# SHADOWS OF BLOW-UP ALGEBRAS

PAOLO ALUFFI

Abstract. Recasting the functorial theory in terms of Segre classes of coherent sheaves would provide a unified treatment of different characteristic classes defined for singular varieties, such as the functorial Chern-Schwartz-MacPherson class and those introduced by Fulton and Fulton-Johnson.

We carry out this program for hypersurfaces, under a technical hypothesis, by producing the Chern-Schwartz-MacPherson class of a hypersurface as the 'shadow' of a cycle determined by the symmetric algebra of its singularity subscheme.

This yields the Chern-Schwartz-MacPherson class of a hypersurface in terms of the Segre class of an explicitly realized coherent sheaf, putting it on the same footing as the classes defined by Fulton and Johnson.

We also discuss in some detail the technical hypothesis needed for the result, identifying situations in which it is necessarily satisfied.

## 1. Introduction

The *blow-up algebras* in the title are the symmetric and Rees algebras of the Jacobian ideal of a hypersurface $X$ in a nonsingular variety $M$; the *shadow* is our terminology for the well-known relation between rational equivalence classes on a variety and classes on a projective bundle over it. The blow-up algebras determine classes in a projective bundle over $X$; in this paper we show that the 'shadows of the blow-up algebras' of $X$ recover familiar characteristic classes of $X$. With suitable positions, the shadow of the *Rees* algebra computes the *Chern-Mather* class of $X$, while (provided $X$ satisfies a certain technical condition) the shadow of its *symmetric* algebra is essentially its *Chern-Schwartz-MacPherson* class (Theorem 3.4).

The work of Claude Sabbah *montre que la théorie des classes de Chern de* [Mac74] *se ramène à une théorie de Chow sur* $T^*M$ ... ([Sab85]), translating the theory of Chern-Schwartz-MacPherson's classes into a study of Lagrangian cycles of $T^*M$, and of classes obtained from them. Our *leitmotiv* consists of a search for a (perhaps more direct) translation of the theory of Chern-Schwartz-MacPherson classes into the language of *Segre classes of coherent sheaves* (in the sense of [FJ80] and [Ful84], Chapter 4).

As is well-known, the category of coherent sheaves on a scheme is equivalent to the category of 'linear fiber spaces' (*Lineare Faserräume*, cf. [Fis67]). The key to the main result in this paper is the observation that the linear fiber space corresponding to the ideal of Jacobian ideal sheaf of a hypersurface $X$ provides an intrinsic realization of the characteristic cycle of $X$, at least if $X$ satisfies our technical hypothesis (the '$\times$-condition'). This is established in §2. The application to characteristic classes is worked out in §3.

In §4 we discuss the $\times$-condition and prove that it holds for several cases in which we have good control on the singularities: indeed, the $\times$-condition is verified for all hypersurfaces whose singularity subscheme embeds *linearly* (in the sense of [Kee93]) in the ambient space; examples of such hypersurfaces abound. From a different viewpoint, the $\times$-condition may be seen as a regularity requirement on the extension of tangent vectors across strata of a stratification of $X$ (cf. §4.6). Since the regularity of extensions of tangent vectors to strata is controlled by conditions—such as the $w_f$ condition of Thom—which hold for *all* Whitney stratifications of *all* hypersurfaces ([Par93] and [BMM94]), it may be that something close to the $\times$-condition (and perhaps sufficient to trigger the arguments in §2 and §3) holds for *all* hypersurfaces. However, it is not hard to produce examples of hypersurfaces which do not satisfy a strong form of the $\times$-condition (cf. §4.5).

The same notion of Segre class of a coherent sheaf used in this paper has been employed before to define intrinsic characteristic classes of schemes: William Fulton and Kent Johnson ([FJ80] and [Ful84], Example 4.2.6 (c)) have developed such a theory *"about as far as Todd's work in the nonsingular case ... "* The protagonist in the work of Fulton and Johnson was the *conormal sheaf* of a scheme in an ambient nonsingular variety; the results we obtain here direct our attention to another naturally defined coherent sheaf: the cokernel of the dual of the map on differentials determined by the embedding. This also leads to the definition of a characteristic class, which we study more carefully elsewhere (cf. [Alu01]). We show here that this new class agrees with Chern-Schwartz-MacPherson's for hypersurfaces satisfying the $\times$-condition (Theorem 3.9).

For a hypersurface, Fulton-Johnson's class agrees with the class of the virtual tangent bundle, as does the class defined by Fulton in Example 4.2.6 (a) of [Ful84], obtained from the Segre class of the *normal cone* of the scheme in a nonsingular variety. While relations between these notions and Chern-Schwartz-MacPherson classes of hypersurfaces and complete intersections have been studied intensely (cf. [Alu94], [Suw97], [BLSS99], [Yok99], [Alu99a], [Alu99b], [Suw00], [Alu00], [PP01] and the very recent [Sch01a] to name a few) we feel that the natural task of finding a common framework for classes defined in the style of Fulton's and Fulton-Johnson's on the one hand, and of Chern-Mather and Chern-Schwartz-MacPherson on the other, is still largely unaccomplished.

The results obtained in this article are the closest we know to a solution of this problem, for the class of hypersurfaces considered here.

Our motivation in pursuing this program resides in the belief that it would be highly worthwhile to uncover any functoriality feature of classes such as Fulton's or Fulton-Johnson's. Chern-Schwartz-MacPherson's classes owe their existence precisely to their excellent functoriality properties; if such functoriality could be transferred to Segre classes (via formulas such as the ones presented in this article), this would offer a new handle on computing Segre classes, arguably one of the most basic invariants in intersection theory.

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## 2. The characteristic cycle of a hypersurface

2.1. We work over an algebraically closed field of characteristic 0. Throughout the paper $M$ will denote a nonsingular irreducible algebraic variety of dimension $m$, and $X$ will be the zero-scheme of a nonzero section $F$ of a line bundle $\mathcal{L}$ on $M$; we will say that $X$ is a *hypersurface* for short. We will assume for simplicity that $X$ is reduced (but see §2.9). The singularity locus of $X$ has an interesting, possibly nonreduced scheme structure. We will denote by $Y$ this *singularity subscheme* of $X$; the precise definition will be recalled in §2.6.

We begin by recalling several well-established notions.

2.2. A *constructible function* on a variety $V$ is a finite linear combination

$$\sum n_W \, 1\!\!1_W$$

where the summation ranges over (closed, irreducible) subvarieties $W \subset V$, $n_W \in \mathbb{Z}$, and $1\!\!1_W$ denotes the function that is the constant 1 on $W$, and 0 outside of $W$. We denote by $C(V)$ the group of constructible functions on $V$. If $f : V_1 \to V_2$ is a proper map, a push-forward $C(f) : C(V_1) \to C(V_2)$ is defined by setting, for $W$ a subvariety of $V_1$ and $p \in V_2$,

$$C(f)(1\!\!1_W)(p) = \chi(f^{-1}(p) \cap W) \quad ,$$

and extending by linearity. Here $\chi$ denotes the topological Euler characteristic when working over $\mathbb{C}$; see [Ken90], §3, for the extension of the theory to arbitrary algebraically closed field of characteristic 0.

With this push-forward, the assignment

$$\mathcal{C} \quad : \quad V \mapsto C(V)$$

yields a covariant functor from algebraic varieties to abelian groups.

2.3. A fundamental result of Robert MacPherson ([Mac74] and [Ken90]) compares this functor to the functor

$$\mathcal{A} \quad : \quad V \mapsto A(V)$$

assigning to a variety its Chow group: there exists a natural transformation

$$c_* \quad : \quad \mathcal{C} \rightsquigarrow \mathcal{A}$$

such that, for $V$ a *nonsingular* variety, the induced group homomorphism

$$C(V) \to A(V)$$

maps $1\!\!1_V$ to the total Chern class of the tangent bundle of $V$:

$$1\!\!1_V \mapsto c(TV) \cap [V] \quad .$$

For arbitrarily singular $V$, one may then define a (total) 'Chern class' in the Chow group of $V$, by setting

$$c_{\mathrm{SM}}(V) := c_*(1\!\!1_V) \quad ;$$

by the above remarks, $c_{\mathrm{SM}}(V) = c(TV) \cap [V]$ if $V$ is nonsingular. Jean-Paul Brasselet and Marie-Hélène Schwartz later discovered that this class defined by MacPherson is in fact Alexander dual to a class previously defined by Schwartz ([Sch65a], [Sch65b]; and [BS81]); nowadays, $c_{\mathrm{SM}}(V)$ is commonly named the *Chern-Schwartz-MacPherson class* of $V$.

2.4. In MacPherson's approach, the natural transformation $c_*$ is defined directly by requiring that

$$c_*(\mathrm{Eu}_V) = c_{\mathrm{Ma}}(V)$$

for all varieties $V$. Here $c_{\mathrm{Ma}}(V)$ stands for the *Chern-Mather* class of $V$, and $\mathrm{Eu}_V$ is the *local Euler obstruction,* a measure of the singularities of $V$. Both these notions were defined in [Mac74], and have been the subject of intense study since; again we refer the reader to [Ken90] for a very readable treatment and for the extension of the theory to arbitrary algebraically closed fields of characteristic 0.

A different approach to the definition of $c_*$ emerged in the work of Claude Sabbah ([Sab85], [Ken90], and [PP01], §1). The natural transformation $c_*$ can be obtained (up to taking a harmless dual) as the composition of two transformations

$$\mathcal{C} \rightsquigarrow \mathcal{L} \rightsquigarrow \mathcal{A} \quad ,$$

where $\mathcal{L}$ denotes the functor assigning to a variety $V$ the group $L(V)$ of *Lagrangian cycles* over $V$, with a suitably defined push-forward. If $V \subset M$ is an embedding of $V$ into a nonsingular variety $M$, the Lagrangian cycles over $V$ are the Lagrangian cycles in the restriction $\mathbb{P}(T^*M)|_V$ of the projectivized cotangent bundle of $M$. As is well known ([Ken90], Lemma 3), every Lagrangian subvariety over $V$ is in fact the projective conormal space $\mathbb{P}(T^*_W M)$ of a closed subvariety $W \subset V$. Hence $L(V)$ is the free abelian group on the set of subvarieties of $V$; the realization of $L(V)$ as a group of cycles in $\mathbb{P}(T^*M)|_V$, for some nonsingular $M$ containing $V$, yields a good notion of push-forward of elements of $L(V)$ (see p. 2829-31 in [Ken90] for details).

The second step $\mathcal{L} \rightsquigarrow \mathcal{A}$ in the above decomposition can be expressed in terms of standard intersection theory, and will be recalled in §3.3. The first step, $\mathcal{C} \rightsquigarrow \mathcal{L}$, is considerably subtler. It is determined by the requirement that, for all (closed, irreducible) subvarieties $W \subset V$, the local Euler obstruction of $W$ correspond (up to a sign) to the *conormal cycle* of $W$ in $M$:

$$(-1)^{\dim W}\, \mathrm{Eu}_W \mapsto [\mathbb{P}(T^*_W M)]$$

For every constructible function $\varphi \in C(V)$ we obtain then a *characteristic cycle*

$$\mathrm{Ch}(\varphi) \in L(V) \quad .$$

The cycle $\mathrm{Ch}(1\!\!1_V)$ (realized as above, that is, in terms of an embedding $V \subset M$) is called the *characteristic cycle of $V$ (in $M$).*

2.5. Summarizing, there are two important cycles associated to a variety $V$ in the projectivized cotangent bundle $\mathbb{P}(T^*M) = \mathrm{Proj}(\mathrm{Sym}_M((\Omega^1_M)^\vee))$ of any nonsingular variety $M$ in which $V$ is embedded:

- the conormal cycle $[\mathbb{P}(T^*_V M)]$, corresponding (up to sign) to the local Euler obstruction of $V$, and to the Chern-Mather class of $V$; and
- the characteristic cycle $\mathrm{Ch}(V)$ of $V$, likewise corresponding to the constant function $1\!\!1_V$ and to the Chern-Schwartz-MacPherson class of $V$.

Our goal here is the explicit realization of a scheme whose cycle is the characteristic cycle of a hypersurface $X$ of a nonsingular variety $M$.

2.6. An analogous realization of the conormal cycle is readily available. The section $F$ of $\mathcal{L}$ defining $X$ determines a section $s$ of the bundle $\mathcal{P}^1_M\mathcal{L}$ of principal parts of $\mathcal{L}$:

$$s : \mathcal{O}_M \to \mathcal{P}^1_M\mathcal{L} \quad ;$$

we let $Y$ denote the zero-scheme of $s$ in $M$, and we call $Y$ the 'singularity subscheme' of $X$. Composing $s$ with the projection to $\mathcal{L}$ recovers $F$:

$$\begin{tikzcd} \mathcal{O}_M \arrow[r, "s"'] \arrow[rr, bend left, "F"] & \mathcal{P}^1_M\mathcal{L} \arrow[r] & \mathcal{L} \end{tikzcd} \quad ;$$

hence $s$ induces a section of $\Omega^1_M \otimes \mathcal{L}$ on $X$, which is natural to name $dF$:

$$dF : \mathcal{O}_X \longrightarrow (\Omega^1_M \otimes \mathcal{L})|_X \quad ;$$

the subscheme $Y$ is the zero-scheme of $dF$ on $X$. It is easily checked that, locally, $dF$ is given by the partial derivatives of $F$ with respect to a set of local parameters for $M$; hence $Y$ is supported on the singular locus of $X$, justifying its name. Locally, we can write (abusing notations):

$$\mathcal{J}_{Y,M} = \left(F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right)$$

for the ideal or $Y$ in $M$; that is, the ideal of $Y$ in $M$ is the Jacobian ideal of $F$, extended by $F$. The ideal of $Y$ in $X$ will be denoted $\mathcal{J}_{Y,X}$; the ideal $\mathcal{J}_{X,M}$ of $X$ in $M$ will be denoted $(F)$ for short (as it is determined by the section $F$ of $\mathcal{L}$ giving $X$).

The blow-up of $X$ along $Y$ affords a realization of the projectivized conormal space of $X$ in $M$. In line with the theme of the paper, we state this (well-known) fact in terms of the *Rees algebra* of $\mathcal{J}_{Y,X}$ over $\mathcal{O}_X$, which we will denote $\mathrm{Rees}_X(Y)$:

**Proposition 2.1.** *The conormal cycle of $X$ in $M$ equals* $[\mathrm{Proj}(\mathrm{Rees}_X(Y))]$.

*Proof.* Recall that we are assuming that $X$ is reduced. The conormal space $T^*_X M$ of $X$ in $M$ is the closure in $T^*M$ of the kernels of the projection

$$(T^*M)_x \longrightarrow (T^*X)_x \longrightarrow 0$$

over nonsingular points $x$ of $X$. In other words, the projectivized conormal space of $X$ is the closure of the image of the section

$$X^{\mathrm{reg}} \longrightarrow \mathbb{P}(T^*M)|_X = \mathbb{P}(T^*M \otimes \mathcal{L})|_X$$

induced on the set $X^{\mathrm{reg}}$ of regular points of $X$ by the section $dF$ determined above. This closure is canonically isomorphic to the blow-up of $X$ along the zero-scheme of $dF$, that is, to the Proj of the Rees algebra of $\mathcal{J}_{Y,X}$. This yields the statement. □

2.7. Our guiding idea is that Proposition 2.1 ought to have a sibling statement realizing the *characteristic cycle* in terms of the *symmetric* algebra $\mathrm{Sym}_X(Y)$ of $\mathcal{J}_{Y,X}$. The best result we know in this direction is Corollary 2.6 below, which gives just such a statement, but subject to a technical condition on the hypersurface. The theorem in this subsection is a less clean, but more encompassing result.

For $d \geq 0$ we let

$$\mathrm{qSym}^d_X(Y) = \frac{\mathcal{J}^d_{Y,M}}{(F)\mathcal{J}^{d-1}_{Y,M}} \quad ;$$

note that $(F)\mathcal{J}^{d-1}_{Y,M} \subset \mathcal{J}^d_{Y,M}$ as $(F) \subset \mathcal{J}_{Y,M}$, and that $\mathrm{qSym}^d_X(Y)$ is an $\mathcal{O}_X$-module since $(F)\mathcal{J}^d_{Y,M} \subset (F)\mathcal{J}^{d-1}_{Y,M}$. We can then consider the $\mathcal{O}_X$-algebra

$$\mathrm{qSym}_X(Y) := \oplus_{d\geq 0}\, \mathrm{qSym}^d_X(Y) \quad .$$

Dualizing the basic section $s : \mathcal{O}_M \to \mathcal{P}^1_M\mathcal{L}$ gives a morphism $(\mathcal{P}^1_M\mathcal{L})^\vee \to \mathcal{O}_M$, surjecting onto $\mathcal{J}_{Y,M}$:

$$(\mathcal{P}^1_M\mathcal{L})^\vee \longrightarrow \mathcal{J}_{Y,M} \longrightarrow 0$$

and such that the composition with $\mathcal{L}^\vee \to (\mathcal{P}^1_M\mathcal{L})^\vee$ is the inclusion $(F) \subset \mathcal{J}_{Y,M}$. This yields a surjection

$$\mathrm{Sym}(\mathcal{P}^1_M\mathcal{L})^\vee|_X \longrightarrow \mathrm{qSym}_X(Y) \longrightarrow 0$$

factoring through $\mathrm{Sym}((\Omega^1_M \otimes \mathcal{L})^\vee|_X)$, and hence an embedding of $\mathrm{Proj}(\mathrm{qSym}_X(Y))$ into $\mathbb{P}(T^*M \otimes \mathcal{L})|_X = \mathbb{P}(T^*M)|_X$. Thus, $\mathrm{Proj}(\mathrm{qSym}_X(Y))$ determines a cycle in $\mathbb{P}(T^*M)|_X$.

**Theorem 2.2.** *The characteristic cycle of $X$ in $M$ equals* $(-1)^{\dim X}[\mathrm{Proj}(\mathrm{qSym}_X(Y))]$.

*Proof.* Consider the blow-up of $M$ along $Y$:

$$\pi : B\ell_Y M = \mathrm{Proj}(\mathcal{O}_M \oplus \mathcal{J}_{Y,M} \oplus \mathcal{J}^2_{Y,M} \oplus \cdots) \longrightarrow M \quad .$$

The surjection of $(\mathcal{P}^1_M\mathcal{L})^\vee$ onto $\mathcal{J}_{Y,M}$ gives an embedding

$$B\ell_Y M \subset \mathbb{P}(\mathcal{P}^1_M\mathcal{L}) \quad .$$

The subscheme $\pi^{-1}(X)$ is contained in

$$\mathbb{P}(T^*M) = \mathbb{P}(T^*M \otimes \mathcal{L}) \subset \mathbb{P}(\mathcal{P}^1_M\mathcal{L}) \quad ,$$

and contains the exceptional divisor $E$ in $B\ell_Y M$.

**Lemma 2.3.** *The characteristic cycle of $X$ in $M$ equals $(-1)^{\dim X}\cdot$ the cycle of the residual of $E$ in $\pi^{-1}(X)$.*

Over $\mathbb{C}$, this statement is Corollary 2.4 in [PP01]; for arbitrary algebraically closed fields of characteristic 0, it follows from Claim 2.1 in [Alu00].

Now the ideal of $\pi^{-1}(X)$ in $B\ell_Y M$ is the pull-back of $(F)$, that is

$$(F) \oplus (F)\mathcal{J}_{Y,M} \oplus (F)\mathcal{J}^2_{Y,M} \oplus (F)\mathcal{J}^3_{Y,M} \oplus \cdots \quad ;$$

therefore, the ideal of the residual is the shift

$$0 \oplus (F) \oplus (F)\mathcal{J}_{Y,M} \oplus (F)\mathcal{J}^2_{Y,M} \oplus \cdots \quad .$$

It follows that the residual of $E$ in $\pi^{-1}(X)$ is the Proj of

$$\mathcal{O}_M \oplus \frac{\mathcal{J}_{Y,M}}{(F)} \oplus \frac{\mathcal{J}^2_{Y,M}}{(F)\mathcal{J}_{Y,M}} \oplus \cdots \oplus \frac{\mathcal{J}^d_{Y,M}}{(F)\mathcal{J}^{d-1}_{Y,M}} \oplus \cdots$$

This algebra agrees with $\mathrm{qSym}_X(Y)$ in degree $\geq 1$, hence its Proj is isomorphic to $\mathrm{Proj}(\mathrm{qSym}_X(Y))$. Therefore, the statement of the theorem follows from Lemma 2.3. □

2.8. We will now identify a technical condition under which the algebra $\mathrm{qSym}_X(Y)$ is nothing but the symmetric algebra of $\mathcal{J}_{Y,X}$, which we will denote $\mathrm{Sym}_X(Y)$. As a consequence of Theorem 2.2, the characteristic cycle of hypersurfaces satisfying this condition is the cycle of the symmetric algebra of their singularity subscheme, as promised in §2.7. In §4 we will show that the condition is automatically verified in several representative situations.

As above, $F$ denotes the section of the line bundle $\mathcal{L}$ on $M$ whose zero-scheme is the hypersurface $X$. For the purpose of this discussion, a *homogeneous, degree $d$ differential operator satisfied by $F$* is a local section of $\mathrm{Sym}^d(\mathcal{P}^1_M\mathcal{L})^\vee$ mapping to 0 in $\mathcal{J}^d_{Y,M}$ via the map induced by the surjection $(\mathcal{P}^1_M\mathcal{L})^\vee \to \mathcal{J}_{Y,M}$ whose existence we pointed out in §2.7. In terms of local parameters $x_1, \ldots, x_n$ on $M$, this object is nothing but a homogeneous polynomial

$$P(T_0, \ldots, T_n)$$

with coefficients (local) functions on $M$, such that

$$P\left(F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right) \equiv 0 \quad ;$$

we will express the condition in this slightly imprecise but more vivid language, leaving to the reader the task of translating it into a global, coordinate-free formulation.

The simplest way to manufacture homogeneous differential operators of degree $d$ satisfied by $F$ is as a product

$$P = P_0 \cdot T_0 + P_1 \cdot T_1 + \cdots + P_n \cdot T_n \quad ,$$

where the $P_i$ are homogeneous polynomials of degree $d-1$ in $T_0, \dots, T_n$, and

$$P_0 \cdot F + P_1 \cdot \frac{\partial F}{\partial x_1} + \cdots + P_1 \cdot \frac{\partial F}{\partial x_n} = 0 \quad .$$

We say that such operators are *trivially* satisfied by $F$. The $\times$*-condition* on $X$ is a softening of this requirement, on operators of sufficiently high degree satisfied by $F$.

**Definition 2.4.** A hypersurface $X$ in $M$ *satisfies the* $\times$*-condition* if there exists a $d_0$ such that every homogeneous differential operator $P$ of degree $d \geq d_0$ and satisfied by $F$ can be written as

$$P = P_0 \cdot T_0 + P_1 \cdot T_1 + \cdots + P_n \cdot T_n \quad ,$$

where the $P_i$ are homogeneous polynomials of degree $d-1$ in $T_0, \dots, T_n$, and

$$P_1 \cdot \frac{\partial F}{\partial x_1} + \cdots + P_n \cdot \frac{\partial F}{\partial x_n} \in (F) \quad .$$

We will discuss this condition further in §4. In this subsection we concentrate on the rôle of the $\times$-condition in the determination of characteristic cycles.

**Proposition 2.5.** *If $X$ satisfies the $\times$-condition, then*

$$\mathrm{Sym}^d_X(Y) \cong \mathrm{qSym}^d_X(Y)$$

*for $d \gg 0$.*

*Proof.* Starting from the natural surjection $\mathrm{Sym}^d \mathcal{J}_{Y,M} \to \mathcal{J}^d_{Y,M}$, we build for $d \geq 1$ the diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_0 \cdot \mathrm{Tors}_{d-1} \arrow[r] \arrow[d] & \mathrm{Tors}_d \arrow[r] \arrow[d] & \mathrm{Disc}_d \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_0 \cdot \mathrm{Sym}^{d-1}\mathcal{J}_{Y,M} \arrow[r] \arrow[d] & \mathrm{Sym}^d \mathcal{J}_{Y,M} \arrow[r] \arrow[d] & \mathrm{Sym}^d(\mathcal{J}_{Y,M}/(F)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \cdot \mathcal{J}^{d-1}_{Y,M} \arrow[r] \arrow[d] & \mathcal{J}^d_{Y,M} \arrow[r] \arrow[d] & \mathcal{J}^d_{Y,M}/(F)\mathcal{J}^{d-1}_{Y,M} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The central column defines $\mathrm{Tors}_d$ by exactness. The left-most column is also exact, as $F$ is a non-zero-divisor. The term $\mathrm{Disc}_d$ is defined so as to fill the diagram.

We have to show that $\mathrm{Disc}_d = 0$ for $d \gg 0$ if and only if $X$ satisfies the $\times$-condition.

Now (cf. for example [Vas94], Chapter 2) $\mathrm{Tors}_d$ can be described as the space of degree-$d$ homogeneous operators satisfied by $F$, modulo those trivially satisfied by $F$. Hence

$$\mathrm{Disc}_d = \frac{\mathrm{Tors}_d}{T_0 \cdot \mathrm{Tors}_{d-1}}$$

is 0 if and only if every degree-$d$ homogeneous operator satisfied by $F$ is equivalent to a multiple of $T_0$ modulo trivial ones. That is, if and only if for every homogeneous $P$ of degree $d$ such that

$$P\left(F, \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right) \equiv 0$$

there exists a $Q$, homogeneous of degree $d-1$ and such that

$$P - T_0 \cdot Q = P_0 \cdot T_0 + P_1 \cdot T_1 + \dots + P_n \cdot T_n$$

with

$$P_0 \cdot F + P_1 \cdot \frac{\partial F}{\partial x_1} + \dots + P_n \cdot \frac{\partial F}{\partial x_n} = 0 \quad .$$

It is straightforward to verify that this latter condition is satisfied for $d \gg 0$ if and only if $X$ satisfies the $\times$-condition. □

**Corollary 2.6.** *If $X$ satisfies the $\times$-condition, then the characteristic cycle of $X$ in $M$ equals* $(-1)^{\dim X}[\mathrm{Proj}(\mathrm{Sym}_X(Y))]$.

*Proof.* If $X$ satisfies the $\times$-condition, then by Proposition 2.5 the algebras $\mathrm{Sym}_X(Y)$ and $\mathrm{qSym}_X(Y)$ are isomorphic in high enough degree, so they have the same Proj. The cycle of this Proj in $\mathbb{P}(T^*M)$ equals the characteristic cycle of $X$ by Theorem 2.2. □

2.9. Given its topological origins, it is not surprising that the Chern-Schwartz-MacPherson theory does not address the case in which the hypersurface $X$ may have multiple components, and we have assumed so in the preceding subsections for simplicity.

It should be noted, however, that $\mathrm{Proj}(\mathrm{Sym}_X(Y))$ is defined, and determines a cycle in $\mathbb{P}(T^*M)$, regardless of whether $X$ is reduced or not. The arguments given above can be traced in this case, and show that (if $X$ satisfies the $\times$-condition) this cycle is nothing but the characteristic cycle of the support $X_{\mathrm{red}}$. We leave the details to the interested reader (cf. §2.1 in [Alu99a]).

## 3. Shadows of blow-up algebras

3.1. In our view, there are two main advantages gained by translating the construction of the characteristic cycle of a variety into terms such as those explored in §2. To begin with, this yields an *intrinsic* description; no reference to the embedding of $X$ in an ambient nonsingular variety $M$ remains, other than offering a convenient place (that is, $\mathbb{P}(T^*M)$) where the characteristic cycle may be lodged. Secondly, the incarnation of the characteristic cycle as a the cycle of a projective Faserraum endows it of a rich structure that is not immediately accessible in other approaches. A projective Faserraum $P$ over $X$ is the *alter ego* of a coherent sheaf on $X$, which manifests itself in terms of a canonical line bundle on $P$, and hence determines a Segre class on $X$. How can this naturally defined class *not* be related to the beautifully natural Chern-Schwartz-MacPherson classes?

3.2. If $\mathcal{E}$ is a locally free sheaf of rank $e+1$ on a scheme $S$, there is a precise structure theorem for the Chow group of the projective bundle

$$\mathbb{P}(\mathcal{E}) := \mathrm{Proj}(\mathrm{Sym}\mathcal{E}^\vee) \xrightarrow{\epsilon} S$$

([Ful84], §3.3): every class $C \in A_r\mathbb{P}(\mathcal{E})$ can be written uniquely as

$$C = \sum_{j=0}^{e} c_1(\mathcal{O}(1))^j \cap \epsilon^*(C_{r-e+j})$$

where $\mathcal{O}(1)$ denotes the tautological line bundle on $\mathbb{P}(\mathcal{E})$, and $C_{r-e+j} \in A_{r-e+j}S$.

Therefore, knowledge of $C$ is equivalent to knowledge of the collection of $e+1$ classes $C_{r-e}, \dots, C_r$ on $S$.

**Definition 3.1.** We say that the class $C_{r-e} + \cdots + C_r \in AS$ is the *shadow* of the class $C$.

As its real world namesake, the shadow neglects some of the information carried by the object that casts it. For example, $c_1(\mathcal{O}(1))^j \cdot [\mathbb{P}(\mathcal{E})]$ has shadow $[S]$ for all $j = 0, \dots, e$. However, a pure-dimensional class $C$ can be reconstructed from its shadow if its dimension is known, as follows immediately from the structure theorem recalled above.

It will be convenient to have a direct way to obtain the shadow of a given class.

**Lemma 3.2.** *The shadow of $C$ is the class*

$$c(\mathcal{E}) \cap \epsilon_* \left(c(\mathcal{O}(-1))^{-1} \cap C\right)$$

*Proof.* Writing $C$ as above, we have

$$\begin{aligned} c(\mathcal{E}) \cap \epsilon_* \left(c(\mathcal{O}(-1))^{-1} \cap C\right) &= c(\mathcal{E}) \cap \epsilon_* \left(c(\mathcal{O}(-1))^{-1} \cap \sum_{j=0}^{e} c_1(\mathcal{O}(1))^j \cap \epsilon^*(C_{r-e+j})\right) \\ &= \sum_{j=0}^{e} c(\mathcal{E}) \cap \epsilon_* \left(\sum_{k\geq j} c_1(\mathcal{O}(1))^k \cap \epsilon^*(C_{r-e+j})\right) \quad . \end{aligned}$$

Since $c_1(\mathcal{O}(1))^k \cap \epsilon^*\alpha = 0$ for $0 \leq k < e$ and any $\alpha \in A_*S$, this says

$$c(\mathcal{E}) \cap \epsilon_* \left(c(\mathcal{O}(-1))^{-1} \cap C\right) = \sum_{j=0}^{e} c(\mathcal{E}) \cap \epsilon_* \left(c(\mathcal{O}(-1))^{-1} \cap \epsilon^*(C_{r-e+j})\right) \quad .$$

Finally, this equals $\sum_{j=0}^{e} C_{r-e+j}$ by [Ful84], Example 3.3.3. □

3.3. As recalled in §2.4, MacPherson's natural transformation $c_*$ can be expressed by a two-step procedure: $(\mathcal{C} \rightsquigarrow \mathcal{L})$ taking the characteristic cycle $\mathrm{Ch}(\varphi)$ of a constructibile function $\varphi$, and $(\mathcal{L} \rightsquigarrow \mathcal{A})$ extracting a rational equivalence class from the characteristic cycle. As the natural habitat of Lagrangian cycles is the projectivized *co*tangent bundle $\mathbb{P}(T^*M)$, we find it convenient to arrange things so as to obtain a class $\check{c}_*(\varphi)$ differing from $c_*(\varphi)$ by the sign of the components of odd dimension:

$$\{\check{c}_*(\varphi)\}_r = (-1)^r \{c_*(\varphi)\}_r$$

in dimension $r$. For example,

$$\check{c}_*(1\!\!1_M) = (-1)^{\dim M} c(T^*M) \cap [M]$$

for the nonsingular ambient $M$.

**Lemma 3.3.** *The class $\check{c}_*(\varphi)$ is the shadow of the characteristic cycle* $\mathrm{Ch}(\varphi)$.

*Proof.* This is formula (12) on p. 67 of [PP01], filtered through Lemma 3.2. As observed in [PP01], this is in agreement with [Mac74]. □

The statement of Lemma 3.3, while implicit in the existing literature, is mysteriously absent in this explicit form relating the transformation $\mathcal{L} \rightsquigarrow \mathcal{A}$ to the structure theorem of the Chow group of projective bundles. This interpretation streamlines the proof that $\mathcal{L} \rightsquigarrow \mathcal{A}$ is a natural transformation; Jörg Schürmann has independently made the same observation [Sch01b].

3.4. We are ready to justify the title of this article. Denote by $\check{c}_{\mathrm{Ma}}(X)$, $\check{c}_{\mathrm{SM}}(X)$ respectively the classes obtained by changing the sign of the components of odd dimension in $c_{\mathrm{Ma}}(X)$, $c_{\mathrm{SM}}(X)$.

**Theorem 3.4.** *Let $X$ be a hypersurface of a nonsingular variety $M$, and let $Y$ be its singularity subscheme. Denote by $\mathrm{Rees}_X(Y)$, $\mathrm{Sym}_X(Y)$ respectively the Rees and symmetric algebras of the ideal $\mathcal{J}_{Y,X}$; these determine cycle classes $[\mathrm{Proj}(\mathrm{Rees}_X(Y))]$, $[\mathrm{Proj}(\mathrm{Sym}_X(Y))]$ in $\mathbb{P}(T^*M)$. Then*

- *the shadow of $[\mathrm{Proj}(\mathrm{Rees}_X(Y))]$ is $(-1)^{\dim X}\check{c}_{\mathrm{Ma}}(X)$;*
- *if $X$ satisfies the $\times$-condition, then the shadow of $[\mathrm{Proj}(\mathrm{Sym}_X(Y))]$ is $(-1)^{\dim X}\check{c}_{\mathrm{SM}}(X)$.*

*Proof.* This now follows from Proposition 2.1, Corollary 2.6, and Lemma 3.3. □

By Theorem 2.2, the extra hypothesis on $X$ in the second statement presented here can be removed, at the price of replacing $[\mathrm{Proj}(\mathrm{Sym}_X(Y))]$ by the less appealing $[\mathrm{Proj}(\mathrm{qSym}_X(Y))]$.

3.5. The careful reader should have noticed that we have not quite fulfilled the promise made in §1 (for the particularly friendly hypersurfaces we are considering) of providing a unified treatment of Chern-Schwartz-MacPherson classes and classes such as Fulton-Johnson's.

The latter are defined by capping the Chern class of the tangent bundle of an ambient space against a class (typically a Segre class) recording all the relevant information about the embedding. As we have tried to stress, the standard approaches to the theory of Chern-Schwartz-MacPherson classes have a definitely different flavor; formulas such as (12) in [PP01] provide a point of contact between the two mind-sets.

Theorem 3.4 makes one further step in the direction of unifying these different philosophies, in that the shadow of a blow-up algebra is obtained (via Lemma 3.3) by capping the Chern classes of a bundle on the ambient space (the cotangent bundle in our case) against a class obtained by exploiting directly the given embedding. However, the situation is still somewhat unsatisfactory, as we have obtained this latter invariant by realizing the characteristic cycle *of a hypersurface* $X$ using in a rather essential way the knowledge of the *normal bundle* $\mathcal{L}$ of $X$: this knowledge is necessary in order to rewrite $[\mathrm{Proj}(\mathrm{Sym}_X(Y))]$, which lives naturally in $\mathbb{P}(T^*M \otimes \mathcal{L})$, as a class in $\mathbb{P}(T^*M)$.

We will now address this point, and obtain a formula reproducing the shadow of the symmetric algebra considered above, but in terms independent of the nature of $X$. The second result in Theorem 3.4 can then be read to state that this formula gives the Chern-Schwartz-MacPherson class of $X$ when $X$ is a hypersurface satisfying the $\times$-condition.

For this operation we define a Segre-class type of invariant, as follows.

Let $Z$ be a subscheme of a nonsingular variety $M$; we assume that $\dim M > \dim Z$. There is a surjection

$$\Omega^1_M|_Z \longrightarrow \Omega^1_X \longrightarrow 0 \quad ,$$

from which we obtain the exact sequence

$$0 \longrightarrow \mathcal{H}om(\Omega^1_Z, \mathcal{O}_Z) \longrightarrow \mathcal{H}om(\Omega^1_M|_Z, \mathcal{O}_Z) \longrightarrow \mathcal{T}_Z M \longrightarrow 0 \quad ,$$

defining the coherent sheaf $\mathcal{T}_Z M$ on $Z$. If $Z$ is nonsingular, then $\mathcal{T}_Z M$ is locally free, and in fact it is the normal bundle of $Z$ in $M$.

Denote by $\pi$ the projection $\mathrm{Proj}(\mathrm{Sym}(\mathcal{T}_Z M)) \to Z$, and by $\mathcal{O}(1)$ the tautological bundle defined on this Proj. The restriction on the dimension of $M$ is made so that the Proj has nonempty fiber over every point of $Z$; alternately, one could add a trivial factor to $\mathcal{T}_Z M$, with the same effect.

**Definition 3.5.** With notations as above, define the *linear checked Segre class*[1] of $Z$ in $M$ to be

$${}^{\ell}\check{s}(Z, M) := \sum_{j\geq 0} c_1(\mathcal{O}(1))^j \cap [\mathrm{Proj}(\mathrm{Sym}\mathcal{T}_Z M)] \quad ,$$

and the *linear Segre class* of $Z$ in $M$ to be

$${}^{\ell}s(Z, M) := \sum_{r\geq 0} (-1)^{\dim Z - r} \{{}^{\ell}\check{s}(Z, M)\}_r \quad ,$$

where $\{\,\}_r$ denotes the component of dimension $r$.

In other words, ${}^{\ell}\check{s}(Z, M)$ is defined as the Segre class of the coherent sheaf $\mathcal{T}_Z M$ (in the sense of [Ful84], Example 4.1.7); the class ${}^{\ell}s(Z, M)$ is obtained from the latter by changing the sign of components of every other codimension in $Z$.

The terminology 'linear' is meant to suggest that this notion should be a 'linear version' of the more conventional, and more refined, notion of Segre class of an embedding (cf. [Ful84], Chapter 4). Both classes pick up data normal to $Z$: the standard Segre class does so starting from the normal *cone* of $Z$ in $M$, which is a nonlinear element of information; our class starts similarly from a coherent sheaf $\mathcal{T}_Z M$, thus a *linear* element of information, obtained essentially (but not quite) by dualizing the conormal sheaf $\mathcal{N}_Z M = \mathcal{J}_Z/\mathcal{J}_Z^2$.

*Example* 3.6. The reader should have no difficulties verifying that if $Z$ is nonsingular and irreducible in $M$, then ${}^{\ell}s(Z, M) = s(Z, M)$.

3.6. The guiding principle behind Fulton's and Fulton-Johnson's class lead us to define a new characteristic class.

Let $Z$ be a scheme embedded in a nonsingular variety $M$; as above, we assume that $\dim M > \dim Z$.

**Definition 3.7.** The *linear Chern class* of $Z$ is the class

$${}^{\ell}c(Z) = c(TM|_Z) \cap {}^{\ell}s(Z, M)$$

in the Chow group $AZ$ of $Z$.

This class is intrinsic of the scheme $Z$, in the sense that it does not depend on the ambient nonsingular variety $M$; the verification of this fact can be found in [Alu01].

[1] The leaner term *coSegre class* is preempted by its use in [Kaw00], with a different meaning.

*Example* 3.8. If $Z$ is nonsingular and irreducible, then ${}^\ell c(Z) = c(TZ) \cap [Z]$ (cf. Example 3.6).

**Theorem 3.9.** *If $X$ is a hypersurface satisfying the $\times$-condition, then the linear class of $X$ equals its Chern-Schwartz-MacPherson class.*

*Proof.* By definition, there is a surjection

$$(\Omega^1_M|_Z)^\vee \longrightarrow \mathcal{T}_Z M \longrightarrow 0$$

for any nonsingular variety $M$ containing $Z$. Thus $\mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_Z}(\mathcal{T}_Z M))$ is a subscheme of $\mathbb{P}(T^*M)$, and hence it determines a cycle $[\mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_Z}(\mathcal{T}_Z M))]$ in $\mathbb{P}(T^*M)$. We will show that, for a hypersurface $X$ satisfying the $\times$-condition, (i) the cycle $(-1)^{\dim X}[\mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{T}_X M))]$ is the characteristic cycle of $X$; and that (ii) the linear class of $X$ computes the (checked) shadow of this cycle. The statement will then follow from Lemma 3.3.

(i) For $X$ is a hypersurface, with ideal $\mathcal{J} = \mathcal{J}_{X,M}$, we have the exact sequence of differentials

$$0 \longrightarrow \mathcal{J}/\mathcal{J}^2 \longrightarrow \Omega^1_M|_X \longrightarrow \Omega^1_X \longrightarrow 0 \quad .$$

Taking $\mathcal{H}om(-,\mathcal{O}_X)$ gives the exact sequence

$$0 \longrightarrow \mathcal{H}om(\Omega^1_X,\mathcal{O}_X) \longrightarrow \mathcal{H}om(\Omega^1_M|_X,\mathcal{O}_X) \longrightarrow \mathcal{H}om(\mathcal{J}/\mathcal{J}^2,\mathcal{O}_X) \quad .$$

A local computation determines the image of the rightmost map as the subsheaf of homomorphisms

$$\mathcal{J}/\mathcal{J}^2 \longrightarrow \mathcal{O}_X$$

factoring through $\mathcal{J}_{Y,X}$. In other words, if $X$ is a hypersurface in $M$ then

$$\mathcal{T}_X M = \mathcal{H}om(\mathcal{J}/\mathcal{J}^2, \mathcal{J}_{Y,X}) = \mathcal{L}^\vee \otimes \mathcal{J}_{Y,X} \quad .$$

It follows that

$$\mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{T}_X M)) = \mathrm{Proj}(\mathrm{Sym}_X(Y)) \quad ;$$

by Corollary 2.6, this shows that $(-1)^{\dim X}[\mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{T}_X M))]$ is the characteristic cycle of $X$ if $X$ satisfies the $\times$-condition, as claimed.

(ii) By Lemma 3.2 and Theorem 3.4, the shadow of this cycle equals

$$\check{c}_{\mathrm{SM}}(X) = (-1)^{\dim X} c(T^*M) \cap \pi_*(c(\mathcal{O}(-1))^{-1} \cap [\mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{T}_X M))]) \quad ,$$

where $\pi$ is the projection $\mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{T}_X M)) \to X$, and $c(\mathcal{O}(-1))$ is the restriction of the universal line subbundle of $T^*M$. However, this latter now equals the universal line subbundle on $\mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{T}_X M))$, so

$$\check{c}_{\mathrm{SM}}(X) = (-1)^{\dim X} c(T^*M) \cap {}^\ell\check{s}(X,M) \quad ,$$

which was our second claim.

Unraveling the signs, $c_{\mathrm{SM}}(X) = c(TM) \cap {}^\ell s(X,M)$, as was to be shown. □

3.7. At this point it is only too natural to pose the problem *to characterize schemes $Z$ for which $(-1)^{\dim Z}[\mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_Z}(\mathcal{T}_Z M)]$ yields a realization of the characteristic cycle of $Z$ in $M$,* and hence whose linear class agrees with the Chern-Schwartz-MacPherson class. The main result of this article is that this *is* the case for hypersurfaces satisfying the technical condition of §2.8 (and of course for all nonsingular irreducible varieties). Examples such as the one in §4.5 leave us skeptical as to the possibility that this equality may hold for arbitrary irreducible varieties.

## 4. The $\times$-files

4.1. The '$\times$' in $\times$-*condition* has been chosen as it reminds us of the prototypical singularities satisfying it: a nodal curve is (i) a hypersurface with a nonsingular singularity subscheme; (ii) a hypersurface with 'quasi-homogeneous' isolated singularities; and (iii) a normal crossing divisor. In this section we show that each of these classes of hypersurfaces satisfies the $\times$-condition. We also comment briefly on relations between the $\times$-condition and other conditions on hypersurface singularities.

4.2. Our proofs that given classes satisfies the $\times$-condition all hinge on the following observation.

Recall ([Kee93]) that an embedding $S \subset T$ is *weakly linear* if the Rees algebra and the symmetric algebra of the ideal of $S$ in $T$ are isomorphic in high degree, that is, if $\mathrm{Proj}(\mathrm{Sym}_T(S))$ is isomorphic to the (Rees) blow-up of $T$ along $S$.

As in the rest of the paper, $X$ is a hypersurface in a nonsingular variety $M$, and $Y$ denotes its singularity subscheme.

**Proposition 4.1.** *If the embedding of $Y$ in $M$ is weakly linear, then $X$ satisfies the $\times$-condition.*

*Proof.* With the notations in the proof of Proposition 2.5, the embedding of $Y$ in $M$ is weakly linear if and only if $\mathrm{Tors}_d = 0$ for $d \gg 0$. This implies $\mathrm{Disc}_d = 0$ for $d \gg 0$, which is equivalent to the $\times$-condition, as observed in that proof. □

This implies immediately the $\times$-condition for the first case listed in §4.1:

**Proposition 4.2.** *If the singularity subscheme of $X$ is nonsingular, then $X$ satisfies the $\times$-condition.*

*Proof.* If $Y$ is nonsingular, then it is regularly embedded in $M$, hence the embedding $Y \subset M$ is linear (by [Mic64]), hence weakly linear. □

We should remind the reader that the requirement that the singularity *subscheme* of a hypersurface be nonsingular is very strong; substantially stronger, for example, than the requirement that the singularity *locus* be nonsingular. Some constraints on this situation are studied in [Alu95], §3. Hypersurfaces whose singularity subscheme is nonsingular are in particular *nice* in the sense of [AB01].

The other classes of hypersurfaces listed in §4.1 also provide examples for which the embedding of $Y$ in $M$ is linear, and hence *a fortiori* weakly linear, as we show in the next two subsections.

4.3. A hypersurface $X$ is *quasi-homogeneous* at a point $p$ if, for a choice of local parameters $x_1, \dots, x_n$ for $M$ at $p$, the local generator $F$ for the ideal of $X$ is a linear combination of monomials $x_1^{m_1} \cdots x_n^{m_n}$ satisfying $r_1 m_1 + \cdots + r_n m_n = 1$, for a fixed $n$-tuple of rational numbers $(r_1, \dots, r_n)$ with $0 \le r_i \le \frac{1}{2}$.

**Proposition 4.3.** *If $X$ has only quasi-homogeneous, isolated singularities, then $X$ satisfies the $\times$-condition.*

*Proof.* As the matter is local, we may assume that $p$ is the only singularity on $X$; thus $Y$ is supported on a single point $p$. If $X$ is quasi-homogeneous at $p$, then it is immediately verified that

$$F \in \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right) \quad \text{and hence} \quad \mathcal{J}_{Y,M} = \left(\frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}\right) \quad .$$

It follows that $Y$ is a complete intersection in $M$, hence $Y \subset M$ is linear, and hence weakly linear. Thus $X$ satisfies the $\times$-condition by Proposition 4.1. $\square$

4.4. Normal crossing divisors provide perhaps the most interesting example.

**Proposition 4.4.** *If $X$ is a divisor with normal crossings and nonsingular components, then $X$ satisfies the $\times$-condition.*

*Proof.* As the $\times$-condition is a local property, we may assume that $M = \operatorname{Spec} R$ is affine, with $R$ regular, and $X$ has ideal $(F)$ with $F = x_1^{m_1} \cdots x_r^{m_r}$, where $m_i > 0$ and $x_1, \dots, x_r$ form part of a system of parameters for $M$ at each point of $X$.

Then $Y$ has ideal $J = \left(\frac{F}{x_1}, \dots, \frac{F}{x_r}\right)$ in $M$.

**Lemma 4.5.** *The embedding of $Y$ in $M$ is linear.*

*Proof.* Let $\mathfrak{q}$ denote the ideal of linear relations on the generators of the ideal $J$ of $Y$, that is, $\mathfrak{q}$ is generated by all

$$\sum_{i=1}^{r} a_i T_i \in R[T_1, \dots, T_r]$$

with $a_i \in R$ such that $\sum a_i \frac{F}{x_i} = 0$. Then we claim that

$$\mathfrak{q} = (x_1 T_1 - x_2 T_2, \dots, x_{r-1} T_{r-1} - x_r T_r) \quad .$$

Indeed, the inclusion $\supset$ is trivial; for $\subset$, consider a linear relation as above:

$$\sum_{i=1}^{r} a_i \frac{F}{x_i} = 0 \quad .$$

Reading this equality mod $x_i^{m_i}$ shows that $x_i | a_i$ for $i = 1, \dots, r$ (as $R$ and $R/(x_i)$ are integral). Writing $a_i = x_i b_i$, the equality implies $\sum b_i = 0$; hence,

$$(b_1, \dots, b_r) = \sum_{i=1}^{r-1} b_i' \cdot (\dots, 1, -1, \dots) \quad \in R^{\oplus r}$$

with $b_i' \in R$, where the entries $1$, $-1$ are at the $i$th, resp. $(i+1)$st place, and all other entries on the right are 0. From this,

$$a_1 T_1 + \dots + a_r T_r = \sum_{i=1}^{r-1} b_i' \cdot (x_i T_i - x_{i+1} T_{i+1}) \in (x_1 T_1 - x_2 T_2, \dots, x_{r-1} T_{r-1} - x_r T_r)$$

as needed.

Now $\mathfrak{q}$ is the ideal of the symmetric algebra of $J$ in $R[T_1, \dots, T_r]$. Since $\mathfrak{q} = (x_1 T_1 - x_2 T_2, \dots, x_{r-1} T_{r-1} - x_r T_r)$ as checked, and this latter is a prime ideal, it follows that the symmetric algebra of $J$ is integral. This implies that $J$ is of linear type, by [Mic64], Proposition 1; so $Y \subset M$ is linear, as claimed. $\square$

The statement of the proposition follows from this lemma, by Proposition 4.1. $\square$

4.5. In view of the cases analyzed in the previous subsections, and of the application to characteristic classes, it is natural to pose the problem of *characterizing the hypersurfaces satisfying the* $\times$*-condition.* The following example hints that the answer to this problem may not be so simple.

For ease of terminology, say that a hypersurface $X$ satisfies the *strong* $\times$-condition if the embedding of its singularity subscheme $Y$ in the ambient $M$ is linear; the types of hypersurfaces studied in §§4.2, 4.3, and 4.4 all satisfy the strong $\times$-condition.

*Example* 4.6. The plane curve $x^4 + x^3y^2 + y^6 = 0$ has an isolated singularity at the origin, and does not satisfy the strong $\times$-condition there.

This is checked by explicit computations, which we performed with `Macaulay 2`, [GS]. In a sense (cf. §4.3), the root of the problem lies in the fact that the Milnor and Tyurina numbers of this singularity *differ.* It may in fact very well be that this curve does *not* satisfy the $\times$-condition at the origin.

4.6. We close with the following remarks, which may clarify the place of the $\times$-condition against the backdrop of better-known conditions on the singularities of a hypersurface.

For the purpose of this discussion assume $M$ is affine, $\mathcal{L} = \mathcal{O}_M$, and for $q \in M$ let $X_q$ denote the hypersurface defined by the ideal $(F - F(q))$; that is, the *level hypersurface* through $q$. The *distinguished stratification* of $X$ is the one for which the closures of the strata are $X$ itself, and the supports of the components of the normal cone of the singularity subscheme $Y$ in the ambient variety $M$; this is a Whitney stratification.

The bundle of principal parts $\mathcal{P}^1_M\mathcal{L}$ has a standard trivialization, described fiberwise so that vectors over $p \in M$ correspond to pairs $(\omega, c)$, where $\omega \in T^*_pM$ and $c$ is a scalar. The basic section $s$ defined in §2.6 is given by $(d_pF, F(p))$ over $p$, and spans a line for $p \notin Y$. Denote by $\mathbb{M}$ the closure of the union of these lines; thus, $\mathbb{M}$ is a 'principal part analog' of the more conventional *relative conormal space* of $F$ (cf. for example [BMM94], §2). Accordingly, fiberwise linear functions on $\mathcal{P}^1_M\mathcal{L}$ vanishing along $\mathbb{M}$ near $p$ are analog to vector fields on $M$ defined near $p$ and vanishing along the relative conormal space; that is, to vector fields $\mathbf{v}$ defined near $p$ and such that $\mathbf{v}(q)$ is tangent to the level hypersurface $X_q$ for $q \notin X$. For lack of better terminology, we refer to fiberwise linear functions on $\mathcal{P}^1_M\mathcal{L}$ as *jet fields,* and say that a jet is *tangent to* $X_q$ at $q$ if it vanishes along the fiber of $\mathbb{M}$ over $q$.

Note also that, by projecting to the $T^*_pM$ component, a tangent vector $v \in T_pM$ determines a jet at $p$.

**Proposition 4.7.** *Assume $X$ satisfies the strong $\times$-condition, $p \in X$, and $v$ is a vector tangent to the distinguished stratum containing $p$. Then $v$ extends to a holomorphic jet field near $p$, tangent to $X_q$ for $q \notin Y$ near $p$.*

*Proof of Proposition 4.7.* We may assume $p \in Y$, and $v \neq 0$.

By construction $\mathbb{M}$ equals the Spec of the Rees algebra of the ideal of $Y$ in $M$; thus the normal cone to $Y$ in $M$ is contained in $\mathbb{M}$, as the Spec of the associated graded ring to the ideal of $Y$. If $S$ is the distinguished stratum containing $p$, the fiber $\mathbb{M}_p$ of $\mathbb{M}$ over $p$ is the fiber of the component of the normal cone of $Y$ corresponding to $S$.

Seeing $v$ as a jet, it defines a hyperplane in the fiber of $\mathcal{P}^1_M\mathcal{L}$ over $p$, containing $\mathbb{M}_p$ since $v$ is tangent to $S$.

Now we use the fact that $X$ satisfies the strong $\times$-condition: this says that the Rees algebra of the ideal of $Y$ in $M$ equals its symmetric algebra; thus, the ideal of $\mathbb{M}$ is generated by fiberwise linear functions on $\mathcal{P}^1_M\mathcal{L}$, that is, by jet fields. The specialization of these jet fields to the fiber over $p$ cut out $\mathbb{M}_p$; hence one of them must agree with $v$ over $p$. This jet field satisfies the requirement in the statement. $\square$

It is known that, under suitable hypotheses, vectors tangent to strata of a stratification of $X$ may extend to vector fields tangent to level hypersurfaces and satisfying some regularity condition. For example, for complex hypersurfaces such extensions may be chosen to be real-analytic and 'rugose', provided that $X$ is (c)-equisingular along the stratum ([Tei77], p. 602). This issue is closely tied to the *$w_f$-condition of Thom,* which guarantees that tangent vectors to strata of a Whitney stratification of $X$ are suitably 'close' to the tangent spaces of nearby level hypersurfaces; extensions can be obtained by projecting the vector to these tangent spaces. The $w_f$-condition is known to hold for every complex Whitney stratification of every hypersurface ([Par93], [BMM94]).

In view of Proposition 4.7, the (strong) $\times$-condition may be viewed as a strong regularity requirement on extensions of tangent vectors near strata of a Whitney stratification of $X$. This suggests that the $\times$-condition may be equivalent to a $w_f$-type condition, and that the techniques in [Par93] or [BMM94] may be apt to establishing the precise range of its validity.

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Max-Planck-Institut für Mathematik, Bonn, Germany

Florida State University, Tallahassee, Florida