

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1 Function spaces

1.A Function spaces: In this paper we always assume that M is simply connected. Let $(\hat{M}, g)$ be a complete Riemannian four manifold whose end is isometric to $(S^3 \times [0,\infty), g|S^3 \times dt^2)$, where $g|S^3$ is a fixed smooth metric on S^3. Let us take a positive number $\delta > 0$, and take a smooth function $w : \hat{M} \mapsto [0,\infty)$ with $w|$ end $(t,m) = t\delta$. Sometimes we denote $w = w(\delta)$ to specify the constant. Recall the weighted Sobolev space $L^p_w(\hat{M})$ is a Banach space with norm $|u|^p = \int_{\hat{M}} exp(w)|u|^p$. Similarly one defines weighted Sobolev k spaces as $(W^p_k)_w(\hat{M}) = \{u : \nabla^i(u) \in L^p_w(\hat{M};\Lambda^*), \quad i \leq k\}$.

Let us take a small $\epsilon \geq 0$ and a pair of positive constants $0 < \delta' \leq \delta$. We put $w' = w(\delta')$ and $w = w(\delta)$. Then one defines particular Banach spaces by:

$$B^0(\hat{M}) = \{u \in L^{4+2\epsilon}_{2w'}(\hat{M}), \quad du \in L^{4+2\epsilon}_{2w}(\hat{M})\},$$
$$B^1(\hat{M}) = \{u \in L^{4+2\epsilon}_{2w}, \quad d(u) \in L^{2+\epsilon}_{w}\}$$
$$B^2(\hat{M}) = L^{2+\epsilon}_w(\hat{M};\Lambda^2_+).$$

1.B AHS complex over $Y \times \mathbf{R}$: In this section one considers the Atiyah Hitchin Singer complex where $\delta' = \delta$ and $d^+ = (1+*)d$:

$$0 \to B^0(\hat{M}) \xrightarrow{d} B^1(\hat{M}) \xrightarrow{d^+} B^2(\hat{M}) \to 0. \tag{1.1}$$

Here we have a simple lemma:

Lemma 1.1 *If the AHS complex above is of Fredholm, then the same is true for the pair* $0 < \delta' < \delta$.

This follows, since the bounds hold:

$$|du|B^1(\hat{M}) = |du|L^{4+2\epsilon}_{2w}(\hat{M}) \geq C|u|L^{4+2\epsilon}_{2w}(\hat{M}) \geq C|u|L^{4+2\epsilon}_{2w'}(\hat{M}).$$

We will choose a smaller $\delta' < \delta$ in $3.D$.

Let (Y, g) be a Riemannian 3 manifold and denote also by g as the product one on $Y \times \mathbf{R}$. Let $\Lambda^1(Y \times \mathbf{R})$ and $\Lambda^2_+(Y \times \mathbf{R})$ be the exterior differentials of 1 forms and self dual 2 forms with respect to g. Then we have the natural identification:

$$\Lambda^1(Y \times \mathbf{R}) = p^*(\Lambda^1(Y)) \oplus p^*(\Lambda^0(Y)),$$
$$\Lambda^2_+(Y \times \mathbf{R}) = p^*(\Lambda^1(Y))$$

where $p : Y \times \mathbf{R} \mapsto Y$ is the projection. The isomorphisms are given by:

$$u + vdt \leftrightarrow (u, v), \quad *_Y u + u \wedge dt \leftrightarrow u.$$

Let us put $X = Y \times \mathbf{R}$. Using L^2 adjoint operator, we get the elliptic operator $P = d^* \oplus d^+ : \Lambda^1(X) \mapsto \Lambda^0(X) \oplus \Lambda^2_+(X)$. Passing through the identification, this is expressed as:

$$P = d^* \oplus d^+ : p^*(\Lambda^0(Y) \oplus \Lambda^1(Y)) \mapsto p^*(\Lambda^0(Y) \oplus \Lambda^1(Y)).$$

Let us use t as the coordinate on $\mathbf{R}$. Then one has the following expression:

$$P = -\frac{d}{dt} + \begin{pmatrix} *_Y d & d \\ d^* & 0 \end{pmatrix} \equiv -\frac{d}{dt} + Q.$$

Notice that Q is an elliptic self adjoint differential operator on $L^2(Y; \Lambda^0(Y) \oplus \Lambda^1(Y))$. Let us define an isometry $(m = 2, 4)$:

$$I : L^{m+\epsilon}(\hat{M}, \Lambda^*) \mapsto L^{m+\epsilon}_w(\hat{M}, \Lambda^*)$$

by $I(u) = exp(-\frac{w}{m+\epsilon})u$. Then a simple calculation shows the equality:

$$I^{-1}PI = P + \frac{1}{m+\epsilon}\frac{dw}{dt} : W^{m+\epsilon}_{k+1}(\hat{M}; \Lambda^1)) \mapsto W^{m+\epsilon}_k(\hat{M}; \Lambda^0 \oplus \Lambda^2_+)$$

$$P(w) \equiv I^{-1}PI = -\frac{d}{dt} + \begin{pmatrix} *_M d & d \\ d^* & 0 \end{pmatrix} + \frac{1}{m+\epsilon}\frac{dw}{dt} \equiv -\frac{d}{dt} + Q_w \quad \text{on the end.}$$

One has a similar expression for P^*. Let us put the operator:

$$Q(\delta) = \begin{pmatrix} *_M d - \frac{\delta}{m+\epsilon} & d \\ d^* & -\frac{\delta}{m+\epsilon} \end{pmatrix} : W^{m+\epsilon}_{k+1}(Y, \Lambda^1 \oplus \Lambda^0) \mapsto W^{m+\epsilon}_k(Y, \Lambda^1 \oplus \Lambda^0).$$

Notice that $Q(-\delta)$ coincides with Q_w on the end. We put $Q(\delta) = P(-\delta)$. There is a similar expression for $P^* = d \oplus (d^+)^*$. In particular $P^*(\delta) = Q(\delta)$.

$(d^+)^*$ is the formal adjoint operator with respect to the L^2 inner product. We will denote by $(d^+)^*_w$ as the one for L^2_w inner product.

1.C Computation of cohomology groups: This section is occupied to verify the following:

Proposition 1.1 *For a small choice of $\delta > 0$ and $0 \leq \epsilon \leq \epsilon_0$, the AHS complex is of Fredholm with $H^0 = H^1 = 0$ and $dim H^2 = b^2_+(M)$.*

Proof : First let us consider $i = 0$. We show that $d : B^0(\hat{M}) \mapsto B^1(\hat{M})$ has closed range. Recall $Q(\delta)$ is an elliptic operator on $L^{m+\epsilon}(S^3)$ defined using $g|S^3$. We show that:

$$P^* = d \oplus (d^+)^* : (W_1^{4+\epsilon})_{2w}(\hat{M}; \Lambda^0 \oplus \Lambda^2_+) \mapsto L^{4+\epsilon}_{2w}(\hat{M}; \Lambda^1)$$

has closed range with finite dimensional kernel.

Sublemma 1.1 *(1) Let us choose a small $\delta > 0$ and any $0 \leq \epsilon \leq \epsilon_0$. We put $p = 4 + 2\epsilon$ or $2 + \epsilon$. Then:*

(1) $P^*(\delta) = Q(\delta) : W_1^p(S^3) \mapsto L^p(S^3)$ satisfies $|Q(\delta)| \geq C > 0$ for some $C = C(\delta)$.

(2) $Q(\delta) + \frac{d}{dt} : W_1^p(S^3 \times \mathbf{R}) \mapsto L^p(S^3 \times \mathbf{R})$ is invertible ([LM]).

Proof : (1) Let us take a family $u_i \in W_1^{4+2\epsilon}(S^3)$ satisfying $|u_i| = 1$ and $|Q(\delta)u_i|L^{4+2\epsilon}(S^3) \to 0$, $i \to \infty$. ¿From the three dimensional Sobolev embedding, the inclusion $W_1^{4+2\epsilon}(S^3) \hookrightarrow L^{4+2\epsilon}(S^3)$ is compact (see [GT] $p167$), and a subsequence of $\{u_i\}_i$ converges to $u_\infty \in L^{4+2\epsilon}(S^3)$. On the other hand, by the elliptic estimate, there is a positive constant $C > 0$ with $|u_i|L^{4+2\epsilon}(S^3) > C$. In particular u_∞ is non zero. Again by the elliptic estimate, u_∞ lies in $W_1^{4+2\epsilon}(S^3)$ and satisfies $Q(\delta)u_\infty = 0$. By elliptic regularity, u_∞ is smooth, but there are no such solution over S^3.

For $p = 2 + \epsilon$ case, the process is similar. This shows the result.

Remark 1.1: In proposition 3.2, we will verify a similar result as (2) above, using different kinds of Banach spaces.

Let us consider (2). Let us put $\Delta = Q(\delta)^2$, $A = -\frac{d}{dt} + Q(\delta)$ and $A^* = \frac{d}{dt} + Q(\delta)$. Then we consider:

$$A^*A = AA^* = -\frac{d^2}{dt^2} + \Delta : W_2^{4+2\epsilon}(S^3 \times \mathbf{R}) \mapsto L^{4+2\epsilon}(S^3 \times \mathbf{R}).$$

It follows from the estimate below that in order to verify (2), It is enough to see that the above operator is invertible:

$$|A^*(u)|L^{4+2\epsilon} = |A^*A(v)|L^{4+2\epsilon} \geq C|v|W_2^{4+2\epsilon} \geq C|u|W_1^{4+2\epsilon}$$

where $v = A^*(AA^*)^{-1}(u) \in W_2^{4+2\epsilon}$.

Proof of proposition: Let us choose a cut off function $\varphi : \hat{M} \mapsto [0,1]$ with $\text{Supp}d\varphi \subset [0,1] \times S^3 \subset \hat{M}$. Then one has the estimate:

$$\begin{aligned}
|u|(W_1^{4+\epsilon})_{2w} &\leq |\varphi u|(W_1^{4+\epsilon})_{2w} + |(1-\varphi)u|(W_1^{4+\epsilon})_{2w} \\
&\leq |\varphi u|(W_1^{4+\epsilon})_{2w} + C|P^*[(1-\varphi)u]|L_{2w}^{4+\epsilon} \\
&\leq C\{|\varphi u|(W_1^{4+\epsilon})_{2w} + |(1-\varphi)P^*(u)|L_{2w}^{4+\epsilon} + |[1-\varphi, P^*]u|L_{2w}^{4+\epsilon}\}, \\
|\varphi u|(W_1^{4+\epsilon})_{2w} &\leq C\{|\varphi P^*(u)|L_{2w}^{4+\epsilon} + |\varphi u|L_{2w}^{4+\epsilon} + |[\varphi, P^*]u|L_{2w}^{4+\epsilon}.
\end{aligned}$$

Notice $\text{Supp}\varphi$, $\text{Supp}[1-\varphi, P^*]$ and $\text{Supp}[\varphi, P^*]$ are all compact.

Since the inclusion $W_1^{4+2\epsilon}(K) \hookrightarrow L^{4+2\epsilon}(K)$ is compact for all $\epsilon \geq 0$ and for any compact subset K, it follows that P^* has closed range with finite dimensional kernel. In particular $d : B^0 \mapsto B^1$ has closed range, since $|df|B^1 = |df|L_{2w}^{4+\epsilon}$. Thus one has verified the result $H^0 = 0$ for $\epsilon \geq 0$.

Next we show $H^1 = 0$.

Sublemma 1.2 (T) *Let f satisfy $df \in L_w^p$. Then there is a constant c uniquely determined by f satisfying the following estimates:*

$$\begin{aligned}
&[\int exp(\frac{4w}{4-p})|f-c|^{\frac{4p}{4-p}}]^{\frac{4-p}{4}} \leq C(p,\delta)\int exp(w)|df|^p, \quad p \in (2,4), \\
&(\int exp(2w)|f-c|^4)^{\frac{1}{2}} \leq C\int exp(w)|df|^2, \\
&|f-c|C^0(S^3 \times \{t\}) \leq C(t)exp(-\frac{\delta}{4+\epsilon}t), \quad p = 4+\epsilon > 4, \quad C(t) \to 0.
\end{aligned}$$

Let us take $u \in H^1$. Thus $d^+(u) = 0$ holds. Notice that for any positive $\mu > 0$, $exp(-\mu w)$ is integrable on $\hat{M}$. Then using Holder's inequality, one sees $d(u) \in L^2(\hat{M}; \Lambda^2)$. $d(u) = 0$ holds by integrating by parts. Let us consider the complex $\{(W_k^p)_{\text{loc}}(\hat{M}; \Lambda^*), d_*\}$ and their cohomology groups H^*. In general in order to get the isomorphism of H^* with the usual de Rham cohomology groups, it is enough to check existence of:

(1) partition of unity in $(W_k^p)_{\text{loc}}$ spaces and

(2) Poincaré lemma in $(W_k^p)_{\text{loc}}(D; \Lambda^*)$ for small disks D (see [W]).

Thus there is $f \in (W_1^{4+2\epsilon})_{\text{loc}}(\hat{M})$ with $df = u$. Then by the above sublemma, u lies in the closure of $d((W_1^{4+2\epsilon})_{2w}(\hat{M})) \subset B^1(\hat{M})$. Since d has closed range, this implies that there is $f \in L_{2w}^{4+2\epsilon}$ with $df = u$. This shows $H^1 = 0$.

Let us consider the case H^2.

Sublemma 1.3 *For all $0 \leq \delta \leq \delta_0$ and all $0 \leq \epsilon \leq \epsilon_0$, $d^+ : (W_1^{2+\epsilon})_w(\hat{M};\Lambda^1) \mapsto L_w^{2+\epsilon}(\hat{M};\Lambda_+^2)$ has closed range with finite dimensional cokernel.*

Proof : Recall that self dual 2 forms over $S^3 \times \mathbf{R}$ can be regarded as 1 forms over S^3. Let $(d^+)^*$ be the formal adjoint operator (without weight function). Then one has the equality:

$$\Delta = d^+ \circ (d^+)^* = -\frac{d^2}{dt^2} + \Delta_1^3 : W_2^{2+\epsilon}(S^3 \times \mathbf{R}; p^*\Lambda^1) \mapsto L^{2+\epsilon}(S^3 \times \mathbf{R}; p^*\Lambda^1)$$

where Δ_1^3 is the Laplace operator on 1 forms over S^3. This operator has closed range with null kernel since Δ_1^3 is invertible over S^3 (see also proposition 3.2).

It is also true for $\Delta_1^3 \pm \delta$ for all small δ. In particular the same argument as function case shows that $d^+ \circ (d^+)^*_w : (W_2^p)_w(\hat{M};\Lambda_+^2) \mapsto L_w^p(\hat{M};\Lambda_+^2)$ also satisfies closed range and finite dimensional kernel. (here one uses compactness of the inclusion $W_2^{2+\epsilon} \hookrightarrow W_1^{2+\epsilon}$).

Let us see d^+ has finite dimensional cokernel. At $\epsilon = 0$, cokernel$d^+ = \ker(d^+)^*_w$ holds using Hilbert space structure. We show the same property for all ϵ. Any element $u \in$ H= $\mathrm{Ker}(d^+)^*_w \cap L_w^2$ lies in also $L_w^{2+\epsilon}$, since it satisfies exponential decay estimate (see sublemma 3.1 or the inequality below).

Let us take $u \in \mathrm{Coker}\Delta \cap L_w^{2+\epsilon}$. We choose a family of cut off functions φ_i with exausting supports. Then each $u_i = \varphi_i u$ lies also in L_w^2. Thus there are $v_i \in H$ and $\alpha_i \in \mathrm{Ker}d_w^* \cap (W_1^2)_w$ with $u_i = d^+(\alpha_i) + v_i \in L_w^2 \cap L_w^{2+\epsilon}$. Now we claim $\alpha_i \in (W_1^{2+\epsilon})_w$. For this it is enough to see $\alpha_i \in L_w^{2+\epsilon}$ by the elliptic estimate. Since Δ has closed range, this implies $u \in H$.

Let us verify the claim. The following argument will be called later as a local to global method. By normalizing one makes $|\alpha_i|(W_1^2)_w = 1$. Recall end$\hat{M}$ is isometric to $S^3 \times [0,\infty)$. By the Sobolev inequality, there is a constant C with:

$$C^{-1}|\alpha_i|L^{2+\epsilon}(S^3 \times [n,n+1]) \leq |\alpha_i|W_1^2(S^3 \times [n,n+1]) \leq exp(-\frac{n\delta}{2}) \leq 1.$$

Then we have inequality:

$$\begin{aligned} |\alpha_i|^{2+\epsilon}L_w^{2+\epsilon} &= \Sigma_n exp(n\delta)|\alpha_i|^{2+\epsilon}L^{2+\epsilon}(S^3 \times [n,n+1]) \\ &\leq C\Sigma_n exp(n\delta)|\alpha_i|^{2+\epsilon}W_1^2(S^3 \times [n,n+1]) \\ &\leq C\Sigma_n exp(n\delta)|\alpha_i|^2 W_1^2(S^3 \times [n,n+1]) \\ &= C|\alpha_i|^2(W_1^2)_w = C|\alpha_i|^{2+\epsilon}(W_1^2)_w. \end{aligned}$$

Thus we get an inequality $|\alpha_i|L_w^{2+\epsilon} \leq C|\alpha_i|(W_1^2)_w$. Again by multiplying constants, one may remove the norm condition above. Thus this inequality holds in general. This verifies the claim.

Sublemma 1.4 *Let $Q' : L^2_w(\hat{M}) \cap$ imd$^+ \mapsto (W^2_1)_w(\hat{M})$ be the inverse to d^+. Then there is an extension $Q : L^{2+\epsilon}_w \cap$ imd$^+ \mapsto (W^{2+\epsilon}_1)_w$.*

Proof : Recall that $d^+((W^{2+\epsilon}_1)_w) \subset L^{2+\epsilon}_w$ has finite codimension for all sufficiently small $0 \leq \delta \leq \delta_0$ and small $0 \leq \epsilon$. Let us denote $H(\delta, \epsilon) = \text{Coker} d^+ \subset L^{2+\epsilon}_w(\hat{M}; \Lambda^2_+)$. Any element $u \in H(0) = H(0,0)$ lies in $(W^{2+\epsilon}_k)_w$ for all k. This follows since u satisfies the exponential decay estimate (see sublemma 3.1) and Sobolev estimate. First we claim $H(0) \subset H(\delta, \epsilon)$. Suppose not. Then there is $u \in H(0)$ satisfying $u = d^+(\alpha)$, $\alpha \in (W^{2+\epsilon}_1)_w$. Let us take another $\delta' < \delta$ and the corresponding weight function w'. We see $\alpha \in (W^2_1)_{w'}$ below. This shows $u = 0$ since $H(\delta, 0)$ is unchanged under small perturbations of $0 \leq \delta \leq \delta_0$. Recall the Holder's inequality $\int ab \leq (\int a^p)^{\frac{1}{p}} (\int b^q)^{\frac{1}{q}}$ for $\frac{1}{p} + \frac{1}{q} = 1$. Then we apply this for

$$ab = exp(w')|\alpha|^2, \quad b^q = exp(w)|\alpha|^{2+\epsilon}.$$

Then we solve $a = exp(w' - q^{-1}w)|\alpha|^{2-q^{-1}(2+\epsilon)}$. Letting $q = \frac{(2+\epsilon)}{2}$, one has $a = exp(w' - q^{-1}w)$. With respect to ϵ, one chooses a sufficiently small $\delta' < \delta$ so that $\delta' - q^{-1}\delta < 0$. Then a^p is integrable. This shows $exp(w')|\alpha|^2$ is also integrable. Thus we get the claim $H(0) \subset H(\delta, \epsilon)$.

Next we show the converse direction. Let us consider $\nabla = d^*_{w'} \oplus d^+ : (W^{2+\epsilon}_1)_{w'} \mapsto L^{2+\epsilon}_{w'}$. We claim that for $\delta' > 0$, there is a constant $C > 0$ with bound:

$$|u|(W^{2+\epsilon}_1)_{w'} \leq C|\nabla(u)|L^{2+\epsilon}_{w'}.$$

Notice that $\Delta_w = d^*_w d : (W^{2+\epsilon}_2)_w \mapsto L^{2+\epsilon}_w$ gives an isomorphism on functions. In particular $d^*_{w'} : (W^{2+\epsilon}_1)_{w'} \mapsto L^{2+\epsilon}_{w'}$ has closed range. Since d^+ also has closed range, it is enough to check $\text{Ker}\nabla = 0$. Let us take $\alpha \in \text{Ker}\nabla$. On the end$\hat{M}$, $I^{-1}\nabla I : W^{2+\epsilon}_1 \mapsto L^{2+\epsilon}$ has the following form where I is in $1.B$:

$$-\frac{d}{dt} + \begin{pmatrix} *_M d + \frac{\delta'}{m+\epsilon} & d \\ d^* & -\delta' \frac{m+\epsilon-1}{m+\epsilon} \end{pmatrix} \equiv -\frac{d}{dt} + Q'(\delta').$$

Notice that $Q'(\delta')$ is a self adjoint elliptic operator. Moreover for small δ', there are no kernels (see [K] lemma 6.3). Thus $I^{-1}(\alpha) \in W^{2+\epsilon}_1$ satisfies the exponential decay (see sublemma 3.1). In particular one sees $\alpha \in (W^{2+\epsilon}_1)_w$.

Now as before one has $d(\alpha) = 0$. Let us put $m = \frac{4}{2-\epsilon}$. Then by sublemma 1.2, there exists $f \in L^{m(2+\epsilon)}_{mw}$ with $df = \alpha$. One may assume $f \in L^2_{w'}$. Since f satisfies $d^*_{w'} \circ df = 0$, this implies $df = \alpha = 0$. This shows the claim.

Suppose $H(0) \neq H(\delta, \epsilon)$. Thus there is $u \in H(\delta, \epsilon)$ but $u \neq H(0)$. As above, one may assume $u \in L^2_{w'}$. Then one may write $u = d^+(\alpha)$, $\alpha \in (W^2_1)_{w'}$ and $d^*_{w'}(\alpha) = 0$. ¿From the Sobolev embedding $(W^2_1)_{\text{loc}} \hookrightarrow L^{2+\epsilon}_{\text{loc}}$, $u_i = d^+(\varphi_i \alpha)$ and

$\mu_i = d^*_{w'}(\varphi_i \alpha)$ converges to u and zero respectively in $L^{2+\epsilon}_{w'}$. ¿From the claim, one sees $\alpha \in (W^{2+\epsilon}_1)_{w'}$. This shows that the image of $H(\delta, \epsilon) \mapsto H(\delta', \epsilon)$ is exactly $H(0)$.

We claim that for all sufficiently small pairs (δ, ϵ), the above maps are all isomorphisms. Suppose not. Then for all small δ, one has non zero elements in $H(\delta, \epsilon) \backslash H(0)$. Let $Q(\delta, \epsilon) : L^{2+\epsilon}_w \mapsto (W^{2+\epsilon}_1)_w$ satisfy $d^+ \circ Q(\delta, \epsilon) |$ imd^+ is the identity and $Q(\delta, \epsilon) | H(\delta, \epsilon) = 0$. We denote $Q = Q(0, \epsilon)$. Passing through the isomorphisms $I(\delta) : (W^{2+\epsilon}_k)_w \cong W^{2+\epsilon}_k$, one may assume that all the operators act on $W^{2+\epsilon}_k$. Let us denote $d^+(\delta) = I(\delta) d^+ I(\delta)^{-1}$. Notice that $d^+(\delta)$ is a continuous operators on $W^{2+\epsilon}_1$ with respect to δ. Then the following holds:

$$|d^+(\delta)Q - id| L^{2+\epsilon} \backslash H(0) \to 0 \quad \delta \to 0.$$

In particular for any sequence δ_i, $\delta_i \to 0$, there are vectors $u_i \in H(\delta_i, \epsilon) \backslash H(0)$, $|u_i| L^{2+\epsilon} = 1$ which satisfy $|d^+(\delta_i) Q(u_i) - u_i| L^{2+\epsilon} \to 0$ $(*)$.

Next let us consider a function:

$$\delta' \in [0, \delta] \mapsto d(\text{ im } d^+(\delta'), \text{ Coker }) = \text{ Sup}_{|u|L^{2+\epsilon}=1} \text{ inf}_f |u - d^+(\delta') f| L^{2+\epsilon}.$$

Clearly this is continuous. In particular there is a positive constant C independent of δ' with:

$$\text{inf}_f |u_i - d^+(\delta') f| L^{2+\epsilon} > C.$$

This contradicts with $(*)$. This shows that for sufficiently small $\epsilon \in [0, \epsilon_0]$ and $0 \leq \delta \leq \delta_0$, $H(\epsilon, \delta) = H(0, 0)$. Notice that one knows $d^+((W^2_1)_w) \subset L^2_w$ has codimension $b^2_+(M)$ ([T]). In particular $d^+((W^{2+\epsilon}_1)_w) \subset L^{2+\epsilon}_w$ has codimension $b^2_+(M)$.

Recall that $\Delta_w = d^*_w d : (W^{2+\epsilon}_2)_w \mapsto L^{2+\epsilon}_w$ gives an isomorphism. Let us take $\alpha \in \text{Ker} d^+ \subset (W^{2+\epsilon}_1)_w (\hat{M}; \Lambda^1)$. Then $d (d^*_w d)^{-1} d^*_w (\alpha) = \alpha$ since $|\nabla| \geq C > 0$. In particular Ker d^+ is equal to im$d$ and $H^1 = 0$. Then by the open mapping theorem, the following map gives an isomorphism between Banach spaces:

$$d \Delta_w^{-1} d^*_w \oplus d^+ : (W^{2+\epsilon}_1)_w \mapsto \text{ Ker} d^+ \subset (W^{2+\epsilon}_1)_w \oplus \text{ im} d^+ \subset L^{2+\epsilon}_w.$$

(Continuity and injectivity are clear. For surjectivity, let us take $(u = df, d^+(v)) \in$ Ker$d^+ \oplus$ im$d$. Then $u - d\Delta_w^{-1} d^*_w (v) + v$ maps to $(u, d^+ v)$).)

Let $H \subset L^{2+\epsilon}_w$ be the finite dimensional subspace which represents the cokernel of d^+. Then since H is finite dimensional, one has a natural isomorphism between Banach spaces $L^{2+\epsilon}_w \cong$ im$d^+ \oplus H$. Let us define a map Q by $Q(d^+(v)) = v - d\Delta_w^{-1} d^*_w (v)$ for $v \in (W^{2+\epsilon}_1)_w$. Then this gives a bounded map:

$$Q : L^{2+\epsilon}_w (\hat{M}; \Lambda^2_+) \cap \text{ im} d^+ \mapsto \text{ Ker} d^*_w \cap (W^{2+\epsilon}_1)_w (\hat{M}; \Lambda^1)$$

which satisfies $d^+Q|\ \mathrm{im} d^+ = \mathrm{id}$. This completes the proof of sublemma 1.4.

Proof of proposition: Using the Sobolev inequality and the above Q, one has the estimate:

$$\begin{aligned}
|u|^{4+2\epsilon} L^{2+\epsilon}_w &\geq C|Q(u)|^{4+2\epsilon}(W^{2+\epsilon}_1)_w \\
&= \frac{C}{2}|Q(u)|^{4+2\epsilon}(W^{2+\epsilon}_1)_w + \frac{C}{2}|Q(u)|^{4+2\epsilon}(W^{2+\epsilon}_1)_w \\
&\geq C'|u|^{4+2\epsilon} L^{2+\epsilon}_w + C'(\Sigma_{i\in\mathbf{N}} exp(i\delta)|Q(u)|^{2+\epsilon} W^{2+\epsilon}_1([i,i+1]\times S^3))^2 \\
&\geq C'|u|^{4+2\epsilon} L^{2+\epsilon}_w + C'(\Sigma_{i\in\mathbf{N}} exp(i\delta)|Q(u)|^{2+\epsilon} L^{4+2\epsilon}([i,i+1]\times S^3))^2 \\
&\geq C'|Q(u)|^{4+2\epsilon} B^1.
\end{aligned}$$

This shows that one has a continuous extension:

$$Q : H^\perp \subset L^{2+\epsilon}_w(\hat{M};\Lambda^2_+) \mapsto B^1$$

satisfying d^+Q is the identity. In particular this shows that $d^+ : B^1 \mapsto L^{2+\epsilon}_w$ has closed range.

Recall $\mathrm{Coker} d^+ : (W^{2+\epsilon}_1)_w \mapsto L^{2+\epsilon}_w$ has $b^2_+(M)$ dimension. Since $Q : L^{2+\epsilon}_w \mapsto (W^{2+\epsilon}_1)_w$ has an extension to $Q : L^{2+\epsilon}_w \mapsto B^1$ with $Q|H = 0$ and $d^+Q|\ \mathrm{im} d^+$ is the identity, it follows $\dim H^2 \leq b^2_+(M)$. Suppose there is a vector $\alpha \in L^{4+2\epsilon}_{2w}$ with $d^+(\alpha) \in H$. Then as before α lies in $L^2_{w'}$. One may assume $\inf_f |\alpha - df| L^2_{w'} = |\alpha| L^2_{w'}$. ¿From this $d^*_{w'}\alpha = 0$ (differentiate $|\alpha + tdf|^2 L^2_{w'}$). Then from the elliptic estimate, it follows $\alpha \in (W^2_1)_{w'}$. This shows $\alpha = 0$. This completes the proof.

Remark 1.2: One gets a relation of the range of Q and an inequality:

$$\begin{aligned}
&\mathrm{im} Q \subset B^1(\hat{M},\epsilon) \cap (W^{2+\epsilon}_1)_w(\hat{M}) \cap \ \mathrm{Ker} d^*_w, \\
&|d^-Qu| L^{2+\epsilon}_w(\hat{M}) \leq C|d^+Qu| L^{2+\epsilon}_w(\hat{M}).
\end{aligned}$$

2 Asymptotic function spaces

2.A Asymptotic Banach spaces: Here we introduce asymptotic function spaces. First we define the asymptotic Banach spaces $\hat{B}^*(\hat{M};1)$ with one bubble. Then we generalize them to another $\hat{B}^*(\hat{M};n)$ with n bubbles. Later when we construct moduli spaces, we will use $\hat{B}^*(\hat{M}) \equiv \hat{B}^*(\hat{M};n)$ for some $n > 0$ as the underlying Banach spaces.

Let $(\hat{M}, g)$ be a cylindrical manifold as before, whose end is isometric to $(S^3 \times [0,\infty), g|S^3 + dt^2)$. Let us take an exaustion by compact subsets:

$$K_0 \subset K_1 \subset \cdots \subset \hat{M}.$$

Here one always assume that the exaustion is of bounded geometry (boundaries are uniformly smooth). For each $l \geq 0$, let us take a cut off function $\varphi_l : \hat{M} \mapsto$

$[0,1]$ satisfying:

$$(1) \quad \varphi_l|\hat{M}\backslash S^3 \times [l,\infty) \equiv 1, \quad (2) \quad \varphi_l|S^3 \times [l+1,\infty) \equiv 0,$$
$$(3) \quad |\varphi_l|C^i \text{ are bounded independently of } l, \quad i \geq 0.$$

By sliding indices, if necessarily, one may assume $\partial K_l \subset S^3 \times [3l,\infty) \subset$ end $\hat{M}$.

Let $K \subset \hat{M}$ be a compact subset with smooth boundary ∂K and int$K$ be its interior. Then here one uses a notation $(W_k^p)_w(K)_0$ to be a Banach space where $C^\infty_{\text{cp}}(\text{ int}K)$ is dense in $(W_k^p)_w(K)_0$ with norm $|u|^p(W_k^p)_w(K)_0 = \Sigma_{i\leq k}\int_K exp(w)|\nabla^i u|^p$. Notice the inclusion $(W_k^p)_w(K)_0 \subset (W_k^p)_w(\hat{M})$. One defines similarly $B^i(K)_0$ using $L^p_w(K)_0$ as before. These also have inclusions $B^i(K)_0 \subset B^i(\hat{M})$.

Let us fix small constants $\epsilon > 0$, $\delta \geq \delta' > 0$ and the corresponding weight functions w and w'. Then we introduce thickened Banach spaces:

$$|u|\hat{L}^4_{2w}(\hat{M}) \equiv \text{ Sup}_{0\leq\epsilon'\leq\epsilon}|u|L^{4+2\epsilon'}_{2w}(\hat{M}).$$

Similarly we define $\hat{L}^2_w = \text{Sup}_{0\leq\epsilon'\leq\epsilon}| \quad |L^{2+\epsilon'}_w(\hat{M})$, or $(\hat{W}^2_k)_w$. Then we put:

$$\bar{B}^0(\hat{M},w,\epsilon) = \bar{B}^0(\hat{M}) = \{u \in \hat{L}^4_{2w'}(\hat{M}), \quad du \in \hat{L}^4_{2w}(\hat{M})\},$$
$$\bar{B}^1(\hat{M}) = \{u \in \hat{L}^4_{2w}(\hat{M}), \quad d(u) \in \hat{L}^2_w(\hat{M})\}, \quad \bar{B}^2(\hat{M}) = \hat{L}^2_w(\hat{M};\Lambda^2_+).$$

Let $w_l : S^3 \times \mathbf{R} \mapsto [0,\infty)$ be a weight function defined by $w_l(m,t) = \delta|l-t|$. We will use the same notation for the restriction $w_l : S^3 \times \mathbf{R}_+ =$ end$\hat{M} \mapsto [0,\infty)$. Moreover let $T_l : S^3 \times [-l,\infty)(\subset S^3 \times \mathbf{R}) \mapsto S^3 \times [0,\infty) \subset$ (end $\hat{M}$) be the linear translation defined as $T_l(m,k) = (m,k+l)$.

Let us consider the set $[u_l]$, $u_l \in \bar{B}^*(K_l)_0$ satisfying the following:

(1) $\varphi_l u_l$ converges to $u_1 \in \bar{B}^*(\hat{M},w,\epsilon)$,

(2) $(1-\varphi_l)u_l \in \bar{B}(\hat{M},w_{2l},\epsilon)$ and $T^*_{2l}((1-\varphi_l)u_l)$ converges to $u_2 \in \bar{B}^*(S^3 \times \mathbf{R},w_0,\epsilon)$.

Then one defines linear spaces $\bar{B}^i(\{K_l\}_l)$ as follows:

$$\bar{B}^i(\{K_l\}_l) = \{[u_l] : \text{ satisfying the above } (1),(2)\}.$$

Let us equip a norm on $\bar{B}^i(\{K_l\}_l)$ as:

$$\begin{aligned}
|[u_l]|\bar{B}^*(\{K_l\}_l) &= \text{ Sup}_l||u_l||_l \\
&\equiv \text{ Sup}_l\{|\varphi_l u_l|\bar{B}^*(\hat{M},w,\epsilon) + |(1-\varphi_l)u_l|\bar{B}^*(\hat{M},w_{2l},\epsilon)\} \\
&= \text{ Sup}_l\{|\varphi_l u_l|\bar{B}^*(\hat{M},w,\epsilon) + |T^*_{2l}(1-\varphi_l)u_l|\bar{B}^*(S^3 \times \mathbf{R},w_0,\epsilon)\}
\end{aligned}$$

Sometimes we use the notation:

$$\begin{aligned}||u_l - u_{l'}||\bar{B}^* \equiv &|\varphi_l u_l - \varphi_{l'} u_{l'}|\bar{B}^*(\hat{M}, w, \epsilon) \\ &+ |T^*_{2l}(1-\varphi_l)u_l - T^*_{2l'}(1-\varphi_{l'})u_{l'}|\bar{B}^*(S^3 \times \mathbf{R}, w_0, \epsilon).\end{aligned}$$

Let us define a vector subspace $\bar{B}^i_0(\{K_l\}_l) \subset \bar{B}^i(\{K_l\}_l)$:

$$\bar{B}^i_0(\{K_l\}_l) = \{[u_l] \in \bar{B}^i(\{K_l\}_l) : ||u_l||_l \bar{B}^i(K_l)_0 \to 0 \text{ as } l \to \infty\}.$$

Lemma 2.1 *With respect to the above metric,*
(1) $\bar{B}^i(\{K_l\}_l)$ is a Banach space, (but is not reflexive) and
(2) $\bar{B}^i_0(\{K_l\}_l)$ is a closed subspace of $\bar{B}^i(\{K_l\}_l)$.

Proof : (1) We check completeness. Suppose a family $[u^\alpha_l] \in \bar{B}^*(\{K_l\}_l)$ consists of a Cauchy sequence. Thus for every $\epsilon > 0$, there is a large α_0 with a uniform bound $||u^\alpha_l - u^\beta_l||_l \bar{B}^* < \epsilon$ for all $\alpha, \beta \geq \alpha_0$ and l. For each l, let $\beta \to \infty$. Then we get some sequence $[u^\infty_l]$. We claim this gives an element in $\bar{B}^*(\{K_l\}_l)$. Let us check convergence property. For any small $\epsilon > 0$, one finds α_0 with the above property. For $[u^{\alpha_0}_l]$, let us take a large l_0 so that $||u^{\alpha_0}_l - u^{\alpha_0}_{l'}||\bar{B}^* \leq \epsilon$ for all $l, l' \geq l_0$. Then for $l, l' \geq l_0$, one gets the estimate:

$$||u^\infty_l - u^\infty_{l'}||\bar{B}^* \leq ||u^\infty_l - u^{\alpha_0}_l||\bar{B}^* + ||u^{\alpha_0}_l - u^{\alpha_0}_{l'}||\bar{B}^* + ||u^{\alpha_0}_{l'} - u^\infty_{l'}||\bar{B}^* \leq 3\epsilon.$$

This shows that $[u^\infty_l]$ satisfies the defining condition above.

(2) Let us take $[u^\alpha_l] \in \bar{B}^i_0(\{K_l\}_l)$, $\alpha \geq 0$, and suppose $\sup_l ||u^\alpha_l - u^\beta_l||_l \bar{B}^i(K_l)_0 \to 0$ as $\alpha, \beta \to \infty$. Then for any $\epsilon > 0$, there is α_0 so that for all $\alpha, \beta \geq \alpha_0$, one has the estimate $\mathrm{Sup}_l ||u^\alpha_l - u^\beta_l||_l \bar{B}^i(K_l)_0 \leq \epsilon$. Thus letting $\beta \to \infty$, one has:

$$\mathrm{Sup}_l ||u^\alpha_l - u^\infty_l||_l \bar{B}^i(K_l)_0 \leq \epsilon.$$

This shows $[u^\infty_l] \in \bar{B}^i_0(\{K_l\}_l)$. This verifies lemma.

Let us put:

$$\hat{B}^i(\{K_l\}_l) \equiv \bar{B}^i(\{K_l\}_l)/\bar{B}^i_0(\{K_l\}_l).$$

One equips the quotient norm on $\hat{B}^i(\{K_l\}_l)$. Now we study some basic properties about $\hat{B}^*(\{K_l\}_l)$.

Let us introduce another norm on $\hat{B}^i(\{K_l\}_l)$:

$$|[u_l]|_1 = \limsup_{l\to\infty} ||u_l||_l \bar{B}^i(K_l)_0.$$

We denote by $| \quad |_0$ the quotient norm before:

$$|[u_l]|_0 = \inf \{ \sup_l ||u_l + v_l||_l \bar{B}^i(K_l)_0 : [v_l] \in \bar{B}^i_0(\{K_l\}_l)\}.$$

Lemma 2.2 $| \quad |_0$ *and* $| \quad |_1$ *are isometric.*

Proof : Let us take $[u_l] \in \hat{B}^i(\{K_l\}_l)$ with $||[u_l]||_1 = 1$. Then for any $\epsilon > 0$, there is l_0 with $1 - \epsilon \leq \sup_{l \geq l_0} ||u_l||_l \bar{B}^i(K_l)_0 \leq 1 + \epsilon$. Let us put:

$$v_l = \begin{cases} -u_l & l \leq l_0 \\ 0 & l > l_0. \end{cases} \tag{2.1}$$

Then $[v_l] \in \bar{B}^i_0(\{K_l\}_l)$, and one has:

$$\sup_l ||u_l + v_l||_l \bar{B}^i(K_l)_0 = \sup_{l \geq l_0} ||u_l||_l \bar{B}^i(K_l)_0 \leq 1 + \epsilon.$$

This shows $||[u_l]||_0 \leq 1$. Suppose $||[u_l]||_0 = 1 - \delta$, $\delta > 0$. Then there is $[v_l] \in \bar{B}^i_0(\{K_l\}_l)$ with $\sup_l ||u_l + v_l||_l \bar{B}^i(K_l)_0 \leq 1 - \frac{\delta}{2}$. Let us take $4\epsilon < \delta$ and a large l with:

$$||v_l||_l \bar{B}^i(K_l)_0 \leq \frac{\delta}{4}, \quad ||u_l||_l \bar{B}^i(K_l)_0 \geq 1 - \epsilon.$$

Then one has the estimates:

$$\begin{aligned} ||u_l||_l \bar{B}^i(K_l)_0 &\leq ||u_l + v_l||_l \bar{B}^i(K_l)_0 + ||v_l||_l \bar{B}^i(K_l)_0 \\ &\leq 1 - \frac{\delta}{2} + \frac{\delta}{4} = 1 - \frac{\delta}{4} < 1 - \epsilon. \end{aligned}$$

This is a contradiction. This shows the claim.

Let us compare $\hat{B}^*(\{K_l\}_l, w, \epsilon)$ and $\bar{B}^*(\hat{M})$. These are different Banach spaces. In fact one has the following:

Proposition 2.1 *There exists a natural isometric embedding:*

$$i : \bar{B}^*(\hat{M}, w, \epsilon) \hookrightarrow \hat{B}^*(\{K_l\}_l, w, \epsilon)$$

with closed image, but i is not surjective.

Proof : Let us take $u \in \bar{B}^*(\hat{M})$ and φ_l be the family of cut off functions in the first paragraph in 2.A. Then one puts $i(u) = [u_l = \varphi_l u]$. $[u_l]$ defines a non zero element in $\hat{B}^*(\{K_l\}_l, w, \epsilon)$. Closedness is clear, since i is isometric by the above lemma (use $| \quad |_1$ norm, also see remark 3.1).

Now let us consider a family $u_l \in \bar{B}^*(K_l)_0$ defined as follows; recall $\hat{M}$ contains an isometric subspace $(S^3 \times [0, \infty), g|S^3 \times dt^2)$. Let us take any $u \in C^\infty_{\text{cp}}(S^3 \times (0, 1))$, and $S_t : S^3 \times [t, t+1] \cong S^3 \times [0, 1]$ be the isometry by the translation. Then one puts $u_l = S^*_{2l}(u) \in C^\infty_{\text{cp}}(S^3 \times (2l, 2l+1))$. Then it is clear $[u_l]$ defines a non zero element in $\hat{B}^*(\{K_l\}_l, w, \epsilon)$, but it is not contained in the image of i. This completes the proof.

It is easy to get the following:

Corollary 2.1 *There is an isomorphism between Banach spaces:*

$$\hat{B}^*(\{K_l\}_l) \cong \bar{B}^*(\hat{M}, w, \epsilon) \oplus \bar{B}^*(S^3 \times \mathbf{R}, w, \epsilon).$$

$\hat{B}^*(\{K_l\}_l)$ defined above, is an extended Banach space containing with one bubble. Let us extend them so that they contain n bubbles, $n \geq 1$. We change indices of $\{K_l\}_l$ by $\{K_{m(l)}\}_l$, $m(l) >> l$, if necessarily. Let us take families of indices $\{a(i,j)\}_{i,j}$ and $\{b(i,j)\}_{i,j}$, $i = 0, 1, \ldots,$ $j = 0, 1, \ldots, n$ satisfying:

(1) $b(i,j) < a(i,j) < b(i,j+1)$ for each j, $a(i,n) = \infty$,
(2) $a(i,j) - b(i,j) \to \infty$,
(3) $b(l,j+1) - a(l,j) = a(l,j) - b(l,j)$,
(4) $\hat{M} \backslash S^3 \times [b(l,n), \infty) \subset K_l$ and $d(b(l,n), \partial K_l) \to \infty$.

$a(l,j)$ and $b(l,j)$ correspond to bottom and top of each wave respectively.

Let us put $w(i,j) = w_{b(i,j)}$ where w_l is in $2.A$. Also we use the linear translations $T(i,j) : S^3 \times \mathbf{R} \mapsto S^3 \times \mathbf{R}$ by $T(i,j)(m,0) = (m, b(i,j))$. Then we consider the set $[u_l]$, $u_l \in \bar{B}^*(K_l, w, \epsilon)_0$ satisfying the following:

(1)' $u_l = \varphi_{a(l,0)} u_l$ converges to $u_1 \in \bar{B}^*(\hat{M}, w, \epsilon)$,

(2)' $[\varphi_{a(l,j)} - \varphi_{a(l,j-1)}]u_l \in \bar{B}^*(K_l, w(l,j), \epsilon)_0$ and $T(l,j)^*([\varphi_{a(l,j)} - \varphi_{a(l,j-1)}]u_l)$ converge to $u_j \in \bar{B}^*(S^3 \times \mathbf{R}, w, \epsilon)$ for $j = 1, \ldots, n$.

Then we have a norm as:

$$\begin{aligned} ||[u_l]|\bar{B}^*(\{K_l\}_l; n) = \ & \mathrm{Sup}_l ||u_l||_l \\ \equiv \ & \mathrm{Sup}_l \{ |\varphi_{a(l,0)} u_l | \bar{B}^*(K_l, w, \epsilon)_0 \\ & \quad + \Sigma_{j=1}^n |[\varphi_{a(l,j)} - \varphi_{a(l,j-1)}]u_l | \bar{B}^*(K_l, w(l,j), \epsilon)_0 \}. \end{aligned}$$

The set of $[u_l]$ as above forms a Banach space $\bar{B}^*(\{K_l\}_l; n)$. Again one gets a closed subspace $\bar{B}_0^*(\{K_l\}_l; n)$ similarly as before. Later we will fix the pair of indices $\{a(l,j)\}$ and $\{T(i,j)\}$ and will not denote them specifically.

Definition 2.1 *Let us fix $n \geq 0$. Then we put:*

$$\hat{B}^*(\hat{M}, w, \epsilon) = \hat{B}^*(\hat{M}, w, \epsilon; n) = \bar{B}^*(\{K_l\}_l; n) / \bar{B}_0^*(\{K_l\}_l; n).$$

As before one gets an isomorphism:

$$\begin{aligned} \hat{B}^*(\hat{M}, \epsilon) \cong \ & \bar{B}^*(\hat{M}, w, \epsilon) \oplus \\ & \bar{B}^*(S^3 \times \mathbf{R}, w, \epsilon) \oplus \cdots \oplus \ (n \text{ times}) \ \bar{B}^*(S^3 \times \mathbf{R}, w, \epsilon). \end{aligned}$$

Notice the isometric filtration:

$$\bar{B}^*(\hat{M}, w, \epsilon) = \hat{B}^*(\hat{M}, w, \epsilon; 0) \subset \hat{B}^*(\hat{M}, w, \epsilon; 1) \subset \cdots \subset \hat{B}^*(\hat{M}, w, \epsilon; n).$$

Let us take a subsequence $k(0) < k(1) < \ldots,\ l << k(l)$. Then one has another exaustion as $K'_0 \subset K'_1 \subset \ldots,\ K'_l = K_{k(l)}$. Then we have two kinds of the natural maps:

$$i : \hat{B}^i(\{K_l\}) \mapsto \hat{B}^i(\{K'_l\}), \quad [u_l] \in \bar{B}^i(K_l)_0 \to [u_l] \in \bar{B}^i(K'_l)_0,$$
$$j : \hat{B}^i(\{K_l\}) \mapsto \hat{B}^i(\{K'_l\}), \quad [u_l] \in \bar{B}^i(K_l)_0 \to [u_{k(l)}] \in \bar{B}^i(K'_l)_0.$$

Lemma 2.3 *Both of i and j give isometric isomorphisms.*

Proof : Clearly i is an isometric injection and j is a surjection. Let us see j is isometric. $|[u_l]| \geq |j([u_l])|$ follows by definition. Take $[u_l] \in \hat{B}^i(\{K_l\})$ and choose subindices $m(l)$ with $\lim_l ||u_{m(l)}||_{m(l)} = |[u_l]|$. For any small $\epsilon > 0$, there exists l_0 so that for all $l \geq l_0$, $||u_{m(l)} - u_{k(l)}|| < \epsilon$. Thus one gets an estimate:

$$||u_{k(l)}||_{k(l)} \geq ||u_{m(l)}||_{m(l)} - ||u_{m(l)} - u_{k(l)}|| \geq ||u_{m(l)}||_{m(l)} - \epsilon.$$

This verifies that j is also isometric.

Now finally one checks that i is a surjection. For simplicity, one sees it for $\hat{B}^*(\{K_l\}_l; 1)$. Let us take $[u_l] \in \hat{B}^*(\{K'_l\}_l; 1)$ such that $\varphi_l u_l$ and $T^*_{2l}(1 - \varphi_l) u_l$ converge to v_1 and v_2 respectively. It is enough to find some $[w_l] \in \hat{B}^*(\{K_l\}_l; 1)$ so that $\varphi_l w_l$ and $T^*_{2l}((1 - \varphi_l) w_l)$ both converge to v_1 and v_2 respectively.

Let $S_l : S^3 \times [0, \infty)(\subset \hat{M}) \cong S^3 \times [-l, \infty)(\subset S^3 \times \mathbf{R})$ be the family of translations, and we take families of cut off functions p_l on $\hat{M}$ and q_l on $S^3 \times \mathbf{R}$ satisfying Supp $p_l \subset \mathrm{Supp} p_{l+1} \cdots \subset \hat{M}$ and Supp $q_l \subset \mathrm{Supp} q_{l+1} \cdots \subset S^3 \times \mathbf{R}$. Moreover one may choose these so that $\mathrm{Supp} p_l \subset K_l$ and $S_l^{-1}(\mathrm{Supp} q_l) \subset K_l$. Then one gets $w_l = p_l v_1 + S^*_{2l}(q_l v_2) \in \bar{B}^*(K_l)_0$ and clearly $[w_l] \in \hat{B}^*(\{K_l\}_l)$ satisfies the desired condition. This completes the proof.

2.B General setting: Here we have the most general form of the asymptotic Banach spaces. Later we will not use Banach spaces defined in this section. For simplicity, we choose zero weight functions $w = w' = 0$.

Let $u_0, u_1, \ldots$ be a family of $u_l \in \bar{B}^i(K_l)_0$ satisfying:

(1) $|u_l - u_{l'}|\bar{B}^i(K) \to 0$, $l' \geq l$ on every compact subset $K \subset K_l \subset K_{l'}$,

(2) $|u_l|\bar{B}^i(K_l)_0$ are uniformly bounded.

We put $\bar{B}(\{K_l\}_l) = \{[u_l]; (1), (2)\}$ as before.

Let $\bar{B}u^*(\{K_l\}_l)$ be a variant consisting of the set $\{[u_l] : (1)', (2)\}$ where we replace the topology in (1) by uniform convergence:

$$(1)' : \quad |u_l - u_{l'}|\bar{B}^i(K_{l'})_0 \to 0 \quad \text{uniformly.}$$

$\bar{B}u^*(\{K_l\}_l)$ is a closed subspace in $\bar{B}^*(\{K_l\}_l)$. $\bar{B}u^*_0(\{K_l\}_l)$ is defined similarly as before. Then the quotient space $\bar{B}^*(\hat{M}) = \bar{B}u^*(\{K_l\}_l)/\bar{B}u_0(\{K_l\}_l)$ naturally isometrically injects:

$$I : \bar{B}^*(\hat{M}) \hookrightarrow \hat{B}^*(\{K_l\}_l) = \bar{B}^*(\{K_l\}_l)/\bar{B}u_0^*(\{K_l\}_l).$$

Lemma 2.4 *Any $[u_l] \in \hat{B}^*(\{K_l\}_l)$ has a decomposition $[u_l] = [w_l] + [b_l]$ where $[w_l] \in \bar{B}u^*(\{K_l\}_l)$ and Supp$b_l \to \infty$ as $l \to \infty$.*

Proof : By the condition, there is $u_\infty \in \bar{B}^*(\hat{M})$ such that u_l converges to u_∞ on each compact subset. Let $\{\varphi_l\}_l$ be the previous family of cut off functions. Let us take any decreasing family $\{\delta_l\}_l$, $\delta_0 >> \delta_1 >> \cdots \to 0$. First take l_0 so that for any $l, l' \geq l_0$, $|u_l - u_{l'}|\bar{B}^*(K_0) < \delta_0$. Next take $l_1 \geq l_0$ so that for any $l, l' \geq l_1$, $|u_l - u_{l'}|\bar{B}^*(K_1) < \delta_1$. Similarly we inductively determine $l_2 << l_3 << \dots$. Then we put:

$$\begin{aligned} \psi_i &= \varphi_0, \quad 0 \leq i \leq l_1, \\ &= \varphi_1, \quad l_1 + 1 \leq i \leq l_2, \\ &= \dots \\ &= \varphi_l, \quad l_l + 1 \leq i \leq l_{l+1}, \\ &= \dots \end{aligned}$$

Then we put $w_l = \psi_l u_l$ and $b_l = (1 - \psi_l)u_l$. Clearly this satisfies the desired condition. This completes the proof.

Corollary 2.2 *$\hat{B}^i(\hat{M})$ contains $\bar{B}^i(\hat{M}) \oplus \bar{B}^i(S^3 \times \mathbf{R}) \cdots \oplus \bar{B}^i(S^3 \times \mathbf{R}) \oplus \dots$ isometrically.*

3 AHS complex

Here we show that the AHS complex between $\hat{B}^*(\{K_l\}_l)$ is of Fredholm. In order to verify it, we will take two steps. First we consider more simple case. Recall Banach spaces $B^*(\hat{M}, w, \epsilon)$ defined in 1.A. Let us fix $n \geq 0$ and put:

$$B^*(\{K_l\}_l, \epsilon) = \{[u_l] : u_l \in B^*(K_0, \epsilon)_0, \text{ satisfying (1)', (2)' in } 2.A\}.$$

We have also its closed subspace $B_0^*(\{K_l\}_l, \epsilon)$. Then we put:

$$\tilde{B}^*(\{K_l\}_l, \epsilon) \equiv B^*(\{K_l\}_l, \epsilon)/B_0^*(\{K_l\}_l, \epsilon).$$

First we verify Fredholmness of the AHS complex for $\tilde{B}^*(\{K_l\}_l)$, and then for $\hat{B}^*(\{K_l\}_l)$.

3.B $\tilde{B}^*$ case: Now one has the bounded complex:

$$0 \to \tilde{B}^0(\{K_l\}_l) \xrightarrow{d} \tilde{B}^1(\{K_l\}_l) \xrightarrow{d^+} \tilde{B}^2(\{K_l\}_l) \to 0 \tag{3.1}$$

Proposition 3.1 *The above complex has closed range.*

Proof: Let us consider $d(\tilde{B}^0) \subset \tilde{B}^1$. Suppose:

$$||[d(u_l^\alpha)] - [w_l]||_1 \to 0, \quad \alpha \to \infty \quad [w_l] \in \tilde{B}^1.$$

We only consider $n = 1$. The general case is similar. Recall the norm $||u_l||_l B^* = |\varphi_l u_l|B^* + |T_{2l}^*(1-\varphi_l)u_l|B^*$. Since $d : (W_1^{4+2\epsilon})_w \mapsto L_w^{4+2\epsilon}(\Lambda^1)$ has closed range with null kernel over both $\hat{M}$ and $S^3 \times \mathbf{R}$, there is a large l_0 determined by $[u_l]$ such that for all $l \geq l_0$, one gets an estimate (see remark 3.1):

$$||du_l||_l B^1 \geq C||u_l||_l B^0. \quad *$$

By definition for small $\epsilon > 0$, there is α_0 with $\lim_{l\to\infty}||d(u_l^\alpha) - w_l||_l B^1 < \epsilon$ for $\alpha \geq \alpha_0$. In particular for each α, there is $l(\alpha)$ such that the set $\{du_{l(\alpha)}^\alpha\}_\alpha$ consists of Cauchy sequence in $|| \quad ||B^1$ topology. Moreover one may assume the estimate $*$ for all $u_{l(\alpha)}^\alpha$. Thus $[v_\alpha = u_{l(\alpha)}^\alpha]$ defines an element in $B^0(\{K_{l(\alpha)}\}_\alpha)$. By the condition, $[dv_\alpha - w_{l(\alpha)}] \in B_0^1(\{K_{l(\alpha)}\})$. Combining with the isomorphism $j : \tilde{B}^*(\{K_l\}_l) \cong \tilde{B}^*(\{K_{l(\alpha)}\}_\alpha)$ in lemma 2.3, this verifies that $d : \tilde{B}^0(\{K_l\}_l) \mapsto \tilde{B}^1(\{K_l\}_l)$ has closed range.

Next let us consider $d^+(\tilde{B}^1) \subset \tilde{B}^2$. The following lemma completes the proof of the proposition.

Lemma 3.1 *There exist a decomposition $\tilde{B}^2 \cong \ imd^+ \oplus H$ and a bounded map $Q : \tilde{B}^2 \mapsto \tilde{B}^1$ with $d^+Q|\ imd^+$ is the identity, where $H = H(0)$ in proposition* 1.1.

Proof : Recall a decomposition $L_w^{2+\epsilon}(\hat{M}; \Lambda_+^2) = \ \text{im}d^+ \oplus H$ where H is of finite dimension, and a bounded map Q with $d^+ \circ Q = \text{id}$:

$$Q : \ \text{im}d^+ \cap L_w^{2+\epsilon}(\hat{M}; \Lambda_+^2) \mapsto B^1(\hat{M}; \epsilon).$$

With respect to the above decomposition, one puts $w = w^1 \oplus w^2$, and $| \quad |_0 \equiv | \quad |_1 \oplus | \quad |_2$ is the Banach norm on $L_w^{2+\epsilon}(M; \Lambda_+^2)$ where $| \quad |_1$ and $| \quad |_2$ are the ones on $L_w^{2+\epsilon}$ and H respectively. Then one has an equivalence $C^{-1}| \quad |_0 \leq | \quad |_1 \leq C| \quad |_0$.

Let us choose a sequence $\{k(l)\}_l$ such that for any $u \in H$, $|u|B^2 = 1$, $\lim_l|(1-\psi_l)u|B^2 = 0$ follows for $\psi_l = \varphi_{k(l)}$. For any element $u = [u_l] \in \tilde{B}^2$, one has a decomposition as $\varphi_l u_l = u_l^1 \oplus u_l^2 \in B^2(\hat{M}, \epsilon)$ where $u_l^1 \in \ \text{im}d^+ \subset L_w^{2+\epsilon}(\hat{M}; \Lambda_+^2)$ and $u_l^2 \in H$. With respect to the family $\{Q(u_l^1)\}_l$, one takes another sequence $k(l)' \geq k(l)$ and puts $\psi_l' = \varphi_{k(l)'}$. Then one may assume $|Q(u_l^1) - \psi_l' Q(u_l^1)|B^1(\hat{M}, \epsilon) \to 0$. In the Banach space $B^2(\{K_{k(l)'}\}_l)$, the sequences $[v_l^1 = d^+(\psi_l' Q u_l^1)]$ and $[\psi_l u_l^2]$ satisfy the condition (1)' in $2.A$, and these defines elements in $\text{im}d^+$, $\tilde{H} \subset \tilde{B}^2(\{K_{k(l)'}\}_l)$ respectively.

Since $d^+ : B^1(S^3 \times \mathbf{R}, \epsilon) \mapsto B^2(S^3 \times \mathbf{R}, \epsilon)$ is surjective, there is a bounded inverse $Q : B^2(S^3 \times \mathbf{R}, \epsilon) \mapsto B^1(S^3 \times \mathbf{R}, \epsilon)$ such that $d^+ \circ Q$ is the identity. Then by choosing a subsequence $\{k(l)'\}_l$ as above, one may assume that the family $[w_l \equiv (1-\psi_l')(T_{2k(l)'}^{-1})^* Q(T_{2l}^*(1-\varphi_l)u_l)]$ gives an element in $\tilde{B}^1(\{K_{k(l)'}\}_l)$.

Thus one assigns $[u_l] \to [d^+\psi_l' Q(u_l^1) + d^+ w_l] \oplus [\psi_l u_l^2]$. Passing through the isometric isomorphisms $i : \tilde{B}^i(\{K_l\}_l) \cong \tilde{B}^i(\{K_{k(l)'}\}_l)$ in lemma 2.3, one gets a map $\tilde{B}^2(\{K_l\}_l) \mapsto \mathrm{im} d^+ \oplus H$. Since the norms are invariant under change of indices, this map gives a one to one onto continuous map. This assignment gives the isomorphism.

If $[u_l^2] \in B_0^2$, then one puts $Q([u_l]) = [\psi_l' Q(u_l) + w_l] \in B^1$. This completes the proof.

Next let us compute the cohomology groups. Let us take a subsequence $k(0) < k(1) < \dots$, $l << k(l)$. Then one has another exaustion as $K_0' \subset K_1' \subset \dots$, $K_l' = K_{k(l)}$, and has the corresponding Banach spaces $\tilde{B}^i(\{K_l\})$ and $\tilde{B}^i(\{K_l'\})$. As before, since these are all isomorphic, one has also isomorphisms between their cohomology groups.

Recall that the AHS complex:

$$0 \to B^0(\hat{M}, g, \epsilon, w) \xrightarrow{d} B^1(\hat{M}, g, \epsilon, w) \xrightarrow{d^+} B^2(\hat{M}, g, \epsilon, w) \to 0 \tag{3.2}$$

has their cohomology groups $H^0 = H^1 = 0$ and $\dim H^2 = b_2^+(M)$. Let us consider the corresponding complex:

$$0 \to \tilde{B}^0(\{K_l\}_l) \xrightarrow{d} \tilde{B}^1(\{K_l\}_l) \xrightarrow{d^+} \tilde{B}^2(\{K_l\}_l) \to 0. \tag{3.3}$$

Here we show that one gets the same result as above. First let us take $u = [u_l] \neq 0 \in H^0$. By definition, there is a positive constant $C > 0$ and a subsequence l with $||u_l||_l B^0(K_l)_0 \geq C$ and $||d(u_l)||_l L^{4+2\epsilon}_{2w}(K_l)_0 \to 0$. This is a contradiction since $d : (W_1^{4+2\epsilon})_{2w} \mapsto L^{4+2\epsilon}_{2w}(\Lambda^1)$ has closed range with null kernel. This shows $H^0 = 0$.

Next let us choose $u = [u_l] \in H^1$. Take a sufficiently large l so that $||d^+(u_l)||_l L_w^{2+\epsilon}(K_l)_0$ is bounded by a small $\epsilon > 0$. Then there is $f_l \in B^0(\hat{M})$ with $||u_l - df_l||_l B^1(\hat{M}) < C\epsilon$. This can be seen as follows; suppose $\varphi_l u_l$ and $T_{2l}^*(1-\varphi_l)u_l$ converge to v_1 and v_2 in $B^1(\hat{M}, w, \epsilon)$ and $B^1(S^3 \times \mathbf{R}, w, \epsilon)$ respectively. Then by the assumption, both of v_i satisfy $d^+(v_i) = 0$. Thus there are $f \in B^0(\hat{M})$ and $g \in B^0(S^3 \times \mathbf{R})$ with $df = v_1$ and $dg = v_2$. Using the notations in the proof of lemma 2.3, one gets $[f_l = p_l f + S_{2l}^*(q_l g)] \in B^0(\{K_l\}_l)$. Clearly $[u_l] = [df_l] \in B^1(\{K_l\}_l)$. This shows $H^1 = 0$.

Let us denote $H^2 = H^2(\tilde{B}^*(\{K_l\}_l))$ and similarly $H^2(B^*(\hat{M}))$. Then there is a natural map (using a family of cut off functions $\{\varphi_l\}$):

$$I : H = H^2(B^*(\hat{M})) \mapsto H^2(\tilde{B}^*(\{K_l\}_l))$$

by $I(u) = [\varphi_l u] \equiv [u_l]$. We see that this gives an isomorphism. Suppose $I(u) = 0$. Then there is $[v_l] \in B^1(\{K_l\}_l)$ with $[u_l - d^+(v_l)] \in B_0^2(\{K_l\}_l)$. In particular for any small $\delta > 0$, there is a large l with $|u_l - d^+(v_l)|L_w^{2+\epsilon} < \delta$. Since $d^+ : B^1(\hat{M}) \mapsto B^2(\hat{M})$ has closed range, this shows $u = 0 \in H$. Thus I is an injection. Surjectivity follows from lemma 3.1. This shows $\dim H^2(\tilde{B}^*) = b_+^2(M)$.

3.C ϵ collapsed Banach spaces: Here we define more general asymptotic Banach spaces $\hat{B}^*(\{K_l, \epsilon_l\}_l)$, and in the next section we study the AHS complex over $\hat{B}^*(\{K_l, \epsilon_l\}_l)$.

Let $K_0 \subset K_1 \subset \cdots \subset \hat{M}$ be an exaustion where $(\hat{M}, g)$ is the cylindrical manifold. One may assume that $(\hat{M} \backslash K_0, g)$ is isometric to $(S^3 \times [0, \infty), g|S^3 + dt^2)$. Let us choose a family of small and positive constants:

$$1 >> \epsilon_0 \geq \epsilon_1 \geq \cdots > 0.$$

Then we have Banach spaces $\hat{L}_w^p(K_l) = \hat{L}_w^p(K_l; \epsilon_l)$ $(\hat{L}_w^p(K_l)_0)$. $\hat{L}_{2w}^4$ is defined by the norm:

$$|u|\hat{L}_{2w}^4(K_l, \epsilon_l) = \ \sup_{0 \leq \epsilon \leq \epsilon_l}\{|u|L_{2w}^{4+2\epsilon}(K_l)\}.$$

$\hat{L}_w^2$ is also similarly defined (see $2.A$). Using these, one obtains similar Banach spaces:

$$\begin{aligned}
&\bar{B}^0(K_l, \epsilon_l)_0 = \{u \in \hat{L}_{2w'}^4(K_l, \epsilon_l)_0, \quad du \in \hat{L}_{2w}^4(K_l, \epsilon_l)_0\}, \\
&\bar{B}^1(K_l, \epsilon_l)_0 = \{u \in \hat{L}_{2w}^4(K_l, \epsilon_l)_0, \quad d(u) \in \hat{L}_w^2(K_l, \epsilon_l)_0\}, \\
&\bar{B}^2(K_l, \epsilon_l)_0 = \hat{L}_w^2(K_l, \epsilon_l; \Lambda_+^2)_0.
\end{aligned}$$

We recall the norms $|| \quad ||_l$ introduced in $2.A$. If one chooses a subfamily $\{\epsilon_l' = \epsilon_{k(l)}\}_l$, $l \leq k(l)$, then the bound holds:

$$||u||_l \bar{B}^i(K_l, \epsilon_l')_0 \leq ||u||_l \bar{B}^i(K_l, \epsilon_l)_0.$$

In order to define asymptotic Banach spaces, we use the formulation used in $2.A$. Let us define:

$$\bar{B}^i(\{K_l, \epsilon_l\}_l) = \{[u_l] : u_l \in \bar{B}^i(K_l, \epsilon_l)_0, \quad (1), (2) \text{ below}\}$$

with the norm $|[u_l]| = \mathrm{Sup}_l ||u_l||_l \bar{B}^i(K_l, \epsilon_l)_0$. Let $\{a(i,j)\}$ and $\{T(i,j)\}$ be the families of indices and translations in $2.A$. Here $[u_l \in \bar{B}^i(K_l)_0]$ is a sequence satisfying:

(1) $|\varphi_l u_l - \varphi_{l'} u_{l'}|\bar{B}^i(K_{l'}, w, \epsilon_{l'})_0 \to 0$ as $l, l' \to \infty$, $l' \geq l$,

(2) $u(l,j) \equiv [\varphi_{a(l,j)} - \varphi_{a(l,j-1)}]u_l \in B^*(K_l, w(l,j), \epsilon_l)_0$ and $|T(l,j)^* u(l,j) - T(l',j)^* u(l',j)| \in \bar{B}^*(S^3 \times \mathbf{R}, w, \epsilon_{l'}) \to 0$, $l' \geq l \to \infty$.

As before one has the closed subspace $\bar{B}^i_0(\{K_l, \epsilon_l\}_l) \subset \bar{B}^i(\{K_l, \epsilon_l\}_l)$. When we take the quotient spaces, we divide into two cases; suppose all ϵ_l are the same $\epsilon > 0$. Then we put the quotient space by:

$$\hat{B}^i(\{K_l\}_l) = \bar{B}^i(\{K_l\}_l)/\bar{B}^i_0(\{K_l\}_l).$$

In this case lemma 2.3 tells us that $\hat{B}^*(\{K_l\}_l)$ doesnot depend on choice of partitions $\{K_l\}_l$, and we denote them as $\hat{B}^*(\hat{M})$.

Let us consider the case, strictly decreasing constants $\epsilon_l > \epsilon_{l+1} > \cdots \to 0$. If we take another partition $\{K'_l\}_l$ with $K_l \subset K'_l$, then we have a natural embedding $\hat{B}^*(\{K_l, \epsilon_l\}_l) \subset \hat{B}^*(\{K'_l, \epsilon_l\}_l)$. Then we take a direct limit and denote as:

$$\hat{B}^*(\hat{M}) = \lim \hat{B}^*(\{K_l\}_l).$$

3.D AHS complex for $\bar{B}^*(\{K_l, \epsilon_l\}_l)$: In this section we consider the AHS complex over $\hat{B}^*(\hat{M}, g)$. When we define $\hat{B}^*(\hat{M}, g)$, we have used a family of non increasing constants $\{\epsilon_l\}_l$. We will study analytic property of the AHS complex $\{\hat{B}^*(\hat{M}, g), d^*\}$ including the case when the family of constants approaches to zero, $\epsilon_0 >> \epsilon_1 >> \ldots \epsilon_l >> \cdots \to 0$. In particular we show that it is of Fredholm and the cohomlogy groups are the expected ones.

We need the following preparation. Let us take pairs $(p, p') \in \{(2, 2+\epsilon), (4, 4+2\epsilon)\}$ and Q be a positive second order self adjoint elliptic operator over S^3. Let us put the Banach spaces:

$$\dot{L}^p = \text{ Max } \{L^p, \quad L^{p'}\}.$$

Namely we have $|u|\dot{L}^p = \max \{|u|L^p, |u|L^{p'}\}$.

When using $\dot{L}^4$ spaces, we restrict the following case; $Q = \Delta_0 + \delta > 0 : \dot{W}^4_2(S^3) \mapsto \dot{L}^4(S^3)$ where Δ_0 is the Laplace operator over functions.

Proposition 3.2

$$\Delta = -\frac{d^2}{dt^2} + Q : \dot{W}^p_2(S^3 \times \mathbf{R}) \mapsto \dot{L}^p(S^3 \times \mathbf{R})$$

gives an isomorphism with a uniform bound:

$$|\Delta(u)|\dot{L}^p(S^3 \times \mathbf{R}) \geq C|u|\dot{W}^p_2(S^3 \times \mathbf{R})$$

where C is independent of choice of small $\epsilon \geq 0$.

Sublemma 3.1

$$\Delta : W_2^p(S^3 \times \mathbf{R}) \mapsto L^p(S^3 \times \mathbf{R})$$

is an isomorphism.

Proof : For $p = 2$, the result is well known, using the exponential decay estimate for vectors with low spectrum.

Let us consider $p = 4$. Let us take $u \in W_2^4(S^3 \times \mathbf{R})$ and put $u_t \in W_2^4(S^3)$ a.e.. Notice that by the Sobolev embedding, u is a continuous section. Suppose $|\Delta(u)|L^4$ is small. We put $f(t) = \int_{S^3} |u_t|^4 vol$. Then we have estimates:

$$\begin{aligned} f(t)' &= 4\int < u', u > |u|^2, \\ f'' &= \int < u'', u > |u|^2 + \int |u'|^2|u|^2 + 2\int | < u', u > |^2 \\ &\geq \int < Qu, u > |u|^2 - \mu f \end{aligned}$$

where $\delta > \mu \geq 0$. Here we use the condition $Q = \Delta_0 + \delta$. We have the following:

$$\begin{aligned} &< \Delta_0 u_t, u_t > |u_t|^2 vol = - * d * du_t u_t^3 vol = -(d * du_t)u_t^3, \\ &\int < \Delta_0 u_t, u_t > u_t^2 = \int 3u_t^2 du_t \wedge * du_t = 3\int |du_t|^2 u_t^2 \geq 0. \end{aligned}$$

Thus we have an estimate $f'' \geq (\delta - \mu)f$. The rest follows from the ordinary method. Namely one can verify the exponential decay for f and Fredholmness for Δ. This completes the proof.

Proof of proposition: First we show that there exists a constant C independent of ϵ with a bound:

$$|\Delta(u)|\dot{L}^p(S^3 \times \mathbf{R}) \geq C|u|\dot{L}^p(S^3 \times \mathbf{R}).$$

Let us take u with $|u|\dot{L}^p(S^3 \times \mathbf{R}) = 1$. When $|u|\dot{L}^p(S^3 \times \mathbf{R}) = |u|L^p(S^3 \times \mathbf{R})$, then the result follows from the above sublemma.

Suppose $|u|\dot{L}^2(S^3 \times \mathbf{R}) = |u|L^{2+\epsilon}(S^3 \times \mathbf{R})$. We decompose:

$$|u|^{2+\epsilon} L^{2+\epsilon}(S^3 \times \mathbf{R}) = \Sigma_n |u|^{2+\epsilon} L^{2+\epsilon}(S^3 \times [n, n+1]).$$

Then by the Sobolev inequality, there is a constant C independent of ϵ and n with a bound:

$$|u|L^{2+\epsilon}(S^3 \times [n, n+1]) \leq C|u|W_1^2(S^3 \times [n, n+1]).$$

Thus one gets an estimate:

$$\Sigma_n |u|^2 L^{2+\epsilon}(S^3 \times [n, n+1]) \leq C\Sigma_n |u|^2 W_1^2(S^3 \times [n, n+1]) = C|u|^2 W_1^2(S^3 \times \mathbf{R}).$$

Here we claim the inequality:

$$\Sigma_n |u|^{2+\epsilon} L^{2+\epsilon}(S^3 \times [n, n+1]) \leq [\Sigma_n |u|^2 L^{2+\epsilon}(S^3 \times [n, n+1])]^{\frac{2+\epsilon}{2}}.$$

¿From this, we get the estimate for $p = 2$:

$$|u| L^{2+\epsilon}(S^3 \times \mathbf{R}) \leq C |u| W_1^2(S^3 \times \mathbf{R}).$$

In fact let us put $b_n = |u|^2 L^{2+\epsilon}(S^3 \times [n, n+1]) \leq 1$ and $\frac{2+\epsilon}{2} = 1 + \delta$. Then we show the inequality $\Sigma_n b_n^{1+\delta} \leq (\Sigma_n b_n)^{1+\delta}$. By normalizing one may assume $\Sigma_n b_n = 1$. Since $b_n \leq 1$, clearly we have $\Sigma_n b_n^{1+\delta} \leq \Sigma_n b_n$. Thus we have the estimate:

$$\Sigma_n b_n^{1+\delta} \leq \Sigma_n b_n = (\Sigma_n b_n)^{1+\delta}.$$

Now we have the desired inequalities:

$$C |\Delta(u)| L^2 \geq |u| W_2^2 \geq |u| W_1^2 \geq C' |u| L^{2+\epsilon}.$$

This implies the estimate $|\Delta(u)| \dot{L}^2 \geq C |u| \dot{L}^2$.

For $p' = 4$, the method is parallel. Here we note the Sobolev embedding $W_1^4 \hookrightarrow L^{4+2\epsilon}$.

Now we want to obtain the inequality $|\Delta(u)| \dot{L}^p \geq C |u| \dot{W}_2^p$. By the elliptic estimate, we have:

$$|u| W_2^{4+2\epsilon} \leq C \{ |u|^2 W_1^{4+2\epsilon} + |\Delta(u)|^2 L^{4+2\epsilon} \}^{\frac{1}{2}}.$$

Suppose $|u_i| \dot{W}_2^p = 1$ and $|\Delta(u_i)| \dot{L}^p \to 0$. If $|u_i| \dot{W}_2^p = |u_i| W_2^p$, this contradicts to $|\Delta(u_i)| L^p \to 0$ and the above sublemma. Thus let $|u_i| \dot{W}_2^p = |u_i| W_2^{4+2\epsilon}$. Combining with $|\Delta(u_i)| L^{4+2\epsilon} \to 0$ and the above elliptic estimate, we have $1 - \delta \leq |u_i| W_1^{4+2\epsilon} \leq 1$. Using the Sobolev embeddings:

$$W_2^2 \hookrightarrow W_1^{2+\epsilon}, \quad W_2^4 \hookrightarrow W_1^{4+2\epsilon}$$

and a similar argument as above, we get the estimate $1 - \delta \leq |u_i| W_1^{4+2\epsilon} \leq C |u_i| W_2^4$. Combining with the estimate $|u_i| W_2^4 \leq C |\Delta(u_i)| L^4$, this is a contradiction. Thus we get the desired estimate:

$$|\Delta(u_i)| \dot{L}^p \geq C |u_i| \dot{W}_2^p.$$

This completes the proof.

Corollary 3.1 *There exists $Q : \hat{L}_w^2(S^3 \times \mathbf{R}) \mapsto (\hat{W}_1^2)_w \cap$ $Kerd_w^*$ with $d^+ Q =$ id.*

Proof : We proceed as in the proof of proposition 1.1. Let us put $\nabla = d^+ \oplus d_w^* : (\hat{W}_1^2)_w \mapsto \hat{L}_w^2$. Then we can see that there is a constant $C > 0$ with $|\nabla| \geq C$.

We can also see $\Delta : (\hat{W}^2_2)_w \mapsto \hat{L}^2_w$ satisfies $|\Delta| \geq C > 0$ on functions. Thus $(\hat{W}^2_1)_w$ decomposes as:

$$(\hat{W}^2_1)_w \cong \text{ im} d \oplus \text{ Ker} d^*_w$$

by $u \to d\Delta^{-1}d^*_w u + u - d\Delta^{-1}d^*_w u$. Since $|\nabla| \geq C > 0$, the bounded map $d^+ : \text{Ker} d^*_w \mapsto \hat{L}^2_w$ satisfies $|d^+| \geq C > 0$. Now we put $Qd^+(v) = v - d\Delta^{-1}d^*_w v$. This completes the proof.

¿From the estimate at the end of senction 1, one gets the following map:

$$Q : \bar{B}^2(S^3 \times \mathbf{R}) \mapsto \bar{B}^1(S^3 \times \mathbf{R}) \cap \text{ Ker} d^*_w$$

with $d^+ Q = \text{id}$.

Remark 3.1: Recall that since Δ over $\hat{M}$ can be written as $d^*_w \circ d$, it follows from the above that $d : (\hat{W}^4_2)_{2w} \mapsto (\hat{W}^4_1)_{2w}$ has closed range.

Let us consider $d : (\hat{W}^4_1)_{2w} \mapsto \hat{L}^4_{2w}$. This has also closed range as follows. ¿From sublemma 3.1, $d : (W^2_1)_{2w} \mapsto L^2_{2w}$ has closed range. Then we have an estimate:

$$|u|^4 L^4_{2w} = \int exp(2w)u^4 \leq C2 \int exp(2w)u^2|du|^2 \leq C|u|^2 L^4_{2w}|du|^2 L^4_{2w}.$$

Thus we get a bound $|u|L^4_{2w} \leq C|du|L^4_{2w}$. Then following the proof of proposition 3.2, we get the desired estimate $|u|\hat{L}^4_{2w} \leq C|du|\hat{L}^4_{2w}$.

Lemma 3.2 *Let us take δ and ϵ_0. Then for $0 < \delta' = \delta'(\delta) < \delta$, $0 \leq \epsilon_l \leq \epsilon_0$, the AHS complex:*

$$0 \to \hat{B}^0(\hat{M}) \xrightarrow{d} \hat{B}^1(\hat{M}) \xrightarrow{d^+} \hat{B}^2(\hat{M}) \to 0 \tag{3.4}$$

has closed range, where $\hat{B}^*(\hat{M}) = \hat{B}^*(\{K_l, \epsilon_l\}_l, w('))$.

Proof : Recall that for $0 \leq \epsilon \leq \epsilon_0$, one gets an isomorphism:

$$d : B^0(\hat{M}) \cong \text{ im} d \subset B^1(\hat{M}).$$

Let $| \quad |^\epsilon_i$, $i = 0, 1$, be the right and left hand sides norms respectively. Then one has equivalences $C^{-1}_\epsilon | \quad |^\epsilon_0 \leq | \quad |^\epsilon_1 \leq C_\epsilon | \quad |^\epsilon_0$.

Sublemma 3.2 *For $d : B^0(\hat{M}; \epsilon) \mapsto B^1(\hat{M}; \epsilon)$, there is a uniform bound of the family of constants $C^{-1} \leq C_\epsilon \leq C$.*

Proof : We show a uniform bound $|f|B^0(\hat{M}; \epsilon) \leq C|df|B^1(\hat{M}; \epsilon)$ for all $0 \leq \epsilon \leq \epsilon_0$, which follows from another one $|f|L^{4+2\epsilon}_{2w'}(\hat{M}) \leq C|df|L^{4+2\epsilon}_{2w}(\hat{M})$. Let us choose $\delta' < \delta$ which will be determined later, and take the corresponding w' with weight δ'. Then there is a constant C independent of ϵ with a bound:

$$| \quad |L^4_{2w'}(\hat{M}) \leq C| \quad |L^{4+2\epsilon}_{2w}(\hat{M}).$$

This follows from the local Holder estimate $L^4_{\text{loc}} \hookrightarrow L^{4+2\epsilon}_{\text{loc}}$ and a local to global method (see below, or proof of proposition 4.1). Then we show another estimate $|u|L^4_{2w'} \leq C|du|L^4_{2w}$. Using Hilbert space structure, one gets easily an estimate $|u|(W^2_1)_w \leq C|du|L^2_w$ (see [K] 6.A). Then we will have inequalities:

$$|u|L^4_{2w'} \leq C|u|(W^2_1)_{w'} \leq C|du|L^2_{w'} \leq |du|L^4_{2w}.$$

Notice local Sobolev embedding $(W^2_1)_{\text{loc}} \hookrightarrow L^4_{\text{loc}}$. Then the first and the last inequalities follow from the next estimates respectively:

$$\Sigma_n a_n^4 exp(2n\delta) \leq (\Sigma_n a_n^2 exp(n\delta))^2,$$

$$\begin{aligned} \Sigma_n a_n^2 exp(n\delta') &= \Sigma_n [a_n^2 exp(n\delta)][exp(n(\delta' - \delta))] \\ &\leq C(\Sigma_n a_n^4 exp(2n\delta))^{\frac{1}{2}}. \end{aligned}$$

Finally we show a uniform bound $|f|L^{4+2\epsilon}_{2w} \leq C|f|(W^4_1)_{2w}$. Notice local Sobolev embedding $|f|L^{4+2\epsilon}_{\text{loc}} \leq C|f|(W^4_1)_{\text{loc}}$. Using this constant C, one proceeds a similar argument as before. By normalizing one may assume $|f|(W^4_1)_{2w} = C^{-1} < 1$. In particular for each n, it follows $a_n \equiv |f|L^{4+2\epsilon}(S^3 \times [n, n+1]) \leq Cb_n \equiv |f|W^4_1(S^3 \times [n, n+1]) \leq 1$. Let us put $1 - \mu = \frac{4}{4+2\epsilon}$. Then one gets the estimate:

$$\begin{aligned} |f|L^{4+2\epsilon}_{2w} &= (\Sigma_n a_n^{4+2\epsilon} exp(2n\delta))^{(4+2\epsilon)^{-1}} \\ &\leq (\Sigma_n a_n^4 exp(2n\delta))^{(4+2\epsilon)^{-1}} \\ &\leq C(\Sigma_n b_n^4 exp(2n\delta))^{(4+2\epsilon)^{-1}} = C|f|^{1-\mu}(W^4_1)_{2w} \\ &= C|f|(W^4_1)_{2w}|f|^{-\mu}(W^4_1)_{2w} \leq C'|f|(W^4_1)_{2w}. \end{aligned}$$

This verifies the claim.

Now suppose a family f_i $\epsilon_i \to 0$, satisfies $|f_i|L^{4+2\epsilon_i}_{2w'} = 1$ and $|df_i|L^{4+2\epsilon_i}_{2w} \to 0$. Then one gets the estimate:

$$\begin{aligned} |f_i|L^{4+2\epsilon}_{2w'} &\leq C|f_i|(W^4_1)_{2w'} \\ &\leq C|f_i|L^4_{2w'} + C\delta_i \\ &\leq C|df_i|L^4_{2w''} + C\delta_i \leq C|df_i|L^{4+2\epsilon_i}_{2w} + C\delta_i \to 0. \end{aligned}$$

This is a contradiction. This completes the proof.

Proof of lemma: Suppose there exists a family $\{[u_l^\alpha]\}_\alpha \subset \hat{B}^0(\{K_l, \epsilon_l\}_l)$ with $|[u_l^\alpha]|\hat{B}^0 = 1$ and $\lim_\alpha |[du_l^\alpha]|\hat{B}^1 \to 0$. Let us take a decreasing family δ_α and assume $\limsup_l ||du_l^\alpha||_l \bar{B}^1 < \delta_\alpha$, while $\limsup_l ||u_l^\alpha||_l \bar{B}^1 = 1$. In particular for some arbitrarily large $l = l(\alpha)$, one has the estimates:

$$||du_l^\alpha||_l \bar{B}^1 \leq \delta_\alpha, \quad ||u_l^\alpha||_l \bar{B}^0 \geq 1 - \delta_\alpha.$$

Then one gets the inequality:

$$1 - \delta_\alpha \leq ||u_l^\alpha||_l \bar{B}^0(K_l, \epsilon_l)_0 \leq C||du_l^\alpha||_l \bar{B}^1(K_l, \epsilon_l)_0 \leq C\delta_\alpha.$$

This is a contradiction. Thus $d : \hat{B}^0(\{K_l, \epsilon_l\}_l) \mapsto \hat{B}^1(\{K_l, \epsilon_l\}_l)$ has closed range.

Now we have another method to verify that d has closed range which also works for d^+.

Let us take any subindices $\{k(l)\}_l$ and consider the restriction map:

$$j : \hat{B}^*(\{K_l, \epsilon_l\}_l) \mapsto \hat{B}^*(\{K_{k(l)}, \epsilon_{k(l)}\}_l),$$
$$j_0 : \bar{B}^*(\{K_l, \epsilon_l\}_l) \mapsto \bar{B}^*(\{K_{k(l)}, \epsilon_{k(l)}\}_l).$$

We claim that j gives an isometric isomorphism between Banach spaces. Since j_0 is surjective, j is also so. Let us take $[u_l] \in \hat{B}^*(\{K_l, \epsilon_l\}_l)$. We choose subindices $\{m(l)\}$ and $\{k(l)'\} \subset \{k(l)\}$ so that $\lim_l ||u_{m(l)}|| = |[u_l]|\hat{B}^*(\{K_l, \epsilon_l\}_l)$ and $\lim_l ||u_{k(l)'}|| = |j([u_l])|\hat{B}^*(\{K_{k(l)}, \epsilon_{k(l)}\}_l)$. By the definition, for any small $\mu > 0$, there is a large l_0 such that for all $l' \geq l \geq l_0$, $||u_{l'} - u_l||\bar{B}^*(K_{l'}, \epsilon_{l'}) \leq \mu$. In particular one gets $||u_{k(l)'} - u_{m(l')}||\bar{B}^*(K_{m(l')}, \epsilon_{m(l')}) \leq \mu$. Then we have estimates:

$$\begin{aligned}
&- ||u_{k(l)'} - u_{m(l')}||\bar{B}^*(K_{m(l')}, \epsilon_{m(l')}) + ||u_{m(l')}||\bar{B}^*(K_{m(l')}, \epsilon_{m(l')}) \\
&\qquad \leq ||u_{k(l)'}||\bar{B}^*(K_{m(l')}, \epsilon_{m(l')}) \\
&\leq ||u_{k(l)'} - u_{m(l')}||\bar{B}^*(K_{m(l')}, \epsilon_{m(l')}) + ||u_{m(l')}||\bar{B}^*(K_{m(l')}, \epsilon_{m(l')}).
\end{aligned}$$

Thus we get the estimate:

$$\begin{aligned}
|[u_l]|\hat{B}^*(\{K_l, \epsilon_l\}_l) &= \lim_l ||u_{m(l')}||\bar{B}^*(K_{m(l')}, \epsilon_{m(l')}) \\
&\leq \lim_l ||u_{k(l)'}||\bar{B}^*(K_{k(l)'}, \epsilon_{k(l)'}) = |j([u_l])|\hat{B}^*(\{K_{k(l)}, \epsilon_{k(l)}\}_l).
\end{aligned}$$

Since j is distance decreasing, this verifies the claim.

Let us take a compactly supported cut off function $\varphi_0 : \hat{M} \mapsto [0, 1]$ with $\varphi_0|K_0 \equiv 1$. One may assume $(\hat{M}\backslash K_0, g)$ is isometric to $(S^3 \times [0, \infty), g|S^3 + dt^2)$. Let us consider $d : \hat{B}^0(\{K_l, \epsilon_l\}_l) \mapsto \hat{B}^1(\{K_l, \epsilon_l\}_l)$. By the above sublemma, one has a uniform bound:

$$C||d(1 - \varphi_0)u_l||\bar{B}^1(K_l; \epsilon_l)_0 \geq ||(1 - \varphi_0)u_l||\bar{B}^0(K_l; \epsilon_l)_0.$$

Let us take a family $[u^\alpha_l] \in \bar{B}^0$ with $|[u^\alpha_l]|\bar{B}^0 = 1$, $|[du^\alpha_l]|\bar{B}^1(\{K_l, \epsilon_l\}_l) \to 0$ as $\alpha \to \infty$. We show that:

$$d : \bar{B}^0(\{K_l, \epsilon_l\}_l) \mapsto \bar{B}^1(\{K_l, \epsilon_l\}_l) \qquad *$$

has closed range asymptotically in the following sense; there exists a subindices $k(l,\alpha)$ and $a(\alpha)$ with $\{[\varphi_0 u^{a(\alpha)}_{k(l,\alpha)}]\}_\alpha$ convergent in $L^{4+2\epsilon_0}(K_0)_0$. Here is the inclusion $\dots \{k(l,m)\}_l \subset \dots \{k(l,1)\}_l \subset \{l\}_l$. Notice that $\{[du^{a(\alpha)}_{k(l,\alpha)}]\}_\alpha$ also converges in $\bar{B}^1$.

We have estimates:

$$\begin{aligned}
|[u^\alpha_l]|\bar{B}^0 &\le |[\varphi_0 u^\alpha_l]|\bar{B}^0 + |[(1-\varphi_0)u^\alpha_l]|\bar{B}^0 \\
&\le |[\varphi_0 u^\alpha_l]|\bar{B}^0 + C|[d(1-\varphi_0)u^\alpha_l]|\bar{B}^1 \\
&\le |[\varphi_0 u^\alpha_l]|\bar{B}^0 + C|[du^\alpha_l]|\bar{B}^1 + C|[d\varphi_0 \wedge u^\alpha_l]|\bar{B}^1 \\
&\le |[\varphi_0 u^\alpha_l]|\bar{B}^0 + C|[du^\alpha_l]|\bar{B}^1 + C|[u^\alpha_l]|\bar{B}^1(\{K_l \cap \text{ Supp } d\varphi_0\}_l) \\
&\le C|[\varphi_0' u^\alpha_l]|\bar{B}^0 + C|[du^\alpha_l]|\bar{B}^1
\end{aligned}$$

where Supp $\varphi_0 \subset$ Supp φ_0'. Let us put $\epsilon = \epsilon_0$. Then $\varphi_0 u^\alpha_l \in L^{4+2\epsilon}(K_0)_0$ for all α, l by the compact embedding $W^4_1(K_0) \hookrightarrow \hat{L}^4(K_0)$. Thus it is enough to see that $\{[\varphi_0 u^{a(\alpha)}_{k(l,\alpha)}]\}_\alpha \subset \bar{B}^0(K_0)_0 \subset L^{4+2\epsilon}(K_0)_0$ is a convergent sequence.

Let us put $v^\alpha_l = \varphi_0 u^\alpha_l$ and choose any decreasing sequence $\delta_i(>0) \to 0$. Since the sets $\{v^\alpha_l\}_l$ consists of Cauchy sequences in $W^4_1(K_0)_0$, they converge to $u^\alpha \in W^4_1(K_0)_0$ for each α. First choose a subsequence $a(i)$ with $|u^{a(i)} - u^{a(j)}|L^{4+2\epsilon}(K_0)_0 <$ $\max\{\delta_i, \delta_j\}$ for all i, j. Then we put $w^l_\alpha = v^{a(l)}_\alpha$, $w^l = u^{a(l)}$.

Next let us choose subindices $\alpha(1,l)$ with $|w^1_{\alpha(1,i)} - w^1|W^4_1(K_0) < \delta_1$ and $|du^{a(1)}_{\alpha(1,i)}|L^{4+\epsilon}_{\text{loc}}(K_0) < \delta_1$ for all i. Then choose subindices $\{\alpha(2,i)\}_i \subset \{\alpha(1,i)\}_i$ with $|w^2_{\alpha(2,i)} - w^2|W^4_1(K_0) < \delta_2$ and $|du^{a(2)}_{\alpha(2,i)}|L^{4+2\epsilon}(K_0) < \delta_2$. We inductively choose $\alpha(l,i)$ as above. Thus we get the family $\{[z^a_l = w^a_{\alpha(a,l)}]\}_l$, and the estimate:

$$\begin{aligned}
|z^i_l - z^j_l|L^{4+2\epsilon}(K_0)_0 &\le |z^i_l - w^i|L^{4+2\epsilon}(K_0)_0 \\
&+ |w^i - w^j|L^{4+2\epsilon}(K_0)_0 + |w^j - z^j_l|L^{4+2\epsilon}(K_0)_0 \le \delta_i + \delta_i + \delta_j.
\end{aligned}$$

We put $\alpha(i,l) = k(l,i)$. Then this shows that $\{[z^i_l = \varphi_0 u^{a(i)}_{k(l,i)}]\}_l$ is a convergent sequence. Notice the inclusion $\{k(l,i)\}_l \subset \{k(l,i-1)\}_l$.

Now let us consider $d : \hat{B}^0 \mapsto \hat{B}^1$. Suppose a family $\{[u^\alpha_l]\}_\alpha \subset \hat{B}^0$ satisfies $|[u^\alpha_l]|\hat{B}^0 = 1$ and $|[du^\alpha_l]|\hat{B}^1 \to 0$. Thus there are arbitrarily large $l = l(\alpha)$ satisfying $||du^\alpha_l||_l \bar{B}^1(K_l) \le \delta_\alpha$ and $||u^\alpha_l||_l \bar{B}^0(K_l) \ge 1 - \delta_\alpha$.

We put $f_l = \varphi_0 u^{a(l)}_{k(l,l)} = z^l_l$ and restrict $\{[z^\alpha_l = \varphi_0 u^\alpha_{k(l,l)}]_l\}_\alpha \subset \hat{B}^0(\{K_{k(l,l)}\}_l) \subset \hat{B}^0(\{K_l\}_l)$. Then we claim that the family converges to $[f_l] \in \hat{B}^0(\{K_{k(l,l)}\}_l)$.

Since we consider $\hat{B}^*$ norm, one may assume $\alpha, \beta \leq l, l'$. By choice, we have conditions:

$$|z_l^\alpha - z_{l'}^\alpha|\bar{B}^0 < 2\delta_\alpha, \quad |z_l^\alpha - z_l^\beta|\bar{B}^0 < 3\max(\delta_\alpha, \delta_\beta).$$

Thus we get the claim. This verifies closedness of d.

Let us consider d^+. Let us take $[u_l^\alpha] \in \hat{B}^1$ satisfying $||[u_l]||\hat{B}^1 = 1$ and $||[du_l^\alpha] - [w_l]||\hat{B}^2 \to 0$ as $\alpha \to \infty$. Notice that the Laplace operator $\Delta : W_2^{2+\epsilon}(S^3; \Lambda^1) \mapsto L^{2+\epsilon}(S^3; \Lambda^1)$ admits a uniform bound $|\Delta| \geq C$ where C is independent of ϵ. In particular $d^+ : (\hat{W}_1^2)_w(S^3 \times \mathbf{R}) \cap$ Ker$d_w^* \mapsto \hat{L}_w^2(S^3 \times \mathbf{R})$ admits a uniform bound $|d^+| \geq C$.

Let us fix a smooth Riemannian metric h on M such that there is an isometry $(\hat{M} \cap \text{ Supp}\varphi_0, g) \cong (\bar{M} \subset M, h)$ where $\hat{M} = \bar{M} \cup_{S^3} S^3 \times [0, \infty)$. Then with respect to h, one constructs another Banach spaces $\hat{B}^*(M, h, \{\epsilon_l\}_l)$.

Sublemma 3.3 *Its cohomology groups are isomorphic to the de Rham's one.*

In fact one can check Poincaré lemma.

Proof of lemma: Thus there is $Q_1 : \text{ im } d^+ \subset \hat{B}^2(M, h) \mapsto \text{ Ker}d^* \subset \hat{B}^1(M, h)$ with $d^+ \circ Q_1 = \text{id}$. Let $(S^3 \times \mathbf{R}, g = g|S^3 + dt^2)$ be the cylinder. Then there is also $Q_2 : \hat{B}^2(S^3 \times \mathbf{R}) \mapsto \hat{B}^1(S^3 \times \mathbf{R})$ with $d^+ \circ Q_2 = \text{id}$.

Let $I : H(0) \subset \hat{B}^2(\hat{M})$ be as before. Then using a cut off function ψ, one may represent any element $u \in H^2$ by a compactly supported form, $u' = u - d^+(\psi Q_2(\psi u)) \in H(0)'$. These forms consists of $H^2(\hat{B}^*(M, h), d_*)$. Let us put $\psi = \varphi_0^{\frac{1}{2}}$, $\psi' = (1 - \varphi_0)^{\frac{1}{2}}$ and:

$$Q' = \psi Q_1 \psi + \psi' Q_2 \psi' : \text{ im}d^+ \subset \hat{B}^2(\hat{M}) \mapsto \hat{B}^1(\hat{M}).$$

We follow the argument in [DS] $p212$.

In corollary 3.1, we have seen that for any $u = [u_l] \in$ image Q_2, each u_l lies in Kerd_w^*. In particular $u_l \in (W_1^2)_{\text{loc}}$. Thus from compact embedding $(W_1^2)_{\text{loc}} \hookrightarrow L_{\text{loc}}^{2+\epsilon}$, it follows $d^+ \circ Q' - \text{ id} : \text{ im}d^+ \subset \hat{B}^2(\hat{M}) \mapsto \text{ im } d^+ \subset \hat{B}^2(\hat{M})$ is compact.

Let $w_1, \dots, w_l$ and $d^+(\alpha_1), \dots, d^+(\alpha_l)$ be the kernel and cokernel sets for d^+Q'. Then we put:

$$Q = (Q' + \Sigma_{i=1}^l (w_i, \quad)\alpha_i)T^{-1} : \text{ im}d^+ \subset \hat{B}^2(\hat{M}) \mapsto \hat{B}^1(\hat{M})$$

where $T = d^+ \circ Q' + \Sigma_{i=1}^l (w_i, \quad)d^+(\alpha_i)$.

Q satisfies $d^+ \circ Q = \text{id}$. This implies the existence of a bounded map:

$$Q : \text{ im}d^+ \subset \hat{B}^2(\hat{M}) \mapsto \hat{B}^1(\hat{M})$$

with $d^+ \circ Q = \text{id}$. Thus d^+ has closed range. This completes the proof.

Proposition 3.3 *The AHS complex between $\hat{B}^i(\hat{M})$ has its cohomology groups as:*

$$H^0(\hat{B}^*(\{K_l,\epsilon_l\}_l)=0, \quad H^1(\hat{B}^*(\{K_l,\epsilon_l\}_l)=0, \quad H^2(\hat{B}^*(\{K_l,\epsilon_l\}_l)\cong H=H(0).$$

In particular $dim H^2 = b^2_+(M)$.

Proof : $H^0=0$ follows from combination of the proof of proposition 1.1 and $3.B$.

Next we claim that there exists $Q=Q(\hat{M}) : \hat{L}^2(\hat{M},\epsilon')\cap$ im$d^+ \mapsto \bar{B}^1(\hat{M},\epsilon')$ with $d^+\circ Q=$ id and $|Q|\leq C$ where C is independent of $0\leq\epsilon'\leq\epsilon$. Assuming this for the moment, we proceed as follows. For H^2, it also follows from $3.B$ and existence of Q above.

Let us consider H^1. For simplicity we consider only the case $n=1$. Then we take $[\alpha_l]\in H^1$. Thus $\lim||d^+(\alpha_l)||_l=0$. Then since $\varphi_l\alpha_l - Q(\hat{M})(d^+(\varphi_l\alpha_l))\in$ Ker$d^+\cap\bar{B}^1(\hat{M},\epsilon_l)$, there is $f_l\in\bar{B}^0(\hat{M},\epsilon_l)$ with $\varphi_l\alpha - df_l = Q(\hat{M})(d^+(\varphi_l\alpha_l))$. Here $|f_l|\bar{B}^0(\hat{M},\epsilon_l)$ are uniformly bounded and $|Q(\hat{M})(d^+(\varphi_l\alpha_l))|\bar{B}^1(\hat{M},\epsilon_l)\to 0$. Similarly we have $g_l\in\bar{B}^0(S^3\times\mathbf{R},\epsilon_l)$ with $T^*_{2l}((1-\varphi_l)\alpha_l)-dg_l = Q(S^3\times\mathbf{R})(T^*_{2l}(d^+(1-\varphi_l)\alpha_l))$.

Let us choose subindices $k(l)>>l$ (with respect to $\{\alpha_l\}_l$) and put $\psi_l=\varphi_{k(l)}$. Then we put $\beta_l=\psi_lf_l+(1-\psi_l)((T^{-1}_{2k(l)})^*g_l)$. Then we have $I([\alpha_l])-[d\beta_l]\in\bar{B}^1_0(\{K_{2k(l)},\epsilon_l\}_l)$. Since $I:\hat{B}^*(\hat{M})\mapsto\hat{B}^*(\hat{M})$ by $[\alpha_l]\in\hat{B}^*(\{K_l,\epsilon_l\}_l)\to[\varphi_l\alpha_l+(T^{-1}_{2k(l)})^*((1-\varphi_l)\alpha_l)]\in\hat{B}^*(\{K_{2k(l)},\epsilon_l\}_l)$ gives an isomorphism, this shows $[\alpha_l]=0\in H^1$.

Now we verify the claim. Recall $\nabla_i=(d^*)_w\oplus d^+ : (\hat{W}^2_1)_w(\hat{M},\epsilon_i)\mapsto\hat{L}^2_w(\hat{M},\epsilon_i)$ satisfies $|\nabla_i|\geq C_i$. It is enough to see uniform bound $C_i\geq C$. Suppose contrary. Let us take $u_i\in(\hat{W}^2_1)_w(\hat{M},\epsilon_i)$ with $|\nabla_i(u_i)|\hat{L}^2_w(\hat{M},\epsilon_i)\to 0$. Notice uniform bound:

$$|\nabla_i((1-\varphi_0)(u_i))|\hat{L}^2_w(\hat{M},\epsilon_i)\geq C|(1-\varphi_0)(u_i)|(\hat{W}^2_1)_w(\hat{M},\epsilon_i). \quad *$$

Then as in the proof of proposition 1.1, we have a familliar estimate:

$$|u_i|(\hat{W}^2_1)_w(\hat{M},\epsilon_i)\leq C\{|\varphi_0u_i|L^{2+\epsilon}(K_0)_0+|\nabla_i(u_i)|\hat{L}^2_w(\hat{M},\epsilon_i)\}.$$

Then using the compact embedding $(W^{2+\epsilon'}_1)_{\rm loc}\hookrightarrow L^{2+\epsilon}_{\rm loc}$, $\epsilon'\leq\epsilon$, we find some $u_\infty\in$ Ker$\nabla\cap(W^2_1)_w$. This must be zero. This shows $|u_i|L^{2+\epsilon}($ Supp$\varphi_0)\to 0$. Combining with the above $*$, one finds $|u_i|\hat{L}^2_w(\hat{M},\epsilon_i)\to 0$. Since $\nabla_i(u_i)\to 0$, it follows from the elliptic estimate, $|u_i|(\hat{W}^2_1)_w(\hat{M},\epsilon_i)\to 0$. This completes the proof.

4 Asymptotic conformality

4.A Quasiconformal mappings: Let g and g' be two Riemannian metrics on the same manifold $\hat{M}$. Then the conformal distance is defined as:

$$d([g_1],[g_2])(x) = \mathrm{Sup}_{\zeta,\psi} \log \{\frac{|\zeta|_2}{|\psi|_2} : \quad |\zeta|_1 = |\psi|_1 = 1\}.$$

Let $D, D' \subset \mathbf{R}^4$ be compact domains with Riemannian metrics g and g' respectively. Then for a quasiconformal homeomorphism $f : (D,g) \cong (D',g')$, it has the following properties:

(1) f is almost everywhere differentiable with $\nabla f \in L^4_{\mathrm{loc}}(D)$. The estimate holds:

$$\int_D |\nabla f|^4 \leq 16K^3 vol(D'), \quad K = \text{ess sup} H(f).$$

(2) ([G]) The derivative ∇f is in fact in $L^{4+\delta}_{\mathrm{loc}}$, where $\delta > 0$ is determined only by the quasiconformal constant, and ∇f is in the sense of $\int_D h\frac{\partial f}{\partial x_i} = -\int_D f\frac{\partial h}{\partial x_i}$.

(3) $d([f^*(g')],[g])$ is a bounded measurable function and for the quasiconformal constants $H(f)$ in the introduction, the following pointwise estimate holds:

$$H(f)(x) = exp(d([f^*(g')],[g])_x).$$

(4) Let $\Lambda^2(g) = \Lambda^+(g) \oplus \Lambda^-(g)$ be the decomposition with respect to g. The projection to the antiself dual parts $pr_-(g) : \Lambda^2 \mapsto \Lambda^2_-(g)$ depends only on the conformal class of g. $\Lambda^-(f^*(g'))$ is determined by a bounded measurable bundle map $\mu = \mu(f^*(g'),g) : \Lambda^-(g) \mapsto \Lambda^+(g)$ so that $\Lambda^-(f^*(g'))$ can be expressed by the graph $\{(x,-\mu(x)) : x \in \Lambda^-\} = \Lambda^-(f^*(g'))$. The following pointwise estimate holds ([DS] $p187$):

$$\frac{1+|\mu|}{1-|\mu|}(x) \leq H(f)(x)^2 \leq (\frac{1+|\mu|}{1-|\mu|}(x))^2.$$

(5) Let us put $\tau = \tau(g, I^*(g')) = pr_-(g) - pr_-(f^*(g'))$. One may choose some canonical basis, $\{\alpha^i_\pm\}_i$ for $\Lambda^2_\pm(g)$ (see [DS] $p186$). Then for $\alpha_\pm \in \{\alpha^i_\pm\}_i \subset \Lambda^\pm(g)$, one has the equalities $pr_-(\alpha_-) = p(\alpha_-)^{-1}\{\alpha_- + \mu(\alpha_-)\}$, $p(a) = 1 - \frac{|\mu(a)|^2}{|a|^2}$, and $-|x|^2 + |\mu(x)|^2 - \mu(x)\alpha^+ = 0$, $pr_-(\alpha_+) = (x,\mu(x))$ where we use g norm. ¿From these inequalies, one gets the pointwise norm estimate:

$$\begin{aligned} |\tau(g,f^*(g'))| &= |pr_-(g) - pr_-(f^*(g'))| \\ &\leq (1-|\mu|^2)^{-\frac{1}{2}}|\mu| \leq H(f)^2 - 1. \end{aligned}$$

Lemma 4.1 (G, DS) *Let $f : (D,g) \mapsto (D',h)$ be a quasi conformal homeomorphism and $p = 1,2$.*

(1) Let α be a form in $L^{\frac{4}{p}+\epsilon}(\Lambda^p(D'))$. Then $f^*(\alpha) \in L^{\frac{4}{p}+\epsilon'}(\Lambda^p(D))$, where $0 \leq \epsilon' = \epsilon(p,\delta,\epsilon) < \delta\epsilon$. Moreover there is $C = C(\epsilon, D, f, g, h)$ with $C \to [4$ (ess sup$H(f))^{\frac{3}{2}}]^{\frac{p}{2}}$ as $\epsilon \to 0$ satisfying:

$$|f^*(\alpha)|L^{\frac{p}{4}+\epsilon'}(D,g) \leq C|\alpha|L^{\frac{p}{4}+\epsilon}(D,h).$$

(2) Let α be a function in $W_1^4(D',h)$. Then

$$\int |f^*(\alpha)|^{4+\epsilon'} vol_g \leq C|\alpha|^{4+\epsilon'} W_1^4(D',h)|\nabla f^{-1}|^4 L^{4+\delta}(D',h).$$

Proof : For (1), $p = 1,2$, one uses conformal invariance of $L^{\frac{4}{p}}$ norms on p forms and $\nabla f \in L^{4+\delta}$. For (2), $p = 0$, one sees it as follows; recall the Sobolev embedding, $L^N \hookrightarrow W_1^4$ for all N (see [GT] $p167$). Then by the Gehring's theorem (2) above, one gets the estimate for a function $\alpha \in W_1^4$:

$$\begin{aligned}
&\int |f^*(\alpha)|^{4+\epsilon'} vol_g \leq \int |f^*(\alpha)|^{4+\epsilon'} f^*|\nabla f^{-1}|^4 |\nabla f|^4 vol_g \\
&\leq C\int |f^*(\alpha)|^{4+\epsilon'} f^*|\nabla f^{-1}|^4 f^*(vol_h) \leq C|\alpha|^{4+\epsilon'} L^N(D',h)|\nabla f^{-1}|^4 L^{4+\delta}(D',h) \\
&\leq C|\alpha|^{4+\epsilon'} W_1^4(D',h)|\nabla f^{-1}|^4 L^{4+\delta}(D',h). \qquad [(\frac{N}{4+\epsilon'})^{-1} + (\frac{4+\delta}{4})^{-1} = 1)]
\end{aligned}$$

Thus one gets (2) above as desired.

Remark 4.1: Let $\varphi : K \cong K'$ be a quasiconformal homeomorphism and $D : K' \cong K''$ be a diffeomorphism. Let C and C' be the quasiconformal constants of φ and $D \circ \varphi$ respectively. If $\nabla\varphi$ lies in $L^{4+\epsilon}_{\text{loc}}$, then $\nabla(D \circ \varphi) \in L^{4+\epsilon}_{\text{loc}}$, even C' is larger than C.

Lemma 4.2 *(see [GT] p 147) Let $f : D \cong D' \subset \mathbf{R}^4$ be a quasiconformal homeomorphism where D is compact. Let us take a smooth Riemannian metric g on D'. Then $f^*(g)$ can be approximated by a family of smooth Riemannian metrics g_i in the sense:*

(1) $g_i \to f^*(g)$ in $L^2(D)$,

(2) quasiconformal constants $K(g_i, f^*(g))$ are uniformly bounded, and

(3) $H(g_i, f^*(g))^2 - 1, \mu(f^*(g), g_i), |pr_-(f^*(g)) - pr_-(g_i)| \to 0$ in all L^n.

Proof : We only consider (3). Let us consider $H(g_i, f^*(g))^2 - 1$. Its pointwise norms are uniformly bounded a.e.. Let us take local orthonomal basis $e_1, \dots, e_4$ with respect to g_i. Then one may write $f^*(g) = \Sigma g^i_{a,b} e^*_a \otimes e^*_b$, $g^i_{a,b} - \delta_{a,b} \to 0$ in L^2. Let us choose sections ζ and ξ with pointwise norms $|\zeta|_{g_i} = |\xi|_{g_i} = 1$. Take small $\delta >> \epsilon > 0$. Then if we choose a sufficiently large i, then the measurable set $D(\delta) \subset D$ satisfying $g^i_{a,b} - \delta_{a,b} > \delta$ on $D(\delta)$ has its measure less than ϵ. Thus we have the estimate $(1-\delta)|\zeta|^2_{g_i} \leq |\zeta|^2 \leq (1+\delta)|\zeta|^2_{g_i}$ on $D \backslash D(\delta)$. Combining with a uniform bound ess sup$\frac{|\zeta|}{|\xi|} \leq K$, we get the estimate:

$$|\frac{|\zeta|^2}{|\xi|^2} - 1|^2 L^2(D) \leq |\frac{|\zeta|^2}{|\xi|^2} - 1|^2 L^2(D \backslash D(\delta)) + (K^2-1)^2 \epsilon$$
$$\leq (\frac{2\delta}{1-\delta}) vol(D) + (K^2-1)^2 \epsilon.$$

This shows $H(g_i, f^*(g))^2 - 1 \to 0$ in L^2. For μ and τ, we have similar inequality using the inequality in (5) above.

Next let us verify L^n convergence. By (2), ess sup $K(g_i, f^*(g))$ are uniformly bounded. ¿From this and above, (3) will follow by the interpolation argument. Recall the intermediate inequality (see [GT] $p146$):

$$|u|L^q \leq \epsilon |u| L^r + \epsilon^{-\frac{1-\lambda}{\lambda}} |u| L^p$$

where $\epsilon > 0$, $\lambda < 1$, $\frac{1}{q} = \frac{\lambda}{p} + \frac{(1-\lambda)}{r}$, $p \leq q \leq r$. We show that if $|u_i|L^\infty \leq C$ and $|u_i|L^2 \to 0$, then $|u_i|L^n \to 0$ for all n. Take any small $\epsilon >> \delta > 0$. Then we choose a sufficiently large i so that $|u_i|L^2 < \delta$. We apply the above estimate as $1 < p \leq 2$, $q = n$, $\lambda = \frac{1}{n}$. With respect to u_i, we choose a sufficiently large r with $|u_i|L^r \leq 2|u_i|L^\infty \leq 2C$. Then these data (q, r, λ) determines $1 < p \leq 2$. Now we have an estimate:

$$|u_i|L^n \leq \epsilon |u_i| L^r + \epsilon^{-\frac{1-\lambda}{\lambda}} |u|_i L^p$$
$$\leq 2C\epsilon + C\epsilon^{-\frac{1-\lambda}{\lambda}} |u_i| L^2 \leq 2C\epsilon + C\epsilon^{-\frac{1-\lambda}{\lambda}} \delta.$$

This implies (3). This completes the proof.

Remark 4.2: We donot know whether the above approximation in (3) could be done in L^∞. However picture of collapsing smooth structure may expect $\mu(f^*(g), g_i)|L^N$ approaches to zero non uniformly with respect to i and N.

4.B Asymptotic morphisms: Here we will introduce asymptotic equivalence relations between two asymptotically quasiconformally homeomorphisms. This generalizes the notion, asymptotic morphism introduced in the introduction.

Let M and N be smooth closed manifolds, and $(\hat{M}, g)$ and $(\hat{N}, g')$ be the corresponding clylindrical ones. Suppose $\check{M}$ and $\check{N}$ are equipped two ACH's to $(\hat{M}, g)$ and $(\hat{N}, g')$ with data respectively:

$$K_0(') \subset \cdots \subset \check{M}(\check{N}), \quad I_l(') : K_l(') \hookrightarrow \hat{M}(\hat{N}).$$

We denote them as $ACH_1 : \check{M} \to (\hat{M}, g)$ and $ACH_2 : \check{N} \to (\hat{N}, g')$ respectively. We identify end$\hat{M}$ and end$\hat{N}$ with $S^3 \times [0, \infty)$.

An *asymptotic morphism* from $\check{M} \to (\hat{M}, g)$ to $\check{N} \to (\hat{N}, g')$ consists of families of quasi conformal homeomorphisms:

$$E_l : K_l \cong K'_l, \quad F_l : I_l(K_l) \cong I'_l(K'_l)$$

satisfying:

(1) $|H(F_l)^2 - 1|L^N_{\text{loc}}, |H(E_l)^2 - 1|L^N_{\text{loc}} \to 0 \quad l \to \infty$,
(2) $\text{Sup}_{x,l} d(F_l(x), x) \leq C$, $I_l(')(x) \in I_l(K_l)$,
(3) $d(F_{l'}(x), F_l(x)), d(E_{l'}(m), E_l(m)) \to 0$ as $l, l' \to \infty$ and
(4) $|\nabla F_l - \nabla F_{l'}|L^{4+\delta}_{\text{loc}}(I_l(K_l)), |\nabla E_l - \nabla E_{l'}|L^{4+\delta}_{\text{loc}}(K_l) \to 0$ for $l' \geq l$.

Let $K_0 \subset K_1 \subset \cdots \subset \check{M}$ be an exaustion as before. Suppose $\check{M}$ is equipped with two asymptotic quasiconformal homeomorphisms to $(\hat{M}, g)$, ACH_1 and to $(\hat{N}, g')$, ACH_2 with data:

$$K_0 \subset \cdots \subset \check{M}, \quad I^1_l : K_l \hookrightarrow \hat{M}, \quad I^2_l : K_l \hookrightarrow \hat{N}.$$

Then we say that these are *self asymptotic equivalent*, if $\{E_l = \text{id}\}$ and $\{F_l = I^2_l \circ (I^1_l)^{-1}\}$ consisit of an asymptotic morphism.

4.C Regularization: Let $F_i : (D, g) \cong (D', g')$ be a family of quasiconformal mapping with $|H(F_i)^2 - 1|L^N_{\text{loc}} \to 0$ for a sufficiently large $N >> 0$. Recall $\tau_i = \tau(g, F_i^*(g')) = pr_-(g) - pr_-(F_i^*(g'))$.

Let α be a smooth 1 form on D' with $d^+(\alpha) = 0$ and $d(\alpha) = w \neq 0$ for all points ([DS] $p196$). Let us take a constant $\delta > 0$.

Lemma 4.3 *Suppose $|F_i^*(w)|L^{2+\delta}_{loc}$ are uniformly bounded. If F_i and ∇F_i converge in C^0_{loc} and L^4_{loc} respectively, then $\nabla F_\infty \equiv lim\ \nabla F_i$ in $L^{4+\delta'}_{loc}$ for some $0 < \delta' < \delta$.*

Proof : By the condition, there are measurable sets $D_i \subset D$ satisfying $m(D_i) \to 0$ and (2) ess $\text{sup}\tau_i|D\backslash D_i < c$. Then there is C with a pointwise bound $|\nabla F_i|^{4+\delta'} \leq C|F_i^*(w)|^{2+\delta}$. Thus $\nabla F_i|D\backslash D_i \in L^{4+\delta'}_{\text{loc}}$. Then we show limit $\nabla F_i \equiv \nabla F_\infty \in L^{4+\delta'}_{\text{loc}}$, $\delta' = \delta'(c)$. Let us choose small $\epsilon > 0$. Then one takes a subsequence $k(l)$ with $m(D(\epsilon)) < \epsilon$, $D(\epsilon) = \cup_l D_{k(l)}$ and $\nabla F_{k(l)}|D\backslash D(\epsilon) \in L^{4+\delta'}_{\text{loc}}$. Recall the inequality $|u|L^q \leq \epsilon|u|L^r + \epsilon^{-\frac{1-\lambda}{\lambda}}|u|L^p$ where $q^{-1} = \lambda/p + (1-\lambda)/r$, $p \leq q \leq r$. Since $|\nabla F_{k(l)}|L^{4+\delta'}_{\text{loc}}(D\backslash D(\epsilon))$ are uniformly bounded, one may apply this inequality to $u = \nabla F_{k(l)} - \nabla F_{k(l')}$. Then one sees that for all $\epsilon > 0$ and some $0 < \delta'' < \delta'$, $|\nabla F_\infty|L^{4+\delta''}_{\text{loc}}(D\backslash D(\epsilon))$ are uniformly bounded. We change the notation δ'' by δ'. Then by letting $\epsilon \to 0$, one knows $\nabla F_\infty \in L^{4+\delta'}_{\text{loc}}$. By changing indices, one may assume $|\nabla F_\infty - \nabla F_i|L^{4+\delta'}_{\text{loc}}(D\backslash D_i) \to 0$. This completes the proof.

Now we consider the converse direction.

Lemma 4.4 *Suppose all F_i above converges in $L^{4+\delta}_{loc}$. Then there are constants C, $0 \leq \epsilon' < \epsilon$ such that for any $u \in \bar{B}^*(D', \epsilon)_{loc}$, there is a bound:*

$$|F^*_\infty(u)|\bar{B}^*(D, \epsilon')_{loc} \leq C|u|\bar{B}^*(D', \epsilon)_{loc}.$$

Proof : As above there are measurable sets $D_i \subset D$ satisfying (1) $m(D_i) \to 0$ and (2) ess sup$\tau_i|D \backslash D_i < c$. Then there is C with a pointwise bound $|\nabla F_i|^4 \leq C$ det (∇F_i) on $D \backslash D_i$. Then as [DS] $p198$, one get the estimate:

$$|F^*_i(u)|\bar{B}^*(D \backslash D_i, \epsilon') \leq C|u|\bar{B}^*(D' \backslash D'_i, \epsilon).$$

Now let us choose a subindices $k(l)$ satisfying $\lim_l m(\cup_{l' \geq l} D_{k(l')}) = 0$. We put $E_l = \cup_{l' \geq l} D_{k(l')} \subset D$. Then for all triple $l'' \geq l' \geq l$, there is a uniform estimate:

$$|(F_{k(l'')} - F_{k(l')})^*(u)|\bar{B}^*(D \backslash E_l, \epsilon') \leq C(|\nabla(F_{k(l'')} - F_{k(l')})|L^{4+\delta}(D))|u|\bar{B}^*(D' \backslash E'_l, \epsilon).$$

This shows $|F^*_\infty(u)|\bar{B}^*(D \backslash E_l, \epsilon') \leq C|u|\bar{B}^*(D' \backslash E'_l, \epsilon)$. Since $m(E_l) \to 0$, this gives the result.

4.D Induced morphisms: Let us take two closed smooth four manifolds M and M'. Suppose $(\hat{M}, g)$ and $(\hat{M}', g')$ admit a uniformly bounded asymptotic morphism, consisting of families of quasiconformal homeomorphisms $\{F_l : (K_l, g) \cong (K'_l, g')\}_l$. Recall $\tau_l = \tau(F^*_l(g'), g) = pr_-(g) - pr_-(F^*_l(g')) \to 0$ in L^n for a sufficiently large n.

Let $\hat{B}^*(\hat{M}(')) = \hat{B}^*(\{K_l(')\}_l, \epsilon('))$ and $(\hat{B}^i(\hat{M}(')), d_i)$ be the AHS complex:

$$0 \to \hat{B}^0(\hat{M}(')) \xrightarrow{d} \hat{B}^1(\hat{M}(')) \xrightarrow{d^+} \hat{B}^2(\hat{M}(')) \to 0. \tag{4.1}$$

Proposition 4.1 *Choosing sufficiently small constants $\epsilon' < \epsilon$ and $\delta' < \delta$, there is an induced morphism:*

$$\bar{F}^* : (\hat{B}^i(\hat{M}, w, \epsilon), d_i) \mapsto (\hat{B}^i(\hat{M}', w', \epsilon'), d_i).$$

Sublemma 4.1 *Let $F : (K, g) \cong (K, h)$ be a quasiconformal homeomorphism between two Riemannian spaces with $\tau(F^*(h), g)|L^n < \alpha$ for some large n. Then there is $c = c(\alpha, F) > 0$ and $\epsilon' < \epsilon$ with:*

$$|[F^*, d^+]u|L^{2+\epsilon'}(K, g) \leq c|du|L^{2+\epsilon}(K, h), \quad c \to 0 \text{ as } \alpha \to 0.$$

Proof: Let $d|\Lambda^2 = d^+ \oplus d^-$ be the decomposition with respect to g, and express $d^+(F^*(h)) = d^+ + \tau d$. Then one has the equality:

$$\begin{aligned}[F^*, d^+] &= F^* \circ d^+(h) - d^+(g) \circ F^* = F^* \circ (1 + *(h))d - d^+(g) \circ F^* \\ &= (1 + F^* * (h)(F^{-1})^*)d \circ F^* - d^+(g) \circ F^* \\ &= (d^+(F^*(h)) - d^+(g)) \circ F^* = \tau d \circ F^*.\end{aligned}$$

Then the following estimates completes the proof:

$$|[F^*, d^+]u|L^{2+\epsilon'}(K_l, g) = |\tau dF^* u|L^{2+\epsilon'}$$
$$\leq |\tau|L^n|dF^*u|L^{2+\epsilon''} \leq C(|\nabla F|L^{4+\delta})|\tau|L^n|du|L^{2+\epsilon}(K_l, h).$$

Proof of proposition: Here we claim the following ; let f and g be functions over $\hat{M}$ with a uniform bound $|f|L^{2+\epsilon'}_{\text{loc}} \leq C|g|L^{2+\epsilon}_{\text{loc}}$, $\epsilon' < \epsilon$. Then for some choice of $\delta' < \delta$, one gets the global bound:

$$|f|L^{2+\epsilon'}_{w'} \leq C|g|L^{2+\epsilon}_w.$$

Proof of claim: We have estimates:

$$\int |f|^{2+\epsilon'} exp(w') \leq \Sigma_n |g_i|^{2+\epsilon'} L^{2+\epsilon}_{\text{loc}} exp(\delta' n)$$
$$= \Sigma_n (|g_i|L^{2+\epsilon}_{\text{loc}} exp(\frac{\delta'' n}{2+\epsilon'}))^{2+\epsilon'} (exp(\delta' - \delta'')n)$$
$$\leq (\Sigma_n |g_i|^{2+\epsilon} L^{2+\epsilon}_{\text{loc}} exp(\delta n))^{p^{-1}} (\Sigma_n exp(qn(\delta' - \delta'')))^{q^{-1}} \leq C|g|^{2+\epsilon'} L^{2+\epsilon}_w$$

where $p = \frac{2+\epsilon}{2+\epsilon'}$, $p^{-1} + q^{-1} = 1$ and $\delta'' = p^{-1}\delta > \delta'$. This verifies the claim.

Proof of proposition: Let us take $[u_l] \in \hat{B}^*(\hat{M})$. Then by the estimate in lemma 4.1, lemma 4.4 and the above claim, one gets:

$$||F_l^*(u_l)||\bar{B}^*(K_l', g', w', \epsilon') \leq C||u_l||\bar{B}^*(K_l, g, w, \epsilon).$$

(More precisely one should remove some small measurable sets, or use F^*_∞. However as far as considering asymptotic sequences, these do not matter and we will denote as above.)

Let us check $[F_l^*(u_l)]$ defines also an element in $\hat{B}^*(\hat{M}')$. Recall $|\nabla(F_l - F_{l'})|L^{4+\delta} \to 0$. Then we have the inequalities:

$$||F_l{}^*(u_l) - F_{l'}^*(u_{l'})||\bar{B}^*(K_l', g', \epsilon') = ||F_{l'}^*[u_l - u_{l'}]|| + ||(F_l - F_{l'})^*(u_l)||$$
$$\leq C(F_l, \epsilon)||u_l - u_{l'}||\bar{B}^*(K_l, g, \epsilon) + \delta(l, l')||u_l|| \to 0.$$

Thus $F^* : \hat{B}^*(\hat{M}, \epsilon) \mapsto \hat{B}^*(\hat{M}', \epsilon')$ gives a bounded map.

Next let us consider the differentials. Notice that $d : \hat{B}^0(\hat{M}) \mapsto \hat{B}^1(\hat{M})$ commutes with F^*.

Next let us consider d^+ and the family $F_l^*(d^+(u_l)) \in \hat{L}^2_{w'}(K_l')_0$. By lemma 4.4 and and the above claim, the family is uniformly bounded. Then we show $||d^+ F_{l'}^* u_{l'} - d^+ F_l^* u_l||\bar{B}^2(K_{l'}, g_{l'}, w', \epsilon') \to 0$. In fact one has the estimates:

$$||d^+ F_{l'}^* u_{l'} - d^+ F_l^* u_l||\bar{B}^2(K_{l'}, g_{l'}, w', \epsilon')$$
$$\leq ||[F_{l'}^*, d^+]u_{l'}|| + ||[F_l^*, d^+]u_l|| + ||F_{l'}^* d^+(u_{l'}) - F_l^* d^+(u_l)||$$
$$\leq \delta(F_l, F_{l'}, \epsilon)\{||du_{l'}||\hat{L}^2_w + ||du_l||\hat{L}^2_w\} + C||du_l - du_{l'}|| + C\delta(l, l')||du_l||.$$

By sublemma 4.1 and $|\tau_l|L^n \to 0$, one may assume that the last term converges to zero. Thus F^* gives a morphism from $\{\hat{B}^*(\hat{M},\epsilon), d^*\}$ to $\{\hat{B}^*(\hat{M}',\epsilon'), d^*\}$. This completes the proof.

Corollary 4.1 *Let $\hat{M}$ and $\hat{M}'$ be asymptotically q.c. equivalent. Let us take $\epsilon > \epsilon' > \epsilon'' > 0$. Then we have the induced maps:*

$$F^* : \hat{B}^*(\hat{M}, w, \epsilon) \mapsto \hat{B}^*(\hat{M}', w', \epsilon'), \quad (F^{-1})^* : \hat{B}^*(\hat{M}', w', \epsilon') \mapsto \hat{B}^*(\hat{M}, w'', \epsilon'')$$

such that the composition $(F^{-1})^* \circ F^*$ gives the identity.

5 Connections and bundles

Let a cylindrical manifold $(\hat{M}, g)$ be as before. For all the later sections, we will use the underlying Banach spaces $\hat{B}^*(\hat{M},\epsilon)$ defined in section 2.A (3.C), where all the constants ϵ_l are the same $\epsilon > 0$.

Let $E \to \hat{M}$ be a G bundle ($G = SU(2)$ or $SO(3)$), and fix a trivialization of E on the end. This determines c_2 or p_1 of E. Let $T \subset \mathbf{R}_+$ be an interval. Then for $g \in G$, we put $T(g) \in Aut(S^3 \times T \times G)$ as the constant gauge transformation. We regard $S^3 \times T \subset S^3 \times \mathbf{R}_+ = \mathrm{end}\hat{M}$.

Let us fix a smooth connection A_0 which is trivial on the end. Then one defines a formal affine Banach space and a Banach Lie group:

$$\begin{aligned} As\mathfrak{A} &= \{A_0 + a : a \in \hat{B}^1\}, \\ As\mathfrak{G} &= \{u = [u_l] : [\nabla_{A_0} u_l] \in \hat{B}^1, u_l \in \bar{B}^0_{\mathrm{loc}}(Aut(E)), *\} \end{aligned}$$

where $*$ is as follows; recall we have families of indices $b(l,j)$, $j = 0, 1, \dots, n$. Let us take a constant $c = c(u) > 0$ and put a family of intervals

$$\begin{aligned} M(l,0) &= S^3 \times [c, b(l,0) - c], \\ M(l,j) &= S^3 \times [b(l,j) + c, b(l,j+1) - c] \subset \ \mathrm{end}\hat{M} \end{aligned}$$

where $j \geq 1$ and $b(l, n+1) = \infty$. Then for some $g = \{g_0(u), \dots, g_n(u)\} \subset G$ and for a locally constant gauge transformation $T(g)_l = \Sigma_{j=0}^n M(l,j) \times g_j$, one has $[u_l - T(g)_l] \in \hat{B}^0$.

Naturally one has stratifications:

$$As\mathfrak{A}_0 \subset As\mathfrak{A}_1 \subset \dots, \quad As\mathfrak{G}_0 \subset As\mathfrak{G}_1 \subset \dots$$

where $As\mathfrak{A}_0 = \{A_0 + a : a \in \bar{B}^1(\hat{M})\}$, $As\mathfrak{G}_0 = \{u \in As\mathfrak{G} : \nabla_{A_0} u \in \bar{B}^0(\hat{M})\}$. It is straightforward to check the following:

Lemma 5.1 *(1) $As\mathfrak{G}$ is a Banach Lie group. Its Lie algebra $As\mathfrak{g} = Lie\ As\mathfrak{G} = \{\sigma \in \hat{B}^0_{loc} : \nabla_{A_0}\sigma \in \hat{B}^1, *\}$ with norm $||\nabla_{A_0}\sigma||\hat{B}^1 + \Sigma_{j=0}^n |h_j(\sigma)|$, $h_j(\sigma) \in \mathfrak{g}$.*

(2) $As\mathfrak{G}$ acts on $As\mathfrak{A}$ by:

$$[u_l](A_0 + [a_l]) = [u_l^*(A_0 + a_l)].$$

Proof : (1) Recall the local Sobolev embedding $C^0 \hookrightarrow W_1^{4+2\epsilon}$. ¿From this, we get the multiplication map:

$$\hat{B}^0(\hat{M};\epsilon) \times \hat{B}^0(\hat{M};\epsilon) \mapsto \hat{B}^0(\hat{M};\epsilon).$$

By the Sobolev embedding, any element $g \in \hat{B}^0(\hat{M};\epsilon)$ consists of a family of continuous sections $g_l : K_l \subset \hat{M} \mapsto \mathrm{Aut}(E)$. Thus one can define exponential map, and passing through it, one gets a manifold structure on $As\mathfrak{G}$.

(2) follows from the Sobolev multiplication properties:

$$\hat{B}^0 \times \hat{L}^p_w \mapsto \hat{L}^p_w, \quad \hat{L}^{4+2\epsilon}_{2w} \times \hat{L}^{4+2\epsilon}_{2w} \mapsto \hat{L}^{2+\epsilon}_w$$

where $p = 4 + 2\epsilon$ or $2 + \epsilon$. Then we check the condition:

$$\begin{aligned}
&||u_l^*(A_0 + a_l) - u_{l'}^*(A_0 + a_{l'})||\bar{B}^1 \\
&\quad \leq C[(1 + ||u_l||\bar{B}^0 + ||u_{l'}||\bar{B}^0)^2 ||u_l - u_{l'}||\bar{B}^0] \\
&\qquad\quad + ||(u_l - u_{l'})^* a_l||\bar{B}^1 + ||u_{l'}^*(a_l - a_{l'})||\bar{B}^1 \\
&\quad \leq C(1 + ||u_l||\bar{B}^0 + ||u_{l'}||\bar{B}^0)^2 \\
&\qquad [||u_l - u_{l'}||\bar{B}^0 + ||u_l - u_{l'}||\bar{B}^0 ||a_l||\bar{B}^1 + ||a_l - a_{l'}||\bar{B}^1] \to 0.
\end{aligned}$$

This completes the proof.

Let us consider a smooth map and its kernel:

$$\begin{aligned}
&F^+ : As\mathfrak{A} \mapsto \hat{B}^2(\hat{M}; \Lambda^2_+ \otimes ad\mathfrak{g}), \quad [A_0 + a_l] \to [F^+_{A_0+a_l}]. \\
&As\hat{\mathfrak{M}} = \{[A_l] \in As\mathfrak{A} | [F^+_{A_l}] \in \bar{B}^2_0(\{K_l\}_l, \epsilon) = 0 \in \hat{B}^2(\{K_l\}_l, \epsilon)\}.
\end{aligned}$$

We call an element $[A_l] \in As\hat{\mathfrak{M}}(\hat{M}, \epsilon)$ an *asymptotic ASD connection.*

$As\mathfrak{G}$, or $As\mathfrak{G}_k$, $0 \leq k \leq n$, also act on $As\hat{\mathfrak{M}}$ or $As\hat{\mathfrak{M}}_k$, and one puts:

$$\begin{aligned}
&As\mathfrak{M} \equiv As\hat{\mathfrak{M}}/As\mathfrak{G}, \quad As\mathfrak{M}_k \equiv As\hat{\mathfrak{M}}_k/As\mathfrak{G}_k, \\
&\mathfrak{M}(\hat{M}) = As\mathfrak{M}_0 \subset As\mathfrak{M}_1 \subset \cdots \subset As\mathfrak{M}_n = As\mathfrak{M}.
\end{aligned}$$

Recall that in order to get the underlying Banach spaces, one has fixed pair of indices and translations $\mathbf{a} = \{\mathbf{a}(\mathbf{l},\mathbf{j})\}$ and $\mathbf{T} = \{\mathbf{T}(\mathbf{l},\mathbf{j})\}$ satisfying some propeorties in $2.A$. Then we get the parametrized asymptotic ASD moduli space:

$$PAs\mathfrak{M}_k(\hat{M}, g, \epsilon) = \cup_{(\mathbf{a},\mathbf{T})} As\mathfrak{M}_k(\hat{M}, g, \epsilon; \mathbf{a}, \mathbf{T}).$$

When we consider a single asymptotic moduli space, then we will not specify about $(\mathbf{a}, \mathbf{T})$.

5.B Regularity: The following lemma is well known.

Lemma 5.2 *Let us take any $[A] \in \mathfrak{M}(\hat{M}, g)$. Then there are positive $\delta_k > 0$ such that one may represent it by a smooth ASD connection A with:*

$$|A| end\hat{M}|C^k(S^3, t) \leq C_k exp(-\delta_k t)$$

identifying $end\hat{M} \cong S^3 \times [0, \infty)$.

Let φ_l be the family of cut off functions over $\hat{M}$ in $2.A$. Then one can assign $I([A]) = [A_l = A_0 + \varphi_l a]$, $A = A_0 + a$ which defines an element in $As\mathfrak{A}(\hat{M})$. It follows from the above exponential decay of A, that $[A_l]$ defines an element in $As\hat{\mathfrak{M}}$. Thus one gets a map $\hat{I} : \hat{\mathfrak{M}}' \mapsto As\hat{\mathfrak{M}}$, where $\hat{\mathfrak{M}}'$ is choice of lift as above for every element in $\mathfrak{M}$.

Let us take $[A_l] \in As\hat{\mathfrak{M}}$.

Lemma 5.3 *(1) Any asymptotic ASD connection $[A_l]$ converges as $l \to \infty$ to $(A(0), \ldots, A(n))$, $\bar{B}^1(\hat{M}, \epsilon)$ and $\bar{B}^1(S^3 \times \mathbf{R}, \epsilon)$ ASD connections respectively.*

(2) Moreover if $[A_l]$ and $[A'_l]$ are $As\mathfrak{G}(\hat{M}, \epsilon)$ equivalent, and if they correspond to $A(j)$ and $A(j)'$ respectively, then each $A(j)$ and $A(j)'$ are $\mathfrak{G}(\hat{M}, \epsilon)$ or $\mathfrak{G}(S^3 \times \mathbf{R}, \epsilon)$ equivalent.

(3) There exists $[g_l] \in As\mathfrak{G}$ so that $[g_l][A_l] \in As\mathfrak{A}$ can be represented by another $[A'_l]$ where each A'_l are given by cutting the same family of smooth ASD connections $(A(0), \ldots, A(n))$.

Proof : We only consider $n = 1$, one bubble case. The general case is similar. Let us take $[A_l] \in As\mathfrak{M}$ and consider the family of $\bar{B}^1$ connections A_l over $\hat{M}$. By definition, $||F^+_{A_l}||\hat{L}^{2+\epsilon}_w(\hat{M}) \to 0$ as $l \to \infty$, and A_l converges to $A(0)$ and $A(1)$ over $\hat{M}$ and $S^3 \times \mathbf{R}$ respectively in $\bar{B}^1$ topology. Each $A(i)$ satisfies the ASD equation with finite L^2 curvature norms. If for some $[g_l] \in As\mathfrak{G}$, $||g_l^* A_l - A'_l||\bar{B}^1 \to 0$, then we get $g_l \to (g(0), g(1)) \in \bar{B}^0 \times \bar{B}^0$, and clearly this satisfies $g(i)^* A(i) = A(i)'$. This verifies (2).

Let $A(i)$ be as above. By the above lemma, there exist gauge transformations $g(0)$ and $g(1)$ over $\hat{M}$ and $S^3 \times \mathbf{R}$ such that $g(i)^*(A(i))$ are smooth ASD connections. We check $g(i) \in \mathfrak{G} \subset As\mathfrak{G}$. On the end one can put $A(i) = d + a_i$ and $g(i)^*(A(i)) = d + a'_i$ where $a_i, a'_i \in \bar{B}^1$. Then one has the estimate:

$$\begin{aligned} |a_i - a'_i|\hat{L}^4_w &= |g(i)^*(A(i)) - A(i)|\hat{L}^4_w = |g(i)^{-1} dg(i) + g(i)^{-1} a_i g(i) - a_i|\hat{L}^4_w \\ &\geq |dg(i)|\hat{L}^4_w - 2|a_i|\hat{L}^4_w. \end{aligned}$$

This shows $dg(i) \in \hat{L}^4_w$ and subtracting some constants, one gets $g(i) \in \bar{B}^0$.

Let S_l be the translations in lemma 2.3. We put $A'_l = d + \varphi_l a'_1 + (1-\varphi_l)S^*_{2l}a'_2$. Since $g(i) \in \bar{B}^0$, there are $g'_i \in G$ so that $|g(0) - g'_0|C^0(S^3 \times \{T\}), |g(1) - g'_1|C^0(S^3 \times \{-T\}) \to 0$ as $T \to \infty$. After multiplying some constant, one may assume $g' = g'_0 = g'_1 \in G$. Again using cut off functions, like $h_l = g' + \varphi_l(g(0) - g') + (1-\varphi_l)S^*_{2l}(g(1) - g')$, one gets an element $[h_l] \in As\mathfrak{G}(\hat{M})$ satisfying $[h_l][A_l] = [A'_l]$. This verifies (3). This completes the proof.

Corollary 5.1 *For any pair $\epsilon > \epsilon' > 0$, the natural inclusions give homeomorphisms:*

$$I : As\mathfrak{M}_k(\hat{M}, g, \epsilon) \cong As\mathfrak{M}_k(\hat{M}, g, \epsilon') \quad 0 \leq k \leq n.$$

Corollary 5.2 *There exists a surjection (essentially an isomorphism):*

$$As\mathfrak{M}(\hat{M}; E) \mapsto \cup_{\{E_l\}}\mathfrak{M}(\hat{M}; E_0) \times \times_{j=1}^n \mathfrak{M}(S^3 \times \mathbf{R}; E_j)$$

where (1) $w_2(E) = w_2(E_0)$ and (2) $p_1(E) = \Sigma_{j=0}^n p_1(E_j)$.

In particular there is a natural injection $I : \mathfrak{M}(\hat{M}, g') \hookrightarrow As\mathfrak{M}(\hat{M})$ which is induced by $\hat{I}$ above.

5.C Local structure of the moduli spaces: Let us consider the action:

$$As\mathfrak{G} \times As\mathfrak{A} \mapsto As\mathfrak{A}.$$

The derivative at (id, A) is given by:

$$(u, a) \mapsto a + d_A(u) \in \hat{B}^1.$$

Lemma 5.4 *For any $A \in As\hat{\mathfrak{M}}(\hat{M})$, $\{\hat{B}^*, d^*_A\}$ is a Fredholm complex with the index $-2p_1(E) - 3(1 + b^2_+(M))$. We denote their cohomology groups by $H^*_A = Kerd^{*+1}_A/\ imd^*_A$.*

Proof : Let us take $A = [A_l] \in As\hat{\mathfrak{M}}(\hat{M})$. First let us see the composition $d^+_A \circ d^0_A = 0$. Since $d^+_A d_A([u_l]) = [F^+_{A_l}, u_l]$, it follows from the Sobolev multiplication $\hat{B}^2 \times \hat{B}^0 \mapsto \hat{B}^2$.

Let us see $\mathrm{im} d^0_A$ is closed. By $5.B$, one may choose $[A'_l]$, $As\mathfrak{G}$ gauge equivalent to $[A_l]$ such that $A_l = \varphi_l A_0 + (1-\varphi_l)S^*_{2l}A_1$, where A_j are smooth ASD connections over $\hat{M}$ and $S^3 \times \mathbf{R}$ respectively. Then for each l, $d_{A'_l} : \bar{B}^0(\hat{M}; ad\mathfrak{g}) \mapsto \bar{B}^1(\hat{M}; \Lambda^1 \otimes ad\mathfrak{g})$ has closed range with finite dimensional kernel (c.f. [K] lemma 1.5). Then as the proofs of proposition 1.1 and lemma 3.2, one sees that $d^0_A : \hat{B}^0 \mapsto \hat{B}^1$ has closed range. Similarly one can check $d^+_A : \hat{B}^1 \mapsto \hat{B}^2$ has closed range. The index computation is done from corollary 2.1, corollary 5.2 and the sum formula. This completes the proof.

Let us take $A \in As\mathfrak{M}$. Then there exists $Q_A : \hat{B}^2 \mapsto \hat{B}^1$ as in the last part of the proof of lemma 3.2. We recall its construction below. First we will construct $Q_A : \text{im}d_A^+ \cap \bar{B}^2(\hat{M}) \mapsto \bar{B}^1(\hat{M})$ with $d_A^+ \circ Q_A = \text{id}$ for an ASD connection A over $\hat{M}$. Similarly we have Q_A over $S^3 \times \mathbf{R}$. Then let $A \in As\mathfrak{M}$ correspond to a family of ASD connections $(A_0, \ldots, A_n)$. As in the last part of the proof of lemma 3.2, we cut and connect Q_{A_j}'s as:

$$Q'_A = \Sigma \psi_j Q_{A_j} \psi'_j : \ \text{im}d_A^+ \cap \hat{B}^2(\{K_l\}_l; \epsilon) \mapsto \hat{B}^1(\{K_l\}_l; \epsilon).$$

Again as in the lemma 3.2, one can modify Q'_A to Q_A so that Q_A satisfies $d_A^+ \circ Q_A = \text{id}$. This induces the desired map $Q_A : \hat{B}^2 \mapsto \hat{B}^1$.

Now let us make the first construction. Let us take the bounded operator:

$$Q_A : \ \text{im}d_A^+ \cap L^2_w(\hat{M}; \Lambda^2_+ \otimes ad\mathfrak{g}) \mapsto \ \text{Ker}(d_A)^*_w \subset (W_1^2)_w(\hat{M}; \Lambda^1 \otimes ad\mathfrak{g})$$

with $d_A^+ \circ Q_A = \text{id}$. We claim that if $u \in L^{2+\epsilon} \cap L^2_w$, then $Q_A(u) \in L^{2+\epsilon}_w$ and the norm of $Q_A(u)$ is uniformly bounded by the one of u. Recall the Sobolev inequality $| \quad |L^{2+\epsilon}_{\text{loc}} \leq C| \quad |(W_1^2)_{\text{loc}}$ where C is independent of choice of small ϵ. Let $\{\varphi_l\}_l$ be the exausting family of cut off functions on $\hat{M}$. Thus $\varphi_l Q_A(u) \in L^{2+\epsilon}_w$. Then we use the Sobolev inequality, the elliptic estimate and the local to global method used in the proof of sublemma 3.2. Thus one gets the following estimate:

$$\begin{aligned} |\varphi_l Q_A(u)|L^{2+\epsilon}_w(\hat{M}) &\leq C|\varphi_l Q_A(u)|L^2_w(\hat{M}) + C|d_A^+(\varphi_l Q_A(u))|L^2_w \\ &\leq C|\varphi_l Q_A(u)|L^2_w(\hat{M}) + C|\varphi_l u|L^2_w(\hat{M}) + C|Q_A(u))|L^2_w(\ \text{Supp}d\varphi_l) \end{aligned}$$

where the last term is uniformly bounded. Moreover $|(\varphi_l - \varphi_{l'})Q_A(u)|L^{2+\epsilon}_w(\hat{M}) \to 0$ as $l, l' \to \infty$. One may choose all the constants uniformly bounded independently of ϵ.

Again by the elliptic estimate, one gets $Q_A : \text{im}d_A^+ \cap \hat{L}^2_w \mapsto (\hat{W}_1^2)_w$. Then by the estimate of the last part of section 1, one gets the desired extension Q_A : $\text{im}d_A^+ \cap \bar{B}^2 \mapsto \bar{B}^1$. This finishes the construction of $Q_A : \text{im}d_A^+ \cap \hat{B}^2 \mapsto \hat{B}^1$.

Thus $\hat{B}^1$ splits as:

$$\hat{B}^1 = \ \text{im}Q_A \oplus \ \text{ker}d_A^+$$

by $u = u' + u - u'$, $u' = Q_A d_A^+(u)$. ¿From this, one has a natural isomorphism:

$$\text{ker}d_A^+ / \ \text{im}d_A \cong \hat{B}^1 / \ \text{im}Q_A + \ \text{im}d_A.$$

In particular $\text{im}Q_A +\ \text{im}d_A \subset \hat{B}^1$ has finite codimension. By adding a finite dimensional subspace $V_A \subset \hat{B}^2$, one has a tangent space of $As\mathfrak{B} = As\mathfrak{A}/As\mathfrak{G}$ at $[A]$:

$$T_{[A]}As\mathfrak{B} \cong \ \text{im}Q_A \oplus V_A.$$

Let us put:

$$CG(A) = \{A + a | a \in \text{ im}Q_A + V_A\},$$
$$I : CG(A) \times As\mathfrak{G} \mapsto As\mathfrak{A} \quad (A', g) \to g^*(A').$$

Then dI is given by:

$$dI = \text{ id } \oplus d_A : (\text{ im}Q_A \oplus V_A) \times As\mathfrak{g} = \text{ Lie}As\mathfrak{G} \mapsto \hat{B}^1.$$

Thus if A is irreducible ($\mathrm{Ker}d_A = 0$), then one gets the following:

Proposition 5.1 *There are neighbourhoods $U \subset As\mathfrak{A}$ of A and $V \subset As\mathfrak{G}$ of id such that:*

$$I : U \cap CG(A) \times V \mapsto As\mathfrak{A},$$
$$I : U \cap CG(A) \cap As\hat{\mathfrak{M}} \times V \mapsto As\hat{\mathfrak{M}}$$

are homeomorphisms respectively.

In particular if d_A^+ is surjective, then at $[A]$, $As\mathfrak{G}$ acts freely on $As\mathfrak{M}$ and $As\mathfrak{M}_k \backslash As\mathfrak{M}_{k-1}$. $As\mathfrak{M}/As\mathfrak{G}$ and $As\mathfrak{M}_k \backslash As\mathfrak{M}_{k-1}/As\mathfrak{G}$ are C^∞ manifolds of dimension $-2p_1(E) - 3(1 + b_+^2(M))$.

Here is one remark. Suppose one is given a uniformly bounded asymptotic equivalence $\{I_l\}_l$ between $(\hat{M}, g)$ and $(\hat{M}', g')$. Let us take $A = [A_l] \in As\mathfrak{M}(\hat{M})$. Let us choose small constants $\epsilon > \epsilon' > \epsilon''$. Then one gets the following commutative diagram:

$$\begin{array}{cccc} d_A^+ & \hat{B}^1(\hat{M}, \epsilon) & \mapsto & \hat{B}^2(\hat{M}, \epsilon) \\ & I^* \quad \downarrow & & I^* \quad \downarrow \\ d_{I^*(A)}^+ & \hat{B}^1(\hat{M}', \epsilon') & \mapsto & \hat{B}^2(\hat{M}', \epsilon') \\ & (I^{-1})^* \quad \downarrow & & (I^{-1})^* \quad \downarrow \\ d_A^+ & \hat{B}^1(\hat{M}, \epsilon'') & \mapsto & \hat{B}^2(\hat{M}, \epsilon''). \end{array} \tag{5.1}$$

¿From this diagram, it follows if $d_{I^*(A)}^+$ is surjective, then for any element $u \in \hat{B}^2(\hat{M}, \epsilon)$, there is $\alpha \in \hat{B}^2(\hat{M}, \epsilon'')$ with $d_A^+(\alpha) = u$.

6 Transversality

Let $(\hat{M}, g)$ be a smooth cylindrical manifold, and $E \to \hat{M}$ be an $SO(3)$ bundle with a fixed trivialization on the end. Thus the bundle determines $w_2(E) \in$

$H^2(M:\mathbf{Z}_2)$ and $p_1(E)\in H^4(M:\mathbf{Z})$. Here we quickly review the construction of $(W_l^2)_w$ ASD moduli space over $(E,\hat{M},g)$.

One has already used an weight function $w:\hat{M}\mapsto[0,\infty)$. Let us choose a smooth connection A_0 over E with $A_0|\text{end}=d$. Then let us put an affine Hilbert space and a Hilbert Lie group ($l\geq 3$):

$$\mathfrak{A}=\{A_0+a : a\in(W_l^2)_w(\hat{M};ad\mathfrak{g}\otimes\Lambda^1)\},$$
$$\mathfrak{G}=\{g\in C^0_{\mathrm{loc}}(AutE):\nabla_{A_0}g\in(W_l^2)_w(M:ad\mathfrak{g}\otimes\Lambda^1)\}.$$

Moreover put $\hat{\mathfrak{M}}=\{A\in\mathfrak{A}:F_A^+=0\}$. $\mathfrak{G}$ acts on both $\mathfrak{A}$ and $\hat{\mathfrak{M}}$, and one denotes their quotient spaces by $\mathfrak{B}=\mathfrak{A}/\mathfrak{G}$ and $\mathfrak{M}=\hat{\mathfrak{M}}/\mathfrak{G}$ respectively. Sometimes one denotes $\mathfrak{M}=\mathfrak{M}(g)$.

Lemma 6.1 *Let $A\in\hat{\mathfrak{M}}$. Then the AHS complex:*

$$\begin{aligned}0\ \to(W_{l+1}^2)_w(\hat{M},g;ad\mathfrak{g})\ &\xrightarrow{d_A}\ (W_l^2)_w(\hat{M},g;ad\mathfrak{g}\otimes\Lambda^1)\\ &\xrightarrow{d_A^+}\ (W_{l-1}^2)_w(\hat{M},g;ad\mathfrak{g}\otimes\Lambda^2_+)\ \to 0.\end{aligned}$$

is of Fredholm with the index $-2p_1(E)-3b^2_+(M)$.

This follows from the usual excision method.

6.B Transversality for $(W_k^2)_w$ spaces: The aim in $6.B$ is to see:

Proposition 6.1 *Suppose $b^2_+(M)\geq 1$. Then for some generic choice of g', $\mathfrak{M}(g')$ is a finite dimensional smooth manifold of dimension $-2p_1(E)-3(1+b^2_+(M))$ (note M is simply connected).*

In order to construct smooth moduli spaces, one uses K.Uhlenbeck's generic metric theorem. The proof is parallel to [FU] as far as one uses Hilbert spaces. Later one will see that this perturbation also can be applicable to L^p_w spaces. Let $(\hat{M},g,w)$ be a cylindrical manifold, and choose a proper map $h:\hat{M}\mapsto[0,\infty)$ with $h(x)\geq\frac{w}{2}(x)$. Then one introduces the following Banach manifold:

$$\mathfrak{C}=\{\phi\in C^l(Gl(T\hat{M})):limsup_i(\Sigma_{j=0}^l e^h|\nabla^j(\phi^*g-g)|S^3\times[i,i+1]=0\}.$$

Let us take $\phi\in\mathfrak{C}$, and put $g'=\phi^*g$. Then one has the AHS complex with respect to g':

$$0\ \to(W_{l+1}^2)_w(\hat{M},g')\ \xrightarrow{d}\ (W_l^2)_w(\hat{M},g')\ \xrightarrow{d^+}\ (W_{l-1}^2)_w(\hat{M},g')\ \to 0.$$

One sees that this is a Fredholm complex with the same index as the unperturbed one.

Lemma 6.2 *Suppose M is indefinite. Then by a small perturbation of the Riemannian metric, there are no orbit of reducible connections in $\mathfrak{M}_k(\hat{M})$.*

In order to verify this, we follows [FU]. Let us take any compactly supported smooth connection A_0 over $\hat{M}$, and consider $\mathfrak{A}(\hat{M}) = \{A_0 + a | a \in (W_l^2)_w(\hat{M}, g)\}$. Then one introduces a $\mathfrak{G}(\hat{M})$ equivariant map:

$$P_+ : \mathfrak{A}(\hat{M}) \times \mathfrak{C} \mapsto (W_{l-1}^2)_w(\hat{M}, g)$$

by $P_+(A, \phi) = P_+(g)(\phi^{-1})^* F_A$, where $P_+(g)$ is the projection to the self dual part with respect to g. Let us put $\bar{\mathfrak{M}}(\hat{M}) = P_+^{-1}(0)$.

Proposition 6.2 (FU) *$\bar{\mathfrak{M}}(\hat{M}) \cap \mathfrak{A}^*(\hat{M}) \times \mathfrak{C} / \mathfrak{G}(\hat{M})$ is a smooth Banach manifold where $\mathfrak{A}^*$ consists of irreducible connections.*

We sketch its proof. First we see that dP_+ is surjective at any (A, φ) with $P_+(A, \varphi) = 0$. Then it follows that $\bar{\mathfrak{M}}(\hat{M})$ is a Banach manifold on which $\mathfrak{G}(\hat{M})$ acts. Then by making slice for the action, one gets the result.

Now dP_+ splits as:

$$dP_+ = d_1 P_+ \oplus d_2 P_+ : (W_l^2)_w(\hat{M}, g) \oplus \mathfrak{c} \mapsto (W_{l-1}^2)_w(\hat{M}, g),$$

where $d_1 P_+(\alpha) = P_+(g)(\varphi^{-1})^* d_A(\alpha) | (u, \varphi)$, and $d_2 P_+(r) = P_+(g)((\varphi^{-1})^*(r^* F))$. We show that the differential of P_+ is surjective.

Let us consider the AHS complex:

$$0 \to (W_{l+1}^2)_w(\hat{M}, g) \xrightarrow{d_A} (W_l^2)_w(\hat{M}, g) \xrightarrow{d_A^+} (W_{l-1}^2)_w(\hat{M}, g) \to 0.$$

Since this is Fredholm, one sees dP_+ has finite codimension. Let us take a representative $u \in \mathrm{Coker} dP_+$ with $(d_A^+)_w^*(u) = 0$. Then one has $d_A(e^w u) = 0$. Then one has the equations, $d_A(F_A) = d_A^*(F_A) = d_A(v) = d_A^*(v) = 0$, where we use $g' = \varphi^*(g)$ metric and $v = \varphi^*(e^w u)$ (here one uses Hilbert space structure of the function space, $(W_{l-1}^2)_w(\hat{M}, g'; \Lambda_+^2)$). Then the same argument as [FU] p 56, shows that on open dense subset of $\hat{M}$, F_A can be expressed as $\alpha \otimes a \in \Lambda_+^2 \otimes Ad(E)$, with $|a| = 1$ (pointwise norm) and $d_A(a) = 0$. By the irreducibility, it follows $a = 0$, which contradicts to non triviality of $p_1(E)$. Thus $dP_+ | (A, \varphi)$ is surjective. This completes the proof.

Corollary 6.1 *For a Baire set of $\phi \in \mathfrak{C}$, there are no reducible connection in $\mathfrak{A}(\hat{M})$ with respect to $\phi^*(g)$.*

Proof : Suppose A_0 is reducible, and denote the corresponding $U(1)$ connection by the same A_0. Let P be a $G = U(1)$ bundle. Then as before one puts:

$$\mathfrak{A}(P) = \{A_0 + a | a \in (W_l^2)_w(\hat{M}, g)\},$$

$$\mathfrak{G}(\hat{M})_0 = \{h \in C^0_{\text{loc}}(Y; Aut(E)) | \nabla_{A_0} h \in (W^2_l)_w(\hat{M}, g'),$$
$$|h - \text{ id}|C^0(S^3 \times [i, i+1]) \to 0\}.$$

Then $\mathfrak{G}(\hat{M})_0$ acts on $\mathfrak{A}(\hat{M})$. One considers $P_+ : \mathfrak{A}(\hat{M}) \times \mathfrak{C} \mapsto (W^2_{l-1})_w(\hat{M}, g)$. dP_+ is surjective at $P_+(A, \varphi) = 0$, since $F_A \neq 0$.

Now one puts $\bar{\mathfrak{M}}(\hat{M}) = P_+^{-1}(0)/\mathfrak{G}(\hat{M})_0$. Then one has a smooth Banach manifold $\bar{\mathfrak{M}}(\hat{M})' = \bar{\mathfrak{M}}(\hat{M})/\mathfrak{G}(\hat{M})_0$. Let us consider a Fredholm map between Banach manifolds $\bar{\pi} : \bar{\mathfrak{M}}(\hat{M})' \mapsto \mathfrak{C}$. Its Fredholm index is the one of the following one:

$$0 \ \to (W^2_{l+1})_w(\hat{M}, g) \ \stackrel{d}{\to} \ (W^2_l)_w(\hat{M}, g) \ \stackrel{d^+}{\longrightarrow} \ (W^2_{l-1})_w(\hat{M}, g) \ \to 0.$$

one has $H^0(AHS) = H^1(AHS) = 0$, and $H^2(AHS) = b^2_+(M)$. This is also the case when one purturbes the Riemannian metrics slightly. In particular, if $b^2_+ > 0$, then the index is negative. Then the Sard Smale theorem shows that for a Baire set of $\mathfrak{C}$, there are no ASD connections over non trivial line bundles. This shows that there are no orbits of reducible ASD connections in $\mathfrak{M}(\hat{M})$ for a Baire set of $\mathfrak{C}$. This completes the proof.

Let us put the projection as $\pi : \bar{\mathfrak{M}}(\hat{M})/\mathfrak{G}(\hat{M}) \mapsto \mathfrak{C}$. Then the direct application of [FU] to this case, shows:

Corollary 6.2 (FU) *Suppose $b^2_+ > 0$. Then a Baire set of $\phi \in \mathfrak{C}$ exists such that $\hat{\mathfrak{M}}(\phi) \equiv \pi^{-1}(\phi)$ are smooth finite dimensional manifolds.*

6.C L^p_w spaces: let g' be a small perturbation of g as above. Thus for any ASD connection A with respect to g', $\text{Ker}\{d_A : \mathfrak{g} = \text{ Lie}\mathfrak{G} \mapsto (W^2_l)_w\} = 0$ and $\text{Coker} d^+_A : (W^2_l)_w \mapsto (W^2_{l-1})_w = 0$. Let us recall that we have defined Banach spaces $B^*(\hat{M})$, $\bar{B}^*(\hat{M})$ in 1.A, 2.A respectively (any element lies in $L^{4+2\epsilon}_{2w}(\hat{L})$ or $L^{2+\epsilon}_w(\hat{L})$). Let A be a $\bar{B}^1$ ASD connection, $A \in \mathfrak{A}(\bar{B}^*) = \{A_0 + a | a \in \bar{B}^1(\hat{M}; ad\mathfrak{g} \otimes \Lambda^1)\}$. One may assume A is smooth.

Lemma 6.3 *For a Baire set of $\phi \in \mathfrak{C}$ and $g' = \phi^*(g)$, the AHS complex:*

$$0 \ \to As\mathfrak{g}(\hat{M}, g') \ \stackrel{d_A}{\longrightarrow} \ \bar{B}^1(\hat{M}, g'; ad\mathfrak{g}) \ \stackrel{d^+_A}{\longrightarrow} \ \bar{B}^2(\hat{M}, g'; ad\mathfrak{g}) \ \to 0.$$

satisfies $Ker d_A = 0$ and $Coker d^+_A = 0$.

Proof : When A is a $B^1(\hat{M}, 0)$ ASD connection, one may use the Hilbert space structure of $B^2(\epsilon = 0)$. Then one can follow a parallel argument as above, and one gets the conclusion that for a generic g', there are no $\text{Ker} d_A : As\mathfrak{g}(\epsilon = 0) \mapsto B^1(\epsilon = 0)$, and no $\text{Coker} d^+_A : B^1(\epsilon = 0) \mapsto B^2(\epsilon = 0)$.

Let us use g' above and take $u \in \mathrm{Ker} d_A : \bar{B}^0(\hat{M}, \epsilon) \mapsto \bar{B}^1(\hat{M}, \epsilon)$. Since u satisfies the elliptic equation $(d_A)^*_w \circ d_A(u) = 0$, u has the exponential decay (see the proof of sublemma 3.1). Thus there is a positive constant $\delta_0 > 0$ and for the corresponding weight function w_0, u lies in $(W_k^{4+2\epsilon})_{2w_0}$ for all k. Thus by the method in the proof of proposition 4.1, $u \in (W_l^4)_{2w}$, $0 < \delta < \delta_0$, which is a contradiction.

Next let us consider $\mathrm{Coker} d_A^+ : \bar{B}^1(\hat{M}, \epsilon) \mapsto \bar{B}^2(\hat{M}, \epsilon)$. In 5.$C$, we have constructed $Q_A : \mathrm{im} d_A^+ \cap \bar{B}^2(\hat{M}, \epsilon) \mapsto \bar{B}^1(\hat{M}, \epsilon)$ from $Q_A : \mathrm{im} d_A^+ \cap L^2_w(\hat{M}) \mapsto (W_1^2)_w(\hat{M})$. In this case, $\mathrm{im} d_A^+ = L^2_w(\hat{M})$. Thus one has $Q_A : \bar{B}^2(\hat{M}, \epsilon) \mapsto \bar{B}^1(\hat{M}, \epsilon)$ with $d_A^+ \circ Q_A = \mathrm{id}$ as desired. This completes the proof.

We want to have transversality for asymptotic Banach spaces $\hat{B}^*(\{K_l\}_l, \epsilon)$. Let us construct a family of perturbed Riemannian metrics $\{g(l)\}_l$. Let g_0 and g_1 be cylindrical metric on $\hat{M}$ and $S^3 \times \mathbf{R}$ respectively, and we take g_i', small perturbations as above. Let $g_i'(N)$ be another metrics with $g_0'(N)|\hat{M}\backslash S^3 \times [N-1, \infty) = g_0'$ and $g_0'(N)|S^3 \times [N, \infty) = g_0$. Similarly $g_1'(N)|S^3 \times [-N+1, \infty) = g_1'$ and $g_1'(N)|S^3 \times (-\infty, -N] = g_1$. Then by regarding $\hat{M} = \hat{M}\backslash S^3 \times [N, \infty) \cup S^3 \times [-N, \infty)$, one gets another smooth Riemannian metric $g(N)$ on $\hat{M}$. This perturbation of Riemannian metrics works when one considers only one bubble at infinity. To treat asymptotic Banach spaces with n bubble, we perturb as follows. We recall a family of indices $\{a(i,j)\}$ and translations $T(i,j)$ $i = 0, 1, \ldots,$ $j = 0, \ldots n$ (2.A). We use the above notation. Then we choose Riemannian family $g_1(i,j)''$ on $S^3 \times [a(i, j-1), a(i,j)]$ satisfying:

$$
\begin{aligned}
&T(i,j)^*\{g_1''(i,j)|S^3 \times [a(i,j-1)+1, a(i,j)-1]\} = g_1' \\
&T(i,j)^*\{g_1''(i,j)|S^3 \times \{a(i,j-1) \cup a(i,j)\}\} = g_1.
\end{aligned}
$$

Then we identify $\hat{M} = \hat{M}\backslash S^3 \times [a(i,0), \infty) \cup S^3 \times [a(i,0), a(i,1)] \cup \ldots S^3 \times [a(i, n-1), a(i,n)] \cup S^3 \times [a(i,n), \infty)$. Correspondingly we make a family of Riemannian metrics on $\hat{M}$ as $g(l) = g_0'(a(l,0)) \cup g_1''(l,1) \cup \cdots \cup g''(l,n)$.

So far we have fixed one Riemannian metric g, in particular when define-ing asymptotic Banach spaces. Here we use a family of Riemannian metrics $\{g(l)\}_l$. Thus we formulate Banach spaces $\bar{B}^*(\{K_l\}_l, \epsilon)$, using norms $||u_l||_l$ by the following:

$$
\begin{aligned}
||[u_l]|\bar{B}^*(\{K_l\}_l; n) = \; & \mathrm{Sup}_l ||u_l||_l \\
\equiv \; & \mathrm{Sup}_l |\varphi_{a(l,0)} u_l | \bar{B}^*(K_l, w, \epsilon, g_0'(a(l,0))) \\
& + \Sigma_{j=1}^n |[\varphi_{a(l,j)} - \varphi_{a(l,j-1)}] u_l | \bar{B}^*(K_l, w(l,j), \epsilon, g_1''(l,j)).
\end{aligned}
$$

Clearly these Banach spaces $\hat{B}^*(\{K_l, g(l)\}, \epsilon)$ are equivalent to the previous

$\hat{B}^*(\{K_l\}_l, \epsilon)$.

Let us take $A = [A_l] \in As\mathfrak{M}(\{K_l, g(l)\}_l, \epsilon)$. We recall corollary 5.2, that there is a natural surjection:

$$As\mathfrak{M}(\{K_l\}_l, \epsilon) \mapsto \cup \mathfrak{M}(\hat{M}, g_0') \times \times_n \mathfrak{M}(S^3 \times \mathbf{R}, g_1')$$

by letting $[A_l] \to \lim_l A_l = (A_0, \dots, A_n)$. Let us consider the AHS complex:

$$0 \ \to As\mathfrak{g}(\{K_l, g(l)\}_l, \epsilon) \ \stackrel{d_A}{\longrightarrow} \ \hat{B}^1(\{K_l, g(l)\}_l, \epsilon) \ \stackrel{d_A^+}{\longrightarrow} \ \hat{B}^2(\{K_l, g(l)\}_l, \epsilon) \ \to 0.$$

We know that the above complex has closed range. Then we show $\mathrm{Ker} d_A = 0$ and $\mathrm{Coker} d_A^+ = 0$ as follows. Suppose A is reducible. Then there is $[u_l] \in As\mathfrak{g} = \mathrm{Lie} As\mathfrak{G}$ with $d_A([u_l]) = 0 \in \hat{B}^1(\hat{M})$. This shows $\lim_l ||d_{A_l}(u_l)||_l \bar{B}^1 = 0$. By choosing sufficiently large l, this contradicts irreducibility of A_j over $\hat{M}$ or $S^3 \times \mathbf{R}$.

Next we show $H_A^2 = 0$. By 5.B, one may choose $[A_l']$, $As\mathfrak{G}$ gauge equivalent to $A = [A_l]$ such that $\lim_l A_l = (A_0, \dots, A_n)$, where A_j are smooth ASD connections over $\hat{M}$ or $S^3 \times \mathbf{R}$. For every ASD connection A_j, there exists:

$$\begin{aligned} &Q_{A_0} : \bar{B}^2(\hat{M}; \Lambda_+^2 \otimes ad\mathfrak{g}) \mapsto \bar{B}^1(\hat{M}; \Lambda^1 \otimes ad\mathfrak{g}), \\ &Q_{A_j} : \bar{B}^2(S^3 \times \mathbf{R}; \Lambda_+^2 \otimes ad\mathfrak{g}) \mapsto \bar{B}^1(S^3 \times \mathbf{R}; \Lambda^1 \otimes ad\mathfrak{g}) \end{aligned}$$

with $d_{A_j}^+ \circ Q_{A_j}$ is the identity. Then a parallel argument as the proof of lemma 3.1 shows that there exists $Q_A : \hat{B}^2 \mapsto \hat{B}^1$ with $d_A^+ \circ Q_A = \mathrm{id}$. In particular $d_{[A_l']}^+ : \hat{B}^1(\hat{M}) \mapsto \hat{B}^2(\hat{M})$ is surjective. This completes the proof.

6.D Perturbation of asymptotic equivalence: Let $K_0 \subset K_1 \subset \dots \subset \hat{M}$ be an exaustion, and take an asymptotic morphism $\{I_l\}_l$ from $(\hat{M}, g)$ to $(\hat{M}', g')$. The morphism is defined using Riemannian metrics, and is not stable under the above perturbation. In order to overcome this, we will use more restrictive perturbation. Let us define:

$$\mathfrak{C} = \{\Phi \in \ \mathrm{Diff}_0^l(\hat{M}) : inf_{\Phi_t} \ \mathrm{Sup}_t[|e^h[\ \mathrm{id} - \Phi_t]|C^0 + \Sigma_{j=1}^l |e^h \nabla^j(\ \mathrm{id} - \Phi_t)|C^0] < \infty\}$$

where we take all the family of diffeomorphisms Φ_t with $\phi_0 = \mathrm{id}$ and $\Phi_1 = \Phi$. Thus we have a norm on $T\mathfrak{C}$ defined by $||\Phi|| = inf_{\Phi_t} \ \mathrm{Sup}_t[|e^h \Phi_t|C^0 + \Sigma_{j=1}^l |e^h \nabla^j \Phi_t|C^0]$.

Since $\hat{M}$ is of bounded geometry, there is a small constant $\epsilon > 0$ such that $\mathfrak{C}(\epsilon) = \{\Phi \in \mathfrak{C} : ||\ \mathrm{id}\ - \Phi|| \leq \epsilon\}$ is a Banach manifold. Then by a parallel argument, one gets the same conclusion as corollary 6.2 for $\mathfrak{C}(\epsilon)$ by taking $d\Phi \in C^l(Gl(T\hat{M}))$.

Let us choose any $\Phi \in \mathfrak{C}(\hat{M}, \epsilon)$ and $\Phi' \in \mathfrak{C}(\hat{M}', \epsilon)$. Let us put:

$$h = (d\Phi)^*(g), \quad h' = (d\Phi')^*(g'), \quad J_l = (\Phi')^{-1} \circ I_l \circ \Phi.$$

Then to give another asymptotic morphism, it is immediate to check the five defining conditions in the introduction for the following triples:

$$\{J_l\}_l : (\hat{M}, h) \mapsto (\hat{M}', h').$$

When we perturb asymptotic Banach spaces as above, we choose $\Phi^0 \in \mathfrak{C}(\hat{M}, \epsilon)$, $\Phi^1 \in \mathfrak{C}(S^3 \times \mathbf{R}, \epsilon)$ and choose paths of diffeomorphisms Φ^0_t and Φ^1_t. Then by a similar method as $6.C$, one obtains a family of diffeomorphisms $\{\Phi(l)\}_l$. By choosing $h(l) = d\Phi(l)^*(g)$ and $J_l = (\Phi(l)')^{-1} I_l \Phi(l)$ as above, one gets asymptotic Banach spaces $\{\hat{B}^*(\{K_l, \Phi(l)\}_l, \epsilon)\}_*$. The same consideration as $6.C$, performs transversality for the AHS complex $\{\hat{B}^*(\{K_l, \Phi(l)\}_l, \epsilon), d^*_A\}_*$.

7 Isomorphisms between moduli spaces

Let us take an asymptotic q.c. equivalence $\{I_l\}_l$ with uniform bound, from $(\hat{M}, g)$ to $(\hat{M}', g')$. Let us choose small constants $\epsilon > \epsilon' > \epsilon''$. Then by proposition 4.1, one gets a continuous map:

$$\{I_l^*\}_l : (As\mathfrak{M}(\hat{M}, g, \epsilon), \mathfrak{M}(\hat{M}, g, \epsilon)) \mapsto (As\mathfrak{M}(\hat{M}', g', \epsilon'), \mathfrak{M}(\hat{M}', g', \epsilon'))$$
$$\{(I_l^{-1})^*\}_l : (As\mathfrak{M}(\hat{M}', g', \epsilon'), \mathfrak{M}(\hat{M}', g', \epsilon')) \mapsto (As\mathfrak{M}(\hat{M}, g, \epsilon''), \mathfrak{M}(\hat{M}, g, \epsilon'')).$$

One has seen that the composition of the above maps gives the identity. By symmetry, one finds that $As\mathfrak{M}(\hat{M}, g)$ and $As\mathfrak{M}(\hat{M}', g')$ are homeomorphic.

Proof of theorem 0.2*:* Suppose $\{I_l\}_l$ consists of locally bounded as. q.c. equivalence, such that for any $[A_l] \in As\mathfrak{M}(\hat{M}('), g('))$,

$$\lim_{r\to 0}\lim\ \sup_{x\to\infty}\ \sup_{k,l}|I_l(^{-1})^*(F_{A_k})|L^2(B_r(x)) = 0.$$

Then we show that for each pair $(\mathbf{a}, \mathbf{T})$, there is another one $(\mathbf{a}', \mathbf{T}')$ such that $\{I_l\}_l$ induces a homeomorphism:

$$\{I_l^*\}_l : As\mathfrak{M}(\hat{M}, g; \epsilon; \mathbf{a}, \mathbf{T}) \cong \mathbf{As\mathfrak{M}(\hat{M}', g'}; \epsilon'; \mathbf{a}', \mathbf{T}'), \quad \epsilon' < \epsilon$$

for all metrics sufficiently near the cylindrical ones. By the construction, this assignment varies continuously with respect to the pair. Thus this will give the desired homeomorphism between parametrized asymptotic ASD moduli spaces.

Let us take any $[A_l] \in As\mathfrak{M}(\hat{M}, g, \epsilon)$. Passing through the surjection $As\mathfrak{M} \mapsto \cup\mathfrak{M}(\hat{M}) \times \times_n \mathfrak{M}(S^3 \times \mathbf{R})$, $[A_l]$ splits as $A_0 \times \times_{j=1}^n A_j$. Let us take two exaustions by compact subsets on $\hat{M}$ and $S^3 \times \mathbf{R}$ respectively. We denote them by the same $K_l \subset \hat{M}$ or $S^3 \times \mathbf{R}$. We consider the restriction $I_l|K_{l_0}$ for $l_0 << l$. Recall the family of translations $\{T(l,j)\}$. When $K_{l_0} \subset S^3 \times \mathbf{R}$ we identify $I_l|K_{l_0}$ with $I_l T(l_0,j)|K_{l_0}$.

For every $l_0 << l$, there exists a gauge transformation $g^j_{l_0} \in \bar{B}^0_{\text{loc}}(Aut(E|K'_{l_0}))$ with:

$$|(g^j_{l_0})^*(I^*_l(A_j|K_{l_0}))|\bar{B}^1(K'_{l_0},\epsilon')_{\text{loc}} \leq C|I^*_l(F_{A_j|K_{l_0}})|\hat{L}^2(K'_{l_0},\epsilon')_{\text{loc}}. \quad *$$
$$|(g^j_{l_0})^*(I^*_l(A_j|K_{l_0}))|(W^{2+\epsilon}_1)_{\text{loc}}(K'_{l_0}) \leq C|I^*_l(F_{A_j|K_{l_0}})|L^2_{\text{loc}}(K'_{l_0}).$$

Recall that there is a family of measurable bundle maps $\tau_l \in L^n(K_l)$ with $|\tau_l|L^n \to 0$. Let us put $A_j(l_0,l) \equiv (g^j_{l_0})^*(I^*_l(A_j|K_{l_0})) = d + a_j(l_0,l)$.

Sublemma 7.1 *([DS] p 244) A subsequence of $\{A_j(l_0,l)\}_l$ converges to a g' ASD connection in $\bar{B}^1(K'_{l_0},\epsilon')_{loc}$ as $l \to \infty$ for every l_0.*

In fact by the condition $|\nabla(I_l - I_{l'})|L^{4+\delta}_{\text{loc}} \to 0$, one does not need to take subsequences.

Let us consider $j = 0$. Then from the sublemma, we have the following data:

(1) pair of subindices $\{k_l\}_l, \{m_l\}_l$, $l << k_l << m_l$,
(2) another indices $\{a'(l,0)\}_l$ and translations $\{T'(l,0)\}_l$

such that these data gives an element in

$$[A_0(l,m_l)] \in As\mathfrak{M}(\{K'_{k_l}, a'(l,0), T'(l,0)\}_l; \epsilon').$$

One can check that this element does not depend on choice of $\{k_l\}_l$ and $\{m_l\}_l$. Moreover a' and T' are essentially determined by $[A_l]$.

For $j \geq 1$, the situation is similar. Let $j = 1$ and put $I(l_0,l) \equiv I_l T(l_0,1)|K_{l_0}$. Recall the exaustion $K_0 \subset K_1 \subset \cdots \subset S^3 \times \mathbf{R}$. Again in this case also, there are data as above with $[A_1(l,m(l,1))] \in As\mathfrak{M}(\{K'_{k(l,1)}, a'(l,1), T'(l,1)\}_l; \epsilon')$. One may assume $\{k(l,1)\}_l$ is a subindex of $\{k_l\}_l$. For $j \geq 2$, we get also $\{k(l,j)\}$, $\{m(l,j)\}_l$, $\{a'(l,j)\}$ and $\{T'(l,j)\}$. Let us put $a' = \cup^n_{j=0}\{a'(l,j)\}_l$ and $T' = \cup^n_{j=0}\{T'(l,j)\}_l$. By gathering and taking subindices, one gets a total element:

$$I(A) = [\Sigma^n_{j=0} A'(l,j)] \in As\mathfrak{M}(\{K'_{k(l,n)}, a', T'\}_l; \epsilon').$$

Thus for $A = [A_l]$, we have chosen $\{k_l\}_l$, a' and T', and made another $I(A)$. As above $I(A)$ is independent of choice of $\{k_l\}$, and a', and T' are essentially deteremined by A. If another $A' \in As\mathfrak{M}(\hat{M}, g, \epsilon)$ is sufficiently near A, then one may choose the same $\{k_l\}_l$ as A, and the corresponding $a'(A')$ and $T'(A')$ are sufficiently near $a' = a'(A)$ and $T = T'(A)$. Thus by taking countable dense subset of $As\mathfrak{M}(\hat{M}, g, \epsilon)$, one can choose the same $\{k_l\}_l$ for all $A \in As\mathfrak{M}(\hat{M}, g, \epsilon)$ by the diagonal method. Moreover the corresponding a' and T' vary continuously with respect to A.

Now we have assigned $[A'_l] \in As\mathfrak{M}(\hat{M}', g', \epsilon')$ to $[A_l] \in As\mathfrak{M}(\hat{M}, g, \epsilon)$. We show that this is well defined. We have chosen $g_l^j \in C^0_{\mathrm{loc}}(Aut(E|K'_l))$ to get $A_j(l)$. Suppose another family $\{h_l^j\}_l$ satisfies the same role as $\{g_l^j\}_l$. Let us put $g'_l = (h_l^j)^{-1} g_l^j$, $A_j(l) = (g_l^j)^*(I_l^*(A_j)) = d + a_j(l)$ and $A'_j(l) = (h_l^j)^*(I_l^*(A_j)) = d + a'_j(l)$. For the moment we omit to denote j. Notice the equality $(g'_l)^*(A'(l)) = (g'_l)^*(h_l^*(I_l^*(A))) = A(l)$. First we have an estimate:

$$||d(g'_l)||\hat{L}^4_{2w} \leq C||(g'_l)^*(A'(l))||\hat{L}^4_{2w} + C||a'_j(l)||\hat{L}^4_{2w} = C\{||a_j(l)||\hat{L}^4_{2w} + ||a'_j(l)||\hat{L}^4_{2w}\}.$$

In particular $||g'_l||\bar{B}^0$ are uniformly bounded. Then by taking a subsequence, one may assume $\{g'_l\}_l$ converges in C^0 topology. Then we have estimates:

$$\begin{aligned}
||d(g'_l - g'_{l'})||\hat{L}^4_{2w} &\leq C\{||(g'_l - g'_{l'})^*[(h_l^*(I_l^*(A)))]||\hat{L}^4_{2w} + ||(g'_l - g'_{l'})^*(a'_j(l))||\hat{L}^4_{2w}\} \\
&\leq C\{||(g'_l)^*(A'(l)) - (g'_{l'})^*(A'(l'))||\hat{L}^4_{2w} \\
&\quad + ||(g'_{l'})^*[A'(l) - A'(l')]||\hat{L}^4_{2w} + |g'_l - g'_{l'}|C^0||a'_j(l)||\hat{L}^4_{2w}\} \\
&\leq C\{||A(l) - A(l')||\hat{L}^4_{2w} + ||A'(l) - A'(l')]||\hat{L}^4_{2w} + |g'_l - g'_{l'}|C^0||a'_j(l)||\hat{L}^4_{2w}\}.
\end{aligned}$$

Thus by the above estimate, one sees $[g'_l] \in As\mathfrak{G}(\hat{M}', g', \epsilon')$. Thus we have verified that the above assignment is independent of choice of local gauge transformations g_l^j.

Next suppose $[A_l], [A'_l] \in As\mathfrak{M}(\hat{M}, g, \epsilon)$ are $As\mathfrak{G}(\hat{M}, g, \epsilon)$ equivalent by $[g_l]$. Let us choose $[h_l]$ and $[h'_l]$ for $[A_l]$ and $[A'_l]$ so that one gets $[A(l)], [A'(l)] \in As\mathfrak{M}(\hat{M}', g', \epsilon')$ as above. Let us put $u_l = g_l \circ I_l$. Then clearly $I_l^*(A'_l) = I_l^*(g_l^*(A_l)) = u_l^*(I_l^*(A_l))$. Thus we compare $[h_l^*(I_l^*(A_l))]$ and $[(h'_l)^* u_l^*(I_l^*(A_l))]$. Then one sees $[(h_l^{-1})(u_l h'_l)] \in As\mathfrak{G}$ by the above argument. Thus $[A(l)]$ and $[A'(l)]$ are $As\mathfrak{G}(\hat{M}', g', \epsilon')$ equivalent.

Now we have a well defined map $I^* : As\mathfrak{M}(\hat{M}, g, \epsilon) \mapsto As\mathfrak{M}(\hat{M}', g', \epsilon')$. By the same way, one gets another map $(I^{-1})^* : As\mathfrak{M}(\hat{M}', g', \epsilon') \mapsto As\mathfrak{M}(\hat{M}, g, \epsilon'')$. Next we see that the composition of these is the identity. Let us take $[A_l] \in As\mathfrak{M}(\hat{M}, g, \epsilon)$. The composition is given as $[g_l^*(A_l)] \in As\mathfrak{M}(\hat{M}, g, \epsilon'')$. Then by the same way as above, one sees $[g_l] \in As\mathfrak{G}(\hat{M}, g, \epsilon'')$. This verifies that the composition $(I^{-1})^* \circ I^*$ gives the identity. One knows also the converse composition $I^* \circ (I^{-1})^* = \mathrm{id}$. Thus combining with corollary 5.1, this completes the proof of theorem 0.2.

8 Large deformation of ASD moduli spaces

Let M be a closed C^∞ oriented four manifold, and for some $R \subset M$ as in the introduction, one gets $\check{M} = M \backslash \bar{R} \subset M$ and $\hat{M} = M\backslash$ pt. Thus $\check{M}$ and $\hat{M}$ are smooth submanifolds of M. Suppose $\check{M}$ is asymptotically quasiconformally homeomorphic to $(\hat{M}, g)$. Let us take an $ACH : \check{M} \to (\hat{M}, g)$, $\{I_l : (K_l, g_l) \mapsto (\hat{M}, g)\}_l$,

Let us take a generic family of Riemannian metrics h_l on M with $|g_l - h_l|K_l|C^\infty \to 0$. Notice that by changing g_l by h_l, the data $I_l : (K_l, h_l) \hookrightarrow (\hat{M}, g)$ satisfies the defining conditions for ACH. Let us take an $SO(3)$ bundle $E \to M$, and construct a family of ASD moduli spaces $\{\mathfrak{M}(M, h_l)\}_l$. Suppose they are all non-empty, and take any elements $[A_l] \in \mathfrak{M}(M, h_l)$. Here we will use Banach spaces $B(\hat{M}, \epsilon)$ in $1.A$. Recall that for an $SO(3)$ bundle $E' \to \hat{M}$ with fixed trivialization on the end, one has a formal affine Banach space $\mathfrak{A} = \{A_0 + a | a \in B^1(\hat{M}, \epsilon)\}$ where A_0 is a smooth compactly supported connection. Using $\mathfrak{A}$, one gets the ASD moduli space $\mathfrak{M}(\hat{M}, \epsilon)$.

Proposition 8.1 *After gauge transformation, a subsequence* $\{A'_l = (I_l^{-1})^*(A_l)\}_l$ *converges to an element* $A_\infty \in \mathfrak{M}(\hat{M}, \epsilon)$.

Proof : Here one may assume that all A_l are in C^∞. After gauge transformation and taking subsequence, one may assume (1) $\{A_l\}$ converges on every compact subset and (2) one has a uniform estimate $|A_l|(W^p_k)_{\text{loc}} \le C|F_{A_l}|L^2_{\text{loc}}$ by taking a trivialization.

Let us consider a family $\{(I_l^{-1})^*(A_l)\}_l$ over $\hat{M}$. By lemma 4.1 (1), one gets the estimate:

$$\begin{aligned}
&|(I_l^{-1})^*(A_l) - (I_{l'}^{-1})^*(A_{l'})|L^{4+2\epsilon'}_{\text{loc}} \\
&\quad \le |(I_l^{-1})^*(A_l) - (I_l^{-1})^*(A_{l'})|L^{4+2\epsilon'}_{\text{loc}} + |(I_l^{-1} - I_{l'}^{-1})^*(A_{l'})|L^{4+2\epsilon'}_{\text{loc}} \\
&\quad \le C(|\nabla I_l^{-1}|L^{4+\delta}_{\text{loc}})|A_l - A_{l'}|L^{4+2\epsilon}_{\text{loc}} + C(|\nabla(I_l^{-1} - I_{l'}^{-1})|L^{4+\delta}_{\text{loc}})|A_{l'}|L^{4+2\epsilon}_{\text{loc}} \to 0.
\end{aligned}$$

$$|(I_l^{-1})^*(A_l)|L^{4+2\epsilon'}_{\text{loc}} \le C(|\nabla I_l^{-1}|L^{4+\delta}_{\text{loc}})|A_l|L^{4+2\epsilon}_{\text{loc}} \le C(|\nabla I_l^{-1}|L^{4+\delta}_{\text{loc}})|F_{A_l}|L^2_{\text{loc}}.$$

Thus the family $\{(I_l^{-1})^*(A_l)\}_l$ converges to A_∞ in $L^{4+2\epsilon'}_{\text{loc}}$ topology. By considering similarly $\{F_{A'_l}\}_l$ also converges in $L^{2+\epsilon'}_{\text{loc}}$. Let $(\quad)^+$ be the projection to the self dual part with respect to g. Then we have the equality $F^+_{A'_l} + \tau_l(F_{A'_l}) = 0$, where $|\tau_l|L^n \to 0$ for large n. Combining with the estimate $|\tau_l F_{A'_l}|L^2 \le C|\tau_l|L^n|F_{A_l}|L^{2+\epsilon}$, $n^{-1} + (2+\epsilon')^{-1} = 1$, we see $F^+_{A_\infty} = 0$. Thus A_∞ is g-ASD connection. After gauge transformation by h, one may assume $h^*(A_\infty)$ satisfies the exponential decy estimate in W^2_k norm. Since $F_{h^*(A_\infty)} = h^*(F_{A_\infty})$, the L^2

norms are unchanged under gauge transformation. In particular F_{A_∞} satisfies exponential decay estimate in L^2 norm.

Let us consider the following estimates:

$$
\begin{aligned}
|(I_l^{-1})^*(A_l)|L^{4+2\epsilon'}_{\text{loc}} &\leq C|F_{A_l}|L^2_{\text{loc}} \\
&= C|I_l^*(F_{(I_l^{-1})^*(A_l)})|L^2_{\text{loc}} \leq C'|F_{(I_l^{-1})^*(A_l)})|L^2_{\text{loc}}, \\
|F_{(I_l^{-1})^*(A_l)}|L^{2+\epsilon'}_{\text{loc}} &= |(I_l^{-1})^*(F_{A_l})|L^{2+\epsilon'}_{\text{loc}} \leq C|F_{A_l}|L^{2+\epsilon}_{\text{loc}} \\
&\leq C'|F_{A_l}|L^2_{\text{loc}} = C'|I_l^*(F_{(I_l^{-1})^*(A_l)})|L^2_{\text{loc}} \leq C''|F_{(I_l^{-1})^*(A_l)}|L^2_{\text{loc}}.
\end{aligned}
$$

¿From these estimate, one gets an element $A_\infty \in \mathfrak{M}(\hat{M}, g, \epsilon')$. This completes the proof.

9 Donaldson's invariant

Let M be an oriented simply connected smooth closed four manifold. Let h be any element in $H_2(M; \mathbf{Z})$. One may represent h by a smoothly embedded Riemann surface. In fact by Hurewicz surjection, h has a representative by a smooth immersion $f : S^2 \mapsto M$. One may assume that the self intersection number of f is zero. For each Whitney pair of self intersection points (p_0, p_1), one has an embedded boundary of a Whitney disk $l = l_0 \cup l_1$, where l_i are arcs with two points $l_0 \cap l_1 = \{p_0, p_1\}$ as the Whitney pair. Remove small neighbourhoods $B_\epsilon(p_i)$, and add a small cylinder $S^1 \times [0, 1]$ along $\partial B_\epsilon(p_i)$ which contains l_0 in the cylinder. The result is an embedded Riemann surface which represents the same homology class as h (notice that $H_2(M; \mathbf{Z})$ is torsion free).

Let $\Sigma_1, \dots, \Sigma_k \subset M$ be embedded surfaces representing homology classes $[\Sigma_i] \in H_2(M; \mathbf{Z})$. Then one can construct $Q(E; M, h_1, \dots, h_m)$, the Donaldson's invariant We follow [FS2]. Let E be an $SO(3)$ bundle over M, and take any embedded Riemann surface $\Sigma \subset M$. Let $\mathfrak{B}^*_k(\Sigma; E|\Sigma) = \mathfrak{A}(E|\Sigma)^*_k / \mathfrak{G}_{k+1}(E|\Sigma)$ be the gauge equivalence classes of irreducible Sobolev k connections. Similarly one has $\tilde{\mathfrak{B}}^*_k(\Sigma; E|\Sigma) = \mathfrak{A}(E|\Sigma)^*_k / \mathfrak{G}_{k+1}(E|\Sigma)_0$, where $\mathfrak{G}_{k+1}(E|\Sigma)_0 \subset \mathfrak{G}_{k+1}(E|\Sigma)$ is a subgroup with $g(x_0)$ is the identity for a fixed point $x_0 \in \Sigma$. Thus one has a fibration:

$$
SO(3) \hookrightarrow \tilde{\mathfrak{B}}^*_k(\Sigma; E|\Sigma) \mapsto \mathfrak{B}^*_k(\Sigma; E|\Sigma)
$$

which gives an $SO(3)$ principal bundle β over $\mathfrak{B}^*_k(\Sigma; E|\Sigma)$. Now one has the following:

Sublemma 9.1 (FS2) *(1) If $< w_2(E), \Sigma > \neq 0$, then β lifts to a $U(2)$ bundle $\tilde{\beta}$.*

(2) For Σ, there is another embedded surface $\Sigma' \subset M$ such that restriction of any irreducible connection A over E to $E|\Sigma$ is also irreducible.

For our application, it is enough to assume the condition (1) in the above sublemma. Thus let us choose a lift $\tilde{\beta}$ over $\mathfrak{B}^*_k(\Sigma; E|\Sigma)$. One has $c_1(\tilde{\beta}) = w_2(\beta)$ mod 2. Let us choose a complex line bundle Λ over $\mathfrak{B}^*_k(\Sigma; E|\Sigma)$ with $c_1(\Lambda) = c_1(\tilde{\beta})$.

By the above sublemma (2), one has the restriction map:

$$r : \mathfrak{M}^*_k(E) \mapsto \mathfrak{B}^*_k(\Sigma; E|\Sigma).$$

One chooses a generic section s of Λ. Then by the transversality argument ([DK] p192), one has a smooth manifold $\mathfrak{M}^*_k(E) \cap r^{-1}(\text{ Ker} s)$ of codimension 2. One may generalize this construction. Suppose $\dim \mathfrak{M}_*(E) = 2l$, and choose l embedded surfaces $\Sigma_1, \dots, \Sigma_l \subset M$. As above one has determinant bundles over tublar neighbourhoods of Σ_i, $\nu(\Sigma_i)$. Then for generic sections, $s_1, \dots, s_l$, one has a smooth submanifold of dimension 0 as:

$$\mathfrak{M}_*(E) \cap r_1^{-1}(\text{ Ker} s_1) \cap \cdots \cap r_l^{-1}(\text{ Ker} s_l).$$

By choosing these embeddings with general positions, one may assume that this 0 dimensional manifold consists of finite oriented points. Thus one may count the number of points so that one obtains an integer. This number is, by definition, $Q(M, E; [\Sigma_1], \dots, [\Sigma_l])$ (modulo constant).

10 Proof of theorem 1

Let X be a simply connected smooth four manifold with even type form $(H_2(X : \mathbf{Z}), \sigma) \cong (H_2(k| - E_8| : \mathbf{Z}) \oplus H_2(l(S^2 \times S^2) : \mathbf{Z}), -kE_8 \oplus lH)$, where σ is the form of X and H is the form of $S^2 \times S^2$. We take an $SO(3)$ bundle $E \to X$. Let $w_2(E) = \hat{\alpha} = \hat{\alpha}_1 \oplus \hat{\alpha}_2$ and take $[\Sigma_i] = h_i^1 \oplus h_i^2 \in H_2(X : \mathbf{Z})$. Then we get the following:

Theorem 10.1 *Suppose all $h_i^1 = 0$, $1 \leq i \leq l$ and $\hat{\alpha}_i \neq 0$, $i = 1, 2$. Then there exists a pair (M, R) with the following property; suppose $\check{M}$ admits an asymptotically conformal equivalence to $(\hat{M}, g)$. Then the Donaldson's invariant vanishes $Q(E, X; [\Sigma_1], \dots, [\Sigma_l]) = 0$.*

Proof : Let us fix a marking of $H^2(X; \mathbf{Z})$ as above, and take a smooth embedding $\check{M} = l(S^2 \times S^2) \backslash \bar{R} \hookrightarrow X$, where $\bar{R}$ is a topologically closed disk, and this embedding is compatible with the forms. By taking any integral lift $\alpha \in H^2(X; \mathbf{Z})$, it gives a lift of the $SO(3)$ bundle on another $U(2)$ bundle $\bar{E}$.

Then for any $h \in H_2(X : \mathbf{Z})$ and its mod 2 reduction $[h]_2$, one has the equality $< w_2(E), [h]_2 >=< c_1(\bar{E}), h >$ mod 2.

$\check{M}$ is an open manifold on which one can equip an asymptotically quasiconformal homeomorphism to $(\hat{M}, g)$. Let us choose any complete Riemannian metrics of bounded geometry h on the interiors of $N = X \backslash \check{M}$.

Let $\Sigma_1, \dots, \Sigma_{m'} \subset \hat{M}$ be embedded surfaces as above and take another $SO(3)$ bundle $E' \to \hat{M}$. Then one has a smooth moduli space $\mathfrak{M}(\hat{M}, E')$ of B^1 ASD connections over E' of dimension $2m$, and $\mathfrak{M}(E', \hat{M}; [\Sigma_1], \dots, [\Sigma_{m'}])$ defined similarly as before. This space has a formal dimension $2(m - m')$.

Now let $\Sigma_1, \dots, \Sigma_m \subset \check{M} \subset X$ be as the theorem. One may assume the general position of these; for any $\Sigma_i, \Sigma_j, \Sigma_k$, one has empty intersection $\Sigma_i \cap \Sigma_j \cap \Sigma_k = \phi$. Suppose one has $Q(E, X; [\Sigma_1], \dots, [\Sigma_m]) \neq 0$. This shows that for a generic choice of Riemannian metric g on X, and also generic choice of these surfaces, one has non empty 0 dimensional moduli space $\mathfrak{M}(E, X; [\Sigma_1], \dots, [\Sigma_m])$ whose algebraic number is equal to the above invariant Q.

Let us choose exausting families of compact subsets as:

$$K_0 \subset K_1 \subset \cdots \subset \check{M}, \quad K'_0 \subset K'_1 \subset \cdots \subset N.$$

Let us choose a family of smooth Riemannian metrics g'_l on X with $(I_l^{-1})^* g'_l \sim g$, $g'_l | K'_l = h | K'_l$ as in the definition of ACH, and choose a family of small constants $\delta_i > 0$, $i = 0, 1, \dots,$ with $\delta_i \to 0$. In order to construct a smooth moduli space $\mathfrak{M}(E, X; [\Sigma_1], \dots, [\Sigma_m])$, one needs to perturbe g'_l slightly. Let us choose a family of generic Riemannian metrics g_l on X with $|g_l - g'_l| W_k^2(K_i \cup K'_i) \leq \delta_l$. Then one has a family of non empty smooth manifolds:

$$\{\mathfrak{M}(E, (X, g_i); [\Sigma_1], \dots, [\Sigma_m])\}_{i \geq 0}.$$

Let us choose any sequence $[A_i] \in \mathfrak{M}(E, (X, g_i); [\Sigma_1], \dots, [\Sigma_m])$, $i = 0, 1, \dots$. Then there is a subsequence which we also denote by $[A_i]$ so that $(I_l^{-1})^*(A_l)$ converges to $[A_\infty] \in \mathfrak{M}(\hat{M}, \epsilon')$, an ASD connection by proposition 8.1. On the other hand, $A_l | N$ converges to a $h-$ASD connection A'_∞ over (N, h) with $F_{A'_\infty} | L^2 < \infty$.

Now let $\{p_1, \dots, p_l\} \subset \check{M}$ be the bubbling points. Then at most $2l$ surfaces can contain these points. Thus by reordering surfaces, one has:

$$A_\infty \in \mathfrak{M}(E', \hat{M}, \epsilon'; [\Sigma_1], \dots, [\Sigma_{m-2l}]).$$

Let us see that A_∞ defines an $SO(3)$ bundle E' on $\hat{M}$ with strictly smaller $|p_1(E')|$ than $|p_1(E)|$. For this, one will see $|F_{A'_\infty}| L^2(N, h) > 0$. One has chosen E so that each $\hat{\alpha}_j \neq 0$ where $w_2(E) = \hat{\alpha} = \Sigma_{j=1}^2 \hat{\alpha}_j \in \oplus_k \mathbf{Z}_2^8 \oplus \oplus_{2l} \mathbf{Z}_2 \cong H^2(X; \mathbf{Z}_2)$.

Let $\alpha_j \in H^2(X;\mathbf{Z})$ be the corresponding lift. Let us take homology classes $h = h_1 + h_2 \in H_2(X;\mathbf{Z})$ such that the mod 2 reduction of $< \alpha_j, h_j >_2 \in \mathbf{Z}_2$ are all non zero $(= 1)$. Notice $w_2(E') = \hat{\alpha}_1 \in H^2(\hat{M};\mathbf{Z}_2)$. Let $\Sigma_j \subset X$ be embedded surfaces which represent h_j. $E|\Sigma_1$ is isomorphic to $E'|\Sigma_1$. In particular one has $w_2(E') \neq 0$. If the family $\{F_{A_i}\}_i$ concentrates at some points $p \in N$, then clearly one has $|p_1(E')| \leq |p_1(E)| - 4$. Now suppose it does not concentrate on N. Let $\Sigma_2 \subset N$ as above. Then one has the equality $< w_2(E), \Sigma_2 >=< c_1(\bar{E}), \Sigma_2 >=< c_1(A_i), \Sigma_2 >\neq 0$ mod 2. Since there are no concentrating points, one has also $< c_1(A'_\infty), \Sigma_2 >\neq 0$. Then clearly lim $\sup_i |F_{A_i}|L^2(N, g_i|N) \geq 4\pi$. Thus in any situation, one has $|p_1(E')| \leq |p_1(E)| - 2$.

Now one has $\mathfrak{M}(E, \hat{M}; [\Sigma_1], \ldots, [\Sigma_{m-2l}])$, non empty moduli space of ASD connections. One may assume that this space contains no reducible orbits. Recall $\dim \mathfrak{M} \leq \dim \operatorname{Ker} d_A^+ / \operatorname{im} d_A - 4 - 2m$, while $\dim \operatorname{Ker} d_A^+ / \operatorname{im} d_A - 2m = 0$. Thus the formal dimension of this space is negative. This is a contradiction. This completes the proof of the theorem.

Proof of theorem 0.1*:* First we will find some X, an $SO(3)$ bundle $E \to X$ and some embedded surface $\Sigma \subset X$ with non zero Donaldson's invariant $Q(X, E, [\Sigma], \ldots, [\Sigma]) \neq 0$. Donaldson verified that one finds such pair (X, E) if X is a simply connected algebraic surface, $|p_1(E)|$ is sufficiently large and $w_2(E)$ is mod 2 reduction of $1,1$ form.

Next one can choose a particular marking $(H_2(X : \mathbf{Z}), \sigma) \cong (H_2(k| - E_8| : \mathbf{Z}) \oplus H_2(l(S^2 \times S^2) : \mathbf{Z}), -kE_8 \oplus lH)$ so that for the corresponding decomposition $[\Sigma] = h_1 + h_2$ and $w_2(E) = \alpha_1 + \alpha_2$, h_1 vanishes and $\alpha_i \neq 0$, if the Picard number of X is sufficiently large. This follows from the existence of automorphisms of even type lattices. For example projective Kummer surface satisfies this condition (see [K] section 3).

The combination of these two steps gives the proof.

11 Asymptotic ASD moduli spaces

11.A Proof of corollary 0.1**:** Recall the statement:

Theorem 11.1 *If the Donaldson's invariants of M and N are different from each other, then $\hat{M}$ and $\hat{N}$ cannot be equivalent.*

For simplicity we assume that $\mathfrak{M}(\hat{M})$ and $\mathfrak{M}(\hat{N})$ are generic ASD moduli spaces of dimension 0. Then the algebraic numbers of them are diffeomorphism type invariant. Now there are examples of M and N so that $\sharp\mathfrak{M}(\hat{M})$ and $\sharp\mathfrak{M}(\hat{N})$ are different. In particular they are non-diffeomorphic.

The proof is now at hand. Suppose contrary to the conclusion of the above theorem. Let $\{I_l\}_l$ gives an equivalence and choose $\epsilon' < \epsilon$. Then one gets homeomorphisms as:

$$\{I_l^*\}_l : \mathfrak{M}(\hat{M}, [\Sigma_i]) \cong As\mathfrak{M}(\hat{M}, [\Sigma_i], \epsilon) \mapsto As\mathfrak{M}(\hat{N}, [\Sigma_i], \epsilon') \cong \mathfrak{M}(\hat{N}, [\Sigma_i]).$$

We have seen in section 7 that the middle map gives a homeomorphism. In particular $\sharp\mathfrak{M}(\hat{M}, [\Sigma_i]) = \sharp\mathfrak{M}(\hat{N}, [\Sigma_i])$. This is a contradiction.

11.B Asymptotic moduli spaces with low energy: Let $\{I_l\}_l$ be an asymptotic morphism from $(\hat{M}, g)$ to $(\hat{M}', g')$.

Let us take isomorphic $SO(3)$ bundles E and E' over $\hat{M}$ and $\hat{M}'$ respectively with $12 \geq -2p_1(E) - 3(1 + b_2^+(M)) = -2p_1(E') - 3(1 + b_2^+(M)) > 0$.

Theorem 11.2 *Suppose the above sutuation. Then the induced morphism:*

$$\{I_l^*\}_l : PAs\mathfrak{M}(E, \hat{M}, g, \epsilon) \cong PAs\mathfrak{M}(E', \hat{M}', g', \epsilon')$$

gives a homeomorphism.

Proof : Let us consider a family $[A_l = I_l^*(A)]$ where $A \in As\mathfrak{M}(\hat{M}, \epsilon)$. Consider a situation; for some sequences $x_0, x_1, \dots$ and small constants $\mu_0 >> \mu_1 >> \dots \to 0$, $|F_{A_l}|L^2(B_{\mu_i}(x_i)) \geq \mu > 0$. Notice that by the properties of I_l, x_i must go to infinity. However in our setting, such situation cannot occur, which follows from compactness of $\mathfrak{M}(E'', S^3 \times \mathbf{R}, \epsilon)$. This completes the proof.

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