STABILITY OF HARMONIC MAPS
AND EIGENVALUES OF LAPLACIAN

Hajime Urakawa

Max-Planck-Institut für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3
Federal Republic of Germany

and

Department of Mathematics
College of General Education
Tohoku University
Kawauchi, Sendai, 980 Japan

MPI/SFB 84-57

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Hajime URAKAWA

Contents

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§o. Introduction and Statement of Results.

The purpose of this article is to give results concerning with the Jacobi operator of a harmonic map which is arisen from the second variational formula of the energy functional of the map. This article is divided into three parts : Chapter I is treated with the estimation of the index and the nullity of a general harmonic map. In chapter II, we will deal with the stability of the identity map of a closed Riemannian manifold, i.e., a compact Riemannian manifold without boundary. Chapter III is devoted into the investigation of the Jacobi operator of the Riemannian submersions with totally geodesic fibers.

More precisely, let (M,g), (N,h) be two Riemannian manifolds of dimension m, n, respectively. We consider the energy functional E on the set $\mathfrak{M}(M,N)$ of all smooth maps ϕ ; $(M,g) \longrightarrow (N,h)$ (cf. [E.L]) :

$$E(\phi) = \frac{1}{2} \int_M \sum_{i=1}^{m} h(\phi_* e_i, \phi_* e_i) *1 ,$$

where $\{e_i\}_{i=1}^m$ is a locally defined orthonormal frame field on M and $*1$ is the volume element of (M,g). A critical point ϕ of E in $\mathfrak{M}(M,N)$ is called to be <u>harmonic</u>. The second variational formula of E was obtained by E.Mazet [Ma] and R.T.Smith [Sm] : For every one-parameter deformation ϕ_t of ϕ with $\phi_o = \phi$, and $\frac{d}{dt} \phi_t \big|_{t=o}$ giving a vector field V along ϕ ,

$$\frac{d^2}{dt^2} E(\phi_t)\Big|_{t=o} = \int_M h(J_\phi V, V) *1 .$$

Here J_ϕ is a second order elliptic differential operator, called a <u>Jacobi operator</u> analogeusly as a Morse theory of geodesics, acting on the space of all vector fields along ϕ . It is known that J_ϕ

has a discrete spectrum when M is a closed manifold. The index of ϕ, denoted by $\mathrm{Index}(\phi)$, is the sum of the multiplicities of the negative eigenvalues of J_ϕ, and the nullity of ϕ, denoted by $\mathrm{Nullity}(\phi)$, is the dimension of the kernel of J_ϕ.

When Ω is a relatively compact domain in a complete Riemannian manifold (M,g), we consider the variation of the energy functional E on the set of all smooth maps $\phi ; \Omega \longrightarrow N$ with the fixed boundary values on $\partial\Omega$. In this case, the second variational formula yields the eigenvalue problem of J_ϕ on Ω with the Dirichlet boundary condition :

$$\begin{cases} J_\phi V = \lambda V & \text{on } \Omega, \\ V = 0 & \text{on } \partial\Omega, \end{cases}$$

where V is a vector field along ϕ. The index of ϕ on Ω, denoted by $\mathrm{Index}_\Omega(\phi)$, is also defined as the sum of the multiplicities of the negative eigenvalues of this eigenvalue problem of J_ϕ, and the nullity of ϕ on Ω, denoted by $\mathrm{Nullity}_\Omega(\phi)$, is the dimension of the zero eigen-space. If $\mathrm{Index}(\phi) = 0$ (resp. $\mathrm{Index}_\Omega(\phi) = 0$), that is, all the eigenvalues of J_ϕ are non-negative, the harmonic map $\phi ; (M,g) \longrightarrow (N,h)$ is called to be stable (resp. stable on Ω).

Main results of chapter I are as follows : The crucial proposition for us, which are the analogue of recent works of P.Bérard and S.Gallot (cf.[B.G]) are :

Proposition 2.1. Let M be a closed manifold and $\phi : (M,g) \longrightarrow (N,h)$, a harmonic map. Then we have

$$\mathrm{Index}(\phi) + \mathrm{Nullity}(\phi) \leqq n \,\mathrm{Inf}\left\{ e^{t\, {}^N R^\phi} Z_M(t) \;;\; 0 < t < \infty \right\},$$

where $n = \dim N$ and ${}^N R^\phi$ is the following quantity :

$$ {}^{N}R^{\phi} := \operatorname{Sup}_{x \in M} \operatorname{Sup}_{v \in T_{\phi(x)}N} \sum_{i=1}^{m} h({}^{N}R(\phi_{*}e_i, v)\phi_{*}e_i, v)/h(v,v), $$

${}^{N}R$ is the curvature tensor of (N,h) (cf. §1). $Z_M(t)$ is the trace of the heat kernel of the Laplace-Beltrami operator Δ_M of (M,g) acting on the space $C^{\infty}(M)$ of all smooth functions on M.

Proposition 2.4. Let Ω be a relatively compact domain in a complete Riemannian manifold (M,g), and $\phi : (M,g) \longrightarrow (N,h)$, a harmonic map. Then we have

$$ \operatorname{Index}(\phi) + \operatorname{Nullity}(\phi) \leqq n \operatorname{Inf}\left\{ e^{t\, {}^{N}R^{\phi}_{\Omega}} Z_{\Omega}(t) \ ; \ 0<t<\infty \right\}, $$

where $n = \dim N$ and the quantity ${}^{N}R^{\phi}_{\Omega}$ is defined by

$$ {}^{N}R^{\phi}_{\Omega} := \operatorname{Sup}_{x \in \Omega} \operatorname{Sup}_{v \in T_{\phi(x)}N} \sum_{i=1}^{m} h({}^{N}R(\phi_{*}e_i, v)\phi_{*}e_i, v)/h(v,v). $$

$Z_{\Omega}(t) := \sum_{i=1}^{\infty} e^{-t\lambda_i(\Omega)}$, where $\lambda_i(\Omega)$, $i=1,2,\ldots,$ are the eigenvalues counted with their multiplicities of the Dirichlet problem of Δ_M for the domain Ω :

$$ \begin{cases} \Delta_M u = \lambda u & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} $$

As applications of these propositions, we have :

Theorem 2.5. Let (M,g) be a closed Riemannian manifold of dimension $m \geqq 2$, whose Ricci curvature Ric_M is bounded below by a positive constant : $Ric_M \geqq (m-1)\delta > 0$. Let $\phi : (M,g) \longrightarrow (N,h)$ be a harmonic map of (M,g) into arbitrary Riemannian manifold (N,h) of dimension n. Then we have :

(i) In case of $m \geqq 3$,

$$ \operatorname{Index}(\phi) + \operatorname{Nullity}(\phi) \leqq n \left(1+\frac{1}{A}\right)^{A} \left\{ 1+(m-1)!\, m^{m-1} A(1+A)^{m-1} \right\}, $$

where $A := {}^N R^{\phi} / m\delta$.

(ii) In case of $m = 2$,

$$\mathrm{Index}(\phi) + \mathrm{Nullity}(\phi) \leqq n\ (1+\tfrac{1}{B})^B \left\{1+4B^2\right\} ,$$

where $B := {}^N R^{\phi} / \delta$.

Remark. The function $(1+\frac{1}{x})^x$ satisfies that $\lim_{x\to 0} (1+\frac{1}{x})^x = 1$ and $(1+\frac{1}{x})^x < e$, $0<x<\infty$.

Theorem 3.1. Let Ω be a relatively compact domain in a complete Riemannian manifold (M,g), and $\phi : (M,g) \longrightarrow (N,h)$ be a harmonic map into arbitrary Riemannian manifold (N,h) of dimension n. Then we have

(i) $\lambda_1(\Omega) \geqq {}^N R^{\phi}_{\Omega} \Longrightarrow \mathrm{Index}_{\Omega}(\phi) = 0$ and $\mathrm{Nullity}_{\Omega}(\phi) \leqq n$.

(ii) $\lambda_1(\Omega) > {}^N R^{\phi}_{\Omega} \Longrightarrow \mathrm{Index}_{\Omega}(\phi) = \mathrm{Nullity}_{\Omega}(\phi) = 0$.

Since $\lambda_1(\Omega)$ grows to infinity and ${}^N R^{\phi}_{\Omega}$ remains still bounded when Ω shrinks to "small", this theorem implies that ϕ is stable on a "small" relatively compact domain in M , which was stated in [Sm].

It is known (cf.[C.L] ,[B.G]) that there exists a constant $C(M,g) > 0$ depending only on (M,g) such that the eigenvalue $\lambda_i(\Omega)$ of the Dirichlet eigenvalue problem of Δ_M for Ω satisfies

$$\lambda_i(\Omega) \geqq C(M,g)\ \mathrm{Vol}(\Omega)^{-2/m}\ i^{2/m} , \quad i=1,2, \ldots .$$

Then we can estimate $\mathrm{Index}_{\Omega}(\phi) + \mathrm{Nullity}_{\Omega}(\phi)$ by the quantity $D := {}^N R^{\phi}_{\Omega}\ C(M,g)^{-1}\ \mathrm{Vol}(\Omega)^{2/m} - 1$ (cf. Theorem 3.4).

In chapter II, we will treat with the Jacobi operator of the identity map. The identity map of a closed Riemannian manifold (M,g) is harmonic and (M,g) is called to be <u>stable</u> (cf.[Na]) if the identity map is stable.

It is known (cf. [Sm], [Na]) that a holomorphic map between Kähler manifolds is always stable. Then the stability of the identity map of a closed Kähler manifold (M,g) yields the Kähler version of a theorem of Lichnérowicz-Obata concerning the first non-zero eigenvalue $\lambda_1(M)$ of the Laplace-Beltrami operator Δ_M :

Theorem 4.2. (M.Obata) Let (M,g) be a closed Kähler manifold whose Ricci curvature Ric_M is bounded below by a positive constant : $Ric_M \geqq \alpha > 0$. Then we have

$$\lambda_1(M) \geqq 2\alpha .$$

When the equality holds, the Lie algebra $\mathfrak{a}$ of the group of holomorphic transformations of M is non-zero.

Remark. In the case that (M,g) is Einstein and Kähler, this theorem was stated in [Ob]. In this case, the equality $\lambda_1(M) = 2\alpha$ holds if and only if $\mathfrak{a} \neq \{0\}$.

Some instability results about Riemannian tori and the canonical deformations of the standard unit sphere (S^{2n+1}, can) are obtained (cf. 5.1 and 5.2). Y.L.Xin [X] showed that every non-constant harmonic map of the standard unit sphere (S^n, can) into arbitrary Riemannian manifold is unstable. On the contrary, we can state :

Proposition 5.6. Every spherical space form $(S^n/G, g)$, where $G \neq \{id\}$ is a finite group acting fixed point freely on S^n, is stable. Here g is the Riemannian metric on S^n/G induced from the standard metric can of S^n with constant curvature 1.

Therefore every closed Riemannian manifold of constant curvature (positive, zero or negative) is stable except only the unit sphere (S^n, can) (cf. Corollary 5.7). The analogous stability theorem for Yang-Mills fields was stated in [B.L, p.223].

In chapter III, we will deal with the Jacobi operator of Riemannian submersions with totally geodesic fibers. The Riemannian submersion $\phi : (M,g) \longrightarrow (N,h)$ with totally geodesic fibers is harmonic (cf. [E.S]). The typical examples are (cf. [B.B]) :

(i) Hopf fibering $\phi_1 : (S^{4n+3}, g) \longrightarrow (HP^n, h)$,

(ii) Hopf fibering $\phi_2 : (S^{2n+1}, g) \longrightarrow (CP^n, h)$,

(iii) The natural projection $\phi : (G/H, g) \longrightarrow (G/K, h)$,

where $G \supset K \supset H$ are compact Lie groups.

For the Riemannian submersion ϕ, we will define the <u>vertical</u> (resp. <u>horizontal</u>) Jacobi operator J^V_ϕ (resp. J^H_ϕ) which satisfy

$$[J^V_\phi , J^H_\phi] = 0 \quad \text{and} \quad J_\phi = J^V_\phi + J^H_\phi$$

(cf. Theorem 6.5). And we can compare Index(ϕ) (resp. Nullity(ϕ)) of the submersion ϕ with Index(id_N) (resp. Nullity(id_N) of the base manifold (N,h) :

Proposition 6.3. Let (M,g) be a closed Riemannian manifold and $\phi : (M,g) \longrightarrow (N,h)$, a Riemannian submersion with totally geodesic fibers. Then we have the inequalities Index(ϕ) $\geqq$ Index(id_N), Nullity(ϕ) $\geqq$ Nullity(id_N) and $\lambda_1(J_\phi) \leqq \lambda_1(J_{id_N})$. In particular, if the base manifold (N,h) is unstable, then the projection ϕ is unstable.

Moreover, following [B.B] , we define the canonical deformation g_t , $0 < t < \infty$, of the Riemannian metric g on M with $g_1 = g$ (cf. §7) such that the projection $\phi : (M, g_t) \longrightarrow (N,h)$ is still a Riemannian submersion with totally geodesic fibers. For this canonical deformation g_t , the Jacobi operator ${}^tJ_\phi$ of ϕ ; $(M,g_t) \longrightarrow (N,h)$ satisfies (cf. Proposition 7.2)

$$ {}^tJ_\phi = t^{-2}J_\phi^V + J_\phi^H . $$

Then we have :

Theorem 7.3. Let (M,g) be a closed Riemannian manifold and ϕ ; $(M,g) \longrightarrow (N,h)$ be a Riemannian submersion with totally geodesic fibers. Let g_t , $0<t<\infty$, be the canonical deformation of g with $g_1 = g$. Then there exists a positive number ε such that

$$ \lambda_1({}^tJ_\phi) = \lambda_1(J_{id_N}) \quad \text{for all } 0<t<\varepsilon. $$

In particular, if (N,h) is stable, then $\phi : (M,g_t) \longrightarrow (N,h)$ is stable for all $0<t<\varepsilon$.

As applications of Proposition 6.3 and Theorem 7.3, we have :

(i) Since (HP^n,h) is unstable (cf. [Sm], [Na]), the submersion $\phi_1 : (S^{4n+3},g) \longrightarrow (HP^n,h)$ is always unstable.

(ii) Since (CP^n,h) is stable, for the canonical deformation g_t, $0<t<\varepsilon$, of g on S^{2n+1} with $g_1 = g$, there exists a positive number ε such that the submersion $\phi_2 : (S^{2n+1},g_t) \longrightarrow (CP^n,h)$ is stable for each $0<t<\varepsilon$ (cf. Proposition 7.4).

On the other hand, when the holomony group of the submersion does not act transitively on the fibers and the base manifold (N,h) is unstable, the index of the submersion ϕ ; $(M,g_t) \longrightarrow (N,h)$ grows to infinity as $t \to \infty$ (cf. Theorem 7.5). This is an extension of results obtained by R.T.Smith in [Sm, Corollary 3.3] .

At last, we will express in terms of Lie algebras, the Jacobi operator of the homogeneous Riemannian submersions (iii) (cf.Theorem 8.11). As an application, we determine the spectrum of the Jacobi operator of the Hopf fibering of S^3 onto $CP^1 = S^2$ (Corollary 8.12).

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Chapter I. The Index and the Nullity of a General Harmonic Map.

§1. Preliminaries.

1.1. In this section, following [E.L], we prepare the second variational formula of the energy functional obtained in [Ma], [Sm].

Let $(M,g),(N,h)$ be two Riemannian manifolds of dimension m,n, respectively. Let $\phi; M\to N$ be a smooth map. Let $E = \phi^{-1}TN$ be the induced bundle by ϕ over M of the tangent bundle TN of N. We denote by $\Gamma(E)$, the space of all sections V of E, that is, $V \in \Gamma(E)$ implies that V is a map of M into E such that $V_x \in T_{\phi(x)}N$ for all $x \in M$. For $X \in \Gamma(TM)$, we define $\phi_* X \in \Gamma(E)$ by $(\phi_* X)_x := \phi_{*x} X_x \in T_{\phi(x)}N$, $x \in M$, where ϕ_{*x} is the differential of ϕ at x. For $Y \in \Gamma(TN)$, we also define $\tilde{Y} \in \Gamma(E)$ by $\tilde{Y}_x := Y_{\phi(x)}$, $x \in M$.

We denote by ∇, ${}^N\nabla$ the Levi-Civita connections of (M,g), (N,h), respectively. Then we give the induced connection $\tilde{\nabla}$ on E by

$$(1.1) \qquad (\tilde{\nabla}_X V)_x := \frac{d}{dt}\, {}^N P_{\phi(\gamma(t))}^{-1}\, V_{\gamma(t)}\Big|_{t=0}, \quad X \in \Gamma(TM),\ V \in \Gamma(E)$$

where $x \in M$, $\gamma(t)$ is a curve through x at $t=0$ whose tangent vector at x is X_x, and ${}^N P_{\phi(\gamma(t))}\,;\ T_{\phi(x)}N \longrightarrow T_{\phi(\gamma(t))}N$, is the parallel displacement along a curve $\phi(\gamma(s))$, $0 \leqq s \leqq t$, given by the Levi-Civita connection ${}^N\nabla$ of (N,h).

We define a tension field $\tau(\phi) \in \Gamma(E)$ of ϕ by

$$\tau(\phi) := \sum_{i=1}^{m} (\tilde{\nabla}_{e_i} \phi_* e_i - \phi_* \nabla_{e_i} e_i),$$

where $\{e_i\}_{i=1}^m$ is a (locally defined) orthonormal frame field on M. We call ϕ to be <u>harmonic</u> if $\tau(\phi) = 0$. For a relatively compact domain Ω in M, the energy $E(\Omega,\phi)$ of ϕ on Ω is defined by

$$E(\Omega,\phi) := \int_{\Omega} e(\phi)(x) *1 ,$$

where $e(\phi)(x) := \frac{1}{2} \sum_{i=1}^{m} h(\phi_* e_i, \phi_* e_i)$ and $*1$ is the volume element of (M,g). We denote $E(\phi) := E(M,\phi)$ when defined. For an element V in $\Gamma(E)$, let ϕ_t ; $M \longrightarrow N$ be a one-parameter family of maps from M into N with $\phi_0 = \phi$, and $\frac{d}{dt} \phi_t(x)\Big|_{t=0} = V_x$, $x \in M$. If $V \in \Gamma(E)$ has a compact support, it is known (cf. [E.S], [E.L], [Ma]) that

$$(1.2) \qquad \frac{d}{dt} E(\phi_t)\Big|_{t=0} = - \int_M h(V, \tau(\phi)) *1 .$$

Moreover, if ϕ ; $(M,g) \longrightarrow (N,h)$ is harmonic and $V \in \Gamma(E)$ has a compact support,

$$(1.3) \qquad \frac{d^2}{dt^2} E(\phi_t)\Big|_{t=0} = \int_M h(V, J_\phi V) *1,$$

where the operator J_ϕ ; $\Gamma(E) \longrightarrow \Gamma(E)$, called the <u>Jacobi</u> <u>operator</u> of ϕ , is a second order elliptic differential operator given by

$$(1.4) \qquad J_\phi V := - \sum_{i=1}^{m} \left\{ \widetilde{\nabla}_{e_i} \widetilde{\nabla}_{e_i} V - \widetilde{\nabla}_{\nabla_{e_i} e_i} V \right\} - \sum_{i=1}^{m} {}^N R(\phi_* e_i, V) \phi_* e_i ,$$

for $V \in \Gamma(E)$. Here ${}^N R$ is the curvature tensor of (N,h) given by

$$(1.5) \qquad {}^N R(X,Y)Z := {}^N\nabla_{[X,Y]} Z - {}^N\nabla_X {}^N\nabla_Y Z + {}^N\nabla_Y {}^N\nabla_X Z ,$$

for $X, Y, Z \in \Gamma(TN)$.

For a relatively compact domain Ω in M, let us consider the Dirichlet eigenvalue problem of J_ϕ as follows :

$$(1.6) \qquad \begin{cases} J_\phi V = \lambda V & \text{on } \Omega , \\ V = 0 & \text{on } \partial\Omega . \end{cases}$$

If M is a closed manifold, we consider the eigenvalue problem of J_ϕ :

$$(1.7) \qquad J_\phi V = \lambda V , \quad V \in \Gamma(E).$$

It is known that the spectra of both problems (1.6), (1.7) consist of discrete eigenvalues with finite multiplicities. The index of ϕ on Ω, denoted by $\mathrm{Index}_\Omega(\phi)$, is defined as the sum of the eigenvalues of the problem (1.6), and the index of ϕ, denoted by $\mathrm{Index}(\phi)$, is also defined as the sum of the eigenvalues of (1.7) when M is a closed manifold. The dimension of the zero eigenspace of (1.6) (resp. (1.7)) is called the nullity of ϕ on Ω (resp. the nullity of ϕ), denoted by $\mathrm{Nullity}_\Omega(\phi)$ (resp. $\mathrm{Nullity}(\phi)$). The harmonic map $\phi; (M,g) \longrightarrow (N,h)$ is stable (resp. stable on Ω) if $\mathrm{Index}(\phi) = 0$ (resp. $\mathrm{Index}_\Omega(\phi) = 0$).

1.2. For the estimation of the index and the nullity of a harmonic map, we have to introduce the quantity ${}^N R^\phi$ or ${}^N R^\phi_\Omega$ as follows. We retain the notations as in 1.1.

Definition 1.1. For a smooth map $\phi; (M,g) \longrightarrow (N,h)$, we define ${}^N R^\phi$ by

$$(1.8) \qquad {}^N R^\phi := \sup_{x \in M} \sup_{v \in T_{\phi(x)}N} \sum_{i=1}^m h({}^N R(\phi_* e_i, v)\phi_* e_i, v)/h(v,v).$$

For a relatively compact domain Ω in M, we define also ${}^N R^\phi_\Omega$ by

$$(1.9) \qquad {}^N R^\phi_\Omega := \sup_{x \in \Omega} \sup_{v \in T_{\phi(x)}N} \sum_{i=1}^m h({}^N R(\phi_* e_i, v)\phi_* e_i, v)/h(v,v).$$

Note that these quantities do not depend on the choice of $\{e_i\}_{i=1}^m$. We have immediately :

Lemma 1.2. Assume that the sectional curvature ${}^N K$ of (N,h)

is bounded above by a positive constant :

$$^N K(\pi) \leqq a \quad \text{for all planes } \pi \text{ in } T_y N,\ y \in N.$$

Then we have

$$(1.10) \qquad {}^N R^{\phi} \leqq 2a\ E^{\infty}(\phi) \quad , \text{ and}$$

$$(1.11) \qquad {}^N R^{\phi}_{\Omega} \leqq 2a\ E^{\infty}(\Omega,\phi)\ .$$

Here $E^{\infty}(\phi) := \sup_{x \in M} e(\phi)(x)$ and $E^{\infty}(\Omega,\phi) := \sup_{x \in \Omega} e(\phi)(x)$.

In fact, it is obvious from that

$$\sum_{i=1}^{m} h({}^N R(\phi_* e_i, v)\phi_* e_i, v) \leqq a \left\{ \sum_{i=1}^{m} h(\phi_* e_i, \phi_* e_i) \right\} h(v,v)$$

at each point of M.

Note that $E(\Omega,\phi) \leqq E^{\infty}(\Omega,\phi)\ \mathrm{Vol}\,\Omega$ and $E(\phi) \leqq E^{\infty}(\phi)\ \mathrm{Vol}\ M$ if $\mathrm{Vol}\ M < \infty$.

Example 1.3. Let $\phi ; (M,g) \longrightarrow (N,h)$ be an isometric immersion. Then $e(\phi)(x) = m/2$ at each point. Therefore

$$(1.12) \qquad E^{\infty}(\phi) = E^{\infty}(\Omega,\phi) = m/2, \quad \text{and}$$

$$(1.12') \qquad {}^N R^{\phi}_{\Omega} \leqq R^{\phi} \leqq ma\ ,$$

for every relatively compact domain Ω in M. In particular, let $\phi ; [0,2\pi] \longrightarrow (N,h)$ be a geodesic with the length L. Then

$$(1.13) \qquad E^{\infty}(\phi) = L^2/8\pi^2\ .$$

Example 1.4. Let $\phi ; (M,g) \longrightarrow (N,h)$ be an Riemannian submersion (cf. §6). Then we can choose an orthonormal local frame $\{e_i\}_{i=1}^{m}$ on M such that $e_i = e_i'$, $1 \leqq i \leqq n$, and $e_i = 0$, $n+1 \leqq i \leqq m$, where $m = \dim M$, $n = \dim N$ and $\{e_i'\}_{i=1}^{n}$ is an orthonormal local frame on N. Then the Ricci curvature of (N,h), $Ric_N(v)$, $v \in T_{\phi(x)}N$, is by definition $\sum_{i=1}^{m} h({}^N R(\phi_* e_i, v)\phi_* e_i, v)/h(v,v)$. Therefore, since ϕ is surjective, we have

$$(1.14) \qquad {}^N R^{\phi} = \sup_{N} Ric_N \qquad \text{and} \qquad {}^N R^{\phi}_{\Omega} = \sup_{\phi(\Omega)} Ric_N \, .$$

§2. The Index and the Nullity of a Harmonic Map from a Closed Manifold.

2.1. Method of Bérard and Gallot. At first, let us recall a method of Bérard and Gallot (cf.[B.G]) how to give estimations of Betti number, dimension of the moduli space of Einstein metrics, and dimension of harmonic spinors. Here let us apply their method to estimate the index and the nullity of a harmonic map.

Let (M,g) be a complete Riemannian manifold of dimension m, and E, a vector bundle over M with an inner product $\langle\cdot,\cdot\rangle$ and a connection $\tilde{\nabla}$ compatible with respect to $\langle\cdot,\cdot\rangle$, that is,

$$\nabla_X\langle s,s'\rangle = \langle\tilde{\nabla}_X s, s'\rangle + \langle s, \tilde{\nabla}_X s'\rangle \quad , \quad X\in\Gamma(TM),\ s\ ,\ s'\in\Gamma(E).$$

Then we can define the rough Laplacian $\bar{\Delta}$ on E in such a way that

$$(2.1) \qquad \bar{\Delta}\, s := \sum_{i=1}^{m}\left\{\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i} s - \tilde{\nabla}_{\nabla_{e_i} e_i} s\right\} \quad , \quad s\in\Gamma(E),$$

where $\{e_i\}_{i=1}^{m}$ is an orthonormal local frame field on M. In case that M is a closed manifold, the eigenvalue problem

$$-\bar{\Delta}s = \lambda\ s\ , \quad s\in\Gamma(E),$$

has a discrete spectrum : $\overline{\lambda_1} \leqq \overline{\lambda_2} \leqq \cdots \leqq \overline{\lambda_i} \leqq \cdots$. Consider the zeta function $\bar{Z}_E(t) := \sum_{i=1}^{\infty} e^{-t\overline{\lambda_i}}$, $t>0$. And let $0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_i \leqq \cdots$ be the spectrum of the Laplace-Beltrami operator Δ_M acting on $C^\infty(M)$. Then we can compare $\bar{Z}_E(t)$ with $Z_M(t) := \sum_{i=0}^{\infty} e^{-t\lambda_i}$:

Theorem (H.Hess, R.Schrader and D.A.Uhlenbrock [H.S.U])

$$\bar{Z}_E(t) \leqq \ell\, Z_M(t) \quad , \quad t>0.$$

Here ℓ is the rank of the vector bundle E.

Now our situation is as follows : The vector bundle E is the induced bundle $\phi^{-1}TN$ over M by a harmonic map ϕ; $(M,g)\longrightarrow(N,h)$. And the Jacobi operator J_ϕ ; $\Gamma(E)\longrightarrow\Gamma(E)$ is of the form (cf.(1.4)) :

$$(2.2) \qquad J_\phi V = -\bar{\Delta} V - \sum_{i=1}^{m} {}^N R(\phi_* e_i, V)\phi_* e_i, \quad V\in\Gamma(E).$$

Here $\bar{\Delta}$ is the rough Laplacian on the bundle $E = \phi^{-1}TN$ and the operator of $\Gamma(E)$ defined by $V\longmapsto\sum_{i=1}^{m} {}^N R(\phi_* e_i, V)\phi_* e_i$, becomes a bundle map of E. Therefore, letting $\tilde{\lambda}_1 \leqq \tilde{\lambda}_2 \leqq \ldots \leqq \tilde{\lambda}_i \leqq \ldots$ be the spectrum of J_ϕ, we have

$$(2.3) \qquad \tilde{\lambda}_i \geqq \overline{\lambda_i} - {}^N R^\phi, \quad i = 1,2,\ldots,$$

by (2.2), definition of ${}^N R^\phi$, and Mini-max Principle of the eigenvalue problem of the elliptic operators. Since Index(ϕ) +Nullity(ϕ) is the number of the non-positive eigenvalues of J_ϕ,

$$(2.4) \qquad \text{Index}(\phi)+\text{Nullity}(\phi) \leqq \sum_{i=1}^{\infty} e^{-t\tilde{\lambda}_i} \leqq e^{t\,{}^N R^\phi}\sum_{i=1}^{\infty} e^{-t\overline{\lambda_i}}, \quad t>0,$$

by (2.3). Here, using a theorem of Hess, Schrader, and Uhlenbrock, we have

$$(2.5) \qquad \text{Index}(\phi)+\text{Nullity}(\phi) \leqq n\, e^{t\,{}^N R} Z_M(t), \quad t>0,$$

noting that the rank of E coincides with $\dim N = n$:

Proposition 2.1. Let M be a closed manifold, and ϕ; $(M,g)\longrightarrow(N,h)$, a harmonic map. Then we have

$$(2.6) \qquad \text{Index}(\phi)+\text{Nullity}(\phi) \leqq n\,\text{Inf}\left\{ e^{t\,{}^N R^\phi} Z_M(t);\ 0<t<\infty\right\}$$

where $n = \dim N$, ${}^N R^\phi$ is the quantity in §1, and $Z_M(t)$ is the trace of the heat kernel of Δ_M acting on $C^\infty(M)$.

Corollary 2.2. The situations are preserved as in Proposition 2.1. Then we have :

(i) $N_R^{\phi} \leqq 0 \Longrightarrow \mathrm{Index}(\phi) = 0$ and $\mathrm{Nullity}(\phi) \leqq n$,

(ii) $N_R^{\phi} < 0 \Longrightarrow \mathrm{Index}(\phi) = \mathrm{Nullity}(\phi) = 0.$

In fact, in the inequality

$$(2.7) \qquad \sum_{i=1}^{\infty} e^{-t\tilde{\lambda}_i} \leqq n\, e^{t N_R^{\phi}} Z_M(t) \quad , \quad t > 0,$$

the assumption $N_R^{\phi} \leqq 0$ implies that the right hand side has a limit smaller than or equal to n as t tends to infinity since $Z_M(t)$ goes to 1 as t goes to infinity. Therefore each eigenvalue $\tilde{\lambda}_i$ of J_ϕ must be non-negative, i.e., $\mathrm{Index}(\phi) = 0$. Moreover, the left hand side of (2.7) is bigger than or equal to $\mathrm{Index}(\phi)+\mathrm{Nullity}(\phi) = \mathrm{Nullity}(\phi)$ for each $t > 0$. Therefore $\mathrm{Nullity}(\phi) \leqq n$. If we assume $N_R^{\phi} > 0$, then the right hand side of (2.7) goes to 0 as t tends to infinity. Therefore we have $\mathrm{Index}(\phi) = \mathrm{Nullity}(\phi) = 0.$

Therefore the problem is reduced to give the estimation of $Z_M(t)$.

2.2. Case of a Domain. The above procedure works well in the case of the Dirichlet eigenvalue problem for a relatively compact domain Ω in a complete Riemannian manifold (M,g).

Certainly, let

$$\bar{\lambda}_1(\Omega) \leqq \bar{\lambda}_2(\Omega) \leqq \cdots \leqq \bar{\lambda}_i(\Omega) \leqq \cdots$$

be the spectrum of the Dirichlet eigenvalue problem of the rough Laplacian $\bar{\Delta}$ (2.1) of a vector bundle E with an inner product $\langle\cdot,\cdot\rangle$ and a connection $\tilde{\nabla}$ compatible with respect to $\langle\cdot,\cdot\rangle$:

$$\begin{cases} -\bar{\Delta} s = \lambda\, s & \text{on } \Omega , \\ \ \ s = 0 & \text{on } \partial\Omega , \end{cases}$$

where s is a section of E on the closure $\bar{\Omega}$ of Ω. Consider the zeta function $\bar{Z}_{E,\Omega}(t)$ defined by

$$\bar{Z}_{E,\Omega}(t) := \sum_{i=1}^{\infty} e^{-t\,\bar{\lambda}_i(\Omega)} \quad , \ t > 0.$$

Similarly, let

$$\lambda_1(\Omega) \leqq \lambda_2(\Omega) \leqq \quad \dots \leqq \lambda_i(\Omega) \leqq \dots ,$$

be the spectrum of the Dirichlet eigenvalue problem of the Laplace-Beltrami operator Δ_M for the domain Ω, and $Z_\Omega(t)$ be the zeta function defined by

$$(2.8) \qquad Z_\Omega(t) := \sum_{i=1}^{\infty} e^{-t\,\lambda_i(\Omega)} \quad , \quad t > 0.$$

Then we have the analogue of a theorem of Hess, Schrader and Uhlenbrock :

Theorem 2.3.

$$(2.9) \qquad \bar{Z}_{E,\Omega}(t) \leqq \ell\, Z_\Omega(t) \quad , \qquad t > 0 ,$$

where ℓ is the rank of E.

Proof. It can be proved in the similar way as the proof in [B.G]. Assume that $s(t,x) \in E_x$, $t > 0$, $x \in \bar{\Omega}$, satisfies the heat equation with the Dirichlet boundary condition :

$$\begin{cases} (\frac{\partial}{\partial t} - \bar{\Delta})\, s(t,x) = 0 & \text{on} \quad (0,\infty)\times\Omega , \\ \qquad\qquad s(t,x) = 0 & \text{on} \quad (0,\infty)\times\partial\Omega . \end{cases}$$

For each $\varepsilon > 0$, let $f_\varepsilon := (\,|s|^2 + \varepsilon^2)^{1/2}$ on $(0,\infty)\times\bar{\Omega}$. Then it can be proved by the same way as in [H.S.U] that

$$\langle -\bar{\Delta} s, s \rangle \leqq f_\varepsilon\,(-\Delta_M\, f_\varepsilon) \qquad \text{on} \quad (0,\infty)\times\Omega .$$

Therefore f_ε satisfies

$$(\frac{\partial}{\partial t} - \Delta_M)\ f_\varepsilon \leqq 0 \qquad \text{on } (0,\infty)\times\Omega .$$

Then we can apply f_ε to the following Maximum Principle of heat kernel

Theorem (Maximum Principle) Let Ω be a relatively compact domain in M, and let $0 < T < \infty$. Assume that u is a real valued continuous function on $[0,T]\times\Omega$ and satisfies the inequality :

$$\frac{\partial}{\partial t} u - \Delta_M u \leqq 0 \qquad \text{on } (0,T)\times\Omega .$$

Then u attains its maximum on the set $\{0\}\times\Omega$ or $[0,T]\times\partial\Omega$.

For proof, see [F,p.2o4].

Then , if $f_\varepsilon(0,x) \leqq f(0,x)+\varepsilon$, then $f_\varepsilon(t,x) \leqq f(t,x)+\varepsilon$. Hence for every integrable section s of E on $\overline{\Omega}$ with the Dirichlet condition $s = 0$ on $\partial\Omega$, we have

$$(2.10) \qquad |(e^{t\overline{\Delta}} s)(x)| \leqq (e^{t\Delta_M}|s|)(x).$$

Therefore applying $s(z) = \sum_{i=1}^{\ell} \delta_{z,y}\, u_j(z)$ to (2.10), where $\delta_{z,y}$ is the Dirac function at y and $\{u_j(z)\}_{j=1}^{\ell}$ is an orthonormal basis of the fiber E_z at each point z in M, and noting $|s(z)| = \ell\,\delta_{z,y}$, we have the desired inequality (2.9). Q.E.D.

We denote the spectrum of the Dirichlet eigenvalue problem of $\mathbb{J}_\phi$ on Ω by

$$(2.11) \qquad \widetilde{\lambda}_1(\Omega) \leqq \widetilde{\lambda}_2(\Omega) \leqq \cdots \leqq \widetilde{\lambda}_i(\Omega) \leqq \cdots ,$$

and define $\widetilde{Z}_\Omega(t) := \sum_{i=1}^{\infty} e^{-t\widetilde{\lambda}_i(\Omega)}$. Then by the similar way as 2.1, we have :

Proposition 2.4. Let Ω be a relatively compact domain in a complete Riemannian manifold (M,g). Let $\phi ; (M,g) \longrightarrow (N,h)$ be a harmonic map. Then we have

$$(2.12) \quad \mathrm{Index}_{\Omega}(\phi)+\mathrm{Nullity}_{\Omega}(\phi) \leqq \widetilde{Z}_{\Omega}(t) \leqq n \,\mathrm{Inf}\left\{e^{t\,{}^{N}R_{\Omega}^{\phi}}\, Z_{\Omega}(t) ; 0<t<\infty\right\},$$

where $n = \dim N$, ${}^{N}R_{\Omega}^{\phi}$ is defined in §1 , and $Z_{\Omega}(t)$ is the zeta function of the Dirichlet eigenvalue of Δ_M on Ω defined by (2.8).

2.3. To apply Proposition 2.1, we make use of the following proposition obtained also by Bérard and Gallot [B.G] :

Proposition (P.Bérard and S.Gallot) Let (M,g) be a closed Riemannian manifold whose Ricci curvature Ric_M is bounded below by a positive constant : $\mathrm{Ric}_M \geqq (m-1)\delta > 0$. Then the trace $Z_M(t)$ of the heat kernel of (M,g) is estimated as

$$(2.13) \quad Z_M(t) \leqq Z_{S^m}(\delta t) ,$$

where $m = \dim M$ and $Z_{S^m}(t)$ is the trace of the heat kernel of the standard unit sphere (S^m, can) of constant curvature 1.

It is known (cf. [B.G.M]) that if $m \geqq 2$,

$$Z_{S^m}(t) = \sum_{k=0}^{\infty} m_k \, e^{-tk(k+m-1)} , \quad t>0 ,$$

where $m_k = \frac{(m+k-2)!}{k!(m-1)!}(m+2k-1)$, $k = 0,1,2,\ldots$.

Then the function $Z_{S^m}(t)$ is estimated as follows :

(i) In case of $m \geqq 3$,

$$(2.14) \quad Z_{S^m}(t) \leqq 1+\sum_{k=1}^{\infty} (mk)^{m-1} e^{-tmk}$$

$$\leqq 1+(m-1)!\,m^{m-1}e^{-tm}(1-e^{-tm})^{-m} ,$$

(ii) In case of $m = 2$,

(2.14') $$Z_{S^2}(t) = \sum_{k=0}^{\infty} (2k+1)\, e^{-tk(k+1)}$$

$$\leqq 1+2 \sum_{k=2}^{\infty} k\, e^{-tk} \leqq 1+2\, e^{-2t}(2-e^{-t})(1-e^{-t})^{-2} .$$

Therefore combining (2.13) with (2.14), we have

(i) in case of $m \geqq 3$,

(2.15) $$\mathrm{Inf}\left\{e^{t\, {}^N R^{\phi}}\, Z_M(t) \;;\; 0<t<\infty\right\}$$

$$\leqq \mathrm{Inf}\left\{e^{t\, {}^N R^{\phi}/\delta}\left\{1+(m-1)!\,m^{m-1} e^{-tm}(1-e^{-tm})^{-m}\right\};\; 0<t<\infty\right\}.$$

Puting $A = {}^N R^{\phi}/m\delta$ and $e^t = 1+\frac{1}{A}$,

the right hand side of (2.15) $\leqq (1+\frac{1}{A})^A\left\{1+(m-1)!\,m^{m-1} A(1+A)^{m-1}\right\}$.

(ii) In case of $m = 2$,

(2.15') $$\mathrm{Inf}\left\{e^{t\, {}^N R^{\phi}} Z_M(t);0<t<\infty\right\} \leqq \mathrm{Inf}\left\{e^{t\, {}^N R^{\phi}/\delta}\left\{1+2e^{-2t}(2-e^{-t})(1-e^{-t})^{-2}\right\} \;;\; 0 < t < \infty\right\} .$$

Letting $B = {}^N R^{\phi}/\delta$ and $e^t = 1+\frac{1}{B}$,

the right hand side of (2.15') $\leqq (1+\frac{1}{B})^B\left\{1+4B^2\right\}$.

Therefore together with (2.6), we have :

Theorem 2.5. Let (M,g) be a closed Riemannian manifold of dimension $m \geqq 2$ whose Ricci curvature Ric_M is bounded below by a positive constant : $Ric_M \geqq (m-1)\delta > 0$. Let ϕ ; $(M,g) \longrightarrow (N,h)$ be a harmonic map of (M,g) into an arbitrary Riemannian manifold (N,h) of dimension n. Then we have :

(i) In case of $m \geqq 3$,

$$\text{Index}(\phi)+\text{Nullity}(\phi) \leqq n \left(1+\frac{1}{A}\right)^{A} \left\{ 1+(m-1)!\, m^{m-1} A(1+A)^{m-1} \right\},$$

where $A := {}^{N}R^{\phi}/m\delta$ and ${}^{N}R^{\phi}$ is the quantity in §1.

(ii) In case of $m = 2$,

$$\text{Index}(\phi)+\text{Nullity}(\phi) \leqq n \left(1+\frac{1}{B}\right)^{B} \left\{ 1+4B^{2} \right\} ,$$

where $B := {}^{N}R^{\phi}/\delta$.

Remark. The function $(1+\frac{1}{x})^{x}$, $x > 0$, satisfies $\lim_{x\to 0}(1+\frac{1}{x})^{x} = 1$, $(1+\frac{1}{x})^{x} < e$, and $\lim_{x\to\infty}(1+\frac{1}{x})^{x} = e$. Therefore, when the quantity ${}^{N}R^{\phi}$ goes to zero, the bounds of the above inequalities in Theorem 2.5 tend to n . In the case that ϕ is the identity map of the n-dimensional flat torus, $\text{Index}(\phi) = 0$ and $\text{Nullity}(\phi) = n$. That is, the above estimate is optimal when ${}^{N}R^{\phi}$ goes to zero.

By the way let us consider the case $M = S^{1} = \mathbb{R}/2\pi\mathbb{Z}$. In this case, we know that

$$Z_{S^1}(t) = 1 + \sum_{k=1}^{\infty} e^{-tk^{2}} .$$

Then we have the estimation of $Z_{S^1}(t)$ ba the same way :

(i) $Z_{S^1}(t) \leqq \dfrac{1+e^{-t}}{1-e^{-t}}$, $t > 0$, and

(ii) $Z_{S^1}(t) \leqq 1+\sqrt{\dfrac{\pi}{t}}$, $t > 0$,

and we have :

Proposition 2.6. Let $\phi ; [0,2\pi] \longrightarrow (N,h)$ be a closed geodesic, that is, $\dot{\phi}(0) = \dot{\phi}(2\pi)$ for the tangent vectors at $\phi(0) = \phi(2\pi)$, in an arbitrary Riemannian manifold (N,h) of dimension n . Then

$$\mathrm{Index}(\phi)+\mathrm{Nullity}(\phi) \leqq n \left(1+\frac{1}{C}\right)^C \mathrm{Min}\left\{ 1+2C,\ 1+\sqrt{\pi}\sqrt{1+C} \right\},$$

where $C := {}^N R^\phi$ defined in §1. In particular, assuming that the sectional curvature ${}^N K$ of (N,h) is bounded above by a positive constant : ${}^N K \leqq a$, the index and the nullity of a closed geodesic $\phi ; [0,2\pi] \longrightarrow (N,h)$ of (N,h) satisfies

$$(2.16) \quad \mathrm{Index}(\phi)+\mathrm{Nullity}(\phi) \leqq n\, e \left\{ 1+\frac{L^2 a}{2\pi^2} \right\}.$$

Remark. The estimate (2.16) is far from the optimal estimate obtained by Morse-Schönberg (cf.[G.K.M]).

2.4. Minimal Isometric Immersions. Let us consider an isometric immersion $\phi ; (M,g) \longrightarrow (N,h)$. Then it is known (cf.[E.S],[E.L]) that ϕ is harmonic if and only if ϕ is minimal. The second variational formula of a volume for an isometric minimal immersion is as follows (cf. [Si]) : Let $F := TM^\perp$ be the normal bundle of M in N which is a subbundle of $E = \phi^{-1}TN$. For a section $V \in \Gamma(F)$, let ϕ_t be a smooth variation of ϕ with $\phi_0 = \phi$ and $V_x = \frac{d}{dt}\phi_t(x)\big|_{t=0}$ $x \in M$. Then

$$\frac{d^2}{dt^2} \mathrm{Vol}(M, \phi_t^* h)\Big|_{t=0} = \int_M h(L_\phi V, V) * 1 .$$

The operator $L_\phi ; \Gamma(F) \longrightarrow \Gamma(F)$ is a second order elliptic differential operator of the form :

$$(2.17) \quad L_\phi V = -\nabla^{\perp 2} V - \mathcal{B}(V) - R^\perp(V), \quad V \in \Gamma(F),$$

where $\nabla^{\perp 2}$ is the rough Laplacian on F given by

$$\nabla^{\perp 2} V := \sum_{i=1}^m \left(\nabla^\perp_{e_i} \nabla^\perp_{e_i} V - \nabla^\perp_{\nabla_{e_i} e_i} V \right),$$

and $\nabla^\perp_X V$ is the normal component of the connection ${}^N\nabla_X V$, $X \in \Gamma(TM)$, $V \in \Gamma(F)$. The operator $\mathcal{B} ; \Gamma(F) \longrightarrow \Gamma(F)$ is defined by

$$\mathcal{B}(V) := \sum_{i=1}^{m} B_{e_i, A^V e_i} ,$$

where B is the second fundamental form of ϕ defined by $B_{X,Y} := ({}^N\nabla_X Y)^{\perp}$, the normal component of ${}^N\nabla_X Y$, $X, Y \in \Gamma(TM)$, and A^V ; $\Gamma(TM) \longrightarrow \Gamma(TM)$ is defined by $h(B_{X,Y}, V) = g(A^V X, Y)$. The operator R^{-} ; $\Gamma(F) \longrightarrow \Gamma(F)$ is the normal component of $\sum_{i=1}^{m} {}^N R(e_i, V) e_i$.

Note that for our Jacobi operator J_ϕ, its normal component $(J_\phi V)^{\perp}$, satisfies

$$(2.18) \qquad (J_\phi V)^{\perp} = -\nabla^{\perp 2} V + \mathcal{B}(V) - R^{\perp}(V), \quad V \in \Gamma(F).$$

Definition 2.7. (i) We denote by S-Index(ϕ) the sum of the multiplicities of the negative eigenvalues of L_ϕ on $\Gamma(F)$, and by S-Nullity(ϕ) the dimension of the kernel of L_ϕ on $\Gamma(F)$.
(ii) Let β (resp. $r^{\perp}$) be the supremum of the maximal eigenvalues of the endomorphism $\mathcal{B}$ (resp. $R^{\perp}$) of the fiber F_x of F where x varies over M.

Note that under the assumption that the sectional curvature ${}^N K$ of (N,h) is bounded above by a positive constant : ${}^N K \leq a$, we have

$$(2.19) \qquad r^{\perp} \leq ma ,$$

where $m = \dim M$ (cf. Lemma 1.2 and Example 1.3). And note that

$$(2.20) \qquad \beta \leq \sup_{x \in M} \sum_{i,j=1}^{m} h(B_{e_i, e_j}, B_{e_i, e_j}).$$

Then by the same way as 2.1 and 2.3, we have :

Proposition 2.8. Let (M,g) be a closed Riemannian manifold and ϕ; $(M,g) \longrightarrow (N,h)$ an isometric minimal immersion. Then

$$\text{S-Index}(\phi)+\text{S-Nullity}(\phi) \leqq (n-m)\ \text{Inf}\left\{ e^{t(\beta+r^{\perp})}\ Z_M(t)\ ;\ 0<t<\infty \right\},$$

where $m = \dim M$, $n = \dim N$, β and $r^{\perp}$ are defined in Definition 2.7, and $Z_M(t)$ is the trace of the heat kernel of the Laplace-Beltrami operator Δ_M of (M,g).

Proposition 2.9. Let (M,g) be a closed Riemannian manifold of dimension $m \geqq 2$ whose Ricci curvature Ric_M is bounded below by a positive constant : $\text{Ric}_M \geqq (m-1)\delta > 0$. Let $\phi ; (M,g) \longrightarrow (N,h)$ be an isometric minimal immersion of (M,g) into an arbitrary Riemannian manifold of dimension n whose sectional curvature ${}^N K$ is bounded above by a positive constant : ${}^N K \leqq a$. Then

(i) In case of $m \geqq 3$,

$$\text{S-Index}(\phi)+\text{S-Nullity}(\phi) \leqq (n-m)(1+\tfrac{1}{A'})^{A'} \left\{ 1+(m-1)!\,m^{m-1}A'(1+A')^{m-}\right.$$

where $A' := (\beta+ma)/m\delta$.

(ii) In case of $m = 2$,

$$\text{S-Index}(\phi)+\text{S-Nullity}(\phi) \leqq (n-2)(1+\tfrac{1}{B'})^{B'} \left\{ 1+4B'^2 \right\},$$

where $B' := (\beta+2a)/\delta$.

Proposition 2.10. Let $\phi ; [0,2] \longrightarrow (N,h)$ be any closed geodesi in an arbitrary Riemannian manifold (N,h). Then

$$\text{S-Index}(\phi)+\text{S-Nullity}(\phi) \leqq (n-1)(1+\tfrac{1}{C})^{C}\ \min\left\{1+2C,\ 1+\sqrt{\pi}\sqrt{1+C}\right\},$$

where $C = {}^N R^{\phi}$ defined in §1. In particular, assume that the sectional survature ${}^N K$ of (N,h) is bounded above by a positive constant : ${}^N K \leqq a$. Then

$$\text{S-Index}(\phi)+\text{S-Nullity}(\phi) \leqq (n-1)\ e\left\{ 1+\frac{L^2 a}{2\pi^2} \right\}.$$

§3. The Index and the Nullity of a Harmonic Map from a Domain.

3.1. We retain the notations as in 2.2. We have :

Theorem 3.1. Let Ω be a relatively compact domain in a complete Riemannian manifold (M,g), $\phi ; (M,g) \longrightarrow (N,h)$, a harmonic map of (M,g) into an arbitrary Riemannian manifold (N,h) of dimension n. Then

(i) $\lambda_1(\Omega) \geqq {}^N R^{\phi}_{\Omega} \Longrightarrow \mathrm{Index}_{\Omega}(\phi) = 0$ and $\mathrm{Nullity}_{\Omega}(\phi) \leqq n$,

(ii) $\lambda_1(\Omega) > {}^N R^{\phi}_{\Omega} \Longrightarrow \mathrm{Index}_{\Omega}(\phi) = \mathrm{Nullity}_{\Omega}(\phi) = 0.$

That is, if $\lambda_1(\Omega) \geqq {}^N R^{\phi}_{\Omega}$, then the harmonic map $\phi ; (M,g) \longrightarrow (N,h)$ is stable on Ω.

Proof. By Proposition 2.4, the zeta function $\widetilde{Z}_{\Omega}(t) = \sum_{i=1}^{\infty} e^{-t\widetilde{\lambda}_i(\Omega)}$ of J_{ϕ} on Ω satisfies

$$\widetilde{Z}_{\Omega}(t) \leqq n\, e^{t\, {}^N R^{\phi}_{\Omega}} Z_{\Omega}(t) = n\, e^{t({}^N R^{\phi}_{\Omega} - \lambda_1(\Omega))}\left\{1 + \sum_{i=2}^{\infty} e^{t(\lambda_1(\Omega) - \lambda_i(\Omega))}\right\},$$

where $\lambda_1(\Omega) \leqq \lambda_2(\Omega) \leqq \dots \leqq \lambda_i(\Omega) \leqq \dots$ is the spectrum of the Dirichlet eigenvalue problem of the Laplace-Beltrami operator Δ_M on Ω. Noting the fact that $\lambda_i(\Omega) > \lambda_1(\Omega)$, $i=2,3,\dots$, the assumption ${}^N R^{\phi}_{\Omega} \leqq \lambda_1(\Omega)$ implies that the limit of the right hand side of the above inequality is less than or equal to n when $t \longrightarrow \infty$. Then $\mathrm{Index}_{\Omega}(\phi) = 0$ and $\mathrm{Nullity}_{\Omega}(\phi) \leqq n$. If ${}^N R^{\phi}_{\Omega} < \lambda_1(\Omega)$, the limit of the right hand side of the inequality is zero when $t \longrightarrow \infty$. Therefore $\mathrm{Index}_{\Omega}(\phi) = \mathrm{Nullity}_{\Omega}(\phi) = 0$. Q.E.D.

Corollary 3.2. Let $B_r(o)$ be a geodesic ball with radius r whose center is a certain point o in the [standard unit sphere / m-dimensional]

(S^m, can) of constant curvature one. We choose the radius r with $0 < r < \pi/2$ in such a way that $\lambda_1(B_r(o)) = m-1$. Then, for every domain Ω in S^m whose volume $Vol(\Omega)$ is less than or equal to the volume $Vol(B_r(o))$, the identity map $id; (S^m, can) \longrightarrow (S^m, can)$ is stable on Ω.

Proof. By Example 1.4, we have ${}^N R^{\phi}_{\Omega} = m-1$ for every domain Ω in S^m. In this case, Theorem 3.1 implies that, if $\lambda_1(\Omega) \geqq m-1$, then the identity map $\phi = id; (S^m, can) \longrightarrow (S^m, can)$ is stable on Ω. By a theorem of P.Bérard and D.Meyer (cf.[B.M]), if $Vol(\Omega) \leqq Vol(B_r(o))$, then $\lambda_1(\Omega) \geqq \lambda_1(B_r(o)) = m-1$. Q.E.D.

It is known that (cf. [C.L], [B.G], [U2]) that there exists a positive constant $C(M,g)$ depending only on (M,g) such that the eigenvalues $\lambda_i(\Omega)$ of the Dirichlet eigenvalue problem of the Laplace-Beltrami operator Δ_M on the domain Ω satisfy

$$(3.1) \qquad \lambda_i(\Omega) \geqq C(M,g)\, Vol(\Omega)^{-2/m}\, i^{2/m}, \quad i=1,2,\ldots,$$

where $m = \dim M$. In particular,

$$(3.2) \qquad \lambda_1(\Omega) \geqq C(M,g)\, Vol(\Omega)^{-2/m}.$$

Then the above Theorem 3.1 implies that

Corollary 3.3. Let Ω be a relatively compact domain in a complete Riemannian manifold (M,g), and $\phi; (M,g) \longrightarrow (N,h)$, a harmonic map. Then

$$C(M,g)\, Vol(\Omega)^{-2/m} \geqq {}^N R^{\phi}_{\Omega} \implies \phi \text{ is stable on } \Omega.$$

In particular, assume that the sectional curvature ${}^N K$ of (N,h) is bounded above by a positive constant : ${}^N K \leqq a$. Then

$$C(M,g)\ \mathrm{Vol}(\Omega)^{-2/m} \geqq 2aE^{\infty}(\Omega,\phi) \Longrightarrow \phi \text{ is stable on } \Omega .$$

If Ω is " small " in (M,g), then $\mathrm{Vol}(\Omega)^{-2/m}$ tends to infinity and ${}^{N}R^{\phi}_{\Omega}$ remains still bounded. Therefore Corollary 3.3 implies that a harmonic map $\phi\ ;\ (M,g)\longrightarrow(N,h)$ is stable on a " sufficiently small" domain Ω in M .

3.2. In this part, we estimate $\mathrm{Index}_{\Omega}(\phi)$ and $\mathrm{Nullity}_{\Omega}(\phi)$. By Proposition 2.4 and (3.1), we have

$$\begin{aligned}\mathrm{Index}_{\Omega}(\phi)+\mathrm{Nullity}_{\Omega}(\phi) &\leqq n\ \mathrm{Inf}\left\{e^{t\,{}^{N}R^{\phi}_{\Omega}}\ Z_{\Omega}(t)\ ;\ 0<t<\infty\right\}\\ &\leqq n\ \mathrm{Inf}\left\{e^{t\,{}^{N}R^{\phi}_{\Omega}} \sum_{k=1}^{\infty} e^{-tC(M,g)\mathrm{Vol}(\Omega)^{-2/m}k^{2/m}}\ ;\ 0<t<\infty\right\}\\ &\leqq n\ \mathrm{Inf}\left\{e^{\frac{a}{b}t} \sum_{k=1}^{\infty} e^{-tk^{2/m}}\ ;\ 0<t<\infty\right\} ,\end{aligned}$$

where we put $m = \dim M$, $n = \dim N$, $a := {}^{N}R^{\phi}_{\Omega}$, and $b := C(M,g)\ \mathrm{Vol}(\Omega)^{-2/m}$. In case of $a \leqq b$, we have Corollary 3.3. So we assume $a > b$. We put $\frac{a}{b} = 1+D$, $D>0$. We express as

$$(3.3)\qquad e^{\frac{a}{b}t} \sum_{k=1}^{\infty} e^{-tk^{2/m}} = e^{(\frac{a}{b}-1)t} \sum_{k=1}^{\infty} e^{-(k^{2/m}-1)t} .$$

(i) In case of $m = 1,2$,

$$\begin{aligned}\text{the right hand side of (3.3)} &\leqq e^{(\frac{a}{b}-1)t} \sum_{k=0}^{\infty} e^{-tk}\\ &= e^{(\frac{a}{b}-1)t}(1-e^{-t})^{-1} .\end{aligned}$$

Putting $e^{t} = 1+\frac{1}{D}$, we have

$$\mathrm{Inf}\left\{e^{\frac{a}{b}t} \sum_{k=1}^{\infty} e^{-tk^{2/m}}\ ;\ 0<t<\infty\right\} \leqq (1+\tfrac{1}{D})^{D}(1+D).$$

(ii) In case of $m \geqq 3$,

$$\sum_{k=1}^{\infty} e^{-t(k^{2/m}-1)} = 1+e^t \sum_{k=2}^{\infty} e^{-tk^{2/m}}$$

$$\leqq 1+e^t \int_1^{\infty} e^{-tx^{2/m}} dx$$

$$= 1+\frac{m}{2} t^{-m/2} \int_t^{\infty} z^{\frac{m}{2}-1} e^{-z} dz$$

$$\leqq \begin{cases} 1+\frac{m}{2} t^{-m/2} p! e^{-t} \sum_{k=0}^{p} \frac{t^k}{k!}, & \text{if } m = 2(p+1),\ p \geqq 1 \\ 1+\frac{m}{2} t^{-(m+1)/2} p! e^{-t} \sum_{k=0}^{p} \frac{t^k}{k!}, & \text{if } m=2p+1,\ p \geqq 1. \end{cases}$$

Putting $e^t = 1+\frac{1}{D}$, we have

$$\operatorname{Inf}\left\{ e^{\frac{a}{b}t} \sum_{k=1}^{\infty} e^{-tk^{2/m}} ;\ 0<t<\infty \right\} \leqq \begin{cases} (1+\frac{1}{D})^D \{1+P(D)\}, & \text{if } m=2(p+1), p \geqq 1 \\ (1+\frac{1}{D})^D \{1+Q(D)\}, & \text{if } m=2p+1,\ p \geqq 1, \end{cases}$$

where

$$(3.4) \qquad P(D) := (p+1)! \sum_{k=0}^{p} \frac{1}{k!} \left\{ \frac{1}{\log(1+\frac{1}{D})} \right\}^{p+1-k}, \quad \text{if } m=2(p+1), p \geqq$$

$$(3.5) \qquad Q(D) := \frac{m}{2}\, p! \sum_{k=0}^{p} \frac{1}{k!} \left\{ \frac{1}{\log(1+\frac{1}{D})} \right\}^{p+1-k}, \quad \text{if } m=2p+1,\ p \geqq 1.$$

(iii) We can give another estimate of $\operatorname{Index}_{\Omega}(\phi)$ and $\operatorname{Nullity}_{\Omega}(\phi)$
In fact, we have

$$\sum_{k=1}^{\infty} e^{-tk^{2/m}} \leqq \int_0^{\infty} e^{-tx^{2/m}} dx = \Gamma(\tfrac{m}{2}+1)\ t^{-m/2} .$$

Therefore we obtain

$$\operatorname{Inf}\left\{ e^{\frac{a}{b}t} \sum_{k=1}^{\infty} e^{-tk^{2/m}} ;\ 0< t<\infty \right\} \leqq \frac{\Gamma(\frac{m}{2}+1) e^{m/2}}{(m/2)^{m/2}} \left(\frac{a}{b}\right)^{m/2} .$$

Summing up, we obtain :

Theorem 3.4. Let Ω be a relatively compact domain in a complete Riemannian manifold (M,g), and ϕ; $(M,g)\longrightarrow(N,h)$, a harmonic map. Then $\mathrm{Index}_{\Omega}(\phi)$ and $\mathrm{Nullity}_{\Omega}(\phi)$ are estimated by the quantity $D:= {}^{N}R^{\phi}_{\Omega}\, C(M,g)^{-1}\, \mathrm{Vol}(\Omega)^{2/m} -1$ as follows :

(i) In case of $m = 1,2$,

$$\mathrm{Index}_{\Omega}(\phi)+\mathrm{Nullity}_{\Omega}(\phi) \leqq n\ (1+\tfrac{1}{D})^{D}\left\{1+D\right\} ,$$

(ii) in case of $m = 2(p+1)$, $p \geqq 1$,

$$\mathrm{Index}_{\Omega}(\phi)+\mathrm{Nullity}_{\Omega}(\phi) \leqq n\ (1+\tfrac{1}{D})^{D}\left\{1+P(D)\right\} ,$$

(iii) in case of $m = 2p+1$, $p \geqq 1$,

$$\mathrm{Index}_{\Omega}(\phi)+\mathrm{Nullity}_{\Omega}(\phi) \leqq n\ (1+\tfrac{1}{D})^{D}\left\{1+Q(D)\right\} .$$

(iv) In all cases $m \geqq 1$,

$$\mathrm{Index}_{\Omega}(\phi)+\mathrm{Nullity}_{\Omega}(\phi) \leqq n\ \frac{(\frac{m}{2}+1)e^{m/2}}{(m/2)^{m/2}}\ (1+D)^{m/2} ,$$

where $P(D)$ and $Q(D)$ are the functions of D given by (3.4), (3.5), respectively , and $m = \dim M$, $n = \dim N$.

Remark. Since the function $f(D) = \dfrac{1}{\log(1+\frac{1}{D})}$ of D satisfies that $f(D)\longrightarrow 0$ as $D\longrightarrow 0$ and $f(D)\sim D$ as $D\rightarrow\infty$, the functions $P(D)$ and $Q(D)$ satisfy

$$\lim_{D\to 0} P(D) = \lim_{D\to 0} Q(D) = 0 \quad , \text{ and}$$

$$P(D)\sim (m/2)!\ D^{m/2} , \quad Q(D)\sim \tfrac{m}{2}(\tfrac{m-1}{2})!\ D^{(m+1)/2} \quad \text{as } D\longrightarrow\infty .$$

3.3. Minimal isometric immersions. We preserve the notations as in 2.4. For a relatively compact domain Ω in a complete Riemannian manifold (M,g), consider the Dirichlet eigenvalue problem of the operator L_ϕ acting sections of $F = TM^\perp$ on Ω :

$$\begin{cases} L_\phi V = \lambda V & \text{on } \Omega, \\ V = 0 & \text{on } \partial\Omega . \end{cases}$$

Definition 3.5. (i) We denote by $S\text{-Index}_\Omega(\phi)$ the sum of the multiplicities of the negative eigenvalues of this problem, and by $S\text{-Nullity}_\Omega(\phi)$, the dimension of the zero eigenspace. (ii) Let $\beta(\Omega)$ (resp. $r^\perp(\Omega)$) be the supremum of the maximal eigenvalue of the endomorphism $\mathfrak{B}$ (resp. $R^\perp$) of the bundle F over the domain Ω (cf. Definition 2.7). Note that $\beta(\Omega) \leqq \beta$ and $r^\perp(\Omega) \leqq r^\perp$ when the right hand sides are finite.

Then by the same reason as 2.4 and 3.2, we have a series of the following propositions :

Proposition 3.5. Let Ω be a relatively compact domain in a complete Riemannian manifold (M,g), and ϕ ; $(M,g) \longrightarrow (N,h)$, a minimal isometric immersion. Then

$$S\text{-Index}_\Omega(\phi) + S\text{-Nullity}_\Omega(\phi) \leqq (n-m) \operatorname{Inf}\left\{ e^{t(\beta(\Omega)+r^\perp(\Omega))} Z_\Omega(t) \; ; \; 0 < t < \infty \right\},$$

where $\beta(\Omega)$ and $r^\perp(\Omega)$ are given in Definition 3.5 (ii) and $Z_\Omega(t)$ is the zeta function of the Dirichlet eigenvalue problem of Δ_M on Ω.

Proposition 3.6. Under the same assumptions of Proposition 3.5,

(i) $\lambda_1(\Omega) \geqq \beta(\Omega) + r^\perp(\Omega) \Longrightarrow S\text{-Index}_\Omega(\phi) = 0$, $S\text{-Nullity}_\Omega(\phi) \leqq n-m$

(ii) $\lambda_1(\Omega) > \beta(\Omega)+r^{\perp}(\Omega) \Longrightarrow$ S-Index$_{\Omega}(\phi)$ = S-Nullity$_{\Omega}(\phi)$ = 0,

where $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet eigenvalue problem of Δ_M on Ω.

Proposition 3.7. Under the same assumptions of Proposition 3.5,

$$C(M,g)\ \mathrm{Vol}(\Omega)^{-2/m} \geqq \beta(\Omega)+r^{\perp}(\Omega) \Longrightarrow \phi \text{ is stable on } \Omega,$$

where $C(M,g)$ is the constant in (3.1).

Proposition 3.8. Under the same assumptions of Proposition 3.5, S-Index$_{\Omega}(\phi)$ and S-Nullity$_{\Omega}(\phi)$ are estimated by the quantity D' given by $D' := \{\beta(\Omega)+r^{\perp}(\Omega)\}\, C(M,g)^{-1}\ \mathrm{Vol}(\Omega)^{2/m} - 1$:

(i) In case of $m = 1,2,$

$$\text{S-Index}_{\Omega}(\phi)+\text{S-Nullity}_{\Omega}(\phi) \leqq (n-m)(1+\tfrac{1}{D'})^{D'}(1+D').$$

(ii) In case of $m = 2(p+1)$, $p \geqq 1$,

$$\text{S-Index}_{\Omega}(\phi)+\text{S-Nullity}_{\Omega}(\phi) \leqq (n-m)(1+\tfrac{1}{D'})^{D'}(1+P(D')),$$

(iii) in case of $m = 2p+1$, $p \geqq 1$,

$$\text{S-Index}_{\Omega}(\phi)+\text{S-Nullity}_{\Omega}(\phi) \leqq (n-m)(1+\tfrac{1}{D'})^{D'}(1+Q(D')) .$$

(iv) In all cases $m \geqq 1$,

$$\text{S-Index}_{\Omega}(\phi)+\text{S-Nullity}_{\Omega}(\phi) \leqq (n-m)\frac{\Gamma(\frac{m}{2}+1)e^{m/2}}{(m/2)^{m/2}}\ (1+D')^{m/2} ,$$

where the functions $P(\cdot)$, $Q(\cdot)$ are the same in Theorem 3.4 , $m = \dim M$ and $n = \dim N$.

Remark. (i) The similar ones as Proposition 3.7 were stated in [Mo],[H],[T2]. (ii) In case of $m = 1$, let $\phi\ ;[0,2\pi] \longrightarrow (N,h)$ be a geodesic in an arbitrary Riemannian manifold (N,h). The i-th eigenvalue $\lambda_i((0,2\pi))$ of the Dirichlet eigenvalue problem of the operator d^2/dx^2 on the interval $(0,2\pi)$ is $i^2/4$, $i=1,2,\ldots$. Then under the assumption that the sectional curvature ${}^N K$ of (N,h) is bounded above by a positive constant : ${}^N K \leqq a$, we have

$$D+1 = D'+1 \leqq L^2 a/\pi^2 ,$$

where L is the length of ϕ. Therefore

$$\text{(I)} \quad \text{Index}_\Omega(\phi)+\text{Nullity}_\Omega(\phi) \leqq \begin{cases} n\sqrt{\dfrac{e\pi}{2}}\, L\dfrac{\sqrt{a}}{\pi} , \\ n \quad , \text{ if } L \leqq \dfrac{\pi}{\sqrt{a}} + \varepsilon_n , \end{cases}$$

where ε_n is a positive constant depending only on $n = \dim N$ (cf. Theorem 3.4 (i),(iv)). And

$$\text{(II)} \quad \text{S-Index}_\Omega(\phi)+\text{S-Nullity}_\Omega(\phi) \leqq \begin{cases} (n-1)\sqrt{\dfrac{e\pi}{2}}\, L\dfrac{\sqrt{a}}{\pi} , \\ n-1 , \text{ if } L \leqq \dfrac{\pi}{\sqrt{a}} + \varepsilon_n , \end{cases}$$

(cf. Proposition 3.8 (i),(iv)). On the other hand, a theorem of M.Morse and I.Schönberg tells us that

$$\text{S-Index}_\Omega(\phi)+\text{S-Nullity}_\Omega(\phi) \leqq (n-1)\left[L\frac{\sqrt{a}}{\pi}\right] ,$$

where $[x]$ expresses the integer part of $x>0$ (cf. [G.K.M, pp.176,142]) When $L \leqq \dfrac{\pi}{\sqrt{a}} + \varepsilon_n$, our estimate (II) is optimal, but in general, it is far from the optimal one of Morse and Schönberg since $\sqrt{\dfrac{e\pi}{2}} = 2.066\cdots$.

Chapter II. Stability of the Identity Map.

§4. Kähler Version of Lichnérowicz-Obata Theorem.

In this chapter, we treat with the Jacobi operator of the identity map. Let (M,g) be a closed Riemannian manifold of dimension m. The identity map $id_M ; (M,g) \longrightarrow (M,g)$ of (M,g) is harmonic (cf. [E.S]), and the Riemannian manifold (M,g) is <u>stable</u> (cf. [Na]) if the identity map id_M is stable. The corresponding Jacobi operator $J := J_{id_M}$ is a differential operator acting on the space $\Gamma(TM)$ of all vector fields on M given by

$$(4.1) \qquad JV = - \sum_{i=1}^{m} (\nabla_{e_i}\nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V) - \rho(V), \qquad V \in \Gamma(TM),$$

where ∇ is the Levi-Civita connection of (M,g), $\rho(V) := \sum_{i=1}^{m} R(e_i, V)e_i$ and $\rho(U,V) := g(\rho(U),V) = \sum_{i=1}^{m} g(R(e_i,U)e_i,V)$ is the Ricci tensor (cf. [Ma], [Sm]). Under the identification of TM with T^*M with respect to the metric g, the Hodge Laplacian $\Delta = d\delta + \delta d$ on $\Gamma(T^*M)$ induces a differential operator, denoted by the same letter and called also as the Hodge Laplacian, on $\Gamma(TM)$, where δ is the codifferential operator of d with respect to the metric g on M. Then the Weitzenböck formula of the Hodge operator Δ tells us that

$$(4.2) \qquad \Delta V = - \sum_{i=1}^{m} (\nabla_{e_i}\nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V) + \rho(V), \qquad V \in \Gamma(TM),$$

and then

$$(4.3) \qquad J = \Delta - 2\rho .$$

Then we have immediately :

Lemma 4.1. Let $\lambda_1^1(M)$ (resp. $\lambda_1(M)$) be the first (resp. first non-zero) eigenvalue of the Hodge Laplacian (resp. the Laplace-

Beltrami operator Δ_M) on 1-forms (resp. smooth functions) on M. Then

(i) (M,g) is stable $\Longrightarrow$ $2 \operatorname{Inf} \operatorname{Ric}_M \leqq \lambda_1^1(M) \leqq \lambda_1(M)$,

(ii) $\lambda_1^1(M) \geqq 2 \operatorname{Sup} \operatorname{Ric}_M$ $\Longrightarrow$ (M,g) is stable ,

where $\operatorname{Inf} \operatorname{Ric}_M$ (resp. $\operatorname{Sup} \operatorname{Ric}_M$) is the infimum (resp. supremum) of the Ricci curvature of (M,g) over M : $\operatorname{Inf} \operatorname{Ric}_M := \operatorname{Inf}\{\rho(u,u);\ u \in TM,\ g(u,u)=1\}$, and $\operatorname{Sup} \operatorname{Ric}_M := \{\rho(u,u);\ u \in TM,\ g(u,u)=1\}$.

Proof. By (4.3), the stability of (M,g) implies that

$$0 \leqq \int_M g(JV,V)*1 = \int_M g(\Delta V,V)*1 - 2\int_M g(\rho(V),V)*1$$

$$\leqq \int_M g(\Delta V,V)*1 - 2(\operatorname{Inf} \operatorname{Ric}_M) \int_M g(V,V)*1 ,$$

which gives the first inequality of (i). Taking V as the gradient of the eigenfunction of Δ_M with the eigenvalue $\lambda_1(M)$, we get the second inequality of (i). The statement (ii) is obvious from (4.3). Q.E.D.

From Lemma 4.1, we obtain :

Theorem 4.2. (M.Obata) Let (M,g) be a closed Kähler manifold whose Ricci curvature Ric_M is bounded below by a positive constant : $\operatorname{Ric}_M \geqq \alpha > 0$. Then the first non-zero eigenvalue $\lambda_1(M)$ of Δ_M on $C^\infty(M)$ satisfies

$$\lambda_1(M) \geqq 2\alpha .$$

When the equality holds, the Lie algebra a of the group of holomorphic transformations of M is non-zero.

Proof. Since every closed Kähler manifold (M,g) is stable (cf. [Sm], [Na]), by Lemma 4.1(i), we have the inequality $\lambda_1(M) \geqq 2\alpha$.

Assume that the equality $\lambda_1(M) = 2\alpha$ holds. We take V as the gradient of the eigenfunction of Δ_M with the eigenvalue 2α. Then $\Delta V = 2\alpha V$. By (4.3), we have

$$\begin{aligned} 2\alpha\int_M g(V,V)*1 &= \int_M g(\Delta V,V)*1 \\ &= \int_M g(JV,V)*1 + 2\int_M g(\rho(V),V)*1 \\ &\geqq 2\alpha\int_M g(V,V)*1 , \end{aligned}$$

since (M,g) is stable and $Ric_M \geqq \alpha$. Hence we have $\int_M g(JV,V)*1 = 0$ and $\int_M g(\rho(V),V)*1 = \alpha\int_M g(V,V)*1$. The former implies $JV = 0$, and then V belongs to a due to a theorem of Lichnérowicz (cf.[L]) since (M,g) is a closed Kähler manifold. Q.E.D.

Remark 1. In [Ob], the above theorem was stated in case of a closed Einstein Kähler manifold (M,g). In this case, i.e., $\rho = \alpha g$, the equality $\lambda_1(M) = 2\alpha$ holds if and only if $a \neq \{0\}$. The author does not know whether or not the equality holds if $a \neq \{0\}$ without the assumtion that (M,g) is Einstein.

Remark 2. A theorem of Lichnérowicz-Obata tells us that for a closed Riemannian manifold (M,g), if $Ric_M \geqq \alpha = (n-1)\delta > 0$, then $\lambda_1(M) \geqq n\delta = \frac{n}{n-1}\alpha$. Note that $\frac{n}{n-1} \leqq 2$ and $\frac{n}{n-1} = 2 \Longleftrightarrow n=2$.

§5. Some examples.

In this section, we give three examples concerning with stability or unstability of closed Riemannian manifolds.

5.1. By (4.1) and Corollary 2.2, we know (cf. [Sm]) that if Ricci curvature Ric_M of a closed Riemannian manifold (M,g) is non-positive, then $\mathrm{Index}(id_M) = 0$ and $\mathrm{Index}(id_M)+\mathrm{Nullity}(id_M) \leqq m = \dim M$. By the similar way as the proof of Proposition 5.6 in [B.G,p.30] noting only the difference of the constant terms of (4.1) and (4.2), we have :

Proposition 5.1. There exists a positive constant $\varepsilon_m > 0$ depending only on m such that for every closed Riemannian manifold (M,g) of dimension m with $Ric_M \leqq \varepsilon_m$, the index and the nullity of the identity map of M satisfies

$$\mathrm{Index}(id_M)+\mathrm{Nullity}(id_M) \leqq m.$$

However one can not expect a positive answer of the following question : " Is there a positive constant $\varepsilon_m > 0$ such that for every closed Riemannian manifold (M,g) of dimension m the assumption $Ric_M \leqq \varepsilon_m$ implies the stability of (M,g), i.e., $\mathrm{Index}(id_M) = 0$? "

In fact, we have the following example :

Example 5.2. Let $T^m = \mathbb{R}^m/\mathbb{Z}^m$ be the m-dimensional torus with the canonical coordinate $(x_1,\ldots,x_m)$. Let $f(x_1)$ be a positive valued smooth function on $\mathbb{R}/\mathbb{Z} = S^1$. Consider the Riemannian metric g_f on T^m defined by

$$g_f := dx_1^2 + f(x_1)^2(dx_2^2+\ldots+dx_m^2) .$$

Lemma 5.3. The vector field $X_1 = f(x_1)\frac{\partial}{\partial x_1}$ on $\mathbb{T}^m$ is a conformal vector field, i.e., the Lie derivative $L_{X_1} g_f$ of g_f by X_1 satisfies $L_{X_1} g_f = \frac{2}{n}\,\mathrm{div}(X_1)\, g_f$, and $X_i = \frac{\partial}{\partial x_i}$, $i=2,\ldots,m$, are Killing, i.e., $L_{X_i} g_f = 0$.

Proof follows from a straightforward computation.

Since for a vector field V on a closed Riemannian manifold (M,g),

$$\int_M g(JV,V)*1 = \int_M \left\{\frac{1}{2}|L_V g|^2 - \mathrm{div}(V)^2\right\}*1 ,$$

where $|L_V g|$ is the norm of $L_V g$ induced by g and $\mathrm{div}(V)$ is the divergence of V (cf. [Y.B]), we have

$$\int_{\mathbb{T}^m} g(JX_1,X_1)*1 = \left(\frac{2}{m} - 1\right) \int_{\mathbb{T}^m} \mathrm{div}(X_1)^2 *1 .$$

Since $\mathrm{div}(X_1) = m\, f'(x_1)$ where $f'(x_1)$ is the derivative of $f(x_1)$, we have :

Proposition 5.4. Let $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ be the m-dimensional torus with the canonical coordinate $(x_1,\ldots,x_m)$. For a positive valued smooth function $f(x_1)$ on $S^1 = \mathbb{R}/\mathbb{Z}$, consider the Riemannian metric g_f on $\mathbb{T}^m$ defined by $g_f = dx_1^{\,2} + f(x_1)^2(dx_2^{\,2}+\ldots+dx_m^{\,2})$. Then, in case of $m \geqq 3$, the Riemannian manifold (T^m, g_f) is stable if and only if the function $f(x_1)$ is constant.

On the other hand the sectional curvature K of the Riemannian manifold $(\mathbb{T}^m, g_f)$ is given (cf. [B.O]) as follows :

For each plane π in the tangent space $T_{(x_1,\ldots,x_m)}\mathbb{T}^m$, let $\left\{x\frac{\partial}{\partial x_1} + v\, ,\; y\frac{\partial}{\partial x_1} + w\right\}$ be an orthonormal basis of π, where $x,y \in \mathbb{R}$, and $v,w \in T_{(x_2,\ldots,x_m)}T^{m-1}$. Then the sectional curvature $K(\pi)$ is

$$K(\Pi) = -\frac{f''(x_1)}{f(x_1)}\left\{x^2 g_f(w,w) - 2xy g_f(w,v) + y^2 g_f(v,v)\right\}$$
$$- \frac{f'(x_1)^2}{f(x_1)^2}\left\{g_f(v,v)g_f(w,w) - g_f(v,w)^2\right\}.$$

Then the sectional curvature K of (T^m, g_f) satisfies that

$$|K| \leqq \frac{|f''|}{f} + \frac{f'^2}{f^2} .$$

For example, we can take a smooth function $f_\varepsilon(x_1)$ on $S^1 = \mathbb{R}/\mathbb{Z}$ as $f_\varepsilon(x_1) := 1 + \varepsilon \sin(2\pi x_1)$, where ε is a small positive constant. Then due to Proposition 5.4, the Riemannian manifold (T^m, g_{f_ε}) , $m \geqq 3$, is unstable, but its sectional curvature K_ε satisfies

$$|K_\varepsilon| \leqq 4\pi^2 \left\{\frac{\varepsilon}{1-\varepsilon} + \frac{\varepsilon^2}{(1-\varepsilon)^2}\right\},$$

which goes to zero as $\varepsilon \to 0$. Therefore we can not take a constant $\varepsilon_m > 0$ such that for every closed Riemannian manifold (M,g) of dimension m, the assumption $Ric_M \leqq \varepsilon_m$ implies the stability of (M,g).

5.2. The next example is the odd dimensional unit sphere S^{2n+1}, $n \geqq 1$. Let $\phi ; (S^{2n+1}, g) \longrightarrow (CP^n, h)$ be the Hopf fibration. Here g is the standard metric on S^{2n+1} of constant curvature one and h is the Fubini-Study metric on CP^n of constant holomorphic sectional curvature 4. Let ξ be the Killing vector field of (S^{2n+1}, g) such that $g(\xi,\xi) = 1$ everywhere S^{2n+1} and ξ is tangent to the fiber $\phi^{-1}(\phi(x))$ at each point x in S^{2n+1}. Let η be the 1-form dual to ξ. Then the projection $\phi ; (S^{2n+1}, g) \longrightarrow (CP^n, h)$ is a Riemannian submersion with totally geodesic fibers (cf. §6) and $g = \phi^* h + \eta \otimes \eta$. Let us consider the canonical variation g_t, $0 < t < \infty$, of the metric g defined by

$$(5.1) \qquad g_t := \phi^* h + t^2 \eta \otimes \eta = g + (t^2 - 1)\eta \otimes \eta .$$

Now let us investigate the stability of (S^{2n+1}, g_t) making use of Lemma 4.1.

(i) The first eigenvalue $\lambda_1^1(g_t)$ of the Hodge Laplacian.
Put $m := 2n+1$. Note that $g_t = s\left\{s^{-1}g + s^{-1}(s^m-1)\eta\otimes\eta\right\}$, where $s := t^{2/m}$
In his paper [T1, Proposition 2.8] , S.Tanno showed that the first eigenvalue $\lambda_1^1(g_t)$ of the Hodge Laplacian on 1-forms is estimated as

$$\lambda_1^1(g_t) \leqq \operatorname{Min}\left\{s^{-1}\cdot 2(m-1)s^{m+1},\ s^{-1}(ms-s(1-s^{-m}))\right\} ,$$

that is,

$$(5.2) \qquad \lambda_1^1(g_t) \leqq \operatorname{Min}\left\{4nt^2,\ 2n+t^{-2}\right\}.$$

(ii) Ricci curvature of (S^{2n+1}, g_t) . Let us recall a work of G.R.Jensen [J]. We denote

$$K := SU(n+1),$$
$$H := S(U(n)\times U(1)) = \left\{\begin{pmatrix}\varepsilon & 0\\ 0 & A\end{pmatrix} \in SU(n+1)\ ;\ \varepsilon \in U(1),\ A \in U(n)\right\},$$
$$H_1 := \left\{\begin{pmatrix}\varepsilon & 0\\ 0 & \gamma I_n\end{pmatrix}\ ;\ \varepsilon \in U(1),\ \gamma = \varepsilon^{-1/n}\right\},$$
$$H_2 := \left\{\begin{pmatrix}1 & 0\\ 0 & A\end{pmatrix}\ ;\ A \in SU(n)\right\} ,$$

where I_n is the unit matrix of order n. Then the natural projection gives the Hopf fibration : $\phi ; S^{2n+1} = K/H_2 \longrightarrow CP^n = K/H$. Let k (resp. h , h_1 , h_2) be the Lie algebra of K (resp. H, H_1 , H_2). Let F be the Killing form of k and m , the orthocomplement of h in k with respect to F. Then we have the orthogonal decomposition of k : $k = h_2 \oplus h_1 \oplus m$. The metrics g_t (5.1) are K-invariant on K/H_2 which come from the $Ad(H_2)$-invariant inner product $\langle\cdot,\cdot\rangle_t$ on $h_1 \oplus m$ such that

$$\langle X_1+X_2, Y_1+Y_2\rangle_t = (4(n+1))^{-1}\left\{\frac{2n}{n+1}t^2 b(X_1,Y_1)+b(X_2,Y_2)\right\},$$

for $X_1, Y_1 \in h_1$, $X_2, Y_2 \in m$, where the inner product b on k is given by $b = -F$. In fact, it is known that the restriction of b to m coincides with $4(n+1)\pi^*h$, and $b(X,X) = 2(n+1)^2/n$ for $X := \sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & -n^{-1}I_n \end{pmatrix}$, and ξ_o is the tangent vector at $o := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in S^{2n+1}$ of the curve $\theta \longmapsto \exp(\theta X)\cdot o$.

We denote by $S_{\tilde{g}}$ the Ricci tensor of the metric $\tilde{g}$ on K/H_2 corresponding to the inner product $4(n+1)\langle\cdot,\cdot\rangle_t$ on m. Then $S_{\tilde{g}}$ is a K-invariant tensor field on K/H_2 which is completely determined by the bilinear form on $h_1 \oplus m$, denoted by the same letter $S_{\tilde{g}}$. Noting that the numbers k, c, r, and $\dim m$ in [J] are given in this case by

$$k = 1/2,\ c = 0,\ r = \dim h_1 = 1, \text{ and } \dim m = 2n,$$

and due to Proposition 11 in [J], the bilinear form $S_{\tilde{g}}$ is given by

$$S_{\tilde{g}}(X_1+X_2, Y_1+Y_2) = \frac{1}{4}\left(\frac{2n}{n+1}\right)t^2\cdot 4(n+1)\langle X_1, Y_1\rangle_t + \left(\frac{1}{2} - \frac{1}{4n}\left(\frac{2n}{n+1}\right)t^2\right)\cdot 4(n+1)\langle X_2, Y_2\rangle_t,$$

$X_1, Y_1 \in h_1$, $X_2, Y_2 \in m$. Therefore, since the infimum $\mathrm{Inf\,Ric}_{\tilde{g}}$ (resp. supremum $\mathrm{Sup\,Ric}_{\tilde{g}}$) of the Ricci curvature of $(S^{2n+1}, \tilde{g})$ is given by $\mathrm{Inf\,Ric}_{\tilde{g}} = \mathrm{Min}\left\{\frac{1}{2} - \frac{t^2}{2(n+1)}, \frac{n}{2(n+1)}t^2\right\}$ (resp. $\mathrm{Sup\,Ric}_{\tilde{g}} = \mathrm{Max}\{\text{"}, \text{"}\}$), the one $\mathrm{Inf\,Ric}_{g_t}$ (resp. $\mathrm{Sup\,Ric}_{g_t}$) of the metric g_t (5.1) is

$$(5.3) \qquad \mathrm{Inf\,Ric}_{g_t} = \mathrm{Min}\left\{2(n+1)-2t^2,\ 2nt^2\right\},$$

(resp. $\mathrm{Sup\,Ric}_{g_t} = \mathrm{Max}\{\text{"},\text{"}\}$). Putting $T := t^2$, let us observe the behavior of $\lambda_1^1(g_t)$ (5.2) and $2\cdot\mathrm{Inf\,Ric}_{g_t}$ (5.3) (cf. Figure 5.1) :

Figure 5.1. The graphs of the functions $4nT, 2n+T^{-1}$, and $4(n+1)-4T$.

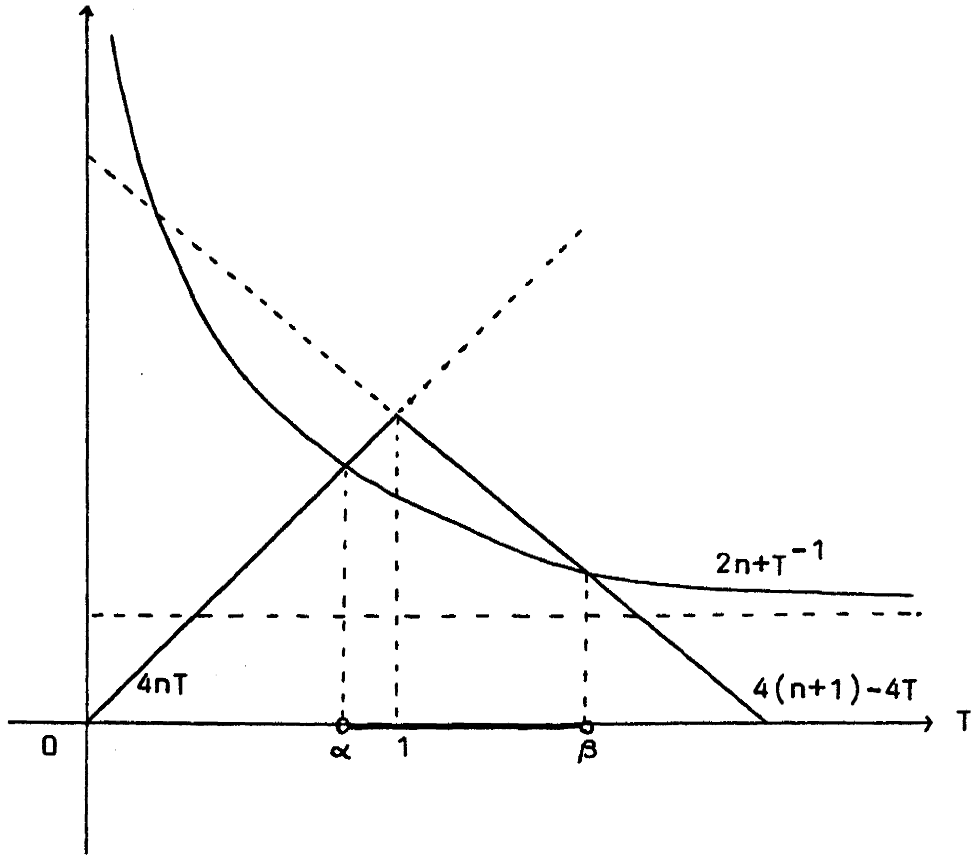

Figure 5.1. The graphs of the functions $4nT, 2n+T^{-1}$, and $4(n+1)-4T$.

Therefore we have :

Proposition 5.5. Let g_t be the canonical variation (5.1) of the standard metric g on S^{2n+1} of constant curvature one with $g_1 = g$: $g_t = g + (t^2-1)\eta\otimes\eta$. Then for every t^2 in the open interval (α,β), the Riemannian manifold (S^{2n+1},g_t) is unstable. Here $\alpha := \frac{n+\sqrt{n^2+4n}}{4n}$ (resp. $\beta := \frac{n+2+\sqrt{n^2+4n}}{4}$) is a root of the equation $4nT = 2n+T^{-1}$ (resp. $4(n+1)-4T = 2n+T^{-1}$).

5.3. The third example is a spherical space form. Here we state the following :

Proposition 5.6. Every spherical space form $(S^n/G,g)$, where $G \neq \{id\}$ is a finite group acting fixed point freely on S^n , is stable. Here the metric g is the Riemannian metric on the quotient space S^n/G induced from the standard metric can on S^n of constant curvature one.

In fact, this follows immediately from Proposition 2.1 in [Sm]. Since $(S^n/G,g)$ is Einstein,i.e., the Ricci tensor ρ of g satisfies $\rho = (n-1)g$, the manifold $(S^n/G,g)$ is stable if and only if the first non-zero eigenvalue $\lambda_1(S^n/G,g)$ of the Laplace-Beltrami operator Δ_M on $C^\infty(S^n/G)$ is bigger than or equal to $2(n-1)$. The eigenvalues of Δ_M of (S^n,can) are given by $k(k+n-1)$, $k=0,1,2,\ldots$, and $k(k+n-1) > 2(n-1)$ if $k \geq 2$. Moreover the eigenfunctions of the first non-zero eigenvalue n with $k=1$ of (S^n,can) are given by $F\cdot id_{S^n}$, where F is a linear map of $\mathbb{R}^{n+1}$ into $\mathbb{R}$ and id_{S^n} is the natural inclusion of S^n into $\mathbb{R}^{n+1}$. Therefore we have only to show that every linear G-invariant function F on $\mathbb{R}^{n+1}$ must be zero. But this follows immediately from the assumption that G acts fixed point freely on S^n. Certainly, $F(x) = \langle x,y \rangle$, $x \in \mathbb{R}^{n+1}$, for some y in $\mathbb{R}^{n+1}$. The G-invariance of F implies that $\gamma \cdot y = y$ for all $\gamma \in G$. Unless F vanishes, the point $y/|y| \in S^n$ must be a fixed point of G.

Since every compact Riemannian manifold of positive constant curvature is as in Proposition 5.6 (cf. [W, Lemma 5.11, p.154]) and every compact Riemannian manifold of constant zero or negative curvature is stable (cf. [Sm]), we have :

Corollary 5.7. Every compact Riemannian manifold of constant curvature is stable except only the standard unit sphere (S^n, can).

Remark. The similar stability theorem for Yang-Mills fields was stated in [B.L, p.223] .

§6. The vertical Jacobi Operator.

6.1. Definition of Riemannian Submersions. Following [O.N], or [B.B], let us recall the definition of the Riemannian submersions. It is known (cf. [E.L,p.127]) that the projection of a Riemannian submersion is harmonic if and only if each fiber of the submersion is a minimal submanifold. In particular, the projection of the Riemannian submersion with totally geodesic fibers is harmonic. The Riemannian submersions are the next simple examples after Riemannian products, but would be rich objects to study. In this chapter, we devote ourself to study the Jacobi operators of the projections of the Riemannian submersions with totally geodesic fibers analogously as in the theory of the Laplace-Beltrami operators (cf.[B.B]).

Definition 6.1. Let (M,g), (N,h) be two closed Riemannian manifolds of dimension m,n, respectively. A map ϕ; $(M,g)\longrightarrow(N,h)$ is a Riemannian submersion (cf.[O.N],[B.B]) if for each point p in M, the tangent space T_pM of M at p has the following orthogonal decomposition $T_pM = H_p \oplus V_p$ with respect to g_p :

(i) The subspace V_p is the kernel of the differential ϕ_{*p} of ϕ at p, which is called as the vertical space.

(ii) The restriction of ϕ_{*p} to the subspace H_p, called the horizotal space, is an isometry of (H_p,g_p) onto $(T_{\phi(p)}N,h_{\phi(p)})$.

In this chapter, we further assume that each fiber $F_p := \phi^{-1}(\phi(p))$ through p admitting the Riemannian metric induced from g is totally geodesic in (M,g).

6.2. Definition of the vertical Jacobi operator. We take an orthonormal local frame field $\{e_i\}_{i=1}^m$ on M such that

(i) $\{e_i\}_{i=1}^n$ is basic associated to an orthonormal local frame field $\{e_i'\}_{i=1}^n$ on N, i.e., e_i, $1\leq i\leq n$, are the horizontal lifts of e_i', $1\leq i\leq n$, and

(ii) $e_i, n+1\leq i\leq m$, are vertical.

Then it is known (cf. [O.N] or [B.B]) that $\nabla_{e_i} e_i$, $1\leq i\leq n$, are basic associated to the vector fields ${}^N\nabla_{e_i'} e_i'$ and $\nabla_{e_i} e_i$, $n+1\leq i\leq m$, are vertical since all the fibers are totally geodesic. In the following we retain the notations in §1.

Definition 6.2. Let ϕ; $(M,g)\longrightarrow(N,h)$ be the Riemannian submersion with totally geodesic fibers and J_ϕ, the Jacobi operator acting on $\Gamma(\phi^{-1}TN)$. We define the <u>vertical</u> Jacobi operator acting on $\Gamma(\phi^{-1}TN)$ by

$$J_\phi^V := -\sum_{i=n+1}^m (\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i} - \tilde{\nabla}_{\nabla_{e_i} e_i}),$$

and the <u>horizontal</u> Jacobi operator acting on $\Gamma(\phi^{-1}TN)$ by $J_\phi^H := J_\phi - J_\phi^V$.

Then it is easy to see that the definitions of J_ϕ^V and J_ϕ^H do not depend on the above choice of the orthonormal local frame field $\{e_i\}_{i=1}^m$ on M (cf. Remark below). Diese definitions are the analogue of the vertical or horizontal Laplacians Δ_V, Δ_H acting on $C^\infty(M)$ defined in [B.B] : $\Delta_V := \sum_{i=n+1}^m (\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i} e_i})$, $\Delta_H := \Delta_M - \Delta_V$, where $\Delta_M := \sum_{i=1}^m (\nabla_{e_i}\nabla_{e_i} - \nabla_{\nabla_{e_i} e_i})$ is the Laplace-Beltrami operator of (M,g). Then Δ_V, Δ_H, and Δ_M are commutative mutually (cf. [B.B, Theorem 1.5]

Each section W in $\Gamma(\phi^{-1}TN)$ can be expressed locally as

(6.1) $W = \sum_i^n f_i \widetilde{e_i'}$

where f_i, $1 \leq i \leq n$, are locally defined smooth functions on M and $\widetilde{e_i'}$, $1 \leq i \leq n$, are local sections of $\phi^{-1}TN$ defined by $\widetilde{e_i'}_x := e_i'_{\phi(x)}$, $x \in M$. Then by definition of $\widetilde{\nabla}$ and $\phi_* e_i = 0$, $n+1 \leq i \leq m$, we have

$$(6.2) \qquad \widetilde{\nabla}_{e_i} W = \sum_{j=1}^{n} \left\{ (e_i f_j)\, \widetilde{e_j'} + f_j \widetilde{\nabla}_{e_i} \widetilde{e_j'} \right\}, \quad 1 \leq i \leq m,$$

$$(6.2') \qquad \widetilde{\nabla}_{e_i} W = \sum_{j=1}^{n} (e_i' f_j)\, \widetilde{e_j'} \qquad , \quad n+1 \leq i \leq m .$$

In particular,

$$(6.3) \qquad J_\phi^{V} W = - \sum_{j=1}^{n} (\Delta_V f_j)\, \widetilde{e_j'} .$$

Remark. (The intrinsic meaning of the vertical Jacobi operator) For each fiber $F_p = \phi^{-1}(\phi(p))$ through $p \in M$, the composition $\phi \circ i_p$; $F_p \longrightarrow N$ of the inclusion i_p of F_p into M and the projection ϕ is constant, so harmonic. The associate Jacobi operator $J_{\phi \circ i_p}$ acting on $\Gamma((\phi \circ i_p)^{-1}TN)$ is well-defined. Then $\Gamma((\phi \circ i_p)^{-1}TN)$ consists of all the restrictions to F_p, $W|_{F_p}$, of elements W in $\Gamma(\phi^{-1}TN)$ and

$$(J_\phi^{V} W)(p) = J_{\phi \circ i_p}(W|_{F_p})(p) , \qquad W \in \Gamma(\phi^{-1}TN).$$

6.3. Fundamental Properties of J_ϕ^{V} and J_ϕ^{H}. Note that, by definitions of $\widetilde{\nabla}$ and $\widetilde{W'}$, $W' \in \Gamma(TN)$,

$$(6.4) \qquad \widetilde{\nabla}_{e_i} W' = \widetilde{{}^N\nabla_{\phi_* e_i} W'} = \begin{cases} \widetilde{{}^N\nabla_{e_i'} W'} , & 1 \leq i \leq n, \\ 0 , & n+1 \leq i \leq m , \end{cases}$$

for $W' \in \Gamma(TN)$. Then we have

$$(6.5) \qquad J_\phi^{V}(\widetilde{W'}) = 0, \quad \text{and} \quad J_\phi^{H}(\widetilde{W'}) = \widetilde{J_{id_N}(W')},$$

for $W' \in \Gamma(TN)$, by (6.4) and definition of J_ϕ^{V} and J_ϕ^{H}. Therefore we obtain :

Proposition 6.3. Let $\phi; (M,g) \longrightarrow (N,h)$ be the Riemannian submersion with totally geodesic fibers. Then

$$\mathrm{Index}(\phi) \geqq \mathrm{Index}(id_N),\ \mathrm{Nullity}(\phi) \geqq \mathrm{Nullity}(id_N),$$

and $\lambda_1(J_\phi) \leqq \lambda_1(J_{id_N})$. In particular, if the base manifold (N,h) is unstable, then the submersion ϕ is unstable.

In fact, suppose that $W' \in \Gamma(TN)$ satisfies $J_{id_N} W' = \lambda W'$. Then the element $\widetilde{W'} \in \Gamma(\phi^{-1}TN)$ satisfies

$$J_\phi \widetilde{W'} = J_\phi{}^H(\widetilde{W'}) = \widetilde{J_{id_N} W'} = \lambda \widetilde{W'},$$

by (6.5). Therefore if λ is the eigenvalue of J_{id_N}, then λ is also the one of J_ϕ. Q.E.D.

Proposition 6.4.

(i) Let $F = F_p$ be the fiber through $p \in M$ of the Riemannian submersion $\phi; (M,g) \longrightarrow (N,h)$ with totally geodesic fibers. For each $W \in \Gamma(\phi^{-1}TN)$, we have

$$\int_F h(J_\phi{}^V W, W)\, dv_F = \sum_{i=n+1}^{m} \int_F h(\widetilde{\nabla}_{e_i} W, \widetilde{\nabla}_{e_i} W)\, dv_F,$$

where dv_F is the volume element on F with respect to the metric g_F induced from the metric g on M.

(ii) Moreover, for each $W \in \Gamma(\phi^{-1}TN)$, $J_\phi{}^V W = 0$ if and only if $W = \widetilde{W'}$ for some $W' \in \Gamma(TN)$.

(iii) Each eigenvalue of $J_\phi{}^V$ is non-negative.

Proof. (i) For each $W \in \Gamma(\phi^{-1}TN)$, we have

$$h(J_\phi{}^V W, W) = -\sum_{i=n+1}^{m} e_i \cdot h(\widetilde{\nabla}_{e_i} W, W) + \sum_{i=n+1}^{m} h(\widetilde{\nabla}_{e_i} W, \widetilde{\nabla}_{e_i} W)$$

$$+ \sum_{i=n+1}^{m} h(\widetilde{\nabla}_{\nabla_{e_i} e_i} W, W).$$

Here there exists an element X in $\Gamma(TF)$ such that $g_F(X,Y) = h(\tilde{\nabla}_Y W, W)$ for each $Y \in \Gamma(TF)$. Then since $\nabla_{e_i} e_i$, $n+1 \leqq i \leqq m$, are vertical

$$\sum_{i=n+1}^{m} \left\{ e_i \cdot h(\tilde{\nabla}_{e_i} W, W) - h(\tilde{\nabla}_{\nabla_{e_i} e_i} W, W) \right\}$$

$$= \sum_{i=n+1}^{m} \left\{ e_i \cdot g_F(X, e_i) - g_F(\nabla_{e_i} e_i, X) \right\}$$

is the gradient of X on (F, g_F). Therefore we have (i).

(ii) By (6.5), we have only to prove that if $J_\phi{}^v W = 0$, then $W = \widetilde{W'}$ for some $W' \in \Gamma(TN)$. Assume that $J_\phi{}^v W = 0$. Then by (i) we have $\tilde{\nabla}_{e_i} W = 0$, $n+1 \leqq i \leqq m$. We choose a local coordinate system $(x_U^1, \ldots, x_U^n)$ on a neighborhood U in N. Then W can be expressed locally as $W_x = \sum_{j=1}^{n} f_{U,j}(x) \left(\frac{\partial}{\partial x_U^j}\right)_{\phi(x)}$, $x \in \phi^{-1}(U)$, where $f_{U,j}$ $C^\infty(\phi^{-1}(U))$. Since $W \in \Gamma(\phi^{-1}TN)$, it satisfies that

$$(6.6) \qquad f_{U,i} = \sum_{j=1}^{n} f_{V,j} \frac{\partial x_U^i}{\partial x_V^j} \qquad \text{on} \quad \phi^{-1}(U) \cap \phi^{-1}(V),$$

for another coordinate system $(x_V^1, \ldots, x_V^n)$ on V. By (6.2'), $0 = \tilde{\nabla}_{e_i} W = \sum_{j=1}^{n} (e_i f_{U,j}) \left(\widetilde{\frac{\partial}{\partial x_U^j}}\right)$. Therefore $e_i f_{U,j} = 0$, $n+1 \leqq i \leqq m$, that is, $f_{U,j}$ are constant along each fiber, which implies that $f_{U,j} = f'_{U,j} \circ \phi$ for some $f'_{U,j} \in C^\infty(U)$. By (6.6), $f'_{U,j}$ satisfies $f'_{U,i} = \sum_{j=1}^{n} f'_{V,j} (\partial x_U^i / \partial x_V^j)$ on $U \cap V$. Therefore $\left\{ \sum_{j=1}^{n} f'_{U,j} \, \partial/\partial x_U^j \right\}$ defines a section W' in $\Gamma(TN)$ such that $W = \widetilde{W'}$. (iii) follows immediately from (i). In fact, suppose that $J_\phi{}^v W = \lambda W$, $0 \neq W \in \Gamma(\phi^{-1}TN)$. Then there exists a fiber F such that $\int_F h(W,W) \, dv_F > 0$. We apply (i) to this fiber F and we have (iii).

Q.E.D.

6.4. Commutatibity of J_ϕ^V, J_ϕ^H and J_ϕ.

Theorem 6.5. Let $\phi ; (M,g) \longrightarrow (N,h)$ be the Riemannian submersion with totally geodesic fibers. Then the operators J_ϕ^V, J_ϕ^H and J_ϕ are commutative each other.

Proof. We have only to prove $J_\phi^V J_\phi^H = J_\phi^H J_\phi^V$. For each $W \in \Gamma(\phi^{-1}TN)$, we have

$$(6.7) \quad J_\phi^H J_\phi^V W = - \sum_{j,k=1}^{n} \Big\{ (\tilde{\nabla}_{e_k}\tilde{\nabla}_{e_k} - \tilde{\nabla}_{\nabla_{e_k} e_k})((\Delta_V f_j)\widetilde{e_j'}) + (\Delta_V f_j)\,{}^N R(e_k', e_j')e_k' \Big\}$$

$$= - \sum_{j,k=1}^{n} \Big\{ e_k^2(\Delta_V f_j)\widetilde{e_j'} + 2e_k(\Delta_V f_j)\tilde{\nabla}_{e_k}\widetilde{e_j'} + (\Delta_V f_j)\tilde{\nabla}_{e_k}\tilde{\nabla}_{e_k}\widetilde{e_j'} - (\nabla_{e_k} e_k)(\Delta_V f_j)\widetilde{e_j'} - (\Delta_V f_j)\tilde{\nabla}_{\nabla_{e_k} e_k}\widetilde{e_j'} \Big\}$$

$$- \sum_{j,k=1}^{n} (\Delta_V f_j)\,{}^N R(e_k', e_j')e_k' ,$$

by definition of J_ϕ^H and (6.3). Since e_k and $\nabla_{e_k} e_k$, $1 \leq k \leq n$, are basic, and Δ_V is commutative with basic vector fields (cf. [B.B, Lemma 1.6]), the first term of the right hand side of (6.7) becomes

$$- \sum_{j,k=1}^{n} \Big\{ \Delta_V(e_k^2 f_j)\widetilde{e_j'} + 2\Delta_V(e_k f_j)\tilde{\nabla}_{e_k}\widetilde{e_j'} + (\Delta_V f_j)\tilde{\nabla}_{e_k}\tilde{\nabla}_{e_k}\widetilde{e_j'} - \Delta_V(\nabla_{e_k} e_k f_j)\widetilde{e_j'} - (\Delta_V f_j)\tilde{\nabla}_{\nabla_{e_k} e_k}\widetilde{e_j'} \Big\}$$

$$= - \sum_{j,k=1}^{n} J_\phi^V \Big\{ (e_k^2 f_j)\widetilde{e_j'} + 2(e_k f_j)\tilde{\nabla}_{e_k}\widetilde{e_j'} + f_j\tilde{\nabla}_{e_k}\tilde{\nabla}_{e_k}\widetilde{e_j'} - (\nabla_{e_k} e_k\, f_j)\widetilde{e_j'} - f_j\tilde{\nabla}_{\nabla_{e_k} e_k}\widetilde{e_j'} \Big\} ,$$

by (6.3) and (6.4). Therefore we obtain

$$J^H J^V W = - \sum_{k=1}^{n} J_\phi^V \Big\{ (\tilde{\nabla}_{e_k}\tilde{\nabla}_{e_k} - \tilde{\nabla}_{\nabla_{e_k} e_k})W - {}^N R(e_k', W)e_k' \Big\}$$

$= J_\phi{}^V J_\phi{}^H W$. Q.E.D.

Therefore we have immediately :

Corollary 6.6. The Hilbert space of all L^2 sections of $\phi^{-1}TN$ with respect to the inner product $(V,W) := \int_M h(V,W)*1$, for sections V, W, has a complete orthonormal basis consisting of the simultaneous eigensections of $J_\phi{}^V$, $J_\phi{}^H$, and J_ϕ.

§7. The Canonical Variation of a Riemannian Submersion.

7.1. Definition of the Canonical Variation. We retain the situations in §6. Let $\phi ; (M,g) \longrightarrow (N,h)$ be the Riemannian submersion with totally geodesic fibers.

Definition 7.1. (cf. [B.B, p.191]) For each positive real number t, let g_t be the unique Riemannian metric on M such that

(i) $g_t(u,v) = g(u,v)$ for $u,v \in H_p$, $p \in M$,

(ii) The subspaces H_p and V_p are orthogonal each other with respect to g_t at each point p in M, and

(iii) $g_t(u,v) = t^2 g(u,v)$ for $u,v \in V_p$, $p \in M$.

Then $\phi ; (M,g_t) \longrightarrow (N,h)$ is a Riemannian submersion with totally geodesic fibers (cf. [B.B , Proposition 5.2]), which is called the <u>canonical variation</u>.

For each $t > 0$, $\{e_1, \ldots, e_n, t^{-1}e_{n+1}, \ldots, t^{-1}e_m\}$ is an orthonormal local frame field on (M,g_t) such that $\{t^{-1}e_i\}_{i=n+1}^{m}$ are vertical and $\{e_i\}_{i=1}^{n}$ are the horizontal lifts of $\{e_i{}'\}_{i=1}^{n}$ with respect to g_t. Then the vertical (resp. horizontal) Jacobi operator ${}^tJ_\phi{}^V$ (resp. ${}^tJ_\phi{}^H$) of the canonical variation $\phi ; (M,g_t) \longrightarrow (N,h)$ satisfies that

$$ {}^tJ_\phi{}^V = t^{-2} J_\phi{}^V \quad , \text{ and } \quad {}^tJ_\phi{}^H = J_\phi{}^H . $$

Therefore we have :

Proposition 7.2. The following formula hold :

$$ {}^tJ_\phi = t^{-2} J_\phi{}^V + J_\phi{}^H = t^{-2} J_\phi + (1-t^{-2}) J_\phi{}^H . $$

Remark. This is the analogue of Proposition 5.3 in [B.B].

7.2. Due to Corollary 6.6 and Proposition 7.2, each eigenvalue of ${}^{t}J_{\phi}$ can be written as

$$(7.1) \quad \lambda + t^{-2}\mu \ ,$$

where λ is the eigenvalue of $J_{\phi}{}^{H}$ and $\mu \geq 0$ is the eigenvalue of $J_{\phi}{}^{V}$. Then the following two cases occur :

(i) $\mu > 0$, or

(ii) $\mu = 0$.

In case of (i), $\lambda + t^{-2}\mu$ goes to infinity when $t \to 0$. In case of (ii), $\lambda + t^{-2}\mu = \lambda$ which does not depend on t. Since the number of the eigenvalues of J_{ϕ} smaller than a given number is finite, there exists a small positive number ε such that for each $0 < t < \varepsilon$, the first eigenvalue $\lambda_1({}^{t}J_{\phi})$ coincides with the smallest eigenvalue of ${}^{t}J_{\phi}$ when the case (ii) occurs. Then we have

$$\lambda_1({}^{t}J_{\phi}) = \mathrm{Min}\ \{\lambda;\ J_{\phi}W = \lambda W \ \text{ and } \ J_{\phi}{}^{V}W = 0 \text{ for some } 0 \neq W \in \Gamma(\phi^{-1}T$$

$$= \lambda_1(J_{id_N}) \ ,$$

because of Propositions 6.4(ii) and 6.3. Therefore we obtain :

Theorem 7.3. Let ϕ; $(M,g) \longrightarrow (N,h)$ be a Riemannian submersion with totally geodesic fibers, and g_t , $0 < t < \infty$, the canonical variatio (cf. Definition 7.1) of g with $g_1 = g$. Then there exists a number $\varepsilon > 0$ such that for each $0 < t < \varepsilon$, we have

$$\lambda_1({}^{t}J_{\phi}) = \lambda_1(J_{id_N}).$$

In particular, if (N,h) is stable, then the submersion ϕ; $(M,g_t) \longrightarrow$ (N,h) is stable for every $0 < t < \varepsilon$.

7.3. The typical examples of the Riemannian submersions with totally geodesic fibers are the homogeneous Riemannian submersions (cf.[B.B,§2]): Let G be a compact connected Lie group, and K H closed subgroups of G. Let g (resp. k, h) be the Lie algebra of G (resp. K, H). We choose subspaces h_1 (resp. p) of k (resp. g) such that

$$k = h \oplus h_1 , \quad \text{with} \quad Ad(H)\, h_1 = h_1 , \quad \text{and}$$

$$g = k \oplus p , \quad \text{with} \quad Ad(K)\, p = p .$$

Put $m := h_1 \oplus p$. Then

$$g = h \oplus m , \quad \text{with} \quad Ad(H)\, m = m .$$

Let $(\cdot,\cdot)_{h_1}$ (resp. $(\cdot,\cdot)_p$) be an $Ad(H)$-invariant (resp. $Ad(K)$-invariant) inner product on h_1 (resp. p). Then we can define an $Ad(H)$-invariant inner product $(\cdot,\cdot)_m$ on m by

$$(X_1+X_2, Y_1+Y_2)_m := (X_1,Y_1)_{h_1}+(X_2,Y_2)_p, \quad X_1,Y_1 \in h_1 , \; X_2,Y_2 \in p.$$

Then the inner product $(\cdot,\cdot)_{h_1}$ (resp. $(\cdot,\cdot)_p$, $(\cdot,\cdot)_m$) gives a K-invariant (resp. G-invariant) Riemannian metric k (resp. h, g) on K/H (resp. G/K , G/H). It is known (cf.[B.B]) that the projection ϕ ; $G/H \ni xH \longmapsto xK \in G/K$ gives the Riemannian submersion of $(G/H,g)$ onto $(G/K,h)$ with totally geodesic fibers $(K/H,k)$.

In particular, these give the Hopf fibrations :

$$\text{(i)} \quad \phi_1 \; ; \; S^{4n+3} = Sp(n+1)/Sp(n) \longrightarrow HP^n = Sp(n+1)/Sp(1)\times Sp(n$$

$$\text{(ii)} \quad \phi_2 \; ; \; S^{2n+1} = SU(n+1)/SU(n) \longrightarrow CP^n = SU(n+1)/S(U(1)\times U(n)).$$

Note that $Sp(n+1)$-invariant (resp. $SU(n+1)$-invariant) metrics h on HP^n (resp. CP^n) are unique up to a constant factor.

Since (HP^n,h) (resp. (CP^n,h)) is unstable (resp. stable) (cf.[Sm], [Na]), we have :

Proposition 7.4.

(i) For each $Sp(n+1)$-invariant metric g on $S^{4n+3} = Sp(n+1)/Sp(n)$, the Riemannian submersion $\phi_1 ; (S^{4n+3},g) \longrightarrow (HP^n,h)$ is unstable.

(ii) For each $SU(n+1)$-invariant metric g on $S^{2n+1} = SU(n+1)/SU(n)$, there exists a number $\varepsilon > 0$ such that for each $0 < t < \varepsilon$, the canonical variation $\phi_2 ; (S^{2n+1},g_t) \longrightarrow (CP^n,h)$ is stable.

The proof follows from Proposition 6.3 and Theorem 7.3.

Remark. Proposition 7.4 asserts that each odd dimensional unit sphere S^{2n+1}, $n \geqq 1$, with the canonical variation g_t, $0 < t < \varepsilon$, admits a non-constant stable harmonic map. On the contrary, Y.L.Xin [X] showed that each non-constant harmonic map from the standard unit sphere (S^m, can), $m \geqq 3$, of constant curvature into arbitrary Riemannian manifold is unstable.

7.4. Next, let us study the case when t goes to infinity. We retain the situations as in 7.1. Let us recall that the <u>holonomy group</u> G of a fiber F of the submersion $\phi ; (M,g) \longrightarrow (N,h)$ with totally geodesic fibers is the group of all isometries of the fiber F induced by the horizontal transports along the horizontal lifts of loops in N based at the projection of F. It is known ([O.N, Theorem 5]) that $G = \{id\}$ if and only if the submersion $\phi ; (M,g) \longrightarrow (N,h)$ is trivial, that is, there exist an isometry τ of (M,g) and a submanifold F of M such that M is the Riemannian product $F \times N$ and $\phi = pr \circ \tau$, where pr is the projection of $F \times N$ onto N.

Theorem 7.5. Let $\phi ; (M,g) \longrightarrow (N,h)$ be the Riemannian submersion with totally geodesic fibers. Assume that the holonomy group G of a fiber F of the submersion $\phi ; (M,g) \longrightarrow (N,h)$ does not act transitively on the fiber, and $\mathrm{Index}(id_N) > 0$. Then the index of the canonical variation $\phi ; (M,g_t) \longrightarrow (N,h)$ goes to infinity when $t \longrightarrow \infty$.

Proof. Let $C^\infty_G(F)$ be the space of all functions f on $C^\infty(F)$ invariant under the actions of G. Since each G-orbit has an open G-invariant tubular neighborhood in M (cf.[Br, Theorem 2.2,p.306]), there exists a non-constant function f in $C^\infty_G(F)$. Then the dimension of $C^\infty_G(F)$ is infinite. Each element f in $C^\infty_G(F)$ can be extended to a function $\tilde{f}$ in the space $C^\infty_v(M)$ of all elements in $C^\infty(M)$ which are invariant under the horizontal transport. Since the parallel transport is isometry, the vertical Laplacian Δ_v preserves $C^\infty_v(M)$ invariant. Therefore there exist an infinite number of the eigenvalues $0 \leqq \mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_i \leqq \cdots$, of Δ_v counted with their multiplicities such that

$$(7.2) \qquad -\Delta_v f_i = \mu_i f_i, \quad 0 \neq f_i \in C^\infty_v(M),\ i = 1,2,\ldots .$$

Now suppose that $\mathrm{Index}(id_N) > 0$, that is, there exists a non-zero element W' in $\Gamma(TN)$ such that $J_{id_N} W' = \lambda W'$ and $\lambda < 0$. Then we have

$$\begin{aligned}
{}^tJ_\phi(f_i\widetilde{W'}) &= (t^{-2}J_\phi{}^V + J_\phi{}^H)(f_i\widetilde{W'}) && \text{(by Proposition 7.2)}\\
&= t^{-2}(-\Delta_v f_i)\widetilde{W'} + f_i\, J_\phi{}^H(\widetilde{W'}) && \text{(by (6.3) and } f_i \in C^\infty_v(M)\text{)}\\
&= (t^{-2}\mu_i + \lambda)(f_i\widetilde{W'}) && \text{(by (6.5) and (7.2))}.
\end{aligned}$$

That is, ${}^tJ_\phi$ has the eigenvalues $t^{-2}\mu_i + \lambda$, $i=1,2,\ldots$. When t goes to infinity, the eigenvalues $t^{-2}\mu_i + \lambda$ tend to the eigenvalue λ. Since $\lambda < 0$, for each $i = 1,2,\ldots$, there exists a number $N > 0$ such that $t^{-2}\mu_i + \lambda < 0$ for $t \geqq N$. Therefore we have the desired conclusion.

Remark. Theorem 7.5 is a generalization of Corollar 3.3 in [Sm]

§8. Homogeneous Riemannian submersions.

8.1. In this section, we consider the homogeneous Riemannian submersions. Our purpose is to express the Jacobi operator of the homogeneous Riemannian submersions in terms of Lie algebras and calculate the spectrum of the Jacobi operator of the Hopf fibration using these results. We retain the situations as in 7.3.

Let G be a compact connected Lie group, and K, H, closed subgroups of G. Let g be the Lie algebra of G consisting of all left invariant vector fields on G. Let k, h be the subalgebras corresponding to K, H. Put $s := \dim G$, $m := \dim G/H$, and $n := \dim G/K$. We choose an $Ad(G)$-invariant inner product $(\cdot,\cdot)$ on g, and h_1 (resp. p), the orthogonal complement of h (resp. k) in k (resp. g). Then

$$k = h \oplus h_1 \quad \text{with} \quad Ad(H)\, h_1 = h_1 , \text{ and}$$
$$g = k \oplus p \quad \text{with} \quad Ad(K)\, p = p .$$

Put $m := h_1 \oplus p$, then

$$g = h \oplus m \quad \text{with} \quad Ad(H)\, m = m.$$

In this section, we always assume the following :

Assumption (A) : We take the inner products $(\cdot,\cdot)_{h_1}$, $(\cdot,\cdot)_p$, $(\cdot,\cdot)_m$ as the restrictions to h_1, p, m of the above $Ad(G)$-invariant inner product $(\cdot,\cdot)$ on g, respectively.

Now we consider the Riemannian submersion ϕ ; $G/H \longrightarrow G/K$ admitting the Riemannian metric g (resp. h) on G/H (resp. G/K) corresponding to the inner product $(\cdot,\cdot)$ on m (resp. p). Since the induced bundle $E := \phi^{-1}T(G/K)$ is identified with the associate bundle $G\times_H p$, which is the space of the equivalence classes of $(x,X) \in G\times p$ under the equivalence relation $(xh, Ad(h)X) \sim (x,X)$, for $h \in H$, we can identify the space $\Gamma(E)$ of its sections with the following space :

Definition 8.1. Let $C^{\infty}(G,p)$ be the space of all smooth maps of G into p. We define the subspace $C^{\infty}_H(G,p)$ of $C^{\infty}(G,p)$ by

$$C^{\infty}_H(G,p) := \left\{ f \in C^{\infty}(G,p) \ ; \ f(xh) = Ad(h^{-1})f(x), \ x \in G, \ h \in H \right\}.$$

The identification Φ of $\Gamma(E)$ with $C^{\infty}_H(G,p)$, $\Phi \ ; \ C^{\infty}_H(G,p) \longrightarrow \Gamma(E)$, is

$$(8.1) \qquad \Phi(f)(xH) := \tau_{x*} f(x)_{\{K\}} \quad , \ x \in G.$$

Here $f(x)_{\{K\}}$ is the tangent vector of G/K at the origin $\{K\}$ corresponding to $f(x) \in p$, and τ_{x*} is the differential of the translation $\tau_x \ ; \ G/K \ni yK \longmapsto xyK \in G/K$. Then it turns out that Φ is an isomorphism of $C^{\infty}_H(G,p)$ onto $\Gamma(E)$. Under the G-actions on $\Gamma(E)$ or $C^{\infty}_H(G,p)$ defined by

$$(\tau_{x*}V)_{yH} := \tau_{x*}V_{x^{-1}yH} \quad , \quad x,y \in G, \quad V \in \Gamma(E),$$

$$(\tau_x f)(y) := f(x^{-1}y) \quad , \quad x,y \in G \ , \ f \in C^{\infty}_H(G,p),$$

Φ is a G-isomorphism, that is,

$$(8.2) \quad \Phi \cdot \tau_x f = \tau_{x*} \Phi(f), \quad x \in G \ , \quad f \in C^{\infty}_H(G,p).$$

Note that the Jacobi operator $J_{\phi} \ ; \ \Gamma(E) \longrightarrow \Gamma(E)$ is G-invariant, that is,

$$(8.3) \qquad J_{\phi}(\tau_{x*}V) = \tau_{x*}(J_{\phi}V) \ , \quad V \in \Gamma(E).$$

In fact, here we denote also by τ_{x*} is the differential of the translation τ_x on G/H or G/K by $x \in G$. Then we have $\tau_{x^{-1}*}\nabla_{e_i} e_i = \nabla_{\tau_{x^{-1}*}e_i} \tau_{x^{-1}*} e_i$, and $\widetilde{\nabla}_{e_i} \tau_{x*} V = \tau_{x*} \widetilde{\nabla}_{\tau_{x^{-1}*} e_i} V$,

for $V \in \Gamma(E)$, $x \in G$, where $\{e_i\}_{i=1}^{m}$ is an orthonormal local frame field on $(G/H,g)$. Because of the expression (1.4) of J_{ϕ} , we have

Furthermore we identify $C_H^\infty(G,p)$ with the subspace $(C^\infty(G)\otimes p)_H$ of the tensor product $C^\infty(G)\otimes p$:

Definition 8.2. $(C^\infty(G)\otimes p)_H$ is by definition the subspace of $C^\infty(G)\otimes p$ consisting of all elements $\sum_{i=1}^{\ell} f_i\otimes X_i \in C^\infty(G)\otimes p$ satisfying

$$\sum_{i=1}^{\ell} R_h f_i \otimes Ad(h)X_i = \sum_{i=1}^{\ell} f_i\otimes X_i \quad \text{for all} \quad h\in H.$$

Here $(R_h f)(x) := f(xh)$, $h\in H$, $x\in G$, $f\in C^\infty(G)$.

Under the G-action of $C^\infty(G)\otimes p$ defined by

$$\tau_x(f\otimes X) := \tau_x f\otimes X , \quad x , y\in G , \quad f\in C^\infty(G) , X\in p ,$$

the subspace $(C^\infty(G)\otimes p)_H$ is a G-submodule. The identification Ψ of $C_H^\infty(G,p)$ with $(C^\infty(G)\otimes p)_H$ is given by

$$(8.4) \qquad \Psi(f) := \sum_{i=1}^{n} f_i\otimes X_i , \qquad f\in C_H^\infty(G,p),$$

where $f(x) = \sum_{i=1}^{n} f_i(x)\, X_i$, $x\in G$, and $\{X_i\}_{i=1}^{n}$ is a fixed orthonormal basis of p with respect to $(\, , \,)$. Then it turns out that Ψ is a G-isomorphism of $C_H^\infty(G,p)$ onto $(C^\infty(G)\otimes p)_H$:

$$(8.5) \qquad \Psi\cdot\tau_x = \tau_x\cdot\Psi \qquad , x\in G.$$

Definition 8.3. Via Φ and Ψ, we can define a G-invariant operator $\widetilde{J}$ on $(C^\infty(G)\otimes p)_H$ from the Jacobi operator J_ϕ in such a way that the following diagram is commutative :

$$\begin{array}{ccccc}
\Gamma(E) & \xrightarrow{\Phi^{-1}} & C_H^\infty(G,p) & \xrightarrow{\Psi} & (C^\infty(G)\otimes p)_H \\
\downarrow J_\phi & & & & \downarrow \widetilde{J} \\
\Gamma(E) & \xrightarrow{\Phi^{-1}} & C^\infty(G,p) & \xrightarrow{\Psi} & (C^\infty(G)\otimes p)_H
\end{array}$$

By (8.2), (8.3) and (8.5), the operator $\tilde{\mathfrak{J}}$ is G-invariant , that is,

$$(8.6) \qquad \tilde{\mathfrak{J}} \cdot \tau_x = \tau_x \cdot \tilde{\mathfrak{J}} \quad , \; x \in G.$$

Therefore the problem to determine the spectrum of $\mathfrak{J}_\phi$ is reduced to the one of the operator $\tilde{\mathfrak{J}}$ on $(C^\infty(G)\otimes p)_H$. The main purpose of this section is to express the operator $\tilde{\mathfrak{J}}$ in terms of the Lie algebra g of G for the above aim (cf. Theorem 8.11).

8.2. For the calculus, we take a neiborhood U in G and a subset N (resp. N_K) of G (resp. K) in such a way that

(i) $N = U \cap \exp(p)$, $N_K = U \cap \exp(h_1)$,

(ii) The map $N \times N_K \ni (y,k) \longmapsto yk \in N \cdot N_K$ is a diffeomorphism,

(iii) The projection π_K of G onto G/K is a diffeomorphism of N onto a neighborhood $\pi_K(N)$ of the origin $\{K\}$ in G/K, and

(iv) the projection π_H of G onto G/H is a diffeomorphism of $N \cdot N_K$ onto a neighborhood $\pi_H(N \cdot N_K)$ of the origin $\{H\}$ in G/H, where $N \cdot N_K := \{yk \; ; \; y \in N, \; k \in N_K\}$.

Now for an element $X \in m = h_1 \oplus p$, define a vector field X^* on the neighborhood $\pi_H(N \cdot N_K)$ of $\{H\}$ in G/H by

$$(8.7) \qquad X^*_{xH} := \tau_{x*} X_{\{H\}} \in T_{xH} G/H \; , \quad x \in N \cdot N_K \; .$$

Similarly, for an element $X \in p$, define a vector field $\bar{X}$ on the neighborhood $\pi_K(N)$ of $\{K\}$ in G/K by

$$(8.8) \qquad \bar{X}_{yK} := \tau_{y*} X_{\{K\}} \in T_{yK} G/K \; , \quad y \in N \; .$$

Let $\{X_i\}_{i=1}^m$ be an orthonormal basis of $(m, (\; , \;))$ such that $\{X_i\}_{i=1}^n$ (resp. $\{X_i\}_{i=n+1}^m$) is a basis of p (resp. h_1). Then $\{X_i^*\}_{i=1}^m$ is an orthonormal frame field on $\pi_H(N \cdot N_K)$ such that X_i^*, $n+1 \leq i \leq m$, are vertical and X_i^*, $1 \leq i \leq n$, are horizontal. And $\{\bar{X}_i\}_{i=1}^n$ is also an orthonormal frame field on $\pi_K(N)$.

Remark. In general, X_i^*, $1 \leq i \leq n$, are not necessarily basic vector fields.

For every $f \in C_H^\infty(G,p)$, we can express $V = \Phi(f) \in \Gamma(E)$ as

$$V_{xH} = \sum_{i=1}^n f_i(x) \; \tau_{x*} X_{i\{K\}} \quad , \quad x \in G ,$$

where $f(x) = \sum_{i=1}^n f_i(x) \; X_i$, $x \in G$. Moreover, putting

$$(8.9) \qquad Ad(k) X_i = \sum_{j=1}^n a_{ij}(k) X_j \quad , \quad k \in K ,$$

$$(8.10) \qquad \tilde{f}_j(ykH) := \sum_{i=1}^n f_i(yk) \; a_{ij}(k) \quad , \quad y \in N \; , \; k \in N_K ,$$

the section V can be expressed on the neighborhood $\pi_H(N \cdot N_K)$ as

$$(8.11) \qquad V = \sum_{j=1}^n \tilde{f}_j \; \widetilde{\bar{X}}_j ,$$

where $\tilde{f}_j$ is a function (8.10) on $\pi_H(N \cdot N_K)$ and $\widetilde{\bar{X}}_j$ is a local section of E corresponding to the vector field $\bar{X}_j$ on $\pi_H(N \cdot N_K)$(cf.1.

Then we have for $X \in m$,

$$(8.12) \qquad \tilde{\nabla}_{X^*} V = \sum_{j=1}^n \left\{ (X^* \tilde{f}_j) \; \widetilde{\bar{X}}_j + \tilde{f}_j \; \tilde{\nabla}_{X^*} \widetilde{\bar{X}}_j \right\} \quad \text{on} \quad \pi_H(N \cdot N_K).$$

Here $(\tilde{\nabla}_{X^*} \widetilde{\bar{X}}_j)_{xH}$, $x \in N \cdot N_K$, is given as follows :

$$(8.13) \qquad (\tilde{\nabla}_{X^*} \widetilde{\bar{X}}_j)_{xH} = ({}^N\nabla_W \bar{X}_j)_{xK} ,$$

where W is a locally defined vector field on G/K satisfying $W_{xK} = \phi_* X^*_{xH}$ (cf. (1.1) or (6.4)), and ${}^N\nabla$ is the Levi-Civita connection of $(G/K,g)$. This vector field W can be actually chosen as follows :

$$(8.14) \qquad W = 0 \quad \text{for} \quad X \in h_1 ,$$

(8.15) $\quad W = \overline{(Ad(k(\cdot))X)}$ (cf. (8.8)), for $X \in p$.

In fact, since $\phi_* X^*_{xH} = 0$ for $X \in h_1$, we have (8.14). For (8.15), let $X \in p$. For a fixed point $x = y(x)k(x)$, $y(x) \in N$, $k(x) \in N_K$, we have

$$\begin{aligned} \phi_* X^*_{xH} &= \tau_{y(x)*}\, \tau_{k(x)*} X_{\{K\}} \\ &= \tau_{y(x)*} (Ad(k(x))X)_{\{K\}} \\ &= \overline{(Ad(k(x))X)}_{y(x)K}, \end{aligned}$$

so we can choose W as in (8.15). By (8.14), we get, for $X \in h_1$,

$$(8.16) \qquad (\widetilde{\nabla}_{X^*} V)_{xH} = \sum_{j=1}^{n} (X^* \tilde{f}_j)(xH)\ (\widetilde{\overline{X}_j})_{xH} .$$

By (8.15), we get in particular, for $X \in p$,

$$(8.17) \qquad (\widetilde{{}^N\nabla_W \overline{X}_j})_{\{H\}} = ({}^N\nabla_{\overline{X}}\ \overline{X}_j)_{\{K\}} .$$

Moreover we get, for $X \in p$,

$$(8.18) \qquad (\widetilde{\nabla}_{X^*} \widetilde{\nabla}_{X^*} \widetilde{\overline{X}}_j)_{\{H\}} = \frac{1}{4}\ ([X,[X,X_j]_p]_p)_{\{K\}} \in T_{\{K\}}G/K ,$$

where X_p is the p-component of X corresponding to the decomposition $g = k + p$.

Proof of (8.18). Let us recall the following :

Lemma 8.4. For every $Y, Z \in p$,

$${}^N\nabla_{\overline{Z}}\ \overline{Y} = \frac{1}{2}\overline{([Z,Y]_p)} \quad \text{, along the curve } \xi(t)K \text{ in } G/K$$

for a sufficiently small t such that $\xi(t) := \exp(tZ)$ belongs to N.

This lemma follows from Theorems 8.1, 10.1 and 13.2 in [No], due to the assumption (A).

$$(8.19) \qquad (\widetilde{\nabla}_{X^*}\widetilde{\nabla}_{X^*}\widetilde{\overline{X}}_j)_{\{H\}} = ({}^N\nabla_W\, {}^N\nabla_W \overline{X}_j)_{\{K\}},$$

where W is in (8.15). Then for the curve $\sigma(t) := \exp(tX)K$ in G/K,

$$\text{the right hand side of } (8.19) = \frac{d}{dt}\, {}^N p_{\sigma(t)}{}^{-1} ({}^N\nabla_W \overline{X}_j)_{\sigma(t)}\Big|_{t=0},$$

where ${}^N p_{\sigma(t)}$ is the parallel transprort of $(G/K, g)$ along the curve $\sigma(t)$. Here $W_{\sigma(t)} = \overline{X}_{\sigma(t)}$, because (8.15) and $\exp(tX) \in N$, and then $k(\sigma(t)) = e$. Then we have

$$({}^N\nabla_W \overline{X}_j)_{\sigma(t)} = ({}^N\nabla_{\overline{X}}\, \overline{X}_j)_{\sigma(t)} = \frac{1}{2}\, \overline{([X, X_j]_p)}_{\sigma(t)},$$

by Lemma 8.4. Therefore

$$\begin{aligned}\text{the right hand side of } (8.19) &= \frac{1}{2}\frac{d}{dt}\, {}^N p_{\sigma(t)}{}^{-1}\, \overline{([X, X_j]_p)}_{\sigma(t)}\Big|_{t=0} \\ &= \frac{1}{2}\, ({}^N\nabla_{\overline{X}} \overline{[X, X_j]_p})_{\{K\}} \\ &= \frac{1}{4}\, ([X, [X, X_j]_p]_p)_{\{K\}},\end{aligned}$$

again by Lemma 8.4, which implies (8.18).

Summing up the above, we have :

Lemma 8.5. For $V = \Phi(f)$, $f \in C_H^\infty(G, p)$, we have

(i) $(\widetilde{\nabla}_{X^*}\widetilde{\nabla}_{X^*} V)_{\{H\}} = \sum_{j=1}^n X^*_{\{H\}}(X^* \widetilde{f}_j)\, \overline{X}_{j\{K\}}$, for $X \in h_1$,

(ii) $(\widetilde{\nabla}_{X^*}\widetilde{\nabla}_{X^*} V)_{\{H\}} = \sum_{j=1}^n X^*_{\{H\}}(X^* \widetilde{f}_j)\, \overline{X}_{j\{K\}} + (X^*_{\{H\}}\widetilde{f}_j)\, \overline{([X, X_j]_p)}_{\{K\}}$
$+ \frac{1}{4}\, \widetilde{f}_j(H)\, \overline{([X, [X, X_j]_p]_p)}_{\{K\}}$,

for $X \in p$.

Our next task is to calculate $X^*_{\{H\}}\widetilde{f}_j$ and $X^*_{\{H\}} X^* \widetilde{f}_j$, for $X \in m$.

Lemma 8.6.

(i) For $X \in h_1$, we have

$$X^*_{\{H\}} \widetilde{f}_j = Xf_j(e) + \sum_{i=1}^{n} f_i(e) ([X, X_i], X_j), \quad \text{and}$$

$$X^*_{\{H\}} X^* \widetilde{f}_j = X^2 f_j(e) + 2 \sum_{i=1}^{n} (Xf_i)(e)([X, X_i], X_j) + \sum_{i=1}^{n} f_i(e)([X, [X, X_i]], X_j).$$

(ii) For $X \in p$, we have

$$X^*_{\{H\}} \widetilde{f}_j = Xf_j(e), \quad \text{and} \quad X^*_{\{H\}} X^* \widetilde{f}_j = X^2 f_j(e).$$

Proof follows immediately from the definition of $\widetilde{f}_j$ (8.9),(8.10) and X^* (8.7).

Lemma 8.7.

$$(\widetilde{\nabla}_{\nabla_{X^*} X^*} V)_H = 0 \quad \text{for all} \quad X \in m, \text{ and } V \in \Gamma(E).$$

Proof. Due to the assumption (A), we have

$$(\nabla_{X^*} X^*)_{\{H\}} = 0 \quad \text{for} \quad X \in m,$$

by Theorems 8.1, 13.1 in [No]. By (8.13) or (1.1), we have Lemma 8.7.

Moreover, it is known (cf. [K.N]) that under the assumption (A), the curvature tensor ${}^N R$ of $(G/K, h)$ is given by

$$-({}^N R(X,Y)Z)_{\{K\}} = \frac{1}{4}[X,[Y,Z]_p]_p - \frac{1}{4}[Y,[X,Z]_p]_p - \frac{1}{2}[[X,Y]_p, Z]_p - [[X,Y]_k, Z], \quad X, Y, Z \in p,$$

where we identify $X \in p$ with the tangent vector $X_{\{K\}} \in T_{\{K\}} G/K$. Then we ge

Lemma 8.8. For $V = \Phi(f)$, $f \in C_H^\infty(G,p)$, we have

$$-\left({}^N R(\phi_* X^*, V)\phi_* X^*\right)_{\{K\}} = \begin{cases} 0 & , \ X \in h_1 , \\ \sum_{i=1}^n f_i(e)\Big\{\frac{1}{4}[X,[X_i,X]_p]_p - \frac{1}{2}[[X,X_i]_p,X]_p \\ \qquad - [[X,X_i]_k,X] \Big\} & , \ X \in p. \end{cases}$$

Summing up Lemmas 8.5 ∼ 8.8, we obtain :

Proposition 8.9. For $V = \Phi(f)$, $f = \sum_{i=1}^n f_i X_i \in C_H^\infty(G,p)$, the evaluation of $J_\phi V$ at the origin $\{H\}$ in G/H is given by

$$\begin{aligned}(J_\phi V)_{\{H\}} = &- \sum_{k=1}^m \sum_{j=1}^n (X_k{}^2 f_j)(e)\ X_{j\{K\}} \\ &- \sum_{k,j=1}^n (X_k f_j)(e)\,[X_k,X_j]_{p\{K\}} \\ &- 2\sum_{k=n+1}^m \sum_{j=1}^n (X_k f_j)(e)\,[X_k,X_j]_{\{K\}} \\ &- \sum_{k=n+1}^m \sum_{j=1}^n f_j(e)\,[X_k,[X_k,X_j]]_{\{K\}} \\ &- \sum_{k,j=1}^n f_j(e)\,[[X_k,X_j]_k,X_k]_{\{K\}} \ .\end{aligned}$$

8.3. Before we state Theorem 8.11, we have to prepare some notations :

Definition 8.10. We define the operators D_i, $i=0,1,\dots,6$, acting on $C^\infty(G)\otimes p$ by

$$D_0 := \sum_{k=1}^s X_k{}^2 \otimes I ,$$

$$D_1 := \sum_{k=1}^m X_k{}^2 \otimes I ,$$

$$D_2 := \sum_{k=1}^n X_k \otimes P_p \cdot \mathrm{ad}(X_k) ,$$

$$D_3 := \sum_{k=n+1}^{m} X_k \otimes ad(X_k),$$

$$D_4 := I \otimes \sum_{k=n+1}^{m} ad(X_k)^2 ,$$

$$D_5 := I \otimes \sum_{k=1}^{n} ad(X_k) \circ P_k \circ ad(X_k),$$

$$D_6 := \sum_{k=m+1}^{s} X_k{}^2 \otimes I ,$$

where P_p , P_k are the projection of $g = k \oplus p$ onto p , k , respectively, $\{X_k\}_{k=1}^{s}$ is an orthonormal basis of $(g, (\, , \,))$ such that $\{X_i\}_{i=1}^{n}$ (resp. $\{X_i\}_{i=n+1}^{m}$, $\{X_i\}_{i=m+1}^{s}$) is a basis of p (resp. h_1, h), I is the identity operator of $C^{\infty}(G)$, p or $C^{\infty}(G) \otimes p$, and $(Xf)(x) := \frac{d}{dt} f(x \exp(tX))\Big|_{t=0}$, for $X \in g$, $f \in C^{\infty}(G)$, and $x \in G$.

It turns out all D_i, $i=0,1,\ldots,6$, do not depend on the choice of the above basis $\{X_k\}_{k=1}^{s}$ and they are G-invariant, i.e., $D_i \cdot \tau_x = \tau_x \cdot D_i$, for all $x \in G$. Noting that

$$R_h \cdot Xf = Ad(h)X \, (R_h f) \quad , \quad f \in C^{\infty}(G), \; h \in H, \text{ and } X \in g ,$$

all D_i preserve the subspace $(C^{\infty}(G) \otimes p)_H$ invariant , because of independency on the choice of the basis $\{X_k\}_{k=1}^{s}$. We also note that

$$(8.20) \qquad D_o = D_1 + D_6 ,$$

$$(8.21) \qquad D_6 = I \otimes \sum_{k=m+1}^{s} ad(X_k)^2 \quad \text{on} \quad (C^{\infty}(G) \otimes p)_H ,$$

because of definitions of $(C^{\infty}(G) \otimes p)_H$ and D_6 . Then by Proposition 8. we obtain :

Theorem 8.11. Let ϕ be the Riemannian submersion of $(G/H, g)'$ onto $(G/K, h)$ whose metrics g , h come from the $Ad(G)$-invariant inner product $(\, , \,)$ on the Lie algebra g . Then the operator $\tilde{J}$ of $(C^{\infty}(G) \otimes p)_H$ corresponding to the Jacobi operator J_{ϕ} of the submersion ϕ coincides with the operator

$$D := - D_0 - D_2 - 2D_3 - D_4 + D_5 + D_6 ,$$

where all D_i are defined in Definition 8.10.

Proof. Proposition 8.9 and (8.21) say that

$$\tilde{J}(\Psi\Phi^{-1}v)(e) = D(\Psi\Phi^{-1}v)(e),$$

for every $v \in \Gamma(E)$. For every $x \in G$, we have

$$\begin{aligned}\tilde{J}(\Psi\Phi^{-1}v)(x) &= \tau_{x^{-1}}\cdot\tilde{J}(\Psi\Phi^{-1}v)(e) \\ &= \tilde{J}(\Psi\Phi^{-1}\tau_{x^{-1}*}v)(e) \\ &= D(\Psi\Phi^{-1}\tau_{x^{-1}*}v)(e) \\ &= \tau_{x^{-1}} D(\Psi\Phi^{-1}v)(e) = D(\Psi\Phi^{-1}v)(e).\end{aligned}$$ Q.E.D.

As applications of Theorem 8.11, we obtain :

Corollary 8.12. Let ϕ be the Riemannian submersion of $(G/H,g$ onto $(G/K,h)$ whose metrics g , h come from the $Ad(G)$-invariant inner product $(\cdot,\cdot)$ on the Lie algebra g. Assume that $(G/K,h)$ is Riemannian symmetric, g is semi-simple , and $(X,Y) := -F(X,Y)$, $X,Y \in g$, where F is the Killing form of g .

(i) Then the operator $\tilde{J}$ of $(C^\infty(G)\otimes p)_H$ corresponding to the Jacobi operator J_ϕ of the submersion ϕ coincides with

$$D := - D_0 - 2D_3 + 2D_6 .$$

Moreover we assume $H = \{id\}$. Then the operator $\tilde{J}$ coincides with

$$D := - D_0 - 2D_3 ,$$

where D_0, D_3 and D_6 are defined in Definition 8.10.

(ii) In particular, the spectrum of the Jacobi operator J_ϕ of the Hopf fibering ϕ; $(SU(2),g) = (S^3,g) \longrightarrow (SU(2)/S(U(1)\times U(1)),h) = (S^2,h)$ is given as follows :

The eigenvalues : $\frac{1}{2}\ell(\ell+1)+i$, $\frac{1}{2}\ell(\ell+1)-i$,

their multiplicities : $2\ell+1$,

where ℓ varies over the set $\{\ell \in \frac{1}{2}\mathbb{Z} \ ; \ \ell \geqq 0\}$, and i varies over the set $\{\ell, \ell-1, \ldots, 1-\ell, -\ell\}$. Then the index and the nullity are given as $\mathrm{Index}(\phi) = 4$ and $\mathrm{Nullity}(\phi) = 7$.

Proof. (i) Since $(G/K,h)$ is symmetric, i.e., $[p,p] \subset k$, we have $D_2 = 0$ and $D_5 = I \otimes \sum_{k=1}^{n} \mathrm{ad}(X_k)^2$. Moreover, we have $D_5 = -\frac{1}{2} I$ and $D_4 + D_6 = -\frac{1}{2} I$, which implies (i). For it follows from that $(\sum_{k=1}^{n} \mathrm{ad}(X_k)^2(X),Y) = \frac{1}{2} F(X,Y)$, and $(\sum_{k=n+1}^{s} \mathrm{ad}(X_k)^2(X),Y) = \frac{1}{2} F(X,Y)$, for $X,Y \in p$ (cf.[T.K, p.212]). The second equality is clear from that $D_6 = 0$ when $H = \{id\}$.

(ii) Let us recall the computation in [U1, §5]. In this case,

$$G = SU(2),$$
$$K = S(U(1)\times U(1)) = \left\{ \begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta} \end{pmatrix}; \ \theta \in \mathbb{R} \right\} ,$$
$$(X,Y) = -4\,\mathrm{Trace}(XY), \quad X,Y \in g = su(2) ,$$
$$k = \{H_1\}_{\mathbb{R}} ,$$
$$p = \{U_\alpha/\sqrt{2} \ , \ V_\alpha/\sqrt{2}\}_{\mathbb{R}} ,$$

where $H_1 := \frac{\sqrt{-1}}{2\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $U_\alpha := 2^{-1}\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$ and $V_\alpha := 2^{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$\{H_1 \ , \ U_\alpha/\sqrt{2} \ , \ V_\alpha/\sqrt{2}\}$ is an orthonormal basis of $(g, (\cdot,\cdot))$. We have only to know the actions of $D_3 = H_1 \otimes \mathrm{ad}(H_1)$ and $D_0 = C \otimes I$ on $C^\infty(G) \otimes p$, where C is the Casimir operator $C := H_1^2 + U_\alpha^2/2 + V_\alpha^2/2$.

A complete orthogonal basis of the space $C^{\infty}_{\mathbb{C}}(G)$ of complex valued smooth functions on G with respect to the inner product $\int_G f(x)\overline{f'(x)}dx$, $f, f' \in C^{\infty}_{\mathbb{C}}(G)$, with the Haar measure dx, is given as follows (Peter-Weyl theorem) : Let $\mathbb{D} := \{ \ell\alpha \ ; \ell \in \frac{1}{2}\mathbb{Z} \ , \ \ell \geq 0 \}$. For $\lambda = \ell\alpha \in \mathbb{D}$, let (V_λ, π^λ) be the irreducible unitary representation of G with highest weight λ, and $\{v_i\}_{i=1}^{d_\lambda}$, $d_\lambda := \dim(V_\lambda)$, an orthonormal basis of V_λ with respect to the G-invariant inner product $((\ , \ ))$ on V_λ. Put $\pi^\lambda_{ij}(x) := ((\pi^\lambda(x)v_i, v_j))$, $1 \leqq i,j \leqq d_\lambda$. Then

$$X\,\pi^\lambda_{ij}(x) = ((\pi^\lambda(x)\pi^\lambda(X)v_i, v_j)), \quad X \in \mathfrak{g} \ , \ 1 \leqq i,j \leqq d_\lambda \ ,$$

and $\{\pi^\lambda_{ij} \ , \lambda \in \mathbb{D} \ , \ 1 \leqq i,j \leqq d_\lambda\}$ is an orthogonal basis of $C^{\infty}_{\mathbb{C}}(G)$.
For $\lambda = \ell\alpha$ with $\ell \in \frac{1}{2}\mathbb{Z}$, $\ell \geqq 0$, V_λ has an orthonormal basis $\{v_m \ ; m = \ell, \ell-1, \ldots, 1-\ell, -\ell\}$ such that

$$\pi^\lambda(H_1)v_m = \frac{\sqrt{-1}}{\sqrt{2}}\, m\, v_m \quad , \text{ for each } m \ .$$

Since $\pi^\lambda(C) = \frac{1}{2}\ell(\ell+1)\, I$ on V_λ, we get

$$H_1\,\pi^\lambda_{ij}(x) = \frac{\sqrt{-1}}{\sqrt{2}}\, i\, \pi^\lambda_{ij}(x),$$

$$C\,\pi^\lambda_{ij}(x) = \frac{1}{2}\ell(\ell+1)\,\pi^\lambda_{ij}(x) \quad , \text{ for } i,j = \ell, \ell-1, \ldots, 1-\ell, -\ell \ .$$

On the other hand,

$$\operatorname{ad}(H_1)(U_\alpha/\sqrt{2}) = \frac{1}{\sqrt{2}}(V_\alpha/\sqrt{2}) \quad \text{and} \quad \operatorname{ad}(H_1)(V_\alpha/\sqrt{2}) = -\frac{1}{\sqrt{2}}(U_\alpha/\sqrt{2}).$$

Then the action of $D_3 = H_1 \otimes \operatorname{ad}(H_1)$ on $\mathcal{J}_\lambda \otimes \mathfrak{p}$, where $\mathcal{J}_\lambda := \{\pi^\lambda_{ij} \ ; \ 1 \leqq i,j \leqq d_\lambda\}_{\mathbb{C}}$ is equivalent to the matrix

$$\begin{pmatrix} \lambda_\ell & & & & \\ & \lambda_{\ell-1} & & & \\ & & \ddots & & \\ & & & \lambda_{1-\ell} & \\ & & & & \lambda_{-\ell} \end{pmatrix} \otimes \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \begin{matrix} 0 & -\lambda_\ell \\ \lambda_\ell & 0 \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} 0 & -\lambda_{-\ell} \\ \lambda_{-\ell} & 0 \end{matrix} \end{pmatrix},$$

where $\lambda_i := \frac{\sqrt{-1}}{\sqrt{2}} i$, $i=\ell,\ell-1,\ldots,1-\ell,-\ell$. Therefore the eigenvalues of D_3 on $\vartheta_\lambda \otimes p$ are given by $\pm \frac{i}{2}$, $i=\ell,\ell-1,\ldots,1-\ell,-\ell$. Hence the spectrum of $D = -D_0 - 2D_3$ is given as in (ii). Q.E.D.

Instead of the assumption of Corollary 8.12, we now assume that $K = H$. In this case, we obtain the formula of $\widetilde{J}$ of the Jacobi operator J_{id} of the identity map of a normally homogeneous space $(G/H,g)$. Here we have $k = h$, $h_1 = 0$, $m = p$ and $D_3 = D_4 = 0$. Then we obtain :

Corollary 8.13. Let $(G/H,g)$ be a normally homogeneous space, that is, the metric g is induced from the Ad(G)-invariant inner product $(\cdot,\cdot)$ on the Lie algebra g. Then the operator $\widetilde{J}$ of $(C^\infty(G)\otimes m)_H$ corresponding to the Jacobi operator J_{id} of the identity map of $(G/H,g)$ coincides with

$$D = -D_0 - D_2 + D_5 + D_6 ,$$

where m is the orthogonal complement of h in g with respect to $(\cdot,\cdot)$ and D_0, D_2, D_5 and D_6 are defined in Definition 8.10.

In particular, assume that $(G/h,g)$ is Riemannian symmetric, g is semi-simple, and $(X,Y) := -F(X,Y)$, X,Y g, where F is the Killing form of g. Then

$$D = -D_0 - I ,$$

where I is the identity map of $(C^\infty(G)\otimes m)_H$.

Proof. The last formula follows from $D_2 = 0$ and $D_5 + D_6 = -I$.

Remark. The last formula $D = -D_0 - I$ for the Jacobi operator of the identity map of a Riemannian symmetric space was stated in [Na].

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Max-Planck-Institut für Mathematik,

Gottfried Claren Straße 26,

5300 Bonn 3, Federal Republic of Germany

and

Department of Mathematics,

College of General Education,

Tohoku University,

Kawauchi, Sendai, 980, Japan